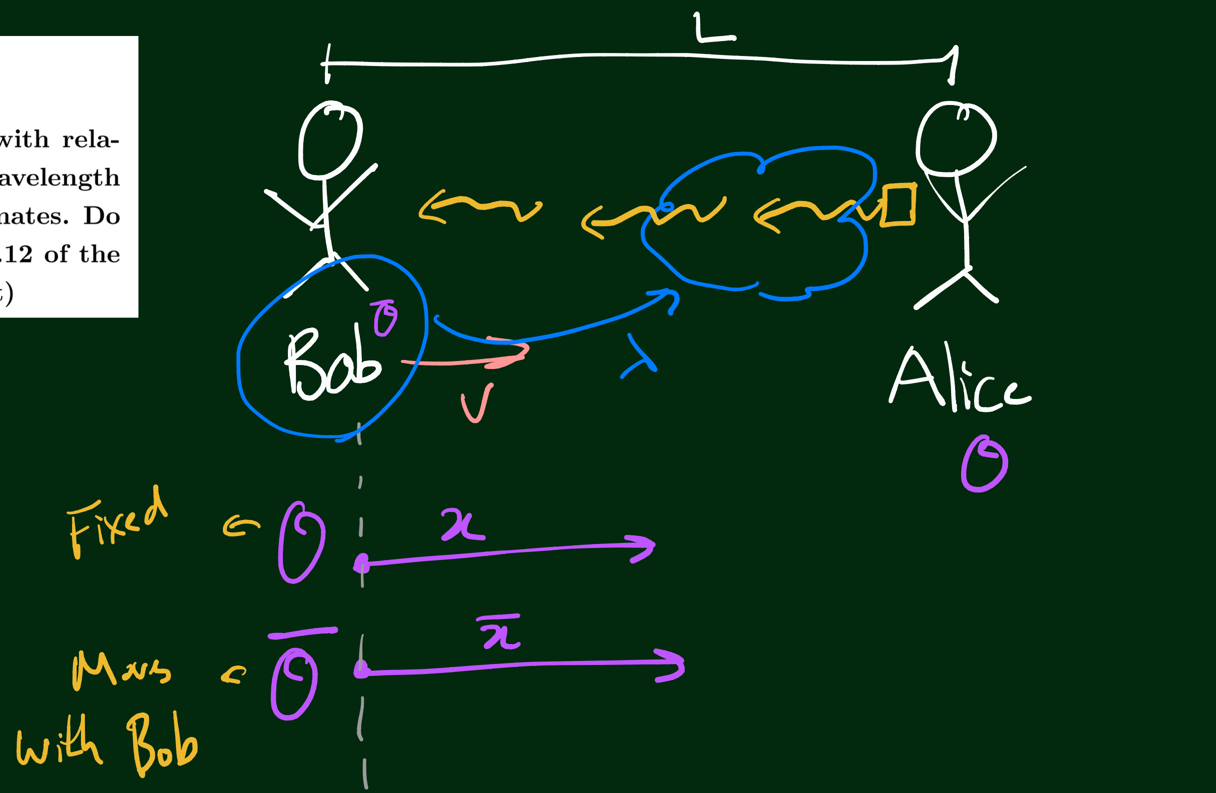
## Exercise 1.1: Doppler effect in Special Relativity

