FRM Part I Exam

By AnalystPrep

Study Notes - Valuation and Risk Models

Last Updated: Jul 20, 2022

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<u>**Reading 45: Measures of Financial Risk**</u>

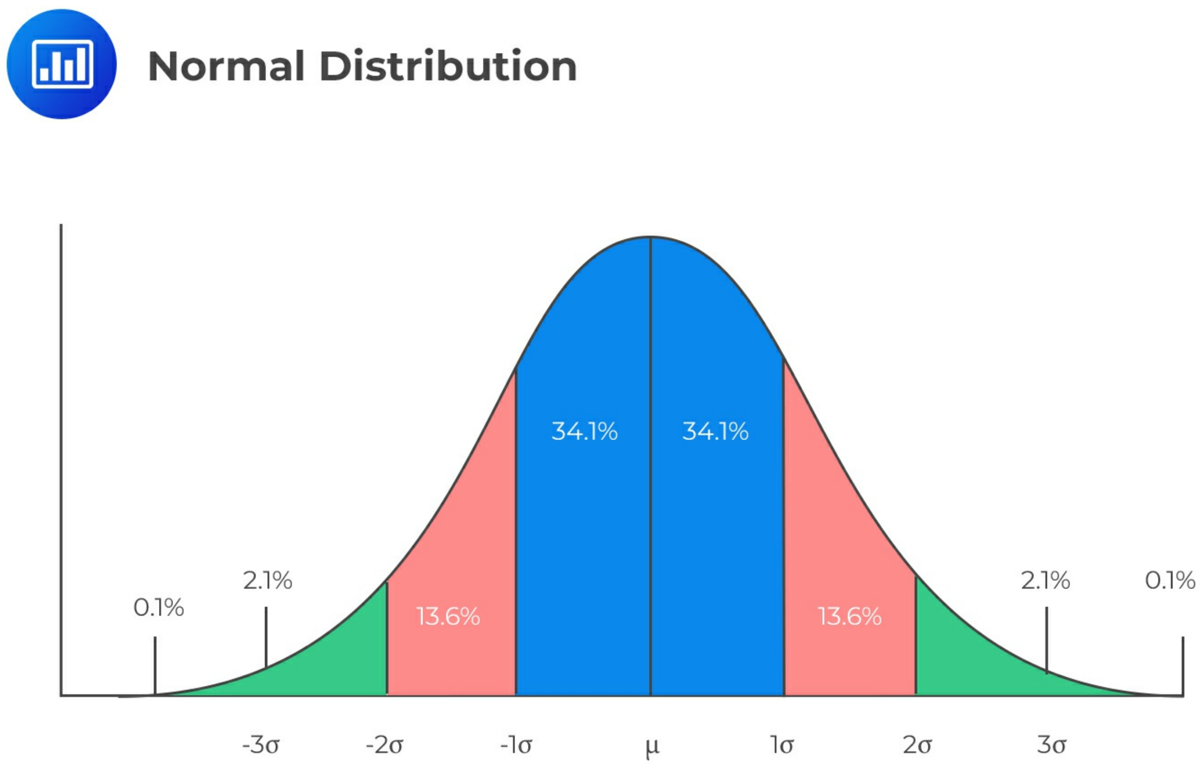
After completing this reading, you should be able to:

- Describe the mean-variance framework and an efficient frontier.
- Compare the normal distribution with the typical distribution of returns of risky financial assets such as equities.
- Define the VaR measure of risk, describe assumptions about return distributions and holding period, and explain the limitations of VaR.
- Explain and calculate Expected Shortfall (ES), and compare and contrast VaR and ES.
- Define the properties of a coherent risk measure and explain the meaning of each property.
- Explain why VaR is not a coherent risk measure.

The Mean-Variance Framework

The mean-variance framework uses the expected mean and standard deviation to measure the financial risk of portfolios. Under this framework, it is necessary to assume that returns follow a specified distribution, usually the normal distribution.

The normal distribution is particularly common because it concentrates most of the data around the mean return. 66.7% of returns occur within plus or minus one standard deviations of the mean. A whopping 95% of the returns occur within plus or minus two standard deviations of the mean.



Investors are generally concerned with downside risk and are therefore interested in probabilities that lie to the left of the expected mean.

Note the expected return does not imply the anticipated return but rather the average returns. On the other hand, the risk is measured using the standard deviation of returns.

Example: Calculating Expected Return and Standard Deviation

The expected returns for an asset with corresponding probabilities are given below:

Return (%)	Probability
10%	0.25
−20%	0.09
15%	0.40
7%	0.06
30%	0.20

Calculate the expected return and standard deviation of the asset return.

Solution

To calculate the expected return, we weight the expected return by their corresponding probability. That is:

$$\bar{R} = \sum_{i=1}^{n} p_i R$$

So for this case,

$$\begin{aligned}\bar{R} &= (0.10 \times 0.25) + (-0.20 \times 0.09) + (0.15 \times 0.40) + (0.07 \times 0.06) + (0.30 \times 0.20) \\ &= 0.1312 = 13.12\%\end{aligned}$$

Recall that the variance for a variable X is given by:

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

Then the variance of the return R is given by:

$$\text{Var}(R) = E(R^2) - [E(R)]^2$$

The standard deviation is equal to the square root of the variance

$$\sigma_R = \sqrt{E(R^2) - [E(R)]^2}$$

Therefore,

$$\begin{aligned}E(R^2) &= (0.10^2 \times 0.25) + ((-0.20)^2 \times 0.09) + (0.15^2 \times 0.40) + (0.07^2 \times 0.06) \\ &\quad + (0.30^2 \times 0.20) \\ &= 0.033394 \\ \Rightarrow \sigma_R &= \sqrt{0.033394 - [0.1312]^2} = 0.1272 = 12.72\%\end{aligned}$$

Combinations of Investments

Consider two investments with respective means μ_1 and μ_2. Suppose that an investor wishes to invest in both investments with a proportion of w_1 in the first investment and w_2 in the second investment. It is safe to state that $w_2 = 1 - w_1$.

The portfolio expected return is equivalent to weighted returns from individual investments. That is:

$$\mu_p = w_1\mu_1 + w_2\mu_2$$

The variance of the portfolio expected return is given by:

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2$$

Where

σ_1: standard deviation of the first investment

σ_2: standard deviation of the second investment

ρ: correlation between investment the first and the second investment

Therefore, the standard deviation of the portfolio is given by:

$$\sigma_P = \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma}$$

Note that the variance of the portfolio can be written as:

$$\sigma_P^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2 w_1 w_2 \text{ Cov } (R_1, R_2)$$

This is true from the fact that:

$$\text{Corr } (R_1, R_2) = \rho = \frac{\text{Cov } (R_1, R_2)}{\sigma_1\sigma_2}$$
$$\Rightarrow \text{ Cov } (R_1, R_2) = \rho\sigma_1\sigma_2$$

Example: Calculating the expected return and the standard deviation of a portfolio

An investor invests in two assets X and Y, with an expected return of 10% and 15%. The investor invests 45% of his funds in asset X and the rest in asset Y. The correlation coefficient is 0.45. Given that the standard deviation of asset X is 15% and Y is 30%, what are the expected return and standard deviations of the portfolio?

Solution

The portfolio expected return is given by:

$$\begin{aligned}\mu_p &= w_X\mu_X + w_Y\mu_Y \\ &= 0.10 \times 0.45 + 0.15 \times 0.55 \\ &= 0.1275 = 12.75\%\end{aligned}$$

The portfolio standard deviation is given by:

$$\begin{aligned}\sigma_P &= \sqrt{w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1 w_2 \sigma_1 \sigma_2} \\ &= \sqrt{0.45^2 \times 0.15^2 + 0.55^2 \times 0.3^2 + 2 \times 0.45 \times 0.45 \times 0.55 \times 0.15 \times 0.3} \\ &= \sqrt{0.041805} = 0.2325 = 23.25\%\end{aligned}$$

Calculating the portfolio expected return and standard deviation can be extended to a portfolio with n investments. The portfolio expected return for n returns is given by:

$$\mu_p = \sum_{i=1}^{n} w_i\mu_i$$

Where μ_i and w_i are the mean return and weight of ith investment

And then the standard deviation of the portfolio is given by:

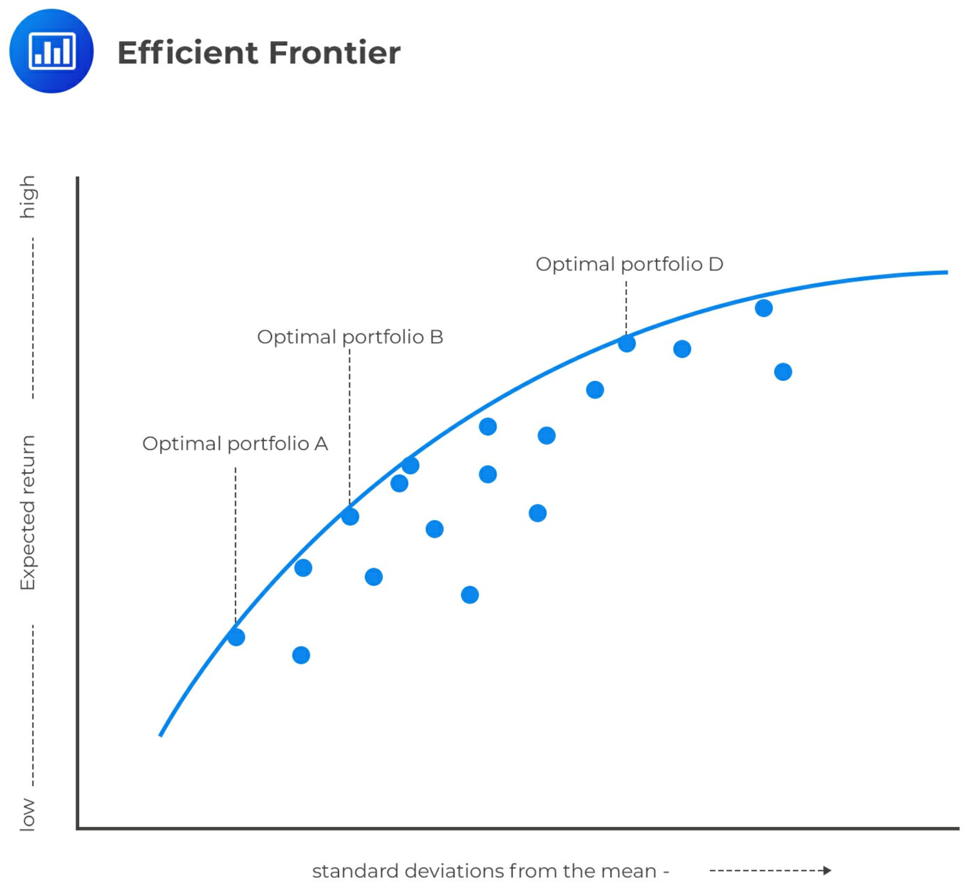
$$\sigma_P = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij} w_i w_j \sigma_i \sigma_j$$

where ρ_{ij} is the correlation coefficient between investments i and j. Other variables are intuitively definitive.

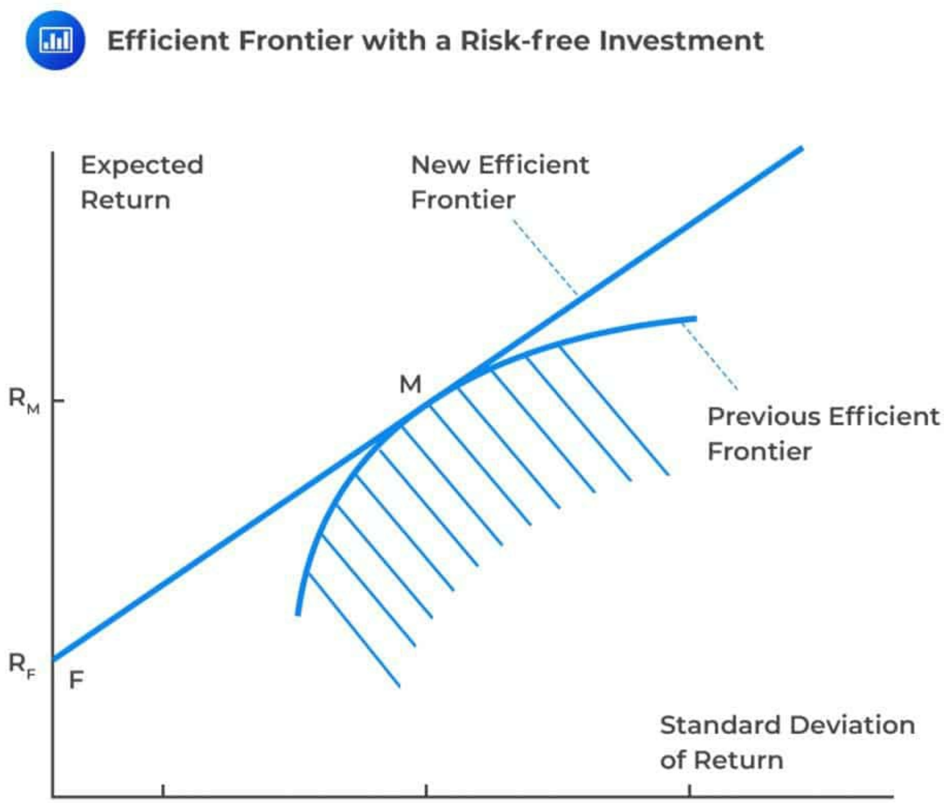
Efficient Frontier

The efficient frontier represents the set of optimal portfolios that offers the highest expected return for a defined level of risk or the lowest risk for a given level of expected return. This concept can be represented on a graph by plotting the expected return (Y-axis) against the standard deviation (X-axis).

For every point on the efficient frontier, at least one portfolio can be constructed from all available investments with the expected risk and return corresponding to that point. Portfolios that do not lie on the efficient frontier are suboptimal: those that lie below the line do not provide enough return for the level of risk. Those that lie on the right of the line have a higher level of risk for the defined rate of return.



Note that, the efficient frontier above considers only the risky assets. Now, consider when we introduce a risk-free investment with a return of R_F. It can be shown that the efficient frontier is a straight line. That is, there is a linear relationship between the expected return and the standard deviation of return.



Denote the risk-free return by R_F (with a standard deviation of 0). Also, let the market portfolio return be R_M, and its standard deviation is σ_M. Let the proportion of funds in a risky portfolio be β and that in risk-free assets, be $1-\beta$. Now using the formula

$$\mu_p = w_1\mu_1 + w_2\mu_2$$

We have $w_1 = 1-\beta, w_2 = \beta, \mu_1 = R_F, \mu_2 = R_M$ so that return from the portfolio is given by

$$\mu_p = R_F(1-\beta) + \beta R_M$$
$$\Rightarrow \beta = \frac{\mu_p - R_F}{R_M - R_F}$$

Also, the standard deviation of a portfolio with two components is given by

$$\sigma = \sqrt{w_1^2\sigma_F^2 + w_2^2\sigma_M^2 + 2\rho w_1 w_2 \sigma_F \sigma_M}$$

But $\sigma_F = 0$

$$\Rightarrow \sigma = \sqrt{0 + w_2^2\sigma_M^2 + 0} = w_2\sigma_M = \beta\sigma_M$$

Therefore,

$$\sigma = \sigma_M\left(\frac{\mu_P - R_F}{R_M - R_F}\right)$$
$$\Rightarrow \sigma = \mu_p\left(\frac{\sigma_M}{R_M - R_F}\right) - \frac{\sigma_M R_F}{R_M - R_F}$$

The efficient frontier involving a risk-free asset also shows that the investor should invest in risky assets (in this case, M) by borrowing and lending at a risk-free rate r_F. For instance, we assume that an investor borrows at the rate r_F so that now we are considering the efficient frontier beyond M. If this is the case, then $\beta > 1$ and the proportion of amount borrowed will be $\beta - 1$, and the total amount available is β multiplied by available funds. Assume now that we invest in risky asset M. Then the expected return is:

$$\beta r_M - (\beta - 1) R_F = (1 - \beta) r_F + \beta r_M$$

The standard deviation can be shown to be $\beta\sigma_M$, which is similar to arguments for the points below point M.

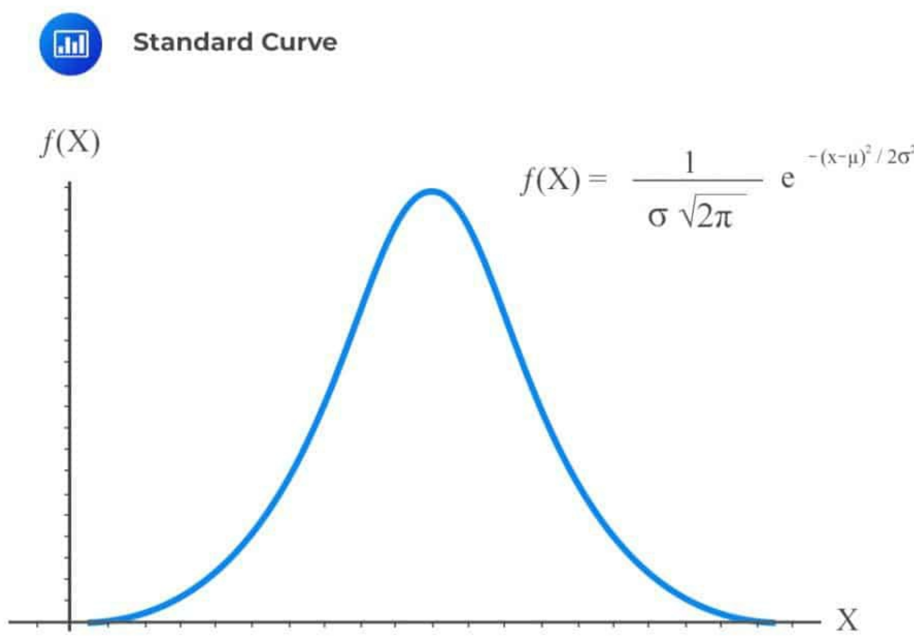
Therefore, it is safe to say risk-averse investors will invest in points on line FM and close to F, and those investors that are risk-seeking will invest on points close to M or even points beyond M on line FM.

The Normal Distribution

The normal distribution, also called Gaussian distribution, is a widely used continuous distribution with two parameters: mean denoted by μ and the standard deviation denoted by σ. The density function of the normal distribution is given by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

The shape of the standard curve is as shown below.



Note that, similar to other probability distributions, the probability that a value lies between a and b is equivalent to the area under the curve between a and b. This can be thought of as the cumulative distribution up to a point b less the cumulative distribution up to point a.

Standard Normal Distribution

A standard normal distribution has a mean of 0 and a standard deviation of 1. In other words, μ=0 and σ=1. As such, the normal distribution density function reduces to:

$$f(x) = f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}}$$

The normal tables give a cumulative distribution of the standard normal distribution. For normal distribution with the mean μ and the standard deviation σ, it can be transformed into z-scores, which gives cumulative probability up to a value x for standard normal. The z-score is defined by:

$$z = \frac{x - \mu}{\sigma}$$

Where z~N(0,1).

