Exercise 6.1
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Exercise 6.2
(LIF in weak gravitational field)

Exercise 6.2: Local inertial frame in a weak gravitational field

A four-dimensional manifold has coordinates (t, x, y, z) and line element

$$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)\,, \tag{6}$$

where $|\phi(t,x,y,z)| \ll 1$ everywhere.

(a) At any point P with coordinates (t_0, x_0, y_0, z_0) find a coordinate transformation to a locally inertial coordinate system, to first order in ϕ. (1 pt)

- We have to perform a change of coordinates such that the metric in the new frame, $\bar{g}_{\mu\nu}$, is locally flat around P
- This means that $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$
- The first derivatives of $\bar{g}_{\mu\nu}$ at P must vanish,

$$\boxed{\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{\partial^2 g_{\mu\nu}}{\partial x^\alpha \partial x^\beta}\Big|_P (x^\alpha - x^\alpha_P)(x^\beta - x^\beta_P) + \mathcal{O}(\Delta x^3)} \tag{7}$$

with

$$\bar{g}_{\bar{\mu}\bar{\nu}} = \frac{\partial x^\mu}{\partial x^{\bar{\mu}}}\frac{\partial x^\nu}{\partial x^{\bar{\nu}}} g_{\mu\nu}\,.$$

We can define

$$\Lambda^\mu{}_{\bar{\nu}} \equiv \frac{\partial x^\mu}{\partial x^{\bar{\nu}}}\,.$$

As η is a diagonal metric we can try with a diagonal transformation matrix Λ leading also to a diagonal metric $\bar{g}_{\mu\nu}$ (this is the simplest initial guess). Thus, we have,

$$\boxed{\begin{aligned} \bar{g}_{00}(x) &= [\Lambda^0{}_{\bar{0}}(x)]^2 g_{00}(x) = -[1+2\phi(x)][\Lambda^0{}_{\bar{0}}(x)]^2 \\ \bar{g}_{ii}(x) &= [\Lambda^i{}_{\bar{i}}(x)]^2 g_{ii}(x) = [1-2\phi(x)][\Lambda^i{}_{\bar{i}}(x)]^2\,. \end{aligned}}$$

In order $\bar{g}_{\mu\nu}$ to take the form (7) the transformation matrix must fulfill the following equations:

$$\begin{aligned} -1 &= -[1+2\phi(P)][\Lambda^0{}_{\bar{0}}(P)]^2 \longrightarrow \Lambda^0{}_{\bar{0}}(P) = 1 - \phi(P) + \mathcal{O}(\phi^2) \\ 1 &= [1-2\phi(P)][\Lambda^i{}_{\bar{i}}(P)]^2 \underset{\text{Taylor expansion}}{\longrightarrow} \Lambda^i{}_{\bar{i}}(P) = 1 + \phi(P) + \mathcal{O}(\phi^2) \end{aligned}$$

and

$$0 = 2\frac{\partial \Lambda^0{}_{\bar{0}}}{\partial x^\mu}\Big|_P g_{00}(P) + [\Lambda^0{}_{\bar{0}}(P)]^2 \frac{\partial g_{00}}{\partial x^\mu}\Big|_P \longrightarrow \frac{\partial \Lambda^0{}_{\bar{0}}}{\partial x^\mu}\Big|_P = -\partial_\mu \phi\Big|_P + \mathcal{O}(\phi^2)$$

$$0 = 2\frac{\partial \Lambda^i{}_{\bar{i}}}{\partial x^\mu}\Big|_P g_{ii}(P) + [\Lambda^i{}_{\bar{i}}(P)]^2 \frac{\partial g_{ii}}{\partial x^\mu}\Big|_P \longrightarrow \frac{\partial \Lambda^i{}_{\bar{i}}}{\partial x^\mu}\Big|_P = \partial_\mu \phi\Big|_P + \mathcal{O}(\phi^2)\,.$$

The elements of the transformation matrix thus read,

$$\Lambda^0{}_{\bar{0}}(x) = 1 - \phi(P) - \partial_\mu\phi\Big|_P (x^\mu - x^\mu_P) + \mathcal{O}(\phi^2)$$

$$\Lambda^i{}_{\bar{i}}(x) = 1 + \phi(P) + \partial_\mu\phi\Big|_P (x^\mu - x^\mu_P) + \mathcal{O}(\phi^2)\,.$$

and with the non-diagonal terms equal to zero.

