Normal, Gaussian and Mean Curvature

Curvature is the amount by which a geometric surface deviates from being flat. Different definitions of curvature exist, the most common being normal, gaussian and mean curvature. All three curvature values are defined locally for a mesh at every vertex and are determined from the principal direction vectors that define the plane tangent to a vertex.

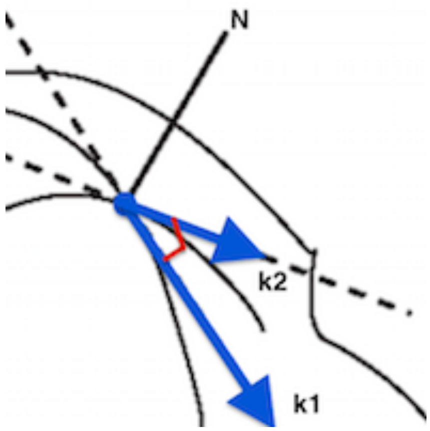


Figure 1: Principal Curvatures

Given the principal directions with maximum curvature k_1 and minimum curvature k_2, normal curvature is defined as linear combination of the two:

$$\mathbf{k_n(v) = k_1 \cos^2ø + k_2 \sin^2ø}$$

(where ø is an offset from the first principal direction) while mean curvature is defined as the arithmetic mean:

$$\mathbf{H(v) = (k_1 + k_2) / 2}$$

and gaussian curvature is defined as the geometric mean:

$$\mathbf{K(v) = k_1\ k_2}$$

Gaussian curvature is often used to classify surface points into three categories: elliptical points ($K > 0$) indicating local convexity, hyperbolic points ($K < 0$) indicating saddle-shaped surfaces, and parabolic points ($K = 0$) separating elliptical and hyperbolic regions. Furthermore, it is an intrinsic measure of curvature that depends only on distances that are measured on the surface and is invariant under isometry. For example, given a flat sheet of paper, the gaussian curvature of the sheet remains zero when it is folded into a cylinder. Normal and mean curvature however are extrinsic properties of the surface that depends on the surface's embedding in space.

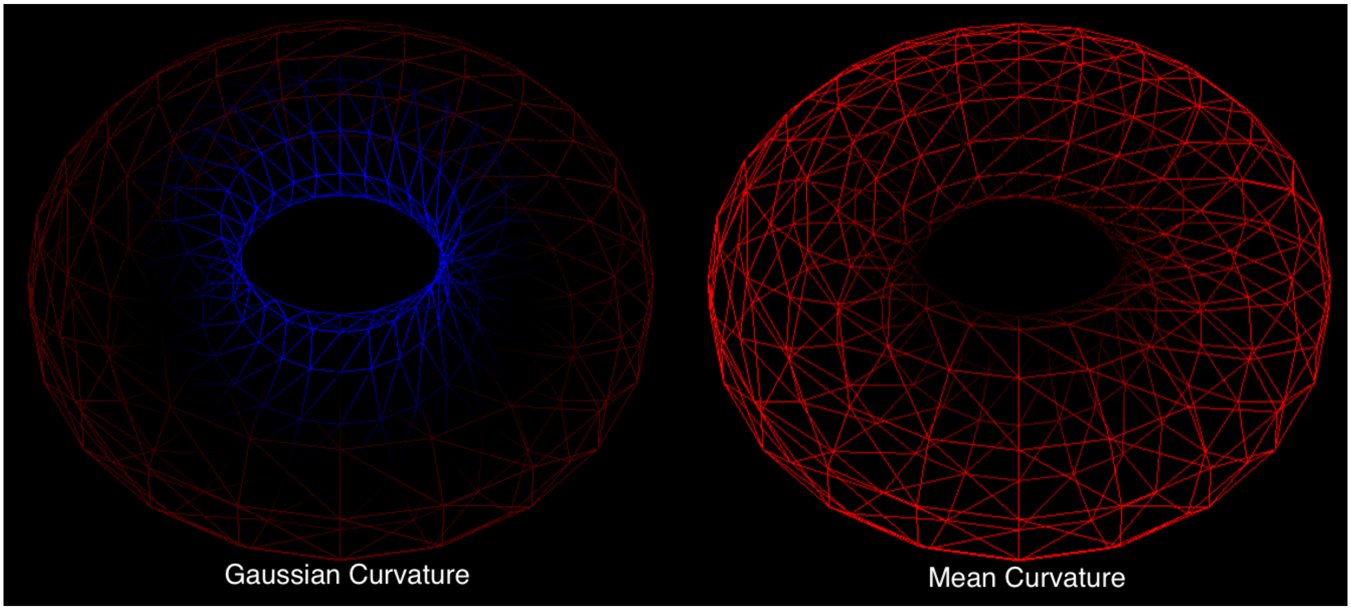


Figure 2: Red indicates positive curvature and blue indicates negative curvature

When the principal directions are not provided, as is generally the case, it is possible to describe the gaussian curvature in a neighborhood around the vertex using the angle defect formula:

$$\mathbf{K(v) = (2\pi - \textstyle\sum_{N(v)} ø_i) / A}$$

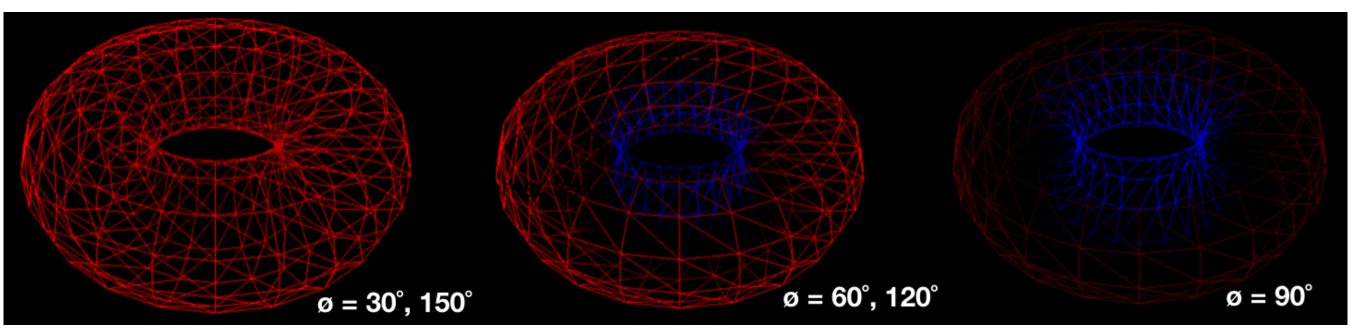
where $ø_i$s denote the angles of the incident triangles at vertex v and A is the barycentric area of the dual cell of v. The absolute mean curvature at vertex v can be computed using the Laplace Beltrami operator:

$$\mathbf{H(v) = |\Delta v| / 2 = | \textstyle\sum_{N(v)} (\cot a_i + \cot \beta_i) (v - v_i) / 2A |}$$

Finally, the principal curvatures can be computed using the relation:

$$\mathbf{k_{1,2} = H(v) +/- \mathrm{sqrt}\, (H(v)^2 - K(v))}$$

and can be used to find the normal curvature.



Implementation: https://github.com/rohan-sawhney/curvature