# Mathematics HL Investigation

How can moments of inertias be mathematically derived for mass distributions with varying degrees of complexity?

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# 1 Introduction

I first became intrigued with the concept of moment of inertia when learning about rotating bodies such as flywheels and ferris wheels. I understood that these bodies were "moving" and stored kinetic energy, but they had no linear movement, leaving me somewhat confused about the general premise for calculating their stored energy. I soon realized that an understanding of rotational movement and a particle based approach could higlight the parallels between linear kinetic energy and rotational kinetic energy which led to the discovery of moments of inertias. Moment of inertias play a major role in rotational dynamics and have a host of applications when it comes to motors, wheels, and machinery. Their ability to serve as a measure of the resistance to angular motion for a body makes them equally important as traditional inertia, which dictates many physical aspects of our daily lives. Pure rolling motion, bridge building, and flywheels in engines are all predicated on the concept of moment of inertia, and thus the idea becomes exceedingly crucial in the study and implementations of engineering. Scientists and mathematicians are continuing to look for methods to calculate more intricate moments of inertias to improve current physical systems and create new practical technology.

This interesting concept lends itself to a a mathematical investigation for deriving moment of inertia formulas for progressively more complex mass distributions. Because of the reliance of moment of inertia on the geometric makeup of a body, I decided to investigate the various approaches that are used to calculate the values and derive some of the most commonly used formulas in modern physics and mathematics.

# 2 Background

## 2.1 Kinetic Energy and Physics Definitions

Kinetic energy is a measure of the energy of a moving body. The formula for the calculation of kinetic energy of a body moving with translational velocity $v$ and a mass $M$, with the assumption that the body is rigid and the center of mass is moving at the same constant velocity is shown in Eq. (1).

$$KE = \frac{1}{2}Mv^2 \tag{1}$$

For a complex body defined by a set of particle-masses with unique energies, the principle of superposition can be applied to produce Eq. (2) in which the kinetic energies of each particle $i$ are summed.

$$KE = \sum_i \frac{1}{2}M_i v_i^2 \tag{2}$$

Thus, the calculation of kinetic energy of a rigid body exhibiting translational motion is based solely on mass and velocity and its geometric configuration has no impact on the quantity. However, for rotational motion, this premise can no longer be accepted because each particle has a unique linear velocity depending on its distance from the axis of rotation. Consider a thin disk rotating about a vertical axis that passes perpendicularly through its center as shown in Fig. 1.

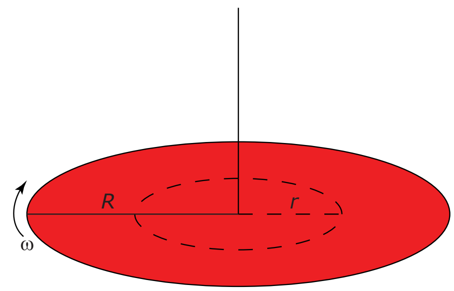


**Figure 1:** A solid disk rotating about a centered, perpendicular axis with a constant angular velocity $\omega$

Although each point of the disk is rotating with a constant angular velocity $\omega$, the linear velocity of each point is different. This can be easily determined using an understanding of rotation and time. The entire disk must undergo one complete revolution in time $\frac{2\pi}{\omega}$, but the distance that a point must travel in that time is dependent on its distance from the axis. For a point at a distance of $r$ from this axis, this distance is simply the circumfrence of a circle with radius $r$, or $2\pi r$. Using the formula $speed = \frac{distance}{time}$, the speed of this particle is defined by Eq. (3). Thus, the rotating disk can actually be thought of as a combination of thin rings each consisting with particles moving at a unique velocity.

$$v = \frac{2\pi r}{\frac{2\pi}{\omega}} = r\omega \tag{3}$$

For better generalization of this formula in instances where the velocity is not directly perpendicular to the axis of rotation, velocity can also be written as a cross product of the angular velocity vector (in the same direction as the axis of rotation) $\boldsymbol{\omega}$ and the position vector of the particle $\boldsymbol{r_i}$ as:

$$\boldsymbol{v_i} = \boldsymbol{\omega} \times \boldsymbol{r_i} \tag{4}$$

and the angular velocity vector as:

$$\boldsymbol{\omega} = \omega \boldsymbol{k} \tag{5}$$

where $\boldsymbol{k}$ is simply a unit vector in the direction of the rotational axis and $\omega$ is the magnitude of the angular velocity. Then, the velocity can simplified to:

$$\boldsymbol{v_i} = \omega |k \times \boldsymbol{r_i}| \tag{6}$$

$|\boldsymbol{k} \times \boldsymbol{r_i}|$ is equivalent to the perpendicular distance between the point and the axis of rotation, which will be proved later. Using this value, we can utilize the traditional formula for kinetic energy to calculate the energy for a single rotating particle as:

$$R.K.E._{i} = \frac{1}{2} m_i \omega^2 |\boldsymbol{k} \times \boldsymbol{r_i}|^2 \tag{7}$$

and the rotational energy for the entire body as:

$$R.K.E. = \sum_i \frac{1}{2} m_i \omega^2 |\boldsymbol{k} \times \boldsymbol{r_i}|^2 \tag{8}$$

To simplify this equation, the moment of inertia of a body is empirically defined as:

$$I = \sum_i m_i |\boldsymbol{k} \times \boldsymbol{r_i}|^2 \tag{9}$$

which for a single particle can be simplified to:

$$I = mr^2 \tag{10}$$

where $r$ is the perpendicular distance from the particle to the axis.