For example, consider a normal distribution with a mean of 4 and a standard deviation of 5. What is the probability that a value X is less than 7? Using standard normal transformation,

$$\begin{aligned} \Pr(X < 7) &= \Pr\left(\frac{X-\mu}{\sigma} < \frac{7-4}{5}\right) = 0.6 \\ &= \Pr(z < 0.6) = \Phi(0.6) = 1 - 0.2743 = 0.7257 \end{aligned}$$

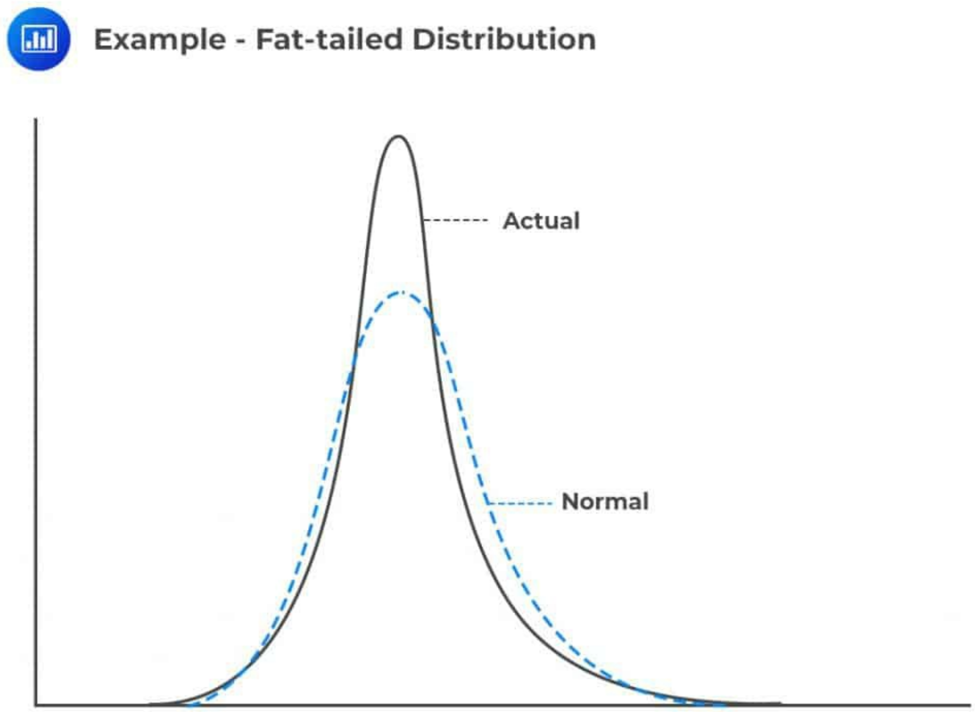
Note that the standard normal table is usually provided in exam.

In this case, the table provided is of negative z-values; as such, if we want to read the probability of $P(z < 0.6)$ then we will be forced to use $1 - P(z < -0.6)$ since the table gives probabilities for negative z-values yet, we want the probability for a positive z-value.

Reference Table: Let Z be a standard normal random variable.

z	P(Z<z)	z	P(Z<z)	z	P(Z<z)	z	P(Z<z)	z	P(Z<z)	z	P(Z<z)
-3	0.0013	-2.50	0.0062	-2.00	0.0228	-1.50	0.0668	-1.00	0.1587	-0.50	0.3085
-2.99	0.0014	-2.49	0.0064	-1.99	0.0233	-1.49	0.0681	-0.99	0.1611	-0.49	0.3121
-2.98	0.0014	-2.48	0.0066	-1.98	0.0239	-1.48	0.0694	-0.98	0.1635	-0.48	0.3156
-2.97	0.0015	-2.47	0.0068	-1.97	0.0244	-1.47	0.0708	-0.97	0.1660	-0.47	0.3192
-2.96	0.0015	-2.46	0.0069	-1.96	0.0250	-1.46	0.0721	-0.96	0.1685	-0.46	0.3228
-2.95	0.0016	-2.45	0.0071	-1.95	0.0256	-1.45	0.0735	-0.95	0.1711	-0.45	0.3264
-2.94	0.0016	-2.44	0.0073	-1.94	0.0262	-1.44	0.0749	-0.94	0.1736	-0.44	0.3300
-2.93	0.0017	-2.43	0.0075	-1.93	0.0268	-1.43	0.0764	-0.93	0.1762	-0.43	0.3336
-2.92	0.0018	-2.42	0.0078	-1.92	0.0274	-1.42	0.0778	-0.92	0.1788	-0.42	0.3372
-2.91	0.0018	-2.41	0.0080	-1.91	0.0281	-1.41	0.0793	-0.91	0.1814	-0.41	0.3409
-2.9	0.0019	-2.40	0.0082	-1.90	0.0287	-1.40	0.0808	-0.90	0.1841	-0.40	0.3446
-2.89	0.0019	-2.39	0.0084	-1.89	0.0294	-1.39	0.0823	-0.89	0.1867	-0.39	0.3483
-2.88	0.0020	-2.38	0.0087	-1.88	0.0301	-1.38	0.0838	-0.88	0.1894	-0.38	0.3520
-2.87	0.0021	-2.37	0.0089	-1.87	0.0307	-1.37	0.0853	-0.87	0.1922	-0.37	0.3557
-2.86	0.0021	-2.36	0.0091	-1.86	0.0314	-1.36	0.0869	-0.86	0.1949	-0.36	0.3594
-2.85	0.0022	-2.35	0.0094	-1.85	0.0322	-1.35	0.0885	-0.85	0.1977	-0.35	0.3632
-2.84	0.0023	-2.34	0.0096	-1.84	0.0329	-1.34	0.0901	-0.84	0.2005	-0.34	0.3669
-2.83	0.0023	-2.33	0.0099	-1.83	0.0336	-1.33	0.0918	-0.83	0.2033	-0.33	0.3707
-2.82	0.0024	-2.32	0.0102	-1.82	0.0344	-1.32	0.0934	-0.82	0.2061	-0.32	0.3745
-2.81	0.0025	-2.31	0.0104	-1.81	0.0351	-1.31	0.0951	-0.81	0.2090	-0.31	0.3783
-2.8	0.0026	-2.30	0.0107	-1.80	0.0359	-1.30	0.0968	-0.80	0.2119	-0.30	0.3821
-2.79	0.0026	-2.29	0.0110	-1.79	0.0367	-1.29	0.0985	-0.79	0.2148	-0.29	0.3859
-2.78	0.0027	-2.28	0.0113	-1.78	0.0375	-1.28	0.1003	-0.78	0.2177	-0.28	0.3897
-2.77	0.0028	-2.27	0.0116	-1.77	0.0384	-1.27	0.1020	-0.77	0.2206	-0.27	0.3936
-2.76	0.0029	-2.26	0.0119	-1.76	0.0392	-1.26	0.1038	-0.76	0.2236	-0.26	0.3974
-2.75	0.0030	-2.25	0.0122	-1.75	0.0401	-1.25	0.1056	-0.75	0.2266	-0.25	0.4013
-2.74	0.0031	-2.24	0.0125	-1.74	0.0409	-1.24	0.1075	-0.74	0.2296	-0.24	0.4052
-2.73	0.0032	-2.23	0.0129	-1.73	0.0418	-1.23	0.1093	-0.73	0.2327	-0.23	0.4090
-2.72	0.0033	-2.22	0.0132	-1.72	0.0427	-1.22	0.1112	-0.72	0.2358	-0.22	0.4129
-2.71	0.0034	-2.21	0.0136	-1.71	0.0436	-1.21	0.1131	-0.71	0.2389	-0.21	0.4168
-2.7	0.0035	-2.20	0.0139	-1.70	0.0446	-1.20	0.1151	-0.70	0.2420	-0.20	0.4207
-2.69	0.0036	-2.19	0.0143	-1.69	0.0455	-1.19	0.1170	-0.69	0.2451	-0.19	0.4247
-2.68	0.0037	-2.18	0.0146	-1.68	0.0465	-1.18	0.1190	-0.68	0.2483	-0.18	0.4286
-2.67	0.0038	-2.17	0.0150	-1.67	0.0475	-1.17	0.1210	-0.67	0.2514	-0.17	0.4325
-2.66	0.0039	-2.16	0.0154	-1.66	0.0485	-1.16	0.1230	-0.66	0.2546	-0.16	0.4364
-2.65	0.0040	-2.15	0.0158	-1.65	0.0495	-1.15	0.1251	-0.65	0.2578	-0.15	0.4404
-2.64	0.0041	-2.14	0.0162	-1.64	0.0505	-1.14	0.1271	-0.64	0.2611	-0.14	0.4443
-2.63	0.0043	-2.13	0.0166	-1.63	0.0516	-1.13	0.1292	-0.63	0.2643	-0.13	0.4483
-2.62	0.0044	-2.12	0.0170	-1.62	0.0526	-1.12	0.1314	-0.62	0.2676	-0.12	0.4522
-2.61	0.0045	-2.11	0.0174	-1.61	0.0537	-1.11	0.1335	-0.61	0.2709	-0.11	0.4562
-2.6	0.0047	-2.10	0.0179	-1.60	0.0548	-1.10	0.1357	-0.60	0.2743	-0.10	0.4602
-2.59	0.0048	-2.09	0.0183	-1.59	0.0559	-1.09	0.1379	-0.59	0.2776	-0.09	0.4641
-2.58	0.0049	-2.08	0.0188	-1.58	0.0571	-1.08	0.1401	-0.58	0.2810	-0.08	0.4681
-2.57	0.0051	-2.07	0.0192	-1.57	0.0582	-1.07	0.1423	-0.57	0.2843	-0.07	0.4721
-2.56	0.0052	-2.06	0.0197	-1.56	0.0594	-1.06	0.1446	-0.56	0.2877	-0.06	0.4761
-2.55	0.0054	-2.05	0.0202	-1.55	0.0606	-1.05	0.1469	-0.55	0.2912	-0.05	0.4801
-2.54	0.0055	-2.04	0.0207	-1.54	0.0618	-1.04	0.1492	-0.54	0.2946	-0.04	0.4840
-2.53	0.0057	-2.03	0.0212	-1.53	0.0630	-1.03	0.1515	-0.53	0.2981	-0.03	0.4880
-2.52	0.0059	-2.02	0.0217	-1.52	0.0643	-1.02	0.1539	-0.52	0.3015	-0.02	0.4920
-2.51	0.0060	-2.01	0.0222	-1.51	0.0655	-1.01	0.1562	-0.51	0.3050	-0.01	0.4960

A normal distribution is usually assumed to apply to financial data because financial analysts are mostly concerned with the mean and standard deviation. However, financial variables have fatter tails than the normal distribution. A large number of portfolio returns also tend to have fatter tails than the normal distribution. For instance, the means created can have fatter tails. Consider the diagram below.



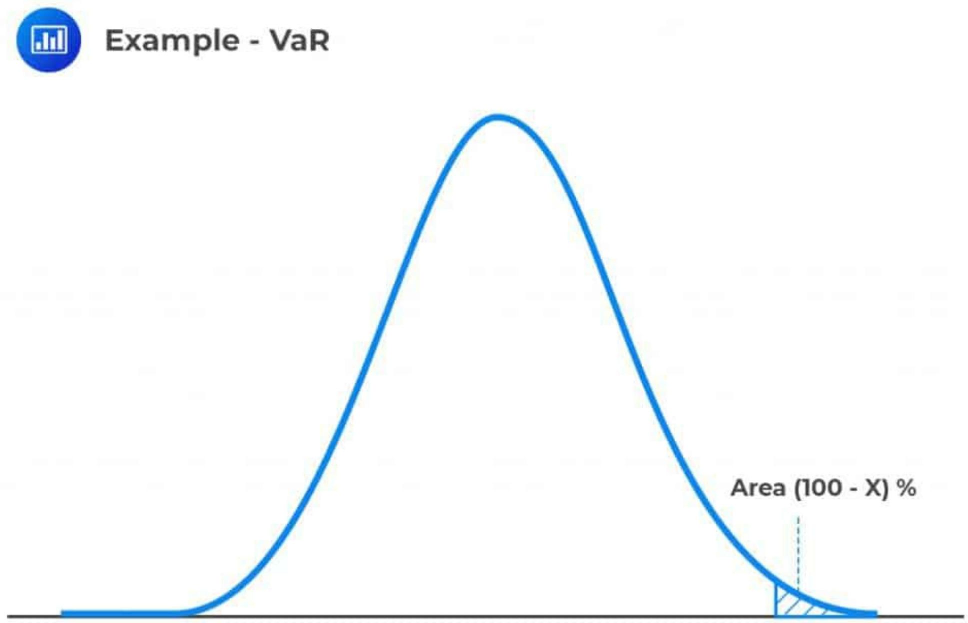
As discussed earlier in this chapter, we have seen that assuming a normal distribution for financial variables (by use of mean and standard deviation) may underestimate the probability of the adverse events.

The standard deviation can be a perfect measure of risk, but it does not capture the tails of the probability distribution.

VaR is a risk measure that is concerned with the occurrence of adverse events and their corresponding probability. VaR is built from two parameters: **the time horizon** and the **confidence level.** Therefore, we can say that VaR is the loss that we do not anticipate to be exceeded over a given time period at a specified confidence level.

For example, consider a time horizon of 30 days and a confidence interval of 98%. Therefore 98% VaR of USD 5 million implies that we are 98% confident that over the next 30 days, the loss will be less than USD 5 million. Similarly, we can say that we are 2% confident that over the next 30 days, the loss will be greater than USD 5 million.

Consider the following loss distribution density function curve:



Consider the following examples:

Example 1: Computing the VaR for normally distributed investment returns

The investment return over a period of time has a normal loss distribution with a mean of -200 and a variance of 300. What is 99% VaR of the loss distribution?

Solution

Denote the VaR level by t, then we need:

$$P(X < t) = 0.99$$

(Note we can also use $P(X > t)=0.01$)

Standardizing the normal distribution with a given mean and standard deviation, we have:

$$P\left(z < \frac{t-(-200)}{300}\right) = 0.99$$
$$\Rightarrow \Phi\left(\frac{t+200}{300}\right) = (0.99)$$
$$\Rightarrow \frac{t+200}{300} = \Phi^{-1}(0.99)$$

Now, $\Phi^{-1}(0.99)$ is the inverse of standard normal cumulative probability. To do this using a standard table, look for 0.99 (or closest value) in the table and read the corresponding vertical and horizontal values and add them. In other words, we are reversing the reading of the standard normal table.

In our cases, consider the following table:

Reference Table: Let Z be a standard normal random variable.

z	P(Z<z)	z	P(Z<z)	z	P(Z<z)	z	P(Z<z)	z	P(Z<z)	z	P(Z<z)
-3	0.0013	-2.50	0.0062	-2.00	0.0228	-1.50	0.0668	-1.00	0.1587	-0.50	0.3085
-2.99	0.0014	-2.49	0.0064	-1.99	0.0233	-1.49	0.0681	-0.99	0.1611	-0.49	0.3121
-2.98	0.0014	-2.48	0.0066	-1.98	0.0239	-1.48	0.0694	-0.98	0.1635	-0.48	0.3156
-2.97	0.0015	-2.47	0.0068	-1.97	0.0244	-1.47	0.0708	-0.97	0.1660	-0.47	0.3192
-2.96	0.0015	-2.46	0.0069	-1.96	0.0250	-1.46	0.0721	-0.96	0.1685	-0.46	0.3228
-2.95	0.0016	-2.45	0.0071	-1.95	0.0256	-1.45	0.0735	-0.95	0.1711	-0.45	0.3264
-2.94	0.0016	-2.44	0.0073	-1.94	0.0262	-1.44	0.0749	-0.94	0.1736	-0.44	0.3300
-2.93	0.0017	-2.43	0.0075	-1.93	0.0268	-1.43	0.0764	-0.93	0.1762	-0.43	0.3336
-2.92	0.0018	-2.42	0.0078	-1.92	0.0274	-1.42	0.0778	-0.92	0.1788	-0.42	0.3372
-2.91	0.0018	-2.41	0.0080	-1.91	0.0281	-1.41	0.0793	-0.91	0.1814	-0.41	0.3409
-2.9	0.0019	-2.40	0.0082	-1.90	0.0287	-1.40	0.0808	-0.90	0.1841	-0.40	0.3446
-2.89	0.0019	-2.39	0.0084	-1.89	0.0294	-1.39	0.0823	-0.89	0.1867	-0.39	0.3483
-2.88	0.0020	-2.38	0.0087	-1.88	0.0301	-1.38	0.0838	-0.88	0.1894	-0.38	0.3520
-2.87	0.0021	-2.37	0.0089	-1.87	0.0307	-1.37	0.0853	-0.87	0.1922	-0.37	0.3557
-2.86	0.0021	-2.36	0.0091	-1.86	0.0314	-1.36	0.0869	-0.86	0.1949	-0.36	0.3594
-2.85	0.0022	-2.35	0.0094	-1.85	0.0322	-1.35	0.0885	-0.85	0.1977	-0.35	0.3632
-2.84	0.0023	-2.34	0.0096	-1.84	0.0329	-1.34	0.0901	-0.84	0.2005	-0.34	0.3669
-2.83	0.0023	-2.33	$\boxed{0.0099}$	-1.83	0.0336	-1.33	0.0918	-0.83	0.2033	-0.33	0.3707
-2.82	0.0024	-2.32	0.0102	-1.82	0.0344	-1.32	0.0934	-0.82	0.2061	-0.32	0.3745
-2.81	0.0025	-2.31	0.0104	-1.81	0.0351	-1.31	0.0951	-0.81	0.2090	-0.31	0.3783
-2.8	0.0026	-2.30	0.0107	-1.80	0.0359	-1.30	0.0968	-0.80	0.2119	-0.30	0.3821
-2.79	0.0026	-2.29	0.0110	-1.79	0.0367	-1.29	0.0985	-0.79	0.2148	-0.29	0.3859
-2.78	0.0027	-2.28	0.0113	-1.78	0.0375	-1.28	0.1003	-0.78	0.2177	-0.28	0.3897
-2.77	0.0028	-2.27	0.0116	-1.77	0.0384	-1.27	0.1020	-0.77	0.2206	-0.27	0.3936
-2.76	0.0029	-2.26	0.0119	-1.76	0.0392	-1.26	0.1038	-0.76	0.2236	-0.26	0.3974
-2.75	0.0030	-2.25	0.0122	-1.75	0.0401	-1.25	0.1056	-0.75	0.2266	-0.25	0.4013
-2.74	0.0031	-2.24	0.0125	-1.74	0.0409	-1.24	0.1075	-0.74	0.2296	-0.24	0.4052
-2.73	0.0032	-2.23	0.0129	-1.73	0.0418	-1.23	0.1093	-0.73	0.2327	-0.23	0.4090
-2.72	0.0033	-2.22	0.0132	-1.72	0.0427	-1.22	0.1112	-0.72	0.2358	-0.22	0.4129
-2.71	0.0034	-2.21	0.0136	-1.71	0.0436	-1.21	0.1131	-0.71	0.2389	-0.21	0.4168
-2.7	0.0035	-2.20	0.0139	-1.70	0.0446	-1.20	0.1151	-0.70	0.2420	-0.20	0.4207
-2.69	0.0036	-2.19	0.0143	-1.69	0.0455	-1.19	0.1170	-0.69	0.2451	-0.19	0.4247
-2.68	0.0037	-2.18	0.0146	-1.68	0.0465	-1.18	0.1190	-0.68	0.2483	-0.18	0.4286
-2.67	0.0038	-2.17	0.0150	-1.67	0.0475	-1.17	0.1210	-0.67	0.2514	-0.17	0.4325
-2.66	0.0039	-2.16	0.0154	-1.66	0.0485	-1.16	0.1230	-0.66	0.2546	-0.16	0.4364
-2.65	0.0040	-2.15	0.0158	-1.65	0.0495	-1.15	0.1251	-0.65	0.2578	-0.15	0.4404
-2.64	0.0041	-2.14	0.0162	-1.64	0.0505	-1.14	0.1271	-0.64	0.2611	-0.14	0.4443
-2.63	0.0043	-2.13	0.0166	-1.63	0.0516	-1.13	0.1292	-0.63	0.2643	-0.13	0.4483
-2.62	0.0044	-2.12	0.0170	-1.62	0.0526	-1.12	0.1314	-0.62	0.2676	-0.12	0.4522
-2.61	0.0045	-2.11	0.0174	-1.61	0.0537	-1.11	0.1335	-0.61	0.2709	-0.11	0.4562
-2.6	0.0047	-2.10	0.0179	-1.60	0.0548	-1.10	0.1357	-0.60	0.2743	-0.10	0.4602
-2.59	0.0048	-2.09	0.0183	-1.59	0.0559	-1.09	0.1379	-0.59	0.2776	-0.09	0.4641
-2.58	0.0049	-2.08	0.0188	-1.58	0.0571	-1.08	0.1401	-0.58	0.2810	-0.08	0.4681
-2.57	0.0051	-2.07	0.0192	-1.57	0.0582	-1.07	0.1423	-0.57	0.2843	-0.07	0.4721
-2.56	0.0052	-2.06	0.0197	-1.56	0.0594	-1.06	0.1446	-0.56	0.2877	-0.06	0.4761
-2.55	0.0054	-2.05	0.0202	-1.55	0.0606	-1.05	0.1469	-0.55	0.2912	-0.05	0.4801
-2.54	0.0055	-2.04	0.0207	-1.54	0.0618	-1.04	0.1492	-0.54	0.2946	-0.04	0.4840
-2.53	0.0057	-2.03	0.0212	-1.53	0.0630	-1.03	0.1515	-0.53	0.2981	-0.03	0.4880
-2.52	0.0059	-2.02	0.0217	-1.52	0.0643	-1.02	0.1539	-0.52	0.3015	-0.02	0.4920
-2.51	0.0060	-2.01	0.0222	-1.51	0.0655	-1.01	0.1562	-0.51	0.3050	-0.01	0.4960

And thus:

$$\Phi^{-1}(0.99) = 2.33$$
$$\therefore \frac{t+200}{300} = 2.33 \Rightarrow t = 499$$

The VaR level is 499 at a 99% confidence level.

