P1.

(a) Given : $\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi \quad = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi$

We will use E-L equations :

(i) $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$ (ii) $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \right) - \frac{\partial \mathcal{L}}{\partial \phi^*} = 0$

$\partial_\mu (\partial^\mu \phi^*) + m^2 \phi^* = 0 \qquad \partial_\mu (\partial^\mu \phi) + m^2 \phi = 0$

$\Rightarrow \quad (\partial_\mu \partial^\mu + m^2) \phi^* = 0 \qquad \Rightarrow \quad (\partial_\mu \partial^\mu + m^2) \phi = 0$

$\therefore$ We can see that we have two copies o- KG eq^n.

(b) (1) Check what happens under global U(1) Symmetry

$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$

$\downarrow \boxed{\phi \to e^{iq\chi} \phi \quad \& \quad \phi^* \to e^{-iq\chi} \phi^*}$

$= (\partial_\mu (e^{-iq\chi} \phi^*))(\partial^\mu (e^{iq\chi} \phi)) - m^2 (\cancel{e^{-iq\chi}} \phi^*)(\cancel{e^{+iq\chi}} \phi)$

$= [(\cancel{\partial_\mu e^{-iq\chi}}) \phi^* + e^{-iq\chi} (\partial_\mu \phi^*)][(\cancel{\partial^\mu e^{iq\chi}}) \phi + e^{iq\chi} (\partial^\mu \phi)] - m^2 \phi^* \phi$

Both = 0 as $q\chi$ is a constant.

$= (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$

which means $\mathcal{L}$ is conserved under U(1) global symmetry.

(2) Check what happens under local U(1) Symmetry

$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$

$\downarrow \boxed{\phi \to e^{iq\chi} \phi \quad \& \quad \phi^* \to e^{-iq\chi} \phi^*}$

$= \partial_\mu (\cancel{e^{-iq\chi}} \phi^*) \; \partial^\mu (\cancel{e^{iq\chi}} \phi) - m^2 \cancel{e^{-iq\chi}} \phi^* \cancel{e^{iq\chi}} \phi$

As χ is a constant.

$$= (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi$$

$$\downarrow \boxed{\phi \to e^{iq\chi(x)}\phi \quad \& \quad \phi^* \to e^{-iq\chi(x)}\phi^*}$$

$$= \underbrace{\partial^\mu\left(e^{-iq\chi(x)}\phi^*\right)}_{\text{I}} \underbrace{\partial_\mu\left(e^{iq\chi(x)}\phi\right)}_{\text{II}} - \underbrace{m^2 \cancel{e^{-iq\chi(x)}} \phi^* \cancel{e^{iq\chi(x)}} \phi}_{\text{mass term unchanged.}}$$

Let us figure out how does term I & II transform

$$\text{II} \quad \partial_\mu\left(e^{iq\chi(x)}\phi\right) = \partial_\mu\left(e^{iq\chi(x)}\right)\phi + e^{iq\chi(x)}(\partial_\mu \phi)$$

$$= iq(\partial_\mu \chi(x))\, e^{iq\chi(x)}\phi + e^{iq\chi(x)}(\partial_\mu \phi)$$

$$= e^{iq\chi(x)}\left[\partial_\mu + iq\partial_\mu \chi\right]\phi$$

Similarly, I $\quad \partial^\mu \phi^* \to e^{-iq\chi(x)}\left[\partial^\mu - iq\partial^\mu \chi\right]\phi^*$

I + II transformation gives us the following first term in the transformation:

$$(\partial^\mu \phi^*)(\partial_\mu \phi) \to (\partial^\mu \phi^*)(\partial_\mu \phi) - iq\,(\partial^\mu \chi(x))(\partial_\mu \phi) + iq\,(\partial^\mu \phi^*)(\partial_\mu \chi(x))\phi$$
$$+ (\partial^\mu \phi)(\partial_\mu \phi^*)\phi^*\phi$$

which is not what we started with.

(3) Looking carefully at I & II transformations, we can guess that if

$$D_\mu = \partial_\mu + iqA_\mu(x) \qquad \& \qquad D^\mu = \partial^\mu - iqA^\mu(x)$$

but this will create additional terms due to $A^\mu(x)$. Just like we have a transf^n for ϕ, ϕ^*, we need a transformation of the following form

$$A_\mu(x) \to A_\mu(x) - \ (\partial_\mu \chi(x))$$

to make sure that the new terms arising from A^μ / A_μ are cancelled out.