The general formula for rotational kinetic energy can then be defined by Eq. (11) using two values, the rotational velocity $\omega$ and the moment of inertia $I$, which for a body rotating at a constant velocity about a fixed axis are constants (similar to $M$ and $v$ for traditional kinetic energy).

$$R.K.E = \frac{1}{2} I \omega^2 \tag{11}$$

Plugging in Eq. (10) and Eq. (3) to the equation for rotational kinetic energy shows that the final units remain the same as that for traditional kinetic energy presented in Eq. (1). Using this information, I will derive moment of inertia formulas for a multitude of common bodies in motion.

## 2.2 Vector Cross Products

The formula for moment of inertia relies on the magnitude of a cross product. The general formula for the magnitude of a cross product is defined as:

$$|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| \sin \theta \tag{12}$$

where $\theta$ is the angle between the two vectors.

This can be proved using basic vector rules:

$$\text{Let } \boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{j} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \boldsymbol{k} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Then, the squared-magnitude of the cross-product can be defined as:

$$\begin{aligned} |\boldsymbol{u} \times \boldsymbol{v}|^2 &= (\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v}) \\ &= (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 \\ &= u_2^2 v_3^2 + u_3^2 v_2^2 + u_1^2 v_3^2 + u_3^2 v_1^2 + u_1^2 v_2^2 + u_2^2 v_1^2 - 2(u_2 u_3 v_2 v_3 + u_1 u_3 v_1 v_3 + u_1 u_2 v_1 v_2) \end{aligned}$$

We can also observe two other intriguing results:

$$\begin{aligned} (\boldsymbol{u} \cdot \boldsymbol{v})^2 &= (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\ &= u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 + 2(u_2 u_3 v_2 v_3 + u_1 u_3 v_1 v_3 + u_1 u_2 v_1 v_2) \end{aligned}$$

$$\begin{aligned} (|\boldsymbol{u}||\boldsymbol{v}|)^2 &= (\sqrt{u_1^2 + u_2^2 + u_3^2} \sqrt{v_1^2 + v_2^2 + v_3^2})^2 \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) \\ &= u_2^2 v_3^2 + u_3^2 v_2^2 + u_1^2 v_3^2 + u_3^2 v_1^2 + u_1^2 v_2^2 + u_2^2 v_1^2 + u_1^2 v_1^2 + u_2^2 v_2^2 + u_3^2 v_3^2 \end{aligned}$$

By subtracting the second result from the third result, we can acquire the first result, leading to the formula:

$$\begin{aligned} |\boldsymbol{u} \times \boldsymbol{v}|^2 &= (|\boldsymbol{u}||\boldsymbol{v}|)^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2 \\ &= (|\boldsymbol{u}||\boldsymbol{v}|)^2 - (|\boldsymbol{u}||\boldsymbol{v}| \cos\theta)^2 \\ &= (|\boldsymbol{u}||\boldsymbol{v}|)^2 - (|\boldsymbol{u}||\boldsymbol{v}|)^2 \cos^2\theta \\ &= (|\boldsymbol{u}||\boldsymbol{v}|)^2 (1 - \cos^2\theta) \\ &= (|\boldsymbol{u}||\boldsymbol{v}|)^2 \sin^2\theta \end{aligned}$$

This result can then be square rooted, resulting in the initial theorem presented in Eq. (12). Because of this formula, the cross product between a unit vector and the position vector can also represent the perpendicular distance between the direction of the unit vector and the specific position. This premise is adopted for this investigation.

## 2.3 Parallel Axis Theorem

To calculate the moment of inertia for more complex bodies, the parallel axis theorem is often utilized to relate the moment of inertia about an axis passing through the center of mass of a body to a parallel axis at a distance $d$ from the central axis. Eq. (13) describes the theorem, where $I_{cm}$ is the moment of inertia through an axis passing through the center of mass, $M$ is the mass of the body, and $I'$ is the resultant moment of inertia through the parallel axis. This equation is applicable for any body, regardless of shape or dimensions.

$$I' = I_{cm} + Md^2 \tag{13}$$

To prove this relationship, consider a lamina with mass $M$ located in the $X - Y$ plane whose center of mass is located at the origin as in Fig. 2. A general point $i$ is depicted and has coordinated $(x_i, y_i)$ and the parallel axis passes through $O$ which has coordinates $(a, b)$.

