GR Exercise sheet 11 Name: Vito Aberham

Nr. 11.1)

(a) CMB temperature $T_\gamma^{(0)} = 2.7255\,K$

Assume black-body spectrum (as it has been measured by Planck): $B_\lambda(\lambda, T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1}$

Peak of energy spectrum ⇒ Wien's displacement law:

$$\lambda_{max} = \frac{2898\,\mu m\,K}{T} = \frac{2898\,\mu m\,K}{2.7255\,K} \approx 1063\,\mu m \approx 1.1\,mm$$

1.1 mm ∈ microwave range ✓

Current energy density of photons

Energy density in Statistical Physics:

$$\rho(t, \vec{x}) = g \int \frac{d^3p}{(2\pi\hbar)^3} f(t, \vec{x}, \vec{p})\, E(t, \vec{x}, \vec{p}) \quad g: \text{degeneracy factor}$$

occupation number $f(t, \vec{x}, \vec{p}) = \frac{1}{\exp\left(\frac{E-\mu}{k_B T}\right) - 1}$ for bosons

Ultra-relativistic case: $\mu = 0$ (chem. potential) & thermal eq.

$$E = \sqrt{m^2c^4 + p^2c^2} \approx \sqrt{p^2c^2} = pc \quad \Rightarrow \quad p = \frac{E}{c}$$

$$\Rightarrow \rho(t, \vec{x}) = \frac{g}{(2\pi\hbar)^3} \int d^3p \frac{E}{e^{E/k_BT} - 1} = \frac{g}{(2\pi\hbar)^3} \int_0^\infty \int_0^{2\pi} \int_0^\pi dp \sin\vartheta\, d\vartheta\, d\varphi \frac{p^2 E}{e^{E/k_BT} - 1}$$

$$= \frac{g}{(2\pi\hbar)^3} \int_0^\infty \int_0^{2\pi} \int_0^\pi \sin\vartheta\, d\varphi\, d\vartheta \frac{E^2 E}{(e^{E/k_BT} - 1)c^3} dE \qquad \Big|\ dp = \frac{1}{c} dE$$

$$= \frac{g\, 4\pi}{(2\pi\hbar c)^3} \int_0^\infty \frac{E^3\, dE}{e^{E/k_BT} - 1} \qquad \text{Substitute } u = \frac{E}{k_B T} \Rightarrow \frac{du}{dE} = \frac{1}{k_B T}$$

$$= \frac{4\pi g}{(2\pi\hbar c)^3} \int_0^\infty (k_B T)^4 \frac{u^3\, du}{e^u - 1} = \frac{4\pi g (k_B T)^4}{(2\pi\hbar c)^3} \underbrace{\int_0^\infty \frac{u^3\, du}{e^u - 1}}_{= \Gamma(4)\zeta(4) = 6 \cdot \frac{\pi^4}{90} = \frac{\pi^4}{15}}$$

$$= \frac{4\pi g\, \pi^4\, k_B^4 T^4}{(2\pi\hbar c)^3 \cdot 15} = \frac{\pi^2}{30} g T^4 k_B^4 \cdot \frac{1}{(\hbar c)^3}$$

For photons $g = 2$ (two polarizations) $\Rightarrow \rho_\gamma = \frac{\pi^2}{15} \frac{(k_B T)^4}{(\hbar c)^3}$

$$\Rightarrow \rho_\gamma^{(0)} = \frac{\pi^2}{15} \frac{(k_B T_\gamma^{(0)})^4}{(\hbar c)^3} = \frac{\pi^2}{15} \frac{(1.38 \cdot 10^{-23} \frac{J}{K} \cdot 2.7255\,K)^4}{(\frac{1}{2\pi} \cdot 6.626 \cdot 10^{-34}\, Js \cdot 3 \cdot 10^8 \frac{m}{s})^3}$$

$$\approx 4.16 \cdot 10^{-14} \frac{J}{m^3}$$

In Planck units: $E_P = 1.22 \cdot 10^{19}\,GeV,\ L_P = 1.61 \cdot 10^{-33}\,cm$

$$\rho_\gamma^{(0)} = 4.16\cdot 10^{-14}\,\frac{(1.602\cdot 10^{-19})^{-1}}{1.22\cdot 10^{28}}\,\frac{E_p}{\left(\frac{10^2}{1.61\cdot 10^{-33}}\right)^3 L_p^3} \qquad E_p = L_p^{-1}$$

$$= 8.8\cdot 10^{-128}\,E_p^4$$

$$= \underline{1.9\cdot 10^{-51}\,\mathrm{GeV}^4}$$

$\boxed{(b)}$ $\Omega_m^{(0)} = 0.314 \qquad H_0 = 67.36\,\frac{km}{s\cdot Mpc} \overset{\text{"}}{=} 2.26\cdot 10^{-18}\,s^{-1} \qquad \rho_r(z) = 1.69\,\rho_\gamma(z)$

$$H^2 = \frac{8\pi}{3}\rho \qquad \rho = \rho_m + \rho_r + \rho_\Lambda + \underbrace{\rho_K}_{=0 \text{ because } 0 \text{ curvature}} = \rho_{m,0}\,a^{-3} + \rho_{r,0}\,a^{-4} + \rho_{\Lambda,0}$$

