Bose Einstein Condensate (2)

Problem Statement

Which of the following is necessary to undergo Bose condensation at low temperatures?

(a) $g(E)/\left(e^{\beta(E-E_{\text{min}})}+1\right)$ is finite as $E \to E_{\text{min}}^{-}$.

(b) $g(E)/\left(e^{\beta(E-E_{\text{min}})}-1\right)$ is finite as $E \to E_{\text{min}}^{-}$.

(c) $E_{\text{min}} \geq 0$.

(d) $\int_{E_{\text{min}}}^{E} g(E')\,/\,(E'-E_{\text{min}})\,dE'$ is convergent at the lower limit E_{min}.

(e) Bose condensation cannot occur in a system whose states are confined to an energy band.

Solution

Looking at all the expressions we have to choose from, E, N are two quantities based on which I can talk about BEC conditions. I will start with N and see if I can make any conclusions, else move on to E. (I also have some motivation on starting with N because it is the form given the options plus, it is the primary parameter that decides whether we have BEC or not. The exact condition is what we wish to derive here.

Starting of with,

$$N = \int_{E_0}^{E_1} \frac{g(E)\,dE}{e^{\beta(E-\mu)}-1} \tag{19}$$

We get a singularity when $\mu \to E$. Another issue is when $\mu > E$ gives us negative N. This means that $\mu < E_0$ (For $E_1 > E_0$) throughout the integral. (We have a similar intuition from what we derived in (a) of this problem.

Recall that the main idea with BEC is that, as $z \to 1$, we start getting more and more particles settling in the ground state which were not accounted for when we use the approximation: $\sum_k \approx \int g(E)\,dE$. This means that, for us to have BEC, this integral must converge!

$$N = \int_{E_0}^{E_1} \frac{g(E)\,dE}{e^{\beta(E-E_0)}-1} < \infty \tag{20}$$

$$\tag{21}$$

I would already go for (d) to be the solution, but I see that the denominator is not agreeing with what I am saying (at least yet).

For $E_1 > E_0$, the place where I the BEC states are sitting in the integral are near E_0. Let $E_{00} = E_0 + \epsilon$ where $\epsilon > 0$ and is a small number.

$$N = \int_{E_0}^{E_{00}} \frac{g(E)\, dE}{e^{\beta(E-E_0)} - 1} + \int_{E_{00}}^{E_1} \frac{g(E)\, dE}{e^{\beta(E-E_0)} - 1} < \infty \tag{22}$$

$$\tag{23}$$

where we can see that the first term will be contributing much more than the second term due to the proximity of E_0, E_{00}

If we Taylor expand the exponential in the first term, the denominator will be $e^{\beta(E-E_0)} \approx 1 + \beta(E - E_0) + \mathcal{O}((E - E_0)^2) - 1$. Ignoring higher terms, we can see that the **case (d)**

$$\int_{E_0}^{E} \frac{g(E)\, dE}{E - E_0} < \infty \tag{24}$$

is our answer. I safely inserted $E_{00} \to E$ as for any $E > E_{00}$, it only helps the convergence of the integral.