Vector Field Design

The goal of vector field design is to describe the smoothest manner in which tangent vectors change when moved from one point to another on discrete surfaces. The changes are required to be independent of the path taken by the vectors, i.e., vectors transported around a loop should end up where they started. The motivation of designing such smoothly varying vector fields stems from applications in texture synthesis and quadrilateral remeshing.

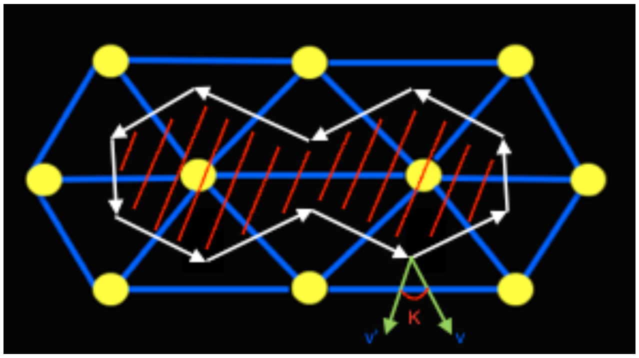

Figure 1: Angle defect of the Levi Civita connection equals the curvature around a loop of dual edges

Crane et al. introduced the notion of “trivial connections” in [1] to design vector fields that are as smooth as possible everywhere except at a prescribed set of singularities. These connections describe how tangent vectors change when moved along a surface such that the difference in angle of the initial and final vectors v and v' when transported along a closed loop is zero. In contrast to the Levi-Civita connections that twist tangent vectors as little as possible but still result in non zero angles around loops (Figure 1), trivial connections define adjustment angles on the dual edges of a mesh to compensate for Levi-Civita's angle defects. The algorithm in [1] computes these adjustment angles to describe how a vector should be rotated when moved across an edge:

1. Find the Basis Cycles: A cycle is a sequence of consistently oriented dual edges that form a loop. A cycle is contractible if it can be continuously contracted to a point and non contractible if it cannot. On a surface of genus g, there are 2g non contractible cycles which can be constructed using the tree-cotree decomposition in [2]. The contractible and non contractible cycles are combined into a matrix A where:

$$A = \begin{bmatrix} d_0{}^T \\ H^T \end{bmatrix} \qquad (d_0)_{ij} = \begin{cases} \pm 1 \text{ if dual edge i is contained in dual cell j} \\ 0 \end{cases} \qquad H_{ij} = \begin{cases} \pm 1 \text{ if dual edge i is non contractible cycle j} \\ 0 \end{cases}$$

2. Compute the Angle Defects: The angle defect of contractible cycles equals the gaussian curvature in the neighborhood around vertices. For non-contractible cycles, the angle defect is computed using the relation:

$$\boldsymbol{\alpha_j = \alpha_i - \theta_{ij} + \theta_{ji}}$$

where α_i is the initial angle in face i, α_j is the new angle in the neighboring face j and $\theta_{ij} + \theta_{ji}$ are the angles between the shared edge e and the fixed reference direction in face i and j. The vector K is used to store the angle defects around contractible cycles and z is used to store the angle defects around non contractible cycles.

3. Set the Singularities: Singularities control the global appearance of the direction field. They can be assigned any arbitrary value k_i as long as $\boldsymbol{\sum_i k_i}$ **= Euler Characteristic** of the mesh. The singularities modify the angle defects around the basis cycles such that $K_i = K_i - 2\, k_i\, \pi$ and $z_i = z_i - 2\, k_i\, \pi$. The modified defects are stored in a single vector $b = [K\ z]^T$

4. Solve for Adjustment Angles: The adjustment angles are computed by solving the convex problem

$$\min_x \|x\|_2 \ \text{ s.t. } Ax = -b$$

using Sparse QR factorization.

