Exercise 7.1
(Gravitational deflection of light in GR)

Exercise 7.1: Gravitational deflection of light in GR

(a) The motion of a free-falling particle around a spherical symmetric and stationary gravitational source of mass M is described by the geodesic equation, with the Schwarzschild metric,

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\varphi^2) \tag{1}$$

and

$$A(r) = \left(1 - \frac{2GM}{r}\right) \qquad B(r) = \left(1 - \frac{2GM}{r}\right)^{-1} \tag{2}$$

Here r is the distance to the center of the source. The solution is valid for $r > 2GM$.

(a) Keeping $A(r)$ and $B(r)$ completely general for a moment, write down the four components of the geodesic equation. (1 pt)

The geodesic equation is $$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\alpha}\frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda} = 0$$

where λ is the affine parameter that we employ to parametrize the path of the free-falling particle.

The only non-null Christoffel symbols associated to the Schwarzschild metric read,

$$\Gamma^t{}_{tr} = \frac{1}{2A}\frac{dA}{dr} \quad ; \quad \Gamma^\theta{}_{\theta r} = \frac{1}{r} \quad ; \quad \Gamma^\theta{}_{\varphi\varphi} = -\frac{1}{2}\sin(2\theta)$$

$$\Gamma^\varphi{}_{\varphi\theta} = \frac{\cos(\theta)}{\sin(\theta)} \quad ; \quad \Gamma^\varphi{}_{\varphi r} = \frac{1}{r}$$

$$\Gamma^r{}_{tt} = \frac{1}{2B}\frac{dA}{dr} \quad ; \quad \Gamma^r{}_{rr} = \frac{1}{2B}\frac{dB}{dr} \quad ; \quad \Gamma^r{}_{\theta\theta} = -\frac{r}{B} \quad ; \quad \Gamma^r{}_{\varphi\varphi} = -\frac{r}{B}\sin^2(\theta)$$

Plug them into the geodesic equation,

$$\begin{aligned}
0 &= \frac{d^2t}{d\lambda^2} + \frac{1}{A}\frac{dA}{dr}\frac{dt}{d\lambda}\frac{dr}{d\lambda} \\
0 &= \frac{d^2r}{d\lambda^2} + \frac{1}{B}\left[\frac{1}{2}\frac{dB}{dr}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\frac{dA}{dr}\left(\frac{dt}{d\lambda}\right)^2 - r\left(\frac{d\theta}{d\lambda}\right)^2 - r\sin^2(\theta)\left(\frac{d\varphi}{d\lambda}\right)^2\right] \\
0 &= \frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{dr}{d\lambda}\frac{d\theta}{d\lambda} - \frac{1}{2}\sin(2\theta)\left(\frac{d\varphi}{d\lambda}\right)^2 \\
0 &= \frac{d^2\varphi}{d\lambda^2} + 2\frac{\cos(\theta)}{\sin(\theta)}\frac{d\varphi}{d\lambda}\frac{d\theta}{d\lambda} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda},
\end{aligned}$$

(b) The spherical symmetry allows us to set $\theta = const. = \pi/2$ with no loss of generality. Use this to simplify the equations obtained in 7.1a and integrate them to get

$$\frac{dt}{d\lambda} = \frac{const.}{A(r)} \equiv \frac{C}{A(r)} \quad ; \quad r^2 \frac{d\varphi}{d\lambda} = const \equiv J. \quad ; \quad B(r)\left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A(r)} = const \equiv -E\,, \tag{11}$$

with λ an affine parameter. (1 pt)

The geodesic equation for φ can be easily solved as follows:

$$\frac{d^2\varphi}{d\lambda^2} + \frac{2}{r}\frac{d\varphi}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \frac{1}{r^2}\frac{d}{d\lambda}\left(r^2\frac{d\varphi}{d\lambda}\right) = 0 \longrightarrow \boxed{r^2\frac{d\varphi}{d\lambda} = const. \equiv J} \tag{12}$$

The one for the time t is also quite simple:

$$\frac{d^2t}{d\lambda^2} + \frac{1}{A}\frac{dA}{dr}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0 \longrightarrow \frac{d^2t}{d\lambda^2} + \frac{1}{A}\frac{dA}{d\lambda}\frac{dt}{d\lambda} = 0$$

$$\frac{d}{d\lambda}\left[\ln\left(\frac{dt}{d\lambda}\right)\right] + \frac{d}{d\lambda}[\ln(A)] = 0 \longrightarrow \boxed{\frac{dt}{d\lambda} = \frac{const.}{A(r)} \equiv \frac{C}{A(r)}}$$

