Exercise 2.1: 4-velocity and 4-acceleration in SR

Prove the formulas of the four-velocity and four-acceleration of a particle measured in some inertial frame,

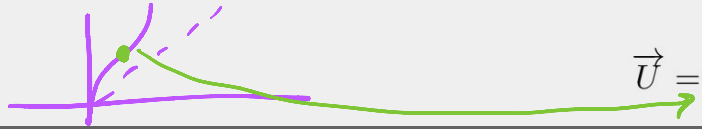
$$U^\alpha = \{\gamma(v), \gamma(v)\mathbf{v}\} \qquad (1)$$

$$a^\mu = \frac{dU^\mu}{d\tau} = \{\gamma(v)\dot\gamma(v), \gamma^2(v)\mathbf{a} + \gamma(v)\dot\gamma(v)\mathbf{v}\} \qquad (2)$$

$$= \gamma^4(v)\{\mathbf{a}\cdot\mathbf{v}, \mathbf{a} + \mathbf{v}\times(\mathbf{v}\times\mathbf{a})\}, \qquad (3)$$

with $\mathrm{v} = \frac{d\mathrm{x}}{dt}$, $\mathrm{a} = \frac{d\mathrm{v}}{dt}$ the three-velocity and three-acceleration measured in that frame, respectively, and the dots being derivatives with respect to the time t. Hint: In the system that is comoving to the particle $\vec U = \vec e_{\bar 0}$. Use the Lorentz transformations to write $\vec U$ in the system in which the particle moves at speed v, i.e. (1), and then perform the derivative of the resulting expression with respect to the proper time τ to obtain the four-acceleration (2). Some geometrical reasoning is needed to write (2) as (3). (2 pt)

The four-velocity can be written as follows in the comoving frame of the particle $(\bar{\mathcal{O}})$,



$$\vec U = \vec e_{\bar 0}\,.$$

In the basis of the inertial frame where the particle moves with velocity $\mathbf{v}$ $(\mathcal{O})$ the four-velocity reads,

$$\vec U = \vec e_{\bar 0} = \Lambda^{\alpha}_{\bar 0}(\mathbf{v})\,\vec e_\alpha\,, \qquad (5)$$

with $\Lambda^{\alpha}_{\bar 0}(\mathbf{v}) = (\Lambda^{-1})^{\bar 0}_{\alpha}(\mathbf{v}) = \Lambda^{\bar 0}_{\alpha}(-\mathbf{v})$. The matrix

$$\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda^{\bar\mu}_{\alpha}(\mathbf{v}) = \begin{pmatrix} \gamma(v) & -\gamma(v)vn_x & -\gamma(v)vn_y & -\gamma(v)vn_z \\ -\gamma(v)vn_x & 1+[\gamma(v)-1]n_x^2 & [\gamma(v)-1]n_xn_y & [\gamma(v)-1]n_xn_z \\ -\gamma(v)vn_y & [\gamma(v)-1]n_xn_y & 1+[\gamma(v)-1]n_y^2 & [\gamma(v)-1]n_yn_z \\ -\gamma(v)vn_z & [\gamma(v)-1]n_xn_z & [\gamma(v)-1]n_yn_z & 1+[\gamma(v)-1]n_z^2 \end{pmatrix}$$

is the Lorentz matrix, with $n_i = v_i/v$. Thus, we have,

$$\boxed{\vec U \to_{\mathcal{O}} \{\gamma(v), \gamma(v)\mathbf{v}\}}$$

$$\frac{dx^\mu}{dt} = \frac{d}{dt}(t, \bar x) = (1, \bar v)$$

$$(x^0)' = \gamma\left(x^0 - \frac{|\bar x|}{c^2}\right)$$

$$\frac{d\tau}{dt} = \frac{1}{\gamma}$$

$$U^\alpha = \frac{d}{d\tau}(t, \bar x) = \left(\frac{dt}{d\tau}, \frac{d\bar x}{dt}\frac{dt}{d\tau}\right) = (\gamma, \bar v\gamma)$$