$$(D_\mu \phi) = (\partial_\mu + iq A_\mu(x))\phi \xrightarrow{U(1)} (\partial_\mu + iq(A_\mu(x) - \partial_\mu(\chi(x))) e^{iq\chi(x)}\phi$$

$$= \partial_\mu(e^{iq\chi(x)}\phi) + iqA_\mu(x) e^{iq\chi(x)}\phi - iq\partial_\mu(\chi(x)) e^{iq\chi(x)}\phi$$

$$= \cancel{iqe^{iq\chi(x)}(\partial_\mu \chi(x))\phi} + e^{iq\chi(x)}\partial_\mu\phi + iqA_\mu(x) e^{iq\chi(x)}\phi$$
$$\cancel{- iq(\partial_\mu \chi(x)) e^{iq\chi(x)}\phi}$$

$$= e^{iq\chi(x)}(\partial_\mu + iqA_\mu(x))\phi$$
$$= e^{iq\chi(x)}(D_\mu\phi)$$

Similarly $\rightarrow$ $(D^\mu \phi^*) \longrightarrow e^{-iq\chi(x)}(D^\mu\phi^*)$

Together $\rightarrow$ $(D_\mu\phi)(D^\mu\phi^*) \xrightarrow{U(1)} \underbrace{\cancel{e^{iq\chi(x)}}(D_\mu\phi)\cancel{e^{-iq\chi(x)}}}_{\text{As long as } [\chi,\phi]=0.}(D^\mu\phi^*)$

This gives us our new Lagrangian,

$$\mathcal{L} = (D^\mu\phi^*)(D_\mu\phi) - m^2\phi^*\phi$$
$$= (\partial^\mu\phi^* - iqA^\mu\phi^*)(\partial_\mu\phi + iqA_\mu\phi) - m^2\phi^*\phi$$
$$= (\partial^\mu\phi^*)(\partial_\mu\phi) - m^2\phi^*\phi$$
$$+ \underbrace{(-iqA^\mu\phi^*(\partial_\mu\phi) + iq(\partial^\mu\phi^*)A_\mu\phi + q^2\phi^*\phi A^\mu A_\mu)}$$

This terms shows the coupling strength between A_μ & ϕ, ϕ^* fields

(4)

$$0 = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi}$$

$$= \partial_\mu(\partial^\mu\phi^* - iqA^\mu\phi^*) - (-m^2\phi^* + q^2\phi^*A^\mu A_\mu)$$

$$= (\partial^\mu \partial_\mu + m^2)\phi^* - \partial_\mu (iqA^\mu \phi^*) - q^2 \phi^* A^\mu A_\mu$$

$$= [\partial^\mu \partial_\mu \phi^* + m^2 \phi^* - iq(\partial_\mu A^\mu)\phi^* - iqA^\mu(\partial_\mu \phi^*) - q^2 \phi^* A^\mu A_\mu]$$

Together

$$= [(\partial_\mu + iqA_\mu)(\partial^\mu - iqA^\mu) + m^2]\phi^*$$

$$\Rightarrow [D_\mu D^\mu + m^2]\phi^* = 0$$

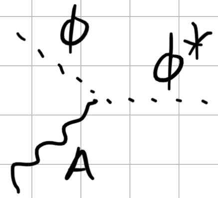
Similarly the other E-L eq^n gives us :

$$[D_\mu D^\mu + m^2]\phi = 0$$

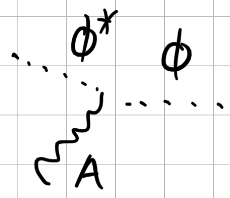
- If ϕ is a particle with charge q propagating in an EM field specified by A^μ, ϕ^* corresponds to its antiparticle.

- The new EOM for ϕ & ϕ^* now comprise of source terms!

  Specifically source terms like this :



&



Rough work

$$(D_\mu D^\mu + m^2)\phi^* = 0$$

$$[(\partial_\mu + iqA_\mu)(\partial^\mu - iqA^\mu) + m^2]\phi^* = 0$$

$$[\partial_\mu \partial^\mu - \partial_\mu(iqA^\mu) + iqA_\mu \partial^\mu + q^2 A_\mu A^\mu + m^2]\phi^* = 0$$

$$[\partial_\mu \partial^\mu + m^2]\phi - iq(\partial_\mu A^\mu)\phi^* + iqA_\mu(\partial^\mu \phi^*) + q^2 A_\mu A^\mu \phi^* = 0$$

(a) Show that $-i\psi_L^T \sigma_2 \psi_L$ is Lorentz-invariant.

