

Exercise 4.1

Exercise 4.1

(a) Show that if a tensor is symmetric in a concrete basis, it will remain symmetric in any other basis. (0.5 pt)

If $A_{\mu\nu}$ is symmetric in a concrete basis $\{\tilde{\omega}^\mu\}$



perform a transformation to $\{\tilde{\omega}^{\bar{\mu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^\mu} \tilde{\omega}^\mu\}$

elements of the tensor \mathbf{A} will change as follows:

$$A_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\nu}{\partial x^{\bar{\nu}}} A_{\mu\nu} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\nu}{\partial x^{\bar{\nu}}} A_{\nu\mu} = \frac{\partial x^\nu}{\partial x^{\bar{\nu}}} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} A_{\nu\mu} = A_{\bar{\nu}\bar{\mu}}$$

(b) Prove that the contraction of a symmetric tensor S with an antisymmetric one A is 0. (0.5 pt)

$$A_{\mu\nu} S^{\mu\nu} = -A_{\nu\mu} S^{\mu\nu} = -A_{\nu\mu} S^{\nu\mu} = -A_{\mu\nu} S^{\mu\nu} \longrightarrow A_{\mu\nu} S^{\mu\nu} = 0$$

Exercise 4.2

Exercise 4.2: Christoffel symbols and covariant derivative

(a) Show that $\Gamma^\beta_{\mu\alpha}$ are the components of the $\binom{1}{1}$ tensor $\nabla \vec{e}_\alpha$ for fixed α , and explain why we do not expect the Christoffel symbols to transform as tensors. (1 pt)

Let us consider a general vector, i.e. a general $\binom{1}{0}$ tensor, \vec{A} .

$$\nabla \vec{A} = \nabla_\beta A^\gamma \tilde{\omega}^\beta \otimes \vec{e}_\gamma$$

For a fixed $\alpha \rightarrow \delta_\alpha^\beta$ is a vector with dummy index β

Let us take now a vector $\vec{e}_\alpha = \delta_\alpha^\beta \vec{e}_\beta$.

$$\nabla \vec{e}_\alpha = \nabla_\beta \delta_\alpha^\gamma \tilde{\omega}^\beta \otimes \vec{e}_\gamma,$$

Computing the first term gives us

$$\nabla_\beta \delta_\alpha^\gamma = \partial_\beta \delta_\alpha^\gamma + \Gamma^\gamma_{\mu\beta} \delta_\alpha^\mu = \Gamma^\gamma_{\alpha\beta}$$

On the other hand,

This means that

$$\nabla \vec{e}_\alpha = \Gamma^\gamma_{\alpha\beta} \tilde{\omega}^\beta \otimes \vec{e}_\gamma.$$

$$\Gamma^{\bar{\gamma}}_{\alpha\bar{\beta}} = \frac{\partial}{\partial x^{\bar{\beta}}} \left(\frac{\partial x^{\bar{\gamma}}}{\partial x^\alpha} \right) + \frac{\partial x^{\bar{\alpha}}}{\partial x^\alpha} \Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}}.$$

$$\begin{aligned} \nabla \vec{e}_\alpha &= \nabla \left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\alpha} \vec{e}_{\bar{\alpha}} \right) = \nabla \left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\alpha} \right) \vec{e}_{\bar{\alpha}} + \frac{\partial x^{\bar{\alpha}}}{\partial x^\alpha} \nabla (\vec{e}_{\bar{\alpha}}) = \\ &= \frac{\partial}{\partial x^{\bar{\beta}}} \left(\frac{\partial x^{\bar{\alpha}}}{\partial x^\alpha} \right) \tilde{\omega}^{\bar{\beta}} \otimes \vec{e}_{\bar{\alpha}} + \frac{\partial x^{\bar{\alpha}}}{\partial x^\alpha} \Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} \tilde{\omega}^{\bar{\beta}} \otimes \vec{e}_{\bar{\gamma}} = \\ &= \left[\frac{\partial}{\partial x^{\bar{\beta}}} \left(\frac{\partial x^{\bar{\gamma}}}{\partial x^\alpha} \right) + \frac{\partial x^{\bar{\alpha}}}{\partial x^\alpha} \Gamma^{\bar{\gamma}}_{\bar{\alpha}\bar{\beta}} \right] \tilde{\omega}^{\bar{\beta}} \otimes \vec{e}_{\bar{\gamma}}. \end{aligned}$$

