Exercise 4.1

Exercise 4.1

(a) Show that if a tensor is symmetric in a concrete basis, it will remain symmetric in any other basis. (0.5 pt)

If $A_{\mu\nu}$ is symmetric in a concrete basis $\{\tilde{\omega}^{\mu}\}$ $\longrightarrow$ perform a transformation to $\{\tilde{\omega}^{\bar{\mu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}\tilde{\omega}^{\mu}\}$

elements of the tensor **A** will change as follows:

$$A_{\bar{\mu}\bar{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}A_{\mu\nu} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}A_{\nu\mu} = \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}A_{\nu\mu} = A_{\bar{\nu}\bar{\mu}}$$

(b) Prove that the contraction of a symmetric tensor S with an antisymmetric one A is 0. (0.5 pt)

$$A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\mu\nu} = -A_{\nu\mu}S^{\nu\mu} = -A_{\mu\nu}S^{\mu\nu} \longrightarrow A_{\mu\nu}S^{\mu\nu} = 0$$

Exercise 4.2

Exercise 4.2: Christoffel symbols and covariant derivative

(a) Show that ${\Gamma^\beta}_{\mu\alpha}$ are the components of the $\binom{1}{1}$ tensor $\nabla\vec{e}_\alpha$ for fixed α, and explain why we do not expect the Christoffel symbols to transform as tensors. (1 pt)

Let us consider a general vector, i.e. a general $\binom{1}{0}$ tensor, $\vec{A}$. $\longrightarrow$ $\nabla\vec{A} = \nabla_\beta A^\gamma \, \tilde{\omega}^\beta \otimes \vec{e}_\gamma$

For a fixed $\alpha \to \delta^\beta_\alpha$ is a vector with dummy index β

Let us take now a vector $\vec{e}_\alpha = \delta^\beta_\alpha \vec{e}_\beta$. $\longrightarrow$ $\nabla\vec{e}_\alpha = \nabla_\beta \delta^\gamma_\alpha \, \tilde{\omega}^\beta \otimes \vec{e}_\gamma$,

Computing the first term gives us $\longrightarrow$ $\nabla_\beta \delta^\gamma_\alpha = \partial_\beta \delta^\gamma_\alpha + {\Gamma^\gamma}_{\mu\beta}\delta^\mu_\alpha = {\Gamma^\gamma}_{\alpha\beta}$

This means that

$$\nabla\vec{e}_\alpha = {\Gamma^\gamma}_{\alpha\beta} \, \tilde{\omega}^\beta \otimes \vec{e}_\gamma \, .$$

On the other hand,

$$\nabla\vec{e}_\alpha = \nabla\left(\frac{\partial x^{\bar\alpha}}{\partial x^\alpha}\vec{e}_{\bar\alpha}\right) = \nabla\left(\frac{\partial x^{\bar\alpha}}{\partial x^\alpha}\right)\vec{e}_{\bar\alpha} + \frac{\partial x^{\bar\alpha}}{\partial x^\alpha}\nabla(\vec{e}_{\bar\alpha}) =$$
$$= \frac{\partial}{\partial x^{\bar\beta}}\left(\frac{\partial x^{\bar\alpha}}{\partial x^\alpha}\right) \tilde{\omega}^{\bar\beta} \otimes \vec{e}_{\bar\alpha} + \frac{\partial x^{\bar\alpha}}{\partial x^\alpha}{\Gamma^{\bar\gamma}}_{\bar\alpha\bar\beta} \, \tilde{\omega}^{\bar\beta} \otimes \vec{e}_{\bar\gamma} =$$
$$= \left[\frac{\partial}{\partial x^{\bar\beta}}\left(\frac{\partial x^{\bar\gamma}}{\partial x^\alpha}\right) + \frac{\partial x^{\bar\alpha}}{\partial x^\alpha}{\Gamma^{\bar\gamma}}_{\bar\alpha\bar\beta}\right] \tilde{\omega}^{\bar\beta} \otimes \vec{e}_{\bar\gamma} \, .$$

$${\Gamma^{\bar\gamma}}_{\alpha\bar\beta} = \frac{\partial}{\partial x^{\bar\beta}}\left(\frac{\partial x^{\bar\gamma}}{\partial x^\alpha}\right) + \frac{\partial x^{\bar\alpha}}{\partial x^\alpha}{\Gamma^{\bar\gamma}}_{\bar\alpha\bar\beta} \, .$$

