Exercise 3.1: One-forms III

Consider the coordinates $u = t - x$, $v = t + x$ in Minkowski space.

(a) Define $\vec{e}_u$ to be the vector connecting the events with coordinates $\{u = 1, v = 0, y = 0, z = 0\}$ and $\{u = 0, v = 0, y = 0, z = 0\}$, and analogously for $\vec{e}_v$. Show that $\vec{e}_u = (\vec{e}_t - \vec{e}_x)/2$, $\vec{e}_v = (\vec{e}_t + \vec{e}_x)/2$, and draw $\vec{e}_u$ and $\vec{e}_v$ in a spacetime diagram of the $t - x$ plane. (0.5 pt)

$(t, x, \cancel{y})$

$\Downarrow$

(t, x)

$u = t - x$
$v = t + x$

Let us consider an event (t, x, y, z) in the basis $\{\vec{e}_u, \vec{e}_v, \vec{e}_y, \vec{e}_z\}$.

$u + v = t - x + t + x$
$= 2t$

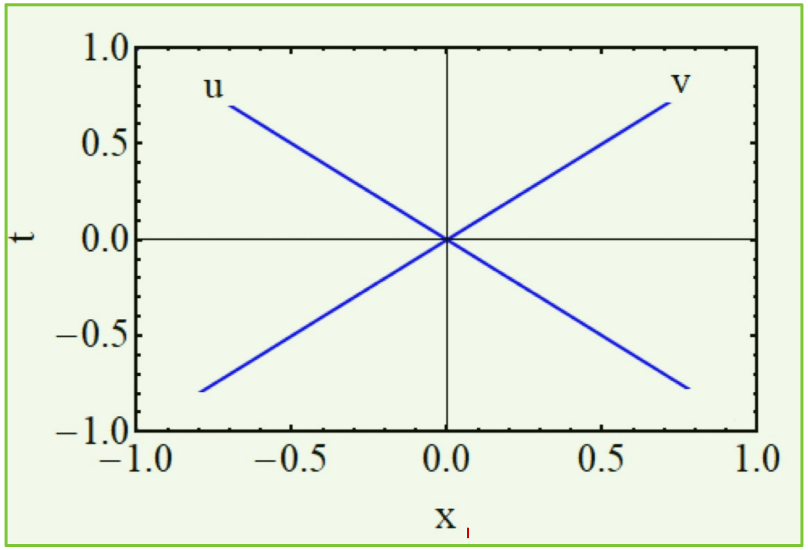
$$\boxed{t = \frac{u+v}{2}}$$

$$(t, x, y, z) = t\vec{e}_t + x\vec{e}_x + y\vec{e}_y + z\vec{e}_z = \left(\frac{u+v}{2}\right)\vec{e}_t + \left(\frac{v-u}{2}\right)\vec{e}_x + y\vec{e}_y + z\vec{e}_z =$$

$$= u\underbrace{\left(\frac{\vec{e}_t - \vec{e}_x}{2}\right)}_{\vec{e}_u} + v\underbrace{\left(\frac{\vec{e}_t + \vec{e}_x}{2}\right)}_{\vec{e}_v} + y\vec{e}_y + z\vec{e}_z\,,$$

$$\vec{e}_u = \frac{\vec{e}_t - \vec{e}_x}{2} \quad ; \quad \vec{e}_v = \frac{\vec{e}_t + \vec{e}_x}{2}\,.$$

$$\vec{e}_u = \frac{1}{2}\vec{e}_t - \frac{1}{2}\vec{e}_x + 0\,\vec{e}_y + 0\,\vec{e}_z$$



ST diagram with $\{\vec{e}_t, \vec{e}_x\}$
& $\{\vec{e}_u, \vec{e}_v\}$ axes

$\vec{e}_u = (1,0,0,0) \leftarrow (u,v,y,z)$

$\vec{e}_u = (\frac{1}{2}, -\frac{1}{2}, 0, 0) \leftarrow (t,x,y,z)$

(b) Show that $\vec{e}_{\bar{\mu}} \equiv \{\vec{e}_u, \vec{e}_v, \vec{e}_y, \vec{e}_z\}$ are a basis for vectors in Minkowski space. **(0.5 pt)**