Now let us take the geodesic equation for r. Using (12) in the last term we find,

$$\frac{d^2r}{d\lambda^2} + \frac{1}{B}\left[\frac{1}{2}\frac{dB}{dr}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\frac{dA}{dr}\left(\frac{dt}{d\lambda}\right)^2 - \frac{J^2}{r^3}\right] = 0$$

which can also be written as

$$\frac{1}{2B\frac{dr}{d\lambda}}\frac{d}{d\lambda}\left[B\left(\frac{dr}{d\lambda}\right)^2\right] + \frac{1}{2B}\frac{C^2}{A^2}\frac{dA}{dr} - \frac{J^2}{Br^3} = 0$$

Multiplying this equation by the denominator of the first term we are lead to

$$\frac{d}{d\lambda}\left[B\left(\frac{dr}{d\lambda}\right)^2\right] + \frac{C^2}{A^2}\frac{dA}{d\lambda} - \frac{2J^2}{r^3}\frac{dr}{d\lambda} = 0$$

The integration of this equation is straightforward and gives:

$$\boxed{B\left(\frac{dr}{d\lambda}\right)^2 - \frac{C^2}{A} + \frac{J^2}{r^2} = const. \equiv -E}$$

(c) Eliminate λ from the integrals of motion obtained in 7.1b to obtain a direct relation between r and φ. Show that

$$\varphi = \pm \int \frac{\sqrt{B(r)}dr}{r^2\sqrt{\frac{C^2}{A(r)J^2} - \frac{E}{J^2} - \frac{1}{r^2}}}. \qquad \textbf{(0.5 pt)} \qquad (15)$$

$$\boxed{r^2\frac{d\varphi}{d\lambda} = const. \equiv J} \longrightarrow d\lambda = \frac{r^2}{J}d\varphi \xrightarrow{\text{Plug in}} B\left(\frac{dr}{d\lambda}\right)^2 - \frac{C^2}{A} + \frac{J^2}{r^2} = const. \equiv -E$$

Get

$$B\frac{J^2}{r^4}\left(\frac{dr}{d\varphi}\right)^2 - \frac{C^2}{A} + \frac{J^2}{r^2} = -E$$

By isolating $d\varphi$ and performing the integral we directly obtain

$$\varphi = \pm \int \frac{\sqrt{B(r)}dr}{r^2\sqrt{\frac{C^2}{A(r)J^2} - \frac{E}{J^2} - \frac{1}{r^2}}}$$

(d) Show that $d\tau^2 = Ed\lambda^2$. What does this impose on E if one considers photons?

We can start from the expression for the line element (1).

As $ds^2 = -d\tau^2$ and the trajectory of the free-falling particle stays at $\theta = const. = \pi/2$ we have,

$$\longrightarrow \quad d\tau^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\varphi^2)$$

$$\longrightarrow \quad d\tau^2 = d\lambda^2\left[A(r)\left(\frac{dt}{d\lambda}\right)^2 - B(r)\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\varphi}{d\lambda}\right)^2\right]$$

$$\boxed{\frac{dt}{d\lambda} = \frac{const.}{A(r)} \equiv \frac{C}{A(r)}}$$

$$\boxed{B\left(\frac{dr}{d\lambda}\right)^2 - \frac{C^2}{A} + \frac{J^2}{r^2} = const. \equiv -E}$$

$$\boxed{r^2\frac{d\varphi}{d\lambda} = const. \equiv J}$$

$$\boxed{d\tau^2 = Ed\lambda^2}$$

(e) Now consider a photon approaching a central mass from infinity with impact parameter b. Denote by r_0 the radius of its closest approach and determine E and J in terms of r_0. (0.5 pt)

For photons $ds^2 = d\tau^2 = 0$, so $\boxed{E = 0}$

due to the result obtained in 7.1d. On the other hand we know that $r(\lambda)$ has a minimum at $r = r_0$, by definition. Hence, the following equation must be fulfilled:

$$\frac{C^2}{A(r_0)} = \frac{J^2}{r_0^2} \longrightarrow \boxed{J^2 = \frac{C^2 r_0^2}{A(r_0)}}$$

