# Exercise 4.1

(a) Show that if a tensor is symmetric in a concrete basis, it will remain symmetric in any other basis. (0.5 pt)

If $A_{\mu\nu}$ is symmetric in a concrete basis $\{\tilde{\omega}^{\mu}\}$ $\longrightarrow$ perform a transformation to $\{\tilde{\omega}^{\bar{\mu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}\tilde{\omega}^{\mu}\}$

elements of the tensor $\mathbf{A}$ will change as follows:

$$A_{\bar{\mu}\bar{\nu}} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}A_{\mu\nu} = \frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}A_{\nu\mu} = \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}A_{\nu\mu} = A_{\bar{\nu}\bar{\mu}}$$

(b) Prove that the contraction of a symmetric tensor S with an antisymmetric one A is 0. (0.5 pt)

$$A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\mu\nu} = -A_{\nu\mu}S^{\nu\mu} = -A_{\mu\nu}S^{\mu\nu} \longrightarrow A_{\mu\nu}S^{\mu\nu} = 0$$

# Exercise 4.2

(a) Show that $\Gamma^{\beta}{}_{\mu\alpha}$ are the components of the $\binom{1}{1}$ tensor $\nabla\vec{e}_{\alpha}$ for fixed $\alpha$, and explain why we do not expect the Christoffel symbols to transform as tensors. (1 pt)

Let us consider a general vector, i.e. a general $\binom{1}{0}$ tensor, $\vec{A}$. $\longrightarrow$ $\nabla\vec{A} = \nabla_{\beta}A^{\gamma}\,\tilde{\omega}^{\beta}\otimes\vec{e}_{\gamma}$

For a fixed $\alpha \to \delta^{\beta}_{\alpha}$ is a vector with dummy index $\beta$

Let us take now a vector $\vec{e}_{\alpha} = \delta^{\beta}_{\alpha}\vec{e}_{\beta}$. $\longrightarrow$ $\nabla\vec{e}_{\alpha} = \nabla_{\beta}\delta^{\gamma}_{\alpha}\,\tilde{\omega}^{\beta}\otimes\vec{e}_{\gamma}$,

Computing the first term gives us $\longrightarrow$ $\nabla_{\beta}\delta^{\gamma}_{\alpha} = \partial_{\beta}\delta^{\gamma}_{\alpha} + \Gamma^{\gamma}{}_{\mu\beta}\delta^{\mu}_{\alpha} = \Gamma^{\gamma}{}_{\alpha\beta}$

On the other hand,

This means that

$$\nabla\vec{e}_{\alpha} = \Gamma^{\gamma}{}_{\alpha\beta}\,\tilde{\omega}^{\beta}\otimes\vec{e}_{\gamma}.$$

$$\nabla\vec{e}_{\alpha} = \nabla\left(\frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\vec{e}_{\bar{\alpha}}\right) = \nabla\left(\frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\right)\vec{e}_{\bar{\alpha}} + \frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\nabla(\vec{e}_{\bar{\alpha}}) =$$

$$= \frac{\partial}{\partial x^{\bar{\beta}}}\left(\frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\right)\tilde{\omega}^{\bar{\beta}}\otimes\vec{e}_{\bar{\alpha}} + \frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}}\,\tilde{\omega}^{\bar{\beta}}\otimes\vec{e}_{\bar{\gamma}} =$$

$$= \left[\frac{\partial}{\partial x^{\bar{\beta}}}\left(\frac{\partial x^{\bar{\gamma}}}{\partial x^{\alpha}}\right) + \frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}}\right]\tilde{\omega}^{\bar{\beta}}\otimes\vec{e}_{\bar{\gamma}}.$$

$$\Gamma^{\bar{\gamma}}{}_{\alpha\bar{\beta}} = \frac{\partial}{\partial x^{\bar{\beta}}}\left(\frac{\partial x^{\bar{\gamma}}}{\partial x^{\alpha}}\right) + \frac{\partial x^{\bar{\alpha}}}{\partial x^{\alpha}}\Gamma^{\bar{\gamma}}{}_{\bar{\alpha}\bar{\beta}}.$$

