12.2 Diagrammatic expansion of partition function for Yukawa theory

We have, a **real scalar field** coupled with **Dirac field.**

The following is the free part of the action

$$S_2[\bar{\psi},\psi,\phi] = \int d^4x \left\{ -\frac{1}{2}\phi\left(-\partial^2 + M^2\right)\phi - i\bar{\psi}(\not{p} + m)\psi \right\} \tag{1}$$

and the interaction term is

$$S_I[\bar{\psi},\psi,\phi] = \int d^4x\{-ig\phi\bar{\psi}\psi\}$$

the partition function with external currents is

$$Z[\bar{\eta},\eta,J] = \int D\bar{\psi}D\psi D\phi \exp\left[iS_2[\bar{\psi},\psi,\phi] + iS_I[\bar{\psi},\psi,\phi] + i\int_x \{\bar{\eta}\psi + \bar{\psi}\eta + J\phi\}\right]$$

where the last term is the $S_\mathrm{S} = \int\left\{\bar{\eta}\psi + \bar{\psi}\eta + J\phi\right\}$ source part.

(a) Without interaction term, $S_I = 0$, show that the partition function can be written as

$$Z_2[\bar{\eta},\eta,J] = \exp\left[i\int d^4x d^4y \bar{\eta}(x)S(x-y)\eta(y)\right]\exp\left[\frac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right]$$

where the scalar and fermionic propagator are given by

$$\Delta(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\frac{1}{p^2 + M^2 - i\epsilon}$$

$$S_{\alpha\beta}(x-y) = -i\int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)}\frac{(-i\not{p} + m\mathbb{I})_{\alpha\beta}}{p^2 + m^2 - i\epsilon}$$

Solution.

Let us use,

$$\Delta^{-1}(x-y) = \left(-\partial^2 + M^2\right)\delta(x-y)$$
$$S^{-1}(x-y) = \left(\not{\partial} + m\right)\delta(x-y)$$

giving us

$$\int_z \Delta^{-1}(x-y)\Delta(z-y) = \delta(x-y)$$
$$\int_z S^{-1}(x-y)S(z-y) = \delta(x-y)$$

which makes sense as we can get our general equation defining the propagator by doing the following steps (we just need the equations above, these steps are just for book-keeping to see we get the result we are used to)

$$\Delta^{-1}(x-y) = \left(-\partial^2 + M^2\right)\delta(x-y)$$
$$\Delta^{-1}(x-y)\Delta(y-z) = \left(-\partial^2 + M^2\right)\delta(x-y)\Delta(y-z)$$
$$\int \Delta^{-1}(x-y)\Delta(y-z) = \int\left(-\partial^2 + M^2\right)\delta(x-y)\Delta(y-z)$$
$$\delta(x-z) = \left(-\partial^2 + M^2\right)\Delta(x-z)$$

Let us write down the action of the free Yukawa theory

$$S_2 + S_s = \int_x \left\{ -\frac{1}{2}\phi(x)\left(-\partial^2 + M^2\right)\phi(x) - i\bar{\psi}(x)(\not{\partial} + m)\psi(x) + \left[\bar{\eta}\psi + \bar{\psi}\eta + J\phi\right](x) \right\}$$

$$= \int_x \int_y \left\{ -\frac{1}{2}\phi(x) \underbrace{\left(-\partial^2 + M^2\right)\delta(x-y)}_{\Delta^{-1}(x-y)} \phi(y) - i\bar{\psi}(x) \underbrace{(\not{\partial} + m)\delta(x-y)}_{S^{-1}(x-y)} \psi(y) + \left[\bar{\eta}\psi + \bar{\psi}\eta + J\phi\right](y)\,\delta(x-y) \right\}$$

$$= \int_x \int_y \left\{ -\frac{1}{2}\phi(x)\,\Delta^{-1}(x-y)\,\phi(y) - i\bar{\psi}(x)\,S^{-1}(x-y)\,\psi(y) + \left[\bar{\eta}\psi + \bar{\psi}\eta + J\phi\right](y)\,\delta(x-y) \right\}$$

From the first to the second line, we basically introduced a delta function to get inverse propagators in the equation (Also will help to complete the square in order to evaluate the Gaussian integrals). The δ function on the source term is basically just for book-keeping, so we can take double spacetime integral over the whole expression.

