

Coordinate Systems, Coordinate transformations and Jacobians

BY ROHAN KULKARNI

For IPSP Freshman Tutorial

1 What is this Jacobian?

Method of substitution during integration

Lets first see a simple example and then analyze it.

Example. Solve $\int_{a=\frac{\pi}{3}}^{b=\pi} \cos(3x+4)dx$ using a substitution. (General form $\int_{x=a}^{x=b} f(x)dx$)

Let $(3x+4)=t$, i.e. $t(x)=(3x+4)$. We do the following steps.

1. Find the new lower limit c :

$$\begin{aligned} t(a) &= c \\ (3a+4) = (\pi+4) &= c \\ c &= \pi+4 \end{aligned}$$

and the new upper limit d :

$$\begin{aligned} t(b) &= d \\ (3b+4) = (3\pi+4) &= d \\ d &= 3\pi+4 \end{aligned}$$

2. Rewrite the function :

$$\int_{c=\pi+4}^{d=3\pi+4} t dx$$

But, we can't do anything more, as we cannot integrate this over x .

3. We change the differential / infinitesimal.

$$\begin{aligned} dt = d(t(x)) &= \frac{dt}{dx} dx \\ dt \left(\frac{dt}{dx} \right)^{-1} &= \left(\frac{dx}{dt} \right)^{-1} \left(\frac{dx}{dt} \right) dt \end{aligned}$$

We know that $\frac{dt}{dx} = \left(\frac{dx}{dt} \right)^{-1}$

$$dt \left(\frac{dx}{dt} \right)^{-1} = dx$$

We have $\frac{dt}{dx} = 3$ which yields $\frac{dx}{dt} = \frac{1}{3}$ and we get

$$dt \left(\frac{1}{3} \right) = dx$$

4. Now we plug in all the 3 points above to get :

$$\begin{aligned}\int_{\frac{\pi}{4}}^{\pi} \cos(3x + 4) dx &= \int_{\pi+4}^{3\pi+4} \cos(t) \frac{1}{3} dt \\ &= \frac{1}{3}(\sin(t))|_{\pi+4}^{3\pi+4}\end{aligned}$$

Analyzing the Jacobian

The graph below has all the 3 functions. **Red** : $y = \cos(x)$, **Green** : $y = \cos(3x + 4)$, **Blue** : $t = 3x + 4$
Analysis :

Jacobian is basically the slope of the **blue** line. You can easily say that it is 3 (Just by counting rise over run). How does it connect with the **red** and **green** graphs?

Ignore the **blue** line now and look carefully on the domain $[0, 2\pi]$ for the **red** graph. You have **one peak-trough** line for a length 2π on the x axis. Now look on the domain $[0 - 0.3, 2\pi - 0.3]$ which still has a length 2π on the x axis but you have **three peak-trough** lines. This information that **three peak-trough** has to transform to **one peak-trough** during this coordinate transformation is stored in the Jacobian.

