Aggregation of Choice Functions on Non-rational Domains*

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Abstract

We consider the problem of aggregating choice functions satisfying the Pareto axiom and Independence of Irrelevant Alternatives (IIA) on a class of domains. We first show that when agents have rational choice functions on all the binary sets (that is, all subsets of cardinality two), a choice aggregator satisfies the Pareto axiom and IIA if and only if it is a dictatorial choice aggregator. Next, we consider two domains of non-rational choice functions, limitedly rational and partially rational domains, and provide the structure of choice aggregators satisfying the Pareto axiom and IIA.

JEL Classification: D71, D82.

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1. Introduction

We first describe the problem, then discuss the existing literature, and finally present our contribution.

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1.1 Description of the problem

We consider the problem of analyzing the structure of choice functions that satisfy *Pareto axiom* and *Independence of Irrelevant Alternatives* (IIA) on various domains. Given a set of alternatives A and a class of 'available' subsets $\mathcal{A}$ of A, a choice function C chooses one element from each set in $\mathcal{A}$. In other words, a choice function represents what an agent would choose from every available set. A choice function is *rational* if it is induced by a (strict) preference (that is, a reflexive, anti-symmetric, complete, and transitive binary relation). More elaborately, a choice function C is rational if there is a preference P such that for every available set B in $\mathcal{A}$, C chooses the best element of B according to the preference P. It is experimentally well-known that agents often make their choices by non-rational choice functions.

We consider a society with an arbitrary number n of at least two agents. Each agent i has a choice function C_i in a set of 'admissible' choice functions $\mathcal{D}(\mathcal{A})$. The social planner wants to aggregate all collections of admissible choice functions $C_1, \ldots, C_n$ to obtain a 'representative' choice function in $\mathcal{D}(\mathcal{A})$ for the society. A mapping that does this job is called a choice aggregator.

A choice aggregator satisfies Pareto axiom if, for every collection of admissible choice functions and every set B in $\mathcal{A}$, the aggregated choice function chooses one of the choices made by the agents (technically, the aggregated choice function chooses either $C_1(B)$ or $C_2(B)$ or $\ldots$ or $C_n(B)$). A choice aggregator satisfies IIA if the aggregated choice for a set B in $\mathcal{A}$ does not depend on the aggregated choice for any other sets in $\mathcal{A}$. In other words, when the aggregated choice function decides the choice for a set B in $\mathcal{A}$, it does not take into consideration what decisions are made for the aggregated choices for other sets in $\mathcal{A}$. Both PO and IIA are considered desirable properties for choice functions (see Kalai et al. (2002), Shelah (2005), Dokow and Holzman (2010)). A choice aggregator is called *dictatorial* if it chooses (as the aggregated one) the choice function of a particular agent at every collection of admissible choice functions of the society.

1.2 Existing literature

We intend to explore the structure of choice aggregators satisfying Pareto axiom and IIA in this paper. This structure depends on the admissible choice functions $\mathcal{D}$. Arrow (1951) considers the problem of preference aggregation, where agents' preferences (instead of choice functions) are aggregated into a preference. He shows that the IIA and Pareto axiom force the aggregator to be dictatorial when at least three alternatives are present. The question arises as to what will happen to the aggregation if one admits non-rational choice functions.

Wilson (1975) considered an aggregation of binary evaluations, and showed that Arrow's theorem applies to a larger class of aggregation problems under certain conditions. Later, Rubinstein and Fishburn (1986) extended Wilson's model for non-binary evaluations. Kalai (2002) considered the problem of aggregating choice functions and proposed a different way to generalize Arrow's theorem. He conjectured that Arrow's theorem extends for choice functions from subsets of arbitrary size (that is, non-binary) when choice functions are symmetric with respect to the alternatives. Shelah (2005) proved a version of Kalai's conjecture when the sizes of the available subsets of alternatives are neither very small nor very large (compared to the total number of alternatives).

1.3 Our contribution

We prove three results in this paper. The first result (Theorem 1) shows that when agents have rational choice functions on $\mathcal{A}$ that contains all the binary sets (that is, all subsets of cardinality two), a choice aggregator satisfies Pareto axiom and IIA if and only if it is a dictatorial choice aggregator. As any preference uniquely corresponds to a choice function on binary sets, Arrow's impossibility theorem follows as a corollary of this result.

Next, we consider domains of non-rational choice functions of some particular structure. First, we consider the situation where the alternatives are presented to the agents following some ordering $\prec$ (as it happens in public stores or for online shopping) and agents have limited time to explore the alternatives. Thus, each agent observes the first κ, where κ is a parameter, alternatives in the ordering $\prec$, and picks the best of them according to her preference. We denote such class of choice functions by $\mathcal{D}_{(\kappa,\prec)}(\mathcal{A})$. We show that as long as all binary subsets of A are available (that is, $\mathcal{A}$ contains all subsets of A of size two) for making choices and agents have time/patience to observe at least two alternatives (that is, $\kappa \geq 2$), the only way to aggregate limitedly rational choice functions is a dictatorial one. This result shows that Arrow's theorem extends to situations where choice functions are non-rational.

