Conformal Parameterization

Parameterizing a triangle mesh means computing a one to one correspondence between a discrete, triangle surface patch and a planar mesh through a piecewise linear map. This amounts to assigning each triangle mesh vertex a pair of coordinates (u, v) indicating its position on the planar mesh. These planar coordinates are particularly useful for mapping textures on a 3D surface in addition to performing mesh operations such as surface fitting and remeshing.

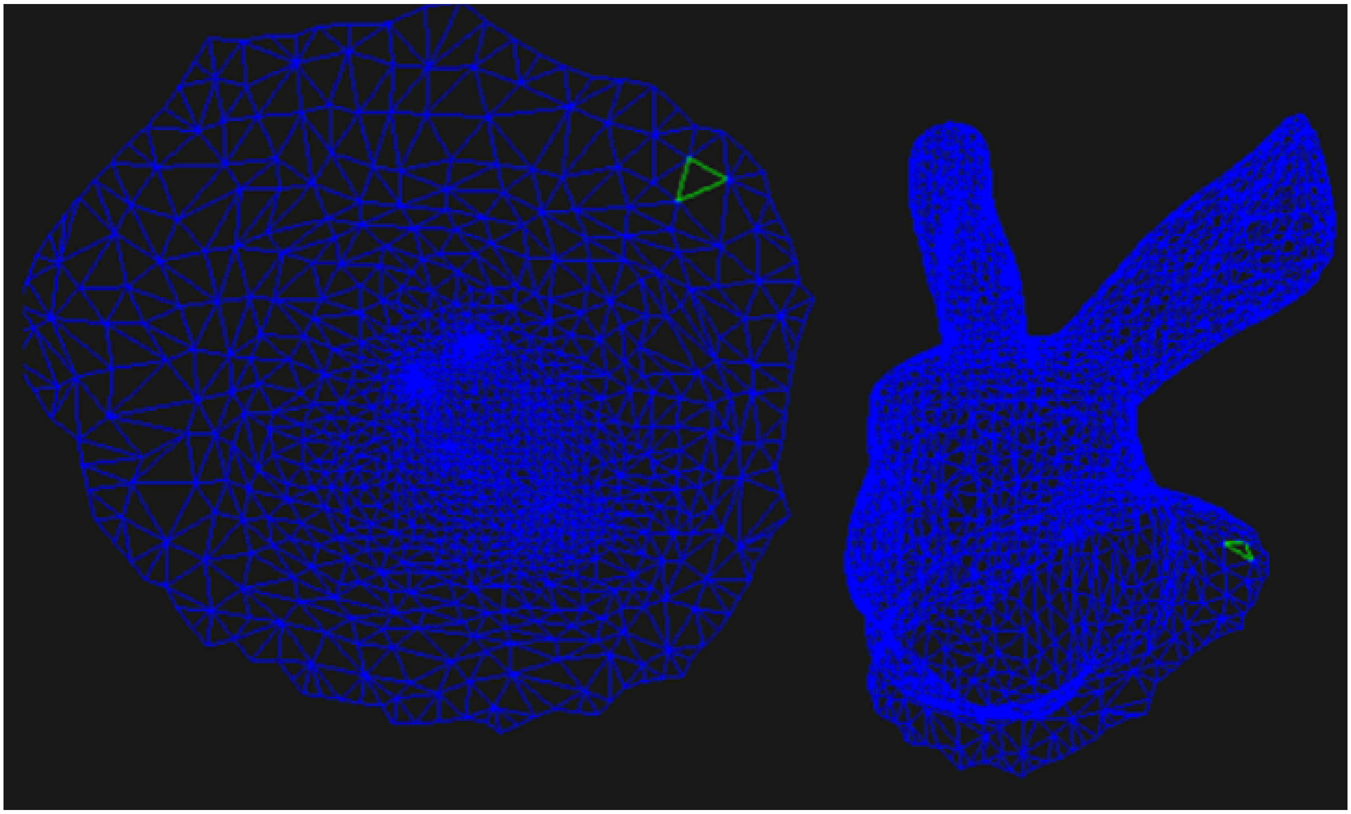

Figure 1: A piecewise-linear map z creates a correspondence between a 3d mesh with boundary and a 2d mesh, mapping each triangle from R^3 to R^2

A consequence of trying to flatten a non flat triangulated surface from 3d to 2d is the introduction of distortions. While it is impossible to flatten a surface isometrically, i.e., preserving both lengths and angles, it is possible to find an angle preserving or conformal parameterization. In their paper Intrinsic Parameterizations of Surface Meshes [1], Desbrun et al. proposed a free boundary method to produce a conformal parameterization that drastically reduces distortion at the relatively small additional cost of finding an eigenvector.

