# Exercise 5.1
# (The gravitational redshift element)

## Exercise 5.1: The gravitational redshift experiment

This exercise aims to understand the *Gedankenexperiment* first suggested by Einstein in 1907, and its important consequences. This idealized experiment goes as follows:

(a) We have an object of rest mass $m$ at the top of a tower of height $h$. The object is initially at rest, but at some point we drop it and it falls freely with constant acceleration $g$. When it reaches the ground its energy is fully transformed into a photon of frequency $\nu$ that is emitted towards the top of the tower. Compute the energy of this photon at the moment of emission. Hint: take into account the effect of the Earth in the conservation equations. (1 pt)

# Exercise 5.2
# (Geodesic equation and affine parameters)

**(a) Consider a non-null curve whose tangent vector is parallel-transported along it. Show that such a vector has constant norm on this geodesic curve. (1 pt)**

Let us parametrize the curve with a parameter $\lambda$ and construct the tangent vector $\vec{V}(\lambda)$ with components $V^\alpha = \frac{dx^\alpha}{d\lambda}$. The norm of this vector reads,

$$\vec{V} \cdot \vec{V} = g(\vec{V}, \vec{V}) = g_{\mu\nu} V^\mu V^\nu = g_{\mu\nu} U^\mu U^\nu \left(\frac{d\tau}{d\lambda}\right)^2 = -\left(\frac{d\tau}{d\lambda}\right)^2 \equiv f(\lambda)$$

$$\boxed{V^\alpha = \frac{dx^\alpha}{d\lambda} = \frac{dx^\alpha}{d\tau}\frac{d\tau}{d\lambda} = U^\alpha \frac{d\tau}{d\lambda}}$$

Let us study now how the norm varies along the curve:

$$\frac{d}{d\lambda}(\vec{V} \cdot \vec{V}) = \frac{df}{d\lambda} = V^\beta \nabla_\beta (g_{\mu\nu} V^\mu V^\nu) = V^\beta g_{\mu\nu} (V^\mu \nabla_\beta V^\nu + V^\nu \nabla_\beta V^\mu)$$

$$\boxed{\text{Remember } \nabla_\beta g_{\mu\nu} = 0}$$

$$\frac{df}{d\lambda} = 2V_\nu V^\beta \nabla_\beta V^\nu = 2V_\nu \frac{dV^\nu}{d\lambda}$$

If $\vec{V}$ is parallel-transported along the curve it is tangent to, $\frac{dV^\nu}{d\lambda} = 0$, and therefore $f = const.$, so $\vec{V}$ has constant norm.

$u^\alpha \nabla_\alpha V^\nu = 0$

(b) Use the previous result to prove that the geodesic equation

$$\frac{dV^\alpha}{d\lambda}=0 \to \frac{d^2x^\nu}{d\lambda^2}+\Gamma^\nu{}_{\alpha\kappa}\frac{dx^\alpha}{d\lambda}\frac{d^2x^\kappa}{d\lambda}=0 \tag{1}$$

only describes the curve under study if it is parametrized with a parameter $\lambda$ related with the proper time $\tau$ by an affine transformation. (0.5 pt)

we know that $\frac{dV^\alpha}{d\lambda}=0$ only applies if and only if $f(\lambda)=const$

i.e. if $\frac{d^2\tau}{d\lambda^2}=0$, so when $\tau=a+b\lambda$, with $a$ and $b$ constants.

(c) How is (1) modified when the curve is parametrized with a non-affine parameter?

We start with the curve parametrized with the proper time $\tau$,

$$\frac{d^2x^\nu}{d\tau^2}+\Gamma^\nu{}_{\alpha\kappa}\frac{dx^\alpha}{d\tau}\frac{d^2x^\kappa}{d\tau}=0\,,$$

perform a change $\tau=\tau(\lambda)$, where here $\lambda$ is a non-affine parameter.

$$\boxed{\frac{d^2x^\nu}{d\lambda^2}+\Gamma^\nu{}_{\alpha\kappa}\frac{dx^\alpha}{d\lambda}\frac{dx^\kappa}{d\lambda}=-\frac{\frac{d^2\lambda}{d\tau^2}}{\left(\frac{d\lambda}{d\tau}\right)^2}\frac{dx^\nu}{d\lambda}} \longrightarrow \boxed{V^\mu\nabla_\mu V^\nu=s(\lambda)V^\nu}$$

where we have employed the following chain rules:

$$\frac{d}{d\tau}=\frac{d\lambda}{d\tau}\frac{d}{d\lambda}\quad;\quad\frac{d^2}{d\tau^2}=\frac{d^2\lambda}{d\tau^2}\frac{d}{d\lambda}+\left(\frac{d\lambda}{d\tau}\right)^2\frac{d^2}{d\lambda^2}\,.$$

(d) Does the form of the geodesic equation remain invariant under coordinate transformations?