Finally, we consider a situation where agents make rational choices for 'small' sets, and for the bigger ones, they choose one of the 'top' alternatives. More elaborately, each agent has a preference P, and for sets with size less than α, where α is a parameter, they choose the best alternative in that set according to P. However, for sets with size more than α, they choose one of the top β alternatives for some parameter β according to P. We call such domains partially rational domains and denote them by $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$. We show that if $\beta < \alpha$ then a choice aggregator satisfies Pareto axiom and IIA on $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ if and only if it is dictatorial, and if $\beta \geq \alpha$ then there exists non-dictatorial choice aggregators on $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$. This result shows that Arrow's impossibility theorem does *not* in general extend to situations when non-rational choice functions are allowed.

To the best of our knowledge, only a few papers address the problem of aggregating non-rational choice functions. Kalai et al. (2002) was the first to conjecture that Arrow's impossibility result could apply even in the absence of complete rationality, provided certain appropriate assumptions are met. Later, Shelah (2005) validated Kalai's conjecture when the domain under consideration is symmetric with respect to the alternatives and the available sets are of a fixed size. A domain of choice functions is symmetric if it is closed under permutation. While the assumption of fixed-sized available sets is not that restrictive, symmetricity is rather a technical condition that limits the practical applicability of the concerned results. We believe analyzing the structure of choice aggregators on non-rational domains of practical importance is a significant contribution to this literature.

2. Preliminaries

Let A be a set of m alternatives, where $m \geq 3$. By $\mathcal{A}$ we denote a set of subsets of A, and for $1 \leq k \leq m$, we denote by $\mathcal{A}_k$ the set of all subsets of A with cardinality k. A *choice function* C on $\mathcal{A}$ is a mapping $C : \mathcal{A} \to A$ such that $C(X) \in X$ for all $X \in \mathcal{A}$. Let $\mathcal{C}(\mathcal{A})$ be the set of all choice functions on $\mathcal{A}$.

Let $N = \{1, \ldots, n\}$ be a finite set of $n \geq 2$ agents. An element C_N of $\mathcal{C}^n(\mathcal{A})$ is called a (choice function) *profile* on $\mathcal{A}$.[1] A subset $\mathcal{D}(\mathcal{A})$ of $\mathcal{C}(\mathcal{A})$ is called a *domain* (of admissible choice functions) on $\mathcal{A}$. When $\mathcal{A} = 2^A$, to simplify the notation, we denote $\mathcal{C}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ by $\mathcal{C}$ and $\mathcal{D}$, respectively. A *choice aggregator* (also called a social welfare function) on $\mathcal{D}^n(\mathcal{A})$ is a mapping $f : \mathcal{D}^n(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$. For ease of presentation, we use the notation $f_X(C_N)$ to denote $f(C_N)(X)$. In other words, for a profile C_N, we denote the choice made by $f(C_N)$ from a set X by $f_X(C_N)$.

A preference on A is a linear order (complete, transitive, and ante-symmetric binary relation) on A. We denote the set of all preferences on A by $\mathbb{L}(A)$. For a preference P, a set of alternatives X, and integer k with $1 \leq k \leq |X|$, we denote by $r_k(X, P)$ the k-th ranked alternative in X

[1]We denote by $\mathcal{C}^n(\mathcal{A})$ the n-fold product of $\mathcal{C}(\mathcal{A})$, that is, the set $(\mathcal{C}(\mathcal{A}))^n$.

according to P, that is, $r_k(X,P) = x$ if $|\{y \in X \mid yPx\}| = k$.[2] We further denote by $R_k(X,P)$ the set $\{r_l(X,P) \mid l \leq k\}$. A choice function $C \in \mathcal{C}(\mathcal{A})$ respects a preference $P \in \mathbb{L}(A)$ at $X \in \mathcal{A}$ if $C(X) = r_1(X,P)$. We call a choice function $C \in \mathcal{C}(\mathcal{A})$ *rational* if there exists a preference $P \in \mathbb{L}(A)$ such that C respects P at all $X \in \mathcal{A}$. We denote the set of all rational choice functions by $\widehat{\mathcal{C}}(\mathcal{A})$.

Next, we introduce some desirable properties of a choice aggregator.

Definition 1. A choice aggregator $f : \mathcal{D}^n(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ satisfies *Independence of Irrelevant Alternative* (IIA) if, for all profiles C_N, C'_N, and all $X \in \mathcal{A}$,

$$[C_i(X) = C'_i(X) \text{ for all } i = 1, \ldots, n] \Rightarrow [f_X(C_N) = f_X(C'_N)].$$

Definition 2. A choice aggregator $f : \mathcal{D}^n(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ satisfies the *Pareto* axiom if, for all profiles C_N and all $X \in \mathcal{A}$,

$$f_X(C_N) \in \cup_{i=1,\ldots,n} C_i(X).$$

PO axiom ta meaningless. Pore eta niye bhabbo

Definition 3. A choice aggregator $f : \mathcal{D}^n(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ is *dictatorial* if there exists an agent $i \in N$ such that for all profiles C_N and all $X \in \mathcal{A}$,

$$f_X(C_N) = C_i(X).$$

Now, we define a crucial property of a domain.