Prove that they are a set of Linearly independent vectors

$$\begin{aligned} \vec{e}_u &= (1/2, -1/2, 0, 0) \\ \vec{e}_v &= (1/2, 1/2, 0, 0) \\ \vec{e}_y &= (0, 0, 1, 0) \\ \vec{e}_z &= (0, 0, 0, 1) \end{aligned} \rightarrow \begin{vmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1/2 \neq 0$$

$\square$

(c) Find the components of the metric tensor on this basis. (1 pt)

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Dual vectors associated to basis $\{\vec{e}_\mu\} = (t,x,y,z)$

$$\mathbf{g} = \eta_{\mu\nu} \tilde{d}x^\mu \otimes \tilde{d}x^\nu = a_{\bar{\mu}\bar{\nu}} \tilde{d}s^{\bar{\mu}} \otimes \tilde{d}s^{\bar{\nu}},$$

Dual vectors associated to basis $\{\vec{e}_{\bar{\mu}}\}$

$a_{\bar{\mu}\bar{\nu}}$

$$\text{i.e. } \tilde{d}x^\nu(\vec{e}_\alpha) = \delta^\nu_\alpha \text{ and } \tilde{d}s^{\bar{\nu}}(\vec{e}_{\bar{\alpha}}) = \delta^{\bar{\nu}}_{\bar{\alpha}}. \rightarrow \tilde{d}s^{\bar{\mu}} = \lambda^{\bar{\mu}}{}_\nu \tilde{d}x^\nu.$$

Can define matrix λ, s.t.

$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

$\vec{A} = a^\mu \vec{e}_\mu$

$\mathbf{A} = a_\mu \boxed{\tilde{\omega}^\mu}$

Plugging this in the 1st eq^n we have:

$\vec{A}\cdot A = (a^{\mu}\vec{e}_{\mu})\cdot(a_{\nu}\, dx^{\nu})$
$= a^{\mu}a_{\mu} = \text{Scalar}$

$$\eta_{\mu\nu}\tilde{d}x^{\mu}\otimes\tilde{d}x^{\nu} = a_{\bar{\mu}\bar{\nu}}\lambda^{\bar{\mu}}{}_{\beta}\lambda^{\bar{\nu}}{}_{\kappa}\tilde{d}x^{\beta}\otimes\tilde{d}x^{\kappa} = a_{\bar{\beta}\bar{\kappa}}\lambda^{\bar{\beta}}{}_{\mu}\lambda^{\bar{\kappa}}{}_{\nu}\tilde{d}x^{\mu}\otimes\tilde{d}x^{\nu}\,,$$

Change dummy indices.

so

$$\eta_{\mu\nu} = a_{\bar{\beta}\bar{\kappa}}\lambda^{\bar{\beta}}{}_{\mu}\lambda^{\bar{\kappa}}{}_{\nu} = \lambda_{\mu}{}^{\bar{\beta}}a_{\bar{\beta}\bar{\kappa}}\lambda^{\bar{\kappa}}{}_{\nu} \longrightarrow \boxed{\eta = \lambda^{\mathrm{T}}a\lambda} \longrightarrow a = (\lambda^{\mathrm{T}})^{-1}\eta\lambda^{-1}\,. \qquad (11)$$

IMP

Making use of $\vec{e}_{\bar{\alpha}} = M_{\bar{\alpha}}{}^{\alpha}\vec{e}_{\alpha}$, with $M = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$,

we get:

$$\tilde{d}s^{\bar{\mu}}(\vec{e}_{\bar{\alpha}}) = \lambda^{\bar{\mu}}{}_{\nu}\tilde{d}x^{\nu}(\vec{e}_{\bar{\alpha}}) = \lambda^{\bar{\mu}}{}_{\nu}\tilde{d}x^{\nu}(M_{\bar{\alpha}}{}^{\alpha}\vec{e}_{\alpha}) = \lambda^{\bar{\mu}}{}_{\nu}M_{\bar{\alpha}}{}^{\alpha}\tilde{d}x^{\nu}(\vec{e}_{\alpha})$$

$$\delta^{\bar{\mu}}_{\bar{\alpha}} = \lambda^{\bar{\mu}}{}_{\nu}M_{\bar{\alpha}}{}^{\alpha}\delta^{\nu}_{\alpha} = \lambda^{\bar{\mu}}{}_{\nu}M_{\bar{\alpha}}{}^{\nu} \longrightarrow \mathbb{1} = \lambda M^{\mathrm{T}} \longrightarrow \boxed{\lambda^{-1} = M^{\mathrm{T}} \text{ and } (\lambda^{\mathrm{T}})^{-1} = M\,.}$$

