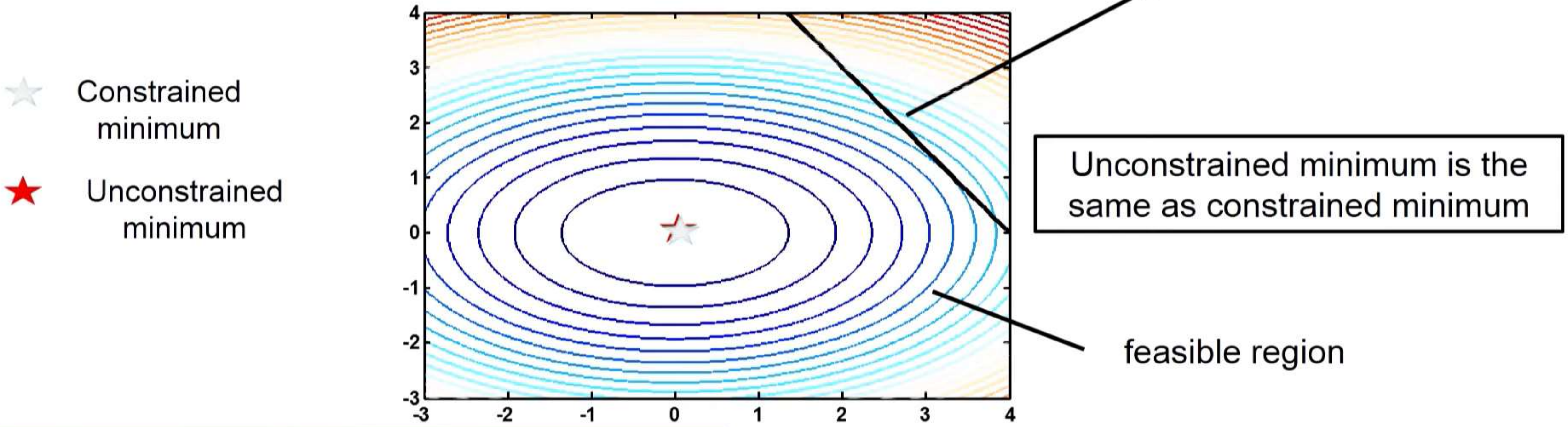
Fundamentals of optimization

Multivariate optimization with constraints

$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

$$st$$

$$3x_1 + 2x_2 \le 12$$



General formulation

Multivariate optimization

$$\min_{\bar{x}} \quad f(\bar{x})$$

$$st$$

$$h_i(\bar{x}) = \bar{0},\ i = 1, \ldots m$$

$$g_j(\bar{x}) \leq \bar{0},\ j = 1, 2 \ldots l$$

Necessary condition for $\overline{x^*}$ to be the minimizer

KKT conditions has to be satisfied

Sufficient condition

$\nabla^2 L(\overline{x^*})$ has to be positive definite

Summary – KKT conditions

Multivariate optimization

When both equality and inequality constraints are present, at the optimum we have

KKT (Karush-Kuhn-Tucker) conditions

$$\nabla f(\overline{x^*}) + \sum_{i=1}^{l} [\nabla h_i(\overline{x^*})]\ \lambda_i^* + \sum_{j=1}^{m} [\nabla g_j(\overline{x^*})]\mu_j^* = 0$$

Gradient of the "Lagrangian function" at x^*

$$L(\overline{x^*}, \lambda^*, \mu^*) = f(\overline{x^*}) + \sum_{i=1}^{l} \lambda_i h(\overline{x^*}) + \sum_{j=1}^{m} \mu_j g_j(\overline{x^*})$$

$$h_i(x^*) = 0, i = 1 \dots l$$

$$\lambda_i \epsilon R, i = 1 \dots l$$

Ensures that the optimum satisfies equality constraints

$$g_j(\overline{x^*}) \leq 0, j = 1 \dots m$$

Ensures that the optimum is in the feasible region

$$\mu_j^*(g_j(\overline{x^*})=0$$

Complementary slackness

$$\mu_j^* \geq 0, j = 1 \dots m$$

- No possibility of improvement near the active constraints

Summary – KKT conditions

Multivariate optimization

- In general it is difficult to use the KKT conditions to solve for the optimum of an inequality constrained problem (than for a problem with equality constraints only) because we do not know *a priori* which constraints are active at the optimum.
- Makes this a combinatorial problem
- KKT conditions are used to verify that a point we have reached is a candidate optimal solution.
- Given a point, it is easy to check which constraints are binding.