Critical density : $\rho_c = \frac{3H^2}{8\pi} \qquad \rho_{c,0} = \frac{3H_0^2}{8\pi}\ \left(= \frac{3H_0^2c^2}{8\pi G}\right)$ in SI

$$\Rightarrow H^2 = \frac{8\pi}{3}(\rho_{m,0}\,a^{-3} + \rho_{r,0}\,a^{-4} + \rho_{\Lambda,0}) = \frac{8\pi\rho_{c,0}}{3}\left(\frac{\rho_{m,0}}{\rho_{c,0}}a^{-3} + \frac{\rho_{r,0}}{\rho_{c,0}}a^{-4} + \frac{\rho_{\Lambda,0}}{\rho_{c,0}}\right)$$

$$\Omega = \frac{\rho}{\rho_c} \searrow \ = \frac{8\pi}{3}\rho_{c,0}\left(\Omega_m^{(0)}a^{-3} + \Omega_r^{(0)}a^{-4} + \Omega_\Lambda^{(0)}\right) = \frac{8\pi}{3}\frac{3H_0^2}{8\pi}\left(\Omega_m^{(0)}a^{-3} + \Omega_r^{(0)}a^{-4} + \Omega_\Lambda^{(0)}\right)$$

$$= H_0^2\left(\Omega_m^{(0)}a^{-3} + \Omega_r^{(0)}a^{-4} + \Omega_\Lambda^{(0)}\right) \qquad a = \frac{1}{1+z}$$

$$= H_0^2\left(\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}\right)$$

$$= \frac{8\pi}{3}\rho(z) = H_0^2\,\frac{\rho(z)}{\rho_c(z)} = H_0^2\,\Omega(z)$$

$$\Rightarrow \Omega(z) = \frac{H^2}{H_0^2} \qquad \text{Components: } \Omega_m(z) = \frac{\rho_m(z)}{\rho_c(z)} = \frac{\rho_m(z)}{3H^2/8\pi}$$

$$= \frac{\cancel{\frac{8\pi}{3}}\rho_m^{(0)}(1+z)^3}{\cancel{\frac{8\pi}{3}}\rho_c^{(0)}\left(\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}\right)}$$

$$= \frac{\Omega_m^{(0)}(1+z)^3}{\left(\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}\right)}$$

$$\Rightarrow \Omega_r(z) = \frac{\Omega_r^{(0)}(1+z)^4}{\left(\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}\right)}$$

$$\Omega_\Lambda(z) = \frac{\Omega_\Lambda^{(0)}}{\left(\Omega_m^{(0)}(1+z)^3 + \Omega_r^{(0)}(1+z)^4 + \Omega_\Lambda^{(0)}\right)}$$

We know $\Omega_m^{(0)} = 0.314 \qquad \Omega_r^{(0)} = \frac{\rho_r^{(0)}}{\rho_c^{(0)}} = \frac{1.69\,\rho_\gamma^{(0)}}{\rho_c^{(0)}}$

