Review

- So far we have discussed linear algebra and matrix theory from a data science perspective
- We will provide some geometric interpretations now
- This section covers the following
  - Vectors
  - Notion of distance
  - Projections
  - Hyperplanes
  - Halfspaces
  - Eigenvalues and eigenvectors

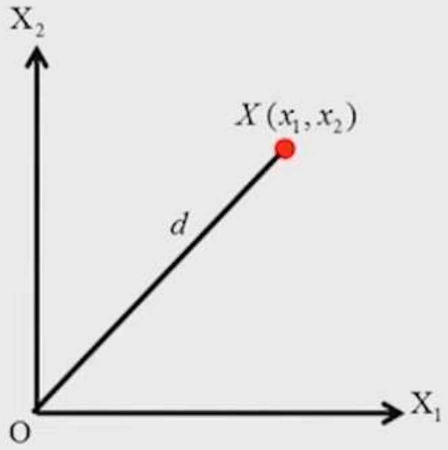
Vectors and lengths

- Consider

$$\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- $\boldsymbol{X}$ is a data point in a 2 dimensional plane with x_1 and x_2 as the distances along the $\boldsymbol{X_1}$ and $\boldsymbol{X_2}$ axes respectively.
- **X** can also be considered as a vector between the origin and the data point
- The length (magnitude) of this vector is

$$d = \sqrt{x_1{}^2 + x_2{}^2}$$



Vectors and lengths: Example

- Consider the point $A = (3,4)$ in a two dimensional plane

$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$d = \sqrt{3^2 + 4^2} = 5\,units$$

- Important: Geometric concepts are easier to visualize in $2D$ or $3D$
- Difficult to do so in the higher dimensions
- However, the fundamental mathematics remain the same irrespective of the dimension of the vector

Vectors and distances

- Consider another example with two points X^1 and X^2

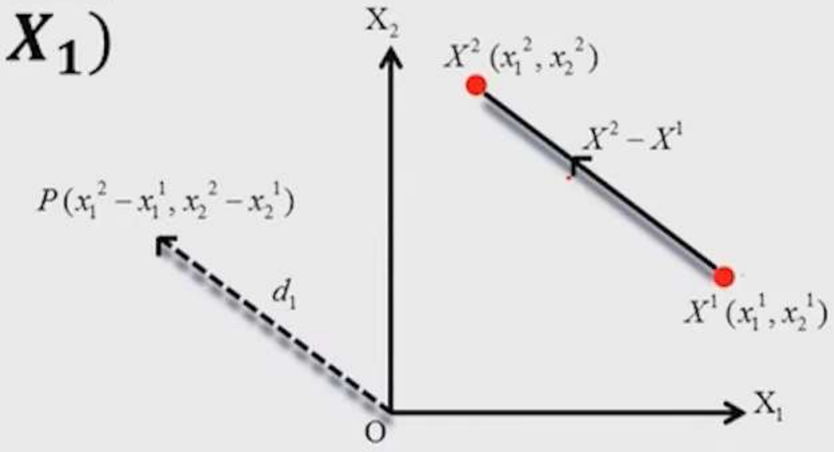
$$\boldsymbol{X^1} = \begin{bmatrix} x_1{}^1 \\ x_2{}^1 \end{bmatrix} \boldsymbol{X^2} = \begin{bmatrix} x_1{}^2 \\ x_2{}^2 \end{bmatrix}$$

- The distance between these two can be calculated

$$l = \left|\boldsymbol{X^2} - \boldsymbol{X^1}\right|_2$$

$$l = \sqrt{(x_1{}^2 \ - x_1{}^1)^2 + (x_2{}^2 \ - x_2{}^1)^2}$$

$$l = \sqrt{(\boldsymbol{X}_2 - \boldsymbol{X_1})^T(\boldsymbol{X}_2 - \boldsymbol{X_1})}$$



Vectors and distances: Example

- What is the distance between points ***A*** and ***B*,** where A *is* (2,7) and B *is* (5,3)
- Using the concept of distance introduced before

$$A = \begin{bmatrix} 2 \\ 7 \end{bmatrix}, B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$l = \sqrt{(5-2)^2 + (3-7)^2}$$

$$l = 5\ units$$

Unit vector

- A unit vector is a vector with magnitude 1 (distance from origin)
- Unit vectors are used to define directions in a coordinate system
- Any vector can be written as a product of a unit vector and a scalar magnitude

$$A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Magnitude of A: $|A| = \sqrt{3^2 + 4^2} = 5$

$$\hat{a} = \frac{A}{|A|} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

Orthogonal vectors

- Two vectors are orthogonal to each other when their dot product is 0
- Dot product (scalar product) of two n dimensional vectors ***A*** and ***B***

$$A.B = \sum_{i=1}^{n} a_i b_i$$

- Thus the vectors A and B are orthogonal to each other if and only if

$$A.B = \sum_{i=1}^{n} a_i b_i = A^T B = 0$$

Orthogonal vectors: Example

- Consider the vectors v_1 and v_2 in 3D space. Identify if they are orthogonal to each other

