Hyperplanes

- Geometrically, hyperplane is a geometric entity whose dimension is one less than that of its ambient space.
- For instance, the hyperplanes for a 3D space are 2D planes and hyperplanes for a 2D space are 1D lines and so on.
- The hyperplane is usually described by an equation as follows

$$X^T n + b = 0$$

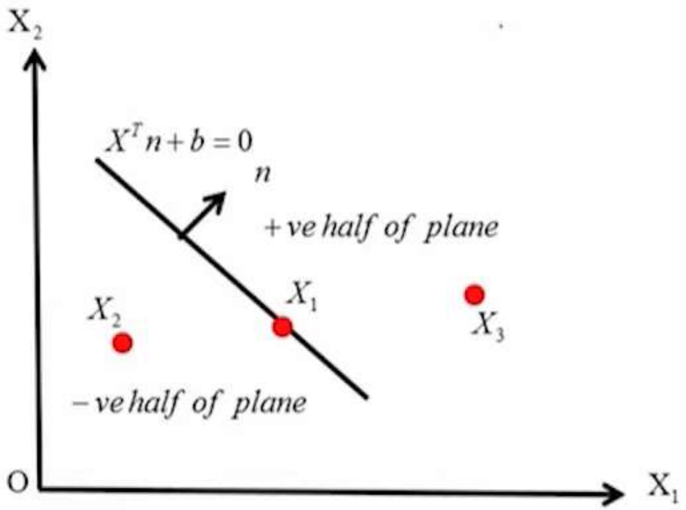
Halfspace

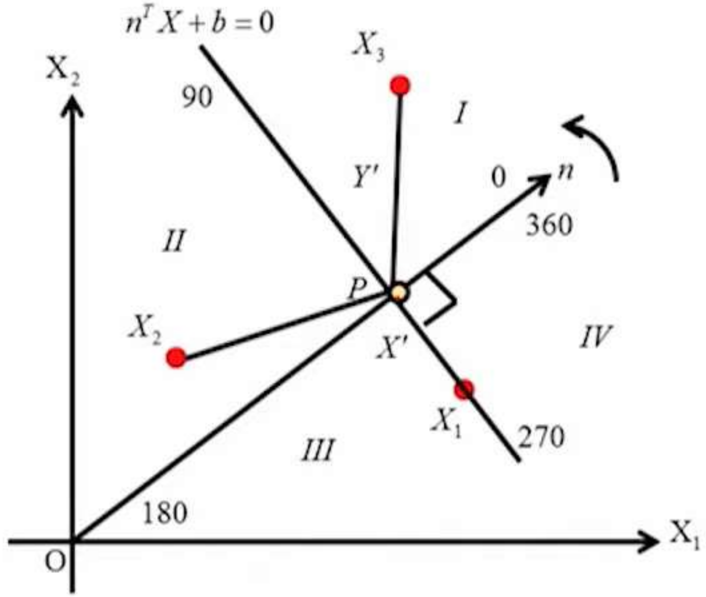
- We can observe that the equation can be evaluated for the two halfspaces
- It can be seen that

$$X^T n + b = 0 \ \forall \ X \in line$$

$$X^T n + b > 0 \ \forall \ X \in subspace\ in\ the\ n\ direction\ (X_3)$$

$$X^T n + b < 0 \ \forall \ X \in subspace\ in\ the\ -n\ direction\ (X_2)$$





Hyperplanes and halfspaces: Example

- Let us consider a 2D geometry with $n = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $b = 4$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X^T n + b = 0$$

$$[x_1 \; x_2] \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 4 = 0$$

$$x_1 + 3x_2 + 4 = 0$$

- The hyperplane is the equation of a line
- The halfspaces corresponding to this hyperplane are

$$x_1 + 3x_2 + 4 > 0 : \text{Positive halfspace}$$

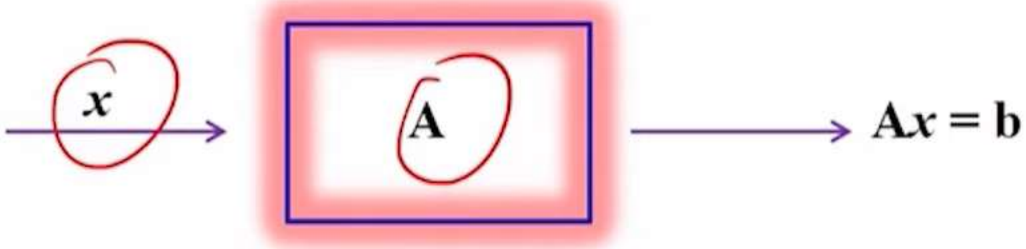
$$x_1 + 3x_2 + 4 < 0 : \text{Negative halfspace}$$