$$\rho_\gamma^{(0)} \overset{11.1)}{=} 4.16\cdot 10^{-14}\,\frac{J}{m^3}$$

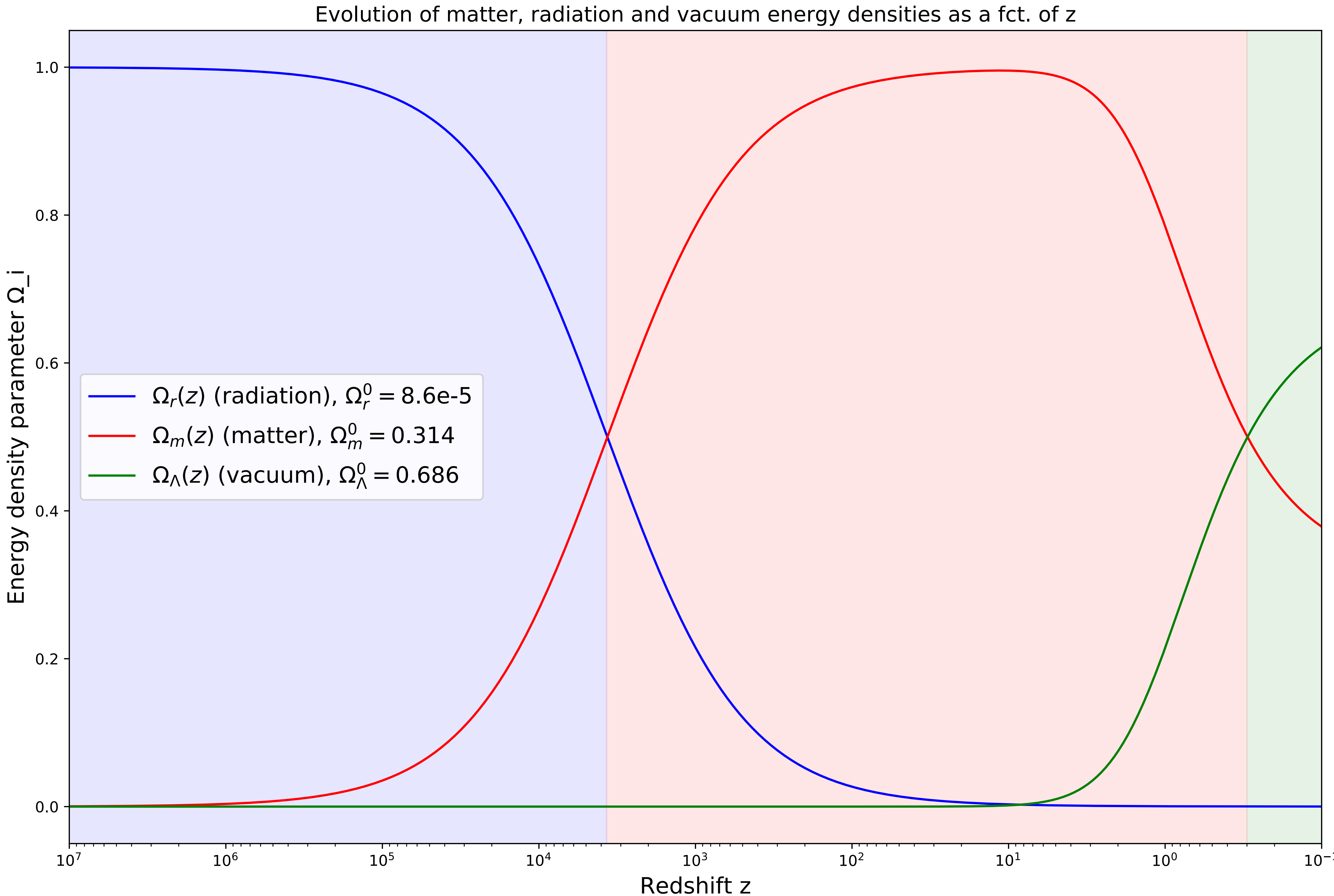
$$\rho_c^{(0)} = \frac{3H_0^2c^2}{8\pi G} = \frac{3\cdot(2.26\cdot 10^{-18}s^{-1})^2(3\cdot 10^8\,\frac{m}{s})^2}{8\pi\cdot 6.67\cdot 10^{-11}\,m^3\,kg^{-1}\,s^{-2}} = 8.2\cdot 10^{-10}\,\frac{J}{m^3}$$

$$\Rightarrow \Omega_r^{(0)} = \frac{1.69\cdot 4.16\cdot 10^{-14}\,\frac{J}{m^3}}{8.2\cdot 10^{-10}\,\frac{J}{m^3}} \simeq 8.6\cdot 10^{-5}$$

$k=0$ (zero curvature) $\Leftrightarrow \rho = \rho_c \Leftrightarrow \Omega = 1 \Rightarrow \Omega_\Lambda^{(0)} = 1 - \Omega_m^{(0)} - \Omega_r^{(0)}$

$$= 1 - 0.314 - 8.6\cdot 10^{-5} \simeq 0.686$$

Plot of the $\Omega_i(z)$ is on the next page:



Evolution of matter, radiation and vacuum energy densities as a fct. of z
Energy density parameter Ω_i
Redshift z
$\Omega_r(z)$ (radiation), $\Omega_r^0 = 8.6\text{e-}5$
$\Omega_m(z)$ (matter), $\Omega_m^0 = 0.314$
$\Omega_\Lambda(z)$ (vacuum), $\Omega_\Lambda^0 = 0.686$
1.0
0.8
0.6
0.4
0.2
0.0
10^7
10^6
10^5
10^4
10^3
10^2
10^1
10^0
10^{-1}

One can see that initially the universe was matter-dominated. After that there was a radiation-dominated epoch, lately followed by the increasing fraction of dark energy. The present-day domination of dark energy accounts for the current period of accelerated expansion.

Nr. 11.2)

(a) Friedmann + pressure eq.: $3H^2 = 8\pi G \rho$ (I)

$$3H^2 + 2\dot H = -8\pi G p \quad \text{(II)}$$

$$H^2 = \frac{8\pi G}{3}\rho \qquad H = \frac{\dot a}{a}, \quad \dot H = \frac{\ddot a a - \dot a^2}{a^2} = \frac{\ddot a}{a} - H^2$$

(II)

$$\Rightarrow 3H^2 + 2\left(\frac{\ddot a}{a}\right) - 2H^2 = H^2 + 2\frac{\ddot a}{a} = -8\pi G p$$

$$\Rightarrow \frac{\ddot a}{a} = -4\pi G p - \frac{1}{2}H^2 = -4\pi G p - \frac{1}{2}\frac{8\pi G}{3}\rho$$

$$= -\frac{8\pi G}{3}\left(\frac{3p}{2} + \frac{1}{2}\rho\right) = -\frac{4\pi G}{3}(\rho + 3p) = \frac{\ddot a}{a}$$

This is the acceleration equation.

