1. (12) Consider a two-level electron system where the electrons do not interact with each other but interact with a single mode of an electromagnetic field. It turns out that if you quantize the electromagnetic field, the the hamiltonian for the system consisting of the electrons and photons can be written in second-quantized form approximately as:

$$H = \hbar\omega b^\dagger b + E_1 a_1^\dagger a_1 + E_2 a_2^\dagger a_2 + C a_1^\dagger a_2 b^\dagger + C a_2^\dagger a_1 b, \tag{1}$$

where E_i are the energy levels, C is the real electron-photon interaction parameter, $a_i^\dagger$, a_i are the creation and annihilation operators for the electron in state i, and $b^\dagger$, b are the creation and annihilation operators for the photon in a particular mode with energy $\hbar\omega$. The fermions and bosons obey the usual commutation relationships, and all fermion operators commute with all boson operators.

(a) Give a brief physical interpretation of each of the five terms in the above Hamiltonian.

(b) Consider the situation where there are either n or $n-1$ photons and at most one electron in the system. Write the Hamiltonian as a 2×2 matrix, H_{mp} in the basis of the two states: $|\phi_1\rangle = \frac{1}{\sqrt{(n-1)!}} a_2^\dagger (b^\dagger)^{n-1}|0\rangle$, $|\phi_2\rangle = \frac{1}{\sqrt{(n)!}} a_1^\dagger (b^\dagger)^{n}|0\rangle$.

(c) To simplify, set $C = \hbar\Omega$, $E_1 = -\hbar\omega_a/2$, $E_2 = \hbar\omega_a/2$. Using these values, show that the energy eigenvalues of your hamiltonian are:

$$E_\pm = \hbar\omega\left(n - \frac{1}{2}\right) \pm \frac{\hbar}{2}\sqrt{(\omega-\omega_a)^2 + 4\Omega_n^2}, \tag{2}$$

where Ω_n is a quantity that depends on n that you must determine.

(d) Now consider the resonant case where $\omega_a = \omega$ and $n = 1$. By using the eigenvectors and eigenvalues of your hamiltonian, give the state of the system as a function of time, given an initial state $|\psi(t=0)\rangle = |\phi_1\rangle$, as a linear combination of $|\phi_1\rangle$ and $|\phi_2\rangle$. You should find that the system oscillates between $|\phi_1\rangle$ and $|\phi_2\rangle$ with a period of $2\pi/\Omega$.

a

$$H = \underbrace{\hbar\omega b^\dagger b}_{} + \underbrace{E_1 a_1^\dagger a_1 + E_2 a_2^\dagger a_2}_{} + \underbrace{C a_1^\dagger a_2 b^\dagger + C a_2^\dagger a_1 b}_{}$$

- Number operator for photons × Energy of photons, keeps track of the energy of all the photons in the system
- Number operator for both the electrons levels. Keeps track of the energy contributed by the electrons in the system. (E_1 & E_2)
- Interaction terms valid between the electron & photons.

- Here is how each of the operators will act on a given state

$$b|n_1, n_2; n\rangle = \sqrt{n}\,|n_1, n_2; n-1\rangle$$
$$b^\dagger|n_1, n_2; n\rangle = \sqrt{n+1}\,|n_1, n_2; n+1\rangle$$
$$a_1|n_1, n_2; n\rangle = n_1(-1)^{n_1}|n_1-1, n_2; n\rangle$$
$$a_1^\dagger|n_1, n_2; n\rangle = (1-n_1)(-1)^{n_1}|n_1+1, n_2; n\rangle$$
$$a_2|n_1, n_2; n\rangle = n_2(-1)^{n_1+n_2}|n_1, n_2-1; n\rangle$$
$$a_2^\dagger|n_1, n_2; n\rangle = (1-n_2)(-1)^{n_1+n_2}|n_1, n_2+1; n\rangle$$

(b)

$$|\phi_1\rangle = \frac{1}{\sqrt{(n-1)!}}\, \underbrace{a_2^\dagger}_{\text{Electron}} \underbrace{(b^\dagger)^{n-1}}_{\text{Photon}} |0\rangle = \frac{1}{\cancel{\sqrt{(n-1)!}}}\cancel{\sqrt{(n-1)!}}\,|0,1;n-1\rangle$$

$$|\phi_2\rangle = \frac{1}{\sqrt{n!}}\, a_1^\dagger (b^\dagger)^{n} |0\rangle = \frac{1}{\cancel{\sqrt{n!}}}\cancel{\sqrt{n!}}\,|1,0;n\rangle = |1,0;n\rangle$$