Definition 4. A domain $\mathcal{D}(\mathcal{A})$ is *dictatorial for n agents* if every choice aggregator $f : \mathcal{D}^n(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ satisfying IIA and the Pareto axiom is dictatorial.

3. Results

In this section, we provide the structure of choice aggregators satisfying the Pareto axiom and IIA on various domains. Theorem 1 says that whenever all binary sets are available for making choices from an admissible choice functions are rational, a choice aggregator satisfies the Pareto axiom and IIA if and only if it is a dictatorial one.

Theorem 1. *For $n \geq 2$ and $\mathcal{A}$ with $\mathcal{A}_2 \subseteq \mathcal{A}$, $\widehat{\mathcal{C}}(\mathcal{A})$ is dictatorial for n agents.*

[2] Note that as P is reflexive, we have xPx.

The proof of this theorem can be found in Appendix A.

In the following two subsections, we consider non-rational choice function domains of practical importance.

3.1 Choice aggregation on limitedly rational domains

Limitedly rational choice functions appear when alternatives are presented to agents following some ordering and agents explore only a certain number of options along the ordering due to limited time/patience.

Let $\prec$ be an ordering of the elements of A and let $\kappa \geq 2$ be an integer. Further, let $\mathcal{A}$ be a set of subsets of A. A choice function C on $\mathcal{A}$ is $(\kappa, \prec)$-limitedly rational if there exists a preference P such that $C(X) = r_1(X_\kappa(\prec), P)$ for all $X \in \mathcal{A}$, where $X_\kappa(\prec) = R_\kappa(X, \prec)$. In such situations, we say that C is *generated* by P. Observe that a $(\kappa, \prec)$-limitedly rational choice function is completely determined by its generating preference. By $\mathcal{D}_{(\kappa,\prec)}(\mathcal{A})$, we denote the set of all $(\kappa, \prec)$-limitedly rational choice functions on $\mathcal{A}$.

Theorem 2. *Let $\mathcal{A} \subseteq 2^A$ be such that $\mathcal{A}_2 \subseteq \mathcal{A}$. Further, let $\kappa \geq 2$ be an integer and $\prec$ an ordering of the elements of A. Then, $\mathcal{D}_{(\kappa,\prec)}(\mathcal{A})$ is dictatorial for n agents.*

Proof. Let $\mathcal{A} \subseteq 2^A$ be such that $\mathcal{A}_2 \subseteq \mathcal{A}$ and $\mathcal{D}_{(\kappa,\prec)}(\mathcal{A})$ be the $(\kappa, \prec)$-limitedly rational domain for an integer $\kappa \geq 2$ and a linear order $\prec$ over A. We show that, for $n \geq 2$, $\mathcal{D}_{(\kappa,\prec)}(\mathcal{A})$ is dictatorial for n agents. Fix $n \in \mathbb{N}$ and f, a choice aggregator on $\mathcal{D}^n_{(\kappa,\prec)}(\mathcal{A})$ satisfying IIA and Pareto. We proceed to prove that f is dictatorial. Note that as $\mathcal{A}_2 \subseteq \mathcal{A}$, we have $\mathcal{D}_{(\kappa,\prec)}(\mathcal{A}_2) = \widehat{\mathcal{C}}(\mathcal{A}_2)$. Thus, by Theorem 1, we have $\mathcal{D}_{(\kappa,\prec)}(\mathcal{A}_2)$ is dictatorial. Therefore, because of IIA, we have the following result: there exists $i \in N$ such that for all $x, y \in A$ and all $(C_1, \ldots, C_n) \in \mathcal{D}^n_{(\kappa,\prec)}(\mathcal{A})$,

$$f_{\{x,y\}}(C_1(\{x,y\}), \ldots, C_n(\{x,y\})) = C_i(\{x,y\}). \tag{1}$$

Fix $(\hat{C}_1, \ldots, \hat{C}_n) \in \mathcal{D}^n_{(\kappa,\prec)}$. The above equation (1) implies that both $\hat{C}_i$ and $f(\hat{C}_1, \ldots, \hat{C}_n))$ are generated from the same preference. As a $(\kappa, \prec)$-limitedly rational choice function is completely determined by the preference it is generated from, we have

$$f(\hat{C}_1, \ldots, \hat{C}_n) = \hat{C}_i,$$

completing the proof of the theorem. ■

Extension: Multiple orderings are there by which the objects are presented to the agents.