(b) Conservation equation $\dot\rho = -3H(\rho + p)$

1) Differentiate (I) wrt time:

$$\frac{d}{dt}(3H^2) = 8\pi G \frac{d}{dt}\rho \Leftrightarrow 2H\dot H = \frac{8\pi G}{3}\dot\rho$$

$$\Leftrightarrow \dot\rho = \frac{3}{4\pi G} H \dot H = \frac{3}{4\pi G} H\left(\frac{-3H^2}{2} - \frac{8\pi G p}{2}\right)$$

$$= \frac{-9}{8\pi G}H^3 - 3pH = \frac{3}{4\pi G}\left(-3H^3 - \frac{1}{2}pH\right)$$

2) Insert conservation equation:

$$-3H(\rho + p) = -\frac{9}{8\pi G}H^3 - 3pH \qquad | :H$$

$$-3(\rho + p) = -\frac{9}{8\pi G}H^2 - 3p$$

$$\rho + p = +\frac{3}{8\pi G}H^2 + p \qquad | \; 3H^2 = 8\pi G\rho$$

3) Insert ~~[illegible]~~ cons. equ. again in step II

to $\dot H = \frac{\ddot a}{a} - H^2 \Rightarrow \dot\rho = \frac{3}{4\pi G} H\left(\frac{\ddot a}{a} - H^2\right)$

$$\Leftrightarrow -3H(\rho + p) = \frac{3}{4\pi G}\left(H\frac{\ddot a}{a} - H^3\right) \Leftrightarrow \rho + p = \frac{1}{4\pi G}\left(-\frac{\ddot a}{a} + \underbrace{H^2}_{\frac{8\pi G}{3}\rho}\right)$$

$\Rightarrow \quad \rho + p = \frac{1}{4\pi G}\left(-\frac{\ddot{a}}{a} + \frac{8\pi G}{3}\rho\right)$

$\Rightarrow \quad \frac{\ddot{a}}{a} = -4\pi G(\rho + p) + \frac{8\pi G}{3}\rho = -\frac{4\pi G}{3}\left(3\rho + 3p - 2\rho\right)$

$= -\frac{4\pi G}{3}(\rho + 3p)$ This is the acceleration eq.

$\Rightarrow$ The three equations are not independent from each other //

$\boxed{\text{c}}$ EOS parameter $w = \frac{p}{\rho}$

$\rho = \frac{E}{V}$ energy density

$\rho = \sum_i \rho_i \qquad i \in \underset{\text{rad.}}{r}, \overset{\text{matter}}{m}, \underset{\text{curvature}}{k}, \overset{\text{dark energy}}{\Lambda}$

$= \frac{1}{V}\sum_i E_i$

$$w = \frac{p}{\rho} = \frac{\sum_i p_i}{\frac{1}{V}\sum_i E_i} = \frac{V\sum_i p_i}{\sum_i E_i} = \frac{V\sum_i p_i}{E} = \frac{V\sum_i p_i \frac{E_i}{E_i}}{E}$$

$$= \sum_i \frac{V}{E_i} p_i E_i \frac{1}{E} = \sum_i \rho_i^{-1} p_i E_i \frac{1}{E} = \sum_i \frac{p_i}{\rho_i}\frac{E_i}{E} = \sum_i w_i \frac{E_i}{E}$$

w_i = EOS parameter for each component

$E_i/E = \varepsilon_i$ = energy fraction of each component

$$\Rightarrow \quad \underline{\underline{w = \sum_i w_i \varepsilon_i}}$$

$\boxed{\text{d}}$ Universe with positive acceleration: $\ddot{a} > 0$

Acceleration eq.: $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \overset{!}{>} 0$

$\overset{a(t)>0}{\Longrightarrow} \quad \rho + 3p \overset{!}{<} 0 \iff \rho < -3p \iff$ [redacted] $1 < -3\frac{p}{\rho} = -3w$

$\Rightarrow \quad 1 < -3w \Rightarrow \underline{\underline{w < -\frac{1}{3}}}$

Universe only filled with pressureless matter & radiation:

$H^2 = H_0^2\left(\Omega_r^{(0)} a^{-4} + \Omega_m^{(0)} a^{-3}\right) \qquad (\rho \sim a^{-3(1+w)})$

$(c) \Rightarrow \quad w = \underbrace{w_r}_{=\frac{1}{3}} \varepsilon_r + w_m \varepsilon_o = \frac{1}{3}\varepsilon_r > 0$ because $\varepsilon_r > 0$

$\llcorner$ pressureless, NR matter $w_m = 0$

Radiation energy is > 0. $\Rightarrow$ Such a universe cannot fulfill this condition because this doesn't represent a case for a fluid with negative pressure.