Example 2: Calculating the VaR for a discrete loss distribution

The loss distribution of investment is as shown below:

Amount of Loss	Probability
USD 10 Million	75%
USD 13 Million	22%
US 17 million	3%

What is the value of the 99% VaR?

Solution

To find the 99%, we need to find the cumulative probability distribution and locate 99%:

Amount of Loss	Probability	Cumulative Probability Range
USD 10 Million	75%	0 to 75%
USD 13 Million	22%	75% to 97%
US 17 million	3%	97% to 100%

Therefore, with a confidence level of 99%, the VaR value is USD 17 million because 99% falls the range of 97% and 100% (the last range).

Note that if we reduce our confidence level to 95%, VaR will change to USD 13 million because 95% falls between 75% to 97% cumulative probability range.

However, if the confidence level is 97%, then we could have two VaR values: USD 13 million and USD 17 million. This will be ambiguous, and so the best estimate is the average of the values, which is USD 15 million.

Limitations of VaR

I. It does not describe the **worst possible** loss. Indeed, as seen from the example above, we would expect the $13 million loss mark to be breached 5 times out of a hundred for a 95% confidence level.

II. VaR does not describe the losses in the left tail. It indicates the probability of a value occurring but **stops short of describing the distribution of losses in the left tail**.

III. Two **arbitrary parameters** are used in its calculation – the confidence level and the holding period. The confidence level indicates the probability of obtaining a value greater than or equal to VaR. The holding period is the time span during which we expect the loss to be incurred, say, a week, month, day, or year. VaR increases at an increasing rate as the confidence level increases. VaR also increases with increases in the holding period.

IV. VaR estimates are subject to both model risk and implementation risk. Model risks arise from incorrect assumptions, while implementation risk is the risk of errors from the implementation process.

Expected Shortfall (ES)

Recall the VaR does not describe the worst possible loss. For instance, if 99% VaR is USD 10 million, we know that we are 1% certain that the loss will exceed USD 10 million. From the VaR level, we cannot say that the loss is greater than 20 million or USD 50 million. Therefore, VaR sets a risk measure equal to a certain percentile of the loss distribution and does not consider the possible losses beyond the VaR level.

Expected shortfall (ES) is a risk measure that considers the expected losses beyond the VaR level. In other words, ES is the expected loss conditional that the loss is greater than the VaR level.

Exam tip: Expected shortfall is also called conditional value at risk (CVaR), average value at risk (AVaR), or expected tail loss (ETL). Think about this as the average loss beyond the VaR.

When the losses are normally distributed with the mean μ and standard deviation σ, then the ES is given by:

$$\text{ES} = \mu + \sigma\left(\frac{e^{-\frac{U^2}{2}}}{(1 - X)\sqrt{2\pi}}\right)$$

Where

X = the confidence level; and

U = the point in the standard normal distribution that has a probability of X% of being exceeded.

Example: Calculating expected shortfall

The investment return over a period of time has a normal loss distribution with a mean of -200 and a variance of 300. What is 99% Expected of the loss distribution?

Solution

We know that:

$$\mathrm{ES} = \mu + \sigma\left(\frac{e^{-\frac{U^2}{2}}}{(1-X)\sqrt{2\pi}}\right)$$

Now, $U = 2.33$

$$\mathrm{ES} = -200 + 300\left(\frac{e^{-\frac{(2.33)^2}{2}}}{(1-0.99)\sqrt{2\pi}}\right) = 592.79$$

ES should always be greater than the VaR level because the ES gives us the average of the values that are in tail exceeding VaR.

Example: Calculating the ES for a discrete loss distribution

The loss distribution of investment is as shown below:

Amount of Loss	Probability
USD 20 Million	2%
USD 17 Million	8%
US 13 million	12%
US 10 million	78%

What is the 95% expected shortfall (ES)?

Solution

At a 95% confidence level, we need to answer the question, "given that we are at 5% of the loss distribution, what is the value of the expected loss?".Now looking at the probability column, it is clear to see that 5% tail distribution consists of a 2% probability that the loss is USD 20 million and 3% that the loss is USD 17 million. Conditioned that we are dealing with tail distribution, then there is a 2/5 chance that the loss is USD 20 million and a 3/5 chance that the loss is USD 17 million. Therefore, the expected shortfall is given by:

$$\frac{2}{5} \times 20 + \frac{3}{5} \times 17 = 18.20$$

Again, note that the 97% VaR is USD 10 million, which is less than ES.

Properties of a Coherent Risk measure

A risk measure summarizes the entire distribution of dollar returns X by one number, $\rho(X)$. There are four desirable properties every risk measure should possess. These are:

I. **Monotonicity:** If $X_1 \leq X_2, \rho(X_1) \geq \rho(X_2)$

Interpretation: If a portfolio has systematically lower values than another, it must have a greater risk in each state of the world. In other words, if a portfolio gives undesirables results than the others, then it must be riskier.

II. **Subadditivity:** $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$

Interpretation: When two portfolios are combined, their total risk should be less than (or equal to) the sum of their risks. Merging of portfolios ought to reduce risk. This property captures the implications of diversification. If two portfolios are perfectly correlated, then the overall risk is the sum of their risk when considered separately. However, if the two portfolios are not perfectly correlated, their overall risk should decrease due to diversification benefits.

III. **Homogeneity:** $\rho(kX) = k\rho(X)$

Interpretation: Increasing the size of a portfolio by a factor k should result in a proportionate scale in its risk measure. For instance, if we increase the portfolio size by a quarter, then the risk should be increased by a quarter.

IV. **Translation invariance:** $\rho(X + h) = \rho(X) - h$

Interpretation: Adding cash h to a portfolio should reduce its risk by h. Like X, h is measured in dollars. This property reflects that more cash acts as a "loss absorber" and can be taken as a replacement for capital.

If a risk measure satisfies all four properties, then it is a coherent risk measure. Expected shortfall is a coherent risk, but VaR is not.

Why VaR is Not a Coherent Risk Measure

Value at risk is not a coherent risk measure because it fails the subadditivity test. Here's an illustration:

Suppose we want to calculate the VaR of a portfolio at 95% confidence over the next year of two zero-coupon bonds (A and B) scheduled to mature in one year. Assume that:

- The current yield on each of the two bonds is 0;
- The bonds have different issuers;
- Each bond has a probability of 4% of defaulting over the next year;
- The event of default in either bond is independent of the other; and
- The recovery rate upon default is 30%.

Given these conditions, the 95% VaR for holding either of the bonds is 0 because the probability of default is less than 5%. Now, what is the probability 'P' that at least one bond defaults?

$$P = 0.04 \times 0.96 + 0.96 \times 0.04 + 0.04 \times 0.04 = 7.84$$

The probability of at least one default is 7.84%, which exceeds 5%.

So if we held a portfolio that consisted of 50% A and 50% B, then the 95% $\text{VaR} = 0.7 \times 0.5 + 0 \times 0.5 = 35$

This violates the subadditivity principle, and VaR is, therefore, not a coherent risk measure.

Practice Question

Ann Conway, FRM, has spent the last several months trying to develop a new risk measure to appraise a set of defaultable zero-coupon bonds owned by her employer. Prior to its use, her supervisor has asked her to demonstrate that it is a coherent risk measure. The results are listed below:

Given:

- X and y are state-contingent payoffs of two different bond portfolios.
- $P(x)$ and $P(y)$ are risk measures for portfolio x and portfolio y.
- K and l are arbitrary constants, with $k > 0$

Which of the following equations shows that Conway's risk measure is **not** coherent?

A. $P(kx) = kP(x)$
B. $P(x) + P(y) \geq P(x+y)$
C. $P(x) \leq P(y)$ if $x \leq y$
D. $P(x+l) = P(x) - l$

The correct answer is **C**.

Option C, as represented above, shows that the risk measure does not satisfy the monotonicity property. Monotonicity requires that $P(x) \geq P(y)$ if $x \leq y$. (If a portfolio has systematically lower values than another, in each state of the world, it must have a greater risk.)

Option A demonstrates that the measure satisfies the homogeneity property.

Option B demonstrates that the measure satisfies the subadditivity property.

Option D demonstrates that the measure satisfies the translation invariance property.

Reading 46: Calculating and Applying VaR

After completing this reading, you should be able to:

- Explain and give examples of linear and nonlinear derivatives.
- Describe and calculate VaR for linear derivatives.
- Describe and explain the historical simulation approach for computing VaR and ES.
- Describe the delta-normal approach for calculating VaR for nonlinear derivatives.
- Describe the limitations of the delta-normal method.
- Explain structured Monte Carlo and stress testing methods for computing VaR, and identify strengths and weaknesses of each approach.
- Describe the implications of correlation breakdown for scenario analysis.
- Describe the worst-case scenario (WCS) analysis and compare WCS to VaR.

Linear and Nonlinear Portfolios

A linear portfolio linearly depends on the changes in the values of its corresponding variables (risk factors). For instance, consider a portfolio consisting of 100 shares, each valued at USD 100. Therefore, the change in portfolio value (ΔP) is attributed to change in stock (share price) which can be denoted by ΔS, and thus the change in portfolio value is given by :

$$\Delta P = 100\Delta S$$

The value of the portfolio is USD 10,000 (=100×100). Now, if we introduce the effect of the interest rate, the change in the value of the portfolio will be given by:

$$\Delta P = 100\Delta r$$

Generally, consider a portfolio consisting of long and short positions in stocks. The change in a linear portfolio is given by:

$$\Delta P = \sum_i n_i \Delta S_i$$

Where:

n_i = number of shares of stock i in the portfolio;

S_i = the price of stock i.

Intuitively, the amount invested in stock i is $n_i S_i$, and the price change is ΔS_i.

Now, if we multiply the above equation by $\frac{S_i}{S_i}$, we have:

$$\Delta P = \sum_i n_i \Delta S_i \times \frac{S_i}{S_i} = \sum_i n_i \Delta S_i \frac{\Delta S_i}{S_i}$$

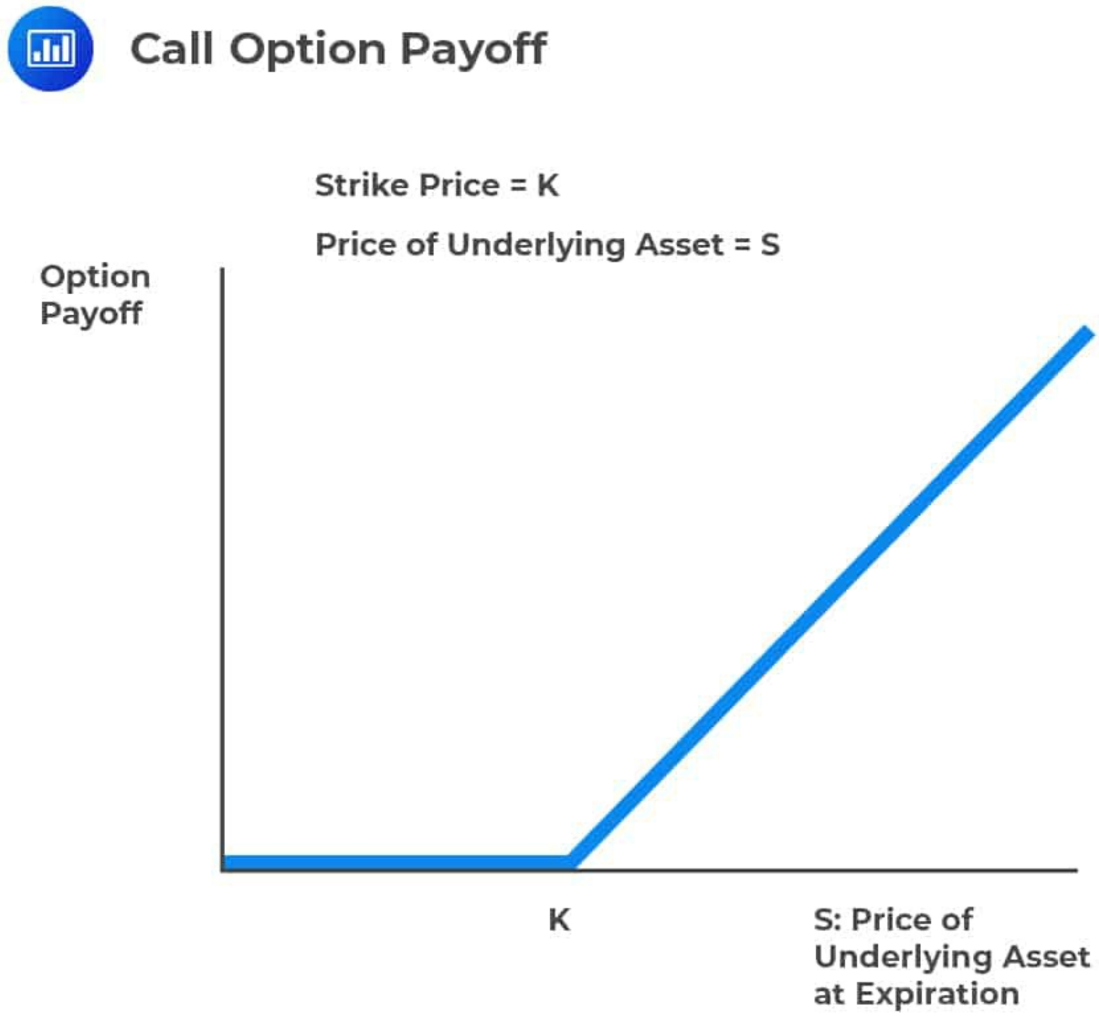
Let $q_i = n_i S_i$ and $\Delta r_i = \frac{\Delta S_i}{S_i}$, then

$$\Delta P = \sum_i q_i \Delta r_i$$

Note that q_i is the amount invested in stock i, and $\Delta r_i = \frac{\Delta S_i}{S_i}$ is the return on stock i.

Therefore, we can say the portfolio change is a linear function of change in stock price or change in stock returns.

Nonlinear portfolios contain complex securities that are not linear. For instance, consider a portfolio made of call options. The payoff from the call option is nonlinear because the payoff is zero if the stock price at maturity is less than the strike price and S-K if the stock price is higher than the strike price K.

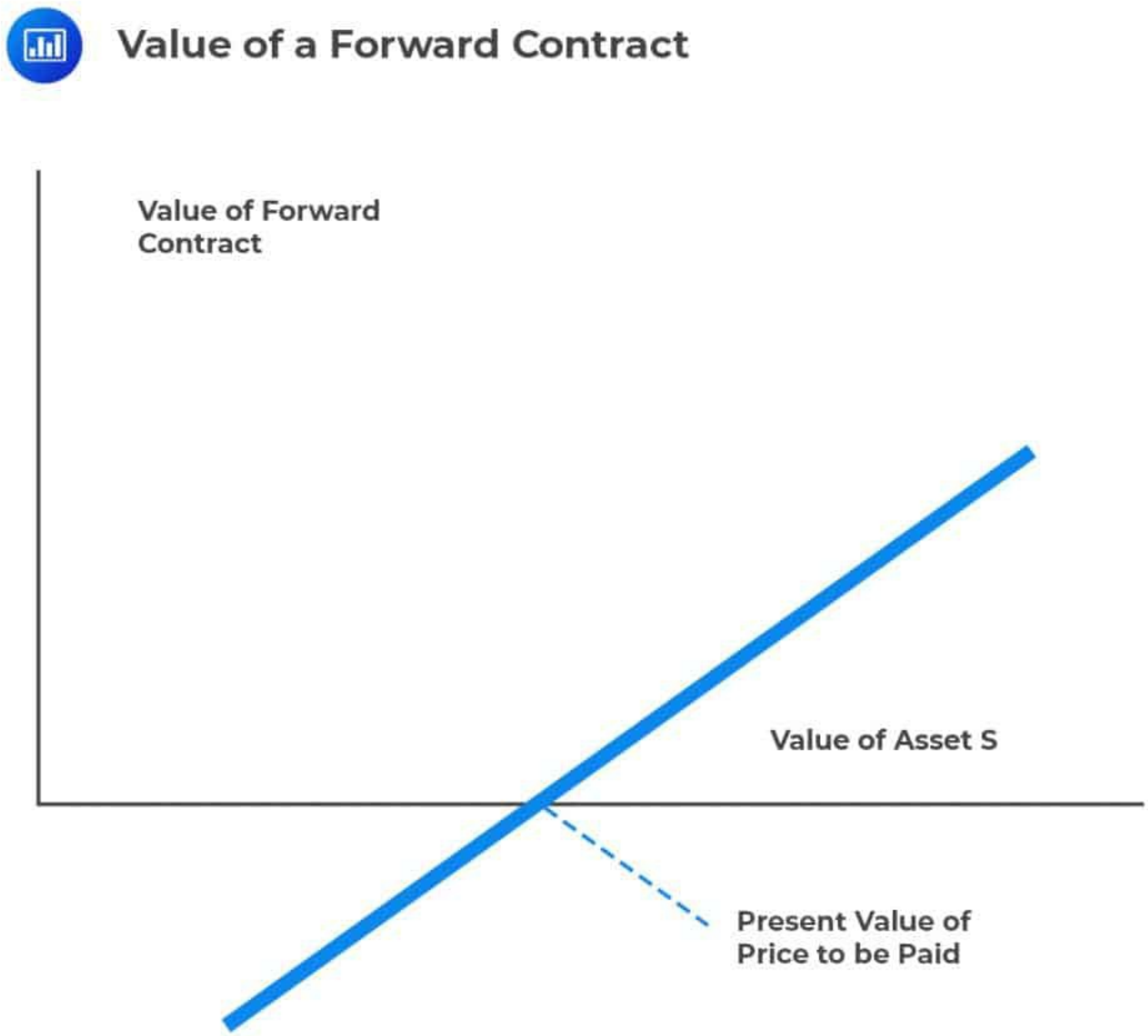


However, a forward contract is an example of a derivative whose value is a linear function of the asset because even if the contracts do give a payoff, the holder is obligated to buy the asset at a future time T at agreed price K. As such, in case the asset provides no income, then the forward contract's value is given by:

$$S - PV(K)$$

Where S is the current asset price, and PV(K) denote the present value of the future price K.