(b) Discover how each expression $V^{\beta}{}_{,\alpha}$ and $V^{\mu}\Gamma^{\beta}{}_{\mu\alpha}$ separately transforms under a change of coordinates. For $\Gamma^{\beta}{}_{\mu\alpha}$ you can begin with

$$\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\mu}{}_{\alpha\beta}\,\vec{e}_{\mu}. \tag{1}$$

Show that neither is the standard tensor law, but that the sum does obey the standard law. (2 pt)

## Tranformation of the Christofel symbols

$$\frac{\partial \vec{e}_{\bar{\alpha}}}{\partial x^{\bar{\beta}}} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\,\vec{e}_{\bar{\mu}}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial}{\partial x^{\beta}}\left(\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\vec{e}_{\alpha}\right) = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^{2}x^{\alpha}}{\partial x^{\beta}\partial x^{\bar{\alpha}}}\vec{e}_{\alpha} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\frac{\partial \vec{e}_{\alpha}}{\partial x^{\beta}} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^{2}x^{\alpha}}{\partial x^{\beta}\partial x^{\bar{\alpha}}}\vec{e}_{\alpha} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

In the first term of the left-hand side we can do $\alpha \to \mu$.

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^{2}x^{\mu}}{\partial x^{\beta}\partial x^{\bar{\alpha}}}\vec{e}_{\mu} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta}\vec{e}_{\mu} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\left(\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^{2}x^{\mu}}{\partial x^{\beta}\partial x^{\bar{\alpha}}} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta}\right)\vec{e}_{\mu} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\vec{e}_{\mu}$$

$$\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial^{2}x^{\mu}}{\partial x^{\beta}\partial x^{\bar{\alpha}}} + \frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}$$

Multiplying both sides by $\frac{\partial x^{\bar{\kappa}}}{\partial x^{\mu}}$ we finally obtain:

$$\frac{\partial x^{\bar{\kappa}}}{\partial x^{\mu}}\frac{\partial^{2}x^{\mu}}{\partial x^{\bar{\beta}}\partial x^{\bar{\alpha}}} + \frac{\partial x^{\bar{\kappa}}}{\partial x^{\mu}}\frac{\partial x^{\beta}}{\partial x^{\bar{\beta}}}\frac{\partial x^{\alpha}}{\partial x^{\bar{\alpha}}}\Gamma^{\mu}{}_{\alpha\beta} = \Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}}\delta^{\bar{\kappa}}_{\mu} = \Gamma^{\bar{\kappa}}{}_{\bar{\alpha}\bar{\beta}}$$

## Tranformation of $V^\mu \Gamma^\beta_{\ \mu\alpha}$

$$V^{\bar\mu}\Gamma^{\bar\beta}_{\ \bar\mu\bar\alpha} = \frac{\partial x^{\bar\mu}}{\partial x^\mu}V^\mu\left[\frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^{\bar\mu}\partial x^{\bar\alpha}} + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\theta}{\partial x^{\bar\mu}}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta_{\ \theta\alpha}\right]$$

$$V^{\bar\mu}\Gamma^{\bar\beta}_{\ \bar\mu\bar\alpha} = \frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^\mu\partial x^{\bar\alpha}}V^\mu + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\theta}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta_{\ \theta\alpha}V^\mu$$

$$\boxed{V^{\bar\mu}\Gamma^{\bar\beta}_{\ \bar\mu\bar\alpha} = \frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^\mu\partial x^{\bar\alpha}}V^\mu + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta_{\ \mu\alpha}V^\mu}$$

## Tranformation of $V^\beta_{\ ,\alpha}$

$$V^{\bar\beta}_{\ ,\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta\right)$$

$$\boxed{V^{\bar\beta}_{\ ,\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial^2 x^{\bar\beta}}{\partial x^\alpha\partial x^\beta}V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta_{\ ,\alpha}}$$

$$V^{\bar\beta}_{\ ,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}_{\ \bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial^2 x^{\bar\beta}}{\partial x^\alpha\partial x^\beta}V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta_{\ ,\alpha} + \frac{\partial x^{\bar\beta}}{\partial x^\sigma}\frac{\partial^2 x^\sigma}{\partial x^\mu\partial x^{\bar\alpha}}V^\mu + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta_{\ \mu\alpha}V^\mu$$

