GR Exercise Sheet #9

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$\boxed{\text{Nr. 9.1}}$

$T^{\mu\nu} = 0$ vacuum : $\Box \bar{h}_{\mu\nu} = 0$ Solution is plane wave

$\bar{h}_{\mu\nu} = A_{\mu\nu} e^{ik_\gamma x^\gamma}$

We are working in Lorenz gauge: $\partial_\mu \bar{h}^{\mu\nu} = \bar{h}^{\mu\nu}{}_{,\mu} = 0$

$\boxed{(a)}$ $\bar{h}_{\mu\nu}{}^{,\mu} = \frac{\partial}{\partial x_\mu} \bar{h}_{\mu\nu} = \frac{\partial}{\partial x_\mu}(A_{\mu\nu} e^{ik_\gamma x^\gamma}) = ik_\gamma A_{\mu\nu} \frac{\partial x^\gamma}{\partial x_\mu} e^{ik_\gamma x^\gamma}$

$= ik_\gamma \delta^{\gamma\mu} A_{\mu\nu} e^{ik_\gamma x^\gamma} = ik^\mu A_{\mu\nu} e^{ik_\gamma x^\gamma} \overset{!}{=} 0$

$\Rightarrow k^\mu A_{\mu\nu} = 0 \rightarrow A_{\mu\nu} \perp k^\mu$

Use gauge freedom ($\bar{h}$ depends on ξ and is not fixed but leaves background $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ unchanged):

$\tilde{x}^\mu = x^\mu + \xi^\mu \quad |\partial_\alpha \xi^\mu| \ll 1$:

Lecture: h changes under general coordinate transformation

as: $h_{\bar{\mu}\bar{\nu}} = h_{\mu\nu} - (\xi_{\mu,\nu} + \xi_{\nu,\mu})$

$\bar{h}_{\bar{\mu}\bar{\nu}} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu} \xi^\alpha{}_{,\alpha}$

Look for ξ such that $\bar{h}_{\bar{\mu}\bar{\nu}}{}^{,\bar{\mu}} = \bar{h}^{new}_{\mu\nu}{}^{,\mu} \overset{\text{Lorenz}}{=} 0$

$\Rightarrow 0 = \bar{h}_{\mu\nu}{}^{,\mu} - \xi_{\mu,\nu}{}^{,\mu} - \xi_{\nu,\mu}{}^{,\mu} + \eta_{\mu\nu} \xi^{\alpha}{}_{,\alpha}{}^{,\mu}$

$= \cancel{\bar{h}_{\mu\nu}{}^{,\mu}} - \xi_{\nu,\mu}{}^{,\mu} - \cancel{\xi_{\mu,\nu}{}^{,\mu}} + \cancel{\xi^{\alpha}{}_{,\alpha}{}^{,\nu}}$

$= -\xi_{\nu,\mu}{}^{,\mu} \Rightarrow \xi_{\nu,\mu}{}^{,\mu} \underset{\text{weak field}}{=} \xi_{\nu;\mu}{}^{;\mu} = \Box \xi_\nu = 0$

This is again a wave equation $\Rightarrow$ Solution given by plane wave: $\Box \xi_\nu = 0 \Rightarrow \Box \xi^\nu = 0 \Rightarrow \xi^\mu = B^\mu(k) e^{ik_\alpha x^\alpha}$

$\xi^\mu{}_{;\nu} = \xi^\mu{}_{,\nu} = ik_\alpha \frac{\partial x^\alpha}{\partial x^\nu} B^\mu e^{ik_\alpha x^\alpha} = ik_\alpha \delta^\alpha_\nu B^\mu e^{ik_\alpha x^\alpha} = ik_\nu B^\mu e^{ik_\alpha x^\alpha}$

$\xi^\mu{}_{;\nu}{}^{;\nu} = \xi^\mu{}_{,\nu}{}^{,\nu} = ik_\nu B^\mu ik_\alpha \frac{\partial x^\alpha}{\partial x_\nu} e^{ik_\alpha x^\alpha} = -B^\mu k_\nu k_\alpha \delta^{\alpha\nu} e^{ik_\alpha x^\alpha}$

$= -B^\mu \underbrace{k_\nu k^\nu}_{=0 \text{ null vector}} e^{ik_\alpha x^\alpha} = 0$ ✓

$\Rightarrow \xi^\mu$ should take the form $\underline{\underline{\xi^\mu = B^\mu(k) e^{ik_\alpha x^\alpha}}}$

$\boxed{(b)}$ We know $\bar{h}^{new}_{\mu\nu} = \bar{h}^{old}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu} \xi^\alpha{}_{,\mu}$

$\Leftrightarrow A^{new}_{\mu\nu} e^{ik_\gamma x^\gamma} = A^{old}_{\mu\nu} e^{ik_\gamma x^\gamma} - (B_\mu e^{ik_\alpha x^\alpha})_{,\nu} - (B_\nu e^{ik_\alpha x^\alpha})_{,\mu}$

$+ \eta_{\mu\nu} (B^\alpha e^{ik_\gamma x^\gamma})_{,\alpha}$

$$= A^{old}_{\mu\nu} e^{ik_\gamma x^\gamma} - \left(B_\mu i k_\alpha \frac{\partial x^\alpha}{\partial x^\nu} + B_\nu i k_\alpha \frac{\partial x^\alpha}{\partial x^\mu}\right) e^{ik_\alpha x^\alpha} + \eta_{\mu\nu} B^\alpha i k_\gamma \frac{\partial x^\gamma}{\partial x^\alpha} e^{ik_\gamma x^\gamma}$$

$$= A^{old}_{\mu\nu} e^{ik_\gamma x^\gamma} - (B_\mu i k_\alpha \delta^\alpha_\nu + B_\nu i k_\alpha \partial^\alpha_\mu) e^{ik_\alpha x^\alpha} + \eta_{\mu\nu} B^\alpha i k_\gamma \delta^\gamma_\alpha e^{ik_\gamma x^\gamma} = A^{old}_{\mu\nu} e^{ik_\gamma x^\gamma} - (B_\mu i k_\nu + B_\nu i k_\mu) e^{ik_\alpha x^\alpha} + \eta_{\mu\nu} B^\alpha i k_\alpha e^{ik_\gamma x^\gamma} = e^{ik_\gamma x^\gamma}\left(A^{old}_{\mu\nu} - (B_\mu i k_\nu + B_\nu i k_\mu) + \eta_{\mu\nu} B^\alpha i k_\alpha\right.$$

$$\Rightarrow A^{new}_{\mu\nu} \cancel{e^{ik_\gamma x^\gamma}} = \cancel{e^{ik_\gamma x^\gamma}} \left(A^{old}_{\mu\nu} - (B_\mu i k_\nu + B_\nu i k_\mu) + \eta_{\mu\nu} B^\alpha k_\alpha \cdot i\right)$$

$$\Rightarrow A^{new}_{\mu\nu} = A^{old}_{\mu\nu} - B_\mu i k_\nu - i k_\mu B_\nu + i k_\alpha B^\alpha \eta_{\mu\nu}$$

Take the trace: $A^{new\,\mu}{}_\mu = g^{\mu\nu} A^{new}_{\mu\nu} = g^{\mu\nu}(A^{old}_{\mu\nu} - B_\mu i k_\nu - i k_\mu B_\nu + i k_\alpha B^\alpha \eta_{\mu\nu})$

$$= A^{old\,\mu}{}_\mu - i B^\nu k_\nu - i B^\mu k_\mu + i k_\alpha B^\alpha \underbrace{\eta^\mu_\mu}_{=4}$$

$$= A^{old\,\mu}{}_\mu - 2 i k_\mu B^\mu + 4 i k_\mu B^\mu$$

$$= A^{old\,\mu}{}_\mu + 2 i k_\mu B^\mu$$

- Traceless: $A^{new\,\mu}{}_\mu = 0 \Leftrightarrow A^{old\,\mu}{}_\mu + 2 i k_\mu B^\mu = 0$
  $\rightarrow 2 i k_\mu B^\mu = -A^{old\,\mu}{}_\mu$
- Transverse: $A^{new}_{\mu\nu} k^\mu = 0$ and 4-velocity $\vec{u}$: $u^\nu A^{new}_{\mu\nu} = 0$

$$u^\nu A^{new}_{\mu\nu} = u^\nu A^{old}_{\mu\nu} - i u^\nu B_\mu k_\nu - i u^\nu B_\nu k_\mu + i u^\nu \eta_{\mu\nu} B^\alpha k_\alpha$$

$$= u^\nu A^{old}_{\mu\nu} - i(u^\nu k_\nu B_\mu + k_\mu u^\nu B_\nu - u_\mu B^\alpha k_\alpha)$$

$$= u^\nu A^{old}_{\mu\nu} - i(u^\nu k_\nu \eta^\nu_\mu B_\nu + k_\mu u^\nu B_\nu - u_\mu k_\alpha \eta^{\alpha\nu} B_\nu)$$

$$= u^\nu A^{old}_{\mu\nu} - i(u^\nu k_\nu \delta^\nu_\mu B_\nu + k_\mu u^\nu B_\nu - u_\mu k^\nu B_\nu)$$

$$= u^\nu A^{old}_{\mu\nu} - i B_\nu \underbrace{(u^\nu k_\nu \delta^\nu_\mu + k_\mu u^\nu - u_\mu k^\nu)}_{= S^\nu_\mu}$$

$$\stackrel{!}{=} 0$$

$$\Rightarrow \quad 0 = i u^\nu A^{old}_{\mu\nu} + S^\nu_\mu B_\nu$$

$$\rightarrow \quad S^\nu_\mu B_\nu = -i u^\nu A^{old}_{\mu\nu}$$

We can now constrain B_ν ~~without further~~ in the TT-gauge:

$$S^\nu_\mu B_\nu = (u^\nu k_\nu \delta^\nu_\mu + k_\mu u^\nu - u_\mu k^\nu) B_\nu = -i u^\nu A^{old}_{\mu\nu}$$

$$= u^\nu k_\nu \delta^\nu_\mu B_\nu + k_\mu u^\nu B_\nu - u_\mu k^\nu B_\nu = -i u^\nu A^{old}_{\mu\nu}$$

