Numerical Integration

Many ordinary differential equations do not have analytic solutions. To obtain a solution therefore requires an approximate approach such as numerical integration. Integration schemes to evaluate a solution to x_{tt} = f/m include:

1. **Explicit Euler -** Given an initial position x_0 and initial velocity v_0, the explicit euler method estimates the solution x(t) by computing the equations $\mathbf{x_{i+1} = x_i + v_i\Delta t}$ and $\mathbf{v_{i+1} = v_i + f_i\Delta t/m}$. This approach has two big disadvantages:
    1. It assumes the force is constant throughout an entire time step and thus overshoots the exact solution, making the error grow relatively quickly with time.
    2. It is only first order accurate. Decreasing the time step by a factor h makes the error only a factor h smaller.
2. **Runga Kutta -** Given an ODE of the form $x_{tt} = f(x, x_t)$, the second order Runga Kutta scheme uses velocities and accelerations at intermediate times $t_{(i+1)/2}$ to estimate the solution at t_{i+1}. Second order accuracy can be achieved with the discretization $\mathbf{x_{i+1} = x_i + v_i'\Delta t}$ and $\mathbf{v_{i+1} = v_i + f_i'\Delta t/m,}$ where v_i' equals $v_{(i+1)/2}$ and f_i' equals $f_{(i+1)/2}$. v_i' and f_i' can be computed using the Euler scheme. The fourth order Runga Kutta has fourth order accuracy and is derived using a similar approach. It is among the most popular integration techniques in the computational sciences but is expensive to compute.
3. **Verlet -** Truncating the taylor series expansion of of x_{i+1} and x_{i-1} to second order and adding the result, the Verlet integration scheme provides an approximation for x_{i+1} given by the equation $\mathbf{x_{i+1} = 2x_i - x_{i-1} + f_i\Delta t^2/m}$. This scheme is simple and has second order accuracy. It is popular in real time applications but it limited to the ODEs $x_{tt} = f(x)$.
4. **Implicit Euler -** The implicit integration approach estimates the solution x(t) by computing the equations $\mathbf{v_{i+1} = v_i + f_{i+1}\Delta t/m}$ and $\mathbf{x_{i+1} = x_i + v_{i+1}\Delta t}$. Substituting the second equation for f_{i+1} in the first and then moving all the future variables v_{i+1} to the right hand side results in a linear system of equations that can be solved using direct or iterative techniques. In contrast to the previous explicit schemes, implicit euler is unconditionally stable as it does not move “blindly into the future.” A potential disadvantage of the scheme is that it introduces numerical damping into the system.