

Problem. Parity transformations of fermion bilinears

$$\psi(t, \vec{x}) \xrightarrow{\mathcal{P}} \eta \gamma^0 \psi(t, -\vec{x}), \quad (1)$$

where $\eta \in \mathbb{C}$ and $|\eta| = 1$.

- a) Derive the transformation properties of the following fermion bilinears under parity:

- a) pseudo-scalar current: $\bar{\psi} i \gamma_5 \psi$
- b) axial-vector current: $\bar{\psi} \gamma^\mu \gamma_5 \psi$

Which structures are invariant under a Parity transformation, and which of them are not? Check it explicitly.

Hints: first of all, check that $\bar{\psi}(t, \vec{x}) \xrightarrow{\mathcal{P}} \eta^* \bar{\psi}(t, -\vec{x}) \gamma^0$. Then, you should make use of the properties of the γ matrices as representations of the Clifford algebra, in particular $(\gamma^0)^2 = 1$; $\{\gamma^0, \gamma^k\} = 0$.

- b) Show explicitly that electromagnetic fermion interactions are invariant under CP transformations.

Hint: remember from the lecture that under charge conjugation: $C\bar{\psi} \gamma^\mu \psi C^{-1} = -\bar{\psi} \gamma^\mu \psi$ and $CA_\mu C^{-1} = -A_\mu$.

- c) Show that the helicity of a dirac fermion changes sign under space reflection, but not under time reversal.

Hint: remember the helicity operator is defined as $\hat{\Lambda} = \frac{\vec{\Sigma} \cdot \vec{p}}{|p|}$ with $\vec{\Sigma} = \frac{1}{2} \begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}$ the spin operator. To check the behavior under time reversal, adopt explicit expressions for the $u(p)$ spinor and restrict to the case $p_x = 0 = p_y$.

[P1.] Parity transformations of Fermion bilinears.

$$\psi(t, \vec{x}) \xrightarrow{P} \eta \gamma^0 \psi(t, -\vec{x})$$

$$\begin{aligned} \eta &\in C \\ |\eta| &= 1 \end{aligned}$$

(a) Derive the transformation properties for the following fermion bilinear under parity

→ Pseudo-scalar current : $\bar{\psi} i \gamma_5 \psi$

→ Axial-vector current : $\bar{\psi} \gamma^\mu \gamma_5 \psi$

Which structures are invariant under a Parity transfo, which aren't?

→ Let's check the hint:

$$\begin{aligned} \bar{\psi}(t, \vec{x}) = \psi^\dagger(t, \vec{x}) \gamma^0 &\xrightarrow{P} (\eta \gamma^0 \psi(t, \vec{x}))^\dagger \gamma^0 \\ &= \eta^* \psi^\dagger(t, -\vec{x}) \gamma^0 \gamma^0 \\ &= \eta^* \bar{\psi}(t, -\vec{x}) \gamma^0 \end{aligned}$$

$$\rightarrow \bar{\psi}(t, \vec{x}) i \gamma_5 \psi(t, \vec{x}) \xrightarrow{P} \eta^* \bar{\psi}(t, -\vec{x}) i \gamma^0 \gamma^0 \gamma^1 \gamma^2 \gamma_5 \gamma^3 \gamma^4 \psi(t, -\vec{x})$$

$$= |\eta|^2 \bar{\psi}(t, -\vec{x}) i (-1) \gamma^5 \psi(t, -\vec{x})$$

$$= |\eta|^2 \bar{\psi}(t, -\vec{x}) i (-1)$$

$$= - \bar{\psi}(t, -\vec{x}) i \gamma^5 \psi(t, -\vec{x})$$

Gets an additional $\xrightarrow{-1}$ sign \Rightarrow Not invariant.

$$\rightarrow \bar{\psi} \gamma^\mu \gamma_5 \psi \xrightarrow{P} \bar{\psi}(t, -\vec{x}) \gamma^0 \gamma^\mu \gamma_5 \gamma^0 \psi(t, -\vec{x}) = \bar{\psi}(t, -\vec{x}) (-(\gamma^\mu \gamma_5)^T) \psi(t, -\vec{x})$$

$$= \begin{cases} -\bar{\psi}(t, -\vec{x}) \gamma^0 \gamma_5 \psi(t, \vec{x}) & \mu = 0 \\ \bar{\psi}(t, \vec{x}) \gamma^i \gamma_5 \psi(t, -\vec{x}) & \mu = i \in \{1, 2, 3\} \end{cases}$$

\rightarrow Spatial comp. are invariant.

- b) Show explicitly that electromagnetic fermion interactions are invariant under CP transformations.

Hint: remember from the lecture that under charge conjugation: $C\bar{\psi}\gamma^\mu\psi C^{-1} = -\bar{\psi}\gamma^\mu\psi$ and $CA_\mu C^{-1} = -A_\mu$.