With $k^\nu B_\nu = -\frac{1}{2i} A^{old\,\mu}{}_{\mu}$ from traceless constraint

$$\Rightarrow \quad u^\nu k_\nu B_\mu + k_\mu u^\nu B_\nu = -i u^\nu A^{old}_{\mu\nu} + U_\mu k^\nu B_\nu$$

$$= -i u^\nu A^{old}_{\mu\nu} - \frac{1}{2i} U_\mu A^{old\,\alpha}{}_{\alpha}$$

$$= -i u^\nu A^{old}_{\mu\nu} + \frac{i}{2} A^{old\,\alpha}{}_{\alpha} u_\mu \qquad | \cdot u^\mu$$

$$u^\mu (u^\nu k_\nu B_\mu + k_\mu u^\nu B_\nu) = u^\mu(-i u^\nu A^{old}_{\mu\nu} + \frac{i}{2} A^{old\,\alpha}{}_{\alpha} u_\mu)$$

$$u^\mu (u^\nu B_\nu \eta^\nu{}_\mu k_\nu + k_\mu u^\nu B_\nu) = -i u^\mu u^\nu A^{old}_{\mu\nu} + \frac{i}{2} A^{old\,\alpha}{}_{\alpha} \underbrace{u^\mu u_\mu}_{=-1}$$

$$(u^\mu k_\mu)(u^\nu B_\nu + u^\nu B_\nu) = -i u^\mu u^\nu A^{old}_{\mu\nu} - \frac{i}{2} A^{old\,\alpha}{}_{\alpha}$$

$$2(u^\mu k_\mu) u^\nu B_\nu = -i u^\mu u^\nu A^{old}_{\mu\nu} - \frac{i}{2} A^{old\,\alpha}{}_{\alpha}$$

$$\Rightarrow \quad u^\nu B_\nu = \frac{1}{2 u^\mu k_\mu}\left(-i u^\mu u^\nu A^{old}_{\mu\nu} - \frac{i}{2} A^{old\,\alpha}{}_{\alpha}\right)$$

So now we know $u^\mu B_\mu$ dependent on u^μ & $A^{old}_{\mu\nu}$:

$$\Rightarrow \quad u^\nu k_\nu B_\mu = -k_\mu (u^\nu B_\nu) - i u^\nu A^{old}_{\mu\nu} + \frac{i}{2} A^{old\,\alpha}{}_{\alpha} u_\mu$$

$$\Rightarrow \quad B_\mu = \frac{1}{u^\nu k_\nu}\left(-k_\mu (u^\nu B_\nu) - i u^\nu A^{old}_{\mu\nu} + \frac{i}{2} A^{old\,\alpha}{}_{\alpha} u_\mu\right)$$

$\vec{u}$ is a constant time-like vector that is chosen.
$A^{(old)}_{\mu\nu}$ is known $\Rightarrow$ we can calculate B_μ for every k_μ via the above relation because $u^\nu B_\nu$ is defined via $\vec{u}$ and $A^{old}{}_{\mu\nu}$ only and the rest also depends on u^ν and $A^{old}{}_{\mu\nu}$ only.

$\Rightarrow$ 4-vector to write the trace-reverse metric in the <u>TT-gauge</u>:

$$\xi_\mu = \frac{1}{u^\nu k_\nu}\left(-k_\mu (u^\nu B_\nu) - i u^\nu A^{old}_{\mu\nu} + \frac{i}{2} A^{old\,\alpha}{}_{\alpha} u_\mu\right) e^{i k_\alpha x^\alpha}$$

where $i k_\alpha x^\alpha = -i(\omega t - \vec{k}\cdot\vec{x})$

$$\Rightarrow \quad \xi_\mu = \frac{1}{u^\nu k_\nu}\left(-k_\mu (u^\nu B_\nu) - i u^\nu A^{old}_{\mu\nu} + \frac{i}{2} A^{old\,\alpha}{}_{\alpha} u_\mu\right) e^{-i(\omega t - \vec{k}\cdot\vec{x})}$$

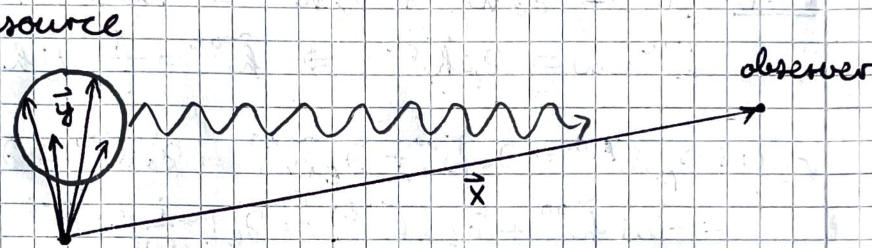
Exercise 9.2) Generation of GW and far-field approx.

(a) Linearized Einstein field equations (Lorentz gauge):

$$\Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}$$

Particular solution: $\bar{h}^{\mu\nu}(t,\vec{x}) = 4G \int \frac{T^{\mu\nu}(t - \frac{1}{c}|\vec{x}-\vec{y}|, \vec{y})}{|\vec{x}-\vec{y}|} d^3y$

Localized source: Source is constrained to a limited volume in space. Far-field approximation: "observer" is far away from the source.



$\vec{y}$ is constrained to the source because $T^{\mu\nu} = 0$ outside.

$\Rightarrow \vec{x} \gg \vec{y} \Rightarrow |\vec{x}-\vec{y}| \approx |\vec{x}|$ ⊛ $\quad (|x+y| = |x||1 + \frac{y}{x}| \approx |x|)$ ⊛

$\Rightarrow \bar{h}^{\mu\nu}(t,\vec{x}) = \frac{4G}{|\vec{x}|} \int T^{\mu\nu}(t - \frac{1}{c}|\vec{x}|, \vec{y})\, d^3y$

(b) FT wrt time → ω-space

$$\mathcal{F}_t[\bar{h}^{\mu\nu}] = \tilde{h}^{\mu\nu} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{h}^{\mu\nu}(t,\vec{x})\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, \frac{4G}{|\vec{x}|} \int T^{\mu\nu}(t - \frac{|\vec{x}|}{c}, \vec{y})\, d^3y\, e^{i\omega t}$$

$$= \frac{4G}{\sqrt{2\pi}|\vec{x}|} \int_{-\infty}^{\infty} dt \int d^3y\, T^{\mu\nu}(t - \frac{|\vec{x}|}{c}, \vec{y})\, e^{i\omega t}$$

$$\mathcal{F}_t[T^{\mu\nu}] = \tilde{T}^{\mu\nu} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt'\, T^{\mu\nu}(t')\, e^{i\omega t'} \qquad t' \to t - \frac{|\vec{x}|}{c}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, T^{\mu\nu}(t - \frac{|\vec{x}|}{c})\, e^{i\omega(t - \frac{|\vec{x}|}{c})}$$

$$\Rightarrow \mathcal{F}_t[\bar{h}^{\mu\nu}] = \frac{4G}{|\vec{x}|} \int d^3y \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt\, T^{\mu\nu}(t - \frac{|\vec{x}|}{c}, \vec{y})\, e^{i\omega(t - \frac{|\vec{x}|}{c})} \right) e^{i\omega\frac{|\vec{x}|}{c}}$$

$$= \frac{4G}{|\vec{x}|} \int d^3y\, \tilde{T}^{\mu\nu}(\omega, \vec{y})\, e^{i\omega\frac{|\vec{x}|}{c}} = \frac{4G\, e^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|} \int d^3y\, \tilde{T}^{\mu\nu}(\omega, \vec{y})$$

⊛ additional content for a):

$$|\vec{x}-\vec{y}| = ((\vec{x}-\vec{y})(\vec{x}-\vec{y}))^{1/2} = (\vec{x}^2 - 2\vec{x}\vec{y} + \vec{y}^2)^{1/2} = (\vec{x}^2 - 2|\vec{x}||\vec{y}|\cos\varphi + \vec{y}^2$$

$$= \left(|\vec{x}|^2 \left(1 - \frac{2|\vec{x}||\vec{y}|}{|\vec{x}|^2}\cos\varphi + \frac{|\vec{y}|^2}{|\vec{x}|^2}\right)\right)^{1/2} = |\vec{x}| \left(1 - 2\cancel{\frac{|\vec{y}|}{|\vec{x}|}} + \cancel{\left(\frac{|\vec{y}|}{|\vec{x}|}\right)^2}\right)^{1/2} \approx |\vec{x}|$$

$|\vec{y}| \ll |\vec{x}|$

(c) Lorenz gauge: $\partial_\mu \bar{h}^{\mu\nu} = 0$

$\partial_\mu \to \{\partial_0, \partial_i\}$ Time derivative in Fourier space:

$$\frac{\partial}{\partial t} f(t) = \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega\right)$$

$$= -i\omega \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega = -i\omega f(t)$$

$$\tilde{h}^{\mu\nu}(\omega, \vec{x}) = \frac{4G e^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\int d^3y \, \tilde{T}^{\mu\nu}(\vec{y}, \omega)$$

$\underline{\nu = l}$: $\partial_\mu \tilde{h}^{\mu l} = \partial_0 \tilde{h}^{0l} + \partial_j \tilde{h}^{jl} = -i\omega \tilde{h}^{0l} + \partial_j \tilde{h}^{jl} \overset{!}{=} 0$

$$\tilde{h}^{0l} \cdot i\omega = \partial_j \tilde{h}^{jl} \Rightarrow \tilde{h}^{0l} = \frac{1}{i\omega}\partial_j \tilde{h}^{jl} = \frac{-i}{\omega}\partial_j \tilde{h}^{jl} \quad \circledast$$

$\underline{\nu = 0}$: $\partial_\mu \tilde{h}^{\mu 0} = \partial_0 \tilde{h}^{00} + \partial_l \tilde{h}^{l0} = \partial_0 \tilde{h}^{00} + \partial_l \tilde{h}^{0l}$

$$= -i\omega \tilde{h}^{00} + \partial_l (\tilde{h}^{0l}) \overset{\circledast}{=} \partial_l\left(-\frac{i}{\omega}\partial_j \tilde{h}^{jl}\right) - i\omega \tilde{h}^{00}$$

$$= -\frac{i}{\omega}\partial_j \partial_l \tilde{h}^{jl} - i\omega \tilde{h}^{00} \overset{!}{=} 0$$

$$\Rightarrow \tilde{h}^{00} = \frac{1}{i\omega}\left(-\frac{i}{\omega}\partial_j \partial_l \tilde{h}^{jl}\right) = \frac{(-i)(-i)}{\omega^2}\partial_j\partial_l \tilde{h}^{jl} = -\frac{1}{\omega^2}\partial_j\partial_l \tilde{h}^{jl}$$

$$\Rightarrow \tilde{h}^{00}(\omega, \vec{x}) = -\frac{1}{\omega^2}\partial_j \partial_l \tilde{h}^{jl}(\omega, \vec{x})$$

$$\tilde{h}^{0l}(\omega, \vec{x}) = -\frac{i}{\omega}\partial_j \tilde{h}^{jl}(\omega, \vec{x})$$

