

# Exercise 5.1

(The gravitational redshift element)

### Exercise 5.1: The gravitational redshift experiment

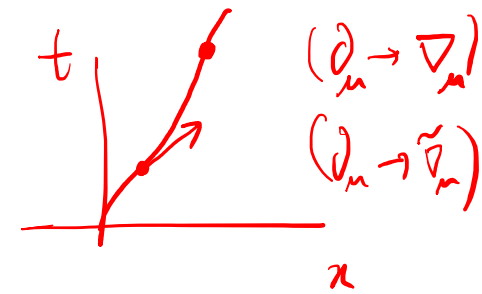
This exercise aims to understand the *Gedankenexperiment* first suggested by Einstein in 1907, and its important consequences. This idealized experiment goes as follows:

(a) We have an object of rest mass  $m$  at the top of a tower of height  $h$ . The object is initially at rest, but at some point we drop it and it falls freely with constant acceleration  $g$ . When it reaches the ground its energy is fully transformed into a photon of frequency  $\nu$  that is emitted towards the top of the tower. Compute the energy of this photon at the moment of emission. Hint: take into account the effect of the Earth in the conservation equations. (1 pt)

## Exercise 5.2

(Geodesic equation and affine parameters)

(a) Consider a non-null curve whose tangent vector is parallel-transported along it. Show that such a vector has constant norm on this geodesic curve. (1 pt)



Let us parametrize the curve with a parameter  $\lambda$  and construct the tangent vector  $\vec{V}(\lambda)$  with components  $V^\alpha = \frac{dx^\alpha}{d\lambda}$ . The norm of this vector reads,

$$\vec{V} \cdot \vec{V} = g(\vec{V}, \vec{V}) = g_{\mu\nu} V^\mu V^\nu = g_{\mu\nu} U^\mu U^\nu \left( \frac{d\tau}{d\lambda} \right)^2 = - \left( \frac{d\tau}{d\lambda} \right)^2 \equiv f(\lambda)$$

$$V^\alpha = \frac{dx^\alpha}{d\lambda} = \frac{dx^\alpha}{d\tau} \frac{d\tau}{d\lambda} = U^\alpha \frac{d\tau}{d\lambda}$$

Let us study now how the norm varies along the curve:

"Like a gradient, projected along the tangent vector in every dim"

$$\frac{d}{d\lambda} (\vec{V} \cdot \vec{V}) = \frac{df}{d\lambda} = V^\beta \nabla_\beta (g_{\mu\nu} V^\mu V^\nu) = V^\beta g_{\mu\nu} (V^\mu \nabla_\beta V^\nu + V^\nu \nabla_\beta V^\mu)$$

$$\text{Remember } \nabla_\beta g_{\mu\nu} = 0$$

$$\frac{df}{d\lambda} = 2V_\nu V^\beta \nabla_\beta V^\nu = 2V_\nu \frac{dV^\nu}{d\lambda}$$

If  $\vec{V}$  is parallel-transported along the curve it is tangent to,  $\frac{dV^\nu}{d\lambda} = 0$ , and therefore  $f = \text{const.}$ , so  $\vec{V}$  has constant norm.

$$u^\alpha \nabla_\alpha V^\nu = 0$$

(b) Use the previous result to prove that the geodesic equation

$$V^\beta \nabla_\beta \left( \frac{dV^\alpha}{d\lambda} \right) = 0 \rightarrow \frac{d^2 x^\nu}{d\lambda^2} + \Gamma^\nu_{\alpha\kappa} \frac{dx^\alpha}{d\lambda} \frac{dx^\kappa}{d\lambda} = 0 \quad (1)$$

only describes the curve under study if it is parametrized with a parameter  $\lambda$  related with the proper time  $\tau$  by an affine transformation. (0.5 pt)

we know that  $\frac{dV^\alpha}{d\lambda} = 0$  only applies if and only if  $f(\lambda) = \text{const}$

i.e. if  $\frac{d^2\tau}{d\lambda^2} = 0$ , so when  $\tau = a + b\lambda$ , with  $a$  and  $b$  constants.