(b) Discover how each expression $V^\beta_{,\alpha}$ and $V^\mu\Gamma^\beta_{\mu\alpha}$ separately transforms under a change of coordinates. For $\Gamma^\beta_{\mu\alpha}$ you can begin with

$$\frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^\mu_{\alpha\beta} \vec{e}_\mu. \tag{1}$$

Show that neither is the standard tensor law, but that the sum does obey the standard law. (2 pt)

Transformation of the Christofel symbols

$$\frac{\partial \vec{e}_{\bar{\alpha}}}{\partial x^{\bar{\beta}}} = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \vec{e}_{\bar{\mu}}$$

$$\frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial}{\partial x^\beta} \left(\frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \vec{e}_\alpha \right) = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \vec{e}_\mu$$

$$\frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial^2 x^\alpha}{\partial x^\beta \partial x^{\bar{\alpha}}} \vec{e}_\alpha + \frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial \vec{e}_\alpha}{\partial x^\beta} = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \vec{e}_\mu$$

$$\frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial^2 x^\alpha}{\partial x^\beta \partial x^{\bar{\alpha}}} \vec{e}_\alpha + \frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\mu_{\alpha\beta} \vec{e}_\mu = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \vec{e}_\mu$$

In the first term of the left-hand side we can do $\alpha \rightarrow \mu$.

$$\frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial^2 x^\mu}{\partial x^\beta \partial x^{\bar{\alpha}}} \vec{e}_\mu + \frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\mu_{\alpha\beta} \vec{e}_\mu = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \vec{e}_\mu$$

$$\left(\frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial^2 x^\mu}{\partial x^\beta \partial x^{\bar{\alpha}}} + \frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\mu_{\alpha\beta} \right) \vec{e}_\mu = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \vec{e}_\mu$$

$$\frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial^2 x^\mu}{\partial x^\beta \partial x^{\bar{\alpha}}} + \frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\mu_{\alpha\beta} = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \frac{\partial x^\mu}{\partial x^{\bar{\mu}}}$$

Multiplying both sides by $\frac{\partial x^{\bar{\kappa}}}{\partial x^\mu}$ we finally obtain:

$$\frac{\partial x^{\bar{\kappa}}}{\partial x^\mu} \frac{\partial^2 x^\mu}{\partial x^{\bar{\beta}} \partial x^{\bar{\alpha}}} + \frac{\partial x^{\bar{\kappa}}}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^{\bar{\beta}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\mu_{\alpha\beta} = \Gamma^{\bar{\mu}}_{\phantom{\bar{\mu}}\bar{\alpha}\bar{\beta}} \delta^{\bar{\kappa}}_{\phantom{\bar{\kappa}}\mu} = \Gamma^{\bar{\kappa}}_{\phantom{\bar{\kappa}}\bar{\alpha}\bar{\beta}}$$

Transformation of $V^\mu \Gamma^\beta_{\mu\alpha}$

$$V^{\bar{\mu}} \Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\bar{\mu}}}{\partial x^\mu} V^\mu \left[\frac{\partial x^{\bar{\beta}}}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x^{\bar{\mu}} \partial x^{\bar{\alpha}}} + \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} \frac{\partial x^\theta}{\partial x^{\bar{\mu}}} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\beta_{\theta\alpha} \right]$$

$$V^{\bar{\mu}} \Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\bar{\beta}}}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x^\mu \partial x^{\bar{\alpha}}} V^\mu + \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} \frac{\partial x^\theta}{\partial x^\mu} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\beta_{\theta\alpha} V^\mu$$

$$V^{\bar{\mu}} \Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^{\bar{\beta}}}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x^\mu \partial x^{\bar{\alpha}}} V^\mu + \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\beta_{\mu\alpha} V^\mu$$