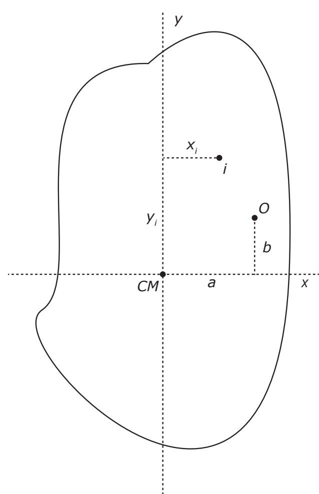


**Figure 2:** A plain lamina in the $X - Y$ plane with its center of mass located at the origin

Because the center of mass is simply a weighted average of the mass of individual points on the lamina and the center of mass coincides with the origin, we can deduce:

$$x_{CM} = \frac{\sum_i m_i x_i}{M} = 0$$
$$y_{CM} = \frac{\sum_i m_i y_i}{M} = 0$$

Using the definition of moment of inertia for a particle and denoting the distance between point $i$ and point $O$ as $r_{iO}$, we can see $I_O = \sum_i m_i r_{iO}^2$. Using the distance formula, we have:

$$r_{iO}^2 = (x_i - a)^2 + (y_i - b)^2$$

Then,

$$\begin{aligned} I_O &= \sum_i m_i r_{iO}^2 \\ &= \sum_i m_i \left[(x_i - a)^2 + (y_i - b)^2\right] \\ &= \sum_i m_i \left[x_i^2 + a^2 - 2x_i a + y_i^2 + b^2 - 2y_i b\right] \end{aligned}$$

Now, using the deductions from the center of mass, the $2x_i a$ and $2y_i b$ terms can be removed, as $\frac{\sum_i m_i x_i}{M} = 0$ and $\frac{\sum_i m_i y_i}{M} = 0$, leaving:

$$I_O = \sum_i m_i(x_i^2 + y_i^2) + \sum_i m_i(a^2 + b^2)$$

Again, the distance formula (and the fact that the center of mass is located at the origin), allows us to see that $(x_i^2 + y_i^2)$ is simply equal to the square of the distance from point $i$ to the origin, or $r_i^2$, and $(a^2 + b^2)$ is equivalent to the square of the distance between point $O$ and the origin, or $d^2$. Then, we have:

$$I_O = \sum_i m_i r_i^2 + \sum_i m_i d^2$$

As we know that the first summation is equivalent to the moment of inertia about the center of mass (the origin) using Eqs. (9) and (10), and $d^2$ is constant, this simplifies to:

$$I_O = I_{CM} + Md^2$$

leaving the initial theorem as presented in Eq. (13).

# 3 Derivations

## 3.1 Hollow Ring/Cylinder

Consider a hollow, uniform ring of radius $R$ and mass $M$ located in the $X - Y$ plane with its center at the origin rotating about the $Z$-axis as shown in Fig. 3.

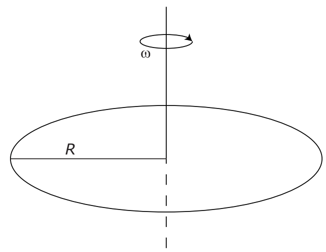


**Figure 3:** A hollow ring rotating about a centered, perpendicular axis with a constant angular velocity $\omega$

For this body, all particles are at a constant perpendicular distance from the axis of rotation, thus **??** can be simplified to:

$$I = \sum_{i=1}^{N} m_i(|\boldsymbol{k}||\boldsymbol{r_i}|\sin\frac{\pi}{2})^2$$

As $\boldsymbol{k}$ is a unit vector, its magnitude is 1 and the magnitude of $\boldsymbol{r_i}$ is simply $R$

$$I = \sum_{i=1}^{N} m_i(1 \times R \times 1)^2$$
$$I = \sum_{i=1}^{N} m_i R^2$$

Since the $R^2$ term is constant for all mass particles, the term can be removed from the summation and the inner term is simply summing over the mass of the body:

$$I = R^2 \sum_{i=1}^{N} m_i$$
$$I = R^2 \times M$$

Thus, the moment of inertia for a hollow ring has been extrapolated leaving only the overall mass and radius in the formula, as shown in Eq. (14).

$$I_{\text{ring}} = MR^2 \tag{14}$$

A similar premise can be adopted for a hollow cylinder, with radius $R$ and mass $M$, centered at the origin and rotating around the $Z$-axis as shown in Fig. 4.