3.2 CHOICE AGGREGATION ON PARTIALLY RATIONAL DOMAINS

Partially rational choice functions appear when agents choose rationally from sets with relatively smaller sizes and may violate rationality to some extent from sets of bigger sizes. Formally, for $\beta \leq \alpha$, a choice function C on $\mathcal{A}$ is (α, β)-partially rational w.r.t. a preference P if for all $X \in \mathcal{A}$, $C(X) = r_1(X, P)$ whenever $|X| \leq \alpha$, and $C(X) \in R_\beta(X, P)$ whenever $|X| > \alpha$. In such situations, we say that C is *generated* by P. We use the following notational terminology: for (α, β)-partially rational choice functions $\bar{C}$, $\hat{C}$, $\tilde{C}$, we denote the corresponding generating preferences by $\bar{P}$, $\hat{P}$, $\tilde{P}$, respectively. Observe that a (α, β)-partially rational choice function C' is not uniquely determined by P' if $\beta > 1$.

REMARK 1. For $\alpha \geq 2$ and $|X| \leq \alpha$, if two (α, β)-partially rational choice functions $\bar{C}$ and $\hat{C}$ agree on all sets of cardinality two, then they agree on X as well. Formally, if $\bar{C}(\{x, y\}) = \hat{C}(\{x, y\})$ for all $x, y \in A$, then $\bar{C}(X) = \hat{C}(X)$.

For $\alpha, \beta \in \{1, \ldots, m\}$ with $\beta \leq \alpha$, we define $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A}) := \{C \in \mathcal{C} \mid C \text{ is } (\alpha, \beta)\text{-rational}\}$. In other words, $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ contains all choice functions that respect some preference P at all $X \in \mathcal{A} \setminus \{X \in \mathcal{A} \mid |X| \leq \alpha\}$, and at any set Y in $\{X \in \mathcal{A} \mid |X| > \alpha\}$, they choose the k^{th} ranked alternative in Y according to P for some $k \leq \beta$. We write $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ as the (α, β)-partially rational domain. Whenever we define a $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ domain on a class of subsets $\mathcal{A}$ of A, we assume that $\mathcal{A}$ has at least one element with cardinality α.

Our next result provides a characterization of all (α, β)-partially rational domains that are dictatorial.

Theorem 3. *Let $\mathcal{A} \subseteq 2^A$ be such that $\mathcal{A}_2 \subseteq \mathcal{A}$. Then, $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ is dictatorial for n agents if and only if $\beta < \alpha$.*

Proof. **(If part)** Let $\mathcal{A} \subseteq 2^A$ be such that $\mathcal{A}_2 \subseteq \mathcal{A}$ and $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ be the (α, β)-partially rational domain for some $\beta < \alpha$. We show that, for $n \geq 2$, $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ is dictatorial for n agents. As $\beta \geq 1$, it must be that $\alpha \geq 2$. Fix $n \in \mathbb{N}$ and f, a choice aggregator on $\mathcal{D}^n_{(\alpha,\beta)}(\mathcal{A})$ satisfying IIA and Pareto. We proceed to prove that f is dictatorial. Note that as $\mathcal{A}_2 \subseteq \mathcal{A}$, we have $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A}_2) = \widehat{\mathcal{C}}(\mathcal{A}_2)$. Thus, by Theorem 1, we have $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A}_2)$ is dictatorial. This together with IIA and Theorem 1 implies the following result: there exists $i \in N$ such that for all $x, y \in A$ and all $(C_1,\dots,C_n) \in \mathcal{D}^n_{(\alpha,\beta)}(\mathcal{A})$,

$$f_{\{x,y\}}(C_1(\{x,y\}),\dots,C_n(\{x,y\})) = C_i(\{x,y\}). \tag{2}$$

WLG we may assume that $i = 1$. We complete the proof of the theorem by showing that for all $X \in \mathcal{A}$ and all $(C_1,\dots,C_n) \in \mathcal{D}^n_{(\alpha,\beta)}(\mathcal{A})$,

$$f_X(C_1(X),\dots,C_n(X)) = C_1(X). \tag{3}$$

Fix $(\hat{C}_1,\dots,\hat{C}_n) \in \mathcal{D}^n_{(\alpha,\beta)}$. First consider $X \in \mathcal{A}$ with $|X| \le \alpha$ and . As $f(\hat{C}_1,\dots,\hat{C}_n)$ respects a preference at X, and $f(\hat{C}_1,\dots,\hat{C}_n)$ and $\hat{C}_1$ agree on all the sets in $\mathcal{A}_2$ (by (2)), we have $f_X(C_1(X),\dots,C_n(X)) = C_1(X)$ (see Remark 1).