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu{}_{\alpha\kappa}\frac{dx^\alpha}{d\tau}\frac{d^2x^\kappa}{d\tau} = 0 \longrightarrow \frac{d}{d\tau}\left(\frac{\partial x^\nu}{\partial x^{\bar\beta}}\frac{dx^{\bar\beta}}{d\tau}\right) + \Gamma^\nu{}_{\alpha\kappa}\frac{dx^{\bar\kappa}}{d\tau}\frac{dx^{\bar\alpha}}{d\tau}\frac{\partial x^\kappa}{\partial x^{\bar\kappa}}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}} = 0$$

Now we can multiply this equation by $\frac{\partial x^{\bar\nu}}{\partial x^\nu}$, and using the transformation law for the Christoffel symbols obtained in Exercise 4.2b we are led to:

$$\frac{d^2x^{\bar\nu}}{d\tau^2} + \frac{\partial x^{\bar\nu}}{\partial x^\nu}\frac{dx^{\bar\beta}}{d\tau}\frac{dx^{\bar\alpha}}{d\tau}\frac{\partial^2 x^\nu}{\partial x^{\bar\alpha}\partial x^{\bar\beta}} + \frac{dx^{\bar\kappa}}{d\tau}\frac{dx^{\bar\alpha}}{d\tau}\left[\Gamma^{\bar\nu}{}_{\bar\alpha\bar\kappa} - \frac{\partial x^{\bar\nu}}{\partial x^\mu}\frac{\partial^2 x^\mu}{\partial x^{\bar\alpha}x^{\bar\kappa}}\right] = 0$$

In the last term we can do $\mu \to \nu$ and $\bar\kappa \to \bar\beta$ to finally get:

$$\frac{d^2x^{\bar\nu}}{d\tau^2} + \Gamma^{\bar\nu}{}_{\bar\alpha\bar\kappa}\frac{dx^{\bar\kappa}}{d\tau}\frac{dx^{\bar\alpha}}{d\tau} = 0$$

The geodesic equation remains invariant under coordinate transformations, as expected.

## (e) Does the geodesic equation work in presence of non-gravitational forces?

No, the geodesic equation governs the motion in spacetime of free-falling (inertial) particles, only subject to gravity. If forces different from gravity are also considered one in general has:

$$\frac{d^2x^\nu}{d\tau^2} + \Gamma^\nu{}_{\alpha\kappa}\frac{dx^\alpha}{d\tau}\frac{d^2x^\kappa}{d\tau} \neq 0\,.$$

# Exercise 5.3
# (Proofs : Covariant derivative of a metric)

(a) $\frac{\partial g}{\partial x^\kappa} = g\, g^{\alpha\beta} g_{\alpha\beta,\kappa}$, with $g$ the determinant of the metric.

$\Sigma^{\mu\nu} = (-1)^{\mu+\nu} M^{\mu\nu}$

$M^{\mu\nu}$ the $(\mu,\nu)$-minor

The determinant of the metric can be written as

$$g = \sum g_{\mu\nu} \Sigma^{\mu\nu}$$

Making use of the derivative chain rule noting that $\Sigma^{\mu\nu}$ does not depend on $g_{\mu\nu}$ we find,

$$\frac{\partial g}{\partial x^\kappa} = \frac{\partial g}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial x^\kappa} = \Sigma^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\kappa}$$

On the other and we have,

$$g^{\mu\nu} = \frac{1}{g} \Sigma^{\mu\nu} \longrightarrow \Sigma^{\mu\nu} = g g^{\mu\nu}$$

$$\boxed{\frac{\partial g}{\partial x^\kappa} = g g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\kappa}}$$

$$\boxed{\text{(b) } \nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, \partial^\mu \phi)}$$

$$\nabla_\mu \nabla^\mu \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = g^{\mu\nu} (\partial_\mu \partial_\nu \phi - \Gamma^\kappa{}_{\mu\nu} \partial_\kappa \phi) \tag{21}$$