γ

$$a^\mu = \frac{dU^\mu}{d\tau} = \{\gamma(v)\dot\gamma(v), \gamma^2(v)\mathbf{a} + \gamma(v)\dot\gamma(v)\mathbf{v}\} \tag{2}$$
$$= \gamma^4(v)\{\mathbf{a}\cdot\mathbf{v}, \mathbf{a} + \mathbf{v}\times(\mathbf{v}\times\mathbf{a})\}, \tag{3}$$

$$\rightarrow \quad \vec{a} = \frac{d\vec{U}}{d\tau} = \frac{dU^\mu}{d\tau}\vec{e}_\mu + \cancel{U^\mu \frac{d\vec{e}_\mu}{d\tau}} = \frac{dU^\mu}{d\tau}\vec{e}_\mu\,,$$

0

$\Gamma^\alpha_{\beta\gamma} = 0$

2 ✓

$^4\vec{U} = U^\mu \vec{e}_\mu$

$A_\mu = (A_\mu)\vec{e}^{\,\mu}$

$$a^\mu = \frac{dU^\mu}{d\tau} = \frac{dU^\mu}{dt}\frac{dt}{d\tau} = \gamma(v)\frac{dU^\mu}{dt} = \{\gamma(v)\dot\gamma(v), \gamma(v)\dot\gamma(v)\mathbf{v} + \gamma^2\mathbf{a}\}\,.$$

$\downarrow$ Road to 3

$$^4\vec{a} = \frac{dU^\mu}{d\tau}\vec{e}_\mu$$

$$\dot\gamma(v) = \frac{d\gamma}{dt} = \frac{d}{dt}\left(\frac{1}{\sqrt{1-v^2}}\right) = -\frac{1}{2}\frac{1}{(1-v^2)^{3/2}}\left(-2\bar{v}\cdot\frac{d\bar{v}}{dt}\right) = \gamma^3(v)\,\bar{v}\cdot\bar{a}$$

Giving us:

$$\frac{1}{\gamma^2} = 1 - v^2$$

$$a^\mu = \{\gamma^4\bar{a}\cdot\bar{v},\ \gamma^2\bar{a} + \gamma^4\bar{a}\bar{v}^2\} = \gamma^4(v)\left\{\bar{a}\cdot\bar{v},\ \frac{\bar{a}}{\gamma^2} + \bar{a}\bar{v}^2\right\}$$
$$= \gamma^4(v)\{\bar{a}\cdot\bar{v},\ \bar{a} - \bar{a}v^2 + (\bar{a}\cdot\bar{v})\bar{v}\}$$

Identity: $\bar{v}\times(\bar{v}\times\bar{a}) = (\bar{v}\cdot\bar{a})\bar{v} - (\bar{v}\cdot\bar{v})\bar{a}$

$$a^\mu = \gamma^4(v)\{\bar{a}\cdot\bar{v},\ \bar{a} + \bar{v}\times(\bar{v}\times\bar{a})\}$$