Now, combine the J source term with $\Delta^{-1}(x-y)$ term and the $\left(\bar{\eta}\psi + \bar{\psi}\eta\right)$ source term with the $S^{-1}(x-y)$ term. Why? Then we can complete the square just like for the case of a Boson and Fermion independently,

$$S_2 + S_s = \int_x \int_y \left\{ -\frac{1}{2}\phi(x)\,\Delta^{-1}(x-y)\,\phi(y) + J\phi(y)\,\delta(x-y) \right\} - \int_x \int_y \left\{ i\bar{\psi}(x)\,S^{-1}(x-y)\,\psi(y) + \left[\bar{\eta}\psi + \bar{\psi}\eta\right](y)\,\delta(x-y)\,G \right\}$$

We can complete the square for the two curly brackets above individually we will get

$$\text{Curly bracket 1} =$$

.

$$\text{Curly bracket 2} = \int_x \int_y \{\}$$

$$\phi'(x) = \phi(x) - \int_z \Delta(x-z)\,J(z)$$

$$i\bar{\psi}'(x) = i\bar{\psi}(x) - \int_z \bar{\eta}(z)\,S(z-x)$$

$$\psi'(x) = \psi(x) - \int_z S(z-x)\,\eta(z)$$

(b) To evaluate the full partition function one can expand in series the interaction term

$$\exp\left[\int d^4x g\phi\bar{\psi}\psi\right] = \sum_{n=0}^{\infty} \frac{g^n}{n!}\left(\int d^4x\phi\bar{\psi}\psi\right)^n$$

The path integral can be formally written as

$$Z[\bar{\eta},\eta,J] = \sum_{n=0}^{\infty} \int D\bar{\psi}D\psi D\phi \frac{g^n}{n!}\left(\int d^4x\phi\bar{\psi}\psi\right)^n e^{iS_2[\bar{\psi},\psi,\phi]+i\int d^4x\{\bar{\eta}\psi+\bar{\psi}\eta+J\phi\}}$$

Convince yourself that this can be re-expressed as exponentiated derivatives acting on the free partition function,

$$Z[\bar{\eta},\eta,J] = \exp\left[g\int_x \left\{\frac{1}{i}\frac{\delta}{\delta J(x)} i\frac{\delta}{\delta\eta(x)}\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right\}\right] Z_2[\bar{\eta},\eta,J]$$

(Note: The minus sign here comes from the anti commutation relations of η and $\bar{\eta}$)

Solution.

$$Z[\bar{\eta},\eta,J] = \sum_{n=0}^{\infty} \int D\bar{\psi}D\psi D\phi \frac{g^n}{n!}\left(\int d^4x\phi\bar{\psi}\psi\right)^n e^{iS_2[\bar{\psi},\psi,\phi]+i\int d^4x\{\bar{\eta}\psi+\bar{\psi}\eta+J\phi\}}$$

use

$$
\begin{aligned}
\exp\left[\int d^4x g\phi\bar{\psi}\psi\right] &= \sum_{n=0}^{\infty}\frac{g^n}{n!}\left(\int_x \phi\bar{\psi}\psi\right)^n \\
&= \left(1+g\left(\int_x \phi\bar{\psi}\psi\right)+\frac{g^2}{2!}\left(\int_x \phi\bar{\psi}\psi\right)^2+\frac{g^3}{3!}\left(\int_x \phi\bar{\psi}\psi\right)^3+\dots\right)
\end{aligned}
$$

giving us

$$
Z[\bar{\eta},\eta,J] = \int D\bar{\psi}D\psi D\phi\left(1+g\left(\int_x \phi\bar{\psi}\psi\right)+\frac{g^2}{2!}\left(\int_x \phi\bar{\psi}\psi\right)^2+\frac{g^3}{3!}\left(\int_x \phi\bar{\psi}\psi\right)^3+\dots\right)e^{iS_2[\bar{\psi},\psi,\phi]+i\int d^4x\{\bar{\eta}\psi+\bar{\psi}\eta+J\phi\}}
$$

Focusing on the parenthesis acting on the $e^{iS_2[\bar{\psi},\psi,\phi]+i\int d^4x\{\bar{\eta}\psi+\bar{\psi}\eta+J\phi\}}$ term, we can think as if the follow differentiation pull out one of the each factor,

$$
\begin{aligned}
\frac{1}{i}\frac{\delta}{\delta J(x)} &\to \phi \\
i\frac{\delta}{\delta\eta(x)} &\to \bar{\psi} \\
\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)} &\to \psi
\end{aligned}
$$

looking at the equation we want, the last substitution will give out the first term (the second relation has an i in the front to cancel out with the $-$ sign that arises from switching the Grassman variables before differentiating), giving us

$$
\begin{aligned}
Z[\bar{\eta},\eta,J] = \int D\bar{\psi}D\psi D\phi &\left\{1+g\int_x\left[\left(i\frac{\delta}{\delta\eta(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]+\frac{g^2}{2!}\left[\left(i\frac{\delta}{\delta\eta(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]^2+\dots\right\}\cdot \\
&e^{iS_2[\bar{\psi},\psi,\phi]+i\int d^4x\{\bar{\eta}\psi+\bar{\psi}\eta+J\phi\}}
\end{aligned}
$$