$\Rightarrow \quad \ddot{a} < 0$ //

(e) Positive acceleration $\Rightarrow$ add fluid with negative pressure. $w_\Lambda = -1 \Rightarrow p_\Lambda = -\rho_\Lambda$

$\Rightarrow$ energy momentum tensor: $T^{\mu\nu} = (\rho + p) u^\mu u^\nu - p g^{\mu\nu}$

$$T^{\mu\nu}_{(\Lambda)} = (\rho_\Lambda + p_\Lambda) u^\mu u^\nu - p_\Lambda g^{\mu\nu} = \cancel{(\rho_\Lambda - \rho_\Lambda)} u^\mu u^\nu - p_\Lambda g^{\mu\nu}$$
$$= \rho_\Lambda g^{\mu\nu}$$

It is $\rho_\Lambda = \rho_0^{(\Lambda)} a^{-3(1+w)} = \rho_0^{(\Lambda)} a^{\overbrace{-3(1-1)}^{=0}} = \rho_\Lambda^{(0)} \cdot 1$

$\Rightarrow$ The energy density of this fluid stays <u>constant</u> during cosmological expansion: $\rho_\Lambda(a) = \rho_\Lambda^{(0)} = \frac{\Lambda}{8\pi G}$

$\Rightarrow T^{\mu\nu} = \frac{\Lambda}{8\pi G} g^{\mu\nu}$

(f) 1st law of thermodynamics: $dU = -p\,dV + dQ + \mu dN$

$\mu dN = 0$ because of no lost particles

$dQ = 0$: adiabatic condition (sufficiently slow expansion)

$\Rightarrow dU = -p\,dV = -w\rho\,dV = +\rho\,dV$

$\Rightarrow$ If the volume increases due to the expansion $V' = Va^3$ the energy U also increases. $U' = \rho V' = \rho V a^3$

$\rightarrow d(\rho V) = \rho\,dV \Rightarrow$ constant energy density in space and time $\Rightarrow$ this cosmological component with $w = -1$ behaves as expected for vacuum

(g) Matter-Λ-dominated epoch

$\Rightarrow H^2 = H_0^2 (\Omega_m^{(0)} a^{-3} + \Omega_\Lambda^{(0)})$ $\qquad \rho = \rho_m + \rho_\Lambda$

Pressure equation: $3H^2 + 2\dot{H} = -8\pi G p$

$$= -8\pi G \sum_i w_i \rho_i$$

$$= -8\pi G \Big(\underbrace{w_m}_{=0} \rho_m + \underbrace{w_\Lambda}_{=-1} \overbrace{\rho_\Lambda}^{\text{const.}\ =\frac{\Lambda}{8\pi G}} \Big) = +8\pi G \frac{\Lambda}{8\pi G} = \Lambda$$

$$\Rightarrow 3H^2 + 2\dot{H} - \Lambda = 0 \Leftrightarrow \frac{dH}{dt} = -\frac{3}{2} H^2 + \frac{1}{2}\Lambda$$

$$\Lambda = 8\pi G \rho_\Lambda = 8\pi G \Omega_\Lambda^{(0)} \rho_{cr}^{(0)} = \cancel{8\pi G} \Omega_\Lambda^{(0)} \frac{3H_0^2}{\cancel{8\pi G}} = 3H_0^2 \Omega_\Lambda^{(0)}$$

$$\Rightarrow \frac{dH}{-\frac{3}{2}H^2 + \frac{3}{2}H_0^2\Omega_\Lambda^{(0)}} = dt \Leftrightarrow \frac{dH}{H_0^2\Omega_\Lambda^{(0)} - H^2} = \frac{3}{2}dt$$

$$\Rightarrow \frac{1}{H_0^2\Omega_\Lambda^{(0)}} \frac{dH}{1 - (H/H_0\sqrt{\Omega_\Lambda^{(0)}})^2} = \frac{3}{2}dt$$

$$\Rightarrow \frac{1}{H_0^2\Omega_\Lambda^{(0)}} \int_{H(t_0)}^{H} \frac{dH'}{1-(H'/H_0\sqrt{\Omega_\Lambda^{(0)}})^2} = \frac{3}{2}\int_{t_0}^{t} dt' = \frac{3}{2}(t-\underbrace{t_0}_{=0})$$

Substitute $x' = \frac{H'}{H_0\sqrt{\Omega_\Lambda^{(0)}}} \Rightarrow dH' = dx' \cdot H_0\sqrt{\Omega_\Lambda^{(0)}}$

$$\Rightarrow \frac{1}{H_0^2\Omega_\Lambda^{(0)}} \cdot H_0\sqrt{\Omega_\Lambda^{(0)}} \underbrace{\int_{x_0}^{x} \frac{dx'}{1-x'^2}}_{} = \frac{3}{2}t$$

$$= \begin{cases} \operatorname{arcoth}(x)\big|_{x_0}^{x} & |x|>1 \\ \operatorname{artanh}(x')\big|_{x_0}^{x} & |x|<1 \end{cases}$$