$\Rightarrow$ all components $\tilde{h}^{0\nu}$ can be expressed in terms of $\tilde{h}^{jl}$ //

(d) Lorenz Gauge: $\partial_\mu \bar{h}^{\mu\nu} = 0$

$$\partial_\mu \tilde{h}^{\mu\nu}(\omega, \vec{x}) = \partial_\mu\left(\frac{4G e^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\int d^3y \, \tilde{T}^{\mu\nu}(\vec{y}, \omega)\right)$$

$\rightarrow \partial_\mu \bar{h}^{\mu\nu} = \ldots$

$$\Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu} \Rightarrow \partial_\mu \Box h^{\mu\nu} = -16\pi G \, \partial_\mu T^{\mu\nu}$$

$$\Leftrightarrow \partial^\alpha \partial_\alpha \partial_\mu h^{\mu\nu} = \Box(\underbrace{\partial_\mu h^{\mu\nu}}_{\overset{!}{=}0}) = -16\pi G \, \partial_\mu T^{\mu\nu}$$

$$\Rightarrow -16\pi G \, \partial_\mu T^{\mu\nu} = 0 \Rightarrow \partial_\mu T^{\mu\nu} = 0$$

$\Rightarrow$ Energy-momentum tensor is conserved at first order (because we're in the linearized case)

(e) 9.26): $\tilde{h}^{\mu\nu}(\omega, \vec{x}) = \frac{4G e^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\int d^3y \, \tilde{T}^{\mu\nu}(\vec{y}, \omega)$

Only need to compute $\tilde{h}^{ij}$:

$$\tilde{h}^{ij}(\omega, \vec{x}) = \frac{4G e^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\int d^3y \, \tilde{T}^{ij}(\vec{y}, \omega)$$

$$\tilde{T}^{ij} = \partial_e(y^i\tilde{T}^{ej}) - (\partial_e\tilde{T}^{ej})y^i = (\partial_e y^i)\tilde{T}^{ej} + \underbrace{y^i\cancel{\partial_e\tilde{T}^{ej}} - \cancel{(\partial_e\tilde{T}^{ej})y^i}}_{=0}$$

$$= \partial_e y^i\,\tilde{T}^{ej} = \frac{\partial y^i}{\partial y^e}\tilde{T}^{ej} = \delta^i_e\tilde{T}^{ej} = \tilde{T}^{ij}$$

$$\Rightarrow \tilde{h}^{ij}(\omega,\vec{x}) = \frac{4Ge^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\int d^3y\,\left(\partial_e(y^i\tilde{T}^{ej}) - (\partial_e\tilde{T}^{ej})y^i\right)$$

$$= \frac{4Ge^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\left\{\int d^3y\,\partial_e(y^i\tilde{T}^{ej}) - \int d^3y\,y^i\partial_e\tilde{T}^{ej}\right\}$$

$$= \frac{4Ge^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\left\{\int d^3y\,\nabla(y^i\tilde{T}^{ej}) - \int d^3y\,y^i\partial_e\tilde{T}^{ej}\right\}$$

$$\overset{\text{Gauß theorem}}{=} \frac{4Ge^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\left\{\oint_S dS\,(y^i\tilde{T}^{ej})\cdot n_e - \int d^3y\,y^i\partial_e\tilde{T}^{ej}\right\}$$

The source is localized and isolated $\Rightarrow$ surface integral vanishes: $\oint_S dS\,(y^i\tilde{T}^{ej})n_i = 0$

$$\Rightarrow \tilde{h}^{ij}(\omega,\vec{x}) = \frac{-4Ge^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\int d^3y\,y^i\partial_e\tilde{T}^{ej}$$

$\partial_\mu T^{\mu\nu} = 0$ in Fourier space: $\partial_\mu\tilde{T}^{\mu\nu} = \partial_0\tilde{T}^{0\nu} + \partial_e\tilde{T}^{e\nu}$

j-comp: $\partial_\mu\tilde{T}^{\mu j} = \partial_0\tilde{T}^{0j} + \partial_e\tilde{T}^{ej} = -i\omega\tilde{T}^{0j} + \partial_e\tilde{T}^{ej} \overset{!}{=} 0$

$\Rightarrow \partial_e\tilde{T}^{ej} = i\omega\tilde{T}^{0j}$

$$\Rightarrow \tilde{h}^{ij}(\omega,\vec{x}) = \frac{4Ge^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}i\omega\int d^3y\,y^i\tilde{T}^{0j}$$

symmetry of LHS & RHS

$$\Rightarrow i\omega\int d^3y\,y^i\tilde{T}^{0j} = 2i\omega\cdot\frac{1}{2}\int d^3y\,y^i\tilde{T}^{0j} = \frac{1}{2}i\omega\left(\int d^3y\,y^i\tilde{T}^{0j} + \int d^3y\,y^j\tilde{T}^{0i}\right)$$

$$= \frac{1}{2}i\omega\int d^3y\,(y^i\tilde{T}^{0j} + y^j\tilde{T}^{0i}) = \frac{1}{2}i\omega\int d^3y\,(\delta^j_e\tilde{T}^{0e}y^i + \delta^i_e\tilde{T}^{0e}y^j)$$

$$= \frac{1}{2}i\omega\left(\int d^3y\,\left(\frac{\partial y^j}{\partial y^e}\tilde{T}^{0e}y^i + \frac{\partial y^i}{\partial y^e}\tilde{T}^{0e}y^j\right)\right)$$

$$= \frac{1}{2}i\omega\left(\int d^3y\,\left(\tilde{T}^{0e}y^i(\partial_e y^j) + \tilde{T}^{0e}y^j(\partial_e y^i)\right)\right)$$

$$= \frac{1}{2}i\omega\left(\int d^3y\,\left(\tilde{T}^{0e}y^i(\partial_e y^j) + \tilde{T}^{0e}y^j(\partial_e y^i) + y^iy^j\partial_e\tilde{T}^{0e} - y^iy^j\partial_e\tilde{T}^{0e}\right)\right)$$

$$= \frac{1}{2}i\omega\left(\underbrace{\int d^3y\,(\partial_e(y^iy^j\tilde{T}^{0e})}_{\substack{\text{surface term}\\ \to\text{vanishes}}} - y^iy^j\partial_e\tilde{T}^{0e}\right)$$

$\oint dS\,y^iy^j\tilde{T}^{0e}n_e = 0$

$$= -\frac{1}{2}i\omega\int d^3y\,y^iy^j\partial_e\tilde{T}^{0e} = -\frac{1}{2}i\omega\int d^3y\,y^iy^j(-i\omega\tilde{T}^{00})$$

$-\partial_e\tilde{T}^{e0} = i\omega\tilde{T}^{00} = -\partial_0\tilde{T}^{00}$ (cons. of $\tilde{T}^{\mu\nu}$)

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$$= +\frac{1}{2}i^2\omega^2\int d^3y\,y^iy^j\tilde{T}^{00} = \frac{-\omega^2}{2}\int d^3y\,y^iy^j\tilde{T}^{00}$$

$$\Rightarrow \tilde{h}^{ij}(\omega,\vec{x}) = +\frac{4G e^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\int d^3y\, y^i \partial_l \tilde{T}^{lj}$$

$$= +\frac{4G e^{i\omega\frac{|\vec{x}|}{c}}}{|\vec{x}|}\left(\frac{-\omega^2}{2}\right)\int d^3y\, y^i y^j \tilde{T}^{00}$$

$$= -\frac{2G e^{i\omega\frac{|\vec{x}|}{c}}\omega^2}{|\vec{x}|}\int d^3y\, y^i y^j \tilde{T}^{00}$$

$\boxed{(f)}$ <u>Quadrupole moment</u>: $I^{ij}(t) = \int d^3y\, y^i y^j \rho(t,\vec{y})$

Weak-field limit: $T^{00} = \rho(t,\vec{y}) = T^{00}(t,\vec{y})$

$$\Rightarrow I^{ij} = \int d^3y\, y^i y^j T^{00}$$

FT: $\tilde{I}^{ij} = \frac{1}{\sqrt{2\pi}}\iint d^3y\, y^i y^j T^{00}\, dt\, e^{i\omega t}$

$$= \int d^3y\, y^i y^j \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{i\omega t} T^{00}\right)$$

$$= \int d^3y\, y^i y^j \tilde{T}^{00}(\omega)$$

e)

$$\Rightarrow \tilde{h}^{ij}(\omega,\vec{x}) = \frac{2G e^{i\omega\frac{|\vec{x}|}{c}}\omega^2}{|\vec{x}|}\tilde{I}^{ij}$$

<u>Back-transformation</u>: $\bar{h}^{ij}(t,\vec{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\tilde{h}(\omega,\vec{x})$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}\left(\frac{-2G e^{i\omega\frac{|\vec{x}|}{c}}\omega^2}{|\vec{x}|}\int d^3y\, y^i y^j \tilde{T}^{00}(\omega,\vec{y})\right)$$

$$= \frac{-2G}{|\vec{x}|}\frac{\omega^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t + i\omega\frac{|\vec{x}|}{c}}\int d^3y\, y^i y^j \tilde{T}^{00}(\omega,\vec{y})$$

$$= \frac{+2G}{\sqrt{2\pi}|\vec{x}|}(-\omega^2)\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-\frac{|\vec{x}|}{c})}\int d^3y\, y^i y^j \tilde{T}^{00}(\omega,\vec{y})$$

$$= +\frac{1}{\sqrt{2\pi}}\frac{2G}{|\vec{x}|} i^2\omega^2 \int_{-\infty}^{\infty} d\omega \int d^3y\, y^i y^j \tilde{T}^{00}(\omega,\vec{y})\, e^{-i\omega(t-\frac{|\vec{x}|}{c})}$$