Fundamentals of optimization

Multivariate optimization-quadratic programming

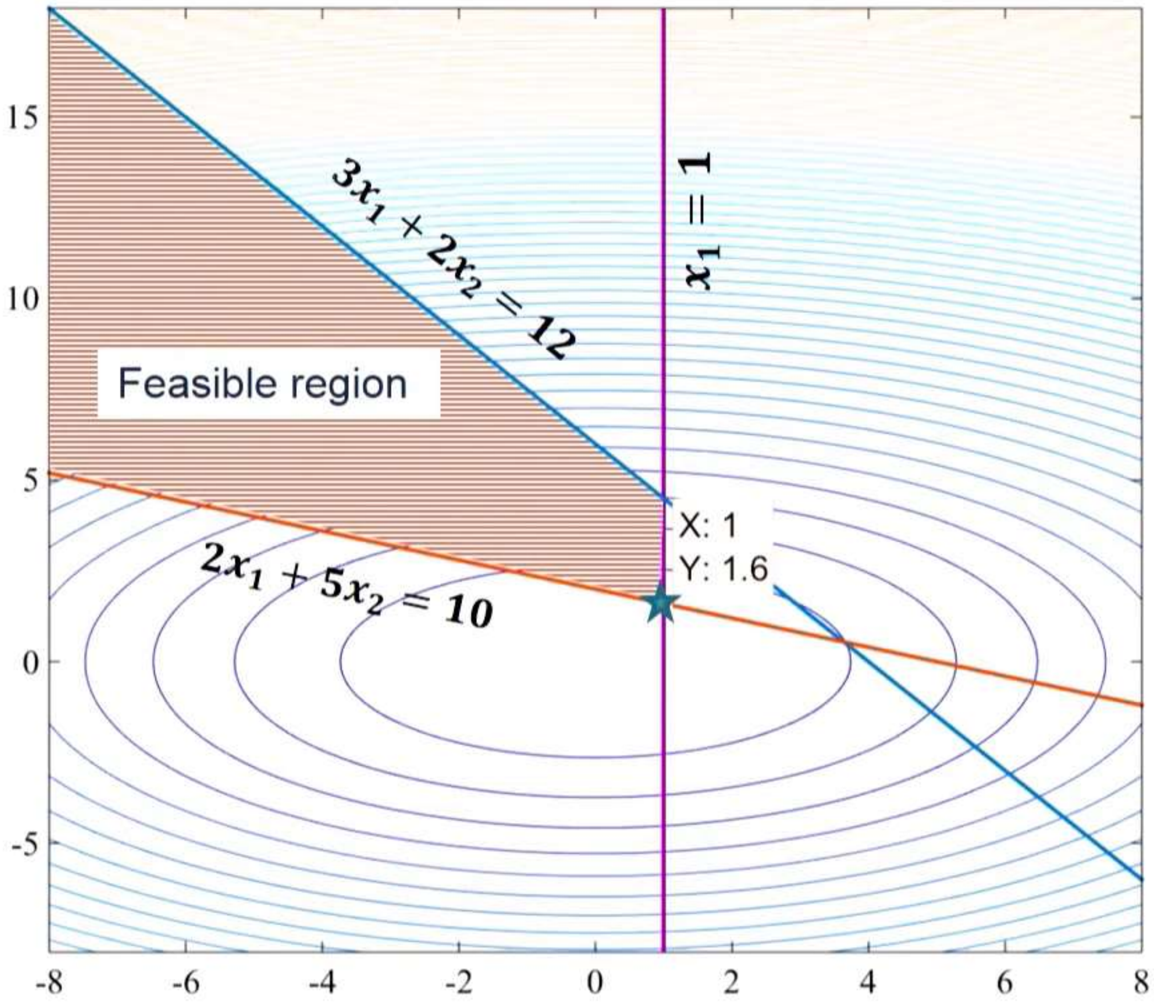
$$\min_{x_1, x_2} \; 2x_1^2 + 4x_2^2$$

$$st$$

$$3x_1 + 2x_2 \le 12$$

$$2x_1 + 5x_2 \ge 10$$

$$x_1 \le 1$$



Fundamentals of optimization

Multivariate optimization-quadratic programming

$$\min_{x_1, x_2} \quad 2x_1^2 + 4x_2^2$$
$$st$$
$$3x_1 + 2x_2 \le 12 \Rightarrow (a)$$
$$2x_1 + 5x_2 \ge 10 \Rightarrow (b)$$
$$x_1 \le 1 \Rightarrow (c)$$

- Lagrangian

$$L(x_1, x_2, \mu_1, \mu_2, \mu_3) = 2x_1^2 + 4x_2^2 + \mu_1(3x_1 + 2x_2 - 12) + \mu_2(10 - 2x_1 - 5x_2) + \mu_3(x_1 - 1)$$

- First order KKT conditions

$$4x_1 + 3\mu_1 - 2\mu_2 + \mu_3 = 0$$
$$8x_2 + 2\mu_1 - 5\mu_2 = 0$$

$$\mu_1(3x_1 + 2x_2 - 12) = 0$$
$$\mu_2(10 - 2x_1 - 5x_2) = 0$$
$$\mu_3(x_3 - 1) = 0$$
$$\mu_i \ge 0$$

Fundamentals of optimization

Multivariate optimization-quadratic programming

Sl.no	Active (A) /Inactive (I) constraints (a)	(b)	(c)	Solution (x, μ)	Possible optima (Y/N)	Remark
1	A	A	A	Infeasible	N	Equations do not have a valid solution.
2	A	A	I	$x = [3.6364 \quad 0.5455]$ $\mu = [-5.2 \quad -1.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied, $\mu_1 < 0$, $\mu_2 < 0$
3	A	I	A	$x = [1 \quad 4.5]$ $\mu = [-18 \quad 0 \quad 50]$	N	$\mu_1 < 0$
4	I	A	A	$x = [1 \quad 1.6]$ $\mu = [0 \quad 2.56 \quad 1.12]$	Y	All constraints and KKT conditions satisfied
5	A	I	I	$x = [3.27 \quad 1.09]$ $\mu = [-4.36 \quad 0 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
6	I	A	I	$x = [1.21 \quad 1.51]$ $\mu = [0 \quad 2.45 \quad 0]$	N	$x_1 \leq 1$ is not satisfied
7	I	I	A	$x = [1 \quad 0]$ $\mu = [0 \quad 0 \quad -4]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied
8	I	I	I	$x = [0 \quad 0]$ $\mu = [0 \quad 0 \quad 0]$	N	$2x_1 + 5x_2 \geq 10$ is not satisfied

Fundamentals of optimization

Solution for each case
Actual Optima