Transformation of $V^\beta_{,\alpha}$

$$V^{\bar{\beta}}_{,\bar{\alpha}} = \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial}{\partial x^\alpha} \left(\frac{\partial x^{\bar{\beta}}}{\partial x^\beta} V^\beta \right)$$

$$V^{\bar{\beta}}_{,\bar{\alpha}} = \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\beta}}}{\partial x^\alpha \partial x^\beta} V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} V^\beta_{,\alpha}$$

$$V^{\bar{\beta}}_{,\bar{\alpha}} + V^{\bar{\mu}} \Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\beta}}}{\partial x^\alpha \partial x^\beta} V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} V^\beta_{,\alpha} + \frac{\partial x^{\bar{\beta}}}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x^\mu \partial x^{\bar{\alpha}}} V^\mu + \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\beta_{\mu\alpha} V^\mu$$

$\sigma \rightarrow \alpha$ and $\mu \rightarrow \beta$

$$V^{\bar{\beta}}_{,\bar{\alpha}} + V^{\bar{\mu}} \Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial^2 x^{\bar{\beta}}}{\partial x^\alpha \partial x^\beta} V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} V^\beta_{,\alpha} - \frac{\partial^2 x^{\bar{\beta}}}{\partial x^\beta \partial x^\alpha} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} V^\beta + \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\beta_{\mu\alpha} V^\mu$$

$$V^{\bar{\beta}}_{,\bar{\alpha}} + V^{\bar{\mu}} \Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} V^\beta_{,\alpha} + \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \Gamma^\beta_{\mu\alpha} V^\mu$$

$$V^{\bar{\beta}}_{,\bar{\alpha}} + V^{\bar{\mu}} \Gamma^{\bar{\beta}}_{\bar{\mu}\bar{\alpha}} = \frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}} \frac{\partial x^{\bar{\beta}}}{\partial x^\beta} \left(V^\beta_{,\alpha} + \Gamma^\beta_{\mu\alpha} V^\mu \right)$$

Therefore, the covariant derivative $V^\beta_{;\alpha} = V^\beta_{,\alpha} + \Gamma^\beta_{\mu\alpha} V^\mu$ is a tensor (of type $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$) because it transforms like tensors do.

Exercise 4.3

Exercise 4.3

Consider the Euclidean space with metric tensor $g = \tilde{dx} \otimes \tilde{dx} + \tilde{dy} \otimes \tilde{dy} + \tilde{dz} \otimes \tilde{dz}$.
(a) Compute the elements of the metric tensor in spherical coordinates, and use them to compute the differential proper volume dV and the squared line element ds^2 . (1 pt)

$ds^2 = d\tilde{x}^2 + d\tilde{y}^2 + d\tilde{z}^2$
 $(0,2)$
 $(0,1) \otimes (0,1) \Rightarrow (0,2)$
 $- \tilde{x}_t \otimes \tilde{x}_t$
 $[\tilde{dx} \ \tilde{dy} \ \tilde{dz}] \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} \tilde{dx} \\ \tilde{dy} \\ \tilde{dz} \end{bmatrix}$
 g (2)

$\Lambda = \begin{pmatrix} \sin(\theta) \cos(\varphi) & r \cos(\theta) \cos(\varphi) & -r \sin(\theta) \sin(\varphi) \\ \sin(\theta) \sin(\varphi) & r \cos(\theta) \sin(\varphi) & r \sin(\theta) \cos(\varphi) \\ \cos(\theta) & -r \sin(\theta) & 0 \end{pmatrix}$
 $\frac{\partial x}{\partial r}$ $\frac{\partial x}{\partial \theta}$ $\frac{\partial x}{\partial \varphi}$
 $A^{\bar{\mu}}_{\bar{\nu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\bar{\mu}}} \frac{\partial x^{\bar{\nu}}}{\partial x^{\bar{\nu}}} A_{\bar{\nu}}$
 $\frac{\partial x}{\partial x^{\bar{\nu}}}$ $\frac{\partial y}{\partial x^{\bar{\nu}}}$ $\frac{\partial z}{\partial x^{\bar{\nu}}}$