(b) Discover how each expression $V^{\beta}{}_{,\alpha}$ and $V^{\mu}\Gamma^{\beta}{}_{\mu\alpha}$ separately transforms under a change of coordinates. For $\Gamma^{\beta}{}_{\mu\alpha}$ you can begin with

$$\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu}\,. \tag{1}$$

Show that neither is the standard tensor law, but that the sum does obey the standard law. (2 pt)

Tranformation of the Christofel symbols

$$\frac{\partial \vec{e}_{\bar{\alpha}}}{\partial x^{\bar{\beta}}} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\vec{e}_{\bar{\mu}}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\vec{e}_{\alpha}\right) = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^2 x^{\alpha}}{\partial x^{\beta}\partial x^{\bar{\alpha}}}\vec{e}_{\alpha} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^2 x^{\alpha}}{\partial x^{\beta}\partial x^{\bar{\alpha}}}\vec{e}_{\alpha} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

In the first term of the left-hand side we can do $\alpha \to \mu$.

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x^{\bar{\alpha}}}\vec{e}_{\mu} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\left(\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x^{\bar{\alpha}}} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta}\right)\vec{e}_{\mu} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x^{\bar{\alpha}}} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}$$

Multiplying both sides by $\frac{\partial x^{\bar{\kappa}}}{\partial x^{\mu}}$ we finally obtain:

$$\boxed{\frac{\partial x^{\bar{\kappa}}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial x^{\bar{\beta}}\partial x^{\bar{\alpha}}} + \frac{\partial x^{\bar{\kappa}}}{\partial x^{\mu}}\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\delta^{\bar{\kappa}}_{\bar{\mu}} = \Gamma^{\bar{\kappa}}{}_{\bar{\alpha}\bar{\beta}}}$$

Tranformation of $V^\mu \Gamma^\beta{}_{\mu\alpha}$

$$V^{\bar\mu}\Gamma^{\bar\beta}{}_{\bar\mu\bar\alpha} = \frac{\partial x^{\bar\mu}}{\partial x^\mu} V^\mu \left[\frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\bar\mu}\partial x^{\bar\alpha}} + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\theta}{\partial x^{\bar\mu}}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta{}_{\theta\alpha}\right]$$

$$V^{\bar\mu}\Gamma^{\bar\beta}{}_{\bar\mu\bar\alpha} = \frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\mu}\partial x^{\bar\alpha}}V^\mu + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\theta}{\partial x^{\mu}}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta{}_{\theta\alpha}V^\mu$$

$$\boxed{V^{\bar\mu}\Gamma^{\bar\beta}{}_{\bar\mu\bar\alpha} = \frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\mu}\partial x^{\bar\alpha}}V^\mu + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta{}_{\mu\alpha}V^\mu}$$

Tranformation of $V^\beta{}_{,\alpha}$

$$V^{\bar\beta}{}_{,\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta\right)$$

$$\boxed{V^{\bar\beta}{}_{,\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial^2 x^{\bar\beta}}{\partial x^\alpha\partial x^\beta}V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta{}_{,\alpha}}$$

$$V^{\bar\beta}{}_{,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}{}_{\bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial^2 x^{\bar\beta}}{\partial x^\alpha\partial x^\beta}V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta{}_{,\alpha} + \frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\mu}\partial x^{\bar\alpha}}V^\mu + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta{}_{\mu\alpha}V^\mu$$