$\vec{e}_{\bar{\alpha}} = \{e_u, e_v, e_y, e_z\}$

$g_{\mu\nu}\vec{e}^{\mu}\otimes\vec{e}^{\nu}$

$\eta = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$

Plugging this result in (11) we are led to:

$$a = M\eta M^{\mathrm{T}} = \begin{pmatrix} 0 & -1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) Show that $\vec{e}_u$ and $\vec{e}_v$ are null and not orthogonal. (They are called a null basis for the $t-x$ plane.) (0.5 pt)

< 0 Timelike

> 0 Spacelike.

Note: Two vectors $\vec{A}$ and $\vec{B}$ are orthogonal if $\mathbf{g}(\vec{A}, \vec{B}) = 0$; a vector $\vec{C}$ is null if $\mathbf{g}(\vec{C}, \vec{C}) = 0$.

$$\mathbf{g}(\vec{e}_u, \vec{e}_u) = a_{\bar{\mu}\bar{\nu}} \tilde{d}s^{\bar{\mu}}(\vec{e}_{\bar{0}}) \otimes \tilde{d}s^{\bar{\nu}}(\vec{e}_{\bar{0}}) = a_{\bar{\mu}\bar{\nu}} \delta^{\bar{\mu}}_{\bar{0}} \delta^{\bar{\nu}}_{\bar{0}} = a_{\bar{0}\bar{0}} = 0\,,$$

$$\mathbf{g}(\vec{e}_v, \vec{e}_v) = a_{\bar{\mu}\bar{\nu}} \tilde{d}s^{\bar{\mu}}(\vec{e}_{\bar{1}}) \otimes \tilde{d}s^{\bar{\nu}}(\vec{e}_{\bar{1}}) = a_{\bar{\mu}\bar{\nu}} \delta^{\bar{\mu}}_{\bar{1}} \delta^{\bar{\nu}}_{\bar{1}} = a_{\bar{1}\bar{1}} = 0\,,$$

$$\mathbf{g}(\vec{e}_u, \vec{e}_v) = a_{\bar{\mu}\bar{\nu}} \tilde{d}s^{\bar{\mu}}(\vec{e}_{\bar{0}}) \otimes \tilde{d}s^{\bar{\nu}}(\vec{e}_{\bar{1}}) = a_{\bar{\mu}\bar{\nu}} \delta^{\bar{\mu}}_{\bar{0}} \delta^{\bar{\nu}}_{\bar{1}} = a_{\bar{0}\bar{1}} = -1/2\,,$$

$$\mathbf{g}(\vec{e}_v, \vec{e}_u) = a_{\bar{\mu}\bar{\nu}} \tilde{d}s^{\bar{\mu}}(\vec{e}_{\bar{1}}) \otimes \tilde{d}s^{\bar{\nu}}(\vec{e}_{\bar{0}}) = a_{\bar{\mu}\bar{\nu}} \delta^{\bar{\mu}}_{\bar{1}} \delta^{\bar{\nu}}_{\bar{0}} = a_{\bar{1}\bar{0}} = 1/2\,,$$

It is clear that $\vec{e}_u$ and $\vec{e}_v$ are null, but not orthogonal.

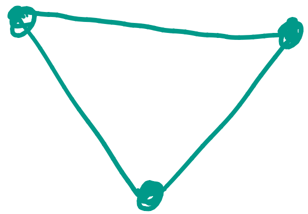
$e_u = (+\frac{1}{2}, \frac{1}{2}, 0, 0)$

$$g(\vec{a}, \vec{b}) = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

E.E : (Field) (Space) = (Energy momentum)