FT: $\frac{\partial^2}{\partial t^2} \to i^2\omega^2$

$$= +\frac{1}{\sqrt{2\pi}}\frac{2G}{|\vec{x}|}\frac{\partial^2}{\partial t^2}\int_{-\infty}^{\infty} d\omega \int d^3y\, y^i y^j \tilde{T}^{00}(\omega,\vec{y})\, e^{-i\omega(t-\frac{|\vec{x}|}{c})}$$

$$= +\frac{2G}{|\vec{x}|}\frac{\partial^2}{\partial t^2}\left(\int d^3y\, y^i y^j \left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-\frac{|\vec{x}|}{c})}\tilde{T}^{00}(\omega,\vec{y})\right)\right)$$

$$= +\frac{2G}{|\vec{x}|}\frac{\partial^2}{\partial t^2}\int d^3y\, y^i y^j T^{00}\left(t-\frac{|\vec{x}|}{c},\vec{y}\right)$$

$$= +\frac{2G}{|\vec{x}|}\frac{\partial^2}{\partial t^2} I^{ij}\left(t-\frac{|\vec{x}|}{c}\right)$$

$$\Rightarrow \bar{h}^{ij}(t,\vec{x}) = +\frac{2G}{|\vec{x}|}\frac{\partial^2}{\partial t^2} I^{ij}\left(t-\frac{|\vec{x}|}{c}\right)$$

Exercise 9.3: Power of gravitational waves

Lorenz gauge: $\partial_\mu \bar{h}^{\mu\nu} = 0$ $\qquad \Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}$

Energy-momentum pseudotensor

$\boxed{(a)}$ From exercise 8.1: Ricci-tensor in terms of the trace reverse: $R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta} = \frac{1}{2}(h_{\mu\beta,\alpha}{}^{\mu} + h_{\alpha\mu,}{}^{\mu}{}_{\beta} - h^{\mu}{}_{\mu,\alpha\beta} - h_{\alpha\beta,}{}^{\mu}{}_{\mu})$

$= \frac{1}{2}(\bar{h}_{\mu\beta,\alpha}{}^{\mu} + \bar{h}_{\alpha\mu,}{}^{\mu}{}_{\beta} - \frac{1}{2}\bar{h}_{\alpha\beta})$ in the limit $|h_{\mu\nu}| \ll 1$

If we take $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ without $|h_{\mu\nu}| \ll 1$, there will be higher-order terms:

$R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta} = \eta^{\mu\nu} R_{\mu\alpha\nu\beta} + h^{\mu\nu} R_{\mu\alpha\nu\beta} + \dots$

$= \frac{1}{2}(\bar{h}_{\mu\beta,\alpha}{}^{\mu} + \bar{h}_{\alpha\mu,}{}^{\mu}{}_{\beta} - \frac{1}{2}\bar{h}^{\alpha\beta}) + R^{(2)}_{\alpha\beta}(\bar{h}) + \dots$

where 1st order is defined as terms of 1st order in $h_{\mu\nu}$ and 1st order in first or second derivatives of $h_{\mu\nu}$; $h_{\mu\nu,\alpha}$ or $h_{\mu\beta,\alpha}{}^{\mu}$ $\Rightarrow \frac{1}{2}(\bar{h}_{\mu\beta,\alpha}{}^{\mu} + \bar{h}_{\alpha\mu,}{}^{\mu}{}_{\beta} - \frac{1}{2}\bar{h}^{\alpha\beta}) = R^{(1)}_{\alpha\beta}(\bar{h})$

$\Rightarrow R_{\alpha\beta}(\bar{h}) = R^{(1)}_{\alpha\beta}(\bar{h}) + R^{(2)}_{\alpha\beta}(\bar{h}) + \dots$

Einstein-tensor:

$g^{\mu\nu} = \eta^{\mu\nu} - [\cancel{\ }] h^{\mu\nu}$
$= \eta^{\mu\nu} - (\bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h})$

$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\, g^{\mu\nu} R_{\mu\nu}$

$= R^{(1)}_{\alpha\beta}(\bar{h}) + R^{(2)}_{\alpha\beta}(\bar{h}) + \dots - \frac{1}{2}(\eta_{\alpha\beta} + h_{\alpha\beta})(\eta^{\mu\nu} - h_{\mu\nu})^{-1}(R^{(1)}_{\mu\nu}(\bar{h}) + R^{(2)}_{\mu\nu}(\bar{h}) + \dots)$

$= \underbrace{R^{(1)}_{\alpha\beta}(\bar{h}) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu} R^{(1)}_{\mu\nu}(\bar{h})}_{G^{(1)}_{\alpha\beta}(\bar{h})} + R^{(2)}_{\alpha\beta}(\bar{h}) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu} R^{(2)}_{\mu\nu}(\bar{h})$

$+ \frac{1}{2}\eta_{\alpha\beta} h^{\mu\nu} R^{(1)}_{\mu\nu}(\bar{h})$

$- \frac{1}{2} h_{\alpha\beta}\eta^{\mu\nu} R^{(1)}_{\mu\nu}(\bar{h})$

$+ O(\bar{h}^3)$

$= G^{(1)}_{\alpha\beta}(\bar{h}) + R^{(2)}_{\alpha\beta}(\bar{h}) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu} R^{(2)}_{\mu\nu}(\bar{h}) + \frac{1}{2}\eta_{\alpha\beta}(\bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h}) \cdot R^{(1)}_{\mu\nu}(\bar{h})$

$+ (-\frac{1}{2})(\bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h})\eta^{\mu\nu} R^{1}_{\mu\nu}(\bar{h}) + O(\bar{h}^3)$

$= G^{(1)}_{\alpha\beta}(\bar{h}) + G^{(2)}_{\alpha\beta}(\bar{h}) + O(\bar{h}^3)$

because $G^{(2)}_{\alpha\beta}(\bar{h}) = R^{2}_{\alpha\beta}(\bar{h}) - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu} R^{(2)}_{\mu\nu}(\bar{h}) + \frac{1}{2}\eta_{\alpha\beta}(\bar{h}^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\bar{h}) \cdot R^{(1)}_{\mu\nu}(\bar{h})$

$- \frac{1}{2}(\bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h})\eta^{\mu\nu} R^{1}_{\mu\nu}(\bar{h})$

is of order $\bar{h}^2$ (only products of $\eta\eta R^{(1)}$ or $\bar{h}\eta R^{(1)}$ or $R^{(2)} \cdot 1$)

$\Rightarrow$ Assume that metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ solves the Einstein's field equation exactly:

$$G_{\mu\nu}(\bar{h}) = 8\pi G\, T_{\mu\nu}$$

$$\Leftrightarrow \quad G^{(1)}_{\mu\nu}(\bar{h}) + G^{(2)}_{\mu\nu}(\bar{h}) + \ldots = 8\pi G\, T_{\mu\nu}$$

$$\Leftrightarrow \quad G^{(1)}_{\mu\nu}(\bar{h}) = 8\pi G\Big(T_{\mu\nu} \underbrace{- \frac{1}{8\pi G}\big(G^{(2)}_{\mu\nu}(\bar{h}) + \ldots\big)}_{=: \text{ energy momentum pseudo-tensor } t_{\mu\nu}}\Big)$$

$$t_{\mu\nu} = -\frac{1}{8\pi G}\left(G^{(2)}_{\mu\nu}(\bar{h}) + G^{(3)}_{\mu\nu}(\bar{h}) - \ldots\right)$$

The linearized field equation with source term $T_{\mu\nu} + t_{\mu\nu}$ is therefore solved by $\bar{h}$. The non-linear terms are now "hidden" in $t_{\mu\nu}$.

$G^{(2)}_{\mu\nu}(\bar{h}) = G^{(2)}_{\mu\nu}(\bar{h}^{(1)})$ because it only depends on products of two $\bar{h}^{(1)}$ or its derivatives.

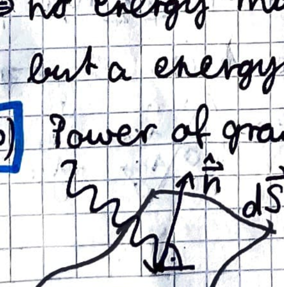
$$\rightarrow \quad t_{\mu\nu} = -\frac{1}{8\pi G}\left(G^{(2)}_{\mu\nu}(\bar{h}^{(1)}) + \ldots\right) \underset{\substack{\uparrow \\ \text{neglect} \\ \text{higher orders}}}{\approx} -\frac{1}{8\pi G}\left(G^{(2)}_{\mu\nu}(\bar{h}^{(1)})\right)$$

$$\Rightarrow \quad \underline{\underline{t_{\mu\nu} = -\frac{G^{(2)}_{\mu\nu}(\bar{h}^{(1)})}{8\pi G}}}$$

This is a pseudotensor because the energy-momentum tensor $T_{\mu\nu}$ contains everything but gravity. Because due to the equivalence principle gravity can be switched off by an appropriate transformation anywhere. But a non-zero tensor like $T_{\mu\nu}$ is non-zero at any point in any coordinates

$\Rightarrow$ no energy-momentum tensor of the gravitational field but a energy-momentum-pseudotensor.