$\sigma \to \alpha$ and $\mu \to \beta$

$$V^{\bar\beta}_{\ ,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}_{\ \bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial^2 x^{\bar\beta}}{\partial x^\alpha\partial x^\beta}V^\beta + \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta_{\ ,\alpha} - \frac{\partial^2 x^{\bar\beta}}{\partial x^\beta\partial x^\alpha}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}V^\beta + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta_{\ \mu\alpha}V^\mu$$

$$V^{\bar\beta}_{\ ,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}_{\ \bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}V^\beta_{\ ,\alpha} + \frac{\partial x^{\bar\beta}}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\Gamma^\beta_{\ \mu\alpha}V^\mu \qquad \boxed{V^{\bar\beta}_{\ ,\bar\alpha} + V^{\bar\mu}\Gamma^{\bar\beta}_{\ \bar\mu\bar\alpha} = \frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\frac{\partial x^{\bar\beta}}{\partial x^\beta}\left(V^\beta_{\ ,\alpha} + \Gamma^\beta_{\ \mu\alpha}V^\mu\right)}$$

Therefore, the covariant derivative $V^\beta_{\ ;\alpha} = V^\beta_{\ ,\alpha} + \Gamma^\beta_{\ \mu\alpha}V^\mu$ is a tensor (of type $\binom{1}{1}$) because it transforms like tensors do.

# Exercise 4.3

Consider the Euclidean space with metric tensor

$$\mathbf{g} = d\tilde{x} \otimes d\tilde{x} + \tilde{d}y \otimes \tilde{d}y + \tilde{d}z \otimes \tilde{d}z. \qquad (2)$$

(a) Compute the elements of the metric tensor in spherical coordinates, and use them to compute the differential *proper* volume $dV$ and the squared line element $ds^2$. (1 pt)

*(handwritten annotations):*
$d\tilde{s} = dx^2 + dy^2 + dz^2$
$[dx\; \tilde{d}y\; \tilde{d}z]\begin{bmatrix}1 & & \\ & 1 & \\ & & 1\end{bmatrix}\begin{bmatrix}dx \\ dy \\ dz\end{bmatrix}$
$(0,1)\otimes(0,1) \Rightarrow (0,2)$
$(0,2)$
$-\tilde{x}t \otimes \tilde{x}t$
$g$

$$\Lambda = \begin{pmatrix} \sin(\theta)\cos(\varphi) & r\cos(\theta)\cos(\varphi) & -r\sin(\theta)\sin(\varphi) \\ \sin(\theta)\sin(\varphi) & r\cos(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{pmatrix}$$

*(handwritten):* $\frac{\partial x}{\partial r}$, $\frac{\partial x}{\partial \theta}$, $\frac{\partial x}{\partial \varphi}$, $\frac{\partial x}{\partial x^{\bar{\nu}}}$, $\frac{\partial y}{\partial x^{\bar{\nu}}}$, $\frac{\partial z}{\partial x^{\bar{\nu}}}$

$$A^{\bar{\mu}}{}_{\bar{\nu}} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\mu}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}} A_{\nu}$$

$$x^{\mu} = (x,y,z) \text{ and } x^{\bar{\mu}} = (r,\theta,\varphi)$$

$$\begin{aligned} x &= r\sin(\theta)\cos(\varphi), \\ y &= r\sin(\theta)\sin(\varphi), \\ z &= r\cos(\theta). \end{aligned}$$

$$g_{\bar{\mu}\bar{\nu}} = \sum_{\mu}\sum_{\nu}\frac{\partial x^{\mu}}{\partial x^{\bar{\mu}}}\frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}g_{\mu\nu}$$

$(0,2)$

$$\Lambda^{\nu}{}_{\bar{\nu}} = \frac{\partial x^{\nu}}{\partial x^{\bar{\nu}}}$$

$$\bar{g} = \Lambda^{T} g \Lambda$$

*(handwritten):* $[\tilde{d}r\; \tilde{d}\theta\; \tilde{d}\varphi] \cdot \begin{bmatrix}dr \\ d\theta \\ d\varphi\end{bmatrix}$

$$\bar{g} = \Lambda^{T}\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2(\theta) \end{pmatrix}$$

