Q5. Given : $O^N_{(2)} = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} O(\vec{R}_i, \vec{P}_i, \vec{S}_i ; \vec{R}_j, \vec{P}_j, \vec{S}_j)$

is a 2-body operator. The second quantized form is given by

$$O^N_{(2)} = \frac{1}{2} \sum_{\vec{k}_\alpha, \vec{k}_\beta} \sum_{\vec{k}'_\alpha \vec{k}'_\beta} \langle \vec{k}_\alpha, \vec{k}_\beta | O(\vec{R}_1, \vec{P}_1, \vec{S}_1 ; \vec{R}_2, \vec{P}_2, \vec{S}_2) | \vec{k}'_\alpha, \vec{k}'_\beta \rangle \, a^\dagger_{\vec{k}_\alpha} a^\dagger_{\vec{k}_\beta} a_{\vec{k}'_\alpha} a_{\vec{k}'_\beta}$$

To show : These two expressions give same matrix elements for $N=2$ bosonic state.

Start with,

$$\langle 1,1 | a^\dagger_{k_\alpha} a^\dagger_{k_\beta} a_{k'_\alpha} a_{k'_\beta} | 1,1 \rangle$$

$$= \langle 0,0 | a_{k_1} a_{k_2} a^\dagger_{k_\alpha} a^\dagger_{k_\beta} a_{k'_\alpha} a_{k'_\beta} a^\dagger_{k'_1} a^\dagger_{k'_2} | 0,0 \rangle \, h_{k_1 k_2} h_{k'_1 k'_2}$$

where $h_{kk'} = \begin{cases} 1/\sqrt{2} & k = k' \\ 1 & k \neq k' \end{cases}$

Following the 1-body operator case, we want to bring creation op to the left & annihilation operators to the right, using standard commutation relations.

$\cancel{a_{k_1} a_{k_2} a^\dagger_{k_\alpha} a^\dagger_{k_\beta} + (a_{k_2} \delta_{k_\alpha k_1} (+ a_{k_1} \delta_{k_2}) a^\dagger_{k_\beta}}$

$[a_k, a^\dagger_{k'}] = \delta_{kk'}$, i.e. $a_k a^\dagger_{k'} - a^\dagger_{k'} a_k = \delta_{kk'}$

$$a_{k_1} a_{k_2} a^\dagger_{k_\alpha} = a_{k_1}\left(a^\dagger_{k_\alpha} a_{k_2} + \delta_{k_2 k_\alpha}\right)$$

$$= a^\dagger_{k_\alpha} a_{k_1} a_{k_2} + a_{k_2}\delta_{k_\alpha k_1} + a_{k_1}\delta_{k_2 k_\alpha}$$

↳ 0 when acts on $|0\rangle$

This eq^n used.

$$a_{k_1} a_{k_2} a^\dagger_{k_\alpha} a^\dagger_{k_\beta} = \left(a_{k_2}\delta_{k_\alpha k_1} + a_{k_1}\delta_{k_2 k_\alpha}\right) a^\dagger_{k_\beta}$$

$$= \delta_{k_\alpha k_1}\delta_{k_2 k_\beta} + \delta_{k_\alpha k_2}\delta_{k_1 k_\beta} + \underbrace{0+0}_{a|0\rangle \text{ terms.}}$$

$$= \delta_{k_\alpha k_1}\delta_{k_2 k_\beta} + \delta_{k_2 k_\alpha}\delta_{k_1 k_\beta}$$

Similarly for the other set of 4 a's

$$a_{k'_\alpha} a_{k'_\beta} a^\dagger_{k'_1} a^\dagger_{k'_\alpha} = \delta_{k'_\alpha k'_1}\delta_{k'_\beta k'_2} + \delta_{k'_\alpha k'_2}\delta_{k'_\beta k'_1}$$

This gives us:

$$h_{k_1 k_2} h_{k'_1 k'_2}\left(\delta_{k_\alpha k_1}\delta_{k_2 k_\beta} + \delta_{k_2 k_\alpha}\delta_{k_1 k_\beta}\right)\left(\delta_{k'_\alpha k'_1}\delta_{k'_\beta k'_2} + \delta_{k'_\alpha k'_2}\delta_{k'_\beta}\delta_{k'_1}\right)$$

Now the other eq^n:

$$\langle \theta^2_{(w)}\rangle = \frac{h_{k_1 k_2} h_{k'_1 k'_2}}{2}\sum_{k'_\alpha k'_\beta}\sum_{k_\alpha k_\alpha}\langle k_\alpha, k_\beta | \theta(\ldots) | k_{\alpha'} k_{\beta'}\rangle$$

$$\left(\delta_{k_\alpha k_1}\delta_{k_2 k_\beta} + \delta_{k_2 k_\alpha}\delta_{k_1 k_\beta}\right)\left(\delta_{k'_\alpha k'_1}\delta_{k'_\beta k'_2} + \delta_{k'_\alpha k'_2}\delta_{k'_\beta}\delta_{k'_1}\right)$$

$$= \frac{h_{k_1 k_2} h_{k'_1 k'_2}}{2}\Big[\langle k_1 k_2 | \theta | k'_1 k'_2\rangle + \langle k_1 k_2 | \theta | k'_2 k'_1\rangle$$
$$+ \langle k_2 k_1 | \theta | k'_1 k'_2\rangle + \langle k_2 k_1 | \theta | k'_2 k'_1\rangle\Big]$$

$$\langle k_1 k_2 | \mathcal{O}^{2}_{(1)} | k_1' k_2' \rangle$$

$$= h_{k_1 k_2} \, h_{k_1' k_2'} \, \frac{1}{\sqrt{2}\sqrt{2}} \left(\langle k_1, k_2 | + \langle k_2 k_1 | \right) \mathcal{O} \left(| k_1' k_2' \rangle + | k_2' k_1' \rangle \right)$$

$$= \frac{h_{k_1' k_2'} \, h_{k_1 k_2}}{2} \Big[\langle k_1 k_2 | \mathcal{O} | k_1' k_2' \rangle + \langle k_1 k_2 | \mathcal{O} | k_2' k_1' \rangle + \langle k_2 k_1 | \mathcal{O} | k_1' k_2' \rangle + \langle k_2 k_1 | \mathcal{O} | k_2' k_1' \rangle$$

agreeing with our previous result. $\square$.