→ We have $\psi_R(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$ for $(0, \frac{1}{2})$ representation

$\psi_L(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \end{bmatrix}$ for $(\frac{1}{2}, 0)$ representation

Recall how these Weyl spinors transforms under r_j & b_j

$$\psi_R \to e^{\frac{1}{2}(ir_j\sigma_j + b_j\sigma_j)}\psi_R = \left(1 + \frac{i}{2}r_j\sigma_j + \frac{1}{2}b_j\sigma_j + \dots\right)\psi_R$$

$$\psi_L \to e^{\frac{1}{2}(ir_j\sigma_j - b_j\sigma_j)}\psi_L = \left(1 + \frac{i}{2}r_j\sigma_j - \frac{1}{2}b_j\sigma_j + \dots\right)\psi_L$$

Giving us the fact that these Spinors infinetesimally transform like:

$$\delta\psi_R = \frac{1}{2}(ir_j + b_j)\sigma_j\psi_R \quad ; \quad \delta\psi_L = \frac{1}{2}(ir_j - b_j)\sigma_j\psi_L$$

$$\delta\psi_R^\dagger = \frac{1}{2}(-ir_j + b_j)\psi_R^\dagger\sigma_j \quad ; \quad \delta\psi_L^\dagger = \frac{1}{2}(-ir_j - b_j)\psi_L^\dagger\sigma_j$$

Note: r_j & b_j are real numbers.

$$\delta(\psi_L^T\sigma_2\psi_L) = \left[(\delta\psi_L^T)\sigma_2\psi_L + \psi_L^T\sigma_2(\delta\psi_L)\right]$$

$$= \left[\frac{1}{2}(ir_j - b_j)\psi_L^T\underbrace{\sigma_j^T\sigma_2}_{-\sigma_2\sigma_j}\psi_L + \psi_L^T\sigma_2\frac{1}{2}(ir_j - b_j)\sigma_j\psi_L\right]$$

$$= \frac{1}{2}\left[\cancel{-(ir_j - b_j)\psi_L^T\sigma_2\sigma_j\psi_L} + \cancel{(ir_j - b_j)\psi_L^T\sigma_2\sigma_j\psi_L}\right]$$

These cancel!

$$= 0 \quad ⊛$$

- This shows that the term $\psi_L^T\sigma_2\psi_L$ is invariant under infinetesimal rotations & boosts.
- This also shows that $\psi_L^\dagger\sigma_2\psi_L^*$ is invariant under such a transformation as it is just the complex conjugate of ⊛

I present a quick proof that $\sigma_j^T \sigma_2 = -\sigma_2 \sigma_j$

- Recall, σ_1, σ_2 are both real. σ_2 is imaginary.

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

So, $\sigma_1^* = \sigma_1$, $\quad \sigma_2^* = -\sigma_2$, $\quad \sigma_3^* = \sigma_3$

$\sigma_1^T = \sigma_1$, $\quad \sigma_2^T = -\sigma_2$, $\quad \sigma_3^T = \sigma_3$

Giving us:

$$\left.\begin{aligned} &\sigma_1^T \sigma_2 = -\sigma_2 \sigma_1, \\ &\sigma_3^T \sigma_2 = \sigma_3 \sigma_2 = -\sigma_2 \sigma_3 \\ \& \quad &\sigma_2^T \sigma_2 = -\sigma_2 \sigma_2 \end{aligned}\right\} \Rightarrow \quad \sigma_j^T \sigma_2 = -\sigma_2 \sigma_j$$

(b) Show that $\frac{m\bar{\Psi}\Psi}{2}$ is not gauge-invariant, i.e. a Majorana particle cannot be charged.

→ Let us take the first mass term in account from $\frac{m}{2}\bar{\Psi}\Psi$

$$\psi_L^T \sigma_2 \psi_L \xrightarrow{U(1)} \psi_L^T e^{i\alpha} \sigma_2 e^{i\alpha} \psi_L = e^{2i\alpha} \psi_L^T \sigma_2 \psi_L$$

Now, second term,

$$\psi_L^\dagger \sigma_2 \psi_L \xrightarrow{U(1)} \psi_L^\dagger \cancel{e^{-i\alpha}} \sigma_2 \cancel{e^{i\alpha}} \psi_L = \psi_L^\dagger \sigma_2 \psi_L$$

So, one of the term is invariant but the other, isn't. Making $m\psi\bar{\psi}$ not-invariant under $U(1)$.

∴ Majorana fermions are not charged.

Another way to see this is from $\psi = \psi_c = -i\gamma_2 \psi^*$ for Majorana fermions. This means under $U(1)$ symmetry they cannot be charged as under symmetry operation

$$\psi \to e^{i\alpha}\psi$$
$$\psi_c \to e^{-i\alpha}\psi_c$$

Which means $\psi = \psi_c$ cannot hold.