*(handwritten: Trig)*

$$\mathbf{g} = \tilde{d}r \otimes \tilde{d}r + r^2[\tilde{d}\theta \otimes \tilde{d}\theta + \sin^2(\theta)\tilde{d}\varphi \otimes \tilde{d}\varphi]$$

*(handwritten): dA = dx dy*

**Squared line element** ⟹
$$ds^2 = d\vec{l}\cdot d\vec{l} = \mathbf{g}(d\vec{l}, d\vec{l}) = g_{\bar{\mu}\bar{\nu}}dx^{\bar{\mu}}dx^{\bar{\nu}} = dr^2 + r^2[d^2\theta + \sin^2(\theta)d\varphi^2]$$

**Differential proper volume** ⟹
$$dV = \sqrt{|-\bar{g}|}\, dr\, d\theta\, d\varphi = r^2\sin(\theta)\, dr\, d\theta\, d\varphi$$

*(handwritten): dA = r dr d\theta*

**(b)** Write the one-form $\tilde{d}f$ in spherical coordinates. (0.5 pt) $\longrightarrow$

$$\tilde{d}f = \frac{\partial f}{\partial x^{\bar{\mu}}}\tilde{d}x^{\bar{\mu}} = \frac{\partial f}{\partial r}\tilde{d}r + \frac{\partial f}{\partial \theta}\tilde{d}\theta + \frac{\partial f}{\partial \varphi}\tilde{d}\varphi$$

MTW

**(c)** Compute the associated gradient vector in spherical coordinates and in terms of the unit vectors $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi)$. (1 pt)

$$\vec{\nabla}_{(\text{cart})} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial}, \frac{\partial}{\partial z}\right)$$
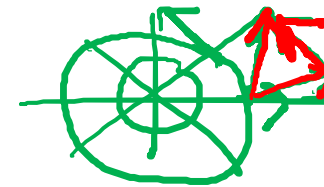
$$\tilde{d}f = \mathbf{g}(\vec{\nabla}f, \tilde{dx}) = \left(g_{\bar{\mu}\bar{\nu}}(\vec{\nabla}f)^{\bar{\mu}}\tilde{d}x^{\bar{\nu}} = \left(\frac{\partial f}{\partial x^{\bar{\nu}}}\right)\tilde{d}x^{\bar{\nu}}$$

$$g_{\bar{\mu}\bar{\nu}}(\vec{\nabla}f)^{\bar{\mu}} = \frac{\partial f}{\partial x^{\bar{\nu}}} \xrightarrow{+g^{-1}} (\vec{\nabla}f)^{\bar{\mu}} = g^{\bar{\mu}\bar{\nu}}\frac{\partial f}{\partial x^{\bar{\nu}}}$$

$$x^{\bar{\nu}} = (r, \theta, \varphi)$$

$$\begin{pmatrix} (\vec{\nabla}f)^r \\ (\vec{\nabla}f)^\theta \\ (\vec{\nabla}f)^\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & r^{-2}\sin^{-2}(\theta) \end{pmatrix} \begin{pmatrix} \frac{\partial f}{\partial r} \\ \frac{\partial f}{\partial \theta} \\ \frac{\partial f}{\partial \varphi} \end{pmatrix}$$

$$\vec{\nabla}f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r^2}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r^2\sin^2(\theta)}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi.$$

$$\begin{aligned}
\vec{e}_r &= |\vec{e}_r|\hat{e}_r = \sqrt{\mathbf{g}(\vec{e}_r, \vec{e}_r)}\,\hat{e}_r = \sqrt{g_{rr}}\,\hat{e}_r = \hat{e}_r, \\
\vec{e}_\theta &= \ldots = \sqrt{g_{\theta\theta}}\,\hat{e}_\theta = r\hat{e}_\theta, \\
\vec{e}_\varphi &= \ldots = \sqrt{g_{\varphi\varphi}}\,\hat{e}_\varphi = r\sin(\theta)\hat{e}_\varphi.
\end{aligned}$$

Unit vectors

$$\vec{\nabla}f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r\sin(\theta)}\frac{\partial f}{\partial \varphi}\hat{e}_\varphi$$