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix}$$

R Code

```
v1=c(1,-2,4)
v2=c(2,5,2)
N=t(v1)%*%v2
```

Console Output

```
> N
     [,1]
[1,]    0
```

- Taking the dot product of the vectors

$$v_1 . v_2 = V_1{}^T V_2 \ = [1 - 2\ 4] \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} = 0$$

- Hence, the vectors are orthogonal

Orthonormal vectors

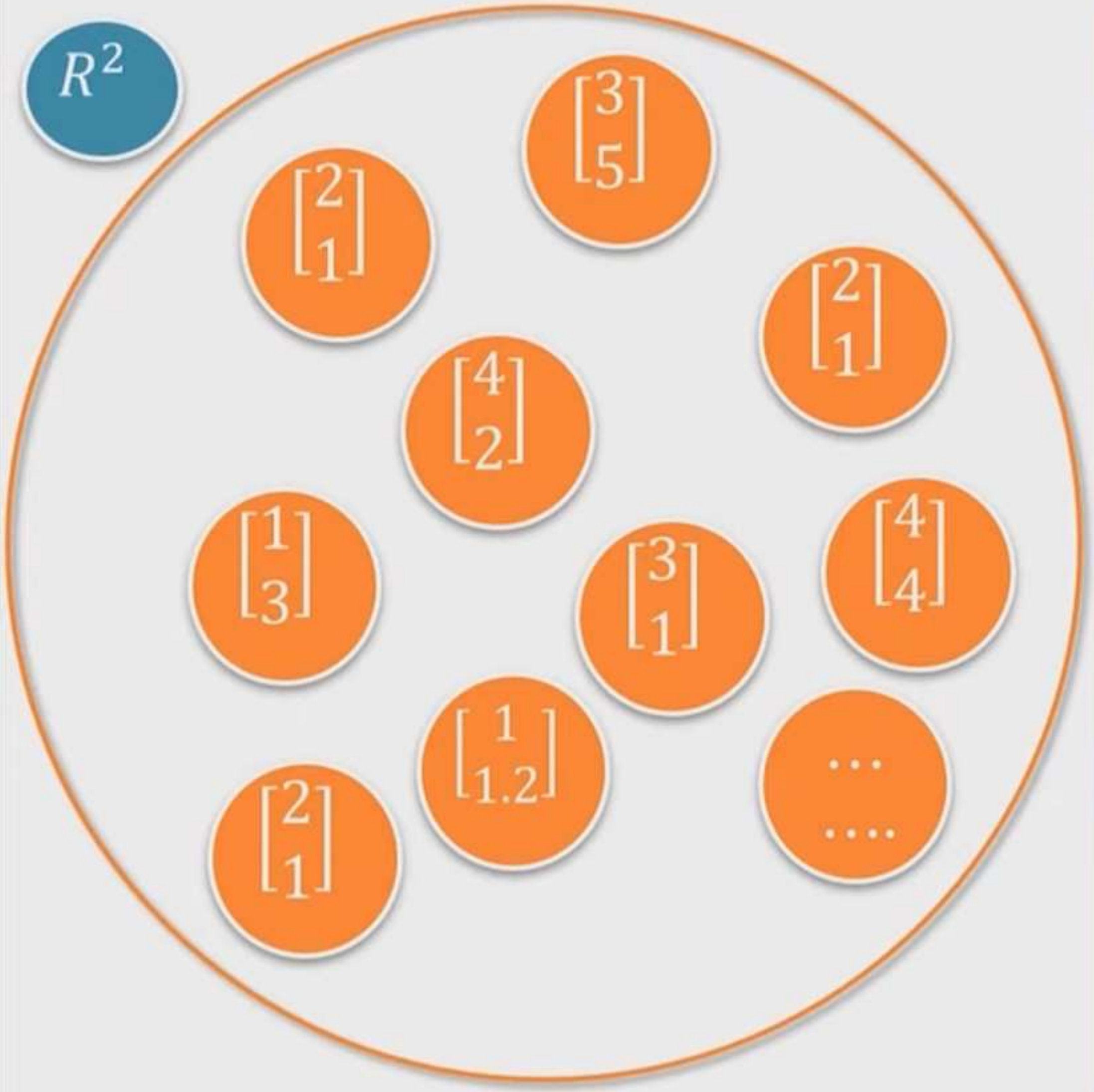
- Orthonormal vectors are orthogonal vectors with unit magnitude
- Example