$\boxed{\text{(b)}}$ Power of gravitational wave crossing $d\vec{S}$



t_{0i} has the same dimension as

$$T_{0i} = \rho v^i \gamma^2 \rightarrow \frac{\text{energy}}{\text{area} \cdot \text{time}}$$

The power is defined as the energy per time:

$\frac{dE}{dt} = P = \iint \vec{j}\, d\vec{S}$ where $\vec{j}$ is the <u>energy current density</u>, and $\vec{S}$ the area through which the current passes

It is $d\vec{S} = \hat{n} \cdot dS$, $\hat{n}$: unit vector $\parallel d\vec{S}$ and normal to S

$\Rightarrow dP = \vec{j}\, d\vec{S} = \underbrace{\vec{j} \cdot \hat{n}}_{\text{scalar product}}\, dS = j_i \hat{n}^i$

We can identify j_i with t_{0i} (total energy would be $E = \int t_{00}\, d^3x$) $\Rightarrow$ $\boxed{dP = t_{0i} \hat{n}^i dS}$

(d) Plane wave: $\bar{h}^{\alpha\beta} = \mathrm{Re}\left[A^{\alpha\beta} e^{ik_\mu x^\mu}\right]$

TT-gauge: $A^{\alpha\beta} k_\beta = 0$, $A^\alpha{}_\alpha = 0 \Rightarrow \bar{h}^{TT}_{\alpha\beta} = \bar{h}^{TT}{}_{\alpha\beta} = h^{TT}_{\alpha\beta}$

and $A_{\alpha\beta} u^\beta = 0$

$$t_{\mu\nu} = -\frac{G^{(2)}_{\mu\nu}(\bar{h}^{(1)})}{8\pi G}$$

Need to calculate $G^{(2)}(\bar{h}^{(1)}) \overset{TT}{=} G^{(2)}(h^{(1)})$

$$G^{(2)}_{\alpha\beta}(\bar{h}^{(1)}) = G^{(2)}_{\alpha\beta}(h^{(1)}) = R^{(2)}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\eta^{\mu\nu}R^{(2)}_{\mu\nu} + \frac{1}{2}\eta_{\alpha\beta}h^{\mu\nu}R^{(1)}_{\mu\nu} - \frac{1}{2}h_{\alpha\beta}\eta^{\mu\nu}R^{(1)}_{\mu\nu}$$

① $R^{(2)}_{\alpha\beta}$ needs to be calculated

$g^{\mu\nu} = (\eta^{\mu\nu} - h^{\mu\nu})$ because $g^{\mu\nu} g_{\mu\alpha} = (\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\mu\alpha} + h_{\mu\alpha})$

$= \eta^{\mu\nu}\eta_{\alpha\mu} - \cancel{h^{\mu\nu} h_{\alpha\mu}} + \eta^{\mu\nu} h_{\alpha\mu} - \eta_{\alpha\mu} h^{\mu\nu} = \delta^\nu_\alpha + h^\nu_\alpha - h^\nu_\alpha$

$= \delta^\nu_\alpha = g^\nu_\alpha \Rightarrow g^{\mu\nu} g_{\mu\alpha} \approx \delta^\nu_\alpha$ with $g^{\mu\nu} = -h^{\mu\nu} + \eta^{\mu\nu}$

<u>Ricci-tensor</u> is defined as: $R_{\alpha\beta} = g^{\mu\nu} R_{\mu\alpha\nu\beta} = R_{\mu\alpha\nu\beta}(\eta^{\mu\nu} - h^{\mu\nu})$

$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\mu \Gamma^\rho_{\rho\nu} - \partial_\rho \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{\rho\nu} - \Gamma^\rho_{\sigma\rho}\Gamma^\sigma_{\mu\nu}$

<u>Christoffels</u>: $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$

$= \frac{1}{2}(\eta^{\rho\sigma} - h^{\rho\sigma})(\partial_\mu h_{\sigma\nu} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu})$

$$\begin{aligned}
\Rightarrow R_{\mu\nu} = {} & \partial_\mu\left(\frac{\eta^{\rho\sigma}}{2}(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\rho\nu,\sigma}) - \frac{h^{\rho\sigma}}{2}(h_{\sigma\nu,\rho} + h_{\sigma\rho,\nu} - h_{\rho\nu,\sigma})\right) \\
& - \partial_\rho\left(\frac{\eta^{\rho\sigma}}{2}(h_{\sigma\nu,\mu} + h_{\sigma\mu,\nu} - h_{\mu\nu,\sigma}) - \frac{h^{\rho\sigma}}{2}(h_{\sigma\nu,\mu} + h_{\sigma\mu,\nu} - h_{\mu\nu,\sigma})\right) \\
& + \left(\frac{1}{2}\left(\frac{\eta^{\rho\kappa}}{2}(h_{\sigma\kappa,\mu} + h_{\mu\kappa,\sigma} - h_{\sigma\mu,\kappa}) - \frac{h^{\rho\kappa}}{2}(h_{\sigma\kappa,\mu} + h_{\mu\kappa,\sigma} - h_{\sigma\mu,\kappa})\right)\right. \\
& \left.\cdot \frac{1}{2}\left(\frac{\eta^{\sigma\lambda}}{2}(h_{\rho\lambda,\nu} + h_{\rho\nu,\lambda} - h_{\rho\nu,\lambda}) - \frac{h^{\sigma\lambda}}{2}(h_{\rho\lambda,\nu} + h_{\lambda\nu,\rho} - h_{\rho\nu,\lambda})\right)\right) \\
& - \left(\frac{1}{2}\left(\frac{\eta^{\rho\kappa}}{2}(h_{\sigma\kappa,\rho} + h_{\rho\kappa,\sigma} - h_{\sigma\rho,\kappa}) - \frac{h^{\rho\kappa}}{2}(h_{\sigma\kappa,\rho} + h_{\rho\kappa,\sigma} - h_{\sigma\rho,\kappa})\right)\right. \\
& \left.\cdot \frac{1}{2}\left(\frac{\eta^{\sigma\lambda}}{2}(h_{\mu\lambda,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda}) - \frac{h^{\sigma\lambda}}{2}(h_{\mu\lambda,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda})\right)\right)
\end{aligned}$$

$= R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu}$

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Keep only terms of order h^2 because $O(h^{(1)})$ will be part of $R_{\alpha\beta}^{(1)}$:

$$R_{\mu\nu}^{(2)} = -\frac{1}{2}\partial_\mu\left(h^{\rho\sigma}(\cancel{h_{\sigma\nu,\rho}} + h_{\sigma\rho,\nu} - \cancel{h_{\rho\nu,\sigma}})\right) + \frac{1}{2}\partial_\rho\left(h^{\rho\sigma}(h_{\sigma\nu,\mu} + h_{\sigma\mu,\nu} - h_{\mu\nu,\sigma})\right)$$

$$+\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\rho\kappa,\mu} + h_{\mu\kappa,\rho} - h_{\rho\mu,\kappa})(h_{\rho\lambda,\nu} + h_{\nu\lambda,\rho} - h_{\rho\nu,\lambda})$$

$$-\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,\rho} + h_{\rho\kappa,\sigma} - h_{\sigma\rho,\kappa})(h_{\mu\lambda,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda})$$

symmetry

$$= -\frac{1}{2}\partial_\mu(h^{\rho\sigma}h_{\sigma\rho,\nu}) + \frac{1}{2}\partial_\rho\left(h^{\rho\sigma}(h_{\sigma\nu,\mu} + h_{\sigma\nu,\mu} + h_{\sigma\mu,\nu} - h_{\mu\nu,\sigma}\right) + \frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}$$

$$(h_{\rho\kappa,\mu} + h_{\mu\kappa,\rho} - h_{\rho\mu,\kappa})(h_{\rho\lambda,\nu} + h_{\nu\lambda,\rho} - h_{\rho\nu,\lambda}) - \frac{1}{4}(\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,\rho} + h_{\rho\kappa,\sigma}$$

$$- h_{\sigma\rho,\kappa})(h_{\mu\lambda,\nu} + h_{\lambda\nu,\mu} - h_{\mu\nu,\lambda})$$

② Ex. 8.1: $R_{\mu\nu}^{(1)} = \frac{1}{2}(h_{\nu\beta,\mu}{}^{\beta} + h_{\mu\beta}{}^{,\beta}{}_{,\nu} - \frac{1}{2}h_{\mu\nu})$

$$R^{(1)}g_{\mu\nu} = \eta_{\mu\nu}h_{\alpha\beta}{}^{,\alpha\beta} + \frac{1}{2}h_{\mu\nu,}{}^{\beta}{}_{\beta}$$

$$G^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}R_{\alpha\beta}^{(2)} + \frac{1}{2}\eta_{\mu\nu}h^{\alpha\beta}R^{(1)}_{\alpha\beta} - \frac{1}{2}h_{\mu\nu}\eta^{\alpha\beta}R^{(1)}_{\alpha\beta}$$

Plane wave in TT-gauge: (assume z-propagation):

$$\bar{h}^{\alpha\beta} = \mathrm{Re}[A^{\alpha\beta}e^{ik_\mu x^\mu}] \quad \text{and} \quad A^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \bar{h}^{\alpha\beta} = h^{\alpha\beta} \quad ; \quad h_{0\nu} = h_{\nu 0} = 0,\ h^{\mu}{}_{\mu} = 0,\ h_{00} = 0$$

Look at the different terms of $G^{(2)}_{\mu\nu}$:

- $h_{\mu\nu}\eta^{\alpha\beta}R_{\alpha\beta}^{(1)} = h_{\mu\nu}(h_{\alpha\beta}{}^{,\alpha\beta} - \cancel{h^{,\alpha}{}_{,\alpha}}) = h_{\mu\nu}(\partial_\beta(\cancel{\partial_\alpha h^{\alpha\beta}})^{-1}) = 0$
- $h^{\alpha\beta}R_{\alpha\beta}^{(1)} = 0$ due to the TT-gauge conditions
- $\eta^{\alpha\beta}R_{\alpha\beta}^{(2)} = -\frac{1}{2}(\partial_\beta)^{-1}(h^{\rho\sigma}(h_{\rho\sigma,\beta})) + \frac{1}{2}\partial_\rho(h^{\rho\sigma}(\cancel{h_{\sigma\beta,}{}^{\beta}} + \cancel{h_{\sigma\beta,}{}^{\beta}} - \cancel{h^{\alpha}{}_{\alpha,\sigma}}))$

$$+\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,}{}^{\alpha} + h^{\alpha}{}_{\kappa,\sigma} - h_{\sigma}{}^{\alpha}{}_{,\kappa})(h_{\rho\lambda,}{}^{\alpha} + h^{\alpha}{}_{\lambda,\rho} - h_{\rho}{}^{\alpha}{}_{,\lambda})$$

$$-\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,\rho} + h_{\rho\kappa,\sigma} - h_{\sigma\rho,\kappa})(\cancel{h_{\alpha\lambda,}{}^{\alpha}} + \cancel{h^{\alpha}{}_{\lambda,}{}^{\alpha}} - \cancel{h^{\alpha}_{\alpha,\lambda}})$$

$$= -\frac{1}{2}\partial^\beta(h^{\rho\sigma}\partial_\beta h_{\rho\sigma}) + \frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,}{}^{\alpha} + h^{\alpha}{}_{\kappa,\sigma} - h_{\sigma}{}^{\alpha}{}_{,\kappa})$$