$\sigma \to \alpha$ and $\mu \to \beta$

$$V^{\bar\beta}{}_{,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}{}_{\bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial^2 x^{\bar\beta}}{\partial x^\alpha\partial x^\beta}V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta{}_{,\alpha} - \frac{\partial^2 x^{\bar\beta}}{\partial x^{\beta}\partial x^{\alpha}}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}V^\beta + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta{}_{\mu\alpha}V^\mu$$

$$V^{\bar\beta}{}_{,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}{}_{\bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta{}_{,\alpha} + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta{}_{\mu\alpha}V^\mu$$

$$\boxed{V^{\bar\beta}{}_{,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}{}_{\bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}\left(V^\beta{}_{,\alpha} + \Gamma^\beta{}_{\mu\alpha}V^\mu\right)}$$

Therefore, the covariant derivative $V^\beta{}_{;\alpha} = V^\beta{}_{,\alpha} + \Gamma^\beta{}_{\mu\alpha}V^\mu$ is a tensor (of type $\binom{1}{1}$) because it transforms like tensors do.

Exercise 4.3

Exercise 4.3

$ds^2 = dx^2 + dy^2 + dz^2$

$[\tilde{d}x\ \tilde{d}y\ \tilde{d}z]\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}\begin{bmatrix}\partial x \\ \partial y \\ \partial z\end{bmatrix}$

Consider the Euclidean space with metric tensor

$(0,1)\otimes(0,1) \Rightarrow (0,2)$

$(0,2)$

$$\mathbf{g} = \tilde{d}x \otimes \tilde{d}x + \tilde{d}y \otimes \tilde{d}y + \tilde{d}z \otimes \tilde{d}z \,. \qquad g \qquad (2)$$

$-\tilde{d}t \otimes \tilde{d}t$

(a) Compute the elements of the metric tensor in spherical coordinates, and use them to compute the differential *proper* volume dV and the squared line element ds^2. (1 pt)

$\frac{\partial x}{\partial r} \quad \frac{\partial x}{\partial \theta} \quad \frac{\partial x}{\partial \varphi}$

$$\Lambda = \begin{pmatrix} \sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix} \begin{matrix} \to \frac{\partial x}{\partial x^{\bar{\nu}}} \\ \to \frac{\partial y}{\partial x^{\bar{\nu}}} \\ \to \frac{\partial z}{\partial x^{\bar{\nu}}} \end{matrix}$$

$A^{\bar{\mu}}{}_{\bar{\nu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}} \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}} A^{\mu}{}_{\nu}$

$$x^{\mu} = (x, y, z) \text{ and } x^{\bar{\mu}} = (r, \theta, \varphi)$$

$$\begin{aligned} x &= r\sin(\theta)\cos(\varphi)\,, \\ y &= r\sin(\theta)\sin(\varphi)\,, \\ z &= r\cos(\theta)\,. \end{aligned}$$

$(0,2)$

$$g_{\bar{\mu}\bar{\nu}} = \sum_{\mu}\sum_{\nu} \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}} \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}} g_{\mu\nu}$$

$$\Lambda^{\nu}{}_{\bar{\nu}} = \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}$$

$$\bar{g} = \Lambda^T g \Lambda$$

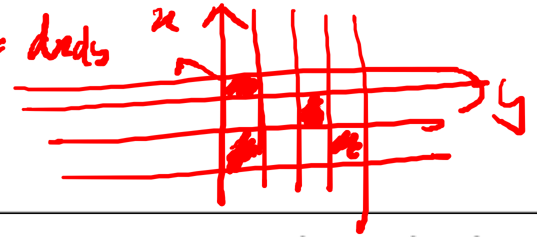
$[\tilde{d}r\ \tilde{d}\theta\ \tilde{d}\varphi]$

