GR Exercise sheet 10 Name: Vito Aberham

$\boxed{\text{Nr. 10.1)}}$

Power that is radiated away by time-dependent matter source. Far field-approx + Lorentz gauge $\partial_\mu \bar{h}^{\mu\nu} = 0$:

$$\bar{h}^{ij}(t, \vec{r}) = \frac{2G}{r} \frac{d^2 I^{ij}}{dt^2}\bigg|_{t=t_R} \qquad t_R = t - \frac{|\vec{x}|}{c} \qquad |\vec{x}| = r$$

$$dP = \frac{1}{32\pi G} \langle \partial_0 \bar{h}^{TT\,je} \partial_0 \bar{h}^{TT}_{je} \rangle dS = n^i \langle t_{0i} \rangle dS = \frac{\overbrace{n^i n_i}^{=1}}{32\pi G} \langle \partial_0 \bar{h}^{TT\,je} \partial_0 \bar{h}^{TT}_{je} \rangle dS$$

$\boxed{\text{(a)}}$ Projector $P_{ij} = \delta_{ij} - n_i n_j \qquad P_{ij} P_{jk} = P_{ik}$

$$\rightarrow \bar{h}^{TT}_{ij} = \left(P^a_i P^b_j - \frac{1}{2} P_{ij} P^{ab}\right) \bar{h}_{ab} = \Lambda_{ij}{}^{ab} \bar{h}_{ab}$$

Insert this into dP:

$$dP = \frac{1}{32\pi G} \langle (\partial_0 \Lambda^{ij}{}_{ab} \bar{h}^{ab})(\partial_0 \Lambda_{ij}{}^{cd} \bar{h}_{cd}) dS$$

time independence of the Proj. P_{ij}

$$\Rightarrow = \frac{1}{32\pi G} \underbrace{\Lambda^{ij}{}_{ab} \Lambda_{ij}{}^{cd}} \langle \partial_0 \bar{h}^{ab} \partial_0 \bar{h}_{cd} \rangle dS$$

$$\Lambda^{ij}{}_{ab} \Lambda_{ij}{}^{cd} = \left(P^i_a P^j_b - \frac{1}{2} P^{ij} P_{ab}\right)\left(P^c_i P^d_j - \frac{1}{2} P_{ij} P^{cd}\right)$$

$$= P^i_a P^j_b P^c_i P^d_j + \frac{1}{4} P^{ij} P_{ab} P_{ij} P^{cd} - \frac{1}{2} P^{ij} P_{ab} P^c_i P^d_j - \frac{1}{2} P^i_a P^j_b P_{ij} P^{cd}$$

$$= P^c_a P^d_b + \frac{1}{4} (P^{ij} P_{ij}) P_{ab} P^{cd} - \frac{1}{2} P^{cd} P_{ab} - \frac{1}{2} P_{ab} P^{cd}$$

$$P^{ij} P_{ij} = \mathrm{Tr}\, P = P^i_i = \underbrace{\delta^i_i}_{=3} - \underbrace{n_i n^i}_{=1} = 3 - 1 = 2$$

$= 1$ because n are unitvectors

$$\Rightarrow \Lambda^{ij}{}_{ab} = P^c_a P^d_b + \frac{2}{4} P_{ab} P^{cd} - 1 \cdot P_{ab} P^{cd}$$

$$= P^c_a P^d_b + \left(\frac{1}{2} - 1\right) P_{ab} P^{cd}$$

$$= P^c_a P^d_b - \frac{1}{2} \underbrace{P_{ab} P^{cd}}$$

$$(\delta_{ab} - n_a n_b)(\delta^{cd} - n^c n^d)$$

$$= \delta_{ab} \delta^{cd} - n_a n_b \delta^{cd} - n^c n^d \delta_{ab} + n_a n_b n^c n^d$$

$$= P^{cd}_{ab}$$

$$= P^c_a P^d_b - \frac{1}{2} P^{cd}_{ab} \,/\!/$$

$$\Rightarrow dP = \frac{1}{32\pi G} \left(P^c_a P^d_b - \frac{1}{2} P^{cd}_{ab}\right) \langle \partial_0 \bar{h}^{ab} \partial_0 \bar{h}_{cd} \rangle dS$$

(b) From a): $\frac{dP}{dS} = \frac{1}{32\pi G}\left(P^c{}_a P^d{}_b - \frac{1}{2} P^{cd}{}_{ab}\right)\langle \partial_0 \bar{h}^{ab}\partial_0 \bar{h}_{cd}\rangle$

①: $\langle \partial_0 \bar{h}^{ab}\partial_0 \bar{h}_{cd}\rangle = \left\langle \partial_0\left(\frac{2G}{r}\frac{d^2 I^{ab}}{dt^2}\right)\Big|_{t=t_R} \partial_0\left(\frac{2G}{r}\frac{d^2 I_{cd}}{dt^2}\right)\Big|_{t=t_R}\right\rangle$

$= \frac{4G^2}{r^2}\left\langle\left(\frac{d}{dt}\frac{d^2}{dt^2}I^{ab}\right)\left(\frac{d}{dt}\frac{d^2}{dt^2}I_{cd}\right)\right\rangle\Big|_{t=t_R}$

$= \frac{4G}{r^2}\left\langle\left(\frac{d^3}{dt^3}I^{ab}\right)\left(\frac{d^3 I_{cd}}{dt^3}\right)\right\rangle\Big|_{t=t_R}$ $\quad t_R = t - \frac{r}{c} \quad$ $I_{cd} = \delta_{ic}\delta_{jd}I^{ij}$

②: $P^c{}_a P^d{}_b - \frac{1}{2}P^{cd}{}_{ab} = (\delta^c{}_a - n^c n_a)(\delta^d{}_b - n^d n_b) - \frac{1}{2}(\delta_{ab} - n_a n_b)(\delta^{cd} - n^c n^d)$

$= \delta^c_a \delta^d_b - \delta^d_b n^c n_a - n^d n_b \delta^c_a + n^c n^d n_a n_b - \frac{1}{2}(\delta_{ab}\delta^{cd} - n_a n_b \delta^{cd}$
$- n^c n^d \delta_{ab} + n_a n_b n^c n^d)$

$I_{cd} \to I^{cd}$ in ① ⇒ multiply by $\delta_{ic}\delta_{jd}$ and relabel $i \to c, j \to d$

⇒ $\delta_{ac}\delta_{bd} - \delta_{bd}n_c n_a - n_d n_b \delta_{ac} + n_c n_d n_a n_b - \frac{1}{2}(\delta_{ab}\delta_{cd} - n_a n_b \delta_{cd}$
$- n_c n_d \delta_{ab} + n_a n_b n_c n_d)$

$= \delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} + \frac{1}{2}n_a n_b n_c n_d - \delta_{bd}n_a n_c - \delta_{ac}n_d n_b + \frac{1}{2}n_a n_b \delta_{cd}$
$+ \frac{1}{2}\delta_{ab}n_c n_d$