Embedding: Pos vector in spherical coords embedded in $\mathbb{R}^3$

$BY: 65$

$$\vec{e}_{\bar{\mu}} = \Lambda^{\nu}_{\bar{\mu}} \vec{e}_{\alpha}$$

$$\vec{e}_{\bar{\mu}} = \Lambda_{\bar{\mu}}{}^{\mu} \vec{e}_{\mu}.$$

$$\begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\varphi) & \sin(\theta)\sin(\varphi) & \cos(\theta) \\ r\cos(\theta)\cos(\varphi) & r\cos(\theta)\sin(\varphi) & -r\sin(\theta) \\ -r\sin(\theta)\sin(\varphi) & r\sin(\theta)\cos(\varphi) & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix}$$

Inverse

$$\vec{e}_\mu = \Lambda_\mu{}^{\bar{\mu}} \vec{e}_{\bar{\mu}}$$

$$\begin{pmatrix} \vec{e}_x \\ \vec{e}_y \\ \vec{e}_z \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\varphi) & \frac{1}{r}\cos(\theta)\cos(\varphi) & -\frac{\sin(\varphi)}{r\sin(\theta)} \\ \sin(\theta)\sin(\varphi) & \frac{1}{r}\cos(\theta)\sin(\varphi) & \frac{\cos(\varphi)}{r\sin(\theta)} \\ \cos(\varphi) & -\frac{\sin(\theta)}{r} & 0 \end{pmatrix} \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix}$$

$$\hat{r} = \sin\theta\cos\varphi \,\vec{e}_x + \sin\theta\sin\varphi \,\vec{e}_y + \cos\theta \,\hat{e}_z$$

$$A_\mu B^\mu = \begin{pmatrix} A_1 & A_2 & A_3 \end{pmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

*(handwritten, top right)*
$$\frac{\partial e^{\alpha}}{\partial x^{\beta}} = \Gamma^{\mu}{}_{\langle\beta} \vec{e}_{\mu}$$

Let us start with the computation of $\Gamma^{\bar{\mu}}{}_{r\theta}$ $\longrightarrow$

$$\frac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}{}_{r\theta} \vec{e}_{\bar{\mu}} .$$

*(handwritten)*
$$= \Gamma^{r}{}_{r\theta} \vec{e}_r + \Gamma^{\theta}{}_{r\theta} \vec{e}_{\theta} + \Gamma^{\varphi}{}_{r\theta} \vec{e}_{\varphi}$$

Use $\vec{e}_r$ result from the previous slide, and differentiate it component-wise

$$\frac{\partial \vec{e}_r}{\partial \theta} = \cos(\theta)\cos(\varphi)\vec{e}_x + \cos(\theta)\sin(\varphi)\vec{e}_y - \sin(\theta)\vec{e}_z = \frac{\vec{e}_{\theta}}{r}$$

$$= \cos(\theta)\cos(\varphi)\left[\sin(\theta)\cos(\varphi)\vec{e}_r + \frac{1}{r}\cos(\theta)\cos(\varphi)\vec{e}_{\theta} - \frac{\sin(\varphi)}{r\sin(\theta)}\vec{e}_{\varphi}\right] +$$

*(handwritten)* $0\vec{e}_{\varphi} + 0\vec{e}_r + \dfrac{\vec{e}_{\theta}}{r}$

$$+ \cos(\theta)\sin(\varphi)\left[\sin(\theta)\sin(\varphi)\vec{e}_r + \frac{1}{r}\cos(\theta)\sin(\varphi)\vec{e}_{\theta} + \frac{\cos(\varphi)}{r\sin(\theta)}\vec{e}_{\varphi}\right] - \sin(\theta)\left[\cos(\theta)\vec{e}_r - \frac{\sin(\theta)}{r}\vec{e}_{\theta}\right] = ...$$

After simplification we get

$$\frac{\partial \vec{e}_r}{\partial \theta} = \Gamma^{\bar{\mu}}{}_{r\theta}\vec{e}_{\bar{\mu}} = \frac{\vec{e}_{\theta}}{r} .$$

$\Longrightarrow$

$$\Gamma^{r}{}_{r\theta} = \Gamma^{\varphi}{}_{r\theta} = 0$$

$$\Gamma^{\theta}{}_{r\theta} = \frac{1}{r}$$

The computation of $\Gamma^{\bar{\mu}}{}_{rr}$ is straightforward.