(b) At what rate does such a frame accelerate with respect to the original coordinates, again to first order in ϕ? Consider that the free-falling observer moves with non-relativistic velocity in the non-inertial system. (1 pt)

The world-line of the free-falling frame measured by an observer located in the non-inertial frame obeys the following geodesic equation,

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\Gamma^i{}_{00}\left(\frac{dt}{d\tau}\right)^2 - \Gamma^i{}_{jl}\frac{dx^j}{d\tau}\frac{dx^l}{d\tau} - 2\Gamma^i{}_{j0}\frac{dx^j}{d\tau}\frac{dt}{d\tau}$$

The Christoffel symbols are of order $\Gamma \sim \mathcal{O}(\phi)$ and since the free-falling body moves with non-relativistic velocity in the non-inertial frame, $d\tau = d\tau + \mathcal{O}(v, \phi)$. Thus, it is clear that the second term in the right-hand side is of order $\mathcal{O}(3)$ and the last one of order $\mathcal{O}(2)$. The leading contribution comes from the first term and is of order $\mathcal{O}(\phi)$. Hence, $\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{00} + \mathcal{O}(\phi^2) = -\partial_i\phi + \mathcal{O}(\phi^2)$

ϕ can be interpreted as the Newtonian gravitational potential.

(c) Compute the elements of the Riemann tensor in the original (non-inertial) frame to first order in ϕ. (1 pt)

The Christoffel symbols are as follows :

$$\begin{aligned} {\Gamma^0}_{00} &= \partial_0\phi + \mathcal{O}(\phi^2) & {\Gamma^i}_{00} &= \partial^i\phi + \mathcal{O}(\phi^2) \\ {\Gamma^0}_{ij} &= -\delta_{ij}\partial_0\phi + \mathcal{O}(\phi^2) & {\Gamma^i}_{0j} &= -\delta^i_j\partial_0\phi + \mathcal{O}(\phi^2) \\ {\Gamma^0}_{0i} &= \partial_i\phi + \mathcal{O}(\phi^2) & {\Gamma^i}_{jl} &= \delta_{jl}\partial^i\phi - \delta^i_j\partial_l\phi - \delta^i_l\partial_j\phi + \mathcal{O}(\phi^2) \end{aligned}$$

Notice that $\partial_i\phi = \partial^i\phi + \mathcal{O}(\phi)$, so partial derivatives with upper and lower indices can be interchanged if we work to first order in ϕ. The components of the Riemann tensor will take the following form if we neglect contributions of order $\mathcal{O}(\phi^2)$ or higher,

$${R^\mu}_{\nu\alpha\beta} = \partial_\alpha{\Gamma^\mu}_{\nu\beta} - \partial_\beta{\Gamma^\mu}_{\nu\alpha} + \mathcal{O}(\phi^2)$$

Thus, we have,

$$\begin{aligned} {R^\mu}_{\nu 00} &= \mathcal{O}(\phi^2) \\ {R^\mu}_{\nu 0i} &= \partial_0{\Gamma^\mu}_{\nu i} - \partial_i{\Gamma^\mu}_{\nu 0} + \mathcal{O}(\phi^2) \\ {R^\mu}_{\nu ij} &= \partial_i{\Gamma^\mu}_{\nu j} - \partial_j{\Gamma^\mu}_{\nu i} + \mathcal{O}(\phi^2) \end{aligned}$$

Equivalently,

$$
\begin{aligned}
R^{\mu}{}_{\nu 00} &= \mathcal{O}(\phi^2) \\
R^{0}{}_{00i} &= \mathcal{O}(\phi^2) \\
R^{0}{}_{j0i} &= -\delta_{ij}\partial_0^2\phi - \partial_i\partial_j\phi + \mathcal{O}(\phi^2) \\
R^{j}{}_{00i} &= -\delta_i^j\partial_0^2\phi - \partial_i\partial^j\phi + \mathcal{O}(\phi^2) \\
R^{j}{}_{l0i} &= \delta_{li}\partial_0\partial^j\phi - \delta_i^j\partial_0\partial_l\phi + \mathcal{O}(\phi^2) \\
R^{0}{}_{0ij} &= \mathcal{O}(\phi^2) \\
R^{0}{}_{lij} &= -\delta_{lj}\partial_i\partial_0\phi + \delta_{li}\partial_j\partial_0\phi + \mathcal{O}(\phi^2) \\
R^{l}{}_{0ij} &= -\delta_j^l\partial_i\partial_0\phi + \delta_i^l\partial_j\partial_0\phi + \mathcal{O}(\phi^2) \\
R^{l}{}_{kij} &= \delta_{kj}\partial_i\partial^l\phi - \delta_j^l\partial_i\partial_k\phi - \delta_{ki}\partial_j\partial^l\phi + \delta_i^l\partial_j\partial_k\phi + \mathcal{O}(\phi^2)
\end{aligned}
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Exercise 6.3
(Killing vectors and conserved quantities)