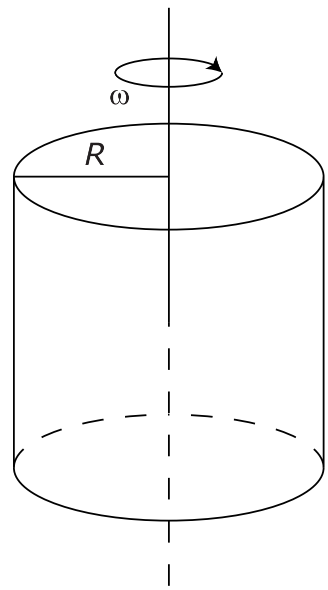


**Figure 4:** A hollow cylinder rotating about a centered, perpendicular axis with a constant angular velocity $\omega$

As the perpendicular distance between each point and the axis of rotation is once again a constant $R$, the final result for the moment of inertia will be identical:

$$I_{\text{hollow cylinder}} = MR^2 \tag{15}$$

## 3.2 Solid Cylinder

For a solid, uniform cylinder of mass $M$, height $L$ and radius $R$, the process becomes more complicated due to the initial problem discussed in the background–different particles having different distances from the axis of rotation. Consider a cylinder as shown in Fig. 5.

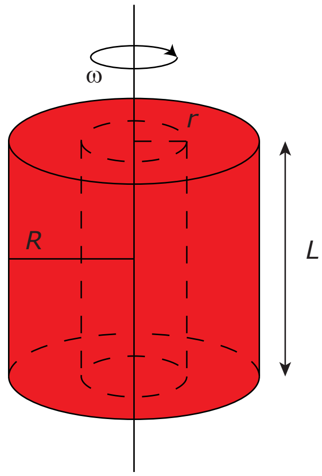


**Figure 5:** A solid cylinder rotating about a centered, perpendicular axis with a constant angular velocity $\omega$. A small shell of radius $r$ has also been depicted for visualization purposes

Assume the cylinder to have a constant density $\rho$, which leads to the determination:

$$density = \frac{Mass}{Volume} = \frac{M}{V}$$
$$V = \pi R^2 L$$
$$\rho = \frac{M}{\pi R^2 L}$$

Consider the thin cylindrical shell formed from all the particles at a distance of $r$ from the axis of rotation, with a thickness $dr$ and a mass $dm$. Each of these shells' moment of inertia can then easily be transformed into an integral:

$$I = \int_0^M r^2 dm \tag{16}$$

as the perpendicular distance is represented by $r$ and the mass elements $dm$ are being summed over for the entire mass of the body. It is important to note, however, that $r$, $dm$ and $r^2$ are not constant in this integral, as they will depend on the specific shell. Thus, we must reduce from two variables down to one to solve for the moment of inertia.

The mass of shell can be represented as:

$$dm = \rho dV \text{ (as mass = volume} \times \text{density)}$$

We can calculate $dV$ using the fact that the cylindrical shell has an infinitesimally small thickness. The shell can be cut and unfolded into a rectangular prism, with a width equal to the circumfrence of the shell, a thickness of $dr$, and a constant length of $L$:

$$V = AL$$
$$dV = dAL$$
$$dV = 2\pi r dr L$$

Plugging this back into our formula for $dm$, we have:

$$dm = \rho dV$$
$$dm = \rho 2\pi r dr L$$

As the distance of the shells from the axis of rotation ranges from 0 to $R$, we must integrate over this interval while removing constants, resulting in:

$$I = \int_0^R 2\pi\rho L r^3 dr$$
$$I = 2\pi\rho L \int_0^R r^3 dr$$
$$I = 2\pi\rho L \left[\frac{r^4}{4}\right]_0^R$$
$$I = 2\pi\rho \frac{R^4}{4}$$

Plugging in the initial value for $\rho$ calculated above, the resulting equation for the moment of inertia of a cylinder (using only the defined quantities of mass and total radius) is determined to be:

$$I_{\text{solid cylinder}} = \frac{1}{2}MR^2 \tag{17}$$

It is also important to note that this final formula for a cylinder is not dependent on the height $L$ of the cylinder, and thus the moment of inertia for a solid disk or cylindrical shape of any height can be calculated with this simple equation.

### 3.3 Solid Sphere

A similar approach can then be employed for other solid shapes that can be broken down into shells or disks, such as a solid sphere with mass $M$ and radius $R$ shown in Fig. 6.

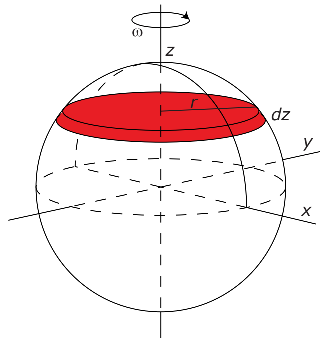


**Figure 6:** A solid sphere centered at the origin rotating about the $Z$-axis with a constant angular velocity $\omega$. A small disk of radius $r$ has also been depicted for visualization purposes

