P1.

(a) Given : $\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi \quad = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi$

We will use E-L equations :

(i) $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$ (ii) $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0$

$\partial_\mu (\partial^\mu \phi^*) + m^2 \phi^* = 0 \qquad \partial_\mu (\partial^\mu \phi) + m^2 \phi = 0$

$\Rightarrow \quad (\partial_\mu \partial^\mu + m^2) \phi^* = 0 \qquad \Rightarrow \quad (\partial_\mu \partial^\mu + m^2) \phi = 0$

∴ We can see that we have two copies o. KG eqⁿ.

(b) (1) Check what happens under global U(1) Symmetry

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

$$\downarrow \boxed{\phi \to e^{iq\chi} \phi \quad \& \quad \phi^* \to e^{-iq\chi} \phi^*}$$

$$= (\partial_\mu (e^{-iq\chi} \phi^*))(\partial^\mu (e^{iq\chi} \phi)) - m^2 (\cancel{e^{-iq\chi}} \phi^*)(\cancel{e^{+iq\chi}} \phi)$$

$$= [(\cancel{\partial_\mu e^{-iq\chi}}) \phi^* + e^{-iq\chi} (\partial_\mu \phi^*)][(\cancel{\partial^\mu e^{iq\chi}}) \phi + e^{iq\chi} (\partial^\mu \phi)] - m^2 \phi^* \phi$$

Both = 0 as $q\chi$ is a constant.

$$= (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

which means $\mathcal{L}$ is conserved under U(1) global symmetry.

(2) Check what happens under local U(1) Symmetry

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

$$\downarrow \boxed{\phi \to e^{iq\chi} \phi \quad \& \quad \phi^* \to e^{-iq\chi} \phi^*}$$

$$= \partial_\mu (\cancel{e^{-iq\chi}} \phi^*) \; \partial^\mu (\cancel{e^{iq\chi}} \phi) - m^2 \cancel{e^{-iq\chi}} \phi^* \cancel{e^{iq\chi}} \phi$$

As χ is a constant.

$$= (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi$$

$$\downarrow \quad \boxed{\phi \to e^{iq\chi(x)}\phi \quad \& \quad \phi^* \to e^{-iq\chi(x)}\phi^*}$$

$$= \underbrace{\partial^\mu (e^{-iq\chi(x)}\phi^*)}_{\text{I}} \underbrace{\partial_\mu (e^{iq\chi(x)}\phi)}_{\text{II}} - \underbrace{m^2 \cancel{e^{-iq\chi(x)}} \phi^* \cancel{e^{iq\chi(x)}} \phi}_{\text{mass term unchanged.}}$$

Let us figure out how does term I & II transform

$$\text{II} \quad \partial_\mu (e^{iq\chi(x)}\phi) = \partial_\mu (e^{iq\chi(x)})\phi + e^{iq\chi(x)}(\partial_\mu \phi)$$

$$= iq(\partial_\mu \chi(x))\, e^{iq\chi(x)}\phi + e^{iq\chi(x)}(\partial_\mu \phi)$$

$$= e^{iq\chi(x)} \left[\partial_\mu + iq\partial_\mu \chi\right]\phi$$

Similarly, I $\quad \partial^\mu \phi^* \to e^{-iq\chi(x)} \left[\partial^\mu - iq\partial^\mu \chi\right]\phi^*$

I + II transformation gives us the following first term in the transformation:

$$(\partial^\mu \phi^*)(\partial_\mu \phi) \to (\partial^\mu \phi^*)(\partial_\mu \phi) - iq(\partial^\mu \chi(x))(\partial_\mu \phi) + iq(\partial^\mu \phi^*)(\partial_\mu \chi(x))\phi$$
$$+ (\partial^\mu \phi)(\partial_\mu \phi^*)\phi^* \phi$$

which is not what we started with.