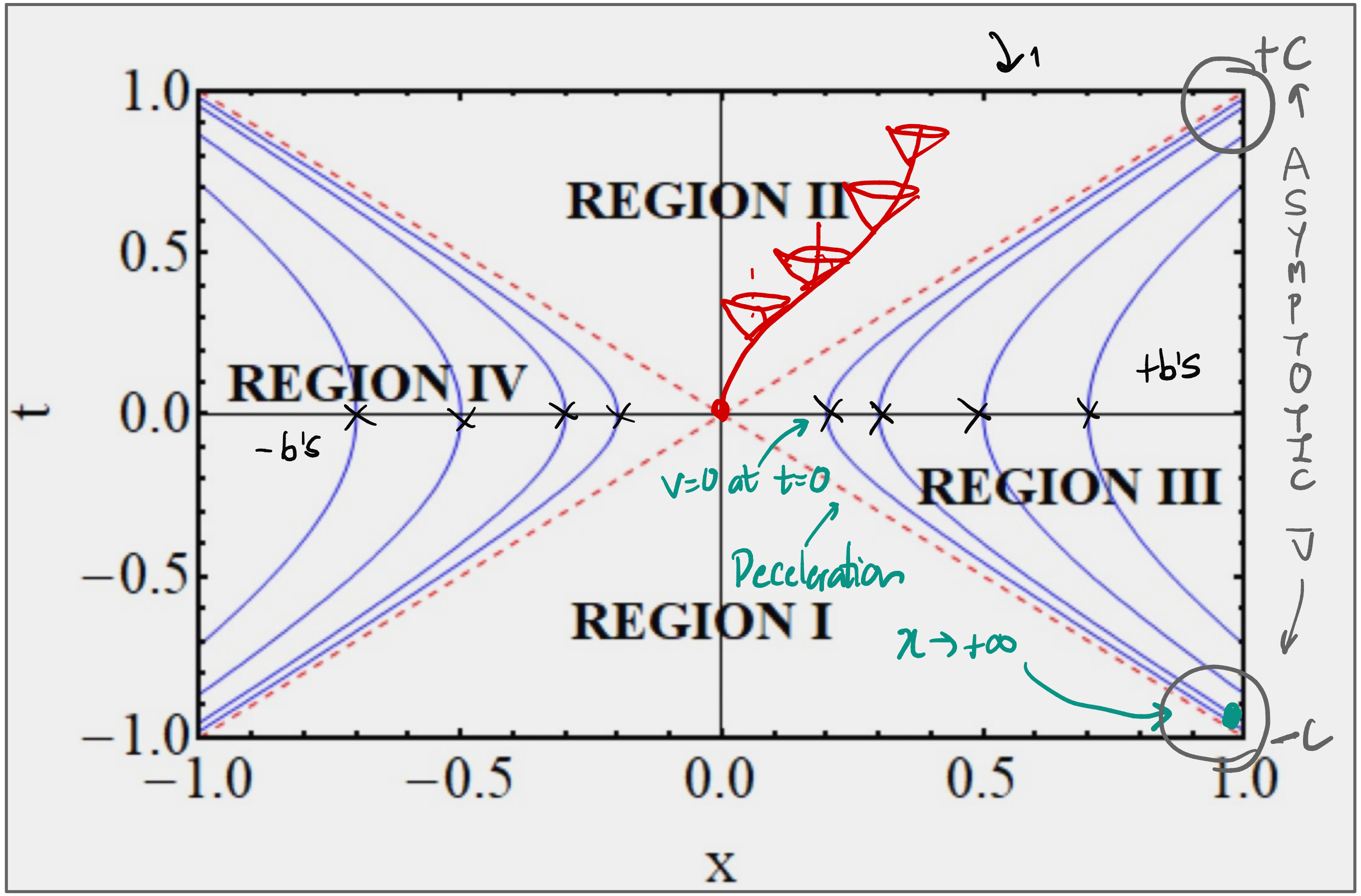
3 ✓

Exercise 2.2: Rindler coordinates

The world line of a particle is described by the parametric equations in some Lorentz frame

$$t(\lambda) = b \sinh\left(\frac{\lambda}{b}\right) \qquad ; \qquad x(\lambda) = b \cosh\left(\frac{\lambda}{b}\right), \tag{15}$$

where λ is the parameter and b is a constant. Describe the motion and compute the particle's four velocity and acceleration components. Show that λ is the proper time along the world line and that the proper acceleration is constant. Interpret b. (2 pt)