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix} / \sqrt{1^2 + (-2)^2 + 4^2}$$

$$v_2 = \begin{bmatrix} 2 \\ 5 \\ 2 \end{bmatrix} / \sqrt{2^2 + 5^2 + 2^2}$$

- Note that we have taken the vectors from the previous example and converted them into unit vectors by dividing them with their magnitudes.
- All orthonormal vectors are orthogonal

Basis vectors



Let us consider two vectors $v_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $v_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$$\begin{bmatrix}2\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\end{bmatrix} = 2v_1 + 1v_2$$

$$\begin{bmatrix}4\\4\end{bmatrix} = 4\begin{bmatrix}1\\0\end{bmatrix} + 4\begin{bmatrix}0\\1\end{bmatrix} = 4v_1 + 4v_2$$

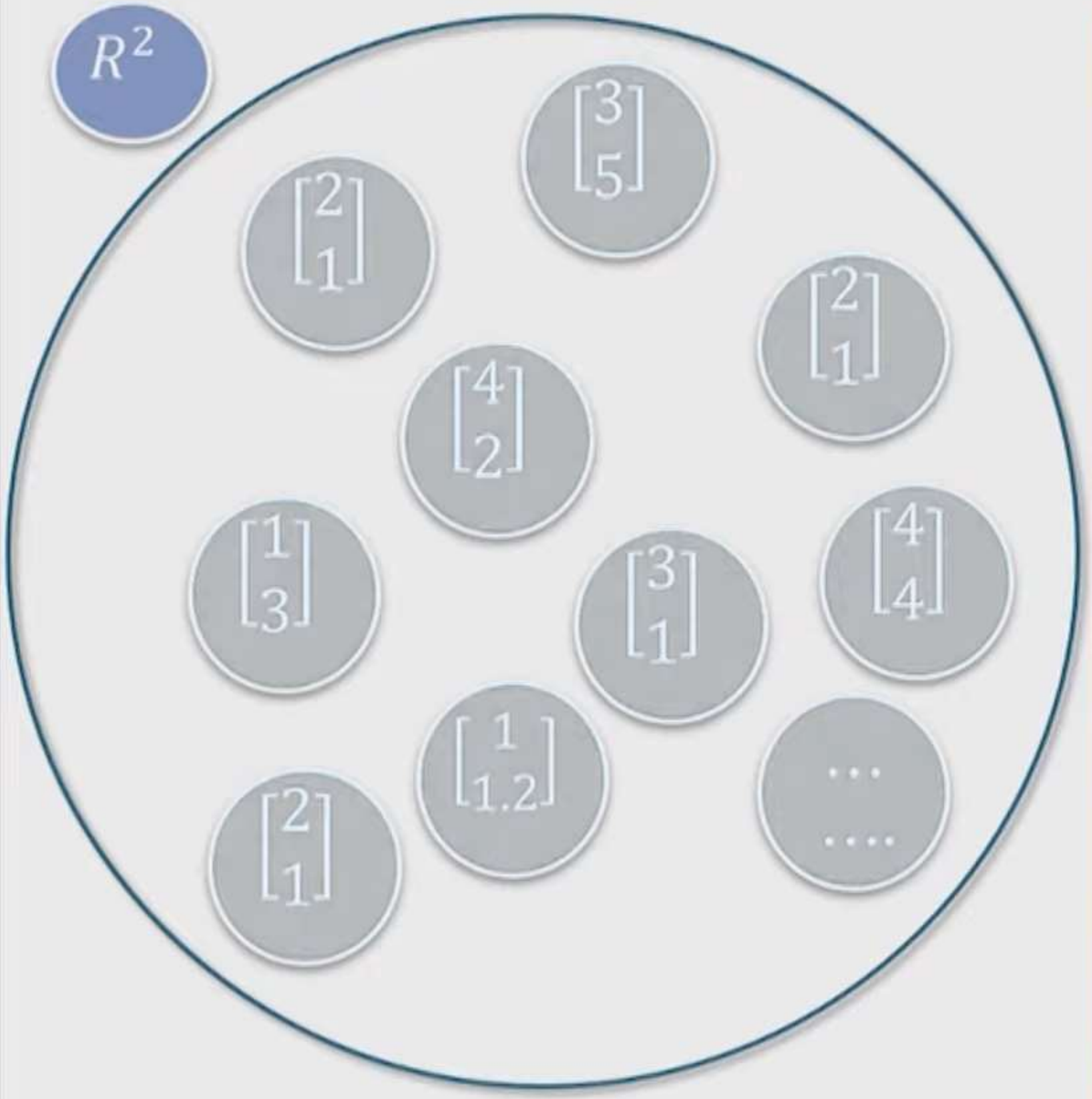
$$\begin{bmatrix}1\\3\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix} = 1v_1 + 3v_2$$

v_1 and v_2 are the basis vectors for R^2

Basis vectors

- Basis vectors are set of vectors that are independent and span the space
- Example:
  - Two vectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - Can span R^2 and are independent and hence form the basis for the R^2 space.