$g_{\mu\nu}$



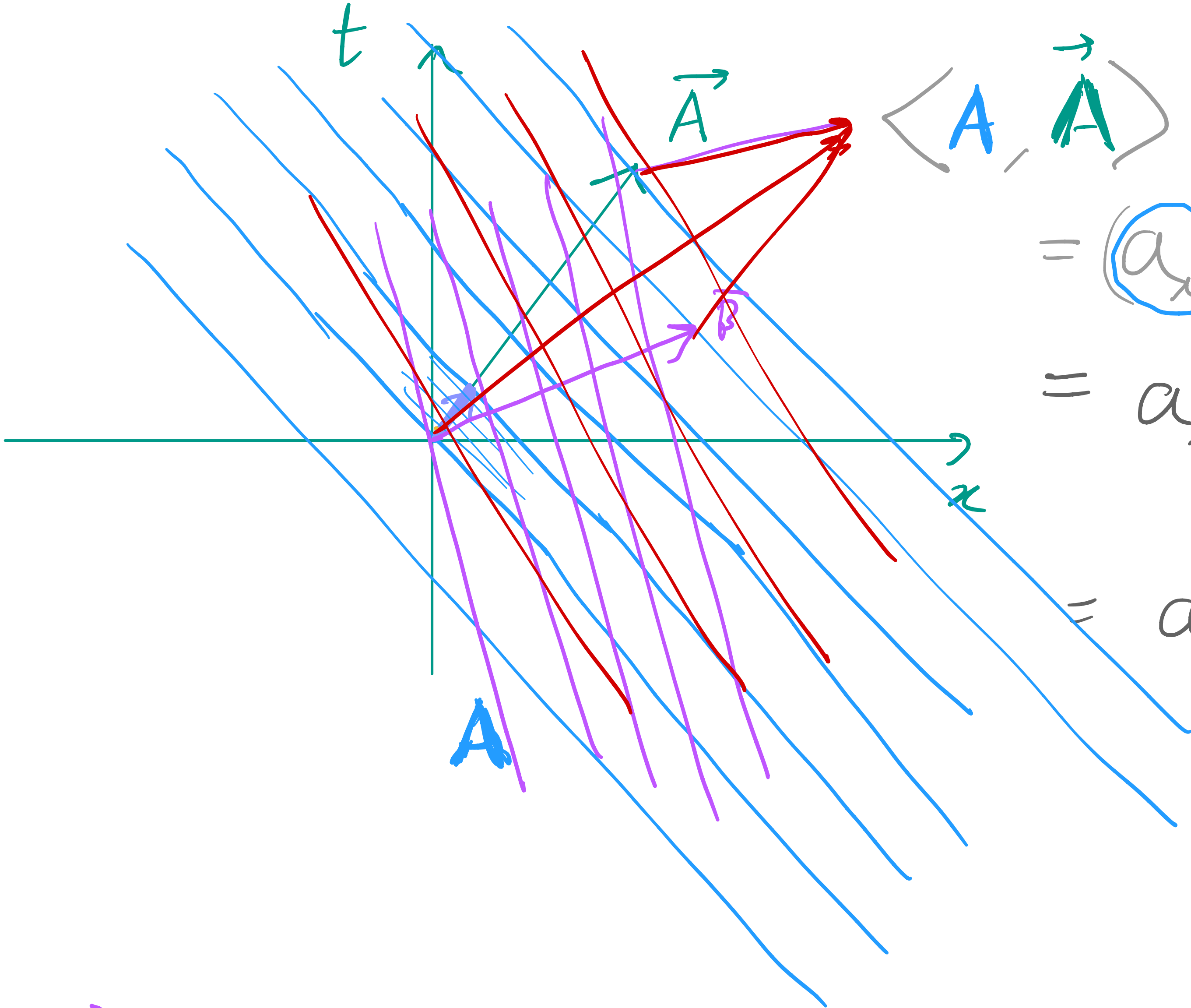
$T_{\mu\nu}$

Geodesic eq^n : (Particle motion)

How does it trajectory look like?

Maxwell's eq^n : (Field)