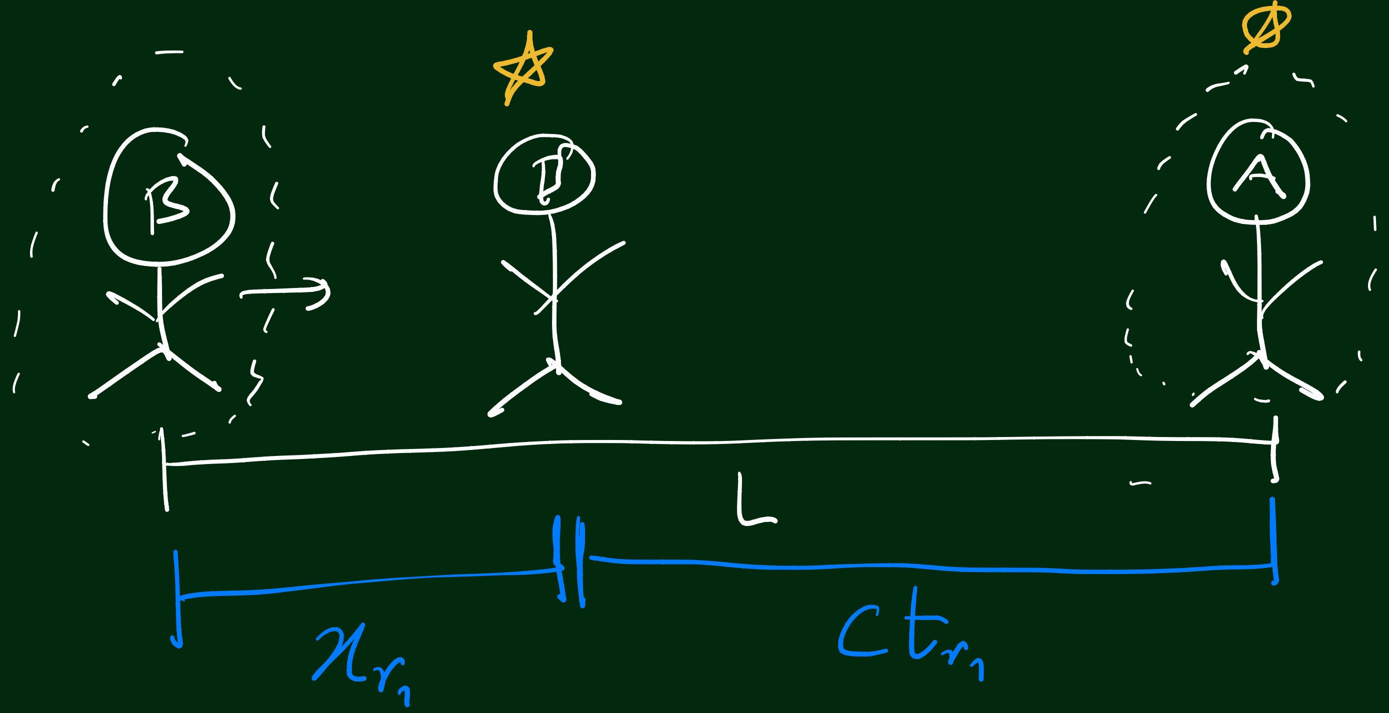
(a) Alice is pointing a laser beam to Bob, who is moving towards Alice with relative velocity $v$. Obtain the formulae that relate the frequency, period and wavelength measured by Bob making use of the Lorentz transformations for the coordinates. Do not use the concept of energy and momentum of the photon (as in Sec. 1.12 of the lecture notes). Do this exercise applying only kinematic arguments. (1.5 pt)




Emission
- ST event of em signal 1 : $(t_{e_1}, x_{e_1}) = (0, L)$
- ST event of em signal 2 : $(t_{e_2}, x_{e_2}) = (T, L)$

Rec
- ST —— " —— signal 1 : $(t_{r_1}, x_{r_1})$ ← ST coordinats at which Bob will recieve the em
- ST —— " —— 2 : $(t_{r_2}, x_{r_2})$

(From O's perspective)




$$x_{r_1} = (vt_{r_1} = L - ct_{r_1})$$

$$vt_{r_1} = L - ct_{r_1} \Rightarrow vt_{r_1} + ct_{r_1} = L \Rightarrow \boxed{t_{r_1} = \frac{L}{v+c}}$$

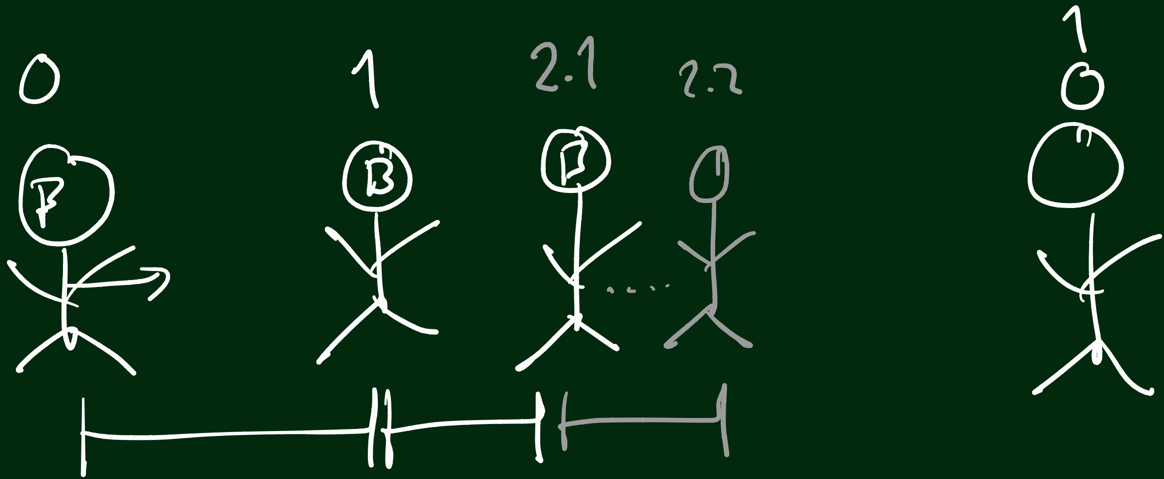
$$\boxed{x_{r_1} = vt_{r_1} = \frac{vL}{v+c}}$$

Coordinate: ST event of the 1st rec. signal.

