Problem. Some commutator identities

Problem Statement

Consider two operators A and B whose commutator is constant $[A, B] = c$.

1. Show that $e^A B e^{-A} = B + c$
2. Evaluate $e^A B^n e^{-A}$
3. Show that $e^A e^B e^{-A} e^{-B} = e^c$

Solution (1)

Such identities are often proved by constructing a function similar to the LHS. Then, differentiate it and formulate some sort of a differential equation. For this problem we can define the function as

$$\mathcal{F}(x) = e^{xA} B e^{-xA} \tag{22}$$

We can see that $\mathcal{F}(0) = B$.

We will be taking the derivative of an operator in the exponential, a quick computation can show us,

$$\begin{aligned}
\frac{de^{xA}}{dx} &= \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{x^n}{n!} A^n\right] && (23)\\
&= \left[\sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} A^n\right] && (24)\\
&= A\left[\sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} A^{n-1}\right] && (25)\\
&= \left[\sum_{n=0}^{\infty} \frac{x^{n-1}}{(n-1)!} A^{n-1}\right] A && (26)\\
&= Ae^{xA} = e^{xA} A && (27)\\
& && (28)
\end{aligned}$$

Now taking a derivative of $\mathcal{F}$, we get

$$\begin{aligned}
\frac{d\mathcal{F}(x)}{dx} &= e^{xA} A B e^{-xA} - e^{xA} B A e^{-xA} && (29)\\
&= e^{xA}\left[A, B\right] e^{-xA} && (30)
\end{aligned}$$

Now using $[A, B] = c$, we have

$$\frac{d\mathcal{F}(x)}{dx} = e^{xA} c e^{-xA} = c \tag{31}$$

Solving this differential is straightforward, just integrate both sides with respect to x, we get,

$$\mathcal{F}(x) = \mathcal{F}(0) + cx = B + cx \tag{32}$$

If we take $x = 1$ in this function, we have proved our statement.

$$\mathcal{F}(1) = e^A B e^{-A} = B + c \tag{33}$$

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Solution (2)

We can effectively use the given hint : $e^A e^{-A} = e^{-A} e^A = 1$

$$e^A B^n e^{-A} = e^A B B^{n-1} e^{-A} \tag{34}$$
$$= e^A B e^{-A} e^A B^{n-1} e^{-A} \tag{35}$$
$$= (B + c)\, e^A B e^{-A} e^A B^{n-2} e^{-A} \tag{36}$$
$$\tag{37}$$

We can see a clear pattern here, for every power of n in B^n, we get one power of $(B + c)$.
Hence,

$$e^A B^n e^{-A} = (B + c)^n \tag{38}$$

Solution 3.

$$e^A e^B e^{-A} e^{-B} = \left(e^A \left(\sum_{n=1}^{\infty} \frac{B^n}{n!} \right) e^{-A} \right) e^{-B} \tag{39}$$
$$= \left(\sum_{n=1}^{\infty} \frac{e^A B^n e^{-A}}{n!} \right) e^{-B} \tag{40}$$
$$\tag{41}$$

We can use the solution.2, to get

$$= \left(\sum_{n=1}^{\infty} \frac{(B + c)^n}{n!} \right) e^{-B} \tag{42}$$
$$= e^{B+c} e^{-B} \tag{43}$$
$$= e^c \tag{44}$$

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