Lorentz force law



$$\langle A, \vec{A}\rangle = (a_\mu dx^\mu)\cdot(a^\nu \vec{e}_\nu)$$

$$= a_\mu a^\nu \underbrace{dx^\mu \vec{e}_\nu}_{\delta^\mu_\nu}$$

$$= a_\mu a^\mu = 5$$

$$= 1$$

$$\langle B, \vec{B}\rangle = b_\mu b^\mu = \underline{1}$$

$$\langle A, \vec{B}\rangle = x$$


θ
v
v

Exercise 3.2: Energy-momentum tensor of dust

(a) Use the energy-momentum tensor of *dust*,

$T^{\mu\nu} =$

$$T^{\alpha\beta} = p^\alpha N^\beta \vec{e}_\alpha \otimes \vec{e}_\beta \Leftarrow \quad \mathbf{T} = \vec{p} \otimes \vec{N}\,,$$

to show that the elements of the energy-momentum tensor of a free (non-interacting) point-like particle is given by

$$T^{\alpha\beta} = mc^2\gamma(v)\,\delta(\mathbf{x}-\mathbf{x}_p)V^\alpha V^\beta\,, \tag{29}$$

with $\vec{U} = \gamma(v)\vec{V}$ and $\mathbf{x}_p = (x_p, y_p, z_p)$ the 4-velocity and spatial location of the particle, respectively, and $v = |\mathbf{V}|$[1]. (1 pt)

A free point-like particle is a particular case of *dust*. In the MCRF of the particle ($\bar{\mathcal{O}}$) there is no flux of energy, nor any flux of three-momentum, as it happens with dust. The 4-momentum and the number-flux 4-vector in $\bar{\mathcal{O}}$ read,

$$\vec{p} \to_{\bar{\mathcal{O}}} mc^2\overbrace{\boxed{(1,0,0,0)}}^{\vec{V}} \quad ; \quad \vec{N} \to_{\bar{\mathcal{O}}} \delta(\bar{\mathbf{x}}-\bar{\mathbf{x}}_p)\overbrace{\boxed{(1,0,0,0)}}^{\vec{V}}$$

After Lorentz transformation $\quad MCRF_{\bar{O}} \longrightarrow O$ (Moves at $-V$ w.r.t $\bar{O}$)

(Basically a γ factor)

$$\vec{p} \to_{\mathcal{O}} mc^2\vec{U} \quad ; \quad \vec{N} \to_{\mathcal{O}} \delta(\bar{\mathbf{x}}-\bar{\mathbf{x}}_p)\vec{U}\,.$$

Hence, the components of $\mathbf{T}$ in $\mathcal{O}$ are

$$T^{\alpha\beta} = p^\alpha N^\beta = mc^2\delta(\bar{\mathbf{x}}-\bar{\mathbf{x}}_p)U^\alpha U^\beta = mc^2\gamma^2(v)\delta(\bar{\mathbf{x}}-\bar{\mathbf{x}}_p)V^\alpha V^\beta\,.$$

Dirac δ still in $\bar{O}$: Bring it to O

$$\int \delta(\bar{\mathbf{x}}-\bar{\mathbf{x}}_p)\,d^3\bar{x} = \int \delta[\Lambda^i{}_j(x^j - x^j_p)]\,d^3x\,,$$

$$(d^3\bar{x} = |\Lambda| d^3x = d^3x.)$$

change of variables $y^i = \Lambda^i_j x^j$, with $d^3y = |\Lambda^i_j| d^3x$

Formal step

$$\Lambda^i_j(\mathbf{v}) = \begin{pmatrix} 1+[\gamma(v)-1]n_x^2 & [\gamma(v)-1]n_x n_y & [\gamma(v)-1]n_x n_z \\ [\gamma(v)-1]n_x n_y & 1+[\gamma(v)-1]n_y^2 & [\gamma(v)-1]n_y n_z \\ [\gamma(v)-1]n_x n_z & [\gamma(v)-1]n_y n_z & 1+[\gamma(v)-1]n_z^2 \end{pmatrix}$$

$$\longrightarrow |\Lambda^i_j(\mathbf{v})| = \ldots = \gamma(v)$$

Or just write informally direct

$$\int \delta(\bar{\mathbf{x}}-\bar{\mathbf{x}}_p)\,d^3\bar{x} = \int \frac{\delta(\mathbf{y}-\mathbf{y}_p)}{\gamma(v)}\,d^3y$$

$$= \int \frac{\delta(\mathbf{x}-\mathbf{x}_p)}{\gamma(v)}\,d^3x = \int \frac{\delta(\mathbf{x}-\mathbf{x}_p)}{\gamma(v)}\,d^3\bar{x}$$

Gives us what we want.