Now assume that $|X| > \alpha$. Assume for contradiction $f_X(\hat{C}_1(X),\dots,\hat{C}_n(X)) \neq \hat{C}_1(X)$. Consider $\tilde{C}_1 \in \mathcal{D}_{(\alpha,\beta)}$ such that it chooses the same alternative from X and $f_X(\hat{C}_1(X),\dots,\hat{C}_n(X))$ does not belong to the top β alternatives in X according to $\tilde{P}_1$. Formally, $\tilde{C}_1$ is such that $\hat{C}_1(X) = \tilde{C}_1(X)$ and $f_X(\hat{C}_1(X),\dots,\hat{C}_n(X)) \notin R_\beta(X,\tilde{P}_1)$. Note that this is possible as $|X| > \alpha$ and $\beta < \alpha$. As $\hat{C}_1(X) = \tilde{C}_1(X)$, by IIA,

$$f_X(\hat{C}_1(X),\dots,\hat{C}_n(X)) = f_X(\tilde{C}_1(X),\dots,\hat{C}_n(X)). \tag{4}$$

Further, by (2), $f(\tilde{C}_1,\dots,\hat{C}_n)$ agrees with $\tilde{C}_1$ on all the sets in $\mathcal{A}_2$. Therefore, it must be that $f(\tilde{C}_1,\dots,\hat{C}_n)$ is generated by the preference $\tilde{P}_1$. Hence, by the definition of $\mathcal{D}_{(\alpha,\beta)}$, $f_X(\tilde{C}_1(X),\dots,\hat{C}_n(X)) \in R_\beta(X,\tilde{P}_1)$. But this is a contradiction to (4) as by the construction of $\tilde{P}_1$, $f_X(\hat{C}_1(X),\dots,\hat{C}_n(X)) \notin R_\beta(X,\tilde{P}_1)$. This shows that $f_X(\hat{C}_1(X),\dots,\hat{C}_n(X)) = \hat{C}_1(X)$. Since $(\hat{C}_1,\dots,\hat{C}_n) \in \mathcal{D}^n_{(\alpha,\beta)}$ is arbitrary, the proof is established.

(Only-if part)) We show that a (α,β)-partially rational domain $\mathcal{D}_{(\alpha,\beta)}(\mathcal{A})$ with $\beta = \alpha$ is non-dictatorial, i.e., there is a choice aggregator f satisfying IIA and Pareto but not dictatorship. Let $Y \in \mathcal{A}$ be such that $|Y| = \alpha$. Consider the following aggregator $g : \mathcal{D}^n_{(\alpha,\beta)} \to \mathcal{D}_{(\alpha,\beta)}$ for n agents:

$$g_X(C_1,C_2,\dots,C_n) = \begin{cases} C_1(X) & \text{if } X \neq Y, \\ C_2(X) & \text{if } X = Y. \end{cases} \tag{5}$$

Clearly, g is non-dictatorial. Also, as it either chooses the choice made by agent 1 or agent 2, it satisfies Pareto. Further, the outcome of g at a set only depends on that set, implying g satisfies IIA. We now show that $g(C_1, C_2, \ldots, C_n) \in \mathcal{D}_{(\alpha,\beta)}$ for all $(C_1, C_2, \ldots, C_n) \in \mathcal{D}^n_{(\alpha,\beta)}$. Fix $(\hat{C}_1, \ldots, \hat{C}_n) \in \mathcal{D}^n_{(\alpha,\beta)}$. From the definition of g (see (5)), we only need to show that $\hat{C}_2(Y) \in R_\beta(Y, \hat{P}_1)$. Note that this follows as $\hat{C}_2(Y) \in Y$ and $|Y| = \alpha = \beta$. The proof is complete. ■

3.3 Weak axiom of revealed preference

A choice function $C : \mathcal{A} \to A$ satisfies the *weak axiom of revealed preference* (WARP) if for all $B, B' \in \mathcal{A}$ and all $a, b \in A$

$$[a, b \in B, C(B) = a, \text{ and } C(B') = b] \implies [a \notin B'].$$

By $\mathcal{D}_{\text{WARP}}(\mathcal{A})$, we denote the set of all choice functions on $\mathcal{A}$ satisfying WARP.

Theorem 4. *Let $\mathcal{A} \subseteq 2^A$ be such that $\mathcal{A}_2, \mathcal{A}_3 \subseteq \mathcal{A}$. Then, $\mathcal{D}_{WARP}(\mathcal{A})$ is dictatorial for n agents.*

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A. Proof of Theorem 1

Proof. We use induction on the number of agents to prove this result. As the base case, we show that the theorem holds for $n = 2$. Consider a choice aggregator f on $\mathcal{S}^2$ satisfying IIA and Pareto. We show that f is dictatorial. Assume $f_{\{a,b\}}(a, b) = a$ for some $a, b \in A$. We show that $f_{\{x,y\}}(x, y) = x$ for all $x, y \in A$. It is equivalent to showing the following two results: (i) $f_{\{a,y\}}(a, y) = a$ for all $y \in A$ and (ii) $f_{\{x,b\}}(x, b) = x$ for all $x \in A$.

