Exercise 4.1

Exercise 4.1

(a) Show that if a tensor is symmetric in a concrete basis, it will remain symmetric in any other basis. (0.5 pt)

If $A_{\mu\nu}$ is symmetric in a concrete basis $\{\tilde{\omega}^{\mu}\}$ $\longrightarrow$ perform a transformation to $\{\tilde{\omega}^{\bar{\mu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}\tilde{\omega}^{\mu}\}$

elements of the tensor $\mathbf{A}$ will change as follows:

$$A_{\bar{\mu}\bar{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}A_{\mu\nu} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}A_{\nu\mu} = \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}A_{\nu\mu} = A_{\bar{\nu}\bar{\mu}}$$

(b) Prove that the contraction of a symmetric tensor S with an antisymmetric one A is 0. (0.5 pt)

$$A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\mu\nu} = -A_{\nu\mu}S^{\nu\mu} = -A_{\mu\nu}S^{\mu\nu} \longrightarrow A_{\mu\nu}S^{\mu\nu} = 0$$

Exercise 4.2

Exercise 4.2: Christoffel symbols and covariant derivative

(a) Show that ${\Gamma^\beta}_{\mu\alpha}$ are the components of the $\binom{1}{1}$ tensor $\nabla\vec{e}_\alpha$ for fixed α, and explain why we do not expect the Christoffel symbols to transform as tensors. (1 pt)

Let us consider a general vector, i.e. a general $\binom{1}{0}$ tensor, $\vec{A}$. $\rightarrow$ $\nabla\vec{A} = \nabla_\beta A^\gamma \, \tilde{\omega}^\beta \otimes \vec{e}_\gamma$

For a fixed $\alpha \to \delta^\beta_\alpha$ is a vector with dummy index β

Let us take now a vector $\vec{e}_\alpha = \delta^\beta_\alpha \vec{e}_\beta$. $\rightarrow$ $\nabla\vec{e}_\alpha = \nabla_\beta \delta^\gamma_\alpha \, \tilde{\omega}^\beta \otimes \vec{e}_\gamma$,

Computing the first term gives us $\rightarrow$ $\nabla_\beta \delta^\gamma_\alpha = \partial_\beta \delta^\gamma_\alpha + {\Gamma^\gamma}_{\mu\beta}\delta^\mu_\alpha = {\Gamma^\gamma}_{\alpha\beta}$

(b) Discover how each expression $V^{\beta}{}_{,\alpha}$ and $V^{\mu}\Gamma^{\beta}{}_{\mu\alpha}$ separately transforms under a change of coordinates. For $\Gamma^{\beta}{}_{\mu\alpha}$ you can begin with

$$\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\mu}{}_{\alpha\beta}\,\vec{e}_{\mu}\,. \tag{1}$$

Show that neither is the standard tensor law, but that the sum does obey the standard law. (2 pt)

Tranformation of the Christofel symbols

$$\frac{\partial \vec{e}_{\bar\alpha}}{\partial x^{\bar\beta}} = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\,\vec{e}_{\bar\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x^{\alpha}}{\partial x^{\bar\alpha}}\vec{e}_{\alpha}\right) = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\frac{\partial x^{\mu}}{\partial x^{\bar\mu}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial^2 x^{\alpha}}{\partial x^{\beta}\partial x^{\bar\alpha}}\vec{e}_{\alpha} + \frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial x^{\alpha}}{\partial x^{\bar\alpha}}\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\frac{\partial x^{\mu}}{\partial x^{\bar\mu}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial^2 x^{\alpha}}{\partial x^{\beta}\partial x^{\bar\alpha}}\vec{e}_{\alpha} + \frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial x^{\alpha}}{\partial x^{\bar\alpha}}\Gamma^{\mu}{}_{\alpha\beta}\,\vec{e}_{\mu} = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\frac{\partial x^{\mu}}{\partial x^{\bar\mu}}\vec{e}_{\mu}$$