World line in inertial frame (0)

$$\rightarrow \quad x^2 - t^2 = b^2,$$

$$x^2 - y^2 = (\text{Const})^2$$

Describe the motion

$$d\tau^2 = -ds^2$$

$$= (dt)^2 - (dx)^2$$

$$= \left(\cancel{b} \cosh\left(\frac{\lambda}{b}\right) d\left(\frac{\lambda}{b}\right)\right)^2$$

$$= \left(\cancel{b} \cosh\left(\frac{\lambda}{b}\right) \cancel{b}\, \frac{1}{\cancel{b}}\, d\lambda\right)^2$$

$$= \left(\cosh\left(\frac{\lambda}{b}\right) d\lambda\right)^2$$

→ The 3-velocity measured by an observer in $\mathcal{O}$

$$dt = \cosh\left(\frac{\lambda}{b}\right) d\lambda \;\circledast \quad ; \qquad dx = \sinh\left(\frac{\lambda}{b}\right) d\lambda\,.$$

Divide

$$v = \frac{dx}{dt} = \tanh\left(\frac{\lambda}{b}\right) \quad ; \quad \frac{dt}{d\lambda} = \gamma = \frac{1}{\sqrt{1-v^2}} = \cosh\left(\frac{\lambda}{b}\right)\,.$$

$\frac{dt}{d\tau}$

3-velocity

→ The relation between the time measured in $\mathcal{O}$ and the proper time of the particle is just

$$d\tau = \frac{dt}{\gamma(v)}$$

$$\boxed{dt = \gamma(v)d\tau = \cosh\left(\frac{\lambda}{b}\right) d\tau\,.} \;\tilde{\circledast}$$

$\circledast$ & $\tilde{\circledast}$

$\Downarrow$

$$\boxed{d\tau = d\lambda}$$

i.e. λ is the proper time measured in the comoving frame of the particle

Using :

(i) $$U^\alpha = \{\gamma(v), \gamma(v)\mathbf{v}\}$$

(ii) $$a^\mu = \frac{dU^\mu}{d\tau} = \{\gamma(v)\dot\gamma(v), \gamma^2(v)\mathbf{a} + \gamma(v)\dot\gamma(v)\mathbf{v}\} = \gamma^4(v)\{\mathbf{a}\cdot\mathbf{v}, \mathbf{a} + \mathbf{v}\times(\mathbf{v}\times\mathbf{a})\}\,,$$

3-acceleration

(iii) $$a = \frac{dv}{dt} = \frac{dv}{d\tau}\frac{d\tau}{dt} = \gamma(v)\frac{dv}{d\tau} = \frac{1}{b\cosh^3\left(\frac{\tau}{b}\right)}\,.$$

$\Downarrow$

$$\boxed{U^\mu = \left\{\cosh\left(\frac{\tau}{b}\right), \sinh\left(\frac{\tau}{b}\right)\right\} \quad ; \quad a^\mu = \frac{1}{b}\left\{\sinh\left(\frac{\tau}{b}\right), \cosh\left(\frac{\tau}{b}\right)\right\},}$$

In order to understand the physical meaning of b let us express the four-acceleration in the comoving frame.

$g_{\mu\nu} u^\mu a^\nu$

As in that frame $\overrightarrow{U} \cdot \overrightarrow{a} = 0$ and $\overrightarrow{U} = \overrightarrow{e}_{\bar{0}}$ we have that $a^{\bar{0}} = 0$, so

$$\overrightarrow{a}^2 = \overrightarrow{a} \cdot \overrightarrow{a} = \eta_{\mu\nu} a^\mu a^\nu = \frac{1}{b^2} = (a^{\bar{1}})^2 \longrightarrow a^{\bar{1}} = \frac{1}{b}.$$

b is the inverse of the proper acceleration & is a constant.