$$(h_{\rho\lambda,}{}^{\alpha} + h^{\alpha}{}_{\lambda,\rho} - h_{\rho}{}^{\alpha}{}_{,\lambda}) = -\frac{1}{2}\underbrace{\partial^\beta(h^{\rho\sigma}ik_\beta h_{\rho\sigma})}_{=0 \text{ because } k^\beta h_\beta = 0} + \frac{1}{4}(\ldots)$$

$$= \frac{1}{4}(h^{\rho\lambda}{}_{,}{}^{\alpha} + h^{\alpha\rho,\lambda} - h^{\lambda\alpha,\rho})(h_{\rho\lambda,}{}^{\alpha} + h^{\alpha}{}_{\lambda,\rho} - h_{\rho}{}^{\alpha}{}_{,\lambda})$$

$$= \frac{1}{4}(\underbrace{h^{\rho\lambda,\alpha}h_{\rho\lambda,}{}^{\alpha}}_{=0} + \cancel{h^{\rho\lambda,\alpha}h^{\alpha}{}_{\lambda,\rho}} - \cancel{h^{\rho\lambda,\alpha}h_{\rho}{}^{\alpha}{}_{,\lambda}} + \cancel{h^{\alpha\rho,\lambda}h_{\rho\lambda,}{}^{\alpha}}$$

$$+ \cancel{h^{\alpha}{}_{\lambda,\rho}h^{\alpha\rho,\lambda}} + \cancel{h_{\rho}{}^{\alpha}{}_{,\lambda}h^{\alpha\rho,\lambda}} - \cancel{h^{\lambda\alpha,\rho}h_{\rho\lambda,}{}^{\alpha}} + (-\cancel{h^{\lambda\alpha,\rho}h_{\rho}{}^{\alpha}{}_{,\lambda}})$$

$$+ \cancel{h^{\lambda\alpha,\rho}h_{\rho}{}^{\alpha}{}_{,\lambda}}) = 0 \quad \text{(because exemplary:}$$

$$h^{\rho\lambda,\alpha}h_{\rho\lambda,\alpha} = i^2 k^\alpha k_\alpha h^{\rho\lambda}h_{\rho\lambda} = 0 \quad \text{because } k^\alpha k_\alpha = 0$$

$$\Rightarrow G^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu} + 0 + 0 + 0 \quad \text{for plane wave + TT-gauge}$$

$$\Rightarrow t_{\mu\nu} = -\frac{R_{\mu\nu}^{(2)}(\bar{h}^{(1)})}{8\pi G} \rightarrow t_{0i} = -\frac{R_{0i}^{(2)}(\bar{h}^{(1)})}{8\pi G} = -\frac{R_{0i}^{(2)}(h^{(1)})}{8\pi G}$$

$$R_{\mu\nu}^{(2)} = -\frac{1}{2}\partial_\mu(h^{\rho\sigma}(h_{\sigma\rho,\nu})) + \frac{1}{2}\partial_\rho((h_{\sigma\nu,\mu}+h_{\sigma\mu,\nu}-h_{\mu\nu,\sigma}))h^{\rho\sigma})$$
$$+\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,\mu}+h_{\mu\kappa,\sigma}-h_{\sigma\mu,\kappa})(h_{\rho\lambda,\nu}+h_{\nu\lambda,\rho}-h_{\rho\nu,\lambda})$$
$$-\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,\rho}+h_{\rho\kappa,\sigma}-h_{\sigma\rho,\kappa})(h_{\mu\lambda,\nu}+h_{\nu\lambda,\mu}-h_{\mu\nu,\lambda})$$

③ $$R_{0i}^{(2)} = -\frac{1}{2}\partial_0(h^{\rho\sigma}h_{\sigma\rho,i}) + \frac{1}{2}\partial_\rho(h_{\sigma i,0}+\cancel{h_{\sigma 0,i}}-\cancel{h_{0i,\sigma}})$$
$$+\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,0}+\cancel{h_{0\kappa,\sigma}}-\cancel{h_{\sigma 0,\kappa}})(h_{\rho\lambda,i}+h_{i\lambda,\rho}-h_{\rho i,\lambda})$$
$$-\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,\rho}+h_{\rho\kappa,\sigma}-h_{\sigma\rho,\kappa})(\cancel{h_{0\lambda,i}}+h_{i\lambda,0}-\cancel{h_{0i,\lambda}})$$
$$= -\frac{1}{2}\partial_0(h^{\rho\sigma}h_{\sigma\rho,i}) + \frac{1}{2}\partial_\rho(h_{\sigma i,0}h^{\rho\sigma}) + \frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,0})(h_{\rho\lambda,i}+h_{i\lambda,\rho}-h_{\rho i,\lambda})$$
$$-\frac{1}{4}\eta^{\rho\kappa}\eta^{\sigma\lambda}(h_{\sigma\kappa,\rho}+h_{\rho\kappa,\sigma}-h_{\sigma\rho,\kappa})h_{i\lambda,0}$$
$$= -\frac{1}{2}\partial_0(h^{\rho\sigma}h_{\sigma\rho,i}) + \frac{1}{2}\partial_\rho(h^{\rho\sigma}h_{\sigma i,0}) + \frac{1}{4}h^{\rho\lambda}{}_{,0}(h_{\rho\lambda,i}+\cancel{h_{i\lambda,\rho}-h_{\rho i,\lambda}})$$
symmetry
$$-\frac{1}{4}\eta^{\sigma\lambda}h_{i\lambda,0}(\cancel{h_{\sigma\kappa,}{}^{\kappa}}+h_{\rho\kappa,\sigma}\eta^{\rho\kappa}-\cancel{h_{\sigma\rho,}{}^{\rho}})$$
$$= -\frac{1}{2}\partial_0(h^{\rho\sigma}h_{\sigma\rho,i}) + \frac{1}{2}\partial_\rho(h^{\rho\sigma}h_{\sigma i,0}) + \frac{1}{4}h^{\rho\lambda}{}_{,0}h_{\rho\lambda,i}$$
$$-\frac{1}{4}\eta^{\sigma\lambda}h_{i\lambda,0}\eta^{\rho\kappa}h_{\rho\kappa,\sigma}$$
$$= -\frac{1}{2}\partial_0(h^{\rho\sigma}h_{\sigma\rho,i}) + \frac{1}{2}\partial_\rho(h^{\rho\sigma}h_{\sigma i,0}) + \frac{1}{4}h^{\rho\lambda}{}_{,0}h_{\rho\lambda,i}$$
$$-\frac{1}{4}\eta^{\sigma\lambda}h_{i\lambda,0}\cancel{h_{\rho}{}^{\rho}{}_{,\sigma}}$$
$$= -\frac{1}{2}\partial_0(h^{\rho\sigma}h_{\sigma\rho,i}) + \frac{1}{2}\partial_\rho(h^{\rho\sigma}h_{\sigma i,0}) + \frac{1}{4}h^{\rho\lambda}{}_{,0}h_{\rho\lambda,i}$$
$$= -\frac{1}{2}(\partial_0 h^{\rho\sigma})h_{\sigma\rho,i} - \frac{1}{2}(\partial_0 h_{\sigma\rho,i})h^{\rho\sigma} + \frac{1}{4}h^{\rho\sigma}{}_{,0}h_{\rho\sigma,i}$$
$$+\frac{1}{2}(\cancel{\partial_\rho h^{\rho\sigma}})h_{\sigma i,0} + \underbrace{\frac{1}{2}(\partial_\rho h_{\sigma i,0})h^{\rho\sigma}}_{\frac{1}{2}h^{\rho\sigma}\partial_\rho\partial_0 h_{\sigma i}\,=\,0 \text{ because of } h^{\rho\sigma}k_\rho = 0}$$ ⊛
$$= -\frac{1}{2}(\partial_0 h^{\rho\sigma})h_{\sigma\rho,i} + \frac{1}{4}h^{\rho\sigma}{}_{,0}h_{\rho\sigma,i} - \frac{1}{2}h^{\rho\sigma}(\partial_0 h_{\sigma\rho,i})$$
$$= -\frac{1}{4}h^{\rho\sigma}{}_{,0}h_{\rho\sigma,i} - \frac{1}{2}h^{\rho\sigma}(\partial_0 h_{\sigma\rho,i})$$
$$= -\frac{1}{4}h^{\rho\sigma}{}_{,0}h_{\rho\sigma,i} - \frac{1}{2}h^{\rho\sigma}h_{\rho\sigma,0i}$$
$$\Rightarrow \quad t_{0i} = -\frac{1}{8\pi G}G_{0i}^{(2)} = -\frac{1}{8\pi G}R_{0i}^{(2)} = -\frac{1}{16\pi G}\left(h^{\rho\sigma}{}_{,0}h_{\rho\sigma,i}+\frac{1}{2}h^{\rho\sigma}h_{\rho\sigma,0i}\right)$$
TT: $h=\bar{h}$, $h_{0\nu}=h_{00}=0$
$$\rightsquigarrow = +\frac{1}{16\pi G}\left(\bar{h}^{ej}{}_{,0}\bar{h}_{ej,i}+\frac{1}{2}\bar{h}^{ej}\bar{h}_{ej,0i}\right)$$

⊛ $$(\partial_\rho\partial_0 h_{\sigma i})h^{\rho\sigma} = h^{\rho\sigma}\partial_\rho\partial_0 A_{\sigma i}e^{ik_\mu x^\mu} = h^{\rho\sigma}\partial_\rho(ik_\mu\delta^\mu_0 A_{\sigma i}e^{ik_\mu x^\mu})$$
$$= h^{\rho\sigma}k_0 i A_{\sigma i}\frac{\partial x^\mu}{\partial x^\rho}e^{ik_\mu x^\mu}ik_\mu = ik_0 i A_{\sigma i}k_\mu\delta^\mu_\rho h^{\rho\sigma}e^{ik_\mu x^\mu}$$
$$= i^2 k_0 A_{\sigma i}e^{ik_\mu x^\mu}(h^{\rho\sigma}k_\rho) = 0$$ due to TT-gauge
$$= 0$$

(e) Time-averaged current density of a plane gravitational wave $\bar{h}^{\alpha\beta} \overset{TT}{=} h^{\alpha\beta} = \mathrm{Re}[A^{\alpha\beta}e^{ik_\mu x^\mu}]$

