Problem : Spin orbit interaction of an electron

Problem Statement

Consider the spinor representation of the spin-orbit interaction for an electron in an atom. The form of interaction is given by

$$W_{S-O} = \frac{e^2}{8\pi\epsilon_0 m_e^2 c^2 R^3} \vec{L} \cdot \vec{S} \tag{66}$$

Where R is magnitude for position operator of the electron, $\vec{L}$ is the orbital angular momentum operator and $\vec{S}$ is the spin operator for the electron.

1. Show that we can write,

$$\vec{L} \cdot \vec{S} = \frac{1}{2} \left(S_+ L_- + S_- L_+ \right) + L_z S_z \tag{67}$$

where,

$$L_\pm \equiv L_x \pm i L_y \tag{68}$$

$$S_\pm \equiv S_x \pm i S_y \tag{69}$$

2. Using this result, write W_{S-O} as a 2×2 matrix spinor operator , where the individual elements are operators involving $L_\pm, L_z, R$, etc. For your basis for the spinor, use the eigenstates of S_z : $|+\rangle , |-\rangle$

Solution 1.

Going from RHS to LHS, we see,

$$\frac{1}{2}(S_+L_- + S_-L_+) + L_zS_z = \frac{1}{2}\left(\left(S_x + iS_y\right)\left(L_x - iL_y\right) + \left(S_x - iS_y\right)\left(L_x + iL_y\right)\right) + L_zS_z \tag{70}$$

$$= \frac{1}{2}\left(S_xL_x - iS_xL_y + iS_yL_x + S_yL_y + S_xL_x + iS_xL_y - iS_yL_x + S_yL_y\right) + L_zS_z \tag{71}$$

$$= \frac{1}{2}\left(2S_xL_x + 2S_yL_y\right) + L_zS_z \tag{72}$$

$$= S_xL_x + S_yL_y + S_zL_z \tag{73}$$

$$= \vec{S} \cdot \vec{L} \tag{74}$$

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Solution 2.

Computing S_+, S_- explicitly,

$$S_+ = S_x + iS_y \tag{75}$$

$$= \frac{\hbar}{2}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\right) \tag{76}$$

$$= \frac{\hbar}{2}\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \tag{77}$$

$$= \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{78}$$

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\begin{aligned}
S_- &= S_x - iS_y && (79)\\
&= \frac{\hbar}{2}\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} - i\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\right) && (80)\\
&= \hbar\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} && (81)\\
& && (82)
\end{aligned}
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This gives us,

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\begin{aligned}
\vec{L}\cdot\vec{S} &= \frac{1}{2}\left(S_+L_- + S_-L_+\right) + L_zS_z && (83)\\
&= \frac{\hbar}{2}\left(\begin{bmatrix}0 & L_-\\ 0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\ L_+ & 0\end{bmatrix}\right) + \frac{\hbar}{2}\begin{bmatrix}L_z & 0\\ 0 & -L_z\end{bmatrix} && (84)\\
&= \frac{\hbar}{2}\begin{bmatrix}L_z & L_-\\ L_+ & -L_z\end{bmatrix} && (85)\\
& && (86)
\end{aligned}
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Giving us,

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\begin{aligned}
W_{S-O} &= \frac{e^2}{8\pi\epsilon_0 m_e^2 c^2 R^3}\frac{\hbar}{2}\begin{bmatrix}L_z & L_-\\ L_+ & -L_z\end{bmatrix} && (87)\\
&= \frac{e^2}{8\pi\epsilon_0 m_e^2 c^2}\frac{\hbar}{2}\begin{bmatrix}L_z/R^3 & L_-/R^3\\ L_+/R^3 & -L_z/R^3\end{bmatrix} && (88)
\end{aligned}
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