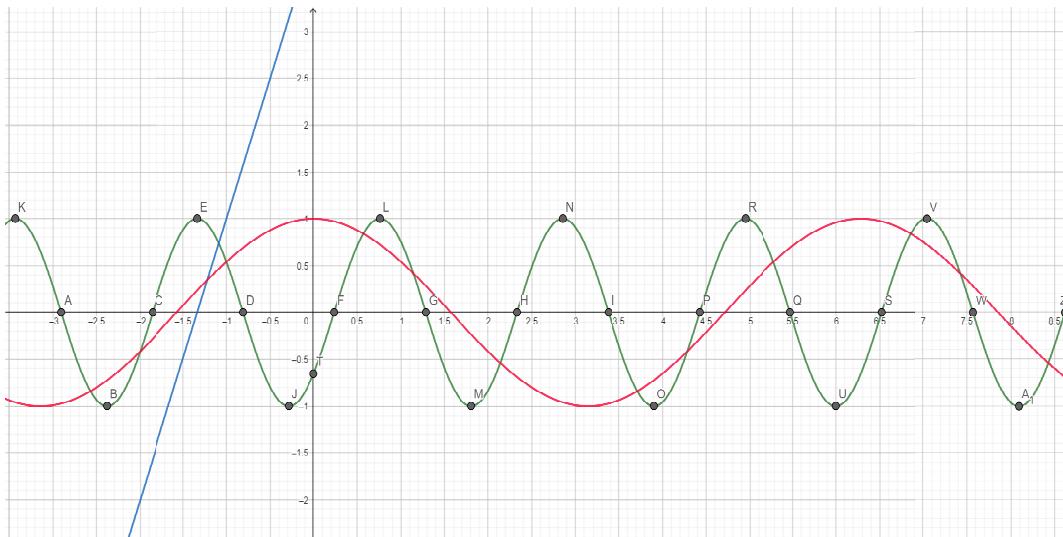


Figure 1. [2]

Plot of all the 3 functions.

Remark.

For the example above the Jacobian was a constant number so the analysis was quite easy. If the Jacobian was not a constant the analysis like we did above becomes difficult but the idea remains exactly the same. Just extrapolate what we learned above to such an example (we will see polar coordinates soon).

Lets start with analyzing this method for a function of one variable.

Whenever we use the method of substitution to solve an integral $\int_a^b f(x)dx$, we change the following things :

1. The limits (boundaries) of integration
2. The function itself

3. The infinitesimal dx

Suppose we substitute a part of it using a new variable t , then we have $t(x)$:

1. The new limits c and d found by solving the following equations :

$$\begin{aligned} a &= x(c) \\ b &= x(d) \end{aligned}$$

2. The function as :

$$f(x) = f(x(t)) \quad (1)$$

3. The infinitesimal dx :

$$dx = \frac{dx}{dt} dt \quad (2)$$

A picture is worth a thousand words (When you can draw one..)

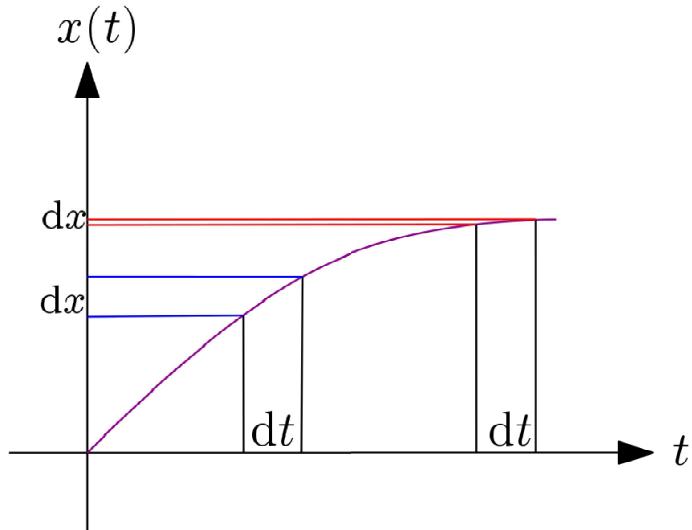


Figure 2.
Why we need to change dx

Suppose x as a function of our new variable t looked like Fig.1. We clearly see that we dt remains constant if we want to integrate this function (by definition of Riemann integration) but dx changes.

This quantity $\frac{dx}{dt}$ is the Jacobian of this coordinate transformation. It tells us how to change the infinitesimal.

So in simple words, when we make a change of a coordinate system, the Jacobian tells us how much the infinitesimal should change.