*Affine transformation.*

(c) How is (1) modified when the curve is parametrized with a non-affine parameter?

We start with the curve parametrized with the proper time  $\tau$ ,

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\kappa} \frac{dx^\alpha}{d\tau} \frac{dx^\kappa}{d\tau} = 0,$$

perform a change  $\tau = \tau(\lambda)$ , where here  $\lambda$  is a non-affine parameter.

$$\frac{d^2 x^\nu}{d\lambda^2} + \Gamma^\nu_{\alpha\kappa} \frac{dx^\alpha}{d\lambda} \frac{dx^\kappa}{d\lambda} = - \frac{\frac{d^2\lambda}{d\tau^2}}{\left(\frac{d\lambda}{d\tau}\right)^2} \frac{dx^\nu}{d\lambda}$$

$$\rightarrow V^\mu \nabla_\mu V^\nu = s(\lambda) V^\nu$$

$$\left(\frac{d\lambda}{d\tau}\right) \cdot \left(\frac{d\lambda}{d\tau}\right)$$

where we have employed the following chain rules:

$$\frac{d}{d\tau} = \frac{d\lambda}{d\tau} \frac{d}{d\lambda} \quad ; \quad \frac{d^2}{d\tau^2} = \frac{d^2\lambda}{d\tau^2} \frac{d}{d\lambda} + \left(\frac{d\lambda}{d\tau}\right)^2 \frac{d^2}{d\lambda^2}.$$

$$V^\mu \nabla_\mu V^\nu = 0$$

(d) Does the form of the geodesic equation remain invariant under coordinate transformations?

$$\frac{d^2 x^\nu}{d\tau^2} + \Gamma^\nu_{\alpha\kappa} \frac{dx^\alpha}{d\tau} \frac{dx^\kappa}{d\tau} = 0 \longrightarrow \frac{d}{d\tau} \left( \frac{\partial x^\nu}{\partial x^{\bar{\beta}}} \frac{dx^{\bar{\beta}}}{d\tau} \right) + \Gamma^\nu_{\alpha\kappa} \frac{dx^{\bar{\kappa}}}{d\tau} \frac{dx^{\bar{\alpha}}}{d\tau} \frac{\partial x^\kappa}{\partial x^{\bar{\kappa}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} = 0$$

Now we can multiply this equation by  $\frac{\partial x^{\bar{\nu}}}{\partial x^\nu}$ , and using the transformation law for the Christoffel symbols obtained in Exercise 4.2b we are led to:

$$\frac{d^2 x^{\bar{\nu}}}{d\tau^2} + \frac{\partial x^{\bar{\nu}}}{\partial x^\nu} \frac{dx^{\bar{\beta}}}{d\tau} \frac{dx^{\bar{\alpha}}}{d\tau} \frac{\partial^2 x^\nu}{\partial x^{\bar{\alpha}} \partial x^{\bar{\beta}}} + \frac{dx^{\bar{\kappa}}}{d\tau} \frac{dx^{\bar{\alpha}}}{d\tau} \left[ \Gamma^{\bar{\nu}}_{\bar{\alpha}\bar{\kappa}} - \frac{\partial x^{\bar{\nu}}}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial x^{\bar{\alpha}} \partial x^{\bar{\kappa}}} \right] = 0$$

In the last term we can do  $\mu \rightarrow \nu$  and  $\bar{\kappa} \rightarrow \bar{\beta}$  to finally get:

$$\frac{d^2 x^{\bar{\nu}}}{d\tau^2} + \Gamma^{\bar{\nu}}_{\bar{\alpha}\bar{\beta}} \frac{dx^{\bar{\kappa}}}{d\tau} \frac{dx^{\bar{\alpha}}}{d\tau} = 0$$

The geodesic equation remains invariant under coordinate transformations, as expected.

(e) Does the geodesic equation work in presence of non-gravitational forces?