Again, assume the cylinder to have a constant density $\rho$, which can be written as:

$$\rho = \frac{M}{\frac{4}{3}\pi R^3}$$

Considering the thin disk with a radius $r$, we can write the infinitesmal moment of inertia for this disk by differentiating Eq. (17):

$$dI = \frac{1}{2}r^2 dm = \frac{1}{2}r^2 \rho dV$$

To calculate $dV$, a similar approach can be adopted by using the relation $V = A \times H$ and the fact that the height of this infinitesmal disk is simply $dz$:

$$dV = \pi r^2 dz$$

We can also relate the value of $z$, representing the height, with the value of $r$, representing the radius of the disk, using basic right-triangle geometry as the distance of all points on the sphere must be a distance of $R$ from the origin:

$$r^2 + z^2 = R^2$$
$$r^2 = R^2 - z^2$$

Plugging our values for $dV$ and $r^2$ back in and integrating from $-R$ to $R$ to travel from the bottom to the top of the sphere:

$$dI = \frac{1}{2}r^2 \rho dV = \frac{1}{2}r^2 \rho \pi r^2 dz$$

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} r^4 dz$$

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^2 - z^2)^2 dz$$

$$I = \frac{1}{2}\rho\pi \int_{-R}^{R} (R^4 + z^4 - 2R^2z^2) dz$$

$$I = \frac{1}{2}\rho\pi \left[\frac{z^5}{5} - \frac{2}{3}R^2z^3 + R^4z\right]_{-R}^{R}$$

$$I = \frac{8}{15}\rho\pi R^5$$

Substituting the initial value for $\rho$, the equation for the moment of inertia of solid cylinder can be derived as:

$$I_{\text{solid sphere}} = \frac{2}{5}MR^2 \tag{18}$$

### 3.4 Solid Plate

For a solid rectangular plate of mass $M$, width $a$ and length $b$, the process has to be modified because of the changing dimensions in two separate directions–along the length and width–so elemental mass areas are utilized instead of volumes to calculate the moment of inertia. Consider the plate depicted in

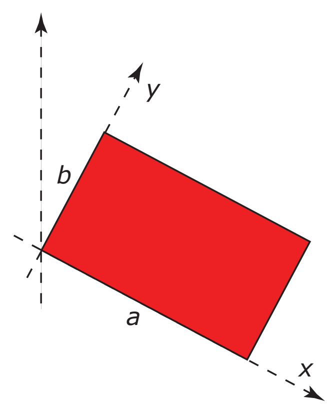


**Figure 7:** A solid rectangular plate in the $X - Y$ plane being rotated about the $Z$-axis, with its corner at the origin

Assume the plate to have a constant per-area density of $\delta$, which means that:

$$A = ab$$

$$M = \delta A$$
$$\delta = \frac{M}{ab}$$
$$dm = \delta dA$$

As the axis of rotation passes through the origin and the entire lamina is located in the $X - Y$ plane, the square of the perpendicular distance $r^2$ of an arbitrary point on the region $(x, y)$ can be written as $x^2 + y^2$ using the distance formula. Thus, the integral for the moment of inertia of the region $D$ becomes:

$$I = \iint_D (x^2 + y^2)\delta dA$$

Then, to further simplify the integral, we must evaluate dA in terms of the cartesian coordinates. Consider a small rectangle with width $dx$ and length $dy$. The area of this elemental mass unit would simply be $length \times width$, or $dxdy$. Thus, the integral is further reduced to:

$$I = \iint_D (x^2 + y^2)\delta dxdy$$

Our limits of integration are simply the length and width of the entire region $D$, leaving a simple, double line integral that can be solved using traditional integration techniques:

$$I = \int_0^b \int_0^a (x^2 + y^2)\delta dxdy$$
$$I = \delta \int_0^b \left[ \frac{x^3}{3} + xy^2 \right]_0^a dy$$
$$I = \delta \int_0^b \left( \frac{a^3}{3} + ay^2 \right) dy$$
$$I = \delta \left[ \frac{a^3 y}{3} + \frac{ay^3}{3} \right]_0^b$$
$$I = \delta \left( \frac{a^3 b}{3} + \frac{ab^3}{3} \right)$$
$$I = \frac{M}{ab} \left( \frac{a^3 b}{3} + \frac{ab^3}{3} \right)$$
$$I = \frac{M}{3} \left( a^2 + b^2 \right)$$

Thus, the moment of inertia of a rectangular lamina about a perpendicular axis through its corner is denoted through this formula. However, to calculate the moment of inertia about a perpendicular axis through the center of this lamina, we must utilize the parallel axis theorem. The distance formula can be used to determine that the distance between these two axes is simply half the length of the diagonal of the plate, or $\frac{\sqrt{a^2+b^2}}{2}$, so the theorem states:

$$I_{corner} = I_{CM} + M\left(\frac{\sqrt{a^2+b^2}}{2}\right)^2$$

$$\frac{M}{3}\left(a^2+b^2\right) = I_{CM} + \frac{M}{4}\left(a^2+b^2\right)$$

The resulting equation for the moment of inertia of a rectangular plate of dimensions $a \times b$ is then determined to be:

$$I_{\text{rectangular plate}} = \frac{M}{12}\left(a^2+b^2\right) \tag{19}$$

This premise can also be adopted for 3-D shapes, using the $(x, y, z)$ coordinate system and $dV$ elements instead of $dA$.