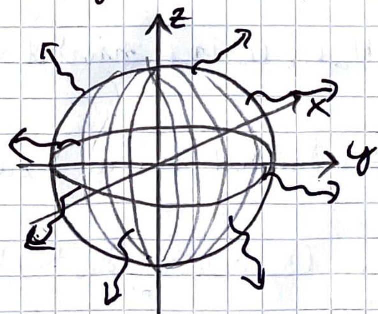
$= \delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} + \frac{1}{2}n_a n_b n_c n_d - \underbrace{(\delta_{bd}n_a n_c + \delta_{ac}n_d n_b)}_{\substack{\text{symmetric in} \\ ac \leftrightarrow bd \\ = 2\delta_{ac}n_b n_d}}$
$+ \frac{1}{2}\underbrace{(n_a n_b \delta_{cd} + n_c n_d \delta_{ab})}_{\substack{\text{symmetric in } ab \leftrightarrow cd \\ = 2 n_c n_d \delta_{ab}}}$

$= \delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - 2\delta_{ac}n_b n_d + \frac{1}{2}n_a n_b n_c n_d + \frac{\cancel{2}}{\cancel{2}}n_c n_d \delta_{ab}$

⇒ $\frac{dP}{dS} = \frac{4G^2}{r^2 \cdot 32\pi G}\left(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - 2\delta_{ac}n_b n_d + \frac{1}{2}n_a n_b n_c n_d\right)\left\langle\frac{d^3 I^{ab}}{dt^3}\frac{d^3 I^{cd}}{dt^3}\right\rangle\Big|_{t_R}$

$= \frac{G}{8\pi r^2}\left(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - 2\delta_{ac}n_b n_d + \frac{1}{2}n_a n_b n_c n_d + n_c n_d \delta_{ab}\right)\left\langle\frac{d^3 I^{ab}}{dt^3}\frac{d^3 I^{cd}}{dt^3}\right\rangle\Big|_{t_R}$

(c) Integrate $\frac{dP}{dS}$ over sphere to get total power



$P = \int_S \frac{dP}{dS}\, dS = P$

In spherical coordinates:

$dS = r^2 \sin\vartheta \, d\vartheta \, d\varphi$

$\int_S \frac{dP}{dS}\, dS = \frac{G}{8\pi}\left\{\int_S \frac{1}{\cancel{r^2}}\left(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - 2\delta_{ac}n_b n_d + \frac{1}{2}n_a n_b n_c n_d + n_c n_d \delta_{ab}\right)\right.$
$\left. \cdot \langle \dddot{I}^{ab}\dddot{I}^{cd}\rangle\Big|_{t=t_R} \cancel{r^2}\sin\vartheta \, d\vartheta \, d\varphi\right\}$

$$= \frac{G}{8\pi}\left\{ \int_S \left(\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd}\right) \langle \dddot{I}^{ab}\dddot{I}^{cd}\rangle\Big|_{t=t_R} \sin\vartheta \, d\vartheta \, d\varphi \right\}$$

$$+ \frac{G}{8\pi}\left\{ \int_S \left(-2\delta_{ac}\, n_b n_d + \frac{1}{2} n_a n_b n_c n_d + n_c n_d \delta_{ab}\right) \langle \dddot{I}^{ab}\dddot{I}^{cd}\rangle\Big|_{t=t_R} \sin\vartheta \, d\vartheta \, d\varphi \right\}$$

It is $\int_S 1 \cdot dS = \int_0^{2\pi} d\varphi \int_0^{\pi} d\vartheta \sin\vartheta = 4\pi$

$\Rightarrow$ The δ don't depend on ϑ or φ so the integration of the first 2 terms will give 4π

Look at the integration of the normal vectors:

1. $\frac{1}{r^2}\int_S n_b n_d \, dS = \int_0^{2\pi}\int_0^{\pi} n_b n_d \sin\vartheta \, d\varphi \, d\vartheta$ $\qquad (n_b) = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} \cdot 1$

Because of symmetry: $\int_0^{2\pi}\int_0^{\pi} \sin^2\vartheta \cos\vartheta \cos\varphi \, d\varphi \, d\vartheta = 0$

$$= \int_0^{2\pi}\int_0^{\pi} n_1 n_3 \sin\vartheta \, d\varphi \, d\vartheta$$

and $\int_0^{2\pi}\int_0^{\pi} n_1 n_2 \, d\varphi \, d\vartheta \sin\vartheta = \int_0^{2\pi}\int_0^{\pi} n_2 n_3 \sin\vartheta \, d\varphi \, d\vartheta = 0$

Only the terms $\int_0^{2\pi}\int_0^{\pi} n_b n^b \sin\vartheta \, d\varphi \, d\vartheta \neq 0$

$$\Rightarrow \int_0^{2\pi}\int_0^{\pi} n_b n_d \sin\vartheta \, d\varphi \, d\vartheta = \frac{\delta_{bd}}{3} \int_0^{2\pi}\int_0^{\pi} n_b n^b \sin\vartheta \, d\varphi \, d\vartheta$$

$3 \rightarrow$ because $\delta_{bb} = 3$

$$= \frac{\delta_{bd}}{3} \int_0^{2\pi}\int_0^{2\pi} \left(\sin^2\vartheta\,(\sin^2\varphi + \cos^2\varphi) + \cos^2\vartheta\right) \sin\vartheta \, d\varphi \, d\vartheta$$

$$= \frac{\delta_{bd}}{3} \int_0^{2\pi}\int_0^{2\pi} \sin\vartheta \, d\varphi \, d\vartheta = \frac{\delta_{bd}}{3} \cdot 4\pi = \frac{4\pi}{3}\delta_{bd}$$

2. $\int_0^{2\pi}\int_0^{\pi} n_a n_b n_c n_d \sin\vartheta \, d\varphi \, d\vartheta$ $\qquad$ Introduce the 3-vectors

$$z^i = \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix}, \quad n_i z^i = \begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix}\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

1
1 2 1
1 3 3 1
1 4 6 4 1

$$\int_0^{2\pi}\int_0^{\pi} n_a n_b n_c n_d \, z^a z^b z^c z^d \sin\vartheta \, d\varphi \, d\vartheta$$

$$= \int_0^{2\pi}\int_0^{\pi} \left(\begin{pmatrix} z^1 \\ z^2 \\ z^3 \end{pmatrix}\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}\right)^4 \sin\vartheta \, d\varphi \, d\vartheta = \int_0^{2\pi}\int_0^{\pi} \left(z^1 \sin\vartheta\cos\varphi + z^2 \sin\vartheta\sin\varphi + z^3\cos\vartheta\right)^4 \sin\vartheta \, d\varphi \, d\vartheta$$

$$= \int_0^{2\pi}\int_0^{\pi} \Big[(z^1 \sin\vartheta\cos\varphi + z^2 \sin\vartheta\sin\varphi)^4 + (z^3\cos\vartheta)^4 + 4(z^1 \sin\vartheta\cos\varphi + z^2\sin\vartheta\sin\varphi)^3 \cdot z^3\cos\vartheta$$
$$+ 6(z^1\sin\vartheta\cos\varphi + z^2\sin\vartheta\sin\varphi)^2 (z^3\cos\vartheta)^2$$
$$+ 4(z^1\sin\vartheta\cos\varphi + z^2\sin\vartheta\sin\varphi)(z^3\cos\vartheta)^3 \Big] \sin\vartheta \, d\varphi \, d\vartheta$$