$\frac{H(t)}{H_0\sqrt{\Omega_\Lambda^{(0)}}} > 1$ because $H > H_0$ in the past and $\sqrt{\Omega_\Lambda^{(0)}} < 1$

$x_0 = \frac{H(t_0)}{H_0\sqrt{\Omega_\Lambda^{(0)}}} \to \infty$ because $H(0)\to\infty$, $a(0)\to 0$
and $H^2(t) = H_0^2(\Omega_m^{(0)}a^{-3} + \Omega_\Lambda^{(0)})$

$$\Rightarrow \int_{\infty}^{x(H(t))} \frac{dx'}{1-x'^2} = \left[\operatorname{arcoth}(x')\right]_{\infty}^{x(H(t))} = \operatorname{arcoth}\left(\frac{H(t)}{H_0\sqrt{\Omega_\Lambda^{(0)}}}\right) - \underbrace{\operatorname{arcoth}(\infty)}_{\to 0}$$

$$\Rightarrow \operatorname{arcoth}\left(\frac{H(t)}{H_0\sqrt{\Omega_\Lambda^{(0)}}}\right) = \frac{H_0^2\Omega_\Lambda^{(0)}}{H_0\sqrt{\Omega_\Lambda^{(0)}}}\frac{3}{2}t = \frac{3}{2}H_0\sqrt{\Omega_\Lambda^{(0)}}\,t$$

$$\Rightarrow H(t) = H_0\sqrt{\Omega_\Lambda^{(0)}}\coth\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda^{(0)}}\,t\right)$$

$\boxed{(h)}$ t-z-relation:

$$H^2 = H_0^2(\Omega_m^{(0)}a^{-3} + \Omega_\Lambda^{(0)}) \Rightarrow H = H_0\sqrt{\Omega_m^{(0)}a^{-3} + \Omega_\Lambda^{(0)}}$$

$$H = \frac{\dot a}{a} = \frac{1}{a}\frac{da}{dt} = \frac{d}{dt}\ln(a)$$

$$\frac{d}{dx}\ln(\sinh(x)) = \frac{\cosh(x)}{\sinh(x)} = \coth(x)$$

$$\overset{(g)}{\Rightarrow} H_0\sqrt{\Omega_\Lambda^{(0)}}\coth\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda^{(0)}}\,t\right) = d\ln(a)$$

$$\Rightarrow \int_{t_0}^{t} H_0\sqrt{\Omega_\Lambda^{(0)}}\coth\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda^{(0)}}\,t'\right)dt' = \int_{a_0}^{a} d\ln(a')$$

$$\frac{2H_0\sqrt{\Omega_\Lambda^{(0)}}}{3H_0\sqrt{\Omega_\Lambda^{(0)}}}\left[\ln\left(\sinh\left(\frac{3}{2}H_0\sqrt{\Omega_\Lambda^{(0)}}\,t'\right)\right)\right]_{t_0}^{t} = \ln\left(\frac{a}{a_0}\right)$$

Use for $t_0 = t_\Lambda$ (time when Λ starts to dominate):

$$\cancel{\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p) = 0 \text{ at } t_\Lambda. \text{ After } t_\Lambda: \frac{\ddot a}{a} > 0}$$

$$\cancel{\Rightarrow \rho + 3p = 0 = \rho_\Lambda + \rho_m + 3(w_\Lambda\rho_\Lambda + w_m\rho_m) = -2\rho_\Lambda + \rho_m}$$

$$\cancel{= \rho_c^{(0)}(\Omega_m^{(0)}a^{-3} - 2\Omega_\Lambda^{(0)}) \Rightarrow}$$

$$\rho_m = \rho_\Lambda \quad \Leftrightarrow \quad \rho_{cr}^{(0)} \Omega_m^{(0)} a^{-3} = \rho_c^{(0)} \Omega_\Lambda^{(0)}$$

$$\Rightarrow \quad a^{-3} = \frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}} \quad \Rightarrow \quad a_\Lambda = \left(\frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}}\right)^{-\frac{1}{3}} = \left(\frac{\Omega_m^{(0)}}{\Omega_\Lambda^{(0)}}\right)^{1/3}$$

$$\Rightarrow \frac{2 \cancel{H_0\sqrt{\Omega_\Lambda^{(0)}}}}{3 \cancel{H_0\sqrt{\Omega_\Lambda^{(0)}}}} \ln\left(\frac{\sinh\left(\frac{3}{2} H_0\sqrt{\Omega_\Lambda^{(0)}}\, t\right)}{\sinh\left(\frac{3}{2} H_0\sqrt{\Omega_\Lambda^{(0)}}\, t_\Lambda\right)}\right) = \ln(a) - \underbrace{\ln(a_0)}_{\approx 0}$$