This is obtained from (14) by setting $E = 0$ and $dr/d\lambda|_{r_0} = 0$.

$$B\left(\frac{dr}{d\lambda}\right)^2 - \frac{C^2}{A} + \frac{J^2}{r^2} = const. \equiv -E$$

(f) Show that the total change in the angle φ of a photon approaching the source from infinity and going again to infinity after passing through the closest radius r_0 is given by the following integral

$$\Delta\varphi = 2\int_{r_0}^{\infty} \frac{\sqrt{B(r)}}{\sqrt{\frac{r^2 A(r_0)}{r_0^2 A(r)} - 1}} \frac{dr}{r} \tag{21}$$

This is the so-called deflection angle. (0.5 pt)

Due to the symmetry of the problem, the total deflection angle is twice the angle subtended by the radius vector from $r \to \infty$ to $r = r_0$. Thus, for an open trajectory, regardless of the object under consideration we have (15),

$$\Delta\varphi = 2\int_{r_0}^{\infty} \frac{\sqrt{B(r)}\,dr}{r^2\sqrt{\frac{C^2}{A(r)J^2} - \frac{E}{J^2} - \frac{1}{r^2}}}\,.$$

If we want to particularize this result for photons we just have to make use of the results obtained in 7.1e, (19) and (20). By doing so we directly obtain (21).

(g) Use the former equation and the approximations for $A(r)$ and $B(r)$ in the Newtonian limit, i.e. $2GM/r \ll 1$, to calculate the deflection angle $\Delta\varphi$ to first order in $2GM/r_0$. Hint: you might need

$$\int \frac{dx}{(x+a)\sqrt{a^2-x^2}} = \frac{-\sqrt{a^2-x^2}}{a(x+a)}. \qquad \textbf{(1.5 pt)} \tag{23}$$

The quantity $r_s \equiv 2GM$ is the Schwarzschild radius. In order to compute the deflection angle (21) and we need to Taylor-expand $\sqrt{B}$ and A to first order in r_s/r,

$$\begin{aligned}\sqrt{B(r)} &= 1+\frac{r_s}{2r}+\mathcal{O}\left(\frac{r_s}{r}\right)^2\\ A(r) &= 1-\frac{r_s}{r}+\mathcal{O}\left(\frac{r_s}{r}\right)^2.\end{aligned}$$

We have,

$$\Delta\varphi = 2\int_{r_0}^{\infty}\frac{\left(1+\frac{r_s}{2r}\right)}{\sqrt{\frac{r^2\left(1-\frac{r_s}{r_0}\right)}{r_0^2\left(1-\frac{r_s}{r}\right)}-1}}\frac{dr}{r} = 2\int_{r_0}^{\infty}\frac{r_0\,dr}{r\sqrt{r^2\left(1-\frac{r_s}{r_0}\right)-r_0^2\left(1-\frac{r_s}{r}\right)}}$$

Now, we perform the change of variables $x = r_0/r$,

$$\Delta\varphi = 2\int_0^1\frac{dx}{\sqrt{1-x^2+\frac{r_s}{r_0}\left(x^3-1\right)}} = 2\int_0^1\frac{dx}{\sqrt{1-x^2}\sqrt{1+\frac{r_s}{r_0}\left(\frac{x^3-1}{1-x^2}\right)}}$$

$$x^3-1=(x^2+x+1)(x-1)\longrightarrow\frac{x^3-1}{1-x^2}=-\left(\frac{x^2+x+1}{1+x}\right)=-\left(1+\frac{x^2}{1+x}\right)$$

and $r_s/r_0 \ll 1$ we can reexpress the integral for the deflection angle as follows,

$$\Delta\varphi = 2\int_0^1\frac{dx}{\sqrt{1-x^2}}\left[1+\frac{r_s}{2r_0}\left(1+\frac{x^2}{1+x}\right)\right]$$

$$\Delta\varphi = 2\left(1+\frac{r_s}{2r_0}\right)\int_0^1\frac{dx}{\sqrt{1-x^2}}+\frac{r_s}{r_0}\int_0^1\frac{dx}{\sqrt{1-x^2}}\left(\frac{x^2}{1+x}\right)$$