$= \ldots$ next page

$$= \int_0^{2\pi}\int_0^{\pi} \Big[(z^1)^4 \sin^4\vartheta\cos^4\varphi + (z^2)^4 \sin^4\vartheta\sin^4\varphi + 4(z^1)^3 \sin^3\vartheta\cos^3\varphi\, z^2 \sin\vartheta\sin\varphi$$
$$+ 4 z^1 \sin\vartheta\cos\varphi (z^2)^3 \sin^3\vartheta\sin^3\varphi + (z^3)^4\cos^4\vartheta$$
$$+ 4\big((z^1)^3 \sin^3\vartheta\cos^3\varphi + (z^2)^3\sin^3\vartheta\sin^3\varphi + 3(z^1)^2\sin^2\vartheta\cos^2\varphi\, z^2 \sin\vartheta\sin\varphi$$
$$+ 3 z^1 \sin\vartheta\cos\varphi\,(z^2)^2\sin^2\vartheta\sin^2\varphi\big) z^3\cos\vartheta$$
$$+ 6(z^3)^2\cos^2\vartheta\big((z^1)^2\sin^2\vartheta\cos^2\varphi + 2z^1z^2\sin^2\vartheta\sin\varphi\cos\varphi + (z^2)^2\sin^2\vartheta\sin^2\varphi\big)$$
$$+ 4 z^1 (z^3)^3 \sin\vartheta\cos^3\vartheta\cos\varphi + 4 z^2 (z^3)^3\sin\vartheta\cos^3\vartheta\sin\varphi\Big]$$
$$\sin\vartheta\, d\varphi\, d\vartheta$$

$$= \int_0^{2\pi}\int_0^{\pi}\Big[(z^1)^4\sin^5\vartheta\cos^4\varphi + (z^2)^4\sin^5\vartheta\sin^4\varphi + 4(z^1)^3\sin^5\vartheta\, z^2\cos^3\varphi\sin\varphi$$
$$+ 4z^1(z^2)^3\sin^5\vartheta\cos\varphi\sin^3\varphi + (z^3)^4\cos^4\vartheta + 4(z^1)^3z^3\sin^4\vartheta\cos\vartheta\cos^3\varphi$$
$$+ 4(z^2)^3z^3\sin^4\vartheta\sin^3\varphi\cos\vartheta + 12(z^1)^2\sin^4\vartheta\cos^2\varphi\sin\varphi\, z^2z^3\cos\vartheta$$
$$+ 12 z^1 z^3 \sin^4\vartheta\cos\vartheta\cos\varphi\sin^2\varphi + 6(z^3)^2\cos^2\vartheta(z^1)^2\sin^3\vartheta\cos^2\varphi$$
$$+ 12 z^1 z^2 (z^3)^2\cos^2\vartheta\sin^3\vartheta\sin\varphi\cos\varphi + 6(z^3)^2(z^2)^2\sin^3\vartheta\cos^2\vartheta\sin^2\varphi$$
$$+ 4z^1(z^3)^3\sin^2\vartheta\cos^3\vartheta\cos\varphi + 4(z^2)(z^3)^3\sin^2\vartheta\cos^3\vartheta\sin\varphi\Big]\,d\varphi\, d\vartheta$$

Due to symmetry many terms will vanish. Mathematica gives the following result (only the $(z_i)^4$ or $(z_i)^2(z_j)^2$ terms contribute):

$$= \frac{4\pi}{5}\left((z^1)^2+(z^2)^2+(z^3)^2\right)^2 = \frac{4\pi}{5}\left((z^1)^2+(z^2)^2+(z^3)^2\right)\cdot\left((z^1)^2+(z^2)^2+(z^3)^2\right)$$

$$= \frac{4\pi}{5}(z_a z^a)(z_c z^c) = \frac{4\pi}{5}(z^a z^b \delta_{ab})(z^c z^d\delta_{cd})$$

$$= \frac{4\pi}{5} z^a z^b z^c z^d \delta_{ab}\delta_{cd} = \int_0^{2\pi}\int_0^{\pi} n_a n_b n_c n_d\, z^a z^b z^c z^d \sin\vartheta\, d\vartheta\, d\varphi$$

$$\overset{z^i=1}{\Longrightarrow} \int_0^{2\pi}\int_0^{\pi} n_a n_b n_c n_d \sin\vartheta\, d\varphi\, d\vartheta = \frac{4\pi}{5}\delta_{ab}\delta_{cd}$$

$$= \frac{4\pi}{15}\left(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}\right) \quad \text{symmetrization}$$

$$\Rightarrow P = \frac{G}{8\pi}\left\{\int_S\!\!\int \left(\delta_{ac}\delta_{bd} - \tfrac{1}{2}\delta_{ab}\delta_{cd}\right)\langle \dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R}\sin\vartheta\, d\vartheta\, d\varphi\right\}$$

$$+ \frac{G}{8\pi}\left\{\int_S\!\!\int \left(-2\delta_{ac}n_b n_d + \tfrac{1}{2} n_a n_b n_c n_d + n_c n_d \delta_{ab}\right)\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R}\sin\vartheta\, d\varphi\, d\vartheta\right.$$

$$= \frac{G}{8\pi}\Big\{ 4\pi\,\delta_{ac}\delta_{bd} - \frac{4\pi}{2}\delta_{ab}\delta_{cd} - 2\delta_{ac}\cdot\frac{4\pi}{3}\delta_{bd} + \frac{1}{2}\frac{4\pi}{15}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) + \frac{4\pi}{3}\delta_{cd}\delta_{ab}\Big\}$$

$$\cdot \langle \dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R} \qquad + \delta_{cd}\delta_{ab}\cdot\frac{1}{3}$$

$$= \frac{G\,4\pi}{8\pi}\Big\{\delta_{ac}\delta_{bd} - \frac{1}{2}\delta_{ab}\delta_{cd} - \frac{2}{3}\delta_{ac}\delta_{bd} + \frac{1}{30}(\delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc} + 10\,\delta_{cd}\delta_{ab})\Big\}\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle_{t_R}$$

$$= \frac{G}{2}\Big\{\frac{1}{30}(30\,\delta_{ac}\delta_{bd} - 15\,\delta_{ab}\delta_{cd} - 20\,\delta_{ac}\delta_{bd} + \delta_{ab}\delta_{cd} + \delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc})\Big\}$$

$$\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R} = \frac{G}{2}\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R}\Big\{\frac{1}{30}\Big(30\,\delta_{ac}\delta_{bd} - 20\,\delta_{ac}\delta_{bd} + \delta_{ac}\delta_{bd} + \underbrace{\delta_{ad}\delta_{bc}}_{=\delta_{ac}\delta_{bd}\text{ because of symmetry}} - 15\,\delta_{ab}\delta_{cd} + 10\,\delta_{ab}\delta_{cd} + \delta_{ab}\delta_{cd}\Big)\Big\}$$

$$= \frac{G}{2}\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R}\Big(\frac{1}{30}\big(12\,\delta_{ac}\delta_{bd} - 4\,\delta_{ab}\delta_{cd}\big)\Big)$$

$$= \frac{G}{2}\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R}\Big(\frac{2}{5}\delta_{ac}\delta_{bd} - \frac{2}{15}\delta_{ab}\delta_{cd}\Big)$$

$$= \frac{\cancel{2}G}{5\,\cancel{2}}\Big(\delta_{ac}\delta_{bd}\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t_R} - \frac{1}{3}\delta_{ab}\delta_{cd}\langle\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t_R}\Big)$$

$$= \frac{G}{5}\Big(\langle\delta_{ac}\delta_{bd}\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R} - \frac{1}{3}\langle\delta_{ab}\delta_{cd}\dddot{I}^{ab}\dddot{I}^{cd}\rangle\big|_{t=t_R}\Big)$$

$$= \frac{G}{5}\Big(\langle\dddot{I}^{ab}\dddot{I}_{ab}\rangle\big|_{t=t_R} - \frac{1}{3}\langle\dddot{I}^{a}{}_{a}\,\dddot{I}^{c}{}_{c}\rangle\big|_{t=t_R}\Big)$$

$$\Rightarrow \boxed{P = \frac{G}{5}\Big\langle\frac{d^3I^{ab}}{dt^3}\frac{d^3I_{ab}}{dt^3} - \frac{1}{3}\frac{d^3I}{dt^3}\frac{d^3I}{dt^3}\Big\rangle\Big|_{t_R}}$$