$\vec{u} \cdot \vec{a}$

$= -u^0 a^0 + u^1 a^1 + u^2 a^2 + u^3 a^3 = 0$

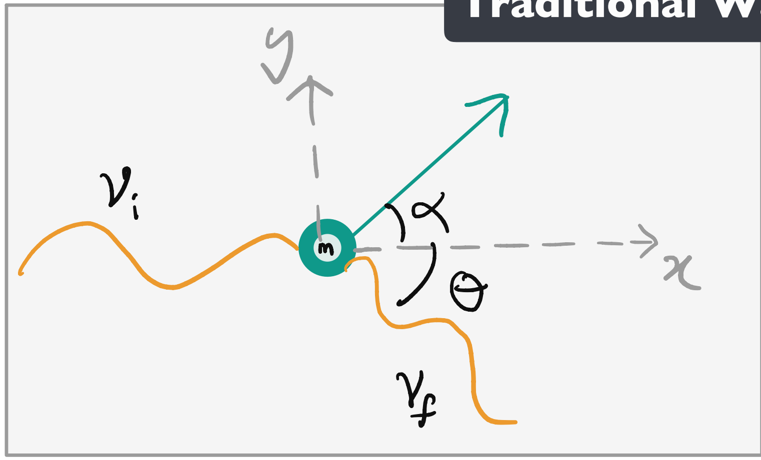
$\bar{a}^\mu = (a^{\bar{0}}, a^{\bar{1}}, \cancel{a^{\bar{2}}, a^{\bar{3}}})$

Exercise 2.3

(a) Let a particle of charge e and rest mass m, initially at rest in the laboratory, scatter a photon of initial frequency ν_i. This is called *Compton scattering*. Suppose the scattered photon comes off at an angle θ from the incident direction. Use the conservation of four-momentum to deduce that the photon's final frequency ν_f is given by

$$\frac{1}{\nu_f} = \frac{1}{\nu_i} + \frac{h}{m}(1-\cos\theta) \qquad \textbf{(1pt)} \tag{23}$$

Traditional Way



$c=1$

Conservation equations

$$
\begin{array}{lrcll}
E_i & h\nu_i + m & = & h\nu_f + \sqrt{m^2+p^2} & E_f \;—\; \circledast \\
(P_x)_{in} & h\nu_i & = & p\,\cos\alpha + h\nu_f\,\cos\theta & (P_x)_f \\
(P_y)_{in} & 0 & = & -p\,\sin\alpha + h\nu_f\,\sin\theta\,, & (P_y)_f
\end{array}
$$

(Before) (After)

$\rightarrow$

$$
\begin{aligned}
h(\nu_i - \nu_f\,\cos\theta) &= p\,\cos\alpha \\
h\nu_f\,\sin\theta &= p\,\sin\alpha\,,
\end{aligned}
$$

$\downarrow$ Squaring & summing

$$h^2(\nu_i^2 + \nu_f^2 - 2\nu_i\nu_f\cos\theta) = p^2\,.$$

$\downarrow$ Substitute p^2 from $\circledast$

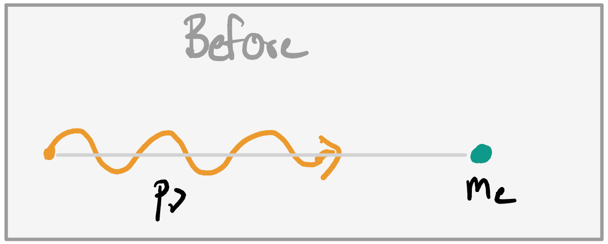
$$h^2(\nu_i^2 + \nu_f^2 - 2\nu_i\nu_f\cos\theta) = h^2(\nu_i-\nu_f)^2 + 2mh(\nu_i-\nu_f).$$

$\downarrow$ Rearrange

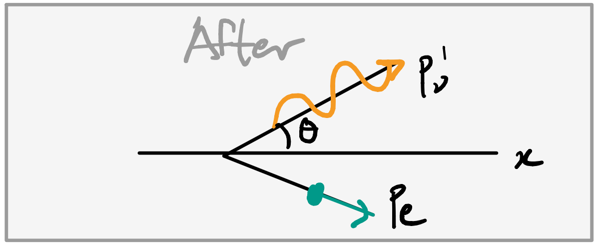
$$\boxed{\frac{1}{\nu_f} = \frac{1}{\nu_i} + \frac{h}{m}(1-\cos\theta)}$$

