# Problem 6

Problem Statement

This problem will require a tiny amount of programming and the use of software capable of graphing numerical results, such as Matlab, Python, Mathematica or Maple. You may work as a team but each student must submit their own copy of the graph. For simplicity let us work in units of frequency, rather than energy, such that $\omega = E/\hbar$. Consider a system of N energy eigenstates $\omega_k = 1/N, 2/N, ...1\ \text{s}^{-1}$ . Suppose you have prepared your state $|\alpha\rangle = \sum_k^N c_k |\omega_k\rangle$ with a mean energy $\omega_0$ and a width $\sigma_\omega$ such that:

$$c_k \propto e^{\left[-\frac{(\omega_k-\omega_0)^2}{2\sigma_\omega^2}\right]} \tag{111}$$

1. Write down the correct expressions for the normalized state and for the correlation amplitude $C(t)$.
2. Now suppose $N = 100, \omega_0 = 0.5\text{s}^{-1}, \sigma_\omega = \text{s}^{-1}$ . On the same graph, plot: 1) $|C(t)|$ for $t \in [0, 100]$s ; 2) the real part of each $c_k(t)$ for the same period.
3. Explain how the correlation time relates to the relative phases of the different eigenkets that you have plotted.
4. Assuming C(t) is a (half)-Gaussian with variance $c_t^2$, evaluate $\sigma_t\sigma_E$. Explain the relationship with a "classical limit"

Solution 1.

For convinience, let us absorb all the $k$ dependence into $c_k$'s. We will take a proportionality constant for the alpha states by $\beta_\alpha$. This gives us,

$$c_k = e^{\left[-\frac{(\omega_k-\omega_0)^2}{2\sigma_\omega^2}\right]} \tag{112}$$

$$|\alpha\rangle = \sum_k^N \beta_\alpha c_k |\omega_k\rangle \tag{113}$$

For normalization we can do,

$$|\alpha\rangle\langle\alpha| = \sum_k^N\sum_l^N \beta_\alpha^2 c_k c_l \langle\omega_k|\omega_l\rangle \tag{114}$$

$$= \sum_k^N \beta_\alpha^2 c_k^\alpha \tag{115}$$

$$= 1 \tag{116}$$

This gives us,

$$\beta_\alpha = \frac{1}{\sqrt{\sum_k^N (c_k)^\alpha}} \tag{117}$$

The original state with this normalization is,

$$|\alpha\rangle = \frac{1}{\sqrt{\sum_{l=1}^N (c_l)^\alpha}} \sum_k^N c_k |\omega_k\rangle \tag{118}$$