(d) Reduced quadrupole moment $\mathcal{I}^{ij} = I^{ij} - \frac{1}{3}\delta^{ij}I$

$$\Rightarrow I^{ij} = \mathcal{I}^{ij} + \frac{1}{3}\delta^{ij}I$$

$$\overset{(c)}{\Rightarrow} P = \frac{G}{5}\Big\langle\frac{d^3}{dt^3}\Big(\mathcal{I}^{ij} + \frac{1}{3}\delta^{ij}I\Big)\frac{d^3}{dt^3}\Big(\mathcal{I}_{ij} + \frac{1}{3}\delta_{ij}I\Big) - \frac{1}{3}\frac{d^3I}{dt^3}\frac{d^3I}{dt^3}\Big\rangle\Big|_{t_R}$$

$$= \frac{G}{5}\Big(\Big\langle\frac{d^3\mathcal{I}^{ij}}{dt^3}\frac{d^3\mathcal{I}_{ij}}{dt^3} + \frac{d^3\mathcal{I}^{ij}}{dt^3}\frac{1}{3}\delta_{ij}\frac{d^3I}{dt^3} + \frac{d^3\mathcal{I}_{ij}}{dt^3}\frac{1}{3}\delta^{ij}\frac{d^3I}{dt^3} + \frac{1}{9}\underbrace{\delta^{ij}\delta_{ij}}_{=\delta^i_i=3}\frac{d^3I}{dt^3}\frac{d^3I}{dt^3}\Big\rangle - \frac{1}{3}\Big\langle\frac{d^3I}{dt^3}\frac{d^3I}{dt^3}\Big\rangle\Big)_{t_R}$$

$$= \frac{G}{5}\Big\langle\frac{d^3\mathcal{I}^{ij}}{dt^3}\frac{d^3\mathcal{I}_{ij}}{dt^3} + \cancel{\frac{1}{3}\frac{d^3I}{dt^3}\frac{d^3I}{dt^3}} - \cancel{\frac{1}{3}\frac{d^3I}{dt^3}\frac{d^3I}{dt^3}} + \frac{1}{3}\frac{d^3I}{dt^3}\frac{d^3}{dt^3}(\delta_{ij}\mathcal{I}^{ij}) + \frac{1}{3}\frac{d^3I}{dt^3}\frac{d^3}{dt^3}(\delta^{ij}\mathcal{I}_{ij})\Big\rangle_{t_R}$$

It is $\delta_{ij}\mathcal{I}^{ij} = \delta_{ij}I^{ij} - \frac{1}{3}\delta^{ij}\delta_{ij}I = I - \frac{\cancel{3}}{3}I = I - I = 0 \quad (\delta^i_i = 3)$

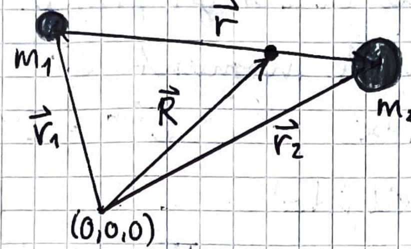
$$\delta^{ij}\mathcal{I}_{ij} = \delta^{ij}I_{ij} - \frac{1}{3}\delta_{ij}\delta^{ij}I = I - I = 0$$

$$\Longrightarrow \boxed{P = \frac{G}{5}\left\langle \frac{d^3 \mathcal{I}^{ij}}{dt^3} \frac{d\mathcal{I}_{ij}}{dt^3} \right\rangle\Bigg|_{t_R}} \qquad t_R = t - \frac{r}{c}$$

Power radiated away from time-dependent matter source.

Nr. 10.2) GW's from binary source + inspiral time

(a)



$$\vec{r} = \vec{r}_2 - \vec{r}_1$$
$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

Newton's theory: On the mass m_1 the following force is acting:

$$m_1 \ddot{\vec{r}}_1 = \frac{G m_1 m_2}{r^3} (\vec{r}_2 - \vec{r}_1) = \vec{F}_1$$
$$m_2 \ddot{\vec{r}}_2 = \frac{G m_1 m_2}{r^3} (\vec{r}_1 - \vec{r}_2) = \vec{F}_2$$

$\Rightarrow$ Newton 3: $\vec{F}_1 = -\vec{F}_2$

Now look at the center of mass: $m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2$

$$= \frac{m_1 \ddot{\vec{r}}_1 + m_2 \ddot{\vec{r}}_2}{m_1 + m_2} (m_1 + m_2) \underset{\substack{m_1, m_2 \\ = \text{const.}}}{=} (m_1 + m_2) \ddot{\vec{R}} = \vec{F}_1 + \vec{F}_2$$
$$= \vec{F}_1 - \vec{F}_1 = 0$$

$\Rightarrow \ddot{\vec{R}} = 0 \Rightarrow$ Velocity vector of the center of mass $\vec{v} = \frac{d\vec{R}}{dt} = \text{const.}$ in order to fulfill $\frac{d}{dt}\frac{d\vec{R}}{dt} = 0$

(b) Reference system that is comoving to center of mass: " ' "

$$\vec{r}_1{}' = \vec{r}_1 - \vec{R} = \frac{\vec{r}_1 m_1 + \vec{r}_1 m_2 - (m_1 \vec{r}_1 + m_2 \vec{r}_2)}{m_1 + m_2}$$
$$= \frac{\vec{r}_1 m_2 - m_2 \vec{r}_2}{m_1 + m_2} = \frac{-m_2 (\vec{r}_2 - \vec{r}_1)}{m_1 + m_2} = \frac{-m_2}{m_1 + m_2} \vec{r}$$

$$\vec{r}_2{}' = \vec{r}_2 - \vec{R} = \frac{\vec{r}_2 m_1 + \cancel{\vec{r}_2 m_2} - m_1 \vec{r}_1 - \cancel{m_2 \vec{r}_2}}{m_1 + m_2} = \frac{m_1 (\vec{r}_2 - \vec{r}_1)}{m_1 + m_2}$$
$$= \frac{+m_1}{m_2 + m_1} \vec{r}$$

(c) Angular momentum of the total system:

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times \frac{m_1 m_2}{m_1 + m_2} \frac{d\vec{r}}{dt} = \vec{r} \times \mu \frac{d\vec{r}}{dt}$$ with