$\hbar = 1$

4 Vector Way



$\Rightarrow$



$p^\mu = (\text{Energy}, \text{Momentum})$

$p_\mu^\gamma = (\nu, \bar{k})$ $\quad (p_\mu^e)$ $\qquad p_\mu^{\gamma'} = (\nu', \bar{k}')$ $\quad p_\mu^{e'}$

Covariant scalar prod:

$a_\mu b_\mu = -a_0 b_0 + \bar{a}.\bar{b}$

$\rightarrow$ Conservation of 4-momentum : $p_\mu^\gamma + p_\mu^e = p_\mu^{\gamma'} + p_\mu^{e'}$

$$\Rightarrow p_\mu^\gamma + p_\mu^e - p_\mu^{\gamma'} = p_\mu^{e'}$$

$\downarrow$ Scalar product with itself.

$$(p_\mu^\gamma + p_\mu^e - p_\mu^{\gamma'})(p_\mu^\gamma + p_\mu^e - p_\mu^{\gamma'}) = p_\mu^{e'} p_\mu^{e'}$$

$\downarrow$

$$p_\mu^\gamma p_\mu^\gamma + p_\mu^e p_\mu^e + p_\mu^{\gamma'} p_\mu^{\gamma'} + 2 p_\mu^\gamma p_\mu^e - 2 p_\mu^e p_\mu^{\gamma'} - 2 p_\mu^\gamma p_\mu^{\gamma'} = p_\mu^{e'} p_\mu^{e'}$$

(Each term here is relativistically invariant & can be computed in any initial frame)

On-shell relation

- $\boxed{p_\mu^\gamma p_\mu^\gamma = (\nu, \bar{k})(\nu, \bar{k}) = m^2}$ $= 0$
- $p_\mu^{\gamma'} p_\mu^{\gamma'} = (\nu', \bar{k}')(\nu', \bar{k}') = m^2 = 0$
- Evaluate $p_\mu^e p_\mu^e$ & $p_\mu^{e'} p_\mu^{e'}$ in their respective rest frames where $p_\mu^e = (m, 0)$

$p_\mu^e p_\mu^e = m^2$; $p_\mu^{e'} p_\mu^{e'} = m^2$

- $p_\mu^\gamma p_\mu^e = \nu m$, $p_\mu^{\gamma'} p_\mu^e = \nu' m$
- $p_\mu^\gamma p_\mu^{\gamma'} = \nu\nu' - \bar{k}.\bar{k}'$

Verify

⊛ becomes :

$$\boxed{m^2 + 2\nu m - 2\nu' m - 2\nu\nu' + 2\bar{k}\cdot\bar{k}' = m^2}$$

$\bar{k}\cdot\bar{k}' = kk'\cos\theta$

$\Downarrow$

$$\boxed{2m(\nu-\nu') - 2\nu\nu'(1-\cos\theta) = 0}$$

$\Downarrow$

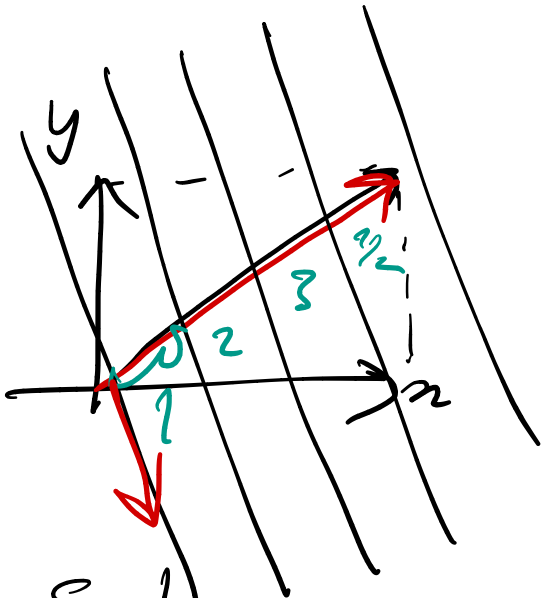
$$\boxed{\frac{(\nu-\nu')}{\nu\nu'} = \frac{1}{m}(1-\cos\theta)}$$

$$\boxed{(\lambda-\lambda') = \frac{1}{m}(1-\cos\theta) = \lambda_c(1-\cos\theta)}$$