The figure below shows the relationship between the value of a forward contract and the underlying asset.



From the figure above, it is correct to say that a person who agrees to buy an asset at a future time is actually in a position to own the asset today but pay for it at a future time. Now, the value of owning the asset today is S, and the present value of what will be paid for the asset at the future time is PV(K). This proves the formula above for the value of the forward contract.

VaR for Linear Derivatives

In general terms, the VaR of a linear derivative can be expressed as:

$$\text{VaR}_{\text{linear derivative}} = \Delta \times \text{VaR}_{\text{Underlying factor}}$$

Where Δ represents the sensitivity of the derivative's price to the price of the underlying asset. It is usually expressed as a percentage.

Example: VaR for linear derivatives

Suppose the permitted lot size of S&P 500 futures contracts is 300 and multiples thereof. What is the VaR of an S&P 500 futures contract?

$$\text{VaR}_{\text{S\&P500futures c.}} = 300 \times \text{VaR}_{\text{S\&P500 index}}$$

Calculating VaR and Expected Shortfall using Historical Simulation

Historical simulation is used to calculate one-day VaR and ES. However, for longer periods T it is assumed that,

$$\text{VaR}(T, X) = \sqrt{T} \times \text{VaR}(1, X)$$
$$\text{ES}(T, X) = \sqrt{T} \times \text{ES}(1, X)$$

$\text{VaR}(T, X)$ = value at risk for a time horizon of T days and confidence level X.

$\text{ES}(T, X)$ = expected shortfall for a time horizon of T days and confidence level X.

The above estimates assume that the portfolios' changes are normally distributed with a mean of zero and independent of each other.

The historical simulation procedure involves the following steps:

i. Identifying market variables/risk factors -The first step involves identifying market variables (risk factors) on which the portfolio value depends. Examples of such variables include commodity prices, equity prices, and volatilities.
ii. Collecting data on the behavior of risk factors - Once risk factors have been identified, the data on the behavior of these risk factors in the past is collected. In this section, we assume that the past period is one from the immediate past.
iii. Creating scenarios - After past data has been collected, scenarios are built by assuming that

each risk factor's change over the next day corresponds to a change observed during one of the past days.

Risk factors are broadly classified into:

i. Those whose past percentage change is to define the future percentage, for example, stock prices and exchange rates; and
ii. Those whose past actual change is used to define an actual change in the future, for example, interest rates and credit spreads.

Example: Illustration of Historical Simulation

A portfolio is assumed to depend on many risk factors. For simplicity, let us assume three risk factors (exchange rates, interest rates, and stock price) over the past 300 days (longer periods are usually considered, such as 500 days). The most recent 301 days of historical data is as follows:

Day	Stock Price (USD)	Exchange rates (USD/CAD)	Interest rate (%)	Portfolio Value (USD millions)
0	30	1.3901	3.51	60.0
1	34	1.4000	2.64	62.5
2	40	1.3921	2.52	61.25
..	..	..	..	..
298	42	1.3876	2.40	65.0
299	40	1.3910	2.45	60.25
300	60	1.4021	2.50	71.25

Assuming that today is the 300$^{\text{th}}$ day, we need to know what will happen between today and tomorrow (301$^{\text{st}}$ day). To achieve this, we use the above data to create 300 scenarios (that is why we have 301-day historical data).

In the first scenario, we will assume that the risk factors behave between the days 300 and 301 in a similar manner as they did between days 0 and 1. For instance, in the first scenario, the stock price increased by 13% $(= \frac{34}{30} - 1)$, and thus the stock price on day 301 is USD 68 $[= (\frac{34}{30} - 1) \times 60]$. For the exchange rate, it increased by 0.7% $(= \frac{1.400}{1.3901} - 1)$, and thus, we expect the exchange rate for the

301st day to be 1.4121 $\left(=1.4021\left[\frac{1.400}{1.3901}-1\right]\right)$. For the interest rate, it decreased by 0.87% (2.64%-3.51%) and thus 301st interest rate is 1.63% (2.50%-0.87%).

The values for the second and subsequent scenarios' risk factors are calculated similarly as the first scenario. For the second scenario, assume that the risk factors behave in a similar manner as they did between days 1 and 2. This will create the following table.

Scenario	Stock Price (USD)	Exchange rates (USD/CAD)	Interest rate (%)	Portfolio Value (USD millions)	Loss
1	68	1.4121	3.37	70.25	1.0
2	71	1.3922	2.38	72.15	0.9
..	..	..	..	..	
299	57.14	1.3910	2.55	71.25	0
300	90	1.4021	2.55	73.25	2.0

The risk factor values for the scenario table are directly calculated from the historical data table. The scenario portfolio values are generated based on the risk factors. We assume that the current portfolio value is USD 71.25 million (300th day's value). After generating the portfolio values, we then calculate the losses while attaching a negative to create a loss distribution.

Assume that the first scenario's portfolio value is 70.25, 72.15 for the second scenario, and so on. Therefore, the loss for the first portfolio is 1.00 (=71.25-70.25), and the second scenario is 0.9 (71.25-72.15), and so on.

In order to calculate the VaR and the expected shortfall, we ought to arrange the scenario losses from the largest to the smallest. Assume that in our example, we wish to calculate one day VaR and ES at a 99% confidence interval. The sorted losses are as follows:

Scenario	Loss
200	3.9
10	3.0
25	2.5
100	2.0
...	...
...	...

In this case, VaR is equivalent to third-worst loss since the third-worst loss is the first percentile point of the distribution, i.e., $\frac{3}{100} = 0.01$. Therefore, VaR=2.5 million.

By definition, the expected shortfall is calculated as the average of the losses that are worse than the VaR. In this case,

$$\text{ES} = \frac{1}{2}(3.9 + 3.0) = 3.45 \text{ million}$$

Example: Calculating VaR using the historical simulation method

The following are hypothetical ten worst returns for an asset B from 120 days of data for 6 months. Find the 1-day 5% VaR and the ES for B.

-3.45%, -14.12%, -15.72%, -10.92%, -5.50%, -3.56%, -6.90%, -2.50%, -5.30%, -4.31%.

Solution

First, we rearrange starting with the worst day, to the least bad day, as shown below:

-15.72%, -14.12%, -10.92%, -6.90%, -5.50%, -5.30%, -4.31%, -3.56%, -3.45%, -2.50%.

The VaR corresponds to the $(5\% \times 120)=6^{\text{th}}$ worst day = -5.30%. However, recall that VaR need not be represented as a negative.

This implies that there is a 95% probability of getting at most 5.3% loss.

The expected shortfall (ES) is calculated as the average of the losses that are worse than the VaR. In this case,

$$\text{ES} = \frac{(15.72\% + 14.12\% + 10.92\% + 6.90\% + 5.50\%)}{5} = 10.63\%$$

Valuing portfolios

Before we look at the delta-normal model, we will biefly look at the full revaluation approach. Under this approach the VaR of a portfolio is established by fully repricing the portfolio under a set of scenarios over a period of time.

However, a full revaluation of a portfolio for many scenarios is a time-consuming activity. One approach to address this challenge is to use Greek letters. The Greek letters are the hedging parameters used by the analysts to quantify and manage risks.

One of the crucial Greek letter deltas (δ) which is defined as:

$$\delta = \frac{\Delta P}{\Delta S}$$

Where ΔS is a small change in risk factors such as stock price and ΔP the corresponding change in the portfolio value. Therefore, delta can be defined as a change of the portfolio value with respect to the change in the risk factor.

For instance, consider stock price as a risk factor. If the delta of a portfolio with respect to the stock price is USD 100, it implies that the portfolio value changes by USD 100 if the stock price changes by 1 USD.

From the delta formula, we express the change in the portfolio value as :

$$\Delta P = \delta \Delta S$$

Generally, if we have multiple risk factors, we find each risk factor's effect and sum it up. That is, if we have *i* risk factors, then change in the portfolio would be:

$$\Delta P = \sum_i \delta_i \Delta S_i$$

However, the delta concept gives relatively accurate estimates in linear portfolios as compared to nonlinear portfolios.

The accuracy of nonlinear portfolios can be enhanced by including another Greek letter gamma (γ) so that the delta-gamma formula is given by:

$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

In case of multiple risk factors, the above equation changes to:

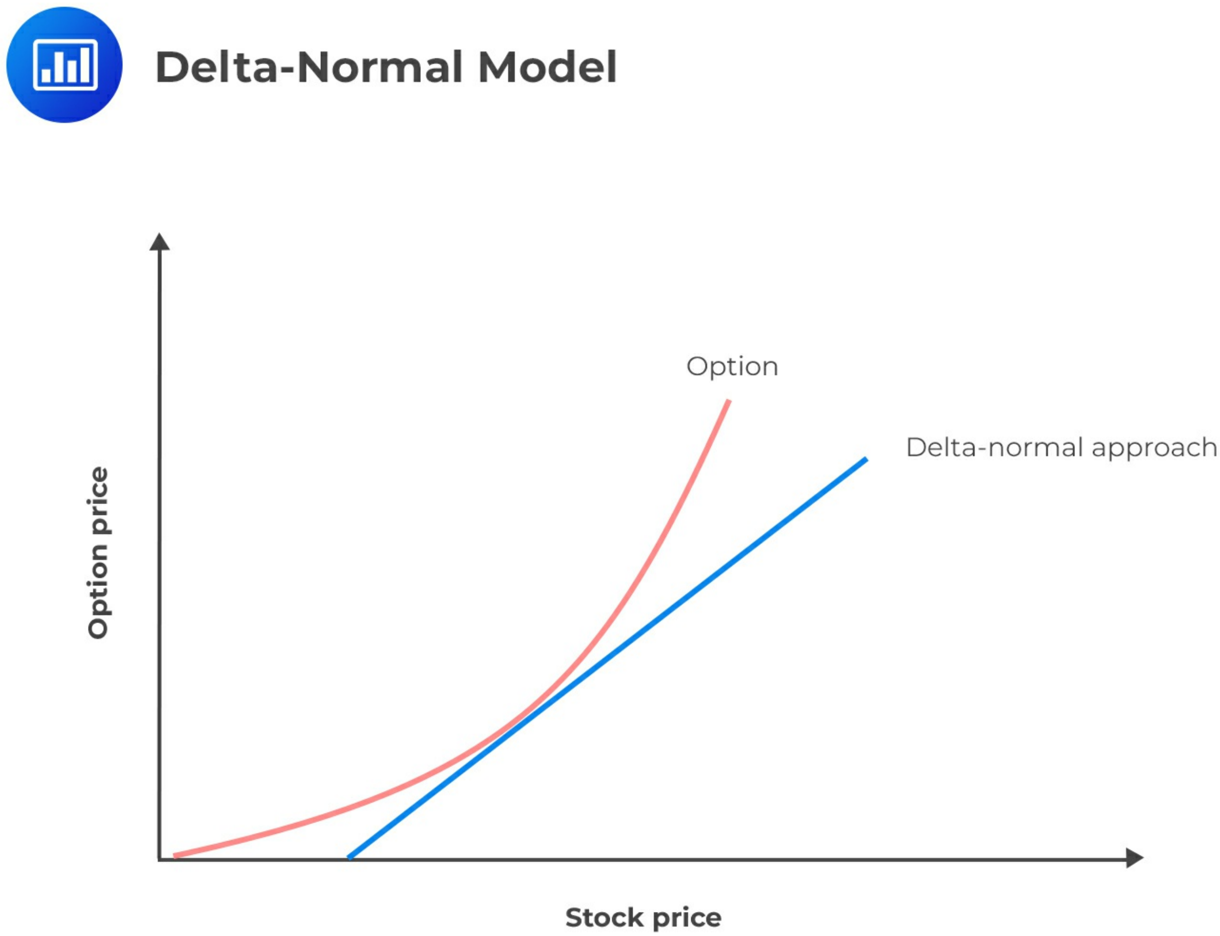
$$\Delta P = \sum_i \delta_i \Delta S_i + \frac{1}{2}\sum_i \gamma_i (\Delta S_i)^2$$

The Delta-Normal Model

The delta-normal model is based on the equation (as seen earlier):

$$\Delta P = \sum_i \delta_i \Delta S_i$$

Recall this equation gives an exact value in a linear portfolio and an approximate value in nonlinear portfolios.



Also, recall that we have two types of risk factors:

1. Those whose past percentage change is to define the future percentage; and
2. Those whose past actual change is used to define an actual change in the future.

To accommodate both types of risk factors, the equation above is written as:

$$\Delta P = \sum_i a_i x_i$$

For risk factors where percentage changes are used, $a_i = \frac{\Delta S_i}{S_i}$ and $x_i = \delta_i S_i$.

And for the risk factors where actual changes are considered $a_i = \Delta S_i$ and $x_i = \delta_i$.

From the resulting equations, the mean and the standard deviation of the change in portfolio value can be calculated as:

$$\mu_P = \sum_{i=1}^{n} a_i \mu_i$$
$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \rho_{ij} \sigma_i \sigma_j$$

Where:

μ_i and σ_i are the mean and standard deviation of x_i, respectively; and

ρ_{ij} is the coefficient of correlation between x_i and x_j.

The formula for the standard deviation can be written as:

$$\sigma_P^2 = a_i^2 \sigma_i^2 + 2 \sum_{i>j} a_i a_j \rho_{ij} \sigma_i \sigma_j$$

Assuming that the portfolio changes are normally distributed, then we can comfortably compute VaR and ES. Recall that the VaR is given by:

$$\text{VaR} = \mu_P + \sigma_P U$$
$$\text{ES} = \mu_P + \sigma_P \left(\frac{e^{-\frac{U^2}{2}}}{(1-X)\sqrt{2\pi}} \right)$$

Where:

X = confidence level

U = point on the normal distribution where X is exceeded

For instance, if $X=95\%$ then $U=\Phi^{-1}(0.05) = -1.645$.

At this point, you might guess where the name "delta-normal" name comes from: the model uses deltas of the risk factors and assumes that the portfolio changes are normally distributed.

A typical assumption is that the mean change in the risk factor is zero. This assumption is sometimes not reasonable but is useful when dealing with short time periods because the mean is less than the standard deviation for short periods when dealing with portfolio value changes. As such, VaR and ES are given by:

$$\text{VaR} = \sigma_P U$$
$$\text{ES} = \sigma_P \left(\frac{e^{-\frac{U^2}{2}}}{(1 - X)\sqrt{2\pi}} \right)$$

Example: Calculating Expected Shortfall and VaR Using Delta-Normal Model

The investment return over a period of time has a normal loss distribution with a mean of -100 and a variance of 400.

Using the delta-normal model, calculate the 99% Expected Shortfall and the 99% VaR of the loss distribution.

Solution

$$\text{ES} = \sigma_p \left(\frac{e^{-\frac{U^2}{2}}}{(1 - x)\sqrt{2\pi}} \right)$$
$$= 20 \left(\frac{e^{-\frac{2.33^2}{2}}}{(1 - 0.99)\sqrt{2\pi}} \right) = 52.85$$

Where X is the confidence level and U is the point on the normal distribution where X is exceeded

The 99% VaR is given by:

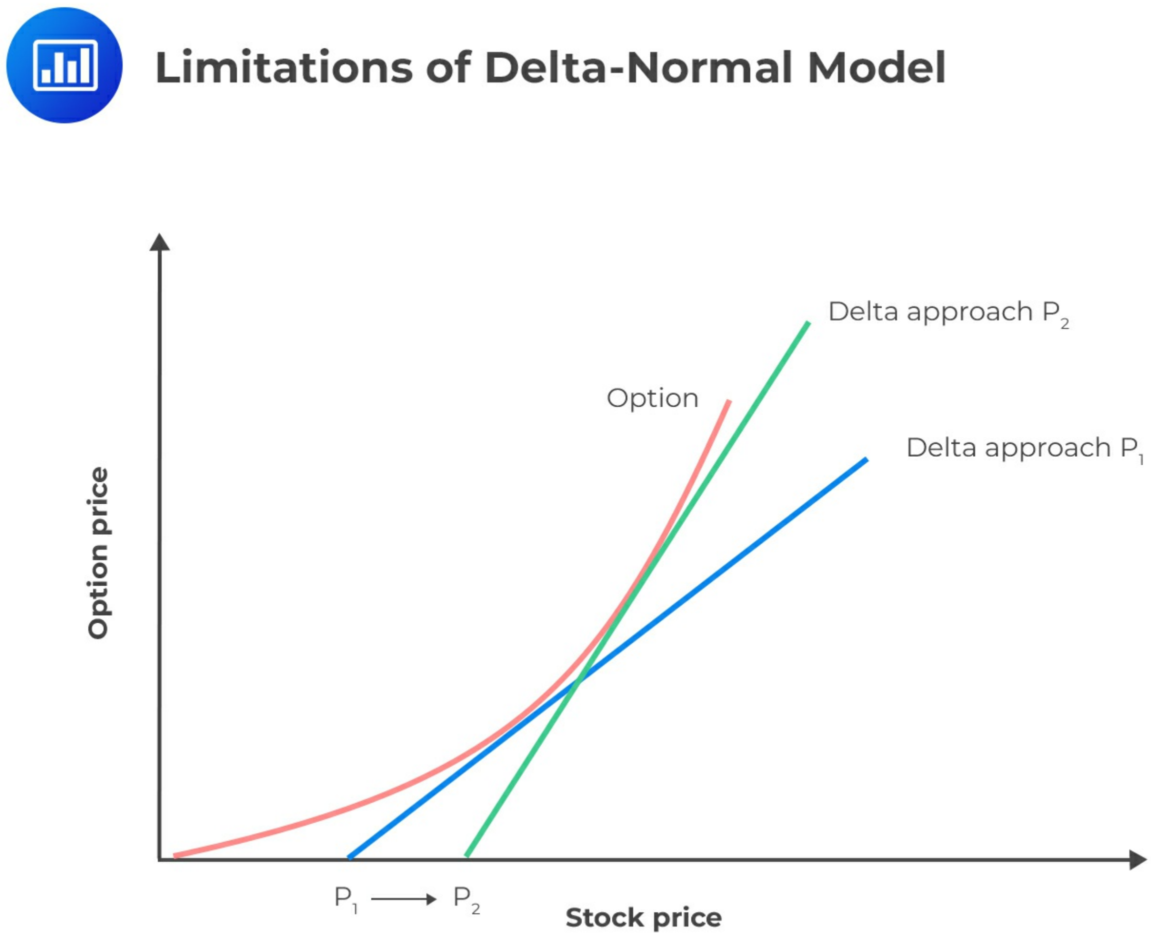
$$\text{VaR} = \sigma_p U = 20 \times (-2.33) = -46.6$$

Limitations of Delta-Normal Model

The method has several disadvantages, chief among them being that:

- It is computationally easy but quite inaccurate compared to other VaR measurement methods. Put more precisely, it may underestimate the occurrence of extreme losses because it relies on the normal distribution.

- This method is accurate for small moves of the underlying, but **quite inaccurate for large moves.** As we can see from the following graph, the slope of the green line is entirely different from the slope of the blue line:



For large changes in a nonlinear derivative, we must use the delta + gamma approximation, or full revaluation as we had discussed earlier.