No, the geodesic equation governs the motion in spacetime of free-falling (inertial) particles, only subject to gravity. If forces different from gravity are also considered one in general has:

$$\underbrace{\frac{d^2 x^\nu}{d\tau^2}}_{St} + \underbrace{\Gamma^\nu_{\alpha\kappa} \frac{dx^\alpha}{d\tau} \frac{d^2 x^\kappa}{d\tau}}_{Gr} \neq 0. \quad \underbrace{q(E + v \times \vec{B})}_{\text{Extra force}}$$

## Exercise 5.3

(Proofs : Covariant derivative of a metric)



$$(a) \quad \frac{\partial g}{\partial x^\kappa} = g g^{\alpha\beta} g_{\alpha\beta,\kappa}, \text{ with } g \text{ the determinant of the metric.}$$

$$\Sigma^{\mu\nu} = (-1)^{\mu+\nu} M^{\mu\nu}$$

$M^{\mu\nu}$  the  $(\mu, \nu)$ -minor

The determinant of the metric can be written as

$$g = \sum g_{\mu\nu} \Sigma^{\mu\nu}$$

Making use of the derivative chain rule

noting that  $\Sigma^{\mu\nu}$  does not depend on  $g_{\mu\nu}$  we find,

$$\frac{\partial g}{\partial x^\kappa} = \frac{\partial g}{\partial g_{\mu\nu}} \frac{\partial g_{\mu\nu}}{\partial x^\kappa} = \Sigma^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\kappa}$$

On the other hand we have,

$$g^{\mu\nu} = \frac{1}{g} \Sigma^{\mu\nu} \longrightarrow \Sigma^{\mu\nu} = g g^{\mu\nu}$$

$$\boxed{\frac{\partial g}{\partial x^\kappa} = g g^{\mu\nu} \frac{\partial g_{\mu\nu}}{\partial x^\kappa}}$$

$$(b) \quad \nabla_\mu \nabla^\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi)$$

$$\nabla_\mu \nabla^\mu \phi = g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = g^{\mu\nu} (\partial_\mu \partial_\nu \phi - \Gamma^\kappa_{\mu\nu} \partial_\kappa \phi) \quad (21)$$

Let us focus on the contraction  $g^{\mu\nu} \Gamma^\kappa_{\mu\nu}$ .

$$g^{\mu\nu} \Gamma^\kappa_{\mu\nu} = g^{\mu\nu} \frac{g^{\kappa\alpha}}{2} (g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}) = g^{\kappa\alpha} (g^{\mu\nu} g_{\alpha\mu,\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu,\alpha})$$

Since  $g^{\beta\nu} g_{\mu\nu} = \delta_\mu^\beta$  we have that  $g^{\mu\nu} g_{\alpha\mu,\nu} = -g_{\alpha\mu} g^{\mu\nu}{}_{,\nu}$  and this can be employed to simplify the first term. For the second one we can make use of the result obtained in 5.4a.

$$\begin{aligned} g^{\mu\nu} \Gamma^\kappa_{\mu\nu} &= - \left( g^{\kappa\alpha} g_{\alpha\mu} g^{\mu\nu}{}_{,\nu} + \frac{1}{2g} g^{,\kappa} \right) = - \left( \delta_\mu^\kappa g^{\mu\nu}{}_{,\nu} + \frac{1}{2g} g^{,\kappa} \right) = - \left( g^{\kappa\nu}{}_{,\nu} + \frac{g^{\kappa\nu}}{2g} g_{,\nu} \right) \\ g^{\mu\nu} \Gamma^\kappa_{\mu\nu} &= - \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\kappa\nu}) \end{aligned} \quad (22)$$

Now we use this result in (21),

$$\begin{aligned} \nabla_\mu \nabla^\mu \phi &= g^{\mu\nu} \partial_\mu \partial_\nu \phi + \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\kappa\nu}) \partial_\kappa \phi = g^{\kappa\nu} \partial_\nu \partial_\kappa \phi + \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\kappa\nu}) \partial_\kappa \phi \\ \nabla_\mu \nabla^\mu \phi &= \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\kappa\nu} \partial_\kappa \phi) = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} \partial^\nu \phi) \end{aligned} \quad (23)$$

$$(c) \nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu\nu}), \text{ with } F^{\mu\nu} \text{ an antisymmetric tensor.}$$