To show (i), take $c \in A$ such that $c \neq a$. Consider two preferences $P_1 \equiv abc \cdots$ and $P_2 \equiv bca \cdots$ such that P_1 and P_2 are identical at all positions below the 3^{rd} rank. Let $\bar{C}_1$ and $\bar{C}_2$ be two choice functions in $\mathcal{D}_{(\alpha,\beta)}$ such that $\bar{C}_1$ and $\bar{C}_1$ respects P_1 and P_2 for all sets with cardinality at most α. This means that for a cardinality two set B other than $\{a, b\}$ and $\{a, c\}$, $\bar{C}_1(B) = \bar{C}_2(B)$ which implies $f_B(\bar{C}_1(B), \bar{C}_2(B)) = \bar{C}_1(B)$. Further, for $\{a, b\}$, as $\bar{C}_1(\{a, b\}) = a$, $\bar{C}_2(\{a, b\}) = b$, and $f_{\{a,b\}}(a, b) = a$, we have $f_{\{a,b\}}(\bar{C}_1(\{a, b\}), \bar{C}_2(\{a, b\})) = \bar{C}_1(\{a, b\})$. These observations together with the fact that $f(\bar{C}_1, \bar{C}_2)$ respects a preference for all cardinality 2 sets, it must be that $f_{\{a,c\}}(\bar{C}_1(\{a, c\}), \bar{C}_2(\{a, c\}) = \bar{C}_1(\{a, c\})$ implying $f_{\{a,c\}}(a, c) = a$. This shows that $f_{\{a,y\}}(a, y) = a$ for all $y \in A$. To show (ii), we can use similar arguments with $\hat{C}_1$ and $\hat{C}_2$ where $\hat{C}_1$ respects $axb \cdots$ and $\hat{C}_2$ respects $bax \cdots$. This completes the proof for the base case.

We now proceed to prove the induction step. Let $n > 2$ be an integer. We consider the following induction hypothesis:

Induction Hypothesis (IH): Assume that the theorem holds for all $k < n$.

We show that the theorem holds for n agents. Consider a choice aggregator $f : \mathcal{S}^n \to \mathcal{S}$ satisfying IIA and PO. Let $N^* = \{1, 3, \ldots, n\}$ and define the choice aggregator $g : \mathcal{S}^{n-1} \to \mathcal{S}$ for the set of agents N^* as follows: For all $C^2_{N^*} = (C^2_1, C^2_3, \ldots, C^2_n) \in \mathcal{S}^{n-1}$,

$$g(C^2_1, C^2_3, \ldots, C^2_n) = f(C^2_1, C^2_1, C^2_3, \ldots, C^2_n).$$

Claim 1. *g satisfies IIA and PO.*

Proof of the claim: Consider a profile $(C^2_1, C^2_3, \ldots, C^2_n) \in \mathcal{S}^{n-1}$ and $X \in \mathcal{A}_2$. By the definition of g, $g(C^2_1, C^2_3, \ldots, C^2_n) = f(C^2_1, C^2_1, C^2_3, \ldots, C^2_n)$. As f satisfies PO, we have $f_X(C^2_1, C^2_1, C^2_3, \ldots, C^2_n) \in \cup_{i \in \{1,3,\ldots,n\}} C^2_i(X)$ implying $g_X(C^2_1, C^2_3, \ldots, C^2_n) \in \cup_{i \in \{1,3,\ldots,n\}} C^2_i(X)$. This shows that g satisfies PO. A similar argument shows that g satisfies IIA. □

Given Claim 2 and IH, we have g is dictatorial. We distinguish two cases based on the dictator of g. Let $k \in N^*$ be the dictator of g.

Case 1: $k \neq 1$

To simplify the notation, WLG we assume that $k = 3$. We show that agent 3 is the dictator for f as well. Because of IIA, it is enough to show that for all $(C_1^2(\{x,y\}), \ldots, C_n^2(\{x,y\})) \in \mathcal{S}^n$ and all $x, y \in A$

$$f_{\{x,y\}}(C_1^2(\{x,y\}), C_2^2(\{x,y\}), C_3^2(\{x,y\}), \ldots, C_n^2(\{x,y\})) = C_3^2(\{x,y\}). \tag{6}$$

Note that by the definition of g, (6) holds when $C_1^2(\{x,y\}) = C_2^2(\{x,y\})$. Consider $(\hat{C}_1^2, \ldots, \hat{C}_n^2)$ and $a, b, c \in A$ such that $\hat{C}_1^2$, $\hat{C}_2^2$, and $\hat{C}_3^2$ respect the preferences $\hat{P}_1 \equiv abc\cdots$, $\hat{P}_2 \equiv bac\cdots$, and $\hat{P}_3 \equiv acb\cdots$, respectively. Thus, we have the following situation

$\cdot$	$\hat{C}_1^2$	$\hat{C}_2^2$	$\hat{C}_3^2$	$\cdots$	$f(\hat{C}_1^2, \ldots, \hat{C}_n^2)$
$\{a,b\}$	a	b	a	$\cdots$	
$\{b,c\}$	b	b	c	$\cdots$	c
$\{a,c\}$	a	a	a	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Note that $f_{\{b,c\}}(\hat{C}_1^2, \ldots, \hat{C}_n^2) = c$ follows as $\hat{C}_1^2(\{b,c\}) = \hat{C}_2^2(\{b,c\})$ and $\hat{C}_3^2(\{b,c\}) = c$. Combining this with the fact that $f(\hat{C}_1^2, \ldots, \hat{C}_n^2)$ respects a preference, we have $f_{\{a,b\}}(\hat{C}_1^2, \ldots, \hat{C}_n^2) = a$. Since a, b, c are arbitrary this shows that (6) holds when $C_1^2(\{x,y\}) \neq C_2^2(\{x,y\})$. This completes the proof for Case 1.