(3) Looking carefully at I & II transformations, we can guess that if

$$D_\mu = \partial_\mu + iqA_\mu(x) \qquad \& \qquad D^\mu = \partial^\mu - iqA^\mu(x)$$

but this will create additional terms due to $A^\mu(x)$. Just like we have a transf^n for ϕ, ϕ^*, we need a transformation of the following form

$$A_\mu(x) \to A_\mu(x) - (\partial_\mu \chi(x))$$

to make sure that the new terms arising from A^μ / A_μ are cancelled out.

$$(D_\mu \phi) = (\partial_\mu + iqA_\mu(x))\phi \xrightarrow{U(1)} (\partial_\mu + iq(A_\mu(x) - \partial_\mu(\chi(x))) e^{iq\chi(x)} \phi$$

$$= \partial_\mu (e^{iq\chi(x)}\phi) + iqA_\mu(x) e^{iq\chi(x)} \phi - iq\partial_\mu(\chi(x)) e^{iq\chi(x)}\phi$$

$$= \cancel{iqe^{iq\chi(x)}(\partial_\mu \chi(x))\phi} + e^{iq\chi(x)}\partial_\mu \phi + iqA_\mu(x) e^{iq\chi(x)}\phi \cancel{- iq(\partial_\mu \chi(x)) e^{iq\chi(x)}\phi}$$

$$= e^{iq\chi(x)} (\partial_\mu + iqA_\mu(x))\phi$$

$$= e^{iq\chi(x)} (D_\mu \phi)$$

Similarly $\rightarrow$ $(D^\mu \phi^*) \longrightarrow e^{-iq\chi(x)} (D^\mu \phi^*)$

Together $\rightarrow$ $(D_\mu \phi)(D^\mu \phi^*) \xrightarrow{U(1)} \underbrace{\cancel{e^{iq\chi(x)}} (D_\mu \phi)}_{\text{As long as } [\chi, \phi] = 0.} \cancel{e^{-iq\chi(x)}} (D^\mu \phi^*)$

This gives us our new Lagrangian,

$$\mathcal{L} = (D^\mu \phi^*)(D_\mu \phi) - m^2 \phi^* \phi$$

$$= (\partial^\mu \phi^* - iqA^\mu \phi^*)(\partial_\mu \phi + iqA_\mu \phi) - m^2 \phi^* \phi$$

$$= (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi$$

$$+ \underbrace{(-iqA^\mu \phi^* (\partial_\mu \phi) + iq(\partial^\mu \phi^*) A_\mu \phi + q^2 \phi^* \phi A^\mu A_\mu)}_{\text{This terms shows the coupling strength between } A_\mu \text{ \& } \phi, \phi^* \text{ fields}}$$

(4)

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi}$$

$$= \partial_\mu (\partial^\mu \phi^* - iqA^\mu \phi^*) - (-m^2 \phi^* + q^2 \phi^* A^\mu A_\mu)$$

$$= (\partial^\mu\partial_\mu + m^2)\phi^* - \partial_\mu(iqA^\mu\phi^*) - q^2\phi^* A^\mu A_\mu$$

$$= [\underline{\partial^\mu\partial_\mu\phi^*} + m^2\phi^* \underline{- iq(\partial_\mu A^\mu)\phi^* - iqA^\mu(\partial_\mu\phi^*) - q^2\phi^* A^\mu A_\mu}]$$

Together

$$= [(\partial_\mu + iqA_\mu)(\partial^\mu - iqA^\mu) + m^2]\phi^*$$

$$\Rightarrow [D_\mu D^\mu + m^2]\phi^* = 0$$

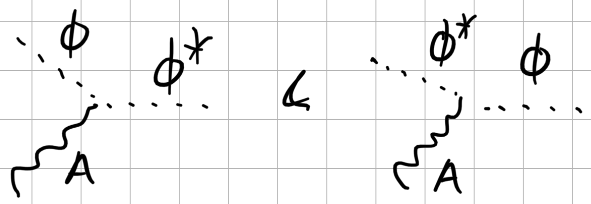
Similarly the other E-L eq^n gives us:

$$[D_\mu D^\mu + m^2]\phi = 0$$

- If ϕ is a particle with charge q propagating in an EM field specified by A^μ, ϕ^* corresponds to its antiparticle.

- The new EOM for ϕ & ϕ^* now comprise of source terms!

  Specifically source terms like this:



Rough work

$$(D_\mu D^\mu + m^2)\phi^* = 0$$

$$[(\partial_\mu + iqA_\mu)(\partial^\mu - iqA^\mu) + m^2]\phi^* = 0$$

$$[\partial_\mu\partial^\mu - \partial_\mu(iqA^\mu) + iqA_\mu\partial^\mu + q^2A_\mu A^\mu + m^2]\phi^* = 0$$

$$[\partial_\mu\partial^\mu + m^2]\phi - iq(\partial_\mu A^\mu)\phi^* + iqA_\mu(\partial^\mu\phi^*) + q^2A_\mu A^\mu\phi^* = 0$$