Problem : Two flavor neutrino oscillation

Problem Statement (i)

In vacuum, the Hamiltonian for propagating neutrinos is simply the *free particle Hamiltonian*, with *eigenvalues* $|\nu_i\rangle$, corresponding to the *eigenstates* with mass m_i. For simplicity, consider the two neutrino case $i = 1, 2$. There exists another *basis*, called the *flavor basis* $|\nu_e\rangle, |\nu_\mu\rangle$, which diagonalizes the *interaction* operators :

- $|\nu_e\rangle$ interacts with electrons
- $|\nu_\mu\rangle$ interacts with muons

These two bases are related by a "rotation"θ :

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle - \sin\theta\,|\nu_2\rangle \tag{89}$$

$$|\nu_\mu\rangle = \sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle \tag{90}$$

Also note that neutrinos are relativistic with $E_i \gg m_i$, their kinetic energy for a fixed momentum is

$$E = \sqrt{p^2c^2 + m^2c^4} \simeq pc\left(1 + \frac{m^2c^2}{2p^2}\right). \tag{91}$$

Show that the **survival probability** of *electron neutrinos* propagating a distance $L = ct$ is given by,

$$P(\nu_e \to \nu_e) = 1 - \sin^2 2\theta \sin^2\left(\Delta m^2 c^4 \frac{L}{4E\hbar c}\right) \tag{92}$$

Solution

We can start by defining state vectors using two different bases, one in the mass (energy) eigenstates basis and one in the flavor eigenstates basis,

$$|\Psi\rangle = c_1|\nu_1\rangle + c_2|\nu_2\rangle = c_e|\nu_e\rangle + c_\mu|\nu_\mu\rangle \tag{93}$$

We assume these to be normalized, giving us

$$|c_1|^2 + |c_2|^2 = 1, \quad |c_e|^2 + |c_\mu|^2 = 1 \tag{94}$$

where c_1, c_2 are the amplitudes for detecting neutrinos in mass state 1 or 2. Similarly c_e, c_μ are the amplitudes for detecting an ν_e or ν_μ respectively.

c_1, c_2 are energy eigenstates for the free particle Hamiltonian. Hence, we can easily define their time evolution using the standard $\mathcal{U} = e^{-iEt}$ time evolution operator. This gives us evolution for these amplitudes as

$$c_1(t) = c_1(0)e^{-iE_1t}, \quad c_2(t) = c_2(0)e^{-iE_2t} \tag{95}$$

We have been given a rotation matrix such that,

$$\begin{bmatrix} |\nu_e\rangle \\ |\nu_\mu\rangle \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} |\nu_1\rangle \\ |\nu_2\rangle \end{bmatrix} \tag{96}$$

Which in terms of amplitudes gives us (by multiplying both sides of the previous equation with $\langle\Psi|$),

$$\begin{bmatrix} c_e(t) \\ c_\mu(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} \tag{97}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c_1(0)e^{-iE_1t} \\ c_2(0)e^{-iE_2t} \end{bmatrix} \tag{98}$$

The next important step would be to assume a boundary condition to solve these differential equations. Let us assume that at $t = 0$ an ν_e is what we have i.e.

$$c_e(0) = 1, \quad c_\mu(0) = 0. \tag{99}$$

If we invert eq. (97), we get

$$\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} c_e(t) \\ c_\mu(t) \end{bmatrix} \tag{100}$$

Plugging $t = 0$ and the initial conditions defined in eq.(99) this we can compute the initial condition on the other two amplitudes by,

$$c_1(0) = \cos\theta, \quad c_2(0) = -\sin(\theta) \tag{101}$$

Plugging this into eq. (98) we get,

$$\begin{bmatrix} c_e(t) \\ c_\mu(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta\, e^{-iE_1 t} \\ -\sin\theta\, e^{-iE_2 t} \end{bmatrix} \tag{102}$$

$$\tag{103}$$

Which gives us two equations,

1. For the time evolution of amplitude of $|\nu_e\rangle$,

$$c_e(t) = \cos^2\theta\, e^{-iE_1 t} + \sin^2\theta\, e^{-iE_2 t} \tag{104}$$

2. For the time evolution of amplitude of $|\nu_\mu\rangle$,

$$c_\mu(t) = \sin\theta\cos\theta\, e^{-iE_1 t} - \sin\theta\cos\theta\, e^{-iE_2 t} \tag{105}$$

$$= \sin\theta\cos\theta \left(e^{-iE_1 t} - e^{-iE_2 t}\right) \tag{106}$$

We can now use the energy momentum relations : $E_1^2 = p^2 + m_1^2$, $E_2^2 = p^2 + m_2^2$ and $LL = t$. The probability that we find an electron flavored neutrino or muon flavored neutrino is given by taking the squar of $c_e(t)$ or $c_\nu(t)$ respectively.

$$|c_e|^2 = 1 - \sin^2(2\theta)\sin^2\left(\frac{(E_2 - E_1)t}{2}\right) \tag{107}$$

$$= 1 - \sin^2(2\theta)\sin^2\left(\frac{(m_2^2 - m_1^2)L}{4E}\right) \tag{108}$$

$$= 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 L}{4E}\right) \tag{109}$$

where $E = p$.

Oops, I just realized that unknowingly I worked in $\hbar = c = 1$ units! No worries, reinstating the units we can get back the needed formula,

$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2 c^4 L}{4E\hbar c}\right) \tag{110}$$

where now, $E = pc$.

Problem Statement (ii)

1. Explain why the observation of neutrino oscillations implies that neutrinos have mass
2. Assuming the three-flavor case looks similar (it does), and knowing that we have detected oscillations between all three flavors, How many energy eigenstates *must* be massive?