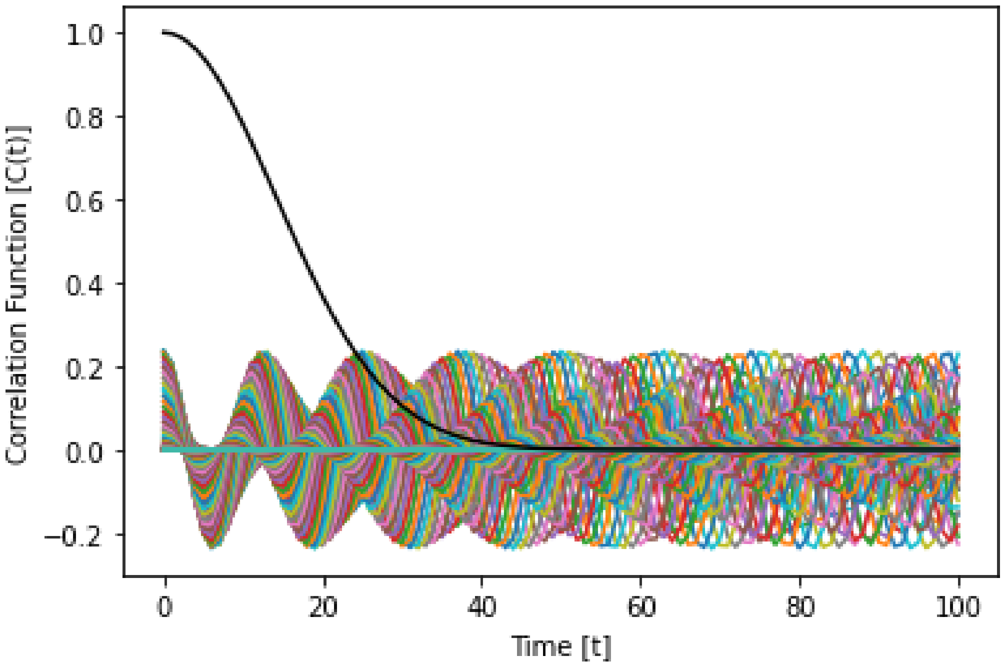
$$\tag{119}$$

Using this, we can now compute $C(t)$,

$$
\begin{aligned}
C(t) &= \langle \alpha, t | \alpha \rangle & (120)\\
&= \langle \alpha | \mathcal{U}(t,0) | \alpha \rangle & (121)\\
&= \frac{1}{\sum_{l=1}^{N} (c_l)^{\alpha}} \left( \sum_{k}^{N} \sum_{m}^{N} c_k c_m e^{i\omega_e t} \langle \alpha_k | \alpha_m \rangle \right) & (122)\\
&= \frac{1}{\sum_{l=1}^{N} (c_l)^{\alpha}} \left( \sum_{k=1}^{N} (c_k)^{\alpha} e^{i\omega_k t} \right) & (123)
\end{aligned}
$$

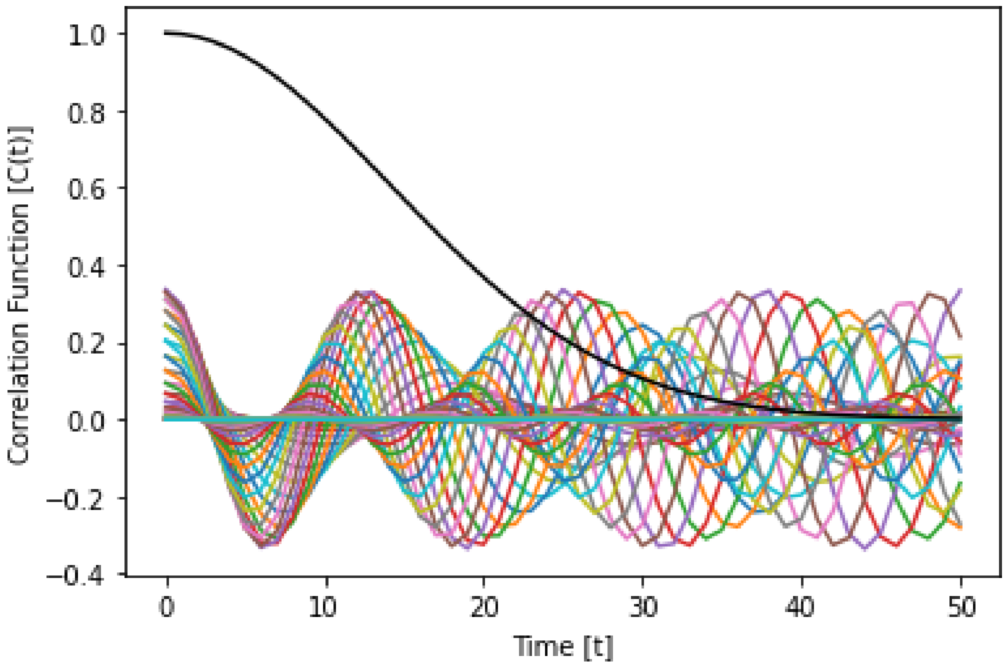
## Solution 2.

For $N = 100$, as asked,




Figuur 1: $N = 100$

For $N = 50$, (helped to interpret as its zoomed in)




Figuur 2: $N = 50$

Solution 3.

It can be seen from the black line, that as the time passes, the correlation function seems to be monotonically decreasing. This same behaviour can be realized if one looks at the colored eigenkets plot, you can see the eigenkets go more and more out of phase as time passes by (more clearly seen in the N=50 figure).

Solution 4.