In the first term of the left-hand side we can do $\alpha \to \mu$.

$$\frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x^{\bar\alpha}}\vec{e}_{\mu} + \frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial x^{\alpha}}{\partial x^{\bar\alpha}}\Gamma^{\mu}{}_{\alpha\beta}\,\vec{e}_{\mu} = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\frac{\partial x^{\mu}}{\partial x^{\bar\mu}}\vec{e}_{\mu}$$

$$\left(\frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x^{\bar\alpha}} + \frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial x^{\alpha}}{\partial x^{\bar\alpha}}\Gamma^{\mu}{}_{\alpha\beta}\right)\vec{e}_{\mu} = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\frac{\partial x^{\mu}}{\partial x^{\bar\mu}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial^2 x^{\mu}}{\partial x^{\beta}\partial x^{\bar\alpha}} + \frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial x^{\alpha}}{\partial x^{\bar\alpha}}\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\frac{\partial x^{\mu}}{\partial x^{\bar\mu}}$$

Multiplying both sides by $\frac{\partial x^{\bar\kappa}}{\partial x^{\mu}}$ we finally obtain:

$$\boxed{\frac{\partial x^{\bar\kappa}}{\partial x^{\mu}}\frac{\partial^2 x^{\mu}}{\partial x^{\bar\beta}\partial x^{\bar\alpha}} + \frac{\partial x^{\bar\kappa}}{\partial x^{\mu}}\frac{\partial x^{\beta}}{\partial x^{\bar\beta}}\frac{\partial x^{\alpha}}{\partial x^{\bar\alpha}}\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\bar\mu}{}_{\bar\alpha\bar\beta}\delta^{\bar\kappa}_{\bar\mu} = \Gamma^{\bar\kappa}{}_{\bar\alpha\bar\beta}}$$

Tranformation of $V^{\mu}\Gamma^{\beta}{}_{\mu\alpha}$

$$V^{\bar{\mu}}\Gamma^{\bar{\beta}}{}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}} V^{\mu} \left[\frac{\partial x^{\bar{\beta}}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x^{\bar{\mu}} \partial x^{\bar{\alpha}}} + \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} \frac{\partial x^{\theta}}{\partial x^{\bar{\mu}}} \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \Gamma^{\beta}{}_{\theta\alpha} \right]$$

$$V^{\bar{\mu}}\Gamma^{\bar{\beta}}{}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\bar{\beta}}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x^{\mu} \partial x^{\bar{\alpha}}} V^{\mu} + \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} \frac{\partial x^{\theta}}{\partial x^{\mu}} \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \Gamma^{\beta}{}_{\theta\alpha} V^{\mu}$$

$$\boxed{V^{\bar{\mu}}\Gamma^{\bar{\beta}}{}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\bar{\beta}}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x^{\mu} \partial x^{\bar{\alpha}}} V^{\mu} + \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \Gamma^{\beta}{}_{\mu\alpha} V^{\mu}}$$

Tranformation of $V^{\beta}{}_{,\alpha}$

$$V^{\bar{\beta}}{}_{,\bar{\alpha}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial}{\partial x^{\alpha}} \left(\frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} V^{\beta} \right)$$

$$\boxed{V^{\bar{\beta}}{}_{,\bar{\alpha}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\beta}}}{\partial x^{\alpha} \partial x^{\beta}} V^{\beta} + \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} V^{\beta}{}_{,\alpha}}$$

$$V^{\bar{\beta}}{}_{,\bar{\alpha}} + V^{\bar{\mu}}\Gamma^{\bar{\beta}}{}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\beta}}}{\partial x^{\alpha} \partial x^{\beta}} V^{\beta} + \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} V^{\beta}{}_{,\alpha} + \frac{\partial x^{\bar{\beta}}}{\partial x^{\sigma}} \frac{\partial^2 x^{\sigma}}{\partial x^{\mu} \partial x^{\bar{\alpha}}} V^{\mu} + \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \Gamma^{\beta}{}_{\mu\alpha} V^{\mu}$$