(b) **Show that the 4-acceleration of dust in Minkowski is $\vec{a} = (0,0,0,0)$. Hint: use the conservation equations for $T^{\mu\nu}$ and N^μ. (1 pt)**

For a general dust fluid $\boxed{T^{\mu\nu} = p^\mu N^\nu} = mc^2 n U^\mu U^\nu$,

$N^\alpha = n(U^0, U^i) = n\vec{U}$ Particle density in MCRF

The covariant conservation equation of the number-flux 4-vector tells us that

$$\partial_\alpha N^\alpha = 0 \longrightarrow \boxed{\partial_0(nU^0) + \partial_i(nU^i) = 0} \, *$$

(Timelike) + (Spacelike)

The 0 component of the conservation equation of the energy-momentum tensor reads,

$B^\beta = \partial_\alpha T^{\alpha\beta} = 0 \Longrightarrow$

$$\partial_\alpha T^{\alpha 0} = 0 \longrightarrow \partial_0 T^{00} + \partial_i T^{i0} = 0 \,,$$

$$\partial_0[n(U^0)^2] + \partial_i(nU^0U^i) = 0 \,,$$

$$nU^0\partial_0 U^0 + \left(U^0\partial_0[nU^0] + \partial_i(nU^i)U^0\right)^* + nU^i\partial_i U^0 = 0 \,.$$

$$U^0\partial_0 U^0 + U^i\partial_i U^0 = 0 \longrightarrow \boxed{a^0 = \frac{dU^0}{d\tau} = U^\alpha\partial_\alpha U^0 = 0}$$

$$\partial_\alpha T^{\alpha i} = 0 \longrightarrow \partial_0 T^{0i} + \partial_j T^{ji} = 0 \,,$$

$$\partial_0[nU^0U^i] + \partial_j(nU^jU^i) = 0 \,,$$

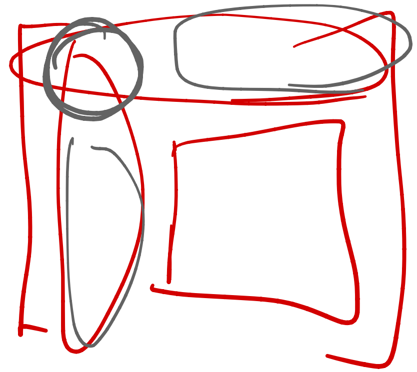
$$nU^0\partial_0 U^i + U^i\partial_0[nU^0] + \partial_j(nU^j)U^i + nU^j\partial_j U^i = 0 \,.$$

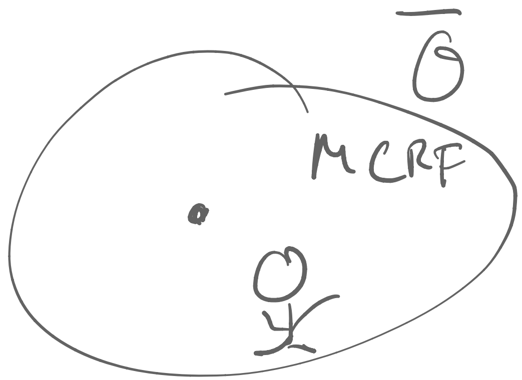
Same, but for i^{th} comp.

$$U^0\partial_0 U^i + U^j\partial_j U^i = 0$$

$$\boxed{a^i = \frac{dU^i}{d\tau} = U^\alpha\partial_\alpha U^i = 0}$$

$T^{\mu\nu} =$





(c) Explain why this is an expected result for *dust* in Minkowski. (0.5 pt)

In the MCRF $\bar{\mathcal{O}}$ there are no momentum nor energy fluxes for *dust*. This means that particles basically remain at the same location and, hence, an inertial frame moving at speed $-\mathbf{v}$ relative to $\bar{\mathcal{O}}$ will see that the dust particles move at uniform velocity, i.e. with no acceleration. This is exactly what we have found above.