1.1 Jacobian in more than one variables

1.1.1 Jacobian in two variables

For a given transformation $x = x(u, v)$ and $y = y(u, v)$ the Jacobian is :

$$J(u, v) = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \right|$$

1.1.2 Jacobian in three variables

For a given transformation $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$ is :

$$J(u, v, w) = J(u, v) = \left| \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} \right|$$

Notation :

$$J(u, v) = J(x, y; u, v) = J\left(\frac{x, y}{u, v}\right) = \left| \frac{\partial(x, y)}{\partial(u, v)} \right|$$

and the way we connect how area / volume elements change under transformation (Well, area and volume elements are made from differentials/infinitesimals) so they change with change of coordinate systems.

$$dx dy \rightarrow |J(u, v)| du dv \quad (3)$$

2 Different important coordinate systems.

2.1 Polar coordinates (2D)

Polar coordinates are another way of representing all the points in \mathbb{R}^2 . We start with our Cartesian coordinates and use a change of variable to write down polar coordinates.

The coordinate transformation is from $(x, y) \mapsto (r, \theta)$.

We define the following coordinates :

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \quad (4)$$

So basically we interpret this diagram

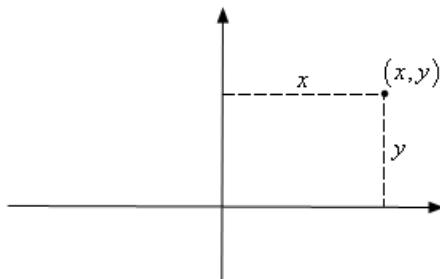


Figure 3. [1]

as

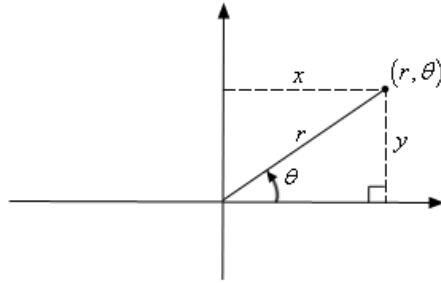


Figure 4. [1]

So if we reverse our coordinate transformation we get :

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

So we basically made the coordinate system change but if we have this under an integral (double integral) we need to take care of the area element. The area element in cartesian coordinate is a tiny area $dxdy$. As we discussed earlier, the differential element/infinitesimals change as we change variables (coordinate transformation) and they change with a factor called the Jacobian.

Look at the figure below, you can easily see that as r increases, the circumference of a circle of radius r is $2\pi r$ and hence the length of an arc spanned by an angle θ is $2\pi r \frac{\theta}{2\pi} = r\theta$.

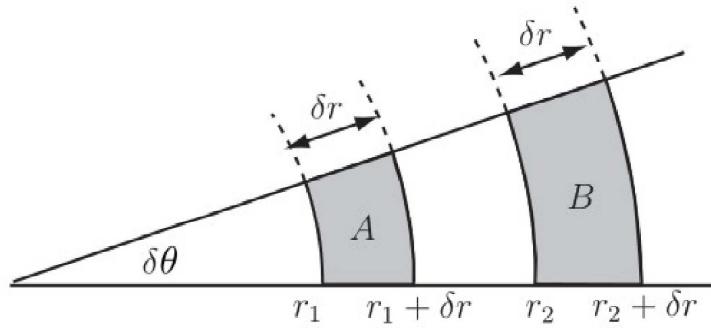


Figure 5. [3]

So basically, these approximate rectangles that you can see have a width dr and length $rd\theta$. So basically $rd\theta dr$ play the same role in polar coordinates as $dxdy$ play in cartesian coordinates.

Jacobian of Polar coordinates with respect to Cartesian coordinates :

$$\begin{aligned} J(r, \theta; x, y) &= \left| \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix} \right| = \cos \theta r \cos \theta - (-r \sin \theta) \times \sin \theta \\ &= r(\cos^2 \theta + \sin^2 \theta) \\ &= r \end{aligned}$$

Bibliography

- [1] Paul Dawkins. Paul's online notes. <http://tutorial.math.lamar.edu/Classes/CalcII/PolarCoordinates.aspx>.
- [2] <https://www.geogebra.org/?lang=en>.
- [3] https://ask.fxplus.ac.uk/tools/HELM/pages/workbooks_1_50_jan2008/Workbook27/27_4_chng_coords.pdf.