Graph Realization Problem

Math 126

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What is a graph realization problem?

- It’s a decision problem in graph theory
- It can be solved in time complexity of O(n^2) with either the Havel-Hakimi Theorem (done recursively) or the Erdős–Gallai theorem(tests whether or not n inequalities are valid)
- It can also be represented using an adjacency matrix with 0s and 1s that correspond to the degree sequence given

Review

- Degree sequence: A sequence of degrees in non-increasing order
  - Sum of the numbers in degree sequence must be even
  - Number of odd degree vertices must be even
  - Sequence must have a repeated number of length > 1
  - Total length of sequence must be of minimum length of the first number in the sequence plus 1
- Theorem 1.1

$$\sum_{v \in V} \deg(v) = 2|E|.$$

Havel-Hakimi Theorem

- The nonincreasing sequence $(a_1,a_2,\ldots,a_n)$ can be graphed if and only if the sequence $(a_2-1,a_3-1,\ldots,a_{a1}+1-1,a_{a1}+2,a_{a1}+3,\ldots,a_n)$ is also possible to graph.
- **IN OTHER WORDS:-**
- Sort the sequence is non-increasing order if applicable.
- Pick the first term, and we call it **a.**
- Remove **a.**
- Subtract **1** from the “**a**” following terms.
- Repeat the steps until we get all **0**s at the sequence.
- 0 at the end means graph is possible, and negative number means not.

Example

Check whether the graph is possible or not from the given sequence:-

(5, 5, 4, 4, 3, 3)

Answer:-

(~~5~~, 5, 4, 4, 3, 3)

(4, 3, 3, 2, 2)

(~~4~~, 3, 3, 2, 2)

(2, 2, 1, 1)

(~~2~~, 2, 1, 1)

(1, 0, 1)

(1, 1, 0)

(0, 0) Graph is **POSSIBLE!**

Another Example...

Given Degree Sequence:- (6, 6, 6, 3, 2, 2, 1, 1, 1)

Step 1:- Remove 6 and subtract 1 from next 6 terms

(5, 5, 2, 1, 1, 0, 1, 1)

Step 2:- Terms are NOT in order! So, we will arrange them in non-increasing order.

(5, 5, 2, 1, 1, 1, 1, 0)

Step 3:- Remove 5 and subtract 1 from next 5 terms

(4, 1, 0, 0, 0, 1, 0)

Step 4:- Terms are NOT in order! So, we will arrange them in non-increasing order.

(4, 1, 1, 0, 0, 0, 0)

Step 5:- Remove 4 and subtract 1 from next 4 terms

(0, 0, -1, -1, 0, 0)

We have the negative numbers in our sequence. This means we can say that graph for this sequence will NOT be possible!

One Last Example:-

Degree Sequence:-

(4, 4, 1, 1, 1, 1)

Answer:-

(~~4~~, 4, 1, 1, 1, 1)

(3, 0, 0, 0, 1)

Arrange them:- (3, 1, 0, 0, 0)

(0, -1, -1, 0)

GRAPH IS NOT POSSIBLE!

Erdős–Gallai theorem

- The **Erdős-Gallai theorem** is one of two ways in solving a graph realization problem.
- It gives necessary and sufficient condition for a finite sequence of natural numbers to be the *degree sequence*[1] of a *simple graph*[2]. (A sequence under this condition is known to be “graphic”).
- **Erdős-Gallai theorem:**
  A sequence of non-negative integers d1 ≥ … ≥ dn can be represented as the degree sequence of a finite simple graph on n vertices if and only if d1 + … + dn is even and

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

  holds for every k in 1 ≤ k ≤ n.

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[1] The **degree sequence** of an undirected graph is the non-increasing sequence of its vertex degrees
[2] A **graph** (sometimes called *undirected graph* for distinguishing from a directed graph, or *simple graph* for distinguishing from a multigraph)

Example

Determine which of the following are graphic sequences:

a) 5, 4, 3, 2, 1, 0
   - Checking the first condition

$$5 + 4 + 3 + 2 + 1 + 0 = 15 \text{ (odd)}$$

Therefore, is not a graphic sequence.

Another example

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min(d_i, k)$$

b) 4, 4, 1, 1, 1, 1

- Checking first condition

  4 + 4 + 1 + 1 + 1 + 1 = 12 (even)

(4, 4, 1, 1, 1, 1) = (d1, d2, d3, d4, d5, d6)

For k = 1 gives

$4 \leq 1(1-1) + (1+1+1+1+1)$

➔ $4 \leq 5$ holds for k = 1

For k = 2 gives

$4 + 4 \leq 2(2-1) + (1+1+1+1)$

➔ $8 \leq 6$ fails to hold for k = 2

Hence, is not a graphic sequence.

Final example

c) 5, 5, 4, 4, 3, 3

- $5 + 5 + 4 + 4 + 3 + 3 = 24$ (even)

$(5, 5, 4, 4, 3, 3) = (d1, d2, d3, d4, d5, d6)$

For $k = 1$ gives

$5 \leq 1(1-1) + (1+1+1+1+1)$

$\rightarrow$ $5 \leq 5$ holds for $k = 1$

For $k = 2$ gives

$5 + 5 \leq 2(2-1) + (2+2+2+2)$

$\rightarrow$ $10 \leq 10$ holds for $k = 2$

For $k = 3$

$5 + 5 + 4 \leq 3(3-1) + (3+3+3)$

$\rightarrow$ $14 \leq 15$ holds for $k = 3$

For $k = 4$

$5 + 5 + 4 + 4 \leq 4(4-1) + (3+3)$

$\rightarrow$ $18 \leq 18$ holds for $k = 4$

For $k = 5$

$5 + 5 + 4 + 4 + 3 \leq 5(5-1) + 3$

$\rightarrow$ $21 \leq 23$ holds for $k = 5$

For $k = 6$

$5 + 5 + 4 + 4 + 3 + 3 \leq 6(6-1)$

$\rightarrow$ $24 \leq 30$ holds for $k = 5$

Is a graphic sequence!

References

https://en.wikipedia.org/wiki/Graph_realization_problem

Erdős–Gallai theorem - Wikipedia

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