Form / Covect

$P_\mu = (E, P_x, P_y, P_z)$

(0, 1) Term



Vector. (1, 0) tensor

$A^\mu = (A^0, A^1, A^2, A^3)$

$A^\mu B_\mu = 3.5$

Vect $\longrightarrow$ $\boxed{\text{Covect}}$ $\longrightarrow$ Scalar.

Exercise 2.4: One-forms I

Given the following vectors in $\mathcal{O}$:

$$\vec{A} \to_{\mathcal{O}} (2,1,1,0),\ \vec{B} \to_{\mathcal{O}} (1,2,0,0),\ \vec{C} \to_{\mathcal{O}} (0,0,1,1)\,,\ \vec{D} \to_{\mathcal{O}} (-3,2,0,0)\,.$$

(a) Show that they are linearly independent. (0.5 pt)

$$\begin{vmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -3 & 2 & 0 & 0 \end{vmatrix} = -8 \neq 0$$

This means that the vectors $\vec{A}, \vec{B}, \vec{C}, \vec{D}$, are linearly independent.

(b) Find the components of $\tilde{p}$ if: $\tilde{p}(\vec{A}) = 1$, $\tilde{p}(\vec{B}) = -1$, $\tilde{p}(\vec{C}) = -1$, $\tilde{p}(\vec{D}) = 0$. pt)

$$\begin{aligned} \tilde{p}(\vec{A}) &= p_\alpha A^\alpha = 1 = 2p_0 + p_1 + p_2 \\ \tilde{p}(\vec{B}) &= p_\alpha B^\alpha = -1 = p_0 + 2p_1 \\ \tilde{p}(\vec{C}) &= p_\alpha C^\alpha = -1 = p_2 + p_3 \\ \tilde{p}(\vec{D}) &= p_\alpha D^\alpha = 0 = -3p_0 + 2p_1 \end{aligned}$$

or, equivalently,

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -3 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

The solution of this linear system reads: $\tilde{p} \to_{\mathcal{O}} (-1/4, -3/8, 15/8, -23/8)$.

(c) Find the value of $\tilde{p}(\vec{E})$ for $\vec{E} \to_{\mathcal{O}} (1,1,0,0)$. (0.5 pt)

$$\tilde{p}(\vec{E}) = p_\alpha E^\alpha = \tilde{p}_0 + \tilde{p}_1 = -5/8\,.$$

(d) Determine whether the one-forms $\tilde{p}, \tilde{q}, \tilde{r}$ and $\tilde{s}$ are linearly independent if $\tilde{q}(\vec{A}) = \tilde{q}(\vec{B}) = 0$, $\tilde{q}(\vec{C}) = 1$, $\tilde{q}(\vec{D}) = -1$, $\tilde{r}(\vec{A}) = 2$, $\tilde{r}(\vec{B}) = \tilde{r}(\vec{C}) = \tilde{r}(\vec{D}) = 0$, $\tilde{s}(\vec{A}) = \tilde{s}(\vec{B}) = -1$, $\tilde{s}(\vec{C}) = \tilde{s}(\vec{D}) = 0$. (1 pt)

$$\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -3 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{36}$$

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -3 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} \tag{37}$$

$$\begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -3 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{38}$$

We find: $\tilde{q} \to_{\mathcal{O}} (1/4, -1/8, -3/8, 11/8)$, $\tilde{r} \to_{\mathcal{O}} (0, 0, -2, -2)$, and $\tilde{s} \to_{\mathcal{O}} (-1/4, -3/8, -1/8, 1/8)$. Now we can compute the determinant:

$$\begin{vmatrix} 1/4 & -1/8 & -3/8 & 11/8 \\ 0 & 0 & 2 & -2 \\ -1/4 & -3/8 & -1/8 & 1/8 \\ -1/4 & -3/8 & 15/8 & -23/8 \end{vmatrix} = -1/4 \neq 0 \tag{39}$$

The one-forms $\tilde{p}, \tilde{q}, \tilde{r}$ and $\tilde{s}$ are linearly independent.