(11) $\langle\phi_1|H|\phi_1\rangle = \langle 0,1;n-1|\,H\,|0,1;n-1\rangle$

$$= \langle 0,1;n-1|\big(\hbar\omega b^\dagger b + E_1 a_1^\dagger a_1 + E_2 a_2^\dagger a_2 + C a_1^\dagger a_2 b^\dagger + C a_2^\dagger a_1 b\big)|0,1;n-1\rangle$$

$$= \langle 0,1;n-1|\big(\hbar\omega\, b^\dagger b\,|0,1;n-1\rangle \;\checkmark\text{(i)} \quad\Big|\; = \hbar\omega\sqrt{n-1}\,b^\dagger\,|0,1;n-2\rangle = \hbar\omega(n-1)\,|0,1;n-1\rangle$$

$$+ E_1\, a_1^\dagger a_1\,|0,1;n-1\rangle \;\times \quad\Big|\; = E_1(0)\,|0,1;n-1\rangle = 0$$

$$+ E_2\, a_2^\dagger a_2\,|0,1;n-1\rangle \;\checkmark\text{(ii)} \quad\Big|\; = E_2(1)\,|0,1;n-1\rangle$$

$$+ C\, a_1^\dagger a_2 b^\dagger\,|0,1;n-1\rangle \;\checkmark\text{(iii)} \quad\Big|\; = C\sqrt{n}\,a_1^\dagger a_2\,|0,1;n\rangle = C\sqrt{n}\,a_1^\dagger\,|0,0;n\rangle = C\sqrt{n}\,|1,0;n\rangle$$

$$+ C\, a_2^\dagger a_1 b\,|0,1;n-1\rangle \times \big) \quad\Big|\; = C\sqrt{n-1}\,a_2^\dagger a_1\,|0,1;n-2\rangle = 0$$

↳ Out of these, (iii) will also go to zero as $\langle 0,1;n-1|1,0;n\rangle = 0$

$$= \langle 0,1;n-1|\,(\hbar\omega(n-1)+E_2)\,|0,1;n-1\rangle = \boxed{(\hbar\omega(n-1)+E_2)} \quad 11$$

(12) $\langle\phi_1|H|\phi_2\rangle = \langle 0,1;n-1|\,H\,|1,0;n\rangle$

$$= \langle 0,1;n-1|\big(\hbar\omega b^\dagger b + E_1 a_1^\dagger a_1 + E_2 a_2^\dagger a_2 + C a_1^\dagger a_2 b^\dagger + C a_2^\dagger a_1 b\big)|1,0;n\rangle$$

$$= \langle 0,1;n-1|\big(\hbar\omega\, b^\dagger b\,|1,0;n\rangle \quad\Big|\; = \hbar\omega\, n\,|1,0;n\rangle \;\text{(i)}$$

$$+ E_1\, a_1^\dagger a_1\,|1,0;n\rangle \quad\Big|\; = E_1\,|1,0;n\rangle \;\text{(ii)}$$

$$+ E_2\, a_2^\dagger a_2\,|1,0;n\rangle \quad\Big|\; = E_2\,|1,0;n\rangle = 0$$

$$+C\; a_1^\dagger a_2 b^\dagger |1,0;n\rangle \quad = C\sqrt{n+1}\; a_1^\dagger a_2 |1,0;n+1\rangle = 0$$

$$+C\; a_2^\dagger a_1 b |1,0;n\rangle) \quad = C\sqrt{n}\; a_2^\dagger a_1 |1,0;n-1\rangle \;(iii)$$

$$= C\sqrt{n}\; a_2^\dagger |0,0;n-1\rangle$$

$$= C\sqrt{n}\; |0,1;n-1\rangle$$

↳ Out of these, (i)(ii) will go to zero as $\langle 0,1;n-1|1,0;n\rangle = 0$

$\langle 0,1;n-1|1,0;n\rangle = 0$

$$= \langle 0,1;n-1|\, C\sqrt{n}\, |0,1;n-1\rangle$$

$$= C\sqrt{n}\, \langle 0,1;n-1|0,1;n-1\rangle = \boxed{C\sqrt{n}}_{12}$$

(21) $\langle \phi_2|H|\phi_1\rangle = \langle 1,0;n|H|0,1;n-1\rangle$

$$= \langle 1,0;n|\,(\hbar\omega b^\dagger b + E_1 a_1^\dagger a_1 + E_2 a_2^\dagger a_2 + C a_1^\dagger a_2 b^\dagger + C a_2^\dagger a_1 b)|0,1;n-1\rangle$$

$$= \langle 1,0;n|\,(\hbar\omega\, b^\dagger b\, |0,1;n-1\rangle \quad = \hbar\omega(n-1)|0,1;n-1\rangle \;(i)$$

$$+ E_1\, a_1^\dagger a_1 |0,1;n-1\rangle \quad = 0$$

$$+ E_2\, a_2^\dagger a_2 |0,1;n-1\rangle \quad = E_2 |0,1;n-1\rangle \;(ii)$$

$$+ C\, a_1^\dagger a_2 b^\dagger |0,1;n-1\rangle \quad = C\sqrt{n}\, a_1^\dagger a_2 |0,1;n\rangle$$

$$\quad = C\sqrt{n}\, a_1^\dagger |0,0;n\rangle$$

$$+ C\, \underbrace{a_2^\dagger a_1 b |0,1;n-1\rangle)}_{=0} \quad = C\sqrt{n}\; |1,0;n\rangle \;(iii)$$