$\sigma \to \alpha$ and $\mu \to \beta$

$$V^{\bar{\beta}}{}_{,\bar{\alpha}} + V^{\bar{\mu}}\Gamma^{\bar{\beta}}{}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\beta}}}{\partial x^{\alpha} \partial x^{\beta}} V^{\beta} + \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} V^{\beta}{}_{,\alpha} - \frac{\partial^2 x^{\bar{\beta}}}{\partial x^{\beta} \partial x^{\alpha}} \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} V^{\beta} + \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \Gamma^{\beta}{}_{\mu\alpha} V^{\mu}$$

$$V^{\bar{\beta}}{}_{,\bar{\alpha}} + V^{\bar{\mu}}\Gamma^{\bar{\beta}}{}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} V^{\beta}{}_{,\alpha} + \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \Gamma^{\beta}{}_{\mu\alpha} V^{\mu}$$

$$\boxed{V^{\bar{\beta}}{}_{,\bar{\alpha}} + V^{\bar{\mu}}\Gamma^{\bar{\beta}}{}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^{\beta}} \left(V^{\beta}{}_{,\alpha} + \Gamma^{\beta}{}_{\mu\alpha} V^{\mu} \right)}$$

Therefore, the covariant derivative $V^{\beta}{}_{;\alpha} = V^{\beta}{}_{,\alpha} + \Gamma^{\beta}{}_{\mu\alpha} V^{\mu}$ is a tensor (of type $\binom{1}{1}$) because it transforms like tensors do.

Exercise 4.3

Exercise 4.3

Consider the Euclidean space, with metric tensor

$$\mathbf{g} = \tilde{d}x \otimes \tilde{d}x + \tilde{d}y \otimes \tilde{d}y + \tilde{d}z \otimes \tilde{d}z \,. \tag{2}$$

(a) Compute the elements of the metric tensor in spherical coordinates, and use them to compute the differential *proper* volume dV and the squared line element ds^2. (1 pt)

$$\Lambda = \begin{pmatrix} \sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix}$$

$$x^\mu = (x, y, z) \text{ and } x^{\bar{\mu}} = (r, \theta, \varphi)$$

$$\begin{aligned} x &= r\sin(\theta)\cos(\varphi)\,, \\ y &= r\sin(\theta)\sin(\varphi)\,, \\ z &= r\cos(\theta)\,. \end{aligned}$$

$$\longrightarrow \quad g_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}}\frac{\partial x^\nu}{\partial x^{\bar{\nu}}} g_{\mu\nu} \quad \xrightarrow{\Lambda^\nu{}_{\bar{\nu}} = \frac{\partial x^\nu}{\partial x^{\bar{\nu}}}} \quad \bar{g} = \Lambda^T g \Lambda$$

$$\bar{g} = \Lambda^T \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{pmatrix} \longrightarrow \mathbf{g} = \tilde{d}r \otimes \tilde{d}r + r^2[\tilde{d}\theta \otimes \tilde{d}\theta + \sin^2(\theta)\tilde{d}\varphi \otimes \tilde{d}\varphi]$$

Squared line element $\Rightarrow$

$$ds^2 = d\vec{l} \cdot d\vec{l} = \mathbf{g}(d\vec{l}, d\vec{l}) = g_{\bar{\mu}\bar{\nu}} dx^{\bar{\mu}} dx^{\bar{\nu}} = dr^2 + r^2[d^2\theta + \sin^2(\theta)d\varphi^2]$$

Differential proper volume $\Rightarrow$

$$dV = \sqrt{-\bar{g}}\; dr\, d\theta\, d\varphi = r^2\sin(\theta)\, dr\, d\theta\, d\varphi$$