The interaction term is given by (for QED) : $\bar{\Psi}\not{A}\Psi$

$$\begin{aligned}\bar{\Psi}\not{A}\Psi &\xrightarrow{P} \bar{\Psi}(t, -\vec{x})\gamma^0 A_0(t, -\vec{x})\Psi(t, \vec{x}) - \bar{\Psi}(t, -\vec{x})\gamma^i(-A_i(t, -\vec{x}))\Psi(t, \vec{x}) \\ &\xrightarrow{C} -\bar{\Psi}(t, -\vec{x})\gamma^0\Psi(t, -\vec{x})(-A_0(t, -\vec{x})) + \bar{\Psi}(t, -\vec{x})\gamma^i\Psi(t, -\vec{x})A_i(t, \vec{x}) \\ &= \bar{\Psi}(t, -\vec{x})\gamma^\mu A_\mu\Psi(t, -\vec{x})\end{aligned}$$

where we used : $\bar{\Psi}\gamma^\mu\Psi \xrightarrow{P} \begin{cases} \bar{\Psi}\gamma^\mu\Psi(t, \vec{x}) & \text{for } \mu=0 \\ -\bar{\Psi}\gamma^\mu\Psi(t, \vec{x}) & \text{else} \end{cases}$

& $A^\mu(t, \vec{x}) \xrightarrow{P} \begin{cases} A^\mu(t, \vec{x}) & \text{for } \mu=0 \\ A^\mu(t, -\vec{x}) & \text{for } \mu=i \in \{1, 2, 3\} \end{cases}$

- c) Show that the helicity of a dirac fermion changes sign under space reflection, but not under time reversal.

Hint: remember the helicity operator is defined as $\hat{\Lambda} = \frac{\vec{\Sigma} \cdot \vec{p}}{|\vec{p}|}$ with $\vec{\Sigma} = \frac{1}{2} \begin{pmatrix} \sigma^i & \\ & \sigma^i \end{pmatrix}$ the spin operator. To check the behavior under time reversal, adopt explicit expressions for the $u(p)$ spinor and restrict to the case $p_x = 0 = p_y$.

- We know $\vec{p} \xrightarrow{Q} -\vec{p}$ but $\hat{\Sigma} \xrightarrow{Q} \hat{\Sigma} \Rightarrow \hat{\Lambda} = \frac{\hat{\Sigma} \cdot \vec{p}}{|\vec{p}|} \xrightarrow{Q} \frac{\hat{\Sigma} \cdot (-\vec{p})}{|-\vec{p}|} = -\hat{\Lambda}$
- Similarly $\vec{p} \xrightarrow{T} -\vec{p}$ and $\hat{\Sigma} \xrightarrow{T} -\hat{\Sigma} \Rightarrow \hat{\Lambda} \xrightarrow{T} \hat{\Lambda}$
- Now, to see the transformation change of $\hat{\Sigma}$ under P & T resp. Consider e.g. $\Sigma^{(3)}$

$$\Sigma^{(3)} = \left(\frac{1}{2} \frac{U_{S=\frac{1}{2}}(P) \overline{U_{S=\frac{1}{2}}(P)}}{2m} - \frac{1}{2} \frac{U_{S=-\frac{1}{2}}(P) \overline{U_{S=-\frac{1}{2}}(P)}}{2m} \right) \delta(P-n) \quad \text{--- } \oplus$$

- We can consider $\vec{p} = \vec{e}_n$ then $U_S(p) = \begin{pmatrix} \sqrt{p_0 + m'} & \chi_s \\ \sqrt{p_0 - m} & \sigma_3 \chi_s \end{pmatrix}$
and $U_S(p) \xrightarrow{Q} U_S(p) \quad ; \quad \overline{U_S(p)} \xrightarrow{Q} \overline{U_S(p)}$

- Since $p' \xrightarrow{Q} p'$, as a consequence $\hat{\Sigma}^{(3)} \xrightarrow{Q} \hat{\Sigma}$ because of \oplus
- In a similar manner because $p_0 \xrightarrow{T} -p_0$

We find $U_S(p) \xrightarrow{T} \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} U_S(p) \quad ; \quad \overline{U_S(p)} \xrightarrow{T} U_S(p) \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} \quad \& \text{ in combo with}$

$$\rightarrow \hat{\Sigma}^{(3)} \xrightarrow{T} \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} \hat{\Sigma}^{(3)} \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix} \frac{1}{2} \begin{pmatrix} \infty^2 & 0 \\ 0 & \infty^2 \end{pmatrix} \begin{pmatrix} 0 & i\sigma^3 \\ i\sigma^3 & 0 \end{pmatrix}$$

$$= -\frac{1}{2} \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^2 \end{pmatrix} = -\widehat{\sum}^{(3)}$$

Analogously we can show $\widehat{\sum} \xrightarrow{\tau} -\widehat{\sum}$