Let us focus on the contraction $g^{\mu\nu}\Gamma^\kappa{}_{\mu\nu}$.

$$g^{\mu\nu}\Gamma^\kappa{}_{\mu\nu} = g^{\mu\nu} \frac{g^{\kappa\alpha}}{2} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) = g^{\kappa\alpha} (g^{\mu\nu} g_{\alpha\mu,\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha})$$

Since $g^{\beta\nu} g_{\mu\nu} = \delta^\beta_\nu$ we have that $g^{\mu\nu} g_{\alpha\mu,\nu} = -g_{\alpha\mu} g^{\mu\nu}{}_{,\nu}$ and this can be employed to simplify the first term. For the second one we can make use of the result obtained in 5.4a.

$$g^{\mu\nu}\Gamma^\kappa{}_{\mu\nu} = -\left( g^{\kappa\alpha} g_{\alpha\mu} g^{\mu\nu}{}_{,\nu} + \frac{1}{2g} g^{,\kappa} \right) = -\left( \delta^\kappa_\mu g^{\mu\nu}{}_{,\nu} + \frac{1}{2g} g^{,\kappa} \right) = -\left( g^{\kappa\nu}{}_{,\nu} + \frac{g^{\kappa\nu}}{2g} g_{,\nu} \right)$$

$$g^{\mu\nu}\Gamma^\kappa{}_{\mu\nu} = -\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, g^{\kappa\nu}) \tag{22}$$

Now we use this result in (21),

$$\nabla_\mu \nabla^\mu \phi = g^{\mu\nu} \partial_\mu \partial_\nu \phi + \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, g^{\kappa\nu}) \partial_\kappa \phi = g^{\kappa\nu} \partial_\nu \partial_\kappa \phi + \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, g^{\kappa\nu}) \partial_\kappa \phi$$

$$\boxed{\nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, g^{\kappa\nu} \partial_\kappa \phi) = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g}\, \partial^\nu \phi)} \tag{23}$$

$$\boxed{\text{(c) } \nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\,F^{\mu\nu}), \text{ with } F^{\mu\nu} \text{ an antisymmetric tensor.}}$$

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} - \Gamma^\mu{}_{\nu\kappa}F^{\kappa\nu} - \Gamma^\nu{}_{\nu\kappa}F^{\mu\kappa} \tag{24}$$

Since the Christoffel symbols are symmetric under the exchange of the lower indices and $F$ is an antisymmetric tensor the second term in the right-hand side vanishes, so

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} - \Gamma^\nu{}_{\nu\kappa}F^{\mu\kappa}\,. \tag{25}$$

The second term can be written as follows,

$$\Gamma^\nu{}_{\nu\kappa}F^{\mu\kappa} = \frac{1}{2}g^{\nu\alpha}(g_{\alpha\nu,\kappa} + g_{\alpha\kappa,\nu} - g_{\nu\kappa,\alpha})F^{\mu\kappa} = \frac{1}{2}g^{\nu\alpha}g_{\alpha\nu,\kappa}F^{\mu\kappa} = \frac{g_{,\kappa}}{2g}F^{\mu\kappa}\,. \tag{26}$$

Thus, we find:

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} - \frac{g_{,\nu}}{2g}F^{\mu\nu}$$

$$\boxed{\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\nu(\sqrt{-g}\,F^{\mu\nu})} \tag{27}$$

# EXTRA - Exercise 5.4
# (Parallel transport in Euclidean space)

(a) Solve the following problem in Euclidean space. Take the vector $(0, 1)$ located at point $(0, 1)$, in Cartesian coordinates. Parallel-transport it along a circle of radius $r = 1$ to the point (1,0), clockwise. Make explicit use of the parallel-transport equation. (0.5 pt)

The parallel-transport equation reads,

$$U^\mu \nabla_\mu V^\nu = 0\,,$$

where $\vec{U}$ is the tangent vector to the path along which we parallel transport the vector $\vec{V}$.