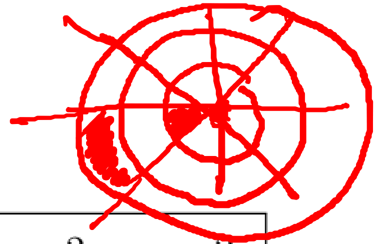
Trig

$$\bar{g} = \Lambda^T \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \begin{bmatrix} \tilde{d}r \\ \tilde{d}\theta \\ \tilde{d}\varphi \end{bmatrix}$$

$$\mathbf{g} = \tilde{d}r \otimes \tilde{d}r + r^2[\tilde{d}\theta \otimes \tilde{d}\theta + \sin^2(\theta)\tilde{d}\varphi \otimes \tilde{d}\varphi]$$

$dA = dx\,dy$





Squared line element

$$ds^2 = d\vec{l} \cdot d\vec{l} = \mathbf{g}(d\vec{l}, d\vec{l}) = g_{\bar{\mu}\bar{\nu}} dx^{\bar{\mu}} dx^{\bar{\nu}} = dr^2 + r^2[d^2\theta + \sin^2(\theta)d\varphi^2]$$

Differential proper volume

$$dV = \sqrt{|-\bar{g}|}\, dr\, d\theta\, d\varphi = r^2\sin(\theta)\, dr\, d\theta\, d\varphi$$

$dA = r\,dr\,d\theta$

(b) Write the one-form $\tilde{d}f$ in spherical coordinates. (0.5 pt)

$$\tilde{d}f \equiv \frac{\partial f}{\partial x^{\bar{\mu}}}\tilde{d}x^{\bar{\mu}} = \frac{\partial f}{\partial r}\tilde{d}r + \frac{\partial f}{\partial \theta}\tilde{d}\theta + \frac{\partial f}{\partial \varphi}\tilde{d}\varphi$$

MTW

(c) Compute the associated gradient vector in spherical coordinates and in terms of the unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$. (1 pt)

$$\vec{\nabla}_{(\text{cart})} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

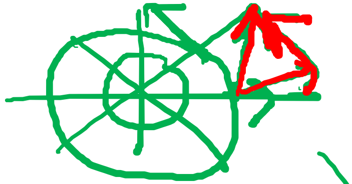
$$\tilde{d}f = \mathbf{g}(\vec{\nabla}f, \tilde{d}x) = g_{\bar{\mu}\bar{\nu}}(\vec{\nabla}f)^{\bar{\mu}}\tilde{d}x^{\bar{\nu}} = \frac{\partial f}{\partial x^{\bar{\nu}}}\tilde{d}x^{\bar{\nu}}$$

$$g_{\bar{\mu}\bar{\nu}}(\vec{\nabla}f)^{\bar{\mu}} = \frac{\partial f}{\partial x^{\bar{\nu}}} \xrightarrow{+g^{-1}} (\vec{\nabla}f)^{\bar{\mu}} = g^{\bar{\mu}\bar{\nu}}\frac{\partial f}{\partial x^{\bar{\nu}}}$$

$x^{\bar{\nu}} = (r, \theta, \varphi)$

$$\begin{bmatrix}\begin{pmatrix}(\vec{\nabla}f)^r \\ (\vec{\nabla}f)^\theta \\ (\vec{\nabla}f)^\varphi\end{pmatrix}\end{bmatrix} = \begin{pmatrix}1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}(\theta)\end{pmatrix}\begin{pmatrix}\frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \varphi}\end{pmatrix}$$

$$\vec{\nabla}f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r^2\sin^2(\theta)}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi\,.$$



$$\begin{aligned}\vec{e}_r &= |\vec{e}_r|\hat{e}_r = \sqrt{\mathbf{g}(\vec{e}_r, \vec{e}_r)}\,\hat{e}_r = \sqrt{g_{rr}}\,\hat{e}_r = \hat{e}_r\,,\\ \vec{e}_\theta &= \ldots = \sqrt{g_{\theta\theta}}\,\hat{e}_\theta = r\hat{e}_\theta\,,\\ \vec{e}_\varphi &= \ldots = \sqrt{g_{\varphi\varphi}}\,\hat{e}_\varphi = r\sin(\theta)\hat{e}_\varphi\,.\end{aligned}$$