Next step: Calculate ST event of 2nd rec. signal.

$$t_{r_2} = \frac{L + cT}{(v+c)} \qquad x_{r_2} = \frac{v}{(v+c)}(L+cT)$$

$$x_{r_2} = \overset{(2.1)}{vT} + \overset{(2.2)}{v\Delta t} = v(T+\Delta t) = (L - c\Delta t)$$



$x_{r_1} = v t_{r_1}$

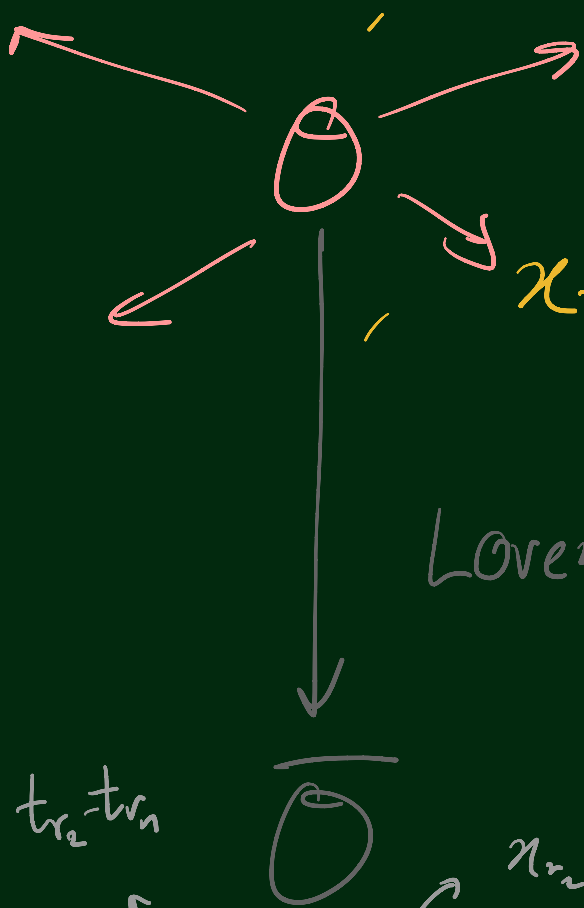
$$t_{r_2} = \frac{L+cT}{c+v}$$

$$t_{r_2} = T + \Delta t$$

$$vT + v\Delta t = L - c\Delta t \Rightarrow v\Delta t + c\Delta t = L - vT \Rightarrow \Delta t(v+c) = L - vt$$

$$\Delta t = \frac{L-vT}{v+c}$$

$$t_{r_1} = \frac{L}{v+c} \qquad x_{r_1} = \frac{L}{v+c}$$



$$t_{r_2} = \frac{L+cT}{v+c} \qquad x_{r_2} = \frac{v(L+ct)}{v+c}$$

Lorentz transfo $\begin{cases} \bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \\ \bar{x} = \gamma(x - vt) \end{cases}$

$t_{r_2} - t_{r_1}$ $\qquad$ $x_{r_2} - x_1$

$$\bar{T} = \bar{t}_{r_2} - \bar{t}_{r_1} = \gamma\left(T - \frac{v}{c^2}X\right)$$

$$= \gamma\left[(t_{r_2} - t_{r_1}) - \frac{v}{c^2}(x_{r_2} - x_{r_1})\right]$$

$$= \gamma \left[ t_{r_2} - t_{r_1} - \frac{v}{c^2} x_{r_2} + \frac{v}{c^2} x_{r_1} \right]$$

$$x_{r_1} = v t_{r_1}$$

$$x_{r_2} = v t_{r_2}$$

$$= \gamma \left[ t_{r_2} - t_{r_1} - \frac{v}{c^2} v t_{r_2} + \frac{v}{c^2} v t_{r_1} \right]$$

$$= \gamma \left[ (t_{r_2} - t_{r_1}) - \frac{v^2}{c^2} t_{r_2} + \frac{v^2}{c^2} t_{r_1} \right]$$

$$= \gamma \left[ (t_{r_2} - t_{r_1}) - (t_{r_2} - t_{r_1}) \frac{v^2}{c^2} \right]$$

$$= \gamma (t_{r_2} - t_{r_1}) \left( 1 - \frac{v^2}{c^2} \right) = \frac{1}{\left( 1 - \frac{v^2}{c^2} \right)^{1/2}} (t_{r_2} - t_{r_1}) \left( 1 - \frac{v^2}{c^2} \right)^{1}$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{1/2} (t_{r_2} - t_{r_1})$$

$$\frac{cT}{v+c} + \frac{L - \cancel{L}}{\cancel{v}+c}$$

$$T = t_{r_2} - t_{r_1} = \sqrt{1 - \frac{v^2}{c^2}}\; (t_{r_2} - t_{r_1})$$

$$= \sqrt{1 - \frac{v^2}{c^2}}\, \frac{cT}{v+c} = \boxed{T \sqrt{1 - \frac{v^2}{c^2}}\, \frac{c}{v+c}}$$