Cartesian coordinates $\longrightarrow$ the Christoffel symbols vanish

$$U^\mu \partial_\mu V^\nu = 0 \Rightarrow \begin{array}{rcl} U^x\partial_x V^x + U^y \partial_y V^x & = & 0 \\ U^x\partial_x V^y + U^y \partial_y V^y & = & 0 \end{array}$$

The points on the path can be parametrized with the angle $\theta$ $\longrightarrow$ $\vec{r} = (x, y) = (\cos\theta, \sin\theta)$

$$\vec{U} = -\frac{d\vec{r}}{d\theta} = (\sin\theta, -\cos\theta) = (y, -x) = (y, -\sqrt{1-y^2})$$

Thus, we have

$$\begin{array}{rcl} y\partial_x V^x - x\partial_y V^x & = & 0 \longrightarrow \dfrac{dV^x}{dy} = 0 \\ y\partial_x V^y - x\partial_y V^y & = & 0 \longrightarrow \dfrac{dV^y}{dy} = 0 \end{array}$$

The parallel-transported vector has constant components in Cartesian coordinates and reads, $\vec{V} = (1, 0)$.

## (b) Do the same exercise in polar coordinates.

In polar coordinates there are some non-null Christoffel symbols, $\Gamma^{r}{}_{\theta\theta} = -r\,. \qquad \Gamma^{\theta}{}_{r\theta} = \dfrac{1}{r}$

$$U^{\alpha}(\partial_{\alpha} V^{\nu} + \Gamma^{\nu}{}_{\alpha\kappa} V^{\kappa}) = 0\,, \quad \Longrightarrow \quad \begin{aligned} U^{\theta}\partial_{\theta}V^{\theta} + U^{r}\partial_{r}V^{\theta} + \frac{1}{r}\left(U^{r}V^{\theta} + U^{\theta}V^{r}\right) &= 0 \\ U^{\theta}\partial_{\theta}V^{r} + U^{r}\partial_{r}V^{r} - rU^{\theta}V^{\theta} &= 0 \end{aligned}$$

$$\begin{pmatrix} \vec{e}_{\theta} \\ \vec{e}_{r} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \vec{e}_{x} \\ \vec{e}_{y} \end{pmatrix} = \begin{pmatrix} -r\sin(\theta) & r\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} \vec{e}_{x} \\ \vec{e}_{y} \end{pmatrix}$$

$$\begin{pmatrix} \vec{e}_{x} \\ \vec{e}_{y} \end{pmatrix} = \begin{pmatrix} -\frac{\sin(\theta)}{r} & \cos(\theta) \\ \frac{\cos(\theta)}{r} & \sin(\theta) \end{pmatrix} \begin{pmatrix} \vec{e}_{\theta} \\ \vec{e}_{r} \end{pmatrix}$$

we find that in polar coordinates $\overrightarrow{U} = -\frac{\overrightarrow{e}_\theta}{r} = -\overrightarrow{e}_\theta$, since $r = 1$ along the path.

$$\begin{aligned} \partial_\theta V^\theta + V^r &= 0 \\ \partial_\theta V^r - V^\theta &= 0 \end{aligned}$$

The solution of this system is

$$V^\theta(\theta) = A\,\cos(\theta) + B\,\sin(\theta) \qquad V^r(\theta) = A\,\sin(\theta) - B\,\cos(\theta)$$

We can fix $A$ and $B$ by using the initial condition, $\overrightarrow{V}_{ini} = \overrightarrow{e}_y(r = 1; \theta = \pi/2) = \overrightarrow{e}_r$.

The vector $\overrightarrow{V}$ along the path reads, $\overrightarrow{V} = \cos(\theta)\overrightarrow{e}_\theta + \sin(\theta)\,\overrightarrow{e}_r$

so $\overrightarrow{V} = \overrightarrow{e}_\theta$ at the end of the transportation process.

# EXTRA - Exercise 5.5
# (Conceptual questions – Parallel transport)

**(a) Consider the following statement:** *If we parallel-transport a vector along a closed loop, i.e. along a path that begins and ends at the same point, the final vector will always coincide with the one we had at the beginning.* **Is this statement true? Explain why. (0.5 pt)**

- Only true $\rightarrow$ Flat spaces (i.e. in spaces where elements of the Riemann tensor vanish)
- For non-zero curvature the two vectors will not coincide
- This is also the reason why covarient derivatives do not commute in curved spaces

**(b) Consider now this statement:** *The comparison of two vectors located at different tangent spaces is univocal and can be carried out by parallel-transporting one of them to the location of the other*. **Is this statement correct? Justify your answer. (0.5 pt)**

- This statement is also false. In curved spaces, the orientation of the parallel-transported vector depends on the path along which the transportation has been carried out.
- This means that we cannot compare two vectors located in two different tangent spaces in a univocal way.