Using Monte Carlo Simulation to Calculate VaR and ES

Monte Carlo approach is similar to that of the historical simulation, but the Monte Carlo simulation produces scenarios by randomly selecting samples from the distribution assumed for the risk factors instead of using historical data. Monte Carlo simulations work for both linear and nonlinear portfolios.

Now, if for instance, we assume that risk factor changes have a multivariate normal distribution (as in delta-normal), Monte Carlo procedure is as follows:

Step 1: Calculate the value of the portfolio today using the current values of the risk factors.

Step 2: Sample once from the multivariate normal probability distribution of Δx_i. Sampling should be consistent with the assumed standard deviations and correlations, which are usually approximated from historical data.

Step 3: Using the sample values of Δx_i, determine the values of the risk factors at the end of the period being considered (such as one day).

Step 4: Revalue the portfolio using these new risk factor values.

Step 5: Subtract the revaluated portfolio from the current portfolio value to determine the loss.

Step 6: Repeat step 2 to step 5 multiple times to come up with a probability distribution for the loss.

For instance, a total of 500 trials are conducted in a Monte Carlo simulation, then 99% VaR for the period under consideration will be the fifth-worst loss, and thus the expected shortfall will be the average of the four losses worse than VaR.

Like other approaches, Monte Carlo simulation computes one-day VaR, and thus the following equations apply when we want to compute T-day time horizon VaR and ES:

$$\text{VaR}(T, X) = \sqrt{T} \times \text{VaR}(1, X)$$
$$\text{ES}(T, X) = \sqrt{T} \times \text{ES}(1, X)$$

Monte Carlo simulation is slow because it is computationally intensive. This can be explained by the fact that portfolios considered are usually huge, and evaluating each one of them in each trial is quite time-consuming. To address this challenge, the delta-gamma approach can be used (as discussed earlier) to determine the change in the portfolio value. This is called partial simulation.

Delta-normal model assumes normal distribution for the risk factors. However, Monte Carlo simulation uses any distribution for the risk factors only if the correlation between the risk factors can be defined.

Estimating the Parameter Values

To execute the Monte Carlo simulation or delta-normal model, an approximation of the standard deviations and correlations of either percentage or actual changes of the risk factors is necessary. The approximation of these parameters is made using recent historical data. More weights can be applied to more recent data using models such as GARCH (1,1), which we will see in the following chapter.

In the case of stressed VaR and ES, standard deviations and correlations should be estimated from the past period, which would be considered stressful to the current portfolio.

Correlation Breakdown

During the stressed market conditions, standard deviations increase as well as correlations. This phenomenon was witnessed during the 2007-2008 financial crisis, where default rates of mortgages increased all over the US.

Therefore, correlations in a high volatility period are quite different from those of normal market conditions. This phenomenon is referred to as a correlation breakdown. Thus, when calculating VaR or ES, risk managers might need to determine what will happen in extreme market conditions.

Worst-case Scenario Analysis

Worst-case scenario analysis focuses on extreme losses at the tail end of the distribution. First, firms assume that an unfavorable event is certain to occur. They then attempt to establish the worst possible outcomes that could come out of it.

WCS analysis dissects the tail further to establish the range of worst-case losses that could be incurred. For example, within the lowest 5% of returns, we can construct a "secondary" distribution that specifies the 1% WCS return.

WCS analysis complements the VaR, and here is how. Recall that the VaR specifies the minimum loss for a given percentage, but it stops short of establishing the severity of losses in the tail. WCS analysis goes a step further to describe the distribution of extreme losses more precisely.

Practice Questions

Question 1

A risk manager wishes to calculate the VaR for a Nikkei futures contract using the historical simulation approach. The current price of the contract is 955, and the multiplier is 250. For the last 300 days, the following return data have been recorded:

-7.8%, -7.0%, -6.2%, -5.2%, -4.6%, -3.2%, -2.0%, …, 3.8%, 4.2%, 4.8%, 5.1%, 6.3%, 6.8%, 7.0%

What is the VaR of the position at 99% using the historical simulation methodology?

A. $12,415

B. $16,713

C. $18,623

D. $14,803

The correct answer is **D.**

The 99% return among 300 observations would be the third-worst observation among the returns $((1 - 0.99) * 300 = 3)$.

Among the returns given above, the third-worst return is -6.2%. As such,

$$\text{VaR}_{(99\%)} = 6.2\% \times 955 \times 250 = 14,802.50$$

A is incorrect. This answer incorrectly uses the fourth-worst observation as the 99% return among 300 observations.

B is incorrect. This answer incorrectly uses the second-worst observation as the 99% return among 300 observations.

C is incorrect. This answer incorrectly uses the worst observation as the 99% return among 300 observations.

Question 2

Bank X and Bank Y are two competing investment banks that are calculating the 1-day 99% VaR for an at-the-money call on a non-dividend-paying stock with the following information:

- Current stock price: USD 100
- Estimated annual stock return volatility: 20%
- Current Black-Scholes-Merton option value: USD 4.80
- Option delta: 0.7

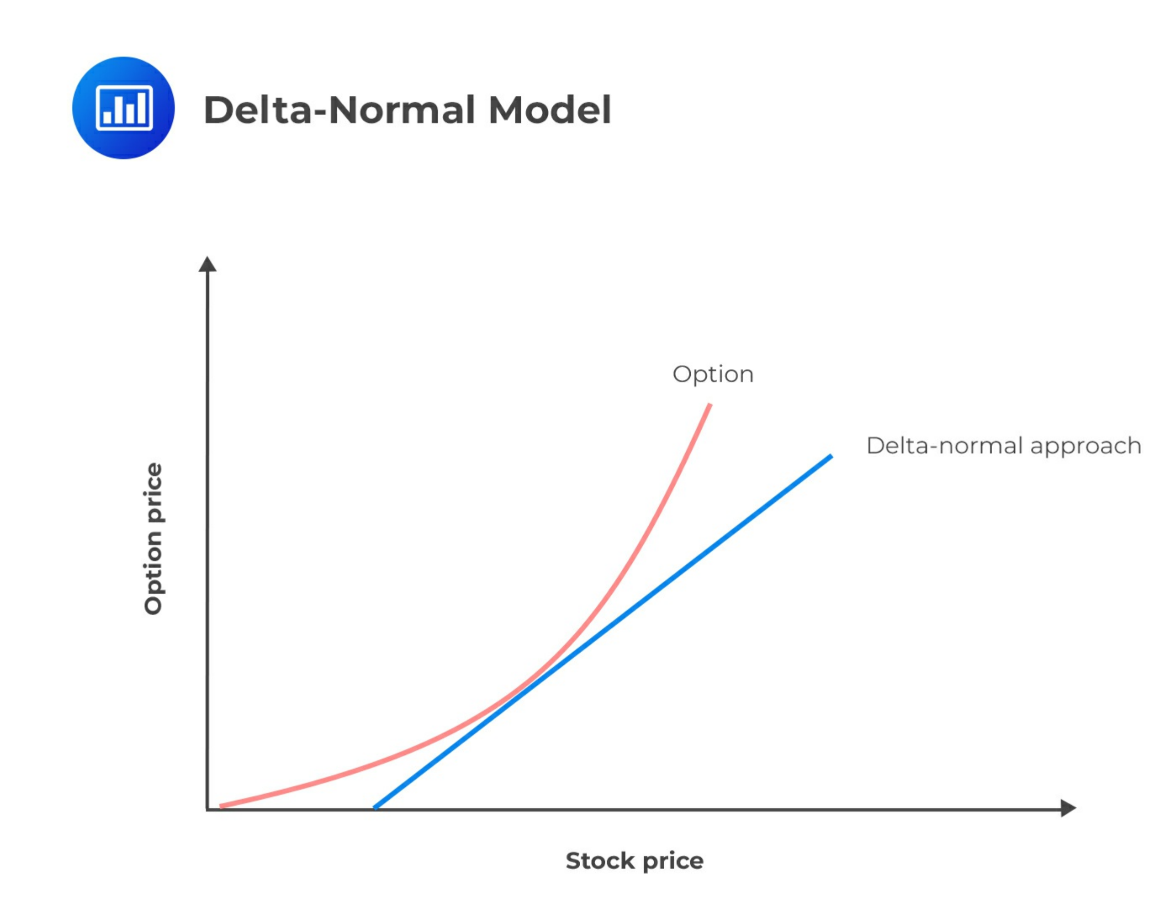
To compute VaR, Bank X uses the linear approximation method, while Bank Y uses a Monte Carlo simulation method for full revaluation. Which bank will estimate a higher value for the 1-day 99% VaR?

A. Bank X
B. Bank Y
C. Both will have the same VaR estimate
D. Insufficient information to determine

The correct answer is **A**.

The option's return function is convex with respect to the value of the underlying; therefore the linear approximation method will always underestimate the true value of the option for any potential change in price. As such, the VaR will always be higher under the linear approximation method than a full revaluation conducted by Monte Carlo simulation analysis. The difference is the bias resulting from the linear approximation, and this bias increases in size with the change in the option price and with the holding period.

As a quick summary, linear approximation **underestimates** true price of the option (see the graph below), and as such, **overestimates** the value-at-risk.



<u>**Reading 47: Measuring and Monitoring Volatility**</u>

After completing this reading, you should be able to:

- Explain how asset return distributions tend to deviate from the normal distribution.
- Explain reasons for fat tails in a return distribution and describe their implications.
- Distinguish between conditional and unconditional distributions, and describe the implications of regime-switching on quantifying volatility.
- Compare and contrast different approaches for estimating conditional volatility.
- Apply the exponentially weighted moving average (EWMA) approach and the GARCH (1,1) model to estimate volatility, and describe alternative approaches to weighting historical return data.
- Apply the GARCH (1,1) model to estimate volatility.
- Explain and apply approaches to estimate long-horizon volatility/VaR and describe the process of mean reversion according to a GARCH (1,1) model.
- Evaluate implied volatility as a predictor of future volatility and its shortcomings.
- Describe an example of updating correlation estimates.

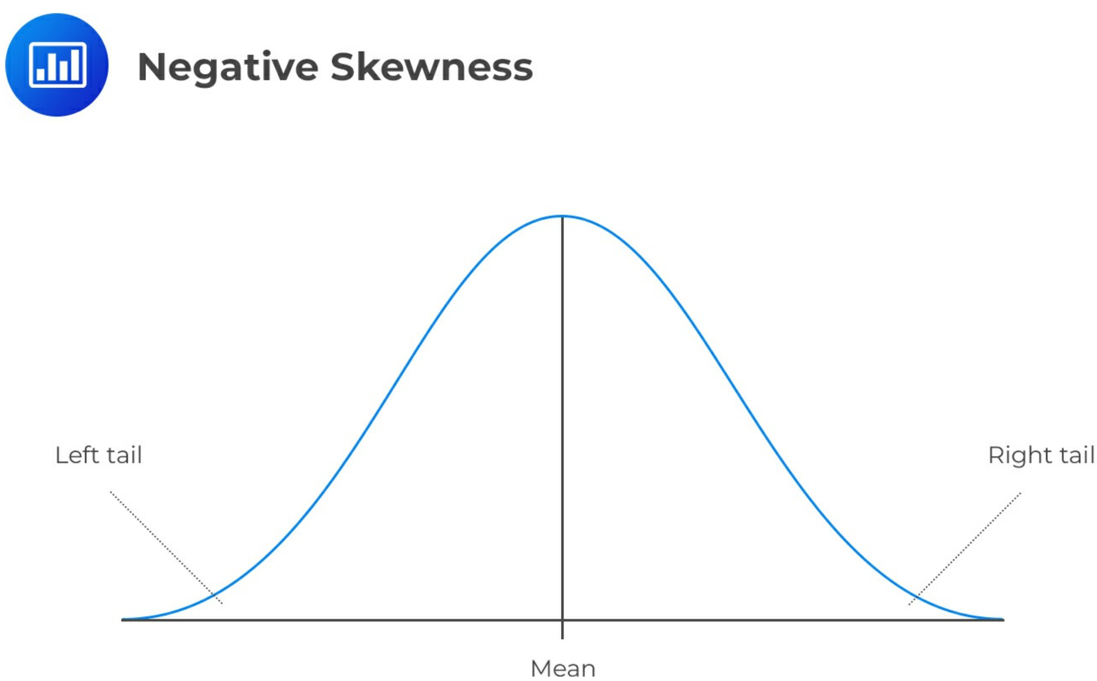
Constant volatility is easily approximated from historical data. However, volatility varies through time. Therefore, an alternative to constant normality of asset returns is to assume that asset returns are normally distributed conditioned on a known volatility. More specifically, high volatility indicates that the daily asset return is normally distributed with high standard deviation, and when the volatility is low, the daily returns are normally distributed with low standard deviation.

Monitoring volatility is made possible through two methodologies (as will be covered later in this chapter): the exponentially weighted moving average (EMWA) model and he GARCH (1,1) model. We will also apply the estimation of the volatility and application of volatility monitoring to correlation.

Deviation of Asset Returns from Normal Distribution

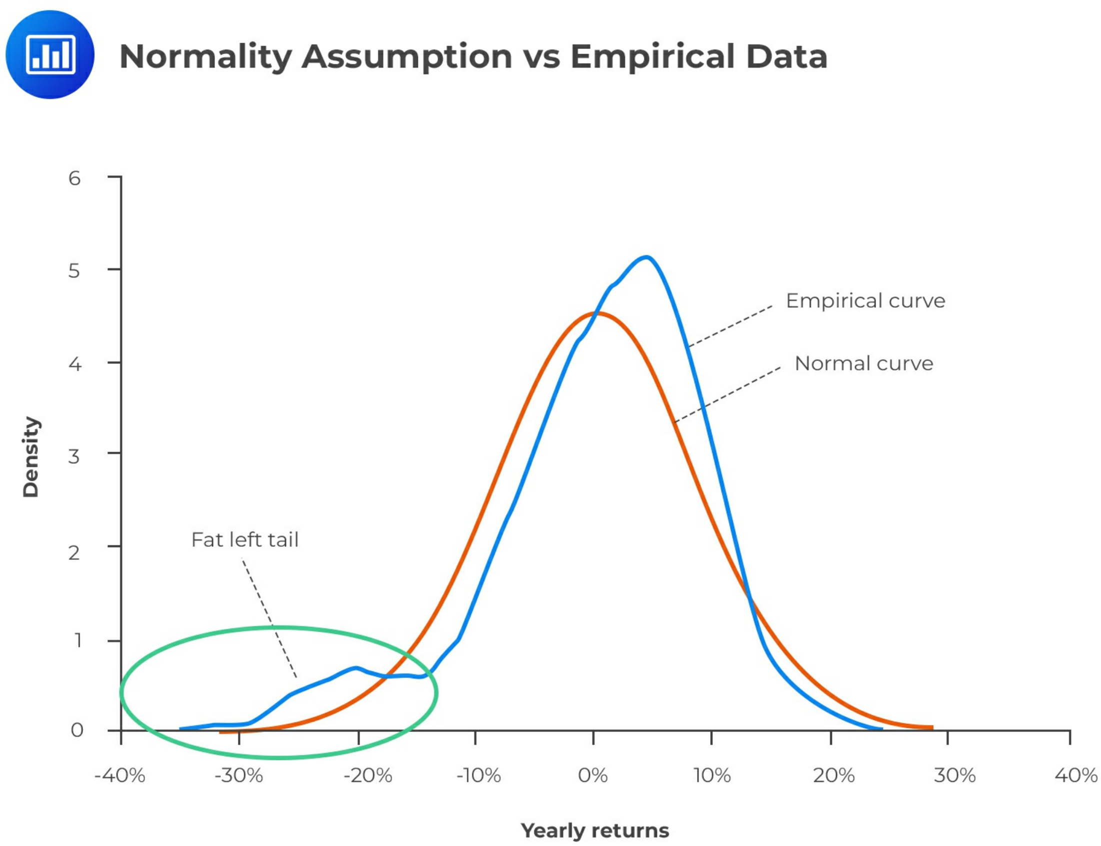
There are several reasons as to why the normality framework has been adopted for most risk estimation attempts. For starters, the normal distribution is relatively easy to implement. We need two parameters – the mean, μ, and the standard deviation, σ, of returns. With these two, it is possible to characterize the distribution fully and even come up with Value at Risk measures at specified levels of confidence. In the real world, however, asset returns are not normally distributed, and this can be observed empirically. Asset return distributions exhibit several attributes of non-normality. These are:

1. **Fat Tails (Negative Skewness):** A defining characteristic of the normal distribution is that most data points are concentrated around the center (mean), with very few points at the tails. The distribution is symmetrical (has equal and opposite halves) with outliers that consistently diminish, resulting in smooth, narrow tails.



2. **Instability of Parameter Values:** Unlike the normal distribution, asset return distributions have unstable parameters (mean and standard deviation). This tendency arises from market conditions that are continually changing. This instability is evident in asset volatility where asset returns turn out to be way more volatile than predictions using the normal distribution would suggest.

3. **The Asset Returns can be Non-symmetrical:** As seen in the figure below, the equity returns lean further to the right than the normal distribution. This phenomenon is known as negative skewness. A direct consequence of negative skewness is that the left slope of equity returns is longer than the left slope of the normal distribution, indicating a greater magnitude of extreme negative events.



Fat Tails and their Implications

For this reading, it's essential to keep in mind that the term 'fat tails' is relative: it refers to the tails of one distribution relative to the normal distribution. If a distribution has fatter tails relative to the normal distribution, it has a similar mean and variance, but probabilities at the extremes are significantly different.

When modeling asset returns, analysts focus on extreme events – those that have a low probability of occurrence but resulting in catastrophic losses. There's no point concentrating on non-extreme events that would reasonably not be expected to cause severe losses.

If we were to assume that the normal distribution holds as far as asset returns are concerned, we would not expect not even a single daily move of 4 standard deviations or more per year. The normal distribution predicts that about 99.7% of outcomes lie within 3 standard deviations from the mean.

In reality, every financial market experience one or more daily price moves of 4 standard deviations or more every year. There's at least one market that experiences a daily price move of 10 or more standard deviations per year. What then does this suggest?

The overriding argument is that the true distribution of asset returns has fatter tails.

Implications of Fatter Tails

There is a higher probability of extreme events than the normal distribution would suggest. This means that reliance on normality results in inaccurately low estimates of VaR. As a result, firms tend to be unprepared for such tumultuous events and are heavily affected, sometimes leading to foreclosure and/or large-scale ripple effects that reverberate and destabilize the entire financial system.

Unconditional and Conditional Normality

Unconditional normal distribution manifests when the mean and standard deviation of asset returns in a model is the same for any given day, regardless of market and economic conditions. In other words, even if there is information about the distribution of asset returns that suggests the existence of different parameters, we ignore that information and assume that the same distribution exists on any given day.