In this, we can take (i),(ii), to be zero, (i) $\langle 1,0;n|0,1;n-1\rangle$

(ii) $\langle 1,0;n|0,1;n-1\rangle$

$$= C\sqrt{n}\; \langle 1,0;n|1,0;n\rangle = \boxed{C\sqrt{n}}_{21}$$

(22) $\langle \phi_2|H|\phi_2\rangle = \langle 1,0;n|H|1,0;n\rangle$

$$= \langle 1,0;n|\,(\hbar\omega\, b^\dagger b\, |1,0;n\rangle \quad = \hbar\omega\, n\, |1,0;n\rangle \;(i)$$

$$+ E_1\, a_1^\dagger a_1 |1,0;n\rangle \quad = E_1 \; |1,0;n\rangle \;(ii)$$

$$
\begin{aligned}
&+ E_2 \quad a_2^\dagger a_2 \, |1,0;n\rangle \Big| = 0 \\
&+ C \quad a_1^\dagger a_2 b^\dagger \, |1,0;n\rangle \Big| = C\sqrt{n+1}\, a_1^\dagger a_2 \, |1,0;n+1\rangle = 0 \\
&+ C \quad a_2^\dagger a_1 b \, |1,0;n\rangle \Big) \Big| = C\sqrt{n} \quad a_2^\dagger a_1 |1,0;n-1\rangle \\
&\qquad\qquad = C\sqrt{n}\, |0,1;n-1\rangle \text{ (iii)}
\end{aligned}
$$

$$
\begin{aligned}
&= \langle 1,0;n| \, (\hbar\omega n + E_1) \, |1,0;n\rangle \\
&= (\hbar\omega n + E_1)
\end{aligned}
$$

$$
H = \begin{bmatrix} (\hbar\omega(n-1) + E_2) & C\sqrt{n} \\ C\sqrt{n} & (\hbar\omega n + E_1) \end{bmatrix}
$$

(c) Using the values given for E_1, E_2, C, we have,

$$
H = \begin{bmatrix} \hbar\omega(n-1) + \frac{\hbar\omega_a}{2} & \hbar\Omega\sqrt{n} \\ \hbar\Omega\sqrt{n} & \hbar\omega n - \frac{\hbar\omega_a}{2} \end{bmatrix} = \hbar \begin{bmatrix} \omega(n-1) + \frac{\omega_a}{2} & \Omega\sqrt{n} \\ \Omega\sqrt{n} & \omega n - \frac{\omega_a}{2} \end{bmatrix}
$$

$$
\det(H - \lambda I) = \left[\left(\left(\hbar\omega n + \frac{\hbar\omega_a}{2} - \hbar\omega\right) - \lambda\right)\left(\left(\hbar\omega n - \frac{\hbar\omega_a}{2}\right) - \lambda\right) - n\hbar^2\Omega^2\right] = 0
$$

$$
\Rightarrow (\hbar\omega n)^2 - \left(\frac{\hbar\omega_a}{2}\right)^2 - (\hbar\omega)^2 n + \hbar^2\frac{\omega\omega_a}{2} + \lambda^2 - \lambda(\hbar\omega(2n-1)) - n(\hbar\Omega)^2 = 0
$$

Mathematica
Solve + Simplify λ

$$
\lambda_\pm = \hbar\omega\left(n - \frac{1}{2}\right) \pm \frac{\hbar}{2}\sqrt{(\omega - \omega_a)^2 + 4\Omega^2 n}
$$

$\Omega_n^2 = \Omega^2 n$

(d) We are given $\omega = \omega_a$, $n = 1$

$$
E_\pm = \lambda_\pm = \frac{\hbar\omega}{2} \pm \hbar\Omega
$$

$$\therefore\ H = \begin{bmatrix} \frac{\hbar\omega}{2} & \hbar\Omega \\ \hbar\Omega & -\hbar\omega/2 \end{bmatrix}$$

$$\left\{ \hbar \begin{bmatrix} \omega(n-1)+\frac{\omega_a}{2} & \Omega\sqrt{n} \\ \Omega\sqrt{n} & \omega n - \frac{\omega_a}{2} \end{bmatrix} \right.$$

Referenc

$$E_- : \quad \begin{bmatrix} \frac{\hbar\omega}{2} - E_- & \hbar\Omega \\ \hbar\omega & -\hbar\omega/2 - E_- \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \hbar\Omega & \hbar\Omega \\ \hbar\Omega & \hbar\Omega \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{i.e. } a = -b \quad \Rightarrow \quad V_{E_-} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$E_+ : \quad \begin{bmatrix} \frac{\hbar\omega}{2} - E_+ & \hbar\Omega \\ \hbar\omega & -\frac{\hbar\omega}{2} - E_+ \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -\hbar\Omega & \hbar\Omega \\ \hbar\omega & -\hbar\Omega \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{b} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{i.e. } \tilde{a} = \tilde{b} \quad \Rightarrow \quad V_{E_+} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$U(t) = e^{-\frac{iHt}{\hbar}} \ ; \quad |\Psi(t=0)\rangle = |\phi_1\rangle \text{ is given.}$$

$$U(t)\,|\Psi(t=0)\rangle = e^{i\left(\frac{\omega}{2}+\Omega\right)t}\,|\phi_1\rangle$$

Which demonstrates the oscillatory nature of this resonant case.