5. Construct the Direction Fields: Starting with an arbitrary face i and initial angle α_i, the angle α_j of the field on face j across each edge is computed using:

$$\alpha_j = \alpha_i - \theta_{ij} + \theta_{ji} - x_k$$

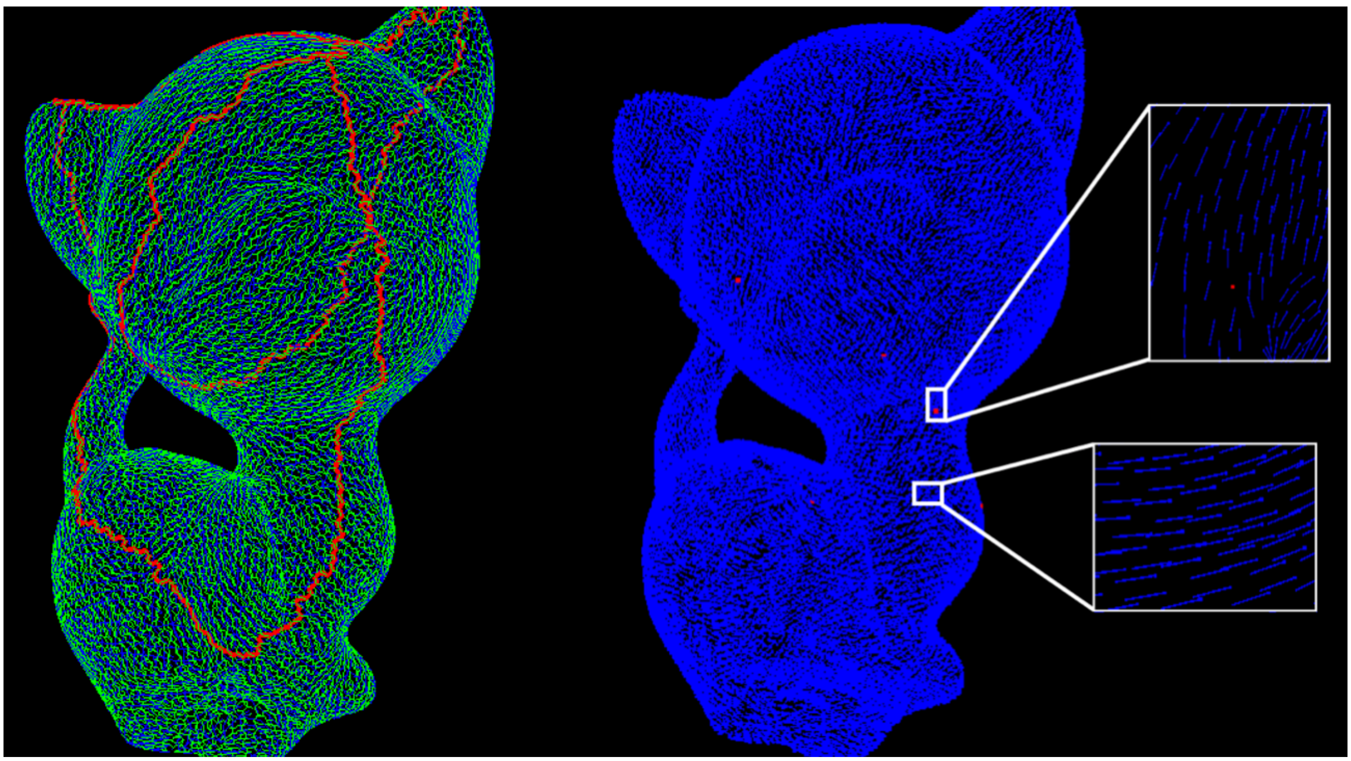

Figure 2: *Left:* Tree-Cotree decomposition of model. The kitten has two non contractible cycles highlighted in red. Right: Smooth direction field with a prescribed set of singularities

Implementation: https://github.com/rohan-sawhney/direction-fields

[1] Crane et al. Trivial Connections on Discrete Surfaces
[2] Eppstein. Dynamic Generators of Topologically Embedded Graphs