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} - \Gamma^\mu_{\nu\kappa} F^{\kappa\nu} - \Gamma^\nu_{\nu\kappa} F^{\mu\kappa} \quad (24)$$

Since the Christoffel symbols are symmetric under the exchange of the lower indices and  $F$  is an antisymmetric tensor the second term in the right-hand side vanishes, so

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} - \Gamma^\nu_{\nu\kappa} F^{\mu\kappa}. \quad (25)$$

The second term can be written as follows,

$$\Gamma^\nu_{\nu\kappa} F^{\mu\kappa} = \frac{1}{2} g^{\nu\alpha} (g_{\alpha\nu,\kappa} + g_{\alpha\kappa,\nu} - g_{\nu\kappa,\alpha}) F^{\mu\kappa} = \frac{1}{2} g^{\nu\alpha} g_{\alpha\nu,\kappa} F^{\mu\kappa} = \frac{g_{,\kappa}}{2g} F^{\mu\kappa}. \quad (26)$$

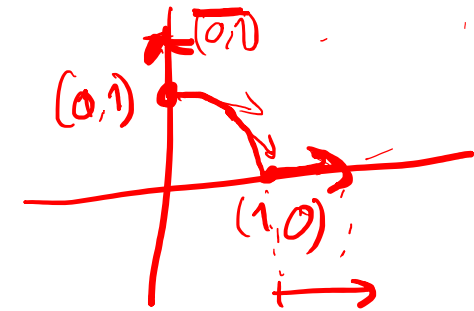
Thus, we find:

$$\nabla_\nu F^{\mu\nu} = \partial_\nu F^{\mu\nu} - \frac{g_{,\nu}}{2g} F^{\mu\nu}$$

$$\boxed{\nabla_\nu F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} F^{\mu\nu})} \quad (27)$$

EXTRA - Exercise 5.4  
(Parallel transport in Euclidean space)

(a) Solve the following problem in Euclidean space. Take the vector  $(0, 1)$  located at point  $(0, 1)$ , in Cartesian coordinates. Parallel-transport it along a circle of radius  $r = 1$  to the point  $(1, 0)$ , clockwise. Make explicit use of the parallel-transport equation. (0.5 pt)



The parallel-transport equation reads,

$$(U^\mu \nabla_\mu) V^\nu = 0,$$

where  $\vec{U}$  is the tangent vector to the path along which we parallel transport the vector  $\vec{V}$ .

Cartesian coordinates  $\longrightarrow$  the Christoffel symbols vanish

$$U^\mu \partial_\mu V^\nu = 0 \quad \longrightarrow \quad \begin{aligned} U^x \partial_x V^x + U^y \partial_y V^x &= 0 \\ U^x \partial_x V^y + U^y \partial_y V^y &= 0 \end{aligned}$$

The points on the path can be parametrized with the angle  $\theta \longrightarrow \vec{r} = (x, y) = (\cos \theta, \sin \theta)$

$$\vec{U} = -\frac{d\vec{r}}{d\theta} = (\sin \theta, -\cos \theta) = (y, -x) = (y, -\sqrt{1-y^2})$$

Thus, we have

$$y \partial_x V^x - x \partial_y V^x = 0 \longrightarrow \frac{dV^x}{dy} = 0$$

$$y \partial_x V^y - x \partial_y V^y = 0 \longrightarrow \frac{dV^y}{dy} = 0$$

The parallel-transported vector has constant components in Cartesian coordinates and reads,  $\vec{V} = (1, 0)$ .

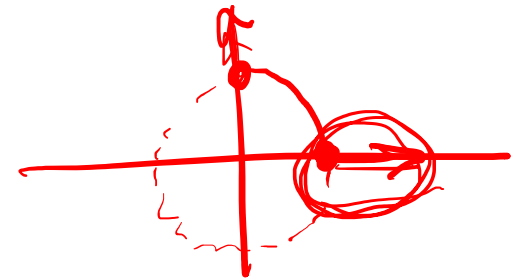
(b) Do the same exercise in polar coordinates.