The term in the curly bracket can get out from the $\int D\bar{\psi}D\psi D\phi$, as it does not have any dependence on those variables.

$$
\begin{aligned}
Z[\bar{\eta},\eta,J] &= \left\{1+g\int_x\left[\left(i\frac{\delta}{\delta\eta(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]+\frac{g^2}{2!}\left[\left(i\frac{\delta}{\delta\eta(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]^2+\dots\right\}\cdot \\
&\quad\int D\bar{\psi}D\psi D\phi\cdot e^{iS_2[\bar{\psi},\psi,\phi]+i\int d^4x\{\bar{\eta}\psi+\bar{\psi}\eta+J\phi\}} \\
&= \left\{\sum_{n=0}^{\infty}\frac{g^n}{n!}\int_x\left[\left(i\frac{\delta}{\delta\eta(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]^n\right\}\underbrace{\int D\bar{\psi}D\psi D\phi\cdot e^{iS_2[\bar{\psi},\psi,\phi]+i\int d^4x\{\bar{\eta}\psi+\bar{\psi}\eta+J\phi\}}}_{=Z_2[\bar{\eta},\eta,J]} \\
&= \exp\left(g\int_x\left[\left(i\frac{\delta}{\delta\eta(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]\right)Z_2[\bar{\eta},\eta,J]
\end{aligned}
$$

(c) Express now the free parts of the partition function also as a Taylor series

$$\begin{aligned} Z_2\left[\bar{\eta},\eta,J\right] &= \sum_{V=0}^{\infty}\frac{1}{V!}\left[\int_x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)g\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]^V \\ &\times\sum_{F=0}^{\infty}\frac{1}{F!}\left[\int_{x'y'}\left(i\bar{\eta}\left(x'\right)\right)\left(\frac{1}{i}S\left(x'-y'\right)\right)\left(i\eta\left(y'\right)\right)\right]^F \quad (2) \\ &\times\sum_{S=0}^{\infty}\frac{1}{S!}\left[\frac{1}{2}\int_{x''y''}\left(iJ\left(x''\right)\right)\left(\frac{1}{i}\Delta\left(x''-y''\right)\right)\left(iJ\left(y''\right)\right)\right]^S \quad (3) \end{aligned}$$

Here, V is the number of **vertices**; F, S are the number of **Fermion and Scalar propagators** respectively. This expression can also be seen as a diagrammatic expansion that contains all possible diagrams allowed. Develop a diagrammatic representations of the partition function similar to what you have learned in the lectures. Compute explicitly the terms with $V = 0$,$F = S = 2$ and $V = 1, S = 1, F = 2$ and draw the corresponding diagrams.

Solution.
Expanding using Taylor series we can write

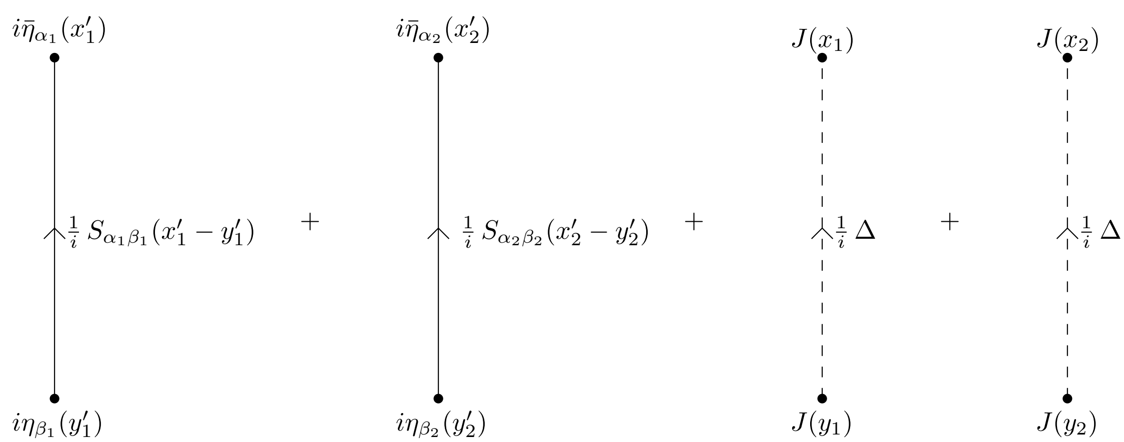
$$\begin{aligned} Z_2[\bar{\eta},\eta,J] &= \exp\left[i\int d^4xd^4y\bar{\eta}(x)S(x-y)\eta(y)\right]\exp\left[\frac{i}{2}\int d^4xd^4yJ(x)\Delta(x-y)J(y)\right] \\ &= \sum_{F=0}^{\infty}\frac{1}{F!}\left[\int_{x'y'}\left(i\bar{\eta}\left(x'\right)\right)\left(\frac{1}{i}S\left(x'-y'\right)\right)\left(i\eta\left(y'\right)\right)\right]^F\times\sum_{S=0}^{\infty}\frac{1}{S!}\left[\frac{1}{2}\int_{x''y''}\left(iJ\left(x''\right)\right)\left(\frac{1}{i}\Delta\left(x''-y''\right)\right)\left(iJ\left(y''\right)\right)\right]^S \end{aligned}$$