Unit vectors

$$\vec{\nabla}f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi$$

(d) Express the basis vectors $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ in terms of $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, and viceversa. (0.75 pt)

Embedding : Pos vector in Spherical Coords embedded in $\mathbb{R}^3$

$\Gamma^{\alpha}_{\beta\gamma}$: 65

$$\vec{e}_{\bar{\mu}} = \Lambda^{\alpha}_{\bar{\mu}} \vec{e}_{\alpha}$$

$$\vec{e}_{\bar{\mu}} = \Lambda_{\bar{\mu}}{}^{\mu} \vec{e}_{\mu} \,.$$

$$\begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\varphi) & \sin(\theta)\sin(\varphi) & \cos(\theta) \\ r\cos(\theta)\cos(\varphi) & r\cos(\theta)\sin(\varphi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix}$$

Inverse

$$\vec{e}_{\mu} = \Lambda_{\mu}^{\bar{\mu}} \vec{e}_{\bar{\mu}}$$

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\varphi) & \frac{1}{r}\cos(\theta)\cos(\varphi) & -\frac{\sin(\varphi)}{r\sin(\theta)} \\ \sin(\theta)\sin(\varphi) & \frac{1}{r}\cos(\theta)\sin(\varphi) & \frac{\cos(\varphi)}{r\sin(\theta)} \\ \cos(\varphi) & -\frac{\sin(\theta)}{r} & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix}$$

$$g_{\mu\nu} x^{\mu} x^{\nu}$$

$$\hat{r} = \sin\theta\cos\varphi\, \vec{e}_x + \sin\theta\sin\varphi\, \vec{e}_y + \cos\theta\, \hat{e}_z$$

$$A_\mu B^\mu = (A_1 \; A_2 \; A_3) \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

(e) Using (1) compute the Chistoffel symbols $\Gamma^{\bar{\mu}}{}_{rr}$ and $\Gamma^{\bar{\mu}}{}_{r\theta}$. (0.75 pt)

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^{\mu}{}_{\alpha\beta}\,\vec{e}_\mu$$

Let us start with the computation of $\Gamma^{\bar{\mu}}{}_{r\theta}$ $\longrightarrow$ $\boxed{\dfrac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}{}_{r\theta}\,\vec{e}_{\bar{\mu}}\,.}$ $= \underbrace{\Gamma^{r}{}_{r\theta}\vec{e}_r + \Gamma^{\theta}{}_{r\theta}\vec{e}_\theta + \Gamma^{\varphi}{}_{r\theta}\vec{e}_\varphi}$

Use $\vec{e}_r$ result from the previous slide, and differentiate it component-wise

$$\frac{\partial \vec{e}_r}{\partial \theta} = \cos(\theta)\cos(\varphi)\vec{e}_x + \cos(\theta)\sin(\varphi)\vec{e}_y - \sin(\theta)\vec{e}_z = \frac{\vec{e}_\theta}{r}$$

$$= \cos(\theta)\cos(\varphi)\left[\sin(\theta)\cos(\varphi)\vec{e}_r + \frac{1}{r}\cos(\theta)\cos(\varphi)\vec{e}_\theta - \frac{\sin(\varphi)}{r\sin(\theta)}\vec{e}_\varphi\right] +$$

$$+\cos(\theta)\sin(\varphi)\left[\sin(\theta)\sin(\varphi)\vec{e}_r + \frac{1}{r}\cos(\theta)\sin(\varphi)\vec{e}_\theta + \frac{\cos(\varphi)}{r\sin(\theta)}\vec{e}_\varphi\right] - \sin(\theta)\left[\cos(\theta)\vec{e}_r - \frac{\sin(\theta)}{r}\vec{e}_\theta\right] = \ldots$$

$$0\vec{e}_\varphi + 0\vec{e}_r + \frac{\vec{e}_\theta}{r}$$

After simplification we get

$$\boxed{\frac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}{}_{r\theta}\,\vec{e}_{\bar{\mu}} = \frac{\vec{e}_\theta}{r}\,.} \quad \Longrightarrow \quad \boxed{\Gamma^{r}{}_{r\theta} = \Gamma^{\varphi}{}_{r\theta} = 0} \qquad \boxed{\Gamma^{\theta}{}_{r\theta} = \frac{1}{r}}$$