Exercise 2.5: One-forms II

Consider the basis $\{\vec{e}_\alpha\}$ of a frame $\mathcal{O}$ and the basis $(\tilde{\lambda}^0, \tilde{\lambda}^1, \tilde{\lambda}^2, \tilde{\lambda}^3)$ for the space of one-forms, where we have

$$\tilde{\lambda}^0 \to_{\mathcal{O}} (1, 1, 0, 0)$$
$$\tilde{\lambda}^1 \to_{\mathcal{O}} (1, -1, 0, 0)$$
$$\tilde{\lambda}^2 \to_{\mathcal{O}} (0, 0, 1, -1)$$
$$\tilde{\lambda}^3 \to_{\mathcal{O}} (0, 0, 1, 1)$$

Note that $\{\tilde{\lambda}^\beta\}$ is not the basis dual to $\{\vec{e}_\alpha\}$.

(a) Show that $\tilde{p} \neq \tilde{p}(\vec{e}_\alpha)\tilde{\lambda}^\alpha$ for arbitrary $\tilde{p}$. (1 pt)

The set of vectors $\{\tilde{\lambda}^\beta\}$ constitutes a basis, since they are linearly independent (you can check that). This means that an arbitrary one-form $\tilde{p}$ can be expressed as

$$\tilde{p} = b_\alpha \tilde{\lambda}^\alpha \,. \tag{40}$$

The vectors $\{\tilde{\lambda}^\beta\}$ can be expressed in terms of the dual vectors $\{\tilde{\omega}^\beta\}$ (the ones that satisfy $\omega^\alpha(\vec{e}_\beta) = \delta^\alpha_\beta$) as $\tilde{\lambda}^\beta = \lambda^\beta_\alpha \tilde{\omega}^\alpha$, so

$$\tilde{p} = b_\alpha \tilde{\lambda}^\alpha = b_\alpha \lambda^\alpha_\kappa \tilde{\omega}^\kappa \,. \tag{41}$$

On the other hand we have:

$$\tilde{p}(\vec{e}_\beta) = b_\alpha \lambda^\alpha_\kappa \delta^\kappa_\beta = b_\alpha \lambda^\alpha_\beta \,. \tag{42}$$

Comparing (41) and (42) we see that $b_\alpha \neq \tilde{p}(\vec{e}_\alpha)$.

(b) Let $\tilde{p} \to_{\mathcal{O}} (1, 1, 1, 1)$. Find numbers l_α such that $\tilde{p} = l_\alpha \tilde{\lambda}^\alpha$. These are the components of $\tilde{p}$ on $\{\tilde{\lambda}^\alpha\}$. (1 pt)

$$\tilde{p} = a_\alpha \tilde{\omega}^\alpha = l_\beta \lambda^\beta_\alpha \tilde{\omega}^\alpha \longrightarrow a_\alpha = l_\beta \lambda^\beta_\alpha \,. \tag{43}$$

This means that

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \\ l_3 \end{pmatrix} \tag{44}$$

Where the columns of the matrix are just the vectors $\{\tilde{\lambda}^\beta\}$. From here we obtain: $(l_0, l_1, l_2, l_3) = (1, 0, 0, 1)$. One can check that:

$$\tilde{p} = l_\alpha \tilde{\lambda}^\alpha = \tilde{\lambda}^0 + \tilde{\lambda}^3 = \tilde{\omega}^0 + \tilde{\omega}^1 + \tilde{\omega}^2 + \tilde{\omega}^3 \longrightarrow_{\mathcal{O}} (1, 1, 1, 1) \,. \tag{45}$$