Putting this in Z, we get what we (3).
Now, we need to talk about diagrams.

- First for $V = 0,\ F = 2,\ S = 2$

$$\begin{aligned} Z_2\left[\bar{\eta},\eta,J\right] &= \frac{1}{0!}\left[\int_x\left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta\eta(x)}\right)g\left(\frac{1}{i}\frac{\delta}{\delta\bar{\eta}(x)}\right)\right]^0 \\ &\times\frac{1}{2!}\left[\int_{x'y'}\left(i\bar{\eta}\left(x'\right)\right)\left(\frac{1}{i}S\left(x'-y'\right)\right)\left(i\eta\left(y'\right)\right)\right]^2 \\ &\times\frac{1}{2!}\left[\frac{1}{2}\int_{x''y''}\left(iJ\left(x''\right)\right)\left(\frac{1}{i}\Delta\left(x''-y''\right)\right)\left(iJ\left(y''\right)\right)\right]^2 \\ &= \frac{1}{2!}\left[\int_{x'y'}\left(i\bar{\eta}\left(x'\right)\right)\left(\frac{1}{i}S\left(x'-y'\right)\right)\left(i\eta\left(y'\right)\right)\right]^2 \\ &\times\frac{1}{2!}\left[\frac{1}{2}\int_{x''y''}\left(iJ\left(x''\right)\right)\left(\frac{1}{i}\Delta\left(x''-y''\right)\right)\left(iJ\left(y''\right)\right)\right]^2 \end{aligned}$$

Diagrammatic representation:



- Now for $V = 1, S = 1, F = 2$

$$
\begin{aligned}
Z_2\left[\bar{\eta}, \eta, J\right] &= \frac{1}{1!}\left[\int_x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta \eta(x)}\right)\left(g\mathbb{I}_{\alpha\beta}\right)\left(\frac{1}{i}\frac{\delta}{\delta \bar{\eta}(x)}\right)\right]^1 \\
&\quad \times \frac{1}{1!}\left[\int_{x'y'} \left(i\bar{\eta}_{\alpha_1}\left(x_1'\right)\right)\left(\frac{1}{i}S_{\alpha_1\beta_1}\left(x_1' - y_1'\right)\right)\left(i\eta_{\beta_1}\left(y_1'\right)\right)\right]^2 \\
&\quad \times \frac{1}{2!}\left[\frac{1}{2}\int_{x''y''} \left(iJ\left(x''\right)\right)\left(\frac{1}{i}\Delta\left(x'' - y''\right)\right)\left(iJ\left(y''\right)\right)\right]^1 \\
&= \left[\int_x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)\left(i\frac{\delta}{\delta \eta(x)}\right) g \left(\frac{1}{i}\frac{\delta}{\delta \bar{\eta}(x)}\right)\right]\left(g\mathbb{I}_{\alpha\beta}\right) \\
&\quad \times \left[\int_{x'y'} \left(i\bar{\eta}_{\alpha_1}\left(x_1'\right)\right)\left(\frac{1}{i}S_{\alpha_1\beta_1}\left(x_1' - y_1'\right)\right)\left(i\eta_{\beta_1}\left(y_1'\right)\right)\right] \\
&\quad \times \left[\int_{x'y'} \left(i\bar{\eta}_{\alpha_2}\left(x_2'\right)\right)\left(\frac{1}{i}S_{\alpha_2\beta_2}\left(x_2' - y_2'\right)\right)\left(i\eta_{\beta_2}\left(y_2'\right)\right)\right] \\
&\quad \times \frac{1}{2}\left[\frac{1}{2}\int_{x''y''} \left(iJ\left(x''\right)\right)\left(\frac{1}{i}\Delta\left(x'' - y''\right)\right)\left(iJ\left(y''\right)\right)\right]
\end{aligned}
$$

Now, we perform the functional derivative

$$
= \left(g\mathbb{I}\right)_{\mu\nu}
$$