Case 2: $k = 1$

The assumption of the case implies that for all $x, y \in A$ and all $(C_3^2(\{x,y\}), \ldots, C_n^2(\{x,y\})) \in \mathcal{S}^{n-2}$

$$f_{\{x,y\}}(x, x, C_3^2(\{x,y\}), \ldots, C_n^2(\{x,y\})) = x. \tag{7}$$

Assume that for some $a, b \in A$ and $\tilde{C}_3^2(\{a,b\}), \ldots, \tilde{C}_n^2(\{a,b\})$,

$$f_{\{a,b\}}(a, b, \tilde{C}_3^2(\{a,b\}), \ldots, \tilde{C}_n^2(\{a,b\})) = a. \tag{8}$$

Claim 2. *(8) holds for all $(C_3^2(\{a,b\}), \ldots, C_n^2(\{a,b\})) \in \mathcal{S}^{n-3}$.*

Proof of the claim: Fix $(\hat{C}_3^2(\{a,b\}), \ldots, \hat{C}_n^2(\{a,b\})) \in \mathcal{S}^{n-3}$. Let $i \in N$ be the agent with minimal index in $\{3, \ldots, n\}$ who has difference choices for the set $\{a,b\}$ in $\tilde{C}_i^2$ and $\hat{C}_i^2$, i.e., for all $l < i$, $\tilde{C}_l^2(\{a,b\}) = \hat{C}_l^2(\{a,b\})$ and $\tilde{C}_i^2(\{a,b\}) \neq \hat{C}_i^2(\{a,b\})$. To simplify the notations, WLG we assume that $i = 3$ and $a = \tilde{C}_3^2(\{a,b\}) \neq \hat{C}_i^2(\{a,b\}) = b$. Consider $c \in A \setminus \{a,b\}$ and $\bar{P}_1, \bar{P}_2, \bar{P}_3 \in \mathcal{S}$ such that $\bar{P}_1 \equiv abc\cdots$, $\bar{P}_2 =\equiv bca\cdots$, and $\bar{P}_3 \equiv cab\cdots$. Let $\bar{C}_p^2$ be the binary choice function corresponding to $\bar{P}_p^2$ for all $p \in \{1,2,3\}$. Thus, (8) and the statement of the case together imply the following situation

$\cdot$	$\bar{C}_1^2$	$\bar{C}_2^2$	$\bar{C}_3^2$	$\tilde{C}_4^2$	$\cdots$	$f(\hat{C}_1^2, \bar{C}_2^2, \bar{C}_3^2, \tilde{C}_4^2 \ldots, \tilde{C}_n^2)$
$\{a,b\}$	a	b	a	$\tilde{C}_4^2(\{a,b\})$	$\cdots$	a
$\{b,c\}$	b	b	c	$\tilde{C}_4^2(\{b,c\})$	$\cdots$	b
$\{a,c\}$	a	c	c	$\tilde{C}_4^2(\{a,c\})$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Here, the choice of $f(\hat{C}_1^2, \bar{C}_2^2, \bar{C}_3^2, \tilde{C}_4^2, \ldots, \tilde{C}_n^2)$ at $\{a,b\}$ follows from (8). As $f(\hat{C}_1^2, \bar{C}_2^2, \bar{C}_3^2, \tilde{C}_4^2 \ldots, \tilde{C}_n^2)$ respects a preference, this implies that

$$f_{\{a,c\}}(a,c,c,\tilde{C}_4^2(\{a,c\}), \ldots, \tilde{C}_n^2(\{a,c\})) = a. \tag{9}$$

Let $\bar{P}'_1, \bar{P}'_2, \bar{P}'_3 \in \mathcal{S}$ such that $\bar{P}'_1 \equiv acb\cdots$, $\bar{P}'_2 \equiv cba\cdots$, and $\bar{P}'_3 \equiv cba\cdots$. Let $\bar{C}_p'^2$ be the binary choice function corresponding to $\bar{P}_p^2$ for all $p \in \{1,2,3\}$. Thus, using (9), we have the following situation

$\cdot$	$\bar{C}_1'^2$	$\bar{C}_2'^2$	$\bar{C}_3'^2$	$\tilde{C}_4^2$	$\cdots$	$f(\hat{C}_1'^2, \bar{C}_2'^2, \bar{C}_3'^2, \tilde{C}_4^2 \ldots, \tilde{C}_n^2)$
$\{a,b\}$	a	b	b	$\tilde{C}_4^2(\{a,b\})$	$\cdots$	
$\{b,c\}$	c	c	c	$\tilde{C}_4^2(\{b,c\})$	$\cdots$	c
$\{a,c\}$	a	c	c	$\tilde{C}_4^2(\{a,c\})$	$\cdots$	a
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

