GradQM : Pset1 Solutions - Rohan Kulkarni

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1 Problem 1

(Sorry forgot to add the question here)

- $|n;+\rangle$ represents the state satisfying $\vec{S}\cdot\vec{n}\,|n;+\rangle = \frac{1}{2}\,|n;+\rangle$
- $|z;\pm\rangle$ represents the basis of S_z,

$$|z;+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |z;-\rangle = \begin{bmatrix}0\\1\end{bmatrix} \tag{1}$$

Hence, our main goal is to derive an expression for $|n;+\rangle$. Let us first define the unit vector $\vec{n}$ using spherical coordinates

$$\vec{n} = \begin{bmatrix}n_x\\n_y\\n_z\end{bmatrix} = \begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix} \tag{2}$$

and

$$\vec{S} = \begin{bmatrix}S_x\\S_y\\S_z\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\end{bmatrix} \tag{3}$$

Where,

$$\sigma_x = (|+\rangle\langle-| + |-\rangle\langle+|) = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \tag{4}$$

$$\sigma_y = i\,(-|+\rangle\langle-| + |-\rangle\langle+|) = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} \tag{5}$$

$$\sigma_z = (|+\rangle\langle+| + |-\rangle\langle-|) = \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} \tag{6}$$

$$\tag{7}$$

We also know ,

$$\vec{S}\cdot\vec{n} = \begin{bmatrix}n_x & n_y & n_z\end{bmatrix}\cdot\begin{bmatrix}S_x\\S_y\\S_z\end{bmatrix} \tag{8}$$

$$= \frac{1}{2}(n_x\sigma_x + n_y\sigma_y + n_z\sigma_z) \tag{9}$$

$$= \frac{1}{2}\left(n_x\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + n_y\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} + n_z\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\right) \tag{10}$$

$$= \frac{1}{2}\begin{bmatrix}n_z & n_x - i\,n_y\\ n_x + i\,n_y & -n_z\end{bmatrix} \tag{11}$$

$$= \frac{1}{2}\begin{bmatrix}\cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi\\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta\end{bmatrix} \tag{12}$$

$$= \frac{1}{2}\begin{bmatrix}\cos\theta & \sin\theta(\cos\phi - i\sin\phi)\\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta\end{bmatrix} \tag{13}$$

$$= \frac{1}{2}\begin{bmatrix}\cos\theta & \sin\theta\,e^{-i\phi}\\ \sin\theta\,e^{i\phi} & -\cos\theta\end{bmatrix} \quad \text{(Euler's formula)} \tag{14}$$

Now that we have an expression for $\vec{S}\cdot\vec{n}$, we need to find the state $|n;+\rangle$ such that

$$\vec{S}\cdot\vec{n}\,|n;+\rangle = +\frac{1}{2}\,|n;+\rangle \tag{15}$$

(One can check that the eigenvalues of $\vec{S}\cdot\vec{n}$ are indeed $\pm\frac{1}{2}$).

We can define $|n;+\rangle$ in as a linear combination of the z states in the following way

$$|n;+\rangle = c_1\,|z;+\rangle + c_2\,|z;-\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We can rearrange the (15) to get the typical equation we solve for eigenvectors.

$$\begin{aligned}
\left(\vec{S}\cdot\vec{n} - \hat{I}\frac{1}{2}\right)|n;+\rangle &= 0 \\
\left(\frac{1}{2}\begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)|n;+\rangle &= 0 \\
\frac{1}{2}\begin{bmatrix} \cos\theta - 1 & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta - 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= 0
\end{aligned}$$

This equation gives us two equations

$$\begin{aligned}
\frac{1}{2}[c_1(\cos\theta - 1) + c_2(\sin\theta\, e^{-i\phi})] &= 0 \\
\frac{1}{2}[c_1(\sin\theta\, e^{i\phi}) - c_2(\cos\theta + 1)] &= 0
\end{aligned}$$

Solving the first one of them gives us

$$\begin{aligned}
c_1(\cos\theta - 1) &= -c_2(\sin\theta\, e^{-i\phi}) \\
c_2 &= \frac{c_1(1-\cos\theta)}{\sin\theta}e^{i\phi}
\end{aligned}$$

We won't solve the second equation because it gives us exactly the same relation between c_1 and c_2. We can use some trigonometric identities to simplify the relation to the following form

$$c_2 = \left(e^{i\phi}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)c_1$$

We have the $|n;+\rangle$ vector in the z-state basis in the following form

$$|n;+\rangle = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 \\ \left(e^{i\phi}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right)c_1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ \left(e^{i\phi}\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) \end{bmatrix}$$

We gotta normalize the state. We start with the normalization condition,

$$\langle n;+|n;+\rangle = 1 \tag{16}$$

$$c_1^*c_1 + c_2^*c_2 = 1 \tag{17}$$

$$|\ c_1\ |^2 + |\ c_2\ |^2 = 1 \tag{18}$$

$$|\ c_1\ |^2\left(1 + \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right) = 1 \tag{19}$$

$$|\ c_1\ |^2\left(\frac{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}}\right) = 1 \tag{20}$$

$$|\ c_1\ |^2 = \cos^2\frac{\theta}{2} \tag{21}$$

Taking the simplest form for c_1 we get

$$c_1 = \cos\frac{\theta}{2}, \quad c_2 = \sin\frac{\theta}{2}e^{i\phi}$$

Which gives us the $|n;+\rangle$ state as follows

$$|n;+\rangle = \cos\frac{\theta}{2}\,|z;+\rangle + \sin\frac{\theta}{2}e^{i\phi}\,|z;-\rangle$$

Recalling that $|z;+\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \quad |z;-\rangle = \begin{bmatrix}0\\1\end{bmatrix}$ we can also write the above equation as

$$|n;+\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}e^{i\phi}\end{bmatrix}$$

in the z-state basis.