5. In our discussions of the Lorentz group, we have been treating our boost generators in a very similar way to the rotation operators. Let's make that explicit. A boost along the x axis sends $x \to \gamma(x - vt)$, and $t \to \gamma(t - vx)$, in units where $c = 1$.

(a) (4) We saw that we can write

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu,$$

where $x^\mu = (ct, x, y, z)$, and we have used the Einstein summation convention where repeated indices are summed over. Show that the Λ can be written as a matrix:

$$\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4}$$

and determine the correct expression for ϕ in terms of v, and check that $\cosh^2\phi - \sinh^2\phi = 1$.

(b) (4) Now show that the set of boosts along the x^1 axis forms a group (i.e. by showing closure, identity, invertibility and associativity).

a) WLOG, lets assume motion in x-direction,

$$x' = \gamma(x - vt)$$
$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad ; \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\underbrace{\begin{bmatrix} x' \\ t' \end{bmatrix}}_{x^\mu} = \begin{bmatrix} \gamma & -v\gamma \\ -\frac{v\gamma}{c^2} & \gamma \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \xrightarrow[c=1]{} \underbrace{\begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix}}_{\Lambda^\mu{}_\nu x^\nu}$$

$$\Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -v\gamma \\ -v\gamma & \gamma \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\to A^2 - B^2 = \gamma^2 - (-v\gamma)^2 = \left(\frac{1}{\sqrt{1-v^2}}\right)^2 - \left(-v \times \frac{1}{\sqrt{1-v^2}}\right)$$

$$= \frac{1}{1-v^2} - \frac{v^2}{1-v^2} = \frac{1-v^2}{1-v^2} = 1$$

- We can have such an identity by taking $A = \cosh\phi$ & $B = \sinh\phi$

$$A^2 - B^2 = 1 = \cosh^2\phi - \sinh^2\phi$$

$$\Rightarrow \left.\begin{aligned} \cosh\phi = \gamma \;&\Rightarrow\; \frac{e^\phi + e^{-\phi}}{2} = \frac{1}{(1-v^2)^{1/2}} \\ \sinh\phi = -v\gamma \;&\Rightarrow\; \frac{e^\phi - e^{-\phi}}{2} = \frac{-v}{(1-v^2)^{1/2}} \end{aligned}\right\} \text{Solving for } \phi \text{ give}$$

$$\phi = \tan^{-1}(v) = \tan^{-1}(\beta)$$

Rapidity

(b)

(i) Identity : For $\phi = 0$, $\Lambda = I$.

(ii) Closure : $\Lambda(\phi_1)\Lambda(\phi_2) = \begin{bmatrix} \cosh\phi_1 & -\sinh\phi_1 & 0 & 0 \\ -\sinh\phi_1 & \cosh\phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\phi_2 & -\sinh\phi_2 & 0 & 0 \\ -\sinh\phi_2 & \cosh\phi_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cosh\phi_1 \cosh\phi_2 + \sinh\phi_1 \sinh\phi_2 & -\cosh\phi_1 \sinh\phi_2 - \sinh\phi_1 \cosh\phi_2 & 0 & 0 \\ -\sinh\phi_1 \cosh\phi_2 - \cosh\phi_1 \sinh\phi_2 & \sinh\phi_1 \sinh\phi_2 + \cosh\phi_1 \cosh\phi_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cosh(\phi_1+\phi_2) & -\sinh(\phi_1+\phi_2) & 0 & 0 \\ -\sinh(\phi_1+\phi_2) & \cosh(\phi_1+\phi_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

hepweb.ucsd.edu

Hyperbolic functions of sums.

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

(iii) Associativity. $\left[\left(\Lambda(\phi_1)\Lambda(\phi_2)\right)\Lambda(\phi_3) = \Lambda(\phi_1)\left(\Lambda(\phi_2)\Lambda(\phi_3)\right)\right]$

LHS : $[\Lambda(\phi_1)\Lambda(\phi_2)]\Lambda(\phi_3) = \Lambda(\phi_1+\phi_2)\Lambda(\phi_3) = \Lambda(\phi_1+\phi_2+\phi_3)$

$= \Lambda(\phi_1 + (\phi_2+\phi_3)) = \Lambda(\phi_1)[\Lambda(\phi_2+\phi_3)] = \text{RHS}$ $_\square$

(iv) Invertibility : $\exists \Lambda^{-1}(\phi) \ \forall \Lambda(\phi)$ s.t $\Lambda^{-1}(\phi)\Lambda(\phi) = I$

Using (ii) Closure , we can see that invertibility is a given as $\Lambda^{-1}(\phi_1) = \Lambda(-\phi_1)$,

as $\Lambda(\phi_1)\Lambda(-\phi_1) = \Lambda(\phi_1 - \phi_1) = \Lambda(0) = I$.

Hence, we can see that the it forms a group