The failure of a map to be conformal can be defined by the expression:

$$\mathbf{E_C(z) = E_D(z) - A(z)}$$

where $E_D(z)$ is the dircihlet energy discretized using the Laplace Beltrami operator L (without the area term):

$$\mathbf{E_D(z) = z^T L z / 2}$$

and A(z) is the signed area of the complex map:

$$A(z) = -i \left(\sum_{e_{ij} \subset \partial} z^*_i z_j - z^*_j z_i\right) / 4$$

The conformal energy $E_C(z)$ is defined as the difference between $E_D(z)$ and $A(z)$. As $E_D(z)$ is bounded from below by $A(z)$ [1], a conformal map is attained by finding the minimal value of $E_D(z)$. This is achieved by solving the optimization problem:

$$\min E_C(z), \quad \text{s.t. } \langle z, 1\rangle = 0 \text{ and } \|z\| = 1$$

The first constraint centers the solution around the origin while the second makes sure the solution doesn't collapse to the origin. The optimization problem above is a disguised eigenvalue problem that can be solved by using the inverse power method. The u,v coordinates per vertex correspond to the real and imaginary parts of the entries of the complex eigenvector with the smallest eigenvalue.

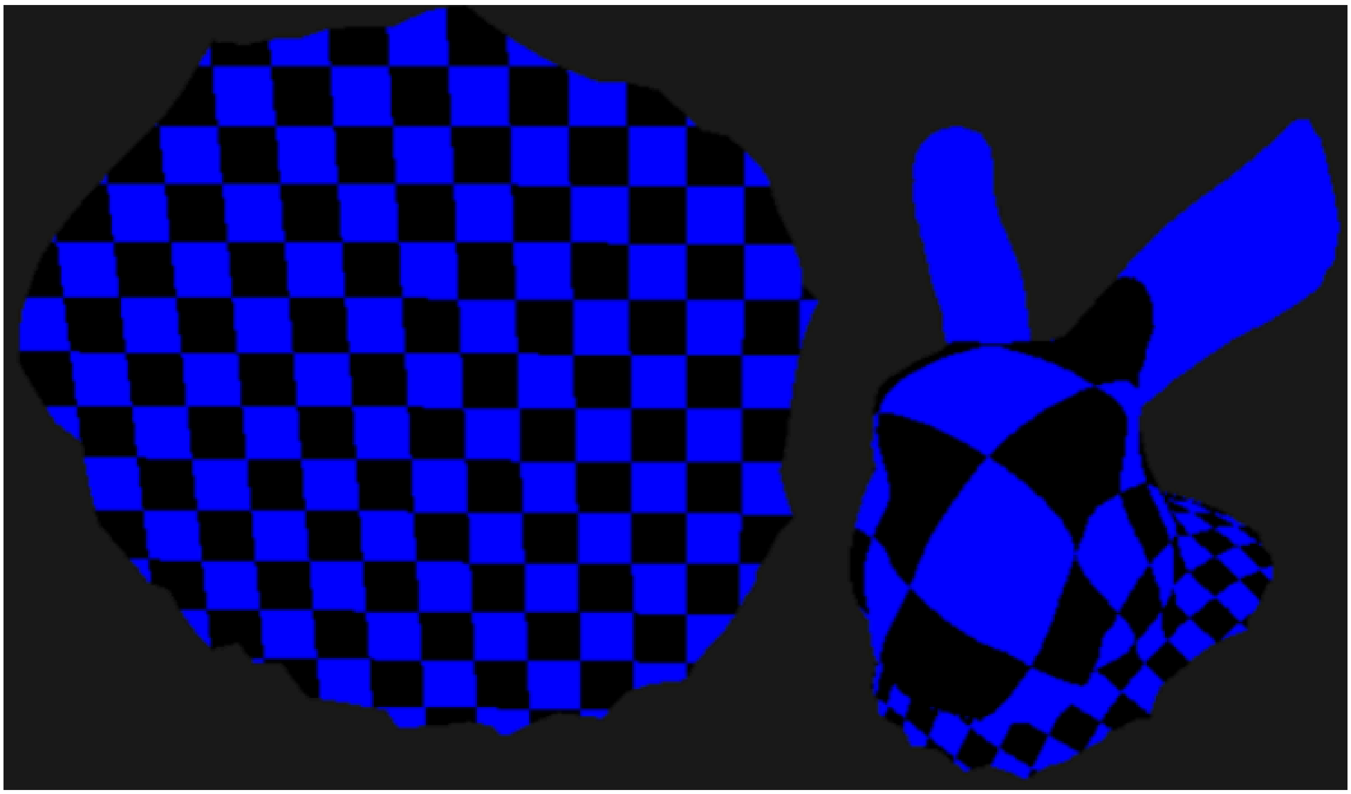

Figure 2: Texture mapping as an application of conformal parameterization

Implementation: https://github.com/rohan-sawhney/conformal-parameterization

[1] Desbrun et al. Intrinsic Parameterizations of Surface Meshes