The last integral seems very complicated by it can be strongly simplified by doing

$$\frac{x^2}{1+x}=\frac{1}{1+x}+x-1.$$

Using this relation we are lead to

$$\Delta\varphi = 2\int_0^1\frac{dx}{\sqrt{1-x^2}}+\frac{r_s}{r_0}\left[\int_0^1\frac{x\,dx}{\sqrt{1-x^2}}+\int_0^1\frac{dx}{\sqrt{1-x^2}\,(1+x)}\right]$$

The first two integrals are trivial. The last one can be solved using the hint. We finally obtain:

$$\Delta\varphi = 2\frac{\pi}{2}+\frac{r_s}{r_0}(1+1)\longrightarrow\boxed{\Delta\varphi=\pi+\frac{2r_s}{r_0}}$$

Exercise 7.2
(Gravitational deflection of light in Newtonian Gravity)

(a) Is the gravitational deflection of light a natural prediction of Newtonian gravity? (1 pt)

- Newton's second law + Newton's law of universal gravitation $\rightarrow$ Bodies fall with the same acceleration rate subject to gravitational fields, just because the inertial and gravitational masses are equal.

- Therefore, according to Newton's theory one should expect the bending of light under the action of a massive object (even in the limit of massless particles).

- He considered light to be formed of very light corpuscles. Newton himself wrote: *Are not the rays of light very small bodies emitted from shining substances? and also, Do not Bodies act upon light at a distance, and by their action bend its rays, and is not this action (cæteris paribus) strongest at the least distance?*

(b) Consider Newton's gravity. Write down the analogous formulas to (11) obtained from the conservation of the angular momentum and energy of a particle that moves under the influence of the gravitational field caused by a spherical object of mas M.

$$\frac{dt}{d\lambda} = \frac{const.}{A(r)} \equiv \frac{C}{A(r)} \quad ; \quad r^2\frac{d\varphi}{d\lambda} = const \equiv J. \quad ; \quad B(r)\left(\frac{dr}{d\lambda}\right)^2 + \frac{J^2}{r^2} - \frac{1}{A(r)} = const \equiv -E\,, \quad (11)$$

- As gravity is a central force the angular momentum is conserved.
  - Motion in a plane $\rightarrow$: $\theta =$ constant, $\phi = \pi/2$ like in Ex.1.

$$\vec{v} = v_r\,\hat{r} + v_\varphi\,\hat{\varphi}$$

$$\vec{L} = mrv_\varphi\hat{\theta}, \text{ with } v_\varphi = r\frac{d\varphi}{dt}$$

The conservation of angular momentum tells $\boxed{mr^2\frac{d\varphi}{dt} = const. \equiv J}$

On the other hand, the (gravitational+kinetic) energy of the particle is also conserved, so

$$-\frac{GMm}{r} + \frac{1}{2}mv^2 = const. \equiv E \longrightarrow -\frac{GMm}{r} + \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\varphi}{dt}\right)^2\right] = E$$

$$\boxed{-\frac{GMm}{r} + \frac{1}{2}m\left[\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{m^2r^2}\right] = E}$$

(c) Write down the integral formula for the deflection angle that gives this angle as function of the distance to the source. (0.5 pt)

$$\boxed{mr^2\frac{d\varphi}{dt} = const. \equiv J} \xrightarrow{\text{to}} dt = mr^2 d\varphi/J \xrightarrow{\text{in}} E = -\frac{GMm}{r} + \frac{J^2}{2mr^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\varphi}\right)^2\right]$$

From it we can isolate $d\varphi$ and integrate with respect to the radial coordinate,

$$\boxed{\Delta\varphi = 2\int_{r_0}^{\infty} \frac{dr}{r\sqrt{\frac{2mr^2}{J^2}\left(E + \frac{GMm}{r}\right) - 1}}}$$

(d) Solve the latter by assuming that light can be described by corpuscles that have velocity c when they pass by the closest point to the source, r_0. (1.5 pt)

Using the energy and angular momentum conservation equations (evaluated both at $r = r_0$, where $v_r = 0$ and $v_\varphi = c$) we find:

$$E = -\frac{GMm}{r_0} + \frac{m}{2}v_\varphi^2(r_0) = -\frac{GMm}{r_0} + \frac{J^2}{2mr_0^2}$$

Let us use this relation in

$$\boxed{\Delta\varphi = 2\int_{r_0}^{\infty} \frac{dr}{r\sqrt{\frac{2mr^2}{J^2}\left(E + \frac{GMm}{r}\right) - 1}}}$$