(d) It is clear that (1) cannot explain the kinematics of accelerated particles in Minkowski. To do so we can add a new piece to the energy-momentum tensor, as follows,

$$T^{\mu\nu} = mc^2 n U^\mu U^\nu + A^{\mu\nu} \,. \tag{3}$$

Show that $-\partial_\mu A^{\mu\nu}/n$ is actually a 4-force and determine the equations that must be satisfied by $A^{\mu\nu}$ if we want the particles to move with uniform positive proper 3-acceleration α in the $\bar{x}$-direction of the MCRF. (1 pt)

Proceeding as in exercise 3.2b we find that in the inertial frame $\mathcal{O}$,

$$mna^0 + \partial_\mu A^{\mu 0} = 0 \qquad ; \qquad mna^i + \partial_\mu A^{\mu i} = 0 \,.$$

It is obvious from the last equations that $F^\nu = -\partial_\mu A^{\mu\nu}/n$ is a 4-force acting on the particles of the system. In the MCRF $U_{\bar{\mu}} a^{\bar{\mu}} = 0$ with $\vec{U} = (1,0,0,0)$, so $a^{\bar{0}} = 0$. On the other hand, $\bar{\mathbf{a}} = (\alpha, 0, 0)$. Thus, we find:

$$\partial_{\bar{\mu}} A^{\bar{\mu}\bar{0}} = \partial_{\bar{\mu}} A^{\bar{\mu}\bar{2}} = \partial_{\bar{\mu}} A^{\bar{\mu}\bar{3}} = 0 \qquad ; \qquad m\alpha = -\frac{1}{n}\partial_\mu A^{\bar{\mu}\bar{1}} \,.$$

Exercise 3.3: Energy-momentum tensor of a perfect fluid

Consider a collection of non-interacting particles with a random distribution of velocities at every point, with no preferred direction in the MCRF. Consider also that the particles do not have the same mass, i.e. there is a joint probability distribution for masses and velocities.

(a) Prove that the energy-momentum tensor is that of a perfect fluid. (2 pt)

Let us take a cell of volume V with N particles, as measured in the MCRF. The number density is $n = N/V$, and we find

$$
\begin{aligned}
T^{00} &= \frac{1}{V}\sum_{n}^{N} E_n = \frac{N}{V} < E_n >= n < \gamma(v) m > \\
T^{0i} &= T^{i0} = \frac{1}{V}\sum_{n}^{N} p_n^i = \frac{N}{V} < p_n^i >= n < \gamma(v) m v^i > \\
T^{ij} &= T^{ij} = \frac{1}{V}\sum_{n}^{N} p_n^i v_n^j = \frac{N}{V} < p_n^i v_n^j >= n < \gamma(v) m v^i v^j > ,
\end{aligned}
$$

Velocity of n^{th} particle in MCRF

→ Because of isotropy $\boxed{\vec{v} = |v| = v}$

$$\boxed{P(\mathbf{v}, m) = P(v, m) = P(v_x^2 + v_y^2 + v_z^2, m)\,.}$$

$$\boxed{< \gamma(v) m v^i >= 0} \quad \& \quad \boxed{< \gamma(v) m v^i v^j >= \delta^{ij} < \gamma(v) m (v^j)^2 >}$$

$$\downarrow$$

$$T^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} n < \gamma(v) m > & 0 & 0 & 0 \\ 0 & \frac{n}{3} < \gamma(v) m v^2 > & 0 & 0 \\ 0 & 0 & \frac{n}{3} < \gamma(v) m v^2 > & 0 \\ 0 & 0 & 0 & \frac{n}{3} < \gamma(v) m v^2 > \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$\downarrow$$

$$\boxed{T^{\mu\nu} = (p+\rho) U^\mu U^\nu + p \eta^{\mu\nu}}$$

Can be chosen $\rightarrow$ $U^\mu = (1,0,0,0)$

(b) Particularize the previous result for a fluid composed by particles with the same mass and velocity v, and show also that for a photon gas $p = \rho/3$. (1 pt)

$$\rightarrow \quad T^{\bar{\mu}\bar{\nu}} = \begin{pmatrix} \gamma n m & 0 & 0 & 0 \\ 0 & \frac{n}{3}\gamma m v^2 & 0 & 0 \\ 0 & 0 & \frac{n}{3}\gamma m v^2 & 0 \\ 0 & 0 & 0 & \frac{n}{3}\gamma m v^2 \end{pmatrix} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p = \frac{\rho}{3} v^2 & 0 & 0 \\ 0 & 0 & p = \frac{\rho}{3} v^2 & 0 \\ 0 & 0 & 0 & p = \frac{\rho}{3} v^2 \end{pmatrix}$$

$\rightarrow$ For photon gas : $v = 1 \Rightarrow p = \frac{\rho}{3}$