### 3.4.1 Polar Coordinates, Jacobian Determinants and Density Functions

Although the moment of inertia of a plate is constantly utilized in physical applications, it is also common for a body to have a density function that dictates the density of each individual point of a lamina. The moment of inertia integral for a 2-D body of this nature rotating about the origin would then be:

$$I = \iint_D (x^2+y^2)\rho(x,y)dA$$

where $\rho(x, y)$ simply represents the density as a function of the cartesian coordinates. A common example of this approach is highlighted for a circular plate with radius $R$ centered at the origin that has the elementary density function $\rho(x, y) = 1 - x^2 - y^2$. Then, the integral would become:

$$I = \iint_D (x^2+y^2)(1-x^2-y^2)dA$$

Using a similar process, this integral could be solved much like a typical rectangular plate, however it is more convenient to use polar coordinates to translate this system from $(x, y) \rightarrow (r, \theta)$. Polar coordinates offer a multitude of simple relationships, including:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

However, the one issue when translating between spaces is the concept of spatial distortion, which affects non-linear measures, such as area and volume. Although $dA = dxdy$ in the $xy$ system, $dA \neq drd\theta$ in the $r\theta$ system. To determine the distortion factor, the Jacobian must be calculated for the translation/mapping of a region from one space to another. The Jacobian of the transformation:

$$(u, v) \rightarrow (x(u, v), y(u, v))$$

is the $2 \times 2$ determinant:

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

Then, the area/volume of a region in the newly-mapped space simply requires the absolute value of the determinant to be multiplied as a factor to account for the distortion:

$$dA = dxdy = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

As $x = r\cos\theta$ and $y = r\sin\theta$, the Jacobian for the polar transformation is:

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

resulting in the derivation:

$$dA = dxdy = rdrd\theta$$

Returning to the original integral for the circular lamina, we have:

$$I = \iint_D (x^2 + y^2)(1 - x^2 - y^2) rdrd\theta$$

and we can substitute $r^2$ for $x^2 + y^2$, resulting in:

$$I = \iint_D (r^2)(1 - r^2) rdrd\theta$$

Our limits for the inner integral would be 0 to $R$, and the outer integral would span the angle around the circle, or 0 to $2\pi$:

$$I = \int_0^{2\pi} \int_0^R (r^2)(1 - r^2) r dr d\theta$$
$$I = \int_0^{2\pi} \int_0^R (r^3 - r^5) dr d\theta$$
$$I = \int_0^{2\pi} \left[ \frac{r^4}{4} - \frac{r^6}{6} \right]_0^R d\theta$$
$$I = \int_0^{2\pi} \left( \frac{R^4}{4} - \frac{R^6}{6} \right) d\theta$$
$$I = \left( \frac{R^4}{4} - \frac{R^6}{6} \right) [\theta]_0^{2\pi}$$
$$I = \pi R^4 \left( \frac{1}{2} - \frac{R^2}{3} \right)$$

Thus, the moment of inertia of this circular lamina with an elementary mass density function can be denoted through a formula that is based solely on its radius. There are obviously significantly more complex regions and density functions for modern day applications, but the premise remains much the same. Once again, the approach offers very simple scaling to 3-D shapes by using the $(r, \theta, z)$ space translation.

# 4 Extension: Moment of Inertia Tensor

Although the work done in this investigation has primarily been for symmetric, 3-D bodies which rely on simple, perpendicular axes of rotation or 2-D laminas located in the $X - Y$ plane, the motion of rotating $3 - D$ bodies can be significantly more complex. This primarily stems from the idea that the angular momentum vector of a 3-D rotating body is often not parallel to the angular velocity vector, even for bodies that exhibit traditional modes of symmetry. This type of multi-dimensional rotation also inhibits the use of the elementary method of determining rotational kinetic energy as described in Eq. (11). To partially alleviate these issues, a moment of inertia tensor (or matrix for traditional purposes) can be utilized which provides a proportionality factor between angular momentum and rotational velocity. The $3 \times 3$ tensor for a rigid object of $N$ point masses $m_k$ can be defined as:

$$\boldsymbol{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

The individual components of the matrix are defined by the following function:

$$I_{ij} = \sum_{k=1}^{N} m_k(\|\boldsymbol{r}_k\|^2\,\delta_{ij} - x_i^{(k)}x_j^{(k)})$$

where:

$i, j$ are equal to $1, 2, 3$ which map to $x, y, z$ for bodies rotating in traditional axes
$\boldsymbol{r}_k = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)})$, which is essentially the position vector to the point mass $m_k$ with respect to the point about which which the tensor is calculated
$\delta_{ij}$ is known as the Kronecker delta, an elementary function which is 1 if the variables are equal and 0 otherwise.

