P4. Given : $X_5 = X_5 + 2\pi R$

(a) Translation in $s(\vec{x})$ direction : $T(s(\vec{x})) = \exp\left(\frac{-i}{\hbar} s(\vec{x}).P_5\right)$

Under the periodic condition :

$$T(s(\vec{x})) = T(s(\vec{x}) + m 2\pi R) \quad , \quad m \in \mathbb{Z}^+$$

$$\Rightarrow \exp\left(\frac{-i}{\hbar} s(\vec{x}).P_5\right) = \exp\left(\frac{-i}{\hbar}(s(\vec{x}) + m2\pi R).P_5\right)$$

i.e. $\exp\left(\frac{-i}{\hbar} m 2\pi R .P_5\right) = 1$

$$\rightarrow \frac{2\pi R m P_5}{\hbar} = 2\pi n \Rightarrow P_5 = \left(\frac{n}{m}\right)\frac{\hbar}{R} \quad , \quad n,m \in \mathbb{Z}^+$$

(b) We have $H = \frac{\vec{P}^2}{2m}$ ~~for~~ as the Hamiltonian of the free particle.

$$\frac{1}{2m}\left[\vec{P}^2, \exp\left(\frac{-i}{\hbar} s(\vec{x}).P_5\right)\right]$$

$$= \frac{1}{2m}\left\{\vec{P}[\vec{P}, U] + [\vec{P}, U]\vec{P}\right\} = \circledast$$

$$[\vec{P}, U] = [\vec{P}, e^{\underbrace{\frac{-i}{\hbar} s(\vec{x}).P_5}_{B(\vec{x})}}] = [P, e^{B(\vec{x})}]$$

from eq^n 13 of script.

$$= i\hbar \frac{\partial}{\partial x}(e^{B(\vec{x})}) = i\hbar\, e^{B(\vec{x})} \frac{\partial}{\partial x} B(\vec{x})$$

$$= \cancel{i\hbar \frac{\partial}{\partial x}}\; i\hbar\, e^{B(\vec{x})} \frac{\partial}{\partial x}\left(\frac{-i}{\hbar} s(\vec{x}) P_5\right)$$

$$= \underbrace{e^{B(\vec{x})}}_{U} \cancel{\hbar}\, s'(\vec{x}) P_5$$

This gives us.

$$\circledast \Rightarrow \frac{1}{2m}\left\{\vec{P}\,\mathcal{U}S'(\vec{x})P_5 + \mathcal{U}S'(\vec{x})P_5\vec{P}\right\} \neq 0$$

As $S'(\vec{X})$ & $\vec{P}$ do not commute!

(c) $$H' = \frac{1}{2m}\left(\vec{P} + P_5\vec{\nabla}X_5\right)^2$$

We know,

$$[\vec{P}, \mathcal{U}] = [\vec{P}, \vec{X}]\,\frac{-i}{\hbar}P_5\vec{\nabla}S(\vec{x})\mathcal{U} = -P_5\vec{\nabla}S(\vec{x})\mathcal{U}$$

$$P\mathcal{U} - \mathcal{U}P \rightarrow P\mathcal{U} = \mathcal{U}P - P_5\vec{\nabla}S(\vec{x})\mathcal{U}$$

$$\mathcal{U}^{+}\left(\vec{P} + P_5\vec{\nabla}X_5\right)\mathcal{U} = \mathcal{U}^{+}\mathcal{U}\vec{P} + \mathcal{U}^{+}\left(-P\vec{\nabla}S(\vec{x})\mathcal{U}\right) + \mathcal{U}^{+}P_5\vec{\nabla}X_5\mathcal{U}$$

We want $\mathcal{U}^{+}(\vec{P} + P_5\vec{\nabla}X_5)\mathcal{U}$ is equal to $(\vec{P} + P_5\vec{\nabla}X_5)$ for ~~the~~ H' to be symmetric under this transl^n op.

$$\mathcal{U}^{+}\mathcal{U}\vec{P} - P_5\underbrace{\cancel{\mathcal{U}^{+}\vec{\nabla}S(\vec{x})\mathcal{U}}}_{\vec{\nabla}S(\vec{x})} + P_5\cancel{\mathcal{U}^{+}\mathcal{U}}X_5 + P_5[\nabla X_5, \mathcal{U}]$$

$$= \cancel{\vec{P}} + \cancel{P_5\vec{\nabla}X_5}$$

If we demand $\vec{\nabla}X_5 \rightarrow \nabla X_5 + \nabla S(\vec{x})$

then we will have invariancy.

- Now we want time ~~invariance~~ evolution to be correctly generated.

$\cancel{\vec{\nabla} X_5 \to \vec{\nabla} X_5 + \vec{\nabla} S(\vec{x})}$

$$i\hbar \frac{d}{dt}|\psi\rangle = H_{final}|\psi\rangle$$

$$i\hbar \frac{d}{dt}(U|\psi\rangle) = i\hbar \frac{dU}{dt}|\psi\rangle + i\hbar U \frac{d}{dt}|\psi\rangle$$

$$= i\hbar \frac{dU}{dt}|\psi\rangle + UH|\psi\rangle$$

$$= H_{final} U|\psi\rangle$$

$$\therefore H_{final} = i\hbar \frac{dU}{dt} + UH = HU + i\hbar \cdot \frac{iP_5}{\hbar} \cdot \frac{\partial S(\vec{x})}{dt} U$$

$$\therefore H_{final} = \frac{(\vec{P} + P_5 \vec{\nabla} X_5)^2}{2m} - P_5 \phi$$

s.t. $\phi \to \phi - \frac{\partial S(x)}{\partial t}$

(d) EM Charge ~ $q \to -P_5$

Vector potential : $\vec{A} \to \vec{\nabla} X_5$

Scalar " $\phi \to \phi$

We have charge quantized in units of $\frac{n\hbar}{mR}$.

Happens after we add a new periodic space dim^n, X_5.