Problem. Spin probabilities

Problem Statement (i)

The matrix operator $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ corresponds to a spin component of an electron, in units of $\hbar/2$. For a state represented by the vector $|\Psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, where α, β are complex numbers, calculate the probability that the spin component is positive. Check your answer by finding the corresponding probability if $|\Psi\rangle$ is prepared with spin orientations,

1. $s_y = \pm\frac{\hbar}{2}$
2. $s_z = \pm\frac{\hbar}{2}$

Solution

The eigenvalues and eigenvectors for σ_y are well known to be,

$$\lambda_1 = +1, \quad |\uparrow_y\rangle = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ i \end{bmatrix} \tag{45}$$

$$\lambda_2 = -1, \quad |\downarrow_y\rangle = \sqrt{\frac{1}{2}} \begin{bmatrix} 1 \\ -i \end{bmatrix} \tag{46}$$

We have been given,

$$|\Psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha |\uparrow_z\rangle + \beta + |\downarrow_z\rangle \tag{47}$$

We want to represent this as a superposition of eigenstates of S_y.
The probability that for $|\Psi\rangle$, the spin is positive $\frac{\hbar}{2}$ in σ_y spin component is given by,

$$\begin{aligned} P &= |\langle\uparrow_y|S_y|\Psi\rangle|^2 && (48) \\ &= \frac{1}{2}\left|\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right|^2 && (49) \\ &= \frac{1}{2}\left|\begin{bmatrix} 1 & -i \end{bmatrix} \begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix}\right|^2 && (50) \\ &= \frac{1}{2}|\alpha - i\beta|^2 && (51) \\ & && (52) \end{aligned}$$

Similarly, the probability that for $|\Psi\rangle$, the spin is negative $-\frac{\hbar}{2}$ in σ_y spin component is given by,

$$\begin{aligned} P &= |\langle\downarrow_y|\sigma_y|\Psi\rangle|^2 && (53) \\ &= \frac{1}{2}\left|\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}\right|^2 && (54) \\ &= \frac{1}{2}\left|\begin{bmatrix} 1 & i \end{bmatrix} \begin{bmatrix} -i\beta \\ i\alpha \end{bmatrix}\right|^2 && (55) \\ &= \frac{1}{2}|\alpha + i\beta|^2 && (56) \\ & && (57) \end{aligned}$$

Similarly, the probability that for $|\Psi\rangle$, the spin is positive $\frac{\hbar}{2}$ in σ_z spin component is given by,

$$P = |\langle \uparrow_z | \sigma_z | \Psi \rangle|^2 \tag{58}$$

$$= \left| \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right|^2 \tag{59}$$

$$= \left| \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \right|^2 \tag{60}$$

$$= |\alpha|^2 \tag{61}$$

Similarly, the probability that for $|\Psi\rangle$, the spin is negative $-\frac{\hbar}{2}$ in σ_z spin component is given by,

$$P = |\langle \downarrow_z | \sigma_z | \Psi \rangle|^2 \tag{62}$$

$$= \left| \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \right|^2 \tag{63}$$

$$= \left| \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ -\beta \end{bmatrix} \right|^2 \tag{64}$$

$$= |-\beta|^2 \tag{65}$$