μ = reduced mass

$$\frac{d\vec{L}}{dt} = \underbrace{\frac{d\vec{r}}{dt} \times \mu \frac{d\vec{r}}{dt}}_{=0 \text{ because } \vec{v} = \frac{d\vec{r}}{dt} \parallel \vec{p} = \mu \frac{d\vec{r}}{dt}} + \overbrace{\vec{r} \times \frac{d^2 \vec{r}}{dt^2}} = \vec{r} \times \frac{1}{\mu} \vec{F} = 0$$ because $\vec{r} \parallel \vec{F}(\vec{r})$ for central forces

$\Rightarrow \frac{d\vec{L}}{dt} = 0 \rightarrow \vec{L}$ is constant $\Rightarrow$ the movement of the two masses happens in the same plane

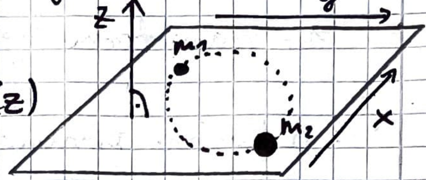
$\boxed{(d)}$ Reduced quadrupole moment in rotating system

$$I^{ij} = I^{ij} - \frac{1}{3}\delta^{ij} I \qquad I^{ij} = \int T^{00} x^i x^j d^3x$$

T^{00} is the energy density $(c=1)$:

$$T^{00}(t, x^n) = \sum_{n=1}^{2} m_n \, \delta(x-x_n)\delta(y-y_n)\delta(z)$$

where $\vec{r}_n = \begin{pmatrix} x_n \\ y_n \\ z \end{pmatrix}$



planar motion (see c))

$$\Rightarrow I^{ij} = \int T^{00}(t, x^n) x^i x^j d^3x = \sum_{n=1}^{2} \iiint m_n \, \delta(x-x_n)\delta(y-y_n)\delta(z)\, dx\, dy\, dz \cdot x^i x^j$$

$$\Rightarrow I^{xx} = \sum_{n=1}^{2} \iiint m_n (x^2)\, \delta(x-x_n)\delta(y-y_n)\delta(z)\, dx\, dy\, dz = \sum_{n=1}^{2} m_n x_n^2$$

$$= m_1 x_1^2 + m_2 x_2^2$$

$$I^{yy} = \sum_{n=1}^{2} \iiint m_n y^2 \delta(x-x_n)\delta(y-y_n)\delta(z)\, dx\, dy\, dz = \sum_{n=1}^{2} m_n y_n^2$$

$$= m_1 y_1^2 + m_2 y_2^2$$

$$I^{zz} = \sum_{n=1}^{2} \iiint \underbrace{m_n z^2 \delta(z) dz}_{=0} \, \delta(x-x_n)\delta(y-y_n)\, dx\, dy = 0$$

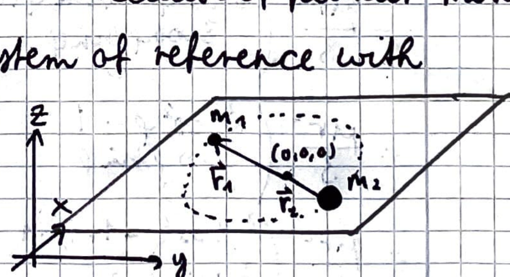
$$I^{xy} = I^{yx} = \sum_{n=1}^{2} \iiint m_n x y \, \delta(x-x_n)\delta(y-y_n)\delta(z)\, dx\, dy\, dz$$

$$= \sum_{n=1}^{2} m_n x_n y_n = m_1 x_1 y_1 + m_2 y_2 x_2 = I^{yx}$$

$I^{xz} = I^{zx} = I^{yz} = I^{zy} = 0$ because of planar motion

Now specify to co-rotating system of reference with origin at the center of mass

From b): $\vec{r}_1{}'$ and $\vec{r}_2{}'$ in this frame:



$$\vec{r}_1{}' = -\frac{m_2}{m_1+m_2}\vec{r}$$

$$\vec{r}_2{}' = \frac{m_1}{m_1+m_2}\vec{r} \qquad \vec{r} = \vec{r}_2 - \vec{r}_1 \Rightarrow r = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$I^{xx} = m_1 x_1'^2 + m_2 x_2'^2 = m_1 \left(\frac{-m_2}{m_1+m_2}\right)^2 (x_2-x_1)^2 + m_2 \left(\frac{m_1}{m_1+m_2}\right)^2 (x_2-x_1)^2$$

$$= \left(m_1 \left(\frac{m_2}{m_1+m_2}\right)^2 + m_2 \left(\frac{m_1}{m_1+m_2}\right)^2\right)(x_2-x_1)^2$$

$$= \frac{m_1 m_2^2 + m_2 m_1^2}{(m_1+m_2)^2}(x_2-x_1)^2 = \frac{(m_1+m_2)m_1 m_2}{(m_1+m_2)^2}(x_2-x_1)^2$$

$$= \frac{m_1 m_2}{m_1+m_2}(x_2-x_1)^2$$

Similarly: $I^{yy} = m_1 y_1'^2 + m_2 y_2'^2 = \dfrac{m_1 m_2}{m_1+m_2}(y_2-y_1)^2$

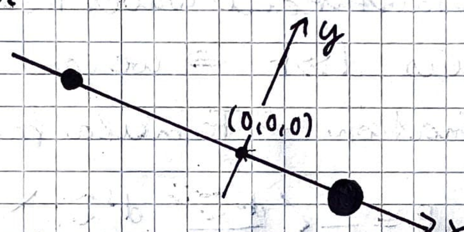
$$\Rightarrow I^{xx} + I^{yy} = \frac{m_1 m_2}{m_1+m_2}\left((x_2-x_1)^2 + (y_2-y_1)^2\right) = \frac{m_1 m_2}{m_1+m_2} r^2$$

$$= \mu r^2$$

$$I^{xy} = m_1 x_1' y_1' + m_2 y_2' x_2' = \frac{-m_1 m_2}{m_1+m_2}(x_2-x_1)\frac{(-m_2)}{m_1+m_2}(y_2-y_1) + \frac{m_2 m_1^2}{(m_1+m_2)^2}(x_2-x_1)(y_2-y_1)$$

$$= \frac{m_1 m_2 \cancel{(m_1+m_2)}}{(m_1+m_2)^{\cancel{2}}}(x_2-x_1)(y_2-y_1)$$

$$= \frac{m_1 m_2}{m_1+m_2}(x_2-x_1)(y_2-y_1) = \mu (x_2-x_1)(y_2-y_1)$$



body-fixed system

Choose the x'-axis such that it connects the two masses

$\Rightarrow$ $y_2 - y_1 = 0$ always and $r = \sqrt{(x_2-x_1)^2 + 0}$

$$= (x_2 - x_1)$$

$\Rightarrow$ $I^{xy} = I^{yx} = 0$ and $I^{xx} = \dfrac{m_1 m_2}{m_1+m_2} r^2$

$$I^{yy} = \frac{m_1 m_2}{m_1+m_2}(y_2-y_1)^2 = 0$$

$$\Rightarrow (I^{ij}) = \begin{pmatrix} I^{xx} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mu r^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow$ $I = \mathrm{Tr}\, I^{ij} = \mu r^2$ $\Rightarrow$ $\tilde{I}^{ij} = I^{ij} - \frac{1}{3}\delta^{ij} I$