$x^\mu = (x, y, z)$ and $x^{\bar{\mu}} = (r, \theta, \varphi)$
 $x = r \sin(\theta) \cos(\varphi)$
 $y = r \sin(\theta) \sin(\varphi)$
 $z = r \cos(\theta)$
 $(0,2)$
 $g_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}} \frac{\partial x^\nu}{\partial x^{\bar{\nu}}} g_{\mu\nu}$
 $\Lambda^{\nu}_{\bar{\nu}} = \frac{\partial x^\nu}{\partial x^{\bar{\nu}}}$
 $\bar{g} = \Lambda^T g \Lambda$

$[\tilde{dr} \ \tilde{d\theta} \ \tilde{d\varphi}]$
 $\bar{g} = \Lambda^T \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2(\theta) \end{pmatrix}$
 $\begin{bmatrix} \tilde{dr} \\ \tilde{d\theta} \\ \tilde{d\varphi} \end{bmatrix}$
 $g = \tilde{dr} \otimes \tilde{dr} + r^2 [\tilde{d\theta} \otimes \tilde{d\theta} + \sin^2(\theta) \tilde{d\varphi} \otimes \tilde{d\varphi}]$



Squared line element $\Rightarrow ds^2 = d\vec{l} \cdot d\vec{l} = \mathbf{g}(d\vec{l}, d\vec{l}) = g_{\bar{\mu}\bar{\nu}} dx^{\bar{\mu}} dx^{\bar{\nu}} = dr^2 + r^2 [d^2\theta + \sin^2(\theta) d\varphi^2]$

Differential proper volume $\Rightarrow dV = \sqrt{|-\bar{g}|} dr d\theta d\varphi = r^2 \sin(\theta) dr d\theta d\varphi$

$dA = r dr d\theta$

(b) Write the one-form $\tilde{d}f$ in spherical coordinates. (0.5 pt)



$$\tilde{d}f \equiv \frac{\partial f}{\partial x^{\bar{\mu}}} \tilde{d}x^{\bar{\mu}} = \frac{\partial f}{\partial r} \tilde{d}r + \frac{\partial f}{\partial \theta} \tilde{d}\theta + \frac{\partial f}{\partial \varphi} \tilde{d}\varphi$$

MTW

(c) Compute the associated gradient vector in spherical coordinates and in terms of the unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$. (1 pt)

$$\vec{\nabla} = \left(\frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} \right)$$

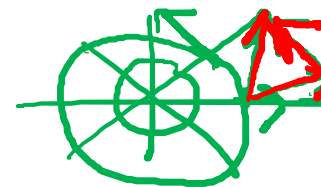
$$\tilde{d}f = \mathbf{g}(\vec{\nabla} f, \tilde{d}x) = g_{\bar{\mu}\bar{\nu}} (\vec{\nabla} f)^{\bar{\mu}} \tilde{d}x^{\bar{\nu}} = \frac{\partial f}{\partial x^{\bar{\nu}}} \tilde{d}x^{\bar{\nu}}$$

$$g_{\bar{\mu}\bar{\nu}} (\vec{\nabla} f)^{\bar{\mu}} = \frac{\partial f}{\partial x^{\bar{\nu}}} \xrightarrow{+g^{-1}} (\vec{\nabla} f)^{\bar{\mu}} = g^{\bar{\mu}\bar{\nu}} \frac{\partial f}{\partial x^{\bar{\nu}}}$$

$$x^{\bar{\nu}} = (r, \theta, \varphi)$$

$$\begin{bmatrix} (\vec{\nabla} f)^r \\ (\vec{\nabla} f)^\theta \\ (\vec{\nabla} f)^\varphi \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2} \sin^{-2}(\theta) \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \varphi} \end{pmatrix}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r^2} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi.$$



$$\begin{aligned} \vec{e}_r &= |\vec{e}_r| \hat{e}_r = \sqrt{\mathbf{g}(\vec{e}_r, \vec{e}_r)} \hat{e}_r = \sqrt{g_{rr}} \hat{e}_r = \hat{e}_r, \\ \vec{e}_\theta &= \dots = \sqrt{g_{\theta\theta}} \hat{e}_\theta = r \hat{e}_\theta, \\ \vec{e}_\varphi &= \dots = \sqrt{g_{\varphi\varphi}} \hat{e}_\varphi = r \sin(\theta) \hat{e}_\varphi. \end{aligned}$$