Exercise 6.3: Killing vectors and conserved quantities

(a) Consider the scalar quantity $u^\alpha \xi_\alpha$, with $\vec{u}$ the four-velocity of a free-falling particle. What equation must be fulfilled by $\vec{\xi}$ if this scalar remains constant along the particle's trajectory? (1.5 pt)

If $u^\alpha \xi_\alpha$ is constant along the particle's trajectory it has to satisfy the following equation: $\dfrac{d}{d\tau}(u^\alpha \xi_\alpha) = 0\,,$

where τ is the proper time of the particle.

This equation can be rewritten as follows,

$$\frac{du^\alpha}{d\tau}\xi_\alpha + u^\alpha \frac{d\xi_\alpha}{d\tau} = 0 \longrightarrow (u^\beta \nabla_\beta u^\alpha)\xi_\alpha + u^\alpha u^\beta \nabla_\beta \xi_\alpha = 0\,.$$

A free particle satisfies the geodesic equation $u^\beta \nabla_\beta u^\alpha = 0$, so we have $\longrightarrow u^\alpha u^\beta \nabla_\beta \xi_\alpha = 0\,.$

Regardless of the 4-velocity of the free-falling particle this equation is automatically fulfilled if $\nabla_\beta \xi_\alpha$ is antisymmetric, i.e. if

$$\boxed{\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta = 0}$$

This is Killing's equation.

(b) Express (6) in spherical coordinates (t, r, θ, φ). (1 pt)

$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)(dx^2+dy^2+dz^2)$

$$\begin{aligned} x &= r\sin(\theta)\cos(\varphi) \\ y &= r\sin(\theta)\sin(\varphi) \\ z &= r\cos(\theta)\,. \end{aligned}$$

Thus, we have,

$$\begin{aligned} dx &= \sin(\theta)\cos(\varphi)dr + r\cos(\theta)\cos(\varphi)d\theta - r\sin(\theta)\sin(\varphi)d\varphi \\ dy &= \sin(\theta)\sin(\varphi)dr + r\cos(\theta)\sin(\varphi)d\theta + r\sin(\theta)\cos(\varphi)d\varphi \\ dz &= \cos(\theta)dr - r\sin(\theta)d\theta\,. \end{aligned}$$

This leads to:

$$\boxed{ds^2 = -(1+2\phi)dt^2 + (1-2\phi)[dr^2 + r^2\{d\theta^2 + \sin^2(\theta)d\varphi^2\}]}$$

(c) Assume now that the metric under consideration is static and spherically symmetric. Find two linearly independent Killing vectors in this particular spacetime, and prove that p_0 and p_φ are conserved along the trajectory of a free-falling particle.

$$\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta = 0 \longrightarrow \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2\Gamma^\alpha{}_{\mu\nu}\xi_\alpha = 0$$

$ds^2 = -(1+2\phi)dt^2 + (1-2\phi)[dr^2 + r^2\{d\theta^2 + \sin^2(\theta)d\varphi^2\}]$

For the spacetime determined by the line element (50) with $\phi = \phi(r)$ there are only nine Christoffel symbols different from zero, namely $\Gamma^\theta{}_{\theta r}, \Gamma^\theta{}_{\varphi\varphi}, \Gamma^0{}_{0r}, \Gamma^\varphi{}_{\varphi r}, \Gamma^\varphi{}_{\varphi\theta}, \Gamma^r{}_{rr}, \Gamma^r{}_{\theta\theta}, \Gamma^r{}_{\varphi\varphi}$ and $\Gamma^r{}_{00}$. Each Killing vector has to satisfy the following set of equations:

$$\begin{aligned}
\partial_0 \xi_0 - \Gamma^r{}_{00}\xi_r &= 0 \\
\partial_r \xi_r - \Gamma^r{}_{rr}\xi_r &= 0 \\
\partial_\theta \xi_\theta - \Gamma^r{}_{\theta\theta}\xi_r &= 0 \\
\partial_\varphi \xi_\varphi - \Gamma^\theta{}_{\varphi\varphi}\xi_\theta - \Gamma^r{}_{\varphi\varphi}\xi_r &= 0 \\
\partial_0 \xi_r + \partial_r \xi_0 - 2\Gamma^0{}_{0r}\xi_0 &= 0
\end{aligned}$$

$$\begin{aligned}
\partial_0 \xi_\theta + \partial_\theta \xi_0 &= 0 \\
\partial_0 \xi_\varphi + \partial_\varphi \xi_0 &= 0 \\
\partial_r \xi_\theta + \partial_\theta \xi_r - 2\Gamma^\theta{}_{r\theta}\xi_\theta &= 0 \\
\partial_r \xi_\varphi + \partial_\varphi \xi_r - 2\Gamma^\varphi{}_{r\varphi}\xi_\varphi &= 0 \\
\partial_\theta \xi_\varphi + \partial_\varphi \xi_\theta - 2\Gamma^\varphi{}_{\varphi\theta}\xi_\varphi &= 0
\end{aligned}$$

Let us see whether there exists a Killing vector with $\xi_r = \xi_\theta = \xi_\varphi = 0$. Using this in the Killing equations we find:

$$\begin{aligned}
\partial_\varphi \xi_0 = \partial_\theta \xi_0 = \partial_0 \xi_0 &= 0 \\
\partial_r \xi_0 - 2\Gamma^0{}_{0r}\xi_0 &= 0\,, \qquad \text{so } \xi_0 = \xi_0(r)
\end{aligned}$$

Using $\Gamma^0{}_{0r} = \frac{\partial_r \phi}{1+2\phi}$ we can solve the last equation and obtain:

$$\tilde{d}\xi = A\,(1+2\phi, 0, 0, 0) \longrightarrow \boxed{\overrightarrow{\xi} = A\,(1, 0, 0, 0)}$$ where A is an integration constant.

Now let us search for another Killing vector, but with $\xi_r = \xi_\theta = \xi_0 = 0$.

The following equations have to be fulfilled,

$$\begin{aligned} \partial_\varphi \xi_\varphi = \partial_\theta \xi_r = \partial_0 \xi_\varphi &= 0 \\ \partial_r \xi_\varphi - 2\Gamma^\varphi{}_{\varphi r}\xi_\varphi &= 0 \\ \partial_\theta \xi_\varphi - 2\Gamma^\varphi{}_{\varphi\theta}\xi_\varphi &= 0\,. \end{aligned}$$

In this case $\xi_\varphi = \xi_\varphi(r,\theta)$. As $\Gamma^\varphi{}_{\varphi r} = 1/r - \partial_r\phi/(1-2\phi)$ and $\Gamma^\varphi{}_{\varphi\theta} = cotg(\theta)$ we find:

$$\begin{aligned} \partial_r \xi_\varphi - \frac{2}{r}\xi_\varphi &= 0 \\ \partial_\theta \xi_\varphi - 2cotg(\theta)\xi_\varphi &= 0\,. \end{aligned}$$

The solution reads,

$$\tilde{d}\xi = B\,(0, 0, 0, r^2 \sin^2(\theta)(1-2\phi)) \longrightarrow \boxed{\overrightarrow{\xi} = B\,(0, 0, 0, 1)}$$

Any linear combination of the two fields will also satisfy the Killing equation

It is clear that along the trajectory of a free-falling particle,

$$u_0 = C_1 \qquad ; \qquad u_\varphi = C_2 \, ,$$

or, equivalently,

$$p_0 = \bar{C}_1 \qquad ; \qquad p_\varphi = \bar{C}_2 \, .$$

(d) Show that these conservation laws can be also obtained directly from the geodesic equation. (1 pt)

The geodesic equation reads, $u^\nu \nabla_\nu u_\mu = 0 \longrightarrow u^\nu \partial_\nu u_\mu = \Gamma^\alpha{}_{\nu\mu} u_\alpha u^\nu$

$$\frac{du_\mu}{d\tau} = \frac{1}{2} g_{\beta\nu,\mu} u^\beta u^\nu \quad \longrightarrow m \frac{dp_\mu}{d\tau} = \frac{1}{2} g_{\beta\nu,\mu} p^\beta p^\nu$$

Since the metric does not depend on time nor the angle φ we find that p_0 and p_φ remain constant along the trajectory of a free-falling particle.