(b) Write the one-form $\tilde{d}f$ in spherical coordinates. (0.5 pt)

$$\tilde{d}f = \frac{\partial f}{\partial x^{\bar{\mu}}}\tilde{d}x^{\bar{\mu}} = \frac{\partial f}{\partial r}\tilde{d}r + \frac{\partial f}{\partial \theta}\tilde{d}\theta + \frac{\partial f}{\partial \varphi}\tilde{d}\varphi$$

(c) Compute the associated gradient vector in spherical coordinates and in terms of the unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$. (1 pt)

$$\tilde{d}f = \mathbf{g}(\vec{\nabla} f, \tilde{d}x) = g_{\bar{\mu}\bar{\nu}}(\vec{\nabla} f)^{\bar{\mu}}\tilde{d}x^{\bar{\nu}} = \frac{\partial f}{\partial x^{\bar{\nu}}}\tilde{d}x^{\bar{\nu}}$$

$$g_{\bar{\mu}\bar{\nu}}(\vec{\nabla} f)^{\bar{\mu}} = \frac{\partial f}{\partial x^{\bar{\nu}}} \longrightarrow (\vec{\nabla} f)^{\bar{\mu}} = g^{\bar{\mu}\bar{\nu}}\frac{\partial f}{\partial x^{\bar{\nu}}}$$

$$\begin{pmatrix} (\vec{\nabla} f)^r \\ (\vec{\nabla} f)^\theta \\ (\vec{\nabla} f)^\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}(\theta) \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \varphi} \end{pmatrix}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r^2\sin^2(\theta)}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi \,.$$

$$\begin{aligned} \vec{e}_r &= |\vec{e}_r|\hat{e}_r = \sqrt{\mathbf{g}(\vec{e}_r, \vec{e}_r)}\;\hat{e}_r = \sqrt{g_{rr}}\;\hat{e}_r = \hat{e}_r \,, \\ \vec{e}_\theta &= \ldots = \sqrt{g_{\theta\theta}}\;\hat{e}_\theta = r\hat{e}_\theta \,, \\ \vec{e}_\varphi &= \ldots = \sqrt{g_{\varphi\varphi}}\;\hat{e}_\varphi = r\sin(\theta)\hat{e}_\varphi \,. \end{aligned}$$

Unit vectors

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi$$

(d) Express the basis vectors $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ in terms of $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, and viceversa. (0.75 pt)

$$\vec{e}_{\bar{\mu}} = \Lambda_{\bar{\mu}}{}^{\mu} \vec{e}_{\mu}.$$

$$\begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\varphi) & \sin(\theta)\sin(\varphi) & \cos(\theta) \\ r\cos(\theta)\cos(\varphi) & r\cos(\theta)\sin(\varphi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix}$$

Inverse

$$\vec{e}_{\mu} = \Lambda_{\mu}{}^{\bar{\mu}} \vec{e}_{\bar{\mu}}$$

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\varphi) & \frac{1}{r}\cos(\theta)\cos(\varphi) & -\frac{\sin(\varphi)}{r\sin(\theta)} \\ \sin(\theta)\sin(\varphi) & \frac{1}{r}\cos(\theta)\sin(\varphi) & \frac{\cos(\varphi)}{r\sin(\theta)} \\ \cos(\varphi) & -\frac{\sin(\theta)}{r} & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix}$$

(e) Using (1) compute the Chistoffel symbols $\Gamma^{\bar{\mu}}{}_{rr}$ and $\Gamma^{\bar{\mu}}{}_{r\theta}$. (0.75 pt)

Let us start with the computation of $\Gamma^{\bar{\mu}}{}_{r\theta}$ $\longrightarrow$ $\boxed{\dfrac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}{}_{r\theta}\, \vec{e}_{\bar{\mu}}\,.}$