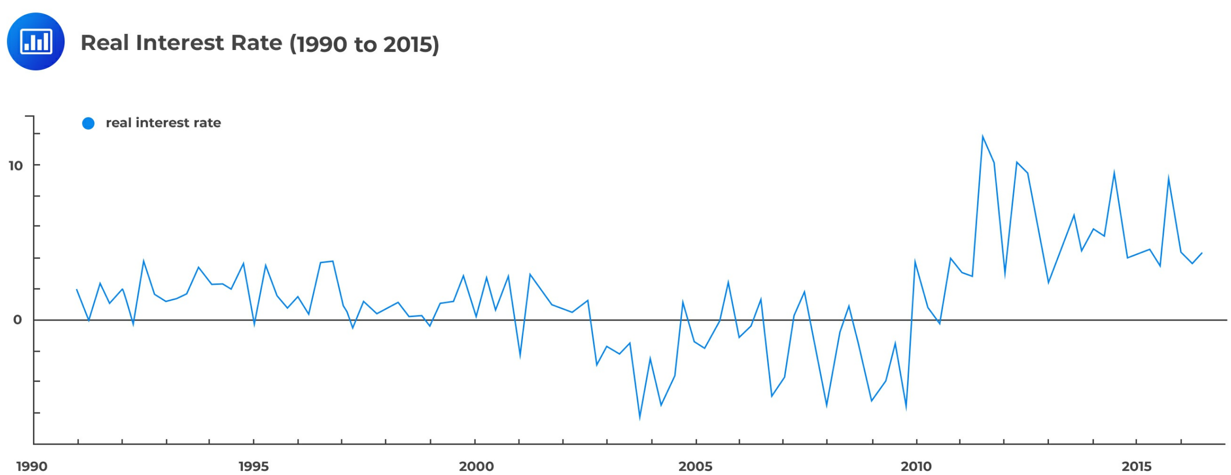
Conditionally normal distribution occurs when a model contains asset returns that are normal every day, but the standard deviation of the returns varies over time. That is the standard deviation high at one in times and low at some other time.

The underpinning idea here is that if we collect daily return data, we will observe the unconditional distribution and that by monitoring volatility, we can estimate the conditional distribution for daily return. For instance, consider that from the (historical) data we have collected, we estimate volatility to be 0.5% per day, but from volatility monitoring, we estimate the volatility to be 1.5% per day. Therefore, it will be more accurate to assume that asset returns are normally distributed with a 1.5% standard deviation and thus use this 1.5% measure to calculate VaR and the Expected Shortfall (ES). In other words, results from monitoring volatility are better than using the results from the fat-tailed distribution or by assuming a normal distribution with volatility from the observed data.

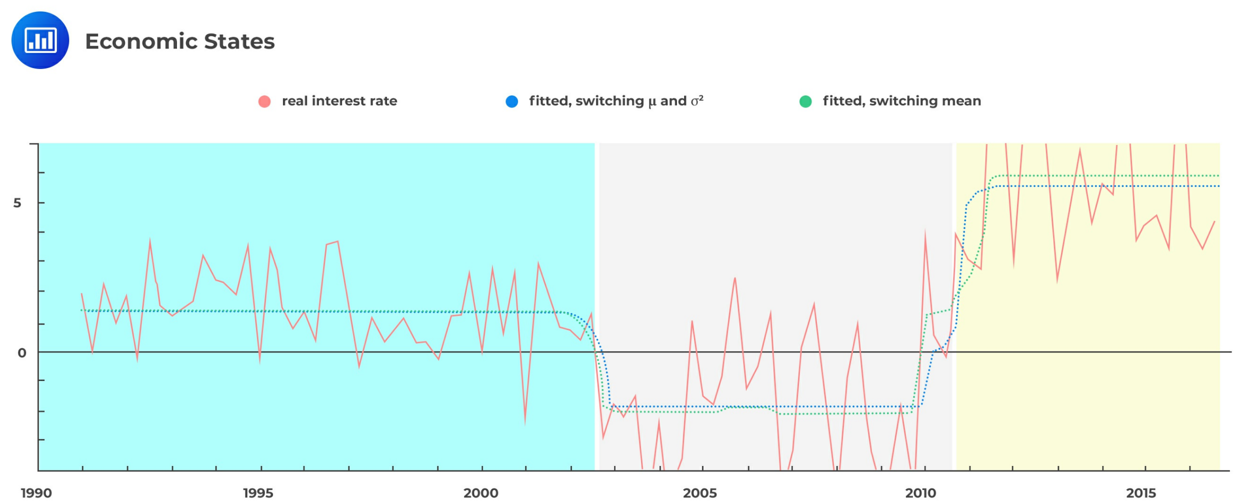
The Implications of Regime Switching on Quantifying Volatility

In an attempt to model asset returns better, analysts may subdivide a specified time period into regimes, where each regime has a clearly noticeable set of parameters that are markedly different from those of other regimes. For example, volatility could rise sharply during a market crash only to stabilize when conditions improve. This is the idea behind the **regime-switching volatility model.**

For illustration, let us use real interest rates of a developed nation between 1990 and 2015:



Given the graph above, it would be difficult to identify different states of the economy. Now, suppose we make use of an econometric model to try and identify the different economic states.



From the econometric model above, we can identify three distinct states of the economy. These are precisely what we would call regimes. As we switch from one regime to another, at least one parameter has to change. So, how exactly does the regime-switching model help to better measure volatility?

The conditional distribution of returns is always normal with either low or high volatility but a constant mean. The regime-switching model captures the conditional normality and, in so doing, helps resolve the fat tails problem.

Measuring Volatility

Conventionally volatility is defined as a change of a variable value over a period of time. In the context risk management, volatility is defined as the standard deviation of an asset return in one day.

The assets return in a day *i* is defined as:

$$r_i = \frac{S_i - S_{i-1}}{S_{i-1}}$$

Where

r_i: return on a day i.

S_i: value of an asset on a close of trading i.

S_{i-1}: value of an asset on a close of trading i-1.

For a day n, the variance (σ_n^2) from the previous m days is given by

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(r_{n-i} - \bar{r})^2$$

And thus the standard deviation is given by:

$$\sigma_n = \sqrt{\left(\frac{1}{m-1}\sum_{i=1}^{m}(r_{n-i} - \bar{r})^2\right)}$$

Where:

$\bar{r}$: average return from the previous m-days defined as:

$$\bar{r} = \frac{1}{m}\sum_{i=1}^{m} r_{n-i}$$

The risk management simplifies the above formulas by:

- **Substituting m-1 by m:** This is justified in that, by using m-1, we obtain an unbiased estimate of volatility, but when we use m, we get the maximum likelihood estimator, which is most likely given any data.
- **Assuming that** $\bar{r} = 0$**:** For a short period data, the standard deviation is far more significant than the mean return, and we need expected return and not the historical mean.

Therefore, defining the variables as before, the formula for the standard deviation changes to:

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} r_{n-i}^2$$

And thus volatility is given by:

$$\sigma_n = \sqrt{\left(\frac{1}{m}\sum_{i=1}^{m} r_{n-i}^2\right)}$$

The square of the volatility is defined as the **variance rate,** which is typically the mean of squared returns.

Example: Calculating the volatility of asset returns

The asset returns over five days are 10%, -5%. 6%, -3%, and 12%. What is the volatility of the asset returns?

Solution

The volatility of the asset returns is given by:

$$\sigma_n = \sqrt{\left(\frac{1}{m}\sum_{i=1}^{m} r_{n-i}^2\right)}$$

Now,

$$\sum_{i=1}^{5} r_{n-i}^2 = 0.1^2 + (-0.05)^2 + 0.06^2 + (-0.03)^2 + 0.12^2 = 0.0314$$

$$\Rightarrow \sigma_n = \sqrt{\frac{1}{5}(0.0314)} = 0.07925 = 7.925\%$$

An alternative method of calculating volatility is to use absolute returns rather than squared returns when calculating volatility. This method is suitable when dealing with non-normal data because it provides an appropriate prediction for fat-tailed distribution. However, the commonly used method uses squared returns, which we have considered and continue to do.

Approximating the Current Volatility

For the effective use of the conditional normal model, it is essential to estimate current volatility.

Current volatility can be estimated using the formula

$$\sigma_n = \sqrt{\left(\frac{1}{m}\sum_{i=1}^{m} r_{n-i}^2\right)}$$

For large values of ?, it would not capture current volatility due to variations of the volatility over the period when data was collected. We can argue that the small data sample might be relatively reliable, but the resulting estimate may not be accurate due to small data.

Alternatively, we can incorporate the effect of the standard error of the estimate. Recall that the standard error of an estimate is the difference between the volatility estimate and the actual value. The standard error is estimated volatility divided by 2(*m*-1) where *m* is the number of observations. That is:

$$S.E.E_{\sigma_n} = \frac{1}{2(m-1)}\sqrt{\left(\frac{1}{m}\sum_{i=1}^{m} r_{n-i}^2\right)}$$

For instance in our example above,

$$S.E.E_{\sigma_n} = \frac{1}{2(4)}(0.07925) \approx 0.01$$

We can then improve the accuracy of our volatility estimate by computing the confidence intervals. Typically, confidence intervals are usually two standard errors from the estimated value. In our case, the confidence interval is 5.925% to 9.925% (=0.07925±2×0.01) per day. We can reduce the confidence interval's width by increasing the size of the observation, but we will still face the disadvantage of the variability of the asset return volatility.

To solve this problem, we use a technique called exponential smoothing, also called an exponentially weighted moving average (EWMA) used by RiskMetrics to estimate volatilities for a wide range of market variables. Also, we use GARCH (1,1) as an exponential smoothing technique.

Exponentially Weighted Moving Average (EWMA)

Recall that from the formula $\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} r_{n-i}^2$, equal weight ($\frac{1}{m}$) is applied to the squared returns. However, in EWMA, the weights given to the squared returns are not equal and must sum up to 1.

Weight in EWMA is defined as weight applied to squared return from the k days ago denoted by λ multiplied by weight applied to squared return from k-1 days ago. λ is a constant positive and must be less than 1.

For example, denote the weight from the recent squared return, that is, squared return from day n-1 by w_0. Therefore, the weight on day n-2 is λw_0, and for the squared return for day n-3 is $\lambda^2 w_0$ and so on. Consider the following table for easy understanding.

Day	Squared Return	Weight
n-1	r_{n-1}^2	w_0
n-2	r_{n-2}^2	λw_0
n-3	r_{n-3}^2	$\lambda^2 w_0$
$\cdots$	$\cdots$	$\cdots$

The suitable value of λ is the one that leads to an estimate that gives the lowest error. Now assume that we have K days of data. Then, the sum of the weight applied is given by:

$$\begin{aligned} & w_0 + w_0\lambda + w_0\lambda^2 + \cdots + w_0\lambda^{k-1} \\ = & w_0(1 + \lambda + \lambda^2 + \cdots + \lambda^{k-1}) \end{aligned}$$

Practically, EWMA can be applied to data spanning one to two years, but data from a long time ago carries less weight; thus, we can assume that data that spans to infinite past is the result of the total weight :

$$w_0 + w_0\lambda + w_0\lambda^2 + w_0\lambda^3 + \cdots$$

The above expression can be compressed to:

$$w_0 \sum_{i=0}^{\infty} \lambda^k$$

The summation above in an infinite series so that:

$$\sum_{i=0}^{\infty} \lambda^k = \frac{1}{1-\lambda}$$
$$\Rightarrow w_0 \sum_{i=0}^{\infty} \lambda^k = w_0 \left(\frac{1}{1-\lambda}\right)$$

However, as stated earlier, the sum of weights in EWMA must be one. Therefore,

$$w_0 \left(\frac{1}{1-\lambda}\right) = 1$$
$$\therefore w_0 = 1 - \lambda$$

By definition of the EWMA model, the estimated volatility on day n is computed by applying the weights to past squared returns. More specifically,

$$\sigma_n^2 = w_0 r_{n-1}^2 + w_0 \lambda r_{n-2}^2 + w_0 \lambda^2 r_{n-3}^2 + \cdots \qquad \text{(Eq1)}$$

Intuitively, the estimated volatility on day n-1 is given by:

$$\sigma_{n-1}^2 = w_0 r_{n-2}^2 + w_0 \lambda r_{n-3}^2 + w_0 \lambda^2 r_{n-4}^2 + \cdots \qquad \text{(Eq2)}$$

If we substitute Eq2 in Eq1 we get:

$$\sigma_n^2 = w_0 r_{n-1}^2 + \lambda \sigma_{n-1}^2$$

But, we know that, w_0=1-λ. Therefore,

$$\sigma_n^2 = (1-\lambda) r_{n-1}^2 + \lambda \sigma_{n-1}^2$$

The last formula results from the EWMA model, which gives the estimate of the variance rate on day n is the weighted average of the estimated variance rate for the previous day (n-1) and the most recent (n-1) observation of the squared return. The equation

$$\sigma_n^2 = (1-\lambda) r_{n-1}^2 + \lambda \sigma_{n-1}^2$$

is termed as adaptive volatility because it incorporates prior information about the volatility into the new information. One merit of the EWMA model is that less data is required once the EWMA model has been built for a particular market variable. Only recent volatility estimates need to be remembered. Moreover, no return history is required.

Example: Calculating volatility using the EWMA model

Assume that you estimate recent volatility to be 3%, with a corresponding return of 2%. Given that λ=0.84, what is the value of new volatility?

Solution

Based on the EWMA model, the variable rate is given by:

$$\begin{aligned}\sigma_n^2 &= (1-\lambda)\, r_{n-1}^2 + \lambda\sigma_{n-1}^2\\ &= (1-0.84)\times 0.02^2 + 0.84\times 0.03^2\\ &= 0.00082\\ \Rightarrow \sigma_n &= \sqrt{0.00082} = 0.02864 = 2.864\%\end{aligned}$$

Determining the Value of λ

RiskMetrics determined the value of λ to be 0.94. This estimate, however, may not be appropriate for today's use, nor values higher than 0.94. Choosing a higher value of λ makes EWMA less responsive to new data. For instance, let λ=0.99. From the equation

$$\sigma_n^2 = (1-0.99)\, r_{n-1}^2 + 0.99\sigma_{n-1}^2$$

it implies that the new volatility is assigned a weight of 99%, and the new squared return is assigned a weight of 1%. Clearly, for a number of days of high volatility will not change the value of volatility significantly. Moreover, using a small value of λ, such 2% will make the volatility estimate to overreact to new information.

There are two methods of determining the value of λ. One of them is to compute realized volatility for a given day using 20 or 30 days of subsequent returns and then determine the value of λ that minimizes the difference between the estimated volatility and the realized volatility.

The second method is the maximum likelihood method, where the λ that maximizes the probability of observed data occurring is determined. For instance, if a trial leads to a value of λ that results in predicting low volatility for a particular day, but the observed data is large, then the estimated value of λ is the best estimate.

Historical Simulation

Similar to EWMA, exponentially declining weights are applied when using historical simulation. Exponentially declining weights are appropriate for determining VaR or expected shortfalls from recent preceding observations because scenarios from the immediate data are more relevant than those from a long time ago.

While weighting in historical simulation, the scenarios are arranged from the largest to the smallest loss, after which the weights are accumulated to determine the VaR.

Other Weighting Schemes

Recall that in the EWMA model, the weights were exponentially declining. An alternative methodology is multivariate density estimation (MDE). In MDE, an evaluation is carried out to determine which periods in the past bear the same feature as the current period, after which weights are assigned to the day's historical data depending on the level of similarity of that day to the present day.

For instance, the volatility of an interest rate varies depending on the level of the interest rates: decreases when the interest rates increases and vice versa. We can calculate the volatility of an interest rate by assigning the weights to interest data that is similar to the present interest rates and decreasing the weight as the difference between past and today's interest rate decreases.

Determining the level of similarity between one period and another is done using conditioning variables. For example, while determining the interest rate weights, we could use GDP growth rates as conditioning variables for the interest rate volatility. For conditional variables $X_1, X_2, \ldots, X_n$ the similarity between today and the previous period is determined by calculating the measure:

$$\sum_{i=1}^{n} a_i(\hat{X}_i - X_i^*)^2$$

Where

X_i^*: the value of X_i today;

$\hat{X}_i$: the historical value of X_i; and

a_i: a constant variable reflecting the importance of the ith variable.

The measure given by the formula becomes smaller as the similarity between current and historical periods increases.

The GARCH Model

The generalized autoregressive conditional heteroscedasticity (GARCH) model is an extension of the EWMA model, where we apply a weight to the recent variance rate estimate and the latest squared return. According to the GARCH(1,1) model, the updated model for the variance rate is given by:

$$\sigma_n^2 = \alpha r_{n-1}^2 + \beta\sigma_{n-1}^2 + \gamma V_L$$

Where:

V_L: long-run average variance rate;

α: weight given to the most recent squared returns;

β: weight given to the previous variance rate estimate; and

γ: weight given to long-run average variance rate.

The sum of the weights must be one so that:

$$\alpha + \beta \leq 1$$

And,

$$\gamma = 1 - \alpha - \beta$$

Note that α and β are positive.

It is easy to see that EWMA is a particular case of GARCH (1,1) where γ=0, α=1-λ, and β = λ. Both GARCH (1,1) and EWMA are called **first-order autoregressive (AR(1)) models** since the forecast for the variance rate depends on the immediately preceding variable.

Similar to the EWMA model, the weights in GARCH (1,1) decline exponentially. Now, by definition of the GARCH (1,1) model, we have:

$$\sigma_n^2 = \alpha r_{n-1}^2 + \beta\sigma_{n-1}^2 + \gamma V_L$$

And,

$$\begin{aligned}\sigma_{n\text{-}1}^2 &= \alpha r_{n-2}^2 + \beta\sigma_{n-2}^2 + \gamma V_L\\ \sigma_{n\text{-}2}^2 &= \alpha r_{n-3}^2 + \beta\sigma_{n-3}^2 + \gamma V_L\\ \sigma_{n\text{-}3}^2 &= \alpha r_{n-4}^2 + \beta\sigma_{n-4}^2 + \gamma V_L\end{aligned}$$

Starting from the equation defining GARCH (1,1), substitute $\sigma_{n\text{-}1}^2, \sigma_{n\text{-}2}^2, \sigma_{n\text{-}3}^2, \ldots$, we define:

$$\begin{aligned} w &= \gamma V_L\\ \Rightarrow V_L &= \frac{\omega}{\gamma}\end{aligned}$$

So,

$$\sigma_n^2 = \omega + \alpha r_{n-1}^2 + \beta r_{n-1}^2$$

Now since $\alpha + \beta + \gamma = 1$ and $V_L = \frac{\omega}{\gamma}$, then

$$V_L = \frac{\omega}{1 - \alpha - \beta}$$

Example: Calculating implied volatility using GARCH(1,1)

Consider a GARCH (1,1) model where ω=0.00005, α=0.15, and β=0.75. Given that the current volatility is 3% and the new return is -5%, (a) what is the long-run variance rate?

Solution

We know that:

$$\begin{aligned}\alpha + \beta + \gamma &= 1\\ \Rightarrow \gamma &= 1 - \alpha - \beta\\ &= 1 - 0.15 - 0.75 = 0.10\end{aligned}$$

Long-run average variance rate is given by:

$$V_L = \frac{\omega}{1 - \alpha - \beta} = \frac{0.00005}{0.10} = 0.0005$$

(b) Assuming the GARCH (1,1) model, what is the new volatility?