Unit vectors

$$\vec{\nabla} f = \frac{\partial f}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin(\theta)} \frac{\partial f}{\partial \varphi} \hat{e}_\varphi$$

(d) Express the basis vectors $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$ in terms of $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$, and viceversa. (0.75 pt)

Embedding: Put vector in Spherical coords, embedded in \mathbb{R}^3
 sy: 6s

$$\vec{e}_{\bar{\mu}} = \Lambda_{\bar{\mu}}^{\alpha} \vec{e}_{\alpha}$$

$$\vec{e}_{\bar{\mu}} = \Lambda_{\bar{\mu}}^{\mu} \vec{e}_{\mu}$$

Inverse

$$\vec{e}_{\mu} = \Lambda_{\mu}^{\bar{\mu}} \vec{e}_{\bar{\mu}}$$

$$\begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix} = \begin{pmatrix} \sin(\theta) \cos(\varphi) & \sin(\theta) \sin(\varphi) & \cos(\theta) \\ r \cos(\theta) \cos(\varphi) & r \cos(\theta) \sin(\varphi) & -r \sin(\theta) \\ -r \sin(\theta) \sin(\varphi) & r \sin(\theta) \cos(\varphi) & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix}$$

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} \sin(\theta) \cos(\varphi) & \frac{1}{r} \cos(\theta) \cos(\varphi) & -\frac{\sin(\varphi)}{r \sin(\theta)} \\ \sin(\theta) \sin(\varphi) & \frac{1}{r} \cos(\theta) \sin(\varphi) & \frac{\cos(\varphi)}{r \sin(\theta)} \\ \cos(\varphi) & -\frac{\sin(\theta)}{r} & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix}$$

$$\hat{r} = \sin\theta \cos\varphi \vec{e}_x + \sin\theta \sin\varphi \vec{e}_y + \cos\theta \vec{e}_z$$

$$A_{\mu}^{\bar{\mu}} = (A_1 \ A_2 \ A_3) \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

(e) Using (1) compute the Christoffel symbols $\Gamma^{\bar{\mu}}_{rr}$ and $\Gamma^{\bar{\mu}}_{r\theta}$. (0.75 pt)

Let us start with the computation of $\Gamma^{\bar{\mu}}_{r\theta}$

$$\frac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}_{r\theta} \vec{e}_{\bar{\mu}}$$

$$= \Gamma^r_{r\theta} \vec{e}_r + \Gamma^\theta_{r\theta} \vec{e}_\theta + \Gamma^\varphi_{r\theta} \vec{e}_\varphi$$

$$\frac{\partial e^\alpha}{\partial x^\beta} = \Gamma^\alpha_{\beta\gamma} e^\gamma$$

Use \vec{e}_r result from the previous slide, and differentiate it component-wise

$$\begin{aligned} \frac{\partial \vec{e}_r}{\partial \theta} &= \cos(\theta) \cos(\varphi) \vec{e}_x + \cos(\theta) \sin(\varphi) \vec{e}_y - \sin(\theta) \vec{e}_z = \frac{\vec{e}_\theta}{r} \\ &= \cos(\theta) \cos(\varphi) \left[\sin(\theta) \cos(\varphi) \vec{e}_r + \frac{1}{r} \cos(\theta) \cos(\varphi) \vec{e}_\theta - \frac{\sin(\varphi)}{r \sin(\theta)} \vec{e}_\varphi \right] + 0 \vec{e}_\varphi + 0 \vec{e}_r + \frac{\vec{e}_\theta}{r} \\ &+ \cos(\theta) \sin(\varphi) \left[\sin(\theta) \sin(\varphi) \vec{e}_r + \frac{1}{r} \cos(\theta) \sin(\varphi) \vec{e}_\theta + \frac{\cos(\varphi)}{r \sin(\theta)} \vec{e}_\varphi \right] - \sin(\theta) \left[\cos(\theta) \vec{e}_r - \frac{\sin(\theta)}{r} \vec{e}_\theta \right] = \dots \end{aligned}$$