$$\frac{\partial \vec{e}_r}{\partial r} = 0 = \Gamma^{\bar{\mu}}{}_{rr} \vec{e}_{\bar{\mu}} \longrightarrow \boxed{\Gamma^{\bar{\mu}}{}_{rr} = 0}$$

(f) Compute all the Christoffel symbols in spherical coordinates using the formula

$$\Gamma^{\bar{\mu}}{}_{\bar{\alpha}\bar{\beta}} = \frac{g^{\bar{\mu}\bar{\sigma}}}{2}\left(g_{\bar{\sigma}\bar{\alpha},\bar{\beta}} + g_{\bar{\sigma}\bar{\beta},\bar{\alpha}} - g_{\bar{\alpha}\bar{\beta},\bar{\sigma}}\right). \qquad \text{(1 pt)}$$

$$\Gamma^{\theta}{}_{r\theta} = \Gamma^{\varphi}{}_{r\varphi} = \frac{1}{r} \qquad \Gamma^{r}{}_{\theta\theta} = -r \qquad \Gamma^{r}{}_{\varphi\varphi} = -r\sin^2(\theta)$$

$$\Gamma^{\theta}{}_{\varphi\varphi} = -\frac{1}{2}\sin(2\theta) \qquad \Gamma^{\varphi}{}_{\varphi\theta} = \frac{\cos(\theta)}{\sin(\theta)}$$

All non-zero Christoffel symbols

$$\Gamma^{r}{}_{\theta\theta} = \frac{g^{r\sigma}}{2}\left(g_{\sigma\theta,\theta} + g_{\sigma\theta,\theta} - g_{\theta\theta,\sigma}\right)$$

$$g = \begin{bmatrix} 1 & & \\ & r^2{}_{\theta\theta} & \\ & & r^2\sin\theta^{k_g} \end{bmatrix}$$

$$\sigma = r$$

$$= \frac{1}{2}\begin{bmatrix} g^{rr}\left(g_{r\theta,\theta} + g_{r\theta,\theta} - g_{\theta\theta,r}\right) \\ \sigma=\theta \leftarrow \quad g^{r\theta}\left(g_{\theta\theta,\theta} + g_{\theta\theta,\theta} - g_{\theta\theta,\theta}\right) \\ \sigma=\varphi \quad g^{r\varphi}\left(g_{\varphi\theta,\theta} + g_{\varphi\theta,\theta} - g_{\theta\theta,\varphi}\right) \end{bmatrix}$$

$$= \frac{1}{2}g^{rr}\left(-g_{\theta\theta,r}\right)$$

$$= -\frac{1}{2}\cdot 1\cdot(2r) = -r$$

(g) Compute the Laplacian of the function $f$, i.e. $\nabla^2 f$, in spherical coordinates. (1 pt)

$$\nabla^2 f = \mathbf{g}(\vec{\nabla}, \vec{\nabla} f) = g_{\bar{\mu}\bar{\nu}} \vec{\nabla}^{\bar{\mu}}(\vec{\nabla} f)^{\bar{\nu}} = \nabla_{\bar{\mu}}(\vec{\nabla} f)^{\bar{\mu}} = \partial_{\bar{\mu}}(\vec{\nabla} f)^{\bar{\mu}} + \Gamma^{\bar{\mu}}{}_{\bar{\kappa}\bar{\mu}}(\vec{\nabla} f)^{\bar{\kappa}}$$

Sum over the indices

$$\nabla^2 f = \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\frac{1}{r^2}\frac{\partial f}{\partial \theta}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r^2 \sin^2(\theta)}\frac{\partial f}{\partial \varphi}\right) + \Gamma^\theta{}_{r\theta}\frac{\partial f}{\partial r} + \Gamma^\varphi{}_{r\varphi}\frac{\partial f}{\partial r} + \Gamma^\varphi{}_{\theta\varphi}\frac{1}{r^2}\frac{\partial f}{\partial \varphi}$$

Use the Christoffel symbols we calculated before

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2(\theta)}\frac{\partial^2 f}{\partial \varphi^2}$$