Use $\vec{e}_r$ result from the previous slide, and differentiate it component-wise

$$\begin{aligned}
\frac{\partial \vec{e}_r}{\partial \theta} &= \cos(\theta)\cos(\varphi)\vec{e}_x + \cos(\theta)\sin(\varphi)\vec{e}_y - \sin(\theta)\vec{e}_z \\
&= \cos(\theta)\cos(\varphi)\left[\sin(\theta)\cos(\varphi)\vec{e}_r + \frac{1}{r}\cos(\theta)\cos(\varphi)\vec{e}_\theta - \frac{\sin(\varphi)}{r\sin(\theta)}\vec{e}_\varphi\right] + \\
&+ \cos(\theta)\sin(\varphi)\left[\sin(\theta)\sin(\varphi)\vec{e}_r + \frac{1}{r}\cos(\theta)\sin(\varphi)\vec{e}_\theta + \frac{\cos(\varphi)}{r\sin(\theta)}\vec{e}_\varphi\right] - \sin(\theta)\left[\cos(\theta)\vec{e}_r - \frac{\sin(\theta)}{r}\vec{e}_\theta\right] = \dots
\end{aligned}$$

After simplification we get

$$\boxed{\frac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}{}_{r\theta}\,\vec{e}_{\bar{\mu}} = \frac{\vec{e}_\theta}{r}\,.} \quad \Longrightarrow \quad \boxed{\Gamma^{r}{}_{r\theta} = \Gamma^{\varphi}{}_{r\theta} = 0} \qquad \boxed{\Gamma^{\theta}{}_{r\theta} = \frac{1}{r}}$$

The computation of $\Gamma^{\bar{\mu}}{}_{rr}$ is straightforward.

$$\boxed{\frac{\partial \overrightarrow{e}_r}{\partial r} = 0 = \Gamma^{\bar{\mu}}{}_{rr} \overrightarrow{e}_{\bar{\mu}} \longrightarrow \boxed{\Gamma^{\bar{\mu}}{}_{rr} = 0}}$$

(f) Compute all the Christoffel symbols in spherical coordinates using the formula

$$\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}} = \frac{g^{\bar{\mu}\bar{\sigma}}}{2}\left(g_{\bar{\sigma}\bar{\alpha},\bar{\beta}} + g_{\bar{\sigma}\bar{\beta},\bar{\alpha}} - g_{\bar{\alpha}\bar{\beta},\bar{\sigma}}\right). \qquad (1\,\text{pt})$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\varphi}{}_{r\varphi} = \frac{1}{r} \qquad \Gamma^{r}{}_{\theta\theta} = -r \qquad \Gamma^{r}{}_{\varphi\varphi} = -r\sin^2(\theta)$$

$$\Gamma^{\theta}{}_{\varphi\varphi} = -\frac{1}{2}\sin(2\theta) \quad \Gamma^{\varphi}{}_{\varphi\theta} = \frac{\cos(\theta)}{\sin(\theta)}$$

All non-zero Christoffel symbols

(g) Compute the Laplacian of the function f, i.e. $\nabla^2 f$, in spherical coordinates. (1 pt)

$$\nabla^2 f = \mathbf{g}(\vec{\nabla}, \vec{\nabla} f) = g_{\bar{\mu}\bar{\nu}} \vec{\nabla}^{\bar{\mu}} (\vec{\nabla} f)^{\bar{\nu}} = \nabla_{\bar{\mu}} (\vec{\nabla} f)^{\bar{\mu}} = \partial_{\bar{\mu}} (\vec{\nabla} f)^{\bar{\mu}} + \Gamma^{\bar{\mu}}{}_{\bar{\kappa}\bar{\mu}} (\vec{\nabla} f)^{\bar{\kappa}}$$

Sum over the indices

$$\nabla^2 f = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r^2}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r^2 \sin^2(\theta)}\frac{\partial f}{\partial \varphi}\right) + \Gamma^{\theta}{}_{r\theta}\frac{\partial f}{\partial r} + \Gamma^{\varphi}{}_{r\varphi}\frac{\partial f}{\partial r} + \Gamma^{\varphi}{}_{\theta\varphi}\frac{1}{r^2}\frac{\partial f}{\partial \varphi}$$

Use the Christoffel symbols we calculated before

$$\boxed{\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \varphi^2}}$$