The computation of ${\Gamma^{\bar{\mu}}}_{rr}$ is straightforward.

$$\boxed{\frac{\partial \overrightarrow{e}_r}{\partial r} = 0 = {\Gamma^{\bar{\mu}}}_{rr}\overrightarrow{e}_{\bar{\mu}} \longrightarrow \boxed{{\Gamma^{\bar{\mu}}}_{rr} = 0}}$$

(f) Compute all the Christoffel symbols in spherical coordinates using the formula

$$\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}} = \frac{g^{\bar{\mu}\bar{\sigma}}}{2}\left(g_{\bar{\sigma}\bar{\alpha},\bar{\beta}} + g_{\bar{\sigma}\bar{\beta},\bar{\alpha}} - g_{\bar{\alpha}\bar{\beta},\bar{\sigma}}\right). \qquad (1\text{ pt})$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\varphi}{}_{r\varphi} = \frac{1}{r} \qquad \Gamma^{r}{}_{\theta\theta} = -r \qquad \Gamma^{r}{}_{\varphi\varphi} = -r\sin^2(\theta)$$

$$\Gamma^{\theta}{}_{\varphi\varphi} = -\frac{1}{2}\sin(2\theta) \qquad \Gamma^{\varphi}{}_{\varphi\theta} = \frac{\cos(\theta)}{\sin(\theta)}$$

All non-zero Christoffel symbols

$$\Gamma^{r}{}_{\theta\theta} = \frac{g^{r\sigma}}{2}\left(g_{\sigma\theta,\theta} + g_{\sigma\theta,\theta} - g_{\theta\theta,\sigma}\right)$$

$$g = \begin{bmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{bmatrix}$$

$\sigma = r$

$$= \frac{1}{2}\Big[g^{rr}\left(\cancel{g_{r\theta,\theta}} + \cancel{g_{r\theta,\theta}} - g_{\theta\theta,r}\right)$$

$\sigma = \theta$

$$+ \cancel{g^{r\theta}\left(g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}\right)}$$

$\sigma = \varphi$

$$+ \cancel{g^{r\varphi}\left(g_{\varphi\theta,\theta} + g_{\varphi\theta,\theta} - g_{\theta\theta,\varphi}\right)} \Big]$$

$$= \frac{1}{2} g^{rr}\left(-g_{\theta\theta,r}\right)$$

$$= -\frac{1}{2}\, 1\, (2r) = -r$$

(g) Compute the Laplacian of the function f, i.e. $\nabla^2 f$, in spherical coordinates. (1 pt)

$$\boxed{\nabla^2 f = \mathbf{g}(\vec{\nabla}, \vec{\nabla} f) = g_{\bar{\mu}\bar{\nu}} \vec{\nabla}^{\bar{\mu}} (\vec{\nabla} f)^{\bar{\nu}} = \nabla_{\bar{\mu}} (\vec{\nabla} f)^{\bar{\mu}} = \partial_{\bar{\mu}} (\vec{\nabla} f)^{\bar{\mu}} + {\Gamma^{\bar{\mu}}}_{\bar{\kappa}\bar{\mu}} (\vec{\nabla} f)^{\bar{\kappa}}}$$

Sum over the indices

$$\boxed{\nabla^2 f = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r^2}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r^2 \sin^2(\theta)}\frac{\partial f}{\partial \varphi}\right) + {\Gamma^{\theta}}_{r\theta}\frac{\partial f}{\partial r} + {\Gamma^{\varphi}}_{r\varphi}\frac{\partial f}{\partial r} + {\Gamma^{\varphi}}_{\theta\varphi}\frac{1}{r^2}\frac{\partial f}{\partial \varphi}}$$

Use the Christoffel symbols we calculated before

$$\boxed{\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \varphi^2}}$$