(a) Show that $-i\psi_L^T \sigma_2 \psi_L$ is Lorentz-invariant.

→ We have $\psi_R(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$ for $(0, \frac{1}{2})$ representation

$\psi_L(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$ for $(\frac{1}{2}, 0)$ representation

Recall how these Weyl spinors transforms under r_j & b_j

$$\psi_R \to e^{\frac{1}{2}(i r_j \sigma_j + b_j \sigma_j)} \psi_R = \left(1 + \frac{i}{2} r_j \sigma_j + \frac{1}{2} b_j \sigma_j + \dots\right) \psi_R$$

$$\psi_L \to e^{\frac{1}{2}(i r_j \sigma_j - b_j \sigma_j)} \psi_L = \left(1 + \frac{i}{2} r_j \sigma_j - \frac{1}{2} b_j \sigma_j + \dots\right) \psi_L$$

Giving us the fact that these Spinors infinetesimally transform like:

$$\delta\psi_R = \frac{1}{2}(i r_j + b_j)\sigma_j \psi_R \quad ; \quad \delta\psi_L = \frac{1}{2}(i r_j - b_j)\sigma_j \psi_L$$

$$\delta\psi_R^\dagger = \frac{1}{2}(-i r_j + b_j)\psi_R^\dagger \sigma_j \quad ; \quad \delta\psi_L^\dagger = \frac{1}{2}(-i r_j - b_j)\psi_L^\dagger \sigma_j$$

Note: r_j & b_j are real numbers.

$$\begin{aligned}
\delta(\psi_L^T \sigma_2 \psi_L) &= \left[(\delta\psi_L^T)\sigma_2 \psi_L + \psi_L^T \sigma_2 (\delta\psi_L)\right] \\
&= \left[\frac{1}{2}(i r_j - b_j)\psi_L^T \underbrace{\sigma_j^T \sigma_2}_{-\sigma_2\sigma_j} \psi_L + \psi_L^T \sigma_2 \frac{1}{2}(i r_j - b_j)\sigma_j \psi_L\right] \\
&= \frac{1}{2}\left[\cancel{-(i r_j - b_j)\psi_L^T \sigma_2 \sigma_j \psi_L} + \cancel{(i r_j - b_j)\psi_L^T \sigma_2 \sigma_j \psi_L}\right] \\
&= 0
\end{aligned}$$

These cancel!

$\circledast$

- This shows that the term $\psi_L^T \sigma_2 \psi_L$ is invariant under infinetesimal rotations & boosts.
- This also shows that $\psi_L^\dagger \sigma_2 \psi_L^*$ is invariant under such a transformation as it is just the complex conjugate of $\circledast$

I present a quick proof that $\sigma_j^T \sigma_2 = -\sigma_2 \sigma_j$

- Recall, σ_1, σ_2 are both real. σ_2 is imaginary.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So, $\sigma_1^* = \sigma_1$, $\sigma_2^* = -\sigma_2$, $\sigma_3^* = \sigma_3$

$\sigma_1^T = \sigma_1$, $\sigma_2^T = -\sigma_2$, $\sigma_3^T = \sigma_3$

Giving us:

$$\left.\begin{aligned} &\sigma_1^T \sigma_2 = -\sigma_2 \sigma_1, \\ &\sigma_3^T \sigma_2 = \sigma_3 \sigma_2 = -\sigma_2 \sigma_3 \\ \& \quad &\sigma_2^T \sigma_2 = -\sigma_2 \sigma_2 \end{aligned}\right\} \Rightarrow \sigma_j^T \sigma_2 = -\sigma_2 \sigma_j$$

(b) Show that $\frac{m\bar{\Psi}\Psi}{2}$ is not gauge-invariant, i.e. a Majorana particle cannot be charged.

→ Let us take the first mass term in account from $\frac{m}{2}\bar{\Psi}\Psi$

$$\psi_L^T \sigma_2 \psi_L \xrightarrow{U(1)} \psi_L^T e^{i\alpha} \sigma_2 e^{i\alpha} \psi_L = e^{2i\alpha} \psi_L^T \sigma_2 \psi_L$$

Now, second term,

$$\psi_L^\dagger \sigma_2 \psi_L \xrightarrow{U(1)} \psi_L^\dagger \cancel{e^{-i\alpha}} \sigma_2 \cancel{e^{i\alpha}} \psi_L = \psi_L^\dagger \sigma_2 \psi_L$$

So, one of the term is invariant but the other, isn't. Making $m\Psi\bar{\Psi}$ not-invariant under $U(1)$.

∴ Majorana fermions are not charged.

Another way to see this is from $\Psi = \Psi_c = -i\gamma_2 \Psi^*$ for Majorana fermions. This means under $U(1)$ symmetry they cannot be charged as under symmetry operation

$$\Psi \to e^{i\alpha}\Psi$$
$$\Psi_c \to e^{-i\alpha}\Psi_c$$

which means $\Psi = \Psi_c$ cannot hold.