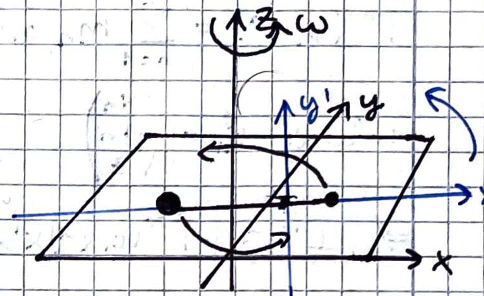
$$(\tilde{I}^{ij}) = \begin{pmatrix} \mu r^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} \mu r^2 & 0 & 0 \\ 0 & \mu r^2 & 0 \\ 0 & 0 & \mu r^2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2\mu r^2 & 0 & 0 \\ 0 & -\mu r^2 & 0 \\ 0 & 0 & -\mu r^2 \end{pmatrix}$$

$\Rightarrow I^{ij}$ in the system that is rotating with the two masses:

$$(I^{ij}) = \frac{1}{3}\begin{pmatrix} 2\mu r^2 & 0 & 0 \\ 0 & -\mu r^2 & 0 \\ 0 & 0 & -\mu r^2 \end{pmatrix}$$

$\boxed{(e)}$ Now I^{ij} in the non-rotating frame $\Rightarrow$ we have to transform from body-fixed to non-fixed system

$\Rightarrow$ rotation around z-axis:



$$\Lambda = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with φ [redacted] time dependent

$$r(\varphi) = \frac{r_0}{1 \pm e\cos\varphi} \quad \text{from Kepler's 1st law}$$

$$\Rightarrow (I^{ij})_{non} = I_{non} = \Lambda I_{non} \Lambda^T$$

$$= \frac{1}{3}\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2\mu r^2 & 0 & 0 \\ 0 & -\mu r^2 & 0 \\ 0 & 0 & -\mu r^2 \end{pmatrix}\begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2\mu r^2\cos\varphi & 2\mu r^2\sin\varphi & 0 \\ \mu r^2\sin\varphi & -\mu r^2\cos\varphi & 0 \\ 0 & 0 & -\mu r^2 \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} 2\mu r^2\cos^2\varphi - \mu r^2\sin^2\varphi & 2\mu r^2\sin\varphi\cos\varphi + \mu r^2\sin\varphi\cos\varphi & 0 \\ 2\mu r^2\cos\varphi\sin\varphi + \mu r^2\sin\varphi\cos\varphi & 2\mu r^2\sin^2\varphi - \mu r^2\cos^2\varphi & 0 \\ 0 & 0 & -\mu r^2 \end{pmatrix}$$

$$= \frac{\mu r^2}{3}\begin{pmatrix} 2\cos^2\varphi - \sin^2\varphi & 2\sin\varphi\cos\varphi + \sin\varphi\cos\varphi & 0 \\ 2\cos\varphi\sin\varphi + \sin\varphi\cos\varphi & 2\sin^2\varphi - \cos^2\varphi & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{\mu r^2}{3}\begin{pmatrix} 3\cos^2\varphi - (\sin^2\varphi + \cos^2\varphi) & 3\sin\varphi\cos\varphi & 0 \\ 3\cos\varphi\sin\varphi & -3\cos^2\varphi + (\sin^2\varphi+\cos^2\varphi)\cdot 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{\mu r^2}{3}\begin{pmatrix} 3\cos^2\varphi - 1 & 3\sin\varphi\cos\varphi & 0 \\ 3\cos\varphi\sin\varphi & -3\cos^2\varphi + 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$\sin\varphi\cos\varphi = \frac{1}{2}\sin(2\varphi)$

$\cos(2\varphi) = 2\cos^2(\varphi) - 1$

$$= \frac{\mu r^2}{3}\begin{pmatrix} \frac{3}{2}(2\cos^2\varphi - 1 + 1) - 1 & \frac{3}{2}\sin(2\varphi) & 0 \\ \frac{3}{2}\sin(2\varphi) & -\frac{3}{2}(2\cos^2\varphi - 1 + 1) + 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{\mu r^2}{3}\begin{pmatrix} \frac{3}{2}(\cos(2\varphi) + 1) - 1 & \frac{3}{2}\sin(2\varphi) & 0 \\ \frac{3}{2}\sin(2\varphi) & -\frac{3}{2}(\cos(2\varphi) + 1) + 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{\mu r^2}{3} \begin{pmatrix} \frac{3}{2} - 1 + \frac{3}{2}\cos(2\varphi) & \frac{3}{2}\sin(2\varphi) & 0 \\ \frac{3}{2}\sin(2\varphi) & -\frac{3}{2} + 2 - \frac{3}{2}\cos(2\varphi) & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$= \frac{\mu r^2}{3} \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{\mu r^2}{3} \begin{pmatrix} \frac{3}{2}\cos(2\varphi) & \frac{3}{2}\sin(2\varphi) & 0 \\ \frac{3}{2}\sin(2\varphi) & -\frac{3}{2}\cos(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{\mu r^2}{3} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{\mu r^2}{2} \begin{pmatrix} \cos(2\varphi) & \sin(2\varphi) & 0 \\ \sin(2\varphi) & -\cos(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= I_{non}^{(non\text{-}oscillating)} + I_{non}^{oscillating}$$

$\boxed{(f)}$ Assume orbit of the two masses is perfectly circular

$\Rightarrow$ eccentricity $e = 0 \Rightarrow r(\varphi) = \frac{r_0}{1 \pm 0} = r_0 = r$

$\varphi = \omega t$

Power of the gravitational waves measured in a sphere of radius $d \gg r \Rightarrow$ we can use the far-field approximation $\Rightarrow$ Formula from Ex. 10.1 d) is applicable:

$$P = \frac{G}{5} \left\langle \frac{d^3 I^{ij}}{dt^3} \frac{d^3 I_{ij}}{dt^3} \right\rangle \Big|_{t=t_R}$$

$t = t - \frac{d}{c}$ With $c \neq 1$: $P = \frac{G}{5c^5}(\dots)$

The $I^{non\text{-}oscillating}$ is not time dependent and will drop out when differentiating

$$\Rightarrow \dot{I} = \frac{dI}{dt} = \frac{\mu r^2}{2} \cdot 2\dot{\varphi} \begin{pmatrix} -\sin(2\varphi) & \cos(2\varphi) & 0 \\ \cos(2\varphi) & \sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \mu r^2 \dot{\varphi} \begin{pmatrix} -\sin(2\varphi) & \cos(2\varphi) & 0 \\ \cos(2\varphi) & \sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\ddot{I} = \frac{d^2 I}{dt^2} = \mu r^2 \underbrace{\ddot{\varphi}}_{\frac{d\omega}{dt} = 0} \begin{pmatrix} -\sin(2\varphi) & \cos(2\varphi) & 0 \\ \cos(2\varphi) & \sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \mu r^2 \cdot 2\dot{\varphi}^2 \begin{pmatrix} -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ -\sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= 2\mu r^2 \dot\varphi^2 \begin{pmatrix} -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ -\sin(2\varphi) & \cos(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\dddot{I} = \frac{d^3 I}{dt} = \begin{pmatrix} \sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot 2\mu r^2 \dot\varphi^2 (2\dot\varphi)$$

$$= 4\mu r^2 \dot\varphi^3 \begin{pmatrix} +\sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \dot\varphi = \frac{d\varphi}{dt} = \omega$$