Solution

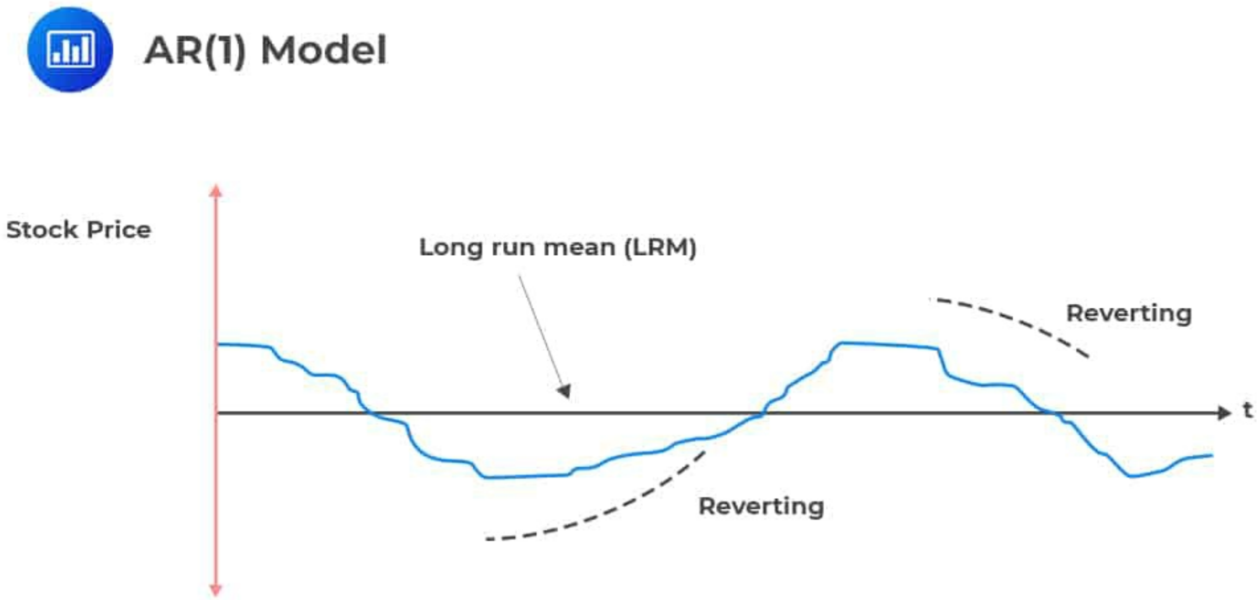
This corresponds to the volatility of 2.24% ($=\sqrt{0.0005}$). Now according to GARCH (1,1) model,

$$\begin{aligned}\sigma_n^2 &= \omega + \alpha r_{n-1}^2 + \beta\sigma_{n-1}^2\\ &= 0.00005 + 0.15 \times (-0.05)^2 + 0.75 \times 0.03^2 = 0.0011\end{aligned}$$

Therefore, the new volatility is 3.32% ($=\sqrt{0.0011}$).

Mean Reversion According to an AR(1) Model

Mean reversion is simply the assumption that a stock's price (or any other variable) will tend to move to its average over time.



The simplest form of mean reversion can be illustrated by a first-order autoregressive process [AR(1)], where the current value is based on the immediately preceding value.

$$X_{t+1} = a + bX_t + e_{t+1}$$

The expected value of X_t as a function of period t information is:

$$E[X_{t+1}] = a + bX_t$$

We can restate this as:

$$E[X_{t+1}] = (1-b) \times \frac{a}{1-b} + bX_t$$

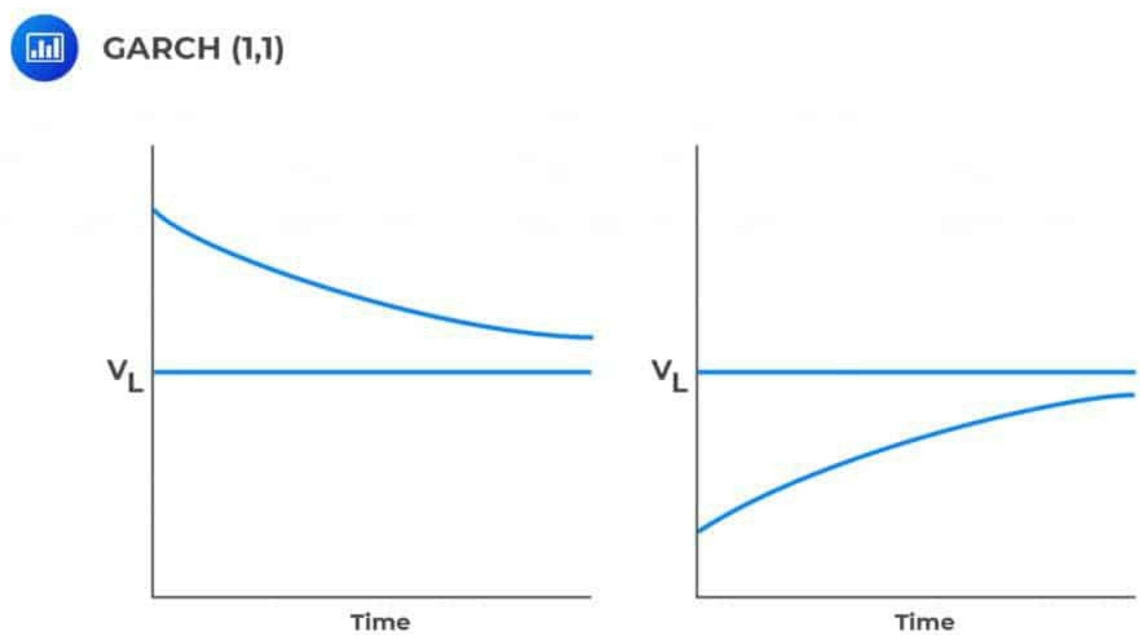
The long-run mean of this model is evaluated as $[\frac{a}{(1-b)}]$. The parameter of utmost interest in this long-run mean equation is b, often termed "the speed of reversion" parameter. There are two scenarios:

- If b = 1, the long-run mean is infinite (i.e., the process is a random walk, nonstationary process with an undefined long-run mean. That implies that the next period's expected value is equal to today's value.
- If b is less than 1, then the process is mean-reverting (i.e., the time series will trend

toward its long-run mean). That implies that when X_t is above the LRM, it will be expected to decline, and if it is below the long-run mean, it is expected to increase in value.

Mean Reversion According to GARCH (1,1)

In GARCH (1,1), the long-run variance rate (V_L) provides a "pull" mechanism towards the long-run average mean. Not that this is absent in the EWMA model due to a lack of the VL factor.



Consider the figure above. When the volatility is above the long-run average mean, V_L "pulls" down toward it. On the other hand, when the volatility is below the long-run mean, V_L pulls it towards the long-run mean. This tendency of the "pull" mechanism is termed as the **mean reversion**.

Therefore, GARCH (1,1) exhibits mean reversion, while EWMA does not. Mean reversion applies to market variables. More importantly, market variables that are traded should not exhibit predictable mean reversion to avoid arbitrage opportunities.

Notably, volatility cannot be traded, and thus it exhibits mean reversion. In this case, mean reversion implies that when the volatility is high, we do not expect to remain in that position forever but rather return to normal levels.

Long Horizon Volatility

Our discussion has been based on approximating one-day volatility. We might want to know what will happen over a longer period, such as one year. To achieve this, we assume that the variance rate over T days is equivalent to T days times variance over one day. That is,

$$\text{T-days variance rate} = \text{one-day variance rate} \times T$$

This means that volatility over days is equivalent to the volatility over one day multiplied by the square root of T days. That is

$$\text{T-days volatility} = \text{one-day volatility} \times \sqrt{T}$$

Recall that this was true also for the VaR.

Example: Calculating long horizon volatility

Assume that the daily variance rate is 0.000025. What is the 30-day volatility?

$$T-\text{day} = \text{one-day volatility} \times \sqrt{T}$$
$$= \sqrt{0.000025} \times \sqrt{30} = 0.0274 = 2.74\%$$

Implied Volatility

Implied volatility is an alternative measure of volatility that is constructed using the option valuation. The options (both put and call) have payouts that are non-linear functions of the price of the underlying asset. For instance, the payout from the put option is given by:

$$\max(K - P_T)$$

Where:

P_T is the price of the underlying asset,

K is the strike price, and

T is the maturity period.

Therefore, the price payout from an option is sensitive to the variance in the asset's return.

The Black-Scholes-Merton model is commonly used for option pricing valuation. The model relates the price of an option to the risk-free rate of interest, the current price of the underlying asset, the strike price, time to maturity, and the variance of return. For instance, the price of the call option can be denoted by:

$$C_t = f(r_f, T, P_t, \sigma^2)$$

Where:

r_f= Risk-free rate of interest

T=Time to maturity

P_t=Current price of the underlying asset

σ^2=Variance of the return

The implied volatility σ relates the price of an option with the other three parameters. The implied volatility is an annualized value but can be converted by dividing by 252, which is an estimated number of trading days in a year. For instance, if annual volatility is 30%, then the daily implied volatility is 1.89% ($= \frac{30\%}{\sqrt{252}}$).

The difference between the implied volatility and historical volatility (such as the one estimated by GARCH(1,1) and EWMA models) is that implied volatility is forward-looking while historical volatility is backward-looking. For example, a one-month implied volatility indicates average volatilities over the next month.

Options are not actively traded, and thus finding reliable volatility is an uphill task. However, risk managers evaluate both implied and historical volatility.

The volatility index (VIX) measures the volatility on the S&P 500 over the coming 30 calendar days. VIX is constructed from a variety of options with different strike prices. VIX applies to a large variety of assets such as gold, but it is only applicable to highly liquid derivative markets and thus not applicable to most financial assets.

Advantages and Disadvantages of Implied Volatility

There are two main advantages of implied volatility over historical volatility:

- It is a forward-looking, predictive measure that reflects the market's consensus; and
- Historical distribution patterns do not restrain it.

On the other hand, implied volatility has its shortcomings:

- It is model dependent;
- Options on the same underlying asset may trade at different implied volatilities. For example, deep out of the money and deep in the money options trade at higher volatility than at the money options;
- It has limited availability because it can only be deduced when there are current market prices; and
- It assumes volatility will remain constant over a period of time, but volatility will change over that same period.

Monitoring Correlation

Recall that in a delta-normal model, correlations between the daily asset returns are needed to compute the VaR or expected shortfall for a linear portfolio. Therefore, it is crucial to monitor correlations.

Updating the correlations is analogous to that of volatilities. Recall that while updating the volatilities, we use the variances. In the case of relationships, we use covariances. Now, assuming that mean of daily returns is zero, then the covariance is defined as the expectation of the product of returns.

According to the EWMA model, updating the correlation between returns X and Y is given by:

$$\text{cov}_n = \lambda \text{cov}_{n-1} + (1-\lambda)\, x_{n-1} y_{n-1}$$

Where:

cov_n: covariance for day n

x_{n-1} and y_{n-1}: values of X and Y at day n.

Recall that for variables X and Y,

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

Where:

Corr (X, Y): correlation between variables X and Y;

Cov (X, Y): covariance between variables X and Y;

σ_X: standard deviation of X; and

σ_Y: standard deviation of Y.

Therefore, if the EWMA model has been used to estimate the standard deviations of the returns, we can comfortably estimate the correlation coefficient. However, the same value of λ is used while updating both variances and covariances for consistency.

Example: Updating the correlation coefficient

Assume that recent volatilities for returns on X and Y are 2% and 5%, respectively. Their corresponding correlation coefficient is 0.4. Moreover, their recent returns are 3% and 4%, respectively.

Assuming λ=0.92, calculate the value of:

i. updated volatilities; and

ii. updated correlation coefficient.

Solution

i. Updated volatilities

Denote the "recent" period by n-1. Now using the formula:

$$\sigma_n^2 = (1 - \lambda) r_{n-1}^2 + \lambda \sigma_{n-1}^2$$

The updated volatility for return X is given by:

$$\sigma_{X_n} = \sqrt{(1 - 0.92) \times 0.03^2 + 0.92 \times 0.02^2} = 0.02098 = 2.098\%$$

And for return Y we have:

$$\sigma_{Y_n} = \sqrt{(1 - 0.92) \times 0.04^2 + 0.92 \times 0.05^2} = 0.04927 = 4.927\%$$

ii. Updated correlation coefficient

Using the formula:

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y}$$

It intuitively means that:

$$\begin{aligned} \text{cov}_{n-1} &= \sigma_{X_{n-1}} \times \sigma_{Y_{n-1}} \times \text{Corr}(X_{n-1}, Y_{n-1}) \\ &= 0.02 \times 0.05 \times 0.4 \\ &= 0.0004 \end{aligned}$$

Now using:

$$\text{cov}_n = \lambda \text{cov}_{n-1} + (1 - \lambda) X_{n-1} Y_{n-1}$$

we have:

$$\text{cov}_n = 0.92 \times 0.0004 + (1 - 0.92) \times 0.03 \times 0.04 = 0.000464$$

And thus, the updated correlation coefficient is given by:

$$\begin{aligned}\text{corr}(x_n, y_n) &= \frac{\text{cov}_n}{\sigma_{X_n}\sigma_{Y_n}} \\ &= \frac{0.000464}{0.02098 \times 0.04927} = 0.4489\end{aligned}$$

Practice Question

The current estimate of daily volatility is 2.3 percent. The closing price of an asset yesterday was CAD 46. Today, the asset ended the day at CAD 47.20. Using log-returns and the exponentially weighted moving average model with $\lambda = 0.94$, determine the updated estimate of volatility.

A. 2.319%

B. 0.0537%

C. 2.317%

D. 2.315%

The correct answer is **C**.

The updated variance is given by:

$$h_t = \lambda(\text{current volatility})^2 + (1-\lambda)(\text{current return})^2$$

Current volatility = 0.023

$$\text{Current log-return} = \ln 47.2 - \ln 46 = 3.85439 - 3.82864 = 0.02575$$
$$ht = 0.94(0.023^2) + (0.06 \times 0.02575^2) = 0.000537$$
$$\text{Updated estimate of volatility} = \sqrt{0.000537} = 0.02317$$

Reading 48: External and Internal Credit Ratings

After completing this reading you should be able to:

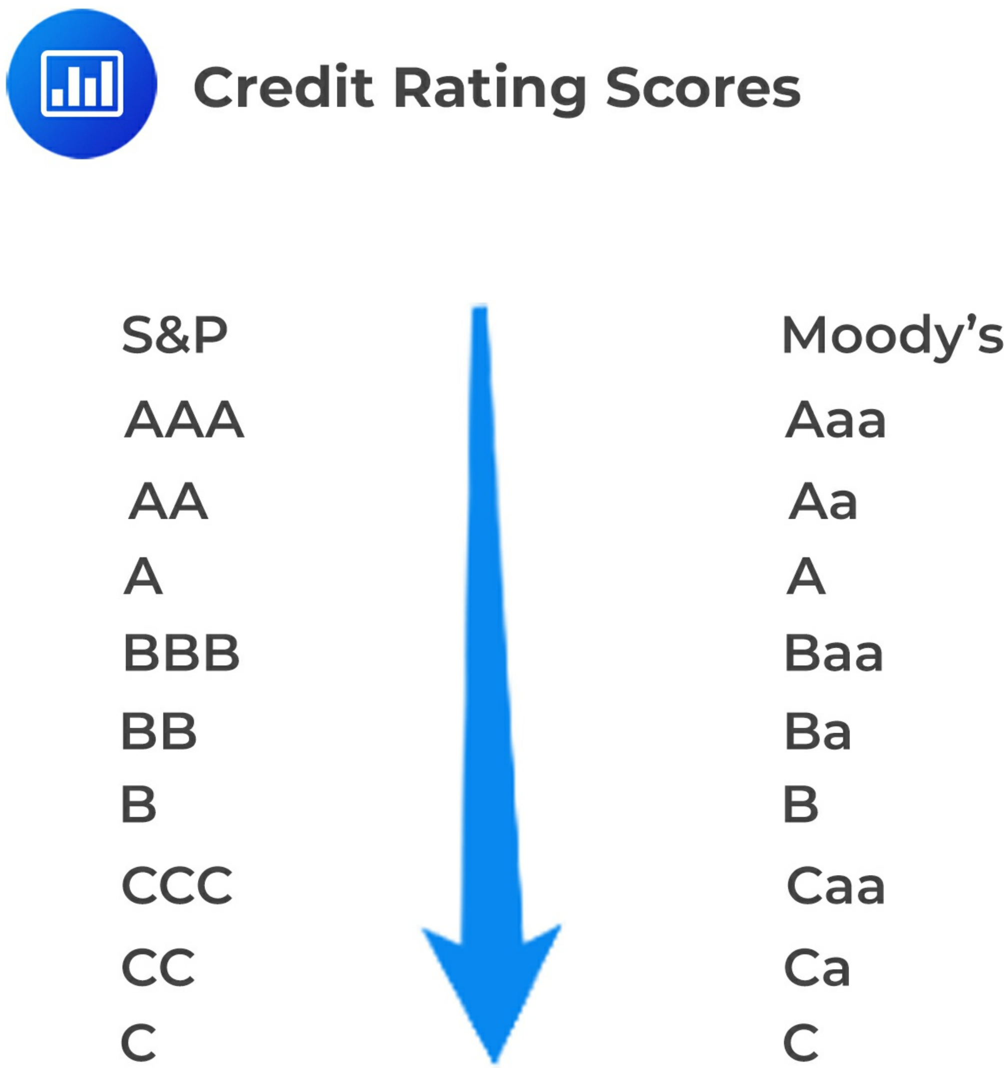
- Describe external rating scales, the rating process, and the link between ratings and default.
- Describe the impact of time horizon, economic cycle, industry, and geography on external ratings.
- Define and use the hazard rate to calculate the unconditional default probability of a credit asset.
- Define recovery rate and calculate the expected loss from a loan.
- Explain and compare the through-the-cycle and point-intime internal ratings approaches.
- Describe alternative methods to credit ratings produced by rating agencies.
- Compare external and internal ratings approaches.
- Describe and interpret a ratings transition matrix and explain its uses.
- Describe the relationships between changes in credit ratings and changes in stock prices, bond prices, and credit default swap spreads.
- Explain historical failures and potential challenges to the use of credit ratings in making investment decisions.

A Description of External Rating Scales and the Rating Process

An external rating scale is a scale used as an ordinal measure of risk. The highest grade on the scale represents the least risky investments, but as we move down the scale, the amount of risk gradually increases (safety decreases).

An **issue-specific credit rating** conveys information about a specific instrument, such as a zero-coupon bond issued by a corporate entity. An **issuer-specific credit rating,** on the other hand, conveys information about the entity behind an issue. The latter usually incorporates a lot more information about the issuer.

Here are S&P's and Moody's credit rating scores for long-term obligations:



The successive move down the scale represents an increase in risk. In the case of Moody's ratings, **Baa and above** are said to be **investment-grade** while those below this level are said to be **non-investment-grade.**

In the case of S&P's, ratings **BBB and above** are investment-grade. All the others are non-investment-grade.

The Rating Process

The process leading up to the issuance of a credit rating follows certain steps. These are:

I. A qualitative analysis of the company, including assessments of the quality of management and competitive aspects
II. A quantitative analysis of financials such as ratio analysis
III. A meeting with the firm's management
IV. A meeting of the rating agency committee assigned to rating the firm
V. A notification is sent to the rated firm detailing the assigned rating
VI. A fee is paid to the rating agency.
VII. The rated firm has a window to appeal the assigned rating or offer new information
VIII. The assigned rating is published

Outlooks and Watchlists

Apart from the ratings themselves, the rating agencies also provide outlooks which shows the changes likely to be experienced over the medium term.

- A positive outlook indicates that a rating is likely to be raised.
- A negative outlook indicates that a rating is likely to be lowered.
- A stable outlook shows that the rating is stationary.
- A developing outlook is an evolving one in which we can't tell the direction of the change.

When a rating is placed on a watchlist, it shows that a very small short-term change is expected.

Rating Stability

Rating stability is necessary since ratings are majorly used by bond traders. If the ratings were to change, then the bond traders are required to trade more frequently and, in this case, they are likely to incur a lot of transaction costs.

Rating stability is important because ratings are also used in financial contracts, and if the ratings vary for different bonds, it would be difficult to administer the underlying contracts.

The Impact of Time Horizon, Economic Cycle, Industry, and Geography on External Ratings

Time Horizon

The probability of default given any rating at the beginning of a cycle increases with the time horizon. Non-investment bonds are the worst hit. Their default probabilities can dramatically increase within a short time.