$$\Rightarrow \quad \frac{\left(\sinh\left(\frac{3}{2} H_0\sqrt{\Omega_\Lambda^{(0)}}\, t\right)\right)^{2/3}}{\underbrace{\left(\sinh\left(\frac{3}{2} H_0\sqrt{\Omega_\Lambda^{(0)}}\, t_\Lambda\right)\right)^{2/3}}_{= a^{-1}(t_\Lambda) = \left(\frac{\Omega_m^{(0)}}{\Omega_\Lambda^{(0)}}\right)^{-1/3}}} = a(t)$$

$$\frac{a(t)}{a(t_\Lambda)} = (\sinh(\dots))^{2/3}$$

$$a = \frac{1}{1+z}$$

$$= \left(\frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}}\right)^{-1/3} \left(\sinh\left(\frac{3}{2} H_0\sqrt{\Omega_\Lambda^{(0)}}\, t\right)\right)^{2/3} = a(t)$$

$$z = a^{-1} - 1$$

$$\Rightarrow \quad z(t) = \left(\frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}}\right)^{+1/3} \left(\sinh\left(\frac{3}{2} H_0\sqrt{\Omega_\Lambda^{(0)}}\, t\right)\right)^{-2/3} - 1$$

$$\Leftrightarrow \left(z \cdot \left(\frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}}\right)^{-1/3} + \left(\frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}}\right)^{-1/3}\right)^{-3/2} = \sinh\left(\frac{3}{2} H_0 \sqrt{\Omega_\Lambda^{(0)}}\, t\right)$$

$$\left((1+z)^{-3/2} \sqrt{\frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}}}\right) = \sinh\left(\frac{3}{2} H_0 \sqrt{\Omega_\Lambda^{(0)}}\, t\right)$$

$$\Rightarrow \quad t(z) = \frac{2}{3 H_0 \sqrt{\Omega_\Lambda^{(0)}}} \operatorname{arsinh}\left(\sqrt{\frac{\Omega_\Lambda^{(0)}}{\Omega_m^{(0)}}}\,(1+z)^{-3/2}\right)$$

Nr. 11.3)

Single perfect fluid with EOS $w = \text{const.}$, already dominant at present

(a) Covariant conservation equation: $T^{\mu\nu}{}_{;\mu} = 0$

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p g^{\mu\nu}$$

In a comoving frame: $u^\mu = (1,0,0,0) \Rightarrow$ only $\nu = 0$ is non-trivial $\Rightarrow T^{\mu 0}{}_{;\mu} = T^{\mu 0}{}_{,\mu} + \Gamma^\mu_{\alpha\mu} T^{\alpha 0}$

$$= \cancel{T^{i0}{}_{,i}} + \underbrace{T^{00}{}_{,0}}_{\dot\rho} + \Gamma^\mu_{\alpha\mu} T^{\alpha 0} = \dot\rho + \Gamma^\mu_{0\mu} T^{00} + \cancel{\Gamma^\mu_{i\mu} T^{i0}}$$

$$= \dot\rho + (\Gamma^0_{00} + \Gamma^i_{0i}) \cdot (\rho + p - p \underbrace{g^{00}}_{=1})$$

FRLW-metric: (diagonal)

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -\frac{a^2}{1-kr^2} & & \\ & & r^2a^2 & \\ 0 & & & r^2a^2\sin^2\theta \end{pmatrix}$$

$\Rightarrow \Gamma^i_{j0} = \Gamma^i_{0j} = H\delta^i_j \rightarrow \Gamma^i_{i0} = H\delta^i_i = 3H$

$\Gamma^0_{00} = 0$

$\Rightarrow T^{\mu 0}{}_{;\mu} = \dot\rho + 3H(\rho + p)$ conservation equation

$= 0$

$$\dot\rho = -3H(\rho+p) = -3H(\rho + w\rho) = -H\rho \cdot 3(1+w)$$

$$\Rightarrow \frac{1}{\rho}\frac{d\rho}{dt} = -\frac{1}{a}\frac{da}{dt} \cdot 3(1+w) \Leftrightarrow \frac{d\rho}{\rho} = -\frac{da}{a} 3(1+w)$$

$$\Rightarrow \int_{\rho_0}^{\rho} \frac{d\rho'}{\rho'} = -\int_{a_0}^{a} \frac{da'}{a} \cdot 3(1+w)$$