Basis vectors are not unique



Consider two vectors $v_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $v_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

$$\begin{bmatrix}2\\1\end{bmatrix} = 1.5\begin{bmatrix}1\\1\end{bmatrix} + (+0.5)\begin{bmatrix}1\\-1\end{bmatrix} = 1.5v_1 + (+0.5)v_2$$

$$\begin{bmatrix}4\\4\end{bmatrix} = 4\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\-1\end{bmatrix} = 4v_1 + 0v_2$$

$$\begin{bmatrix}1\\3\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} + (-1)\begin{bmatrix}1\\-1\end{bmatrix} = 2v_1 + (-1)v_2$$

Hence, this v_1 and v_2 are also basis vectors for R^2

Basis vectors

$$\begin{bmatrix}6\\5\\8\\11\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}9\\4\\7\\10\end{bmatrix}, \begin{bmatrix}-3\\1\\1\\1\end{bmatrix}, \begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix},$$
$$\begin{bmatrix}14\\7\\12\\17\end{bmatrix}, \begin{bmatrix}11\\8\\13\\18\end{bmatrix}, \begin{bmatrix}7\\0\\1\\2\end{bmatrix}, \begin{bmatrix}2\\-3\\-4\\-5\end{bmatrix}, \begin{bmatrix}7\\7\\11\\15\end{bmatrix}, \ldots\ldots$$

Consider two vectors $v_1 = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ and $v_2 = \begin{bmatrix}4\\1\\2\\3\end{bmatrix}$

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix} = 1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + 0\begin{bmatrix}4\\1\\2\\3\end{bmatrix} = 1v_1 + 0v_2$$

$$\begin{bmatrix}7\\7\\11\\15\end{bmatrix} = 3\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + 1\begin{bmatrix}4\\1\\2\\3\end{bmatrix} = 3v_1 + 1v_2$$

Finding basis vectors

$$\begin{bmatrix}6\\5\\8\\11\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}9\\4\\7\\10\end{bmatrix},\begin{bmatrix}-3\\1\\1\\1\end{bmatrix},\begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix},$$
$$\begin{bmatrix}14\\7\\12\\17\end{bmatrix},\begin{bmatrix}11\\8\\13\\18\end{bmatrix},\begin{bmatrix}7\\0\\1\\2\end{bmatrix},\begin{bmatrix}2\\-3\\-4\\-5\end{bmatrix},\begin{bmatrix}7\\7\\11\\15\end{bmatrix}$$

To find basis vectors of the given set of vectors, arrange the vectors as shown below

$$\begin{bmatrix}6 & 1 & 9 & -3 & 3 & 14 & 11 & 7 & 2 & 7\\ 5 & 2 & 4 & 1 & -1 & 7 & 8 & 0 & -3 & 7\\ 8 & 3 & 7 & 1 & -1 & 12 & 13 & 1 & -4 & 11\\ 11 & 4 & 10 & 1 & -1 & 17 & 18 & 2 & -5 & 15\end{bmatrix}$$

Finding basis vectors

$$\begin{bmatrix}6\\5\\8\\11\end{bmatrix},\begin{bmatrix}1\\2\\3\\4\end{bmatrix},\begin{bmatrix}9\\4\\7\\10\end{bmatrix},\begin{bmatrix}-3\\1\\1\\1\end{bmatrix},\begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix},$$
$$\begin{bmatrix}14\\7\\12\\17\end{bmatrix},\begin{bmatrix}11\\8\\13\\18\end{bmatrix},\begin{bmatrix}7\\0\\1\\2\end{bmatrix},\begin{bmatrix}2\\-3\\-4\\-5\end{bmatrix},\begin{bmatrix}7\\7\\11\\15\end{bmatrix}$$

Evaluate the rank of the matrix

$$\begin{bmatrix}6 & 1 & 9 & -3 & 3 & 14 & 11 & 7 & 2 & 7\\ 5 & 2 & 4 & 1 & -1 & 7 & 8 & 0 & -3 & 7\\ 8 & 3 & 7 & 1 & -1 & 12 & 13 & 1 & -4 & 11\\ 11 & 4 & 10 & 1 & -1 & 17 & 18 & 2 & -5 & 15\end{bmatrix}$$

Rank of the matrix is 2

Any two independent columns can be picked from the above matrix as basis vectors