We can write the deflection angle as follows,

$$\Delta\varphi = 2\int_{r_0}^{\infty} \frac{dr/r}{\sqrt{\frac{r^2}{r_0^2} - 1 + \frac{m^2 r_s r}{J^2}\left(1 - \frac{r}{r_0}\right)}}$$

Using $J = mr_0c$ we can rewrite the latter in a much compact way,

$$\Delta\varphi = 2\int_{r_0}^{\infty} \frac{r_0\,dr/r^2}{\sqrt{1 - \frac{r_0^2}{r^2} + \frac{r_s}{r}\left(1 - \frac{r}{r_0}\right)}}$$

As in exercise 7.1 we can now perform the change of variables $x = r_0/r$,

$$\Delta\varphi = 2\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1 - \frac{r_s}{r_0}\frac{1}{1+x}}} \simeq 2\int_0^1 \frac{dx}{\sqrt{1-x^2}}\left(1 + \frac{r_s}{2r_0}\frac{1}{1+x}\right)$$

$$\boxed{\Delta\varphi = \pi + \frac{r_s}{r_0}}$$

(e) Does this result coincide with the relativistic one? What is the deflection angle (in Newton's theory and GR) of a light ray subject to the gravitational field of the Sun if it passes very close to its surface? (0.5 pt)

→ The Schwarzschild radius of the Sun is $r_s \simeq 2.95$ km, and its radius $r_0 \simeq 6.957 \cdot 10^5$ km.

→ have already seen that the deflection angle predicted by Newton's theory does not coincide witht the one found in GR,

$$\begin{aligned} \Delta\varphi_{\mathrm{NG}} - \pi &= \frac{r_s}{r_0} = 0.87'' \\ \Delta\varphi_{\mathrm{GR}} - \pi &= 2\frac{r_s}{r_0} = 2(\Delta\varphi_{\mathrm{NG}} - \pi) = 1.74'' \end{aligned}$$

The deflection angle of light rays passing very close to the Sun was first measured by the two expeditions of 1919 led by Arthur Eddington and Frank Dyson. They measured $(1.61 \pm 0.40)''$ and $(1.98 \pm 0.16)''$, respectively, which were $\sim 2\sigma$ and $\sim 7\sigma$ away from the Newtonian value, and in agreement with the GR one at $< 1\sigma$ and $\sim 1.5\sigma$.

Bonus Question 1

Consider the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu}$ the Minkowski metric and $|h_{\mu\nu}| \ll 1$. Are the following statements true or false? Argue why.

(a) In a weak gravitational field the metric takes always this form. (0.5 pt)

False. Even in a purely flat spacetime the components of the metric tensor can be very different from the Minkowski ones depending on the coordinate system that we choose. For instance, if we use spherical coordinates in flat spacetime we will have

$$ds^2\Big|_{\text{flat}} = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + dr^2 + r^2\{d\theta^2 + \sin^2(\theta)d\varphi^2\} \longrightarrow g_{\mu\nu} \neq \eta_{\mu\nu}\,. \tag{36}$$

Thus, this statement does not hold neither in flat spacetime nor in the presence of weak gravitational fields.

(b) $h_{\mu\nu}$ is a tensor. (0.5 pt)

False. In order to determine whether $h_{\mu\nu}$ is or not a tensor we have to see how it transforms under a change of coordinates. Under a general transformation Λ we have:

$$\bar{g} = \Lambda^T g \Lambda = \Lambda^T (\eta + h)\Lambda = \Lambda^T \eta \Lambda + \Lambda^T h \Lambda = \eta + \bar{h}\,. \tag{37}$$

We can define $\Lambda \equiv \Lambda^{\mathrm{L}} + \tilde{\Lambda}$, with Λ^{L} being the Lorentz matrix. As $(\Lambda^{\mathrm{L}})^T \eta \Lambda^{\mathrm{L}} = \eta$ we find,

$$\bar{h} = \tilde{\Lambda}^T \eta \tilde{\Lambda} + \Lambda^T h \Lambda\,. \tag{38}$$

Thus, $h_{\mu\nu}$ does not transform as a tensor under a general change of coordinates (it only transforms as a tensor when $\Lambda = \Lambda^{\mathrm{L}}$, so when $\tilde{\Lambda} = 0$). Therefore, we must conclude that it is not a tensor.