From (d) it is known: $t_{0i} = +\frac{1}{16\pi G}\left(\bar{h}^{lj}{}_{,0}\bar{h}_{lj,i} + \frac{1}{2}\bar{h}^{lj}\bar{h}_{lj,0i}\right)$

$$\Rightarrow t_{0i} = \frac{1}{16\pi G}\Big(\partial_0 \mathrm{Re}[A^{lj}e^{ik_\mu x^\mu}]\,\partial_i \mathrm{Re}[A_{lj}e^{ik_\mu x^\mu}] + \frac{1}{2}\mathrm{Re}[A^{lj}e^{ik_\mu x^\mu}] \cdot \partial_0\partial_i \mathrm{Re}[A_{lj}e^{ik_\mu x^\mu}]\Big)$$

$$\Rightarrow 16\pi G\, t_{0i} = \mathrm{Re}\Big(ik_\mu \frac{\partial x^\mu}{\partial x^0}A^{lj}e^{ik_\mu x^\mu}\Big)\mathrm{Re}\Big(ik_\mu \frac{\partial x^\mu}{\partial x^i}A_{lj}e^{ik_\mu x^\mu}\Big) + \frac{1}{2}\mathrm{Re}[A^{lj}e^{ik_\mu x^\mu}]\,\mathrm{Re}\Big[i^2\Big(k_\mu\frac{\partial x^\mu}{\partial x^0}\Big)\Big(\frac{\partial x^\mu}{\partial x^i}k_\mu\Big)(A^{lj})^{-1}e^{ik_\mu x^\mu}\Big]\Big]$$

$$= +k_0k_i\,\mathrm{Re}[iA^{lj}e^{ik_\mu x^\mu}]\,\mathrm{Re}[iA_{lj}e^{ik_\mu x^\mu}] - \frac{1}{2}k_0k_i\,\mathrm{Re}[A^{lj}e^{ik_\mu x^\mu}]\,\mathrm{Re}[A_{lj}e^{ik_\mu x^\mu}]$$

For a complex number z it is: $\mathrm{Re}(z) = \frac{z+\bar{z}}{2}$ wher "bar" denotes complex conjugated

$$\Rightarrow \mathrm{Re}[A^{lj}e^{ik_\mu x^\mu}] = \frac{1}{2}\left(A^{lj}e^{ik_\mu x^\mu} + \bar{A}^{lj}e^{-ik_\mu x^\mu}\right)$$

$$\Rightarrow t_{0i}\cdot 16\pi G = +k_0k_i\frac{1}{2}\left(iA^{lj}e^{ik_\mu x^\mu} - i\bar{A}^{lj}e^{-ik_\mu x^\mu}\right)\frac{1}{2}\left(iA_{lj}e^{ik_\mu x^\mu} - i\bar{A}_{lj}e^{-ik_\mu x^\mu}\right) - \frac{1}{2}k_0k_i\cdot\frac{1}{4}\left(A^{lj}e^{ik_\mu x^\mu} + \bar{A}^{lj}e^{-ik_\mu x^\mu}\right)\left(A_{lj}e^{ik_\mu x^\mu} + \bar{A}_{lj}e^{-ik_\mu x^\mu}\right)$$

$$= \frac{k_0k_i}{4}\left(i^2A_{lj}A^{lj}e^{2ik_\mu x^\mu} + i^2\bar{A}^{lj}\bar{A}_{lj}e^{-2ik_\mu x^\mu} - i^2\bar{A}^{lj}A_{lj}e^{-ik_\mu x^\mu}e^{ik_\mu x^\mu} - i^2A^{lj}\bar{A}_{lj}e^{ik_\mu x^\mu}e^{-ik_\mu x^\mu}\right) - \frac{k_0k_i}{8}\Big(A^{lj}A_{lj}e^{ik_\mu x^\mu}e^{ik_\mu x^\mu} + \bar{A}^{lj}\bar{A}_{lj}e^{-ik_\mu x^\mu}e^{-ik_\mu x^\mu} + A^{lj}\bar{A}_{lj}e^{ik_\mu x^\mu}e^{-ik_\mu x^\mu} + \bar{A}^{lj}A_{lj}e^{-ik_\mu x^\mu}e^{ik_\mu x^\mu}\Big)$$

$$= \frac{k_0k_i}{4}\left(-A_{lj}A^{lj}e^{2ik_\mu x^\mu} - \bar{A}^{lj}\bar{A}_{lj}e^{-2ik_\mu x^\mu} + \bar{A}^{lj}A_{lj} + A^{lj}\bar{A}_{lj}\right) - \frac{k_0k_i}{8}\left(A^{lj}A_{lj}e^{2ik_\mu x^\mu} + \bar{A}^{lj}\bar{A}_{lj}e^{-2ik_\mu x^\mu} + A^{lj}\bar{A}_{lj} + \bar{A}^{lj}A_{lj}\right)$$

$$= \left(\frac{2k_0k_i}{8} + \frac{k_0k_i}{8}\right)\left(-A_{lj}A^{lj}e^{2ik_\mu x^\mu} - \bar{A}^{lj}\bar{A}_{lj}e^{-2ik_\mu x^\mu}\right) + \left(\bar{A}^{lj}A_{lj} + A^{lj}\bar{A}_{lj}\right)\left(\frac{k_0k_i}{4} - \frac{k_0k_i}{8}\right)$$

$A_{lj}\bar{A}^{lj} = A^{lj}\bar{A}_{lj} = |z|^2$

$$= -\frac{3k_0k_i}{8}\left(A_{lj}A^{lj}e^{2ik_\mu x^\mu} + \bar{A}_{lj}\bar{A}^{lj}e^{-2ik_\mu x^\mu}\right) + 2A_{lj}\bar{A}^{lj}\frac{k_0k_i}{8}$$

$$\overset{\circledast}{=} -\frac{3k_0k_i}{4}\left(\left(\frac{e^{2ik_\mu x^\mu} + e^{-2ik_\mu x^\mu}}{2}\right)\left(\frac{A^{lj}A_{lj} + \bar{A}^{lj}\bar{A}_{lj}}{2}\right) - \left(\frac{e^{2ik_\mu x^\mu} - e^{-2ik_\mu x^\mu}}{2i}\right)\left(\frac{A^{lj}A_{lj} - \bar{A}^{lj}\bar{A}_{lj}}{2i}\right)\right) + \frac{A_{lj}\bar{A}^{lj}}{4}k_0k_i$$

$\Big|\ \frac{z-\bar{z}}{2i} = \mathrm{Im}(z)$

$$\circledast\ \left(\frac{e^{2ik_\mu x^\mu} + e^{-2ik_\mu x^\mu}}{2}\right)\left(\frac{A^{lj}A_{lj} + \bar{A}^{lj}\bar{A}_{lj}}{2}\right) - \left(\frac{e^{2ik_\mu x^\mu} - e^{-2ik_\mu x^\mu}}{2i}\right)\left(\frac{A^{lj}A_{lj} - \bar{A}^{lj}\bar{A}_{lj}}{2i}\right)$$

$$= \frac{1}{4}\Big(A^{\ell j}A_{\ell j}e^{2ik_\mu x^\mu} + \bar{A}^{\ell j}\bar{A}_{\ell j}e^{-2ik_\mu x^\mu} + \cancel{A^{\ell j}\bar{A}_{\ell j}e^{-2ik_\mu x^\mu}} + \cancel{\bar{A}^{\ell j}A_{\ell j}e^{2ik_\mu x^\mu}}$$
$$+ A^{\ell j}A_{\ell j}e^{2ik_\mu x^\mu} + \bar{A}^{\ell j}\bar{A}_{\ell j}e^{-2ik_\mu x^\mu} - \cancel{A^{\ell j}\bar{A}_{\ell j}e^{-2ik_\mu x^\mu}} - \cancel{\bar{A}^{\ell j}A_{\ell j}e^{2ik_\mu x^\mu}}\Big)$$
$$= \frac{2}{4}\left(A^{\ell j}A_{\ell j}e^{2ik_\mu x^\mu} + \bar{A}^{\ell j}\bar{A}_{\ell j}e^{-2ik_\mu x^\mu}\right)$$
$$= \frac{1}{2}\left(A^{\ell j}A_{\ell j}e^{2ik_\mu x^\mu} + \bar{A}^{\ell j}\bar{A}_{\ell j}e^{-2ik_\mu x^\mu}\right) \checkmark$$

$$\Rightarrow \quad 16\pi G\, t_{0i} = -\frac{3k_0k_i}{4}\Big(\mathrm{Re}\{A^{\ell j}A_{\ell j}\}\cos(2k_\mu x^\mu) - \mathrm{Im}\{A^{\ell j}A_{\ell j}\}\cdot\sin(2k_\mu x^\mu)\Big) + \frac{k_0k_i}{4}A_{\ell j}\bar{A}^{\ell j}$$

$$2k_\mu x^\mu = 2(-\omega t + k_i x^i) = 2(k_i x^i - \omega t)$$
$$\Rightarrow \cos(2k_\mu x^\mu) = \cos(2(k_i x^i - \omega t)) \quad ; \sin(2(k_\mu x^\mu)) = \sin((k_i x^i - \omega t)\cdot 2)$$

Time average: T – period

$$16\pi G\cdot\langle t_{0i}\rangle = -\frac{3k_0k_i}{4}\Big(\mathrm{Re}\{A^{\ell j}A_{\ell j}\}\int_0^T \cos(2(k_i x^i) - 2\omega t)\,dt - \mathrm{Im}\{A^{\ell j}A_{\ell j}\}\int_0^T \sin(2(k_i x^i) - 2\omega t)\,dt\Big) + \left\langle\frac{k_0k_i}{4}A_{\ell j}\bar{A}^{\ell j}\right\rangle$$

$$= -\frac{3k_0k_i}{4}\Big(\mathrm{Re}\{A^{\ell j}A_{\ell j}\}\left[\cos(2k_i x^i) - \cos(2k_i x^i - \underbrace{2\omega\cdot T}_{=2\cdot 2\pi})\right] - \mathrm{Im}\{A^{\ell j}A_{\ell j}\}\left[-\sin(2k_i x^i) + \sin(2k_i x^i - 4\pi)\right]$$
$$+ \frac{k_0k_i}{4}\langle A_{\ell j}\bar{A}^{\ell j}\rangle$$