Eigenvalues and eigenvectors

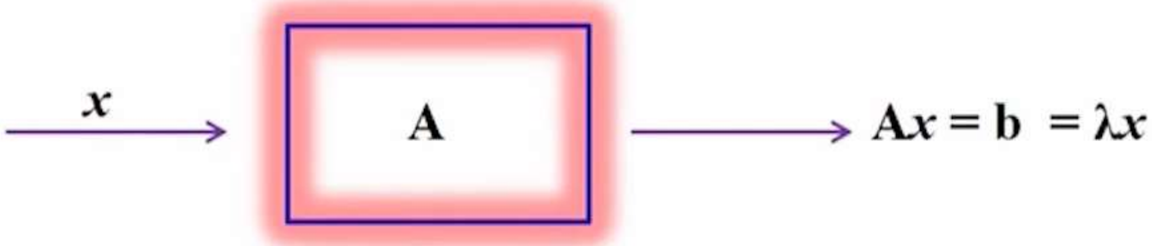
- We have previously seen linear equations of the form
  $Ax = b$
- What is the geometrical interpretation of this equation?
- We can make an interpretation as follows
  - When vector x is operated on by A, we obtain a new vector b with a different orientation

Eigenvalues and eigenvectors

- Operator representation



- The newly obtained b vector represents a new orientation. So we ask the following question
- Are there directions for a matrix A such that when the matrix operates on these directions they maintain their orientation save for multiplication by a scalar (positive or negative)?
- That is

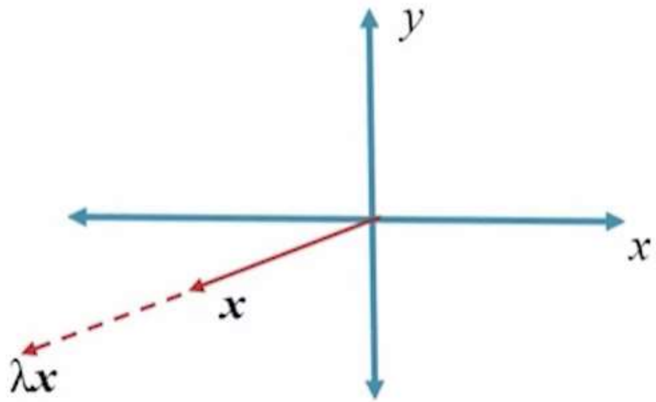


Eigenvalues and eigenvectors

- The mathematical formulation of our question is

$$Ax = \lambda x$$

- The constant λ $(positive)$ represents the amount of stretch or shrinkage the attributes x go through in the x direction
- The solutions (x) are known as eigenvectors and their corresponding λ are eigenvalues



Eigenvalues and eigenvectors

- We can find the eigenvalues as follows

$$Ax = \lambda x \quad A(n \times n); x(n \times 1)$$

$$Ax - \lambda I x = 0$$

$$(A - \lambda I)x = 0$$

- Thus the eigenvalues of the equation can be identified using

$$|A - \lambda I| = 0$$

- Substituting the eigenvalues in the original equation will help us find solutions for the eigenvector x

Eigenvalues and eigenvectors: Examples

- Consider the following example with the given A matrix

$$A = \begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \end{bmatrix}$$

$$|A - \lambda I| = \left|\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = \left|\begin{bmatrix} 8-\lambda & 7 \\ 2 & 3-\lambda \end{bmatrix}\right| = 0$$

$$(8-\lambda)(3-\lambda) - 14 = 0$$

$$\lambda^2 - 11\lambda + 10 = 0$$

$$\lambda = (10,1)$$

R Code

```
A = matrix(c(8,7,2,3), 2, 2, byrow=TRUE)
ev =eigen(A)
values = ev$values
```

Console output

```
> values
[1] 10  1
```

- Thus we identify two eigenvalues and now we proceed to find the corresponding eigenvectors

Eigenvalues and eigenvectors: Examples

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\lambda = 1$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 7x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = 0$$

- Thus the eigenvector (unit) corresponding to $\lambda = 1$is

$$X = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Eigenvalues and eigenvectors: Examples

- $\lambda = 10$

$$\begin{bmatrix} 8 & 7 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

$$\begin{bmatrix} 8x_1 + 7x_2 \\ 2x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 \\ 10x_2 \end{bmatrix}$$

$$7x_2 = 2x_1$$

- Thus the eigenvector (unit) corresponding to $\lambda = 10$

$$X = \begin{bmatrix} \frac{7}{\sqrt{53}} \\ \frac{2}{\sqrt{53}} \end{bmatrix}$$

R Code

```
A = matrix(c(8,7,2,3), 2, 2, byrow=TRUE)
ev =eigen(A)
vectors <- ev$vectors
> vectors
          [,1]       [,2]
[1,] 0.9615239 -0.7071068
[2,] 0.2747211  0.7071068
```

Summary

$Ax = b$

- Geometric interpretation

$Ax = \lambda x$

- Eigenvalue-eigenvector equation

λ

- N eigenvalues from $|A - \lambda I| = 0$

x

- Eigenvectors, generally expressed in unit vector form