$$= 4\mu r^2 \omega^3 \begin{pmatrix} \sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{d^3 I^{ij}}{dt^3}\frac{d^3 I_{ij}}{dt^3} = \dddot{I}^{ij}\dddot{I}_{ij} = tr(\dddot{I}\cdot\dddot{I}) = \delta^{ia}(\dddot{I}^{ij}\dddot{I}_{ja})$$

$$\Rightarrow tr(\dddot{I}\dddot{I}) = tr\left((4\mu r^2\omega^3)^2 \begin{pmatrix} \sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \sin(2\varphi) & -\cos(2\varphi) & 0 \\ -\cos(2\varphi) & -\sin(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$= (4\mu r^2\omega^3)^2 \left(tr \begin{pmatrix} \sin^2(2\varphi)+\cos^2(2\varphi) & -\cos(2\varphi)\sin(2\varphi)+\sin(2\varphi)\cos(2\varphi) & 0 \\ -\cos(2\varphi)\sin(2\varphi)+\sin(2\varphi)\cos(2\varphi) & \cos^2(2\varphi)+\sin^2(2\varphi) & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)$$

$$= 16\mu^2 r^4\omega^6 \, tr \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 16\mu^2 r^4 \omega^6 \cdot 2 = 32\mu^2 r^4\omega^6$$

$\langle \dddot{I}^{ij}\dddot{I}_{ij} \rangle$ is an average over one period $T = \frac{2\pi}{\omega}$

$$P = \frac{G}{5}\cdot 32\mu^2 r^4\omega^6 \Big|_{t=t_R} = \frac{32G}{5}\frac{(m_1 m_2)^2}{(m_1+m_2)^2} r^4\omega^6 \Big|_{t=t_R} = \frac{32G}{5}\frac{(m_1m_2)^2}{(m_1+m_2)} r^4(t-\frac{d}{c})\omega^6$$

$\boxed{(g)}$ Kepler's 3rd law: $\frac{4\pi^2 r^3}{T^2} = G(m_1+m_2)$ $(T_1 = T_2)$

because $\frac{m_1 v^2}{r} = \frac{Gm_1m_2}{r^2}$ and $\frac{m_2 v^2}{r} = \frac{Gm_1 m_2}{r^2}$

and $T = \frac{2\pi r}{v}$: $T = \frac{2\pi r^{3/2}}{\sqrt{G(m_1+m_2)}}$

$$\Rightarrow \omega = \frac{2\pi}{T} = \frac{\sqrt{G(m_1+m_2)}\,2\pi}{2\pi r^{3/2}} = \sqrt{\frac{G(m_1+m_2)}{r^3}}$$

$$\Rightarrow P = \frac{G}{5}\cdot 32 \frac{(m_1m_2)^2}{(m_1+m_2)^2} r^4 \left(\sqrt{\frac{G(m_1+m_2)}{r^3}}\right)^6$$

$$= \frac{32}{5} G \frac{(m_1m_2)^2}{(m_1+m_2)^2} r^4 \frac{G^3(m_1+m_2)^3}{r^9} = \frac{32G^4}{5}\frac{(m_1m_2)^2(m_1+m_2)}{r^5}$$

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With c: $\boxed{P = \frac{32G^4}{5c^5}\frac{m_1^2 m_2^2 (m_1+m_2)}{r^5}}$ — $r = r(t-\frac{d}{c})$ retarded time

$\boxed{\text{(h)}}$ P is radiated away by the binary system

$\Rightarrow$ It loses energy $\Rightarrow$ r slowly decreases and the period T decreases

$$E = \frac{1}{2}\mu v^2 - E_{pot} = \frac{1}{2}\mu v^2 - \frac{G m_1 m_2}{r}$$

Circular orbit: $v = \omega r$

$$\Rightarrow v^2 = \omega^2 r^2 \overset{g)}{=} \left(\frac{G(m_1+m_2)}{r^3}\right) r^2 = \frac{G(m_1+m_2)}{r}$$

$$\Rightarrow E = \frac{1}{2}\frac{m_1 m_2}{m_1+m_2}\frac{G(m_1+m_2)}{r} - \frac{G m_1 m_2}{r} = \frac{1}{2} m_1 m_2 G \frac{1}{r} - \frac{G m_1 m_2}{r}$$

$$= -\frac{1}{2}\frac{G m_1 m_2}{r}$$

Energy loss $(r = r(t))$ $\rightarrow$ smaller radius & period

$\rightarrow$ faster orbital velocity

$\rightarrow$ binary system emits <u>more</u> GW

$$\Rightarrow P = -\frac{dE}{dt} = -\left(+\frac{1}{2}\frac{G m_1 m_2}{r^2}\frac{dr}{dt}\right) = -\frac{1}{2}\frac{G m_1 m_2}{r^2}\frac{dr}{dt}$$

$$\Rightarrow \frac{32 G^4}{5c^5}\frac{(m_1 m_2)^2 (m_1+m_2)}{r^5} = -\frac{1}{2}\frac{G m_1 m_2}{r^2}\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = -\frac{64 G^3}{5c^5}\frac{m_1 m_2 (m_1+m_2)}{r^3}$$

$$\Leftrightarrow r^3 \frac{dr}{dt}\cdot 4 = -\frac{64\cdot 4\, G^3}{5c^5}\frac{m_1 m_2 (m_1+m_2)}{1}$$

$$\frac{d}{dt}(r^4) = -\frac{256 G^3}{5c^5} m_1 m_2 (m_1+m_2)$$

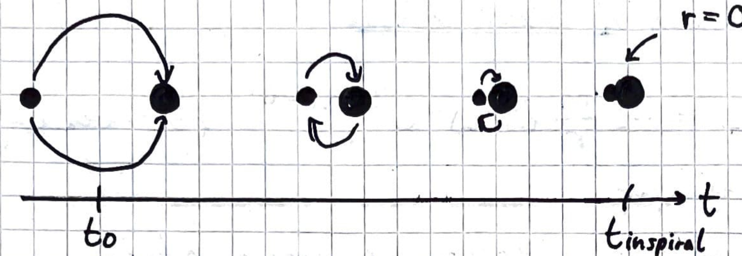
$$\Rightarrow \int_{t_0=0}^{t} \frac{d}{dt'}(r^4)\, dt' = -\int_{t_0}^{t} \frac{256 G^3}{5c^5} m_1 m_2 (m_1+m_2) \qquad | r_0 = r(t=0)$$

$$\Leftrightarrow r^4(t) - r_0^4 = -\frac{256 G^3}{5c^5} m_1 m_2 (m_1+m_2)\cdot t + 0$$

$$r^4(t) = r_0^4 - \frac{256 G^3}{5c^4}(m_1 m_2)(m_1+m_2)\cdot t$$

$$\Rightarrow r(t) = r_0\left(1 - \frac{256 G^3}{5c^4}(m_1 m_2)(m_1+m_2)\frac{t}{r_0^4}\right)^{1/4}$$

Inspiral time: time at which the two masses will collide: $r(t_{inspiral}) = 0$



Define $t_{inspiral} = \dfrac{5c^5\, r_0^{\,4}}{256\, G^3\, m_1 m_2 (m_1 + m_2)}$

because for $t_{inspiral} = t$ $\quad r = 0$:

$$r(t) = r_0 \left(1 - \frac{t}{t_{inspiral}}\right)$$

$\boxed{\text{(i)}}$ Sun-Earth system: Take $r_0 = 1\,\text{AU} = 1.5 \cdot 10^{11}\,\text{m}$

$$\Rightarrow t_{inspiral} = \frac{5 \cdot (3 \cdot 10^8 \frac{m}{s})^5 \cdot (1.5 \cdot 10^{11} m)^4}{256 \left(6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}\right)^3 (M_E M_\odot)(M_E + M_\odot)}$$