Because $\sin(x+2\pi) = \sin(x) = \sin(x-2\pi)$ and $\cos(x) = \cos(x-2\pi)$

$$= -\frac{3k_0k_i}{4}\Big(\mathrm{Re}\{A^{\ell j}A_{\ell j}\}(\cancel{\cos(2k_i x^i) - \cos(2k_i x^i)}) - \mathrm{Im}\{A^{\ell j}A_{\ell j}\}(\cancel{\sin(2k_i x^i) - \cos(2k_i x^i)})\Big)$$
$$+ \frac{k_0k_i}{4}\langle A_{\ell j}\bar{A}^{\ell j}\rangle$$

$$= \frac{k_0k_i}{4}\langle A_{\ell j}\bar{A}^{\ell j}\rangle = 16\pi G\langle t_{0i}\rangle$$

$$\Rightarrow \langle t_{0i}\rangle = \frac{1}{16\pi G}\frac{k_0k_i}{4}\langle A_{\ell j}\bar{A}^{\ell j}\rangle = \underline{\underline{\frac{k_0k_i}{64\pi G}\langle A_{\ell j}\bar{A}^{\ell j}\rangle}}$$

$\hat{n}_i = \frac{k_i}{k_0}$ unit vector parallel to k_i: $\hat{n}_i\hat{n}^i = \frac{k_ik^i}{k_0^2} = \frac{\overbrace{k_\mu k^\mu}^{=0} - k_0k^0}{k_0^2}$

$$= \frac{0 + k_0^2}{k_0^2} = 1$$

$$\langle t_{0i}\rangle = \frac{k_0 k_i}{64\pi G}\langle A_{\ell j}\bar{A}^{\ell j}\rangle = \frac{1}{64\pi G}\left(\frac{k_i}{k_0}\right)k_0^2\,\langle A_{\ell j}\bar{A}^{\ell j}\rangle$$

$$= \frac{n_i}{64\pi G}\underbrace{\langle k_0^2 A_{\ell j}\bar{A}^{\ell j}\rangle}_{=\otimes}$$

$$\otimes = \langle k_0^2 A_{\ell j}\bar{A}^{\ell j}\rangle = k_0^2\left\langle \frac{1}{2}\left(2\bar{A}^{\ell j}A_{\ell j} - A_{\ell j}A^{\ell j}e^{2ik_\mu x^\mu} - \bar{A}_{\ell j}\bar{A}^{\ell j}e^{-2ik_\mu x^\mu}\right)\right\rangle$$

$$= k_0^2\cdot\frac{1}{2}\left\langle\left(iA_{\ell j}e^{ik_\mu x^\mu} - i\bar{A}_{\ell j}e^{ik_\mu x^\mu}\right)\left(iA^{\ell j}e^{ik_\mu x^\mu} - i\bar{A}^{\ell j}e^{-ik_\mu x^\mu}\right)\right\rangle$$

$$= \frac{4k_0^2}{2}\left\langle \mathrm{Re}\{iA_{\ell j}e^{ik_\mu x^\mu}\}\,\mathrm{Re}\{iA^{\ell j}e^{ik_\mu x^\mu}\}\right\rangle$$

$$= \frac{4}{2}\left\langle \mathrm{Re}\{ik_0 A_{\ell j}e^{ik_\mu x^\mu}\}\,\mathrm{Re}\{ik_0 A^{\ell j}e^{ik_\mu x^\mu}\}\right\rangle$$

$$= \frac{4}{2}\left\langle \mathrm{Re}\left\{i\frac{\partial x^\mu}{\partial x^0}(k^\mu)^{-1}A_{\ell j}e^{ik_\mu x^\mu}\right)\mathrm{Re}\left\{i\frac{\partial x^\mu}{\partial x^0}k_\mu e^{ik_\mu x^\mu}\right\}\right\rangle$$

$$= \frac{4}{2}\left\langle \frac{\partial}{\partial x^0}\mathrm{Re}\{A_{\ell j}e^{ik_\mu x^\mu}\}\,\frac{\partial}{\partial x^0}\mathrm{Re}\{A^{\ell j}e^{ik_\mu x^\mu}\}\right\rangle$$

$$= \frac{4}{2}\left\langle \frac{\partial}{\partial x^0}(h_{\ell j})\,\frac{\partial}{\partial x^0}(h^{\ell j})\right\rangle = \frac{1}{2}\langle \partial_0 h_{\ell j}\,\partial_0 h^{\ell j}\rangle$$

$$\underset{\text{gauge}}{\overset{TT}{=}} \frac{4}{2}\langle \partial_0 \bar{h}^{j\ell}\,\partial_0 \bar{h}_{j\ell}\rangle = 2\langle \partial_0 \bar{h}^{j\ell}\,\partial_0 \bar{h}_{j\ell}\rangle$$

$$\Rightarrow \underline{\underline{\langle t_{0i}\rangle = \frac{k_0 k_i}{64\pi G}\langle A_{\ell j}\bar{A}^{\ell j}\rangle = \frac{n_i}{32\pi G}\langle \partial_0 \bar{h}^{j\ell}\,\bar{h}_{j\ell}\rangle}}$$

BRUNNEN

Questions

Question 1)

Amplitude of gravitational waves:

$M_{BH} = 10 M_\odot$ in the Andromeda Galaxy, $d = 765$ kpc

Quadrupole moment: $I^{em} = \int T^{00} x^e x^m d^3x$

Tensor virial theorem: $\int T^{em} d^3x = -\frac{1}{2}\Omega^2 I^{em}$

From Ex. 9.2: $\bar{h}^{ij}(t,\vec{x}) = +\frac{2G}{|\vec{x}|}\frac{\partial^2}{\partial t^2} I^{ij}\left(t - \frac{|\vec{x}|}{c}\right)$

Fourier space: $\tilde{h}^{ij} = \frac{-2G}{|\vec{x}|}\omega^2 e^{i\omega\frac{|\vec{x}|}{c}} I^{ij}$ for the generation of gravitational waves.

The Quadrupole of the BH ~~hole~~ should be of order

$MR^2 \to R$ = radius of BH $= R_S = \frac{2GM}{c^2}$

$|\vec{x}| = d$

$$\Rightarrow |h| = +\frac{2G}{d}\omega^2 \underbrace{\left|e^{i\omega\frac{d}{c}}\right|}_{=1} \cdot MR^2 = \frac{2G}{d} M \underbrace{(\omega R)^2}_{\text{orbital velocity } v} = \frac{2G}{d} M v^2$$

Virial theorem ($|E_{pot}| \sim E_{kin}$)

$$\Rightarrow v^2 \sim |\Phi_{source}| = \frac{GM}{R_S}$$

$$\frac{GM}{d} = |\Phi_{earth}|$$

$$\Rightarrow |h| \approx |\Phi_{source}||\Phi_{earth}| = |\Phi_{source} \cdot \Phi_{earth}|$$

$$= \left|\frac{GM}{R_S} \cdot \frac{GM}{d}\right| = \frac{G^2 M^2}{R_S \cdot d}$$

$G = 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}$

$$\overset{G=1}{=} \frac{M^2}{R_S \cdot d}$$

Planck units: $M_p = 2.17 \cdot 10^{-5}$ g

$L_p = 1.61 \cdot 10^{-33}$ cm

$$M_{BH} = \frac{10 M_\odot}{M_p} M_p = \frac{10 \cdot 2 \cdot 10^{33} g}{2.17 \cdot 10^{-5} g} M_p = 9.2 \cdot 10^{38} M_p$$

$$R_S = \frac{2 \cdot 6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2} \cdot 2 \cdot 10^{30} kg \cdot 10}{\left(3 \cdot 10^8 \frac{m}{s}\right)^2} \approx 3 \cdot 10^6 \text{ cm}$$

$$= \frac{3 \cdot 10^6 \text{ cm}}{1.61 \cdot 10^{-33} \text{ cm}} L_p = 1.9 \cdot 10^{39} L_p$$

$d = 765\,\text{kpc} = 2.4 \cdot 10^{24}\,\text{cm} = \dfrac{2.4 \cdot 10^{24}\,\text{cm}}{1.61 \cdot 10^{-33}\,\text{cm}} = 1.5 \cdot 10^{57}\,L_p$

Natural units: $c = \hbar = k_B = G = 1 \Rightarrow L_p = M_p^{-1}$

$$\Rightarrow |h| = \frac{M^2}{R_S \cdot d} = \frac{(9.2 \cdot 10^{38})\,M_p^2}{(1.9 \cdot 10^{39}) \cdot (1.5 \cdot 10^{57})\,L_p^2}$$

Planck energy

$$= 3.2 \cdot 10^{-58}\,M_p^4 = 3.2 \cdot 10^{-58}\,E_p^4$$

$$= \cancel{7}1 \cdot 10^{54}\,eV^4 = 4.7 \cdot 10^{-21}\,J^4 \quad \text{not } m?$$

Question 2

$\boxed{(a)}$ Birkhoff theorem: The Schwarzschild solution

$$ds^2 = -dt^2\left(1 - \frac{2M}{r}\right) + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega^2$$ is the unique spherically symmetric solution to Einstein's equations in vacuum. It describes a static metric but the object itself doesn't have to be static.

$\boxed{(b)}$ A pulsating spherically symmetric star cannot produce gravitational waves: Because of the spherical symmetry Birkhoff's theorem applies in this case: Because only the location of the stellar surface changes but its relative position in spherical coordinates (ϑ, φ) does not, the metric outside of the star must be given by the Schwarzschild metric. And because this metric is static there is no wave solution. These objects can therefore not emit GW.

$\boxed{(c)}$ Rotating star with homogeneous and constant energy density $\rho = T^{00}$. In the lecture we have seen that $\bar{h}_{ij} = -2\Omega^2 I_{ij} \frac{e^{i\Omega r}}{r}$ $\quad I_{ij} = e^{-i\Omega t} \int \rho x_i x_j \, d^3x$

$\overset{\text{sph.}}{\underset{\text{symmetry}}{=}} g(r)\delta_{ij} = \left(\int \rho r^4 dr\right)\delta_{ij}$ in the far field-approx.

There should be a time-varying quadrupole moment for GW to be present. Because $\Omega \neq 0$ and $I_{ij} \neq 0$

$\Rightarrow$ It generates GW.