ρ_0, a_0 at present ($a_0 = 1$)

$$\Leftrightarrow \ln(\rho) - \ln(\rho_0) = -3(1+w)(\ln(a) - \ln(a_0))$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = -3(1+w)\ln\left(\frac{a}{a_0}\right)$$

$$\rho = \rho_0 e^{-3(1+w)\ln\left(\frac{a}{a_0}\right)} \quad \Big|\ e^{-x\ln a} = a^{-x}$$

$$= \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}$$

Friedm. eq. $\Rightarrow H^2 = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)} = H_0^2 \Omega^{(0)} \left(\frac{a}{a_0}\right)^{-3(1+w)}$

$$\Rightarrow H(a) = \sqrt{\frac{8\pi G\rho_0}{3}} \left(\frac{a}{a_0}\right)^{-\frac{3(1+w)}{2}} = H_0\sqrt{\Omega^{(0)}}\left(\frac{a}{a_0}\right)^{-\frac{3(1+w)}{2}}$$

$$= H_0\sqrt{\Omega^{(0)}}\, a^{-3/2(1+w)}$$

$\boxed{(b)}$ How long for universe to be infinitely large?

$\Rightarrow a \to \infty$

$H = \frac{da}{a\,dt} = H_0\sqrt{\Omega^{(0)}}\left(\frac{a}{a_0}\right)^{-\frac{3(1+w)}{2}}$ $\quad a_0 = 1$ at present

$= H_0\sqrt{\Omega^{(0)}}\, a^{-\frac{3(1+w)}{2}}$

$\Rightarrow H_0\sqrt{\Omega^{(0)}}\,dt = a^{-1}a^{+\frac{3(1+w)}{2}}da = a^{\frac{1}{2}(1+3w)}da$

$\Rightarrow \int_{t_0}^{t} H_0\sqrt{\Omega^{(0)}}\,dt = \int_{a_0}^{a} a'^{\frac{1}{2}(1+3w)}\,da'$

$H_0\sqrt{\Omega^{(0)}}(t-t_0) = \left[\frac{a'^{\frac{1}{2}(1+3w)+1}}{\frac{1}{2}(1+3w)+1}\right]_{a_0=1}^{a}$

$= \frac{1}{\frac{1}{2}(1+3w)+1}\left(a^{\frac{1}{2}(1+3w)+1} - 1\right)$

$= \frac{\left(a^{\frac{3}{2}(1+w)} - 1\right)}{\frac{3}{2}(1+w)}$

$\Rightarrow t = t_0 + \frac{2\left(H_0\sqrt{\Omega^{(0)}}\right)^{-1}}{3(w+1)}\left(a^{\frac{3}{2}(w+1)} - 1\right)$

$= t_0 + 2\left(3H_0\sqrt{\Omega^{(0)}}(w+1)\right)^{-1}\left(a^{3/2(w+1)} - 1\right)$

① Phantom dark energy: $w_{ph} < -1 \Rightarrow w_{ph} + 1 < 0$

Infinitely large: $a \to \infty \Rightarrow a^{3/2(w_{ph}+1)} = \frac{1}{a^{\frac{3}{2}|w_{ph}+1|}} \xrightarrow{a\to\infty} 0$

$\Rightarrow t = t_{Big\,Rip} = t_0 + 2\left(3H_0\sqrt{\Omega^{(0)}}(w_{ph}+1)\right)^{-1}(-1)$

$= t_0 + \cancel{(-1)(-1)}\; 2\left(3H_0\sqrt{\Omega^{(0)}}\,|w_{ph}+1|\right)^{-1}$

$= t_0 + \frac{2}{3H_0\sqrt{\Omega^{(0)}}\,|w_{ph}+1|}$

② Quintessence dark energy: $w > -1,\ w < -\frac{1}{3}$

$\Rightarrow a^{3/2(1+w_Q)} = a^{\frac{3}{2}|1+w_Q|} \xrightarrow{a\to\infty} \infty$

$\Rightarrow t_{Big\,Rip} \to \infty \quad \Rightarrow \nexists$ big rip

③ Λ: $w = -1 \quad \Rightarrow \quad 1 + w_\Lambda = 0 \quad \Rightarrow t = t_0 + \frac{2(\overbrace{1-1}^{=0})}{3H_0\sqrt{\Omega^{(0)}}(\underbrace{1-1}_{=0})}$

$\to \infty$

$\Rightarrow \nexists$ Big rip

Question: Einstein's static universe

(a) $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

$$\Leftrightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left(T_{\mu\nu} + \frac{\Lambda}{8\pi G} g_{\mu\nu} \right)$$
$$= 8\pi G \left(T_{\mu\nu(m)} + T_{\mu\nu(\Lambda)} \right)$$

$$T_{\mu\nu(\Lambda)} = \frac{\Lambda}{8\pi G} g_{\mu\nu}$$

FLRW-metric: $$g_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -\frac{a^2}{1+kr^2} & & \\ & & r^2 a^2 & \\ 0 & & & r^2 a^2 \sin^2\vartheta \end{pmatrix}$$

$$\Rightarrow \rho_\Lambda = T_{00(\Lambda)} = \frac{\Lambda}{8\pi G} g_{00}$$
$$= \frac{\Lambda}{8\pi G}$$

(b) /