$2 \cdot 10^{30}\,kg$

$5.97 \cdot 10^{24}\,kg$

$$\approx 3.39 \cdot 10^{30}\, s$$
$$= \underline{\underline{1.1 \cdot 10^{23}\, yr}}$$

Binary pulsar: $m_1 = 1.44 M_\odot \quad m_2 = 1.39 M_\odot$

Kepler 3: $T = \dfrac{2\pi r^{3/2}}{\sqrt{G(m_1 + m_2)}} \Rightarrow r = r_0 = \left(\dfrac{T \sqrt{G(m_1+m_2)}}{2\pi}\right)^{2/3}$

$$= \left(\frac{T^{2/3} (G(m_1+m_2))^{1/3}}{(2\pi)^{2/3}}\right) = \left(\frac{T^2\, G(m_1+m_2)}{4\pi^2}\right)^{1/3} \qquad \Big| \; T = 7.75\,h = 2.79 \cdot 10^4\, s$$

$$= \left(\frac{(2.79 \cdot 10^4 s)^2 (6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2})(1.44 + 1.39) \cdot 2 \cdot 10^{30} kg)}{4\pi^2}\right)^{1/3}$$

$$\approx 2 \cdot 10^9\, m$$

$$\Rightarrow t_{inspiral} = \frac{5 \cdot (3 \cdot 10^8 \frac{m}{s})^5 \cdot (2 \cdot 10^9 m)^4}{256 \cdot \left(6.67 \cdot 10^{-11} \frac{m^3}{kg\, s^2}\right)^3 (1.44 \cdot 1.39 (M_\odot)^2)(1.44 + 1.39) M_\odot}$$

$$\approx 5.65 \cdot 10^{16} s = 1.78 \cdot 10^9\, yr$$
$$= \underline{\underline{1.78\, Gyr}}$$

Question: Frequency of light & GWs

(a) $h^{TT}_{xx} = h_+ = -2\frac{m}{r} l_0^2 \omega^2 e^{2i\omega(r-t)}$ for binary

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

TT gauge + z-direction of propagation:

$$h \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

$$\Rightarrow ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 + 2g_{12}\, dx^1 dx^2$$

+ polarization: $h_{\mu\nu}^{TT} = h^{TT}_{\mu\nu +} + \cancel{h^{TT}_{\mu\nu \times}}$

$$= h^{TT}_{\mu\nu}$$

$\Rightarrow h_{xy} = 0 \quad \Rightarrow ds^2 = g_{00}\, dt^2 + g_{11}\, dx^2 + g_{22}\, dy^2 + g_{33}\, dz^2$

$$\Rightarrow ds^2 = -dt^2 + (1 + h_+(z-t))dx^2 + (1 - h_+(t-z))dy^2 + dz^2$$

(b) Consider a freely falling particle → equivalence principle: can choose a frame where the particle is at rest. Free particles are governed by the Geodesic equation:

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0$$

Particle initially at rest: $u^\mu \rightarrow \{1,0,0,0\}$

$$\Rightarrow \left.\frac{du^\alpha}{d\tau}\right|_{\tau_0} = -\Gamma^\alpha_{00} \cdot 1 = -\Gamma^\alpha_{00}$$

$$= -\frac{g^{\alpha\kappa}}{2}(g_{\kappa 0,0} + g_{0\kappa,0} - g_{00,\kappa}) = -\frac{g^{\alpha\kappa}}{2}(h_{\kappa 0,0} + h_{0\kappa,0} - h_{00,\kappa})$$

TT gauge: $h_{\kappa 0} = 0 = h_{0\kappa} = 0 = h_{00,\kappa}$

$\Rightarrow \left.\frac{du^\alpha}{d\tau}\right|_{\tau_0} = 0 \quad \Rightarrow$ particle is not accelerated at τ_0

$\Rightarrow$ particle has no velocity at $\tau_0 + \delta\tau \Rightarrow$ particle is at rest at $\tau_0 + \delta\tau \Rightarrow$ particle has no velocity at $\tau_0 + \delta\tau + \Delta\tau$

$\Rightarrow$ particle remains at rest, i.e. at constant coordinate position $\Rightarrow$ coordinate distances are not affected by the GW $\Rightarrow$ TT gauge corresponds to freely falling reference frame

Two freely falling particles at $x_1 \wedge x_2$ with $y_1 = y_2 = 0$ and $z_1 = z_2 = 0$: proper distance $s = \int |ds^2|^{1/2}$ ($dt^2 = 0$)

Take $\varepsilon = x_2 - x_1$ and $x_1 = 0$

$$\Rightarrow \quad s = \int_0^\varepsilon dx\, |g_{xx}|^{1/2} \approx |g_{xx}(x=0)|^{1/2} \cdot \varepsilon = |1 + h_{xx}^{TT}(x=0)|^{1/2} \varepsilon$$

$$\overset{h_{xx}^{TT} \ll 1}{\approx} \left(1 + \tfrac{1}{2} h_{xx}^{TT}(x=0)\right)\varepsilon = \left(1 + \tfrac{1}{2} h_+(x=0)\right)\varepsilon$$

$\Rightarrow$ the GW causes a change of the proper distance in contrast to the coordinate distance

(c) flight from x_A to x_B $\quad d = x_B - x_A$

$\frac{\nu_{A'}}{\nu_A} \rightarrow \frac{\nu_B}{\nu_A}$ $\quad$ A': return at A $\quad$ A: start of light emission

At first consider 1 photon, which travels on null geodesics

$ds^2 = 0$ ($dy = dz$ → now light propagates along x)

$$= -dt^2 + (1 + h_+)dx^2 \Rightarrow dt = \sqrt{(1+h_+)}\, dx$$

$$t_B = t_A + \int_0^d |ds|^{1/2} = t_A + c \cdot \int_{t_A}^{t_B} dt$$

$$= t_A + \int_0^d \sqrt{1 + h_+(t)}\, dx$$

$$t_B = t_A + \int_0^d \left(1 + \tfrac{1}{2} h_+(t_A + x)\right) dx = t_A + d + \tfrac{1}{2} \int_0^d h_+(t_A + x)\, dx$$

$$\Rightarrow \frac{dt_B}{dt_A} = 1 + \frac{1}{2} \int_0^d \frac{d}{dt_A} h_+(t_A + x)\, dx$$

$x = t_A - t$

$$= 1 + \frac{1}{2} \int_0^d \frac{d}{dt_A} \left(h_+(t_A + x)\right) dt_A$$

$$= 1 + \tfrac{1}{2}\left(h_+(t_A + d) - h_+(t_A)\right)$$

$$= 1 + \frac{1}{2}\left(h_+(t_B) - h_+(t_A)\right)$$

$$= 1 + \frac{1}{2}\left(h_+(x_B) - h_+(x_A)\right)$$

Dt is $\dfrac{dt_B}{dt_A} = \dfrac{v_B}{v_A} = 1 + \frac{1}{2}\left(h_+(x_B) - h_+(x_A)\right)$