Change $c \to v_s$

$$= T \sqrt{1 - \frac{v^2}{c^2}}\; \frac{1}{1 + \frac{v}{c}} = T \sqrt{\frac{\left(1 - \frac{v}{c}\right)\cancel{\left(1 + \frac{v}{c}\right)}}{\left(1 + \frac{v}{c}\right)^{\cancel{2}}}}$$

$$= \boxed{T \sqrt{\frac{\left(1 - \frac{v}{c}\right)}{\left(1 + \frac{v}{c}\right)}}}$$

$$\bar{\nu} = \nu \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \quad \Rightarrow \quad \bar{\lambda} = \lambda \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

(b) **Repeat the previous exercise, but considering that Alice sends a sound wave with velocity $v_s$ in her reference frame $\mathcal{O}$, instead of an electromagnetic signal. (1.5 pt)**

## EM signal

$$t_{r_1} = \frac{L}{v+c} \quad ; \quad t_{r_2} = \frac{L+cT}{v+c}$$

Time dilation

$$\bar{T} = T\sqrt{1-\frac{v^2}{c^2}}\,\frac{c}{c+v}$$

## Sound

$$t_{r_1} = \frac{L}{v+v_s} \quad ; \quad t_{r_2} = \frac{L+cT}{v+v_s}$$

Time dilation

$$\bar{T} = T\sqrt{1-\frac{v^2}{c^2}}\,\frac{v_s}{v_s+v}$$

$$\bar{\nu} = \nu\,\frac{1+\frac{v}{v_s}}{\sqrt{1-\frac{v^2}{c^2}}}$$

- Goal: Compute $\bar{\lambda}$ (measured by Bob) $\bar{O}: \bar{v}_s$

$$\bar{v}_s = \frac{v + v_s}{1 + \frac{v v_s}{c^2}} = \frac{\bar{\lambda}}{\bar{T}} \rightarrow \boxed{\bar{\lambda} = \frac{\bar{T}(v + v_s)}{1 + \frac{v v_s}{c^2}}}$$

$$\bar{\lambda} = \frac{\lambda \sqrt{1 - \frac{v^2}{c^2}}}{1 + \frac{v v_s}{c^2}} \Rightarrow \bar{\nu} = \nu \frac{1 + \frac{v}{v_s}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

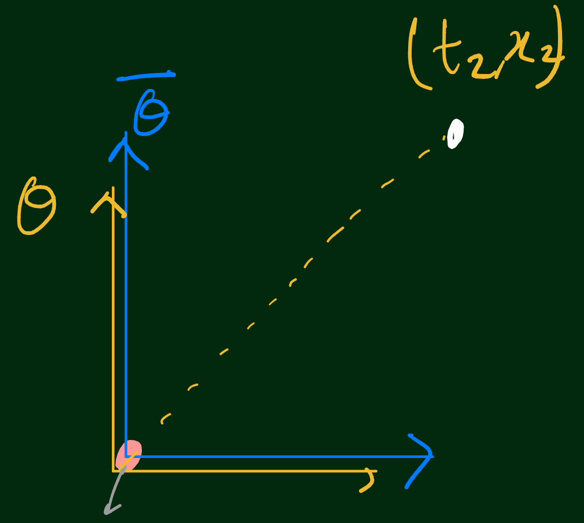
(c) **Consider the results obtained in 1.1a and 1.1b. Do you recover the expected results in the non-relativistic limit? (1 pt)**

Speed = freq × wavelength

EM signal $\Rightarrow$ $\bar{\lambda} = \lambda\,(1)$

Sound : $\bar{\lambda} = \lambda$ , $\bar{\nu} = \nu\left(1 + \frac{v}{v_s}\right)$

$$\bar{t} = \gamma(t - vx)$$
$$\bar{x} = \gamma(x - vt)$$




Event 1: $(t_1, x_1) = (\bar{t}_1, \bar{x}_1) = (0,0)$

Want to know: Under what conditions can we find RF $\bar{O}$ s.t.:

$\bar{x}_2 = \bar{x}_1 = 0$

$$\bar{t}_2 = \gamma(t_2 - vx_2)$$
$$x_2 = vt_2$$

$$\Rightarrow \bar{t}_2 = \gamma(t_2 - v^2 t_2)$$
$$= \gamma t_2 (1 - v^2)$$