For this investigation, we will assume that the point of reference for the tensor is the origin, so $\|\boldsymbol{r}_k\|^2 = x_k^2 + y_k^2 + z_k^2$. Based on these definitions, the tensor is symmetric (as $I_{ij} = I_{ji}$). For the diagonal elements of the matrix, the kronecker delta function will evaluate to 1, resulting in the assertions:

$$I_{xx} = \sum_{k=1}^{N} m_k((x_k^2 + y_k^2 + z_k^2) - x_k^2) = \sum_{k=1}^{N} m_k(y_k^2 + z_k^2)$$

$$I_{yy} = \sum_{k=1}^{N} m_k((x_k^2 + y_k^2 + z_k^2) - y_k^2) = \sum_{k=1}^{N} m_k(x_k^2 + z_k^2)$$

$$I_{zz} = \sum_{k=1}^{N} m_k((x_k^2 + y_k^2 + z_k^2) - z_k^2) = \sum_{k=1}^{N} m_k(x_k^2 + y_k^2)$$

This can also be thought of as rotation about the $x, y, z$ axes respectively as the distance is essentially the perpendicular distance from these axes. For the off-diagonal elements, the kronecker delta function will instead evaluate to 0, resulting in the following relationships:

$$I_{xy} = I_{yx} = -\sum_{k=1}^{N} m_k x_k y_k$$

$$I_{xz} = I_{zx} = -\sum_{k=1}^{N} m_k x_k z_k$$

$$I_{yz} = I_{zy} = -\sum_{k=1}^{N} m_k y_k z_k$$

The initial moment of inertia tensor can also be simplified using these defined quantities for easier understanding:

$$
\boldsymbol{I} = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}
$$

What is even more intriguing about this seemingly complicated setup is that it can be simplifed into a similar setup as the mass density approach used in the solid plate section with elemental units of volume. If the region defining the object is $V$ and the density function is $\rho(x, y, z)$, the moment of inertia tensor can be defined as:

$$
\boldsymbol{I} = \iiint_V \rho(x, y, z)(\|\boldsymbol{r}_k\|^2 \boldsymbol{E}_3 - \boldsymbol{r} \otimes \boldsymbol{r}) dxdydz \tag{20}
$$

In this definition, $\boldsymbol{E}_3$ is the $3 \times 3$ identity vector and $\otimes$ represents the outer product of 2 vectors, defined as $(\boldsymbol{u} \otimes \boldsymbol{v})_{ij} = u_i v_j$. This results in an identical layout to the individualized definitions presented earlier, but is more streamlined and easier to read. The integral also closely mimics the structure presented for the plate by replacing $dV$ with $dxdydz$ and multiplying with the density function to produce the elemental mass $dm$.

Although this finding has endlessly complex implementations, the basic premise can be illustrated through the example of a cube with side length $a$, mass $M$ and constant density $\rho$ whose vertex is located at the origin and sides extend in the positive $x, y, z$ directions. We can immediately determine that:

$$
\begin{aligned}
\rho(x, y, z) &= \rho \\
V &= a^3 \\
\rho &= \frac{M}{a^3}
\end{aligned}
$$

Our linearized integral results in:

$$
\boldsymbol{I} = \iiint_V \rho \left( (x_k^2 + y_k^2 + z_k^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} \otimes \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix} \right) dxdydz
$$

$$
\boldsymbol{I} = \iiint_V \rho \left( \begin{bmatrix} x_k^2 + y_k^2 + z_k^2 & 0 & 0 \\ 0 & x_k^2 + y_k^2 + z_k^2 & 0 \\ 0 & 0 & x_k^2 + y_k^2 + z_k^2 \end{bmatrix} - \begin{bmatrix} x_k^2 & x_k y_k & x_k z_k \\ x_k y_k & y_k^2 & y_k z_k \\ x_k z_k & y_k z_k & z_k^2 \end{bmatrix} \right) dxdydz
$$

$$
\boldsymbol{I} = \iiint_V \rho \left( \begin{bmatrix} y_k^2 + z_k^2 & -x_k y_k & -x_k z_k \\ -x_k y_k & x_k^2 + z_k^2 & -y_k z_k \\ -x_k z_k & -y_k z_k & x_k^2 + y_k^2 \end{bmatrix} \right) dxdydz
$$