After simplification we get

$$\frac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}_{r\theta} \vec{e}_{\bar{\mu}} = \frac{\vec{e}_\theta}{r}$$



$$\Gamma^r_{r\theta} = \Gamma^\varphi_{r\theta} = 0$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r}$$

The computation of $\Gamma^{\bar{\mu}}_{rr}$ is straightforward.

$$\frac{\partial \vec{e}_r}{\partial r} = 0 = \Gamma^{\bar{\mu}}_{rr} \vec{e}_{\bar{\mu}} \longrightarrow \boxed{\Gamma^{\bar{\mu}}_{rr} = 0}$$

(f) Compute all the Christoffel symbols in spherical coordinates using the formula

$$\Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}} = \frac{g^{\bar{\mu}\bar{\sigma}}}{2} (g_{\bar{\sigma}\bar{\alpha},\bar{\beta}} + g_{\bar{\sigma}\bar{\beta},\bar{\alpha}} - g_{\bar{\alpha}\bar{\beta},\bar{\sigma}}). \quad (1 \text{ pt})$$

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\varphi}_{r\varphi} = \frac{1}{r} \quad \Gamma^r_{\theta\theta} = -r \quad \Gamma^r_{\varphi\varphi} = -r \sin^2(\theta)$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\frac{1}{2} \sin(2\theta) \quad \Gamma^{\varphi}_{\varphi\theta} = \frac{\cos(\theta)}{\sin(\theta)}$$

All non-zero Christoffel symbols

$$\Gamma^r_{\theta\theta} = \frac{g^{r\sigma}}{2} (g_{\sigma\theta,\theta} + g_{\sigma\theta,\theta} - g_{\theta\theta,\sigma})$$

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} g^{rr} (g_{r\theta,\theta} + g_{r\theta,\theta} - g_{\theta\theta,r}) \\ g^{r\theta} (g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}) \\ g^{r\varphi} (g_{\theta\theta,\varphi} + g_{\theta\theta,\varphi} - g_{\theta\theta,\varphi}) \end{bmatrix}$$

$$= \frac{1}{2} g^{rr} (-g_{\theta\theta,r})$$

$$= -\frac{1}{2} 1 (2r) = -r$$

(g) Compute the Laplacian of the function f , i.e. $\nabla^2 f$, in spherical coordinates. (1 pt)

$$\nabla^2 f = \mathbf{g}(\vec{\nabla}, \vec{\nabla} f) = g_{\bar{\mu}\bar{\nu}} \vec{\nabla}^{\bar{\mu}} (\vec{\nabla} f)^{\bar{\nu}} = \nabla_{\bar{\mu}} (\vec{\nabla} f)^{\bar{\mu}} = \partial_{\bar{\mu}} (\vec{\nabla} f)^{\bar{\mu}} + \Gamma^{\bar{\mu}}_{\bar{\kappa}\bar{\mu}} (\vec{\nabla} f)^{\bar{\kappa}}$$

Sum over the indices

$$\nabla^2 f = \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{1}{r^2} \frac{\partial f}{\partial \theta} \right) + \frac{\partial}{\partial \varphi} \left(\frac{1}{r^2 \sin^2(\theta)} \frac{\partial f}{\partial \varphi} \right) + \Gamma^{\theta}_{r\theta} \frac{\partial f}{\partial r} + \Gamma^{\varphi}_{r\varphi} \frac{\partial f}{\partial r} + \Gamma^{\varphi}_{\theta\varphi} \frac{1}{r^2} \frac{\partial f}{\partial \varphi}$$

Use the Christoffel symbols we calculated before

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \varphi^2}$$