In polar coordinates there are some non-null Christoffel symbols,  $\Gamma^r_{\theta\theta} = -r$ ,  $\Gamma^\theta_{r\theta} = \frac{1}{r}$

$$U^\alpha (\underbrace{\partial_\alpha V^\nu}_{\nabla_\alpha V^\nu} + \Gamma^\nu_{\alpha\kappa} V^\kappa) = 0, \quad \begin{matrix} \nu=\theta \\ \nu=r \end{matrix} \Rightarrow \begin{aligned} U^\theta \partial_\theta V^\theta + U^r \partial_r V^\theta + \frac{1}{r} (U^r V^\theta + U^\theta V^r) &= 0 \\ U^\theta \partial_\theta V^r + U^r \partial_r V^r - r U^\theta V^\theta &= 0 \end{aligned}$$

$$\begin{pmatrix} \vec{e}_\theta \\ \vec{e}_r \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix} = \begin{pmatrix} -r \sin(\theta) & r \cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix}$$

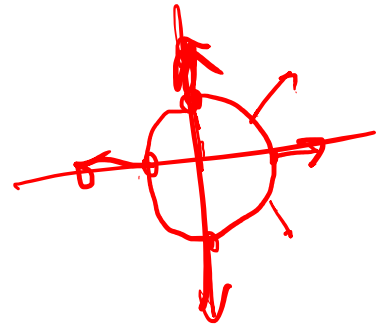
$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \end{pmatrix} = \begin{pmatrix} -\frac{\sin(\theta)}{r} & \cos(\theta) \\ \frac{\cos(\theta)}{r} & \sin(\theta) \end{pmatrix} \begin{pmatrix} \vec{e}_\theta \\ \vec{e}_r \end{pmatrix}$$



we find that in polar coordinates  $\vec{U} = -\frac{\vec{e}_\theta}{r} = -\vec{e}_\theta$ , since  $r = 1$  along the path.

$$\partial_\theta V^\theta + V^r = 0$$

$$\partial_\theta V^r - V^\theta = 0$$



The solution of this system is

$$V^\theta(\theta) = A \cos(\theta) + B \sin(\theta)$$

$$V^r(\theta) = A \sin(\theta) - B \cos(\theta)$$

We can fix  $A$  and  $B$  by using the initial condition,  $\vec{V}_{ini} = \vec{e}_y(r = 1; \theta = \pi/2) = \vec{e}_r$ .

The vector  $\vec{V}$  along the path reads,

$$\vec{V} = \underbrace{\cancel{\cos(\theta)}}_{\sin \theta} \vec{e}_\theta + \underbrace{\cancel{\sin(\theta)}}_{\cos \theta} \vec{e}_r$$

$r=1, \theta=0$

$0 \qquad 1$

so  $\vec{V} = \vec{e}_r$  at the end of the transportation process.

# EXTRA - Exercise 5.5

(Conceptual questions – Parallel transport)



**(a) Consider the following statement:** *If we parallel-transport a vector along a closed loop, i.e. along a path that begins and ends at the same point, the final vector will always coincide with the one we had at the beginning. Is this statement true? Explain why. (0.5 pt)*

- Only true  $\rightarrow$  Flat spaces (i.e. in spaces where elements of the Riemann tensor vanish)
- For non-zero curvature the two vectors will not coincide
- This is also the reason why covariant derivatives do not commute in curved spaces



**(b) Consider now this statement:** *The comparison of two vectors located at different tangent spaces is univocal and can be carried out by parallel-transporting one of them to the location of the other. Is this statement correct? Justify your answer. (0.5 pt)*

- This statement is also false. In curved spaces, the orientation of the parallel-transported vector depends on the path along which the transportation has been carried out.
- This means that we cannot compare two vectors located in two different tangent spaces in a univocal way.