The limits for our integrals simply span the length of the sides, or 0 to $a$, transforming the problem into a multi-line integral in matrix form and allowing us to remove the $k$ subscripts defining each individual mass element:

$$\boldsymbol{I} = \rho \int_0^a \int_0^a \int_0^a \left( \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -xy & x^2 + z^2 & -yz \\ -xz & -yz & x^2 + y^2 \end{bmatrix} \right) dxdydz$$

$$\boldsymbol{I} = \rho \int_0^a \int_0^a \int_0^a \left( \begin{bmatrix} (y^2 + z^2)\,[x]_0^a & \left[\frac{-x^2 y}{2}\right]_0^a & \left[\frac{-x^2 z}{2}\right]_0^a \\ \left[\frac{-x^2 y}{2}\right]_0^a & \left[\frac{x^3}{3} + xz^2\right]_0^a & [-xyz]_0^a \\ \left[\frac{-x^2 z}{2}\right]_0^a & [-xyz]_0^a & \left[\frac{x^3}{3} + xy^2\right]_0^a \end{bmatrix} \right) dxdydz$$

$$\boldsymbol{I} = \rho \int_0^a \int_0^a \left( \begin{bmatrix} ay^2 + az^2 & \frac{-a^2 y}{2} & \frac{-a^2 z}{2} \\ \frac{-a^2 y}{2} & \frac{a^3}{3} + az^2 & -ayz \\ \frac{-a^2 z}{2} & -ayz & \frac{a^3}{3} + ay^2 \end{bmatrix} \right) dydz$$

$$\boldsymbol{I} = \rho \int_0^a \left( \begin{bmatrix} \left[\frac{ay^3}{3} + ayz^2\right]_0^a & \left[\frac{-a^2 y^2}{4}\right]_0^a & \left[\frac{-a^2 zy}{2}\right]_0^a \\ \left[\frac{-a^2 y^2}{4}\right]_0^a & \left[\frac{a^3 y}{3} + ayz^2\right]_0^a & \left[\frac{-ay^2 z}{2}\right]_0^a \\ \left[\frac{-a^2 zy}{2}\right]_0^a & \left[\frac{-ay^2 z}{2}\right]_0^a & \left[\frac{a^3 y}{3} + \frac{ay^3}{3}\right]_0^a \end{bmatrix} \right) dz$$

$$\boldsymbol{I} = \rho \int_0^a \left( \begin{bmatrix} \frac{a^4}{3} + a^2 z^2 & \frac{-a^4}{4} & \frac{-a^3 z}{2} \\ \frac{-a^4}{4} & \frac{a^4}{3} + a^2 z^2 & \frac{-a^3 z}{2} \\ \frac{-a^3 z}{2} & \frac{-a^3 z}{2} & \frac{2a^4}{3} \end{bmatrix} \right) dz$$

$$\boldsymbol{I} = \rho \begin{bmatrix} \left[\frac{a^4 z}{3} + \frac{a^2 z^3}{3}\right]_0^a & \left[\frac{-a^4 z}{4}\right]_0^a & \left[\frac{-a^3 z^2}{4}\right]_0^a \\ \left[\frac{-a^4 z}{4}\right]_0^a & \left[\frac{a^4 z}{3} + \frac{a^2 z^3}{3}\right]_0^a & \left[\frac{-a^3 z^2}{4}\right]_0^a \\ \left[\frac{-a^3 z^2}{4}\right]_0^a & \left[\frac{-a^3 z^2}{4}\right]_0^a & \left[\frac{2a^4 z}{3}\right]_0^a \end{bmatrix}$$

Simplyifing, substituting for $\rho$, and reducing results in a final moment of inertia tensor of:

$$\boldsymbol{I}_{\text{uniform cube}} = \begin{bmatrix} \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{bmatrix}$$

We notice that the diagonal moment of inertia values ($I_{xx}, I_{yy}, I_{zz}$) are identical, as well as the off-diagonal values. This is somewhat intuitive because the cube has a uniform density and is situated symmetrically with respect to the individual axes. It is important to keep in mind that the moment of inertia is a measure of the difficulty of rotating a body about a given axis (much like inertia is the tendency of a body to resist translational motion), so this matrix allows us to easily determine the angular momentum or energy of a body rotating about any axis passing through the origin (as this was the assumed reference point). To calculate the moment of inertia about any axis passing through the reference point, we can simply use the relation described in Eq. (21), where $\hat{\omega}$ represents the unit angular velocity vector for a given axis of rotation.

$$I = \hat{\omega}^T \boldsymbol{I} \hat{\omega} \tag{21}$$

The proof for this result is left out, as it is outside the scope of the investigation, but the equation can be used to quickly determine the moment of inertia for the cube about its diagonal, something which was difficult to do using traditional methods. The unit vector for this axis is simply $\begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$, so:

$$I = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 & -\frac{1}{4}Ma^2 \\ -\frac{1}{4}Ma^2 & -\frac{1}{4}Ma^2 & \frac{2}{3}Ma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{6\sqrt{3}}Ma^2 & \frac{1}{6\sqrt{3}}Ma^2 & \frac{1}{6\sqrt{3}}Ma^2 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} = \frac{1}{6}Ma^2$$

In this simple manner, the moment of inertia can easily be calculated for any potential axis, and the entire approach can easily be adopted for other bodies, not only cubes. This matrix can also be shifted in order to make the center of mass and reference point coincide, which eliminates the off-diagonal values entirely and results in a matrix with only 3 integral values.

# 5 Conclusion