As $f(\hat{C}_1'^2, \bar{C}_2'^2, \bar{C}_3'^2, \tilde{C}_4^2 \ldots, \tilde{C}_n^2)$ respects a preference, it must be that

$$f_{\{a,b\}}(a,b,b,\tilde{C}_4^2(\{a,b\}), \ldots, \tilde{C}_n^2(\{a,b\})) = a.$$

Therefore,

$$f_{\{a,b\}}(a,b,\hat{C}_3^2(\{a,b\}),\tilde{C}_4^2(\{a,b\}), \ldots, \tilde{C}_n^2(\{a,b\})) = a.$$

Continuing in this manner we can show that

$$f_{\{a,b\}}(a,b,\hat{C}_3^2(\{a,b\}),\hat{C}_4^2(\{a,b\}), \ldots, \hat{C}_n^2(\{a,b\})) = a.$$

Since $(\hat{C}_3^2(\{a,b\}), \hat{C}_4^2(\{a,b\}), \ldots, \hat{C}_n^2(\{a,b\}))$ is arbitrary the claim is established. □

We now complete the proof of the induction step by showing that for all (x,y) where $x,y \in A$ and all $(C_3^2(\{x,y\}), \ldots, C_n^2(\{x,y\})) \in \mathcal{S}^{n-2}$,

$$f_{\{x,y\}}(x, y, C_3^2(\{x,y\}), \ldots, C_n^2(\{x,y\})) = x. \tag{10}$$

First consider $x = a$ and $y = c$ for some $c \in A \setminus \{a\}$ in (10). Consider $\check{P}_1 \equiv abc\cdots$ and $\check{P}_2 \equiv bca\cdots$. Further, let $\check{C}_l^2$ be the binary choice function generated by $\check{P}_l$ for all $l \in \{1,2\}$. Fix $(\check{C}_3^2, \ldots, \check{C}_3^2) \in \mathcal{S}^{n-2}$. Then we have the following situation.

$\cdot$	$\check{C}_1^2$	$\check{C}_2^2$	$\check{C}_3^2$	$\cdots$	$f(\check{C}_1^2, \ldots, \check{C}_n^2)$
$\{a,b\}$	a	b	$\check{C}_3^2(\{a,b\})$	$\cdots$	a
$\{b,c\}$	b	b	$\check{C}_3^2(\{b,c\})$	$\cdots$	b
$\{a,c\}$	a	c	$\check{C}_3^2(\{a,c\})$	$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Here, $f_{a,b}(a, b, \check{C}_3^2(\{a,b\}), \ldots, \check{C}_n^2(\{a,b\})) = a$ by (2). As $f(\check{C}_1^2, \check{C}_2^2, \check{C}_3^2, \ldots, \check{C}_n^2)$ respects a preference, it must be that $f_{a,c}(a, c, \check{C}_3^2(\{a,c\}), \ldots, \check{C}_n^2(\{a,c\})) = a$. Since $(\check{C}_3^2, \ldots, \check{C}_3^2) \in \mathcal{S}^{n-2}$ is arbitrary. This shows that (10) holds for all (a,y).

Now consider $x = d$ and $y = b$ for some $d \in A \setminus \{b\}$ in (10). Consider $\grave{P}_1 \equiv dab\cdots$ and $\grave{P}_2 \equiv bda\cdots$. Further, let $\grave{C}_l^2$ be the binary choice function generated by $\grave{P}_l$ for all $l \in \{1,2\}$. Fix $(\grave{C}_3^2, \ldots, \grave{C}_3^2) \in \mathcal{S}^{n-2}$. Then we have the following situation.

$\cdot$	$\grave{C}_1^2$	$\grave{C}_2^2$	$\grave{C}_3^2$	$\cdots$	$f(\grave{C}_1^2, \ldots, \grave{C}_n^2)$
$\{a,b\}$	a	b	$\grave{C}_3^2(\{a,b\})$	$\cdots$	a
$\{b,d\}$	d	b	$\grave{C}_3^2(\{b,c\})$	$\cdots$	
$\{a,d\}$	d	d	$\grave{C}_3^2(\{a,c\})$	$\cdots$	d
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Here, $f_{a,b}(a, b, \grave{C}_3^2(\{a,b\}), \ldots, \grave{C}_n^2(\{a,b\})) = a$ by (2). As $f(\grave{C}_1^2, \grave{C}_2^2, \grave{C}_3^2, \ldots, \grave{C}_n^2)$ respects a preference, it must be that $f_{b,d}(a, c, \grave{C}_3^2(\{a,c\}), \ldots, \grave{C}_n^2(\{a,c\})) = d$. Since $(\grave{C}_3^2, \ldots, \grave{C}_3^2) \in \mathcal{S}^{n-2}$ is arbitrary. This shows that (10) holds for all (x,b).

Next, from (x,b) we can show that (10) holds for $(x,y) \neq (b,a)$ in the same way we showed that (10) holds for (a,y) from (a,b). For the case (b,a), consider $c \in A \setminus \{a,b\}$, and first we show that (10) for (a,c). Then, we show that (10) for (b,c). Finally, we show that (10) for (b,a). ■