Economic Cycle

Since ratings are generally produced with an eye on a long-term period, they must take into account any economic/industrial cycle on the horizon. Rating agencies make efforts to incorporate the effects associated with an economic cycle in their ratings. Although this practice is generally valid, it can lead to underestimation or overestimation of default if the predicted economic cycle doesn't play out exactly as expected. Put precisely, the probability of default can be underestimated if an economic recession occurs, or overestimated if an economic boom occurs. In addition, the default rate of lower-grade bonds is correlated with the economic cycle, while the default rate of high-grade bonds is fairly stable.

Industry and Geographic Consistency

Two firms in different industries – say, banking and manufacturing – could have the same rating, but the probability of default may be higher for one of the firms than for the other. What does that mean? The implication here is that for a given rating category, default rates can vary from industry to industry. However, there's little evidence to support the notion that geographic location has a similar effect.

Hazard Rate

Consider a firm defaulting in a very short time, that is, δt.

The task is to answer the question, "What is the conditional probability of a firm defaulting between time t and time $t + \delta t$ given that there is no default before time t?"

We can denote this by $h\delta t$, where h is the rate at which defaults are happening at time t .

Unconditional default probabilities can be calculated using the hazard rates.

Suppose that $\bar{h}$ is the average hazard rate between time 0 and time t.

Then, the unconditional probability between time 0 and t is

$$1 - \exp(-\bar{h}t)$$

and the survival probability to time t is therefore given by

$$\exp(-\bar{h}t)$$

and the unconditional probability between time t_1 and t_2 is given by the expression;

$$\exp(-\bar{h}_1 t_1) - \exp(-\bar{h}_2 t_2)$$

Example: Calculating Default Probabilities Given Hazard Rates

Suppose you have been given a constant hazard of 0.05,

Calculate:

a. The probability of default by the end of 2 years.
b. The unconditional probability of defaulting during the 3rd year.
c. The conditional probability of defaulting in the 3rd year, given that it has survived until the end of the second year.

Solution

a. The probability of default at the end of the 2nd year is given by:

$$1 - \exp(-ht)$$
$$= 1 - \exp(-0.05 \times 2) = 0.09516$$

b. The unconditional probability of default during the 3rd year.

$$\exp(-0.05 \times 2) - \exp(-0.05 \times 3) = 0.04413$$

c. The conditional probability of defaulting in the 3rd year, given that it has survived until the end of the second year is given by:

$$\frac{\text{Unconditional probability of a default occurring during the third year}}{\text{Probability of surviving to the end of the second year}}$$
$$= \frac{0.04413}{1 - 0.09516} = 0.04877$$

Recovery Rates

In the event that a firm runs bankruptcy or defaults, it may pay part of the amount of the total loan to the lender. This amount that is repaid, expressed as a percentage, is known as the recovery rate.

Since the loan is not fully repaid, then we can calculate the expected loss from the loan over a given period of time as;

$$\text{Expected Loss} = \text{Probability of Default} \times \text{Loss Given Default}$$
$$\text{EL} = \text{PD} \times \text{LGD}$$

But since $\text{LGD} = 1 - \text{Recovery Rate}$

Then, the expected loss from a loan is also calculated as

$$\text{EL} = \text{PD} \times (1 - \text{Recovery Rate})$$

For example, if the recovery rate is 70%, then

$$\text{LGD} = 100\% - 70\% = 30\%.$$

Suppose the debt instrument has a notional value of $100 million, and that there is a 1% probability of default, then the expected loss when the loan defaults is $0.3 million.

Comparing the Through-the-Cycle and Point-in-Time Internal Ratings Approaches

Point-in-Time Internal Ratings

Point-in-time ratings, also called **at-the-point internal ratings**, evaluate the **current situation** of a customer by taking into account both cyclical and permanent effects. As such, they are known to react promptly to changes in the customer's current financial situation.

Point-in-time ratings**,** try to assess the customer's quantitative financial data (e.g. balance sheet information), qualitative factors (e.g. quality of management), and information about the state of the economic cycle. Using statistical procedures such as scoring models, all that information is transformed into rating categories.

Point-in-time ratings**, are only valid for the short-term or medium term**, and that's largely because they take into account cyclic information. They are usually valid for a period not exceeding one year.

Through-the-Cycle Internal Ratings

Through-the-cycle (ttc) internal ratings try to evaluate the permanent component of default risk. Unlike point-in-time ratings, they are said to be nearly independent of cyclical changes in the creditworthiness of the borrower. They are not affected by credit cycles, i.e. they are through-the-cycle. As a result, they are less volatile than at-the-point ratings and are valid for a much longer period (exceeding one year).

Advantages of ttc ratings include:

I. They are much more stable over time compared to at-the-point ratings
II. Because of their low volatility, ttc ratings help financial institutions to better manage

customers. Too many rating changes necessitate changes in the way a bank handles a customer, including the products the bank is ready to offer.

One of the **disadvantages** of ttc ratings over at-the-point ratings is that they can at times be too conservative if the stress scenarios used to develop the rating are frequently materially different from the firm's current condition. If the firm's current condition is worse than the stress scenarios simulated, then the ratings may be too optimistic. In fact, ttc ratings have very low default prediction in the short-term.

Alternative Methods to Credit Ratings Produced by Rating Agencies

Apart from the commonly known rating agencies, that is, Moody's, S&P, and Fitch, we have some organizations such as KMV and Kamakura which use some models to come up with default probabilities and hence can then use probabilities to provide important information to clients.

Factors considered include:

- The amount of debt the firm has in its capital structure.
- The market value of the firm's equity.
- The volatility of the firm's equity.

In the underlying model, a company defaults if the value of its debt exceeds the value of its assets.

Suppose v is the value of the asset and d is the value of the debt, the firm defaults when $v < d$.

The value of the equity, at a future point in time, is:

$$\text{Equity} = \max(v\text{-}d, 0)$$

This implies that equity in a company is a call option on the assets of the firm with a strike price equal to the face value of the debt. The firm defaults if the option is not exercised.

The estimates provided by KMV and Kamakura are point-in-time estimates which are only valid for the short/medium term.

Comparing External and Internal Ratings Approaches

External ratings are produced by independent rating agencies and aim at revealing the financial stability of both lenders and borrowers. For example, Moody's periodically releases ratings for big banks around the globe. Such ratings are important because banks usually rely on customer deposits and money raised through the issuance of various assets such as bonds to sustain lending. The funds raised this way to create a pool of money that is then loaned to borrowers in smaller chunks. Thus, depositors and bond owners use such ratings to assess the riskiness of giving their money to the bank.

Sometimes, however, banks also need their own ratings so as to undertake an independent assessment of the creditworthiness of a specific borrower – either an individual or a corporate. That's where **internal credit ratings** come in.

In modern times, internal credit ratings are usually developed based on the techniques used to develop external credit ratings. Such methodology consists of identifying the most meaningful financial ratios and risk factors. These ratios and factors are then assigned weights such that the final rating estimate is close to what a rating agency analyst would come up with. The same indicators are used, albeit with a few adjustments depending on whether the borrower is an individual or a corporate.

One way of carrying out an internal rating is by use of a statistical technique known as the Altman's Z-score. The following ratios need to be provided when using this technique:

i. X_1 :Working capital to total assets
ii. X_2 :Retained earnings to total assets
iii. X_3 :Earnings before interest and taxes to total assets
iv. X_4 :Market value of equity to book value of total liabilities
v. X_5 :Sales to total asset

Using the discriminant analysis, the Z-score is given by:

$$Z = 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5.$$

A Z-score above 3 means that the firm is not likely to default and when the Z-score is below 3, then the firm is likely to default.

Nowadays, machine learning algorithms use more than five input variables as compared to Altman's Z-score. Also, the functions used in machine learning algorithms can be non-linear.

Some of the factors that have contributed to the increased sophistication of modern internal credit ratings are:

I. The ever-growing use of external credit rating agency language in financial markets
II. Enforcement of capital requirements such as Basel II

Banks should also ensure that they back-test their procedures for calculating internal ratings. Back-testing requires atleast ten years of data. If the default statistics show that firms with higher ratings have performed better than those with low ratings, a bank can then have some confidence in its rating methodology.

Internal ratings have two main uses:

I. Assessing the creditworthiness of a customer during the loan application process
II. To determine the value of inputs used in the modeling of capital required as per the existing regulations, e.g. Basel II

For these reasons, internal ratings have to be calibrated. This involves establishing a link between the internal rating scale and tables displaying the cumulative probabilities of default. The timeline of such tables must capture all maturities, from, say, 1 year to 30 years. Sometimes, it may be necessary to build different transition matrices that are specific to the asset classes owned by the bank.

Ratings Transition Matrices and Their Uses

A rating transition matrix gives the probability of a firm ending up in a certain rating category at some point in the future, given a specific starting point. The matrix, which is basically a table, uses historical data to show exactly how bonds that begin, say, a 5-year period with an Aa rating, change their rating status from one year to the next. Most matrices show one-year transition probabilities.

Transition matrices demonstrate that the higher the credit rating, the lower the probability of default.

The table below presents an example of a rating transition matrix according to S&P's rating categories:

One-year transition matrix

Initial		Rating	at	year	end			
Rating	AAA	AA	A	BBB	BB	B	CCC	Default
AAA	90.81%	8.33%	0.68%	0.06%	0.12%	0.00%	0.00%	0.00%
AA	0.70%	90.65%	7.79%	0.64%	0.06%	0.14%	0.02%	0.00%
A	0.09%	2.27%	91.05%	5.52%	0.74%	0.26%	0.01%	0.06%
BBB	0.02%	0.33%	5.95%	86.93%	5.30%	1.17%	0.12%	0.18%
BB	0.03%	0.14%	0.67%	7.73%	80.53%	8.84%	1.00%	1.06%
B	0.00%	0.11%	0.24%	0.43%	6.48%	83.46%	4.07%	5.20%
CCC	0.22%	0.00%	0.22%	1.30%	2.38%	11.24%	64.86%	19.79%

Exam tips:

- Each row corresponds to an initial rating
- Each column corresponds to a rating at the end of 1 year. For example, a bond initially rated BB has an 8.84% chance of moving to a B rating by the end of the year.
- The sum of the probabilities of all possible destinations, given an initial rating, is equal to 1 (100%)
- You will need to recall the rules of probability from mathematics to come up with n-year transition probabilities, where $n>1$.
- Credit ratings are more stable over a one-year horizon. Stability decreases with longer horizons.

The Impact of Rating Changes on Bond Price, Stock Prices

and Credit Default Swap Spreads

Bonds

There's overwhelming evidence that a rating downgrade triggers a decrease in bond prices. In fact, bond prices sometimes decrease just because there's a strong possibility of a downgrade. Anxious investors tend to sell bonds whose credit quality is declining.

A rating upgrade triggers an increase in bond prices, although there's relatively less market evidence to support this conclusion.

Therefore, the underperformance of bonds whose credit quality has been downgraded is more statistically significant compared to the over-performance of bonds recently upgraded.

Stocks

There's moderate evidence to support the view that a rating downgrade will lead to a stock price decrease. A ratings upgrade, on the other hand, is somewhat likely to trigger an increase in stock prices.

In practice, the relationship between changes in rating and stock prices can be quite complex and will usually be heavily impacted by the reason behind the changes. Furthermore, downgrades tend to have more impact on the stock price compared to upgrades.

Credit Default Swap Spreads

The impact of rating changes on credit default swap spreads has been examined based on outlooks, watchlists, and rating changes. It has been concluded that according to watchlists, reviews for downgrades contain significant information, but this is not the case for downgrades and negative outlooks. On the other hand, positive rating events proved to be much less significant.

In general terms, credit default swap changes seem to anticipate rating changes. The research findings show that credit spread changes provide vital information in estimating the probability of negative credit rating changes.

Historical Failures and Potential Challenges to the Use of Credit Ratings in Making Investment Decisions

During the run-up to the 2007-2008 crisis, rating agencies became much more involved in the rating of structured products created from portfolios of subprime mortgages.

Rating of structured products relied much more on the model in use. The three common models used were S&P, Fitch, and Moody's. S&P and Fitch based their ratings on the probability that the structured product would give a loss. On the other hand, Moody's based its ratings on expected loss as a percent of the principal. However, the inputs to their models, i.e. the correlations between the defaults on different mortgages, seemed too optimistic. Furthermore, they developed their ratings of structured products from other structured products.

Creators of structured products came to understand the models used by rating agencies and hence they could create the structured products in a manner that they would achieve the ratings they desired. In case where the desired ratings were not achieved, these structured products could be adjusted until the desired ratings are achieved. Creators of structured products could also pay rating agencies to give structured products higher ratings. Even though the rating companies knew about the decline of the leading standards and rising fraud and that their independence was being interfered with, they did not pay attention to this since they found working on structured products to be more profitable.

What followed is that most of the structured products created from mortgages defaulted during the 2007-2008 crisis period. This ruined the reputation of the rating agencies. Currently, rating agencies are subject to more oversight than during the pre-crisis period. Furthermore, bank supervisors no longer use rating agencies to determine regulatory capital.

Questions

Question 1

You have been given the following one-year transition matrix:

Rating From	Rating To			
	A	B	CCC	Default
A	80%	10%	10%	0%
B	5%	85%	5%	5%
CCC	0%	10%	70%	20%

Determine the probability that a B –rated firm will default over a two-year period.

A. 5%

B. 4.25%

C. 1%

D. 10.25%

The correct answer is **D.**

Required probability = Sum of probabilities of all possible paths that could lead to a rating of D (default) after two years.

In other words, in how many ways can a B-rated firm default over a two-year period? The following are the possible paths:

Path	Probability
B→ default	0.05
B→ B → default	0.85 x 0.05= 0.0425
B→ CCC → default	0.05 x 0.20= 0.01
Total	0.1025

Question 2

ABC Co., currently rated BBB, has an outstanding bond trading in the market. Suppose the company is upgraded to A. What will be the most likely effect on the bond's price?

A. Positive and stronger than the negative effect triggered by a bond downgrade
B. Negative and stronger than the positive effect triggered by a bond downgrade
C. Positive and weaker than the negative effect triggered by a bond downgrade
D. Positive and as strong as the negative effect triggered by a bond downgrade

The correct answer is **C.**

Rating downgrades tend to have more impact on the stock price compared to upgrades. This can be explained by the fact that firms tend to release good news a lot more often than bad news, and thus the expectations among investors are generally positive. Negative news is usually unexpected and unanticipated, triggering a stronger downward effect.

Reading 49: Country Risk

After completing this reading you should be able to:

- Explain how a country's position in the economic growth life cycle, political risk, legal risk, and economic structure affect its risk exposure.
- Evaluate composite measures of risk that incorporate all major types of country risk.
- Compare instances of sovereign default in both foreign currency debt and local currency debt, and explain common causes of sovereign defaults.
- Describe the consequences of sovereign default.
- Describe factors that influence the level of sovereign default risk; explain and assess how rating agencies measure sovereign default risks.
- Describe characteristics of sovereign credit spreads and sovereign credit default swap (CDS) and compare the use of sovereign spreads to credit ratings.

Country risk could be attributed to the following:

- A country's position in the economic growth life cycle.
- Differences in political risk.
- The legal system.
- The country's economic structure.

A Country's Position in the Economic Growth Cycle

The value of all goods and services produced within a country's borders is known as GDP. When we look at the economic growth rate with regard to an increase in the price of goods and services, then this is called real GDP. The GDP, therefore, gives us a clear snapshot of the economic performance of a given country.

Countries in early growth are more exposed to risk than larger, more mature countries. For example, a global recession hits small, emerging markets harder than it does mature markets. For instance, while countries like Japan and the U.K. experienced a 1-2% dip in GDP following a recession, early growth economies like Kenya and Panama can record a dip as high as 5%.

Emerging markets are also hit hard in case of an economic shock. Even in the face of a robust legal framework and good governance, there's an upper cap on the powers that countries have over their risk exposure. Some risks may simply be unavoidable. This is why it's important to thoroughly analyze a country's risk profile before making critical investment decisions.

A country's competitive advantage depends on:

- Factors of production;
- Demand for the products of given industries;
- The availability of other related industries that produce high-quality products; and
- The structure and management of the industries.

Differences in Political Risk

The political environment in a country can have a major bearing on its risk exposure. This can particularly be explained in four main ways:

I. **Continuous vs. Discontinuous Risk:** Countries deeply rooted in democracy and free speech have **continuous risk** in the sense that rules and regulations governing are continuously being changed and amended. As such, regulations enacted for the long-term may remain in place only for a while before new laws and bills are introduced and signed into law. This means that an investor may have to contend with new capital requirements, for example, at a time when they are faced with a crippling cash crunch.

   In authoritarian states, on the other hand, dictatorial policies create **discontinuous risk**. That means it may be difficult to amend rules and regulations, however good or bad they might be, from the investor's perspective. In practice, the strength and reliability associated with stable policies are usually undermined by corruption and ineffective legal systems.

II. **Corruption and Side costs:** The rules and regulations governing business and operations in a country are only as good as the systems put in place to enforce them. High levels of corruption make it easy to circumvent regulations or ignore them outright. An investor defending a certain move or arguing against an unfavorable regulation may have to part with huge sums of money to get the job done.

III. **Physical violence:** Internal conflicts or civil war expose investors to both physical harm and high operational costs, including high insurance costs and depreciation of physical assets.

IV. **Expropriation risk:** Expropriation risk is the risk that a government may seize ownership of a firm's assets or impose certain rights that collectively reduce the firm's value. This could also happen when firms are subjected to specific taxes either by virtue of their presence in a country or the nature of their core business. Compensation for expropriation may be well below the value of rights or assets relinquished. Mining firms would be a good example of firms constantly under expropriation risk.

Legal Risk

Legal risk has much to do with the enforcement of property/contractual rights and fidelity to the rule of law. Laxity toward enforcement of rules and regulations not only disadvantages current investors; it also serves to discourage potential investors from coming in. At best, those who decide to invest in the face of a legal system's failure have to incorporate similar outcomes in their expectations for the future.

Legal risk is also a function of how efficiently the system operates. For example, if enforcing a contractual right takes years in a given country, investors will most likely shun that country.

Businesses and individual investors would not tolerate legal limbo for too long.

Some countries are known to have (and enforce) very robust property rights, especially in North America. Others have very weak enforcement, especially African and South American countries.

The Country's Economic Structure

It is important that we pay attention to the economic risks associated with investing in a foreign country. While GDP per capita and the real GDP growth rate give us a rough picture of economic risk, there is a need to assess the country's competitive advantages and its level of economic diversification.

A country that depends too much on one product or service, for example, exposes investors to additional risk. A decline in the price or demand of the product or service can create severe economic shocks that may reverberate well beyond the companies immediately affected. For instance, a country whose oil proceeds amount to, say, 50% of the GDP exposes all investors within the country to economic pain if the price of oil tumbles. In most cases, prices of other commodities shoot up.

Composite Measures of Risk: Risk Services

A good measure of country risk should incorporate all the dimensions of such risk, whether political, financial, or economic. Collectively these risks make up what we call total risk. There are several professional organizations around the world that offer country risk measurement services. These include:

I. Political Risk Services (PRS)-PRS is a subscription-based service that offers members extensive risk analysis of over 100 countries based on three core dimensions: political, financial, and economical. These three dimensions are made up of 22 distinct variables. PRS gives both dimensional and composite scores. The maximum score is 100, while the minimum score is zero. For example, in its report dated July 2015, Switzerland was rated the least risky country with a score of 88.5. Syria took the bottom spot with a score of 35.3
II. Euromoney- Euromoney is an organization that uses surveys of 400 prominent economists to come up with a composite risk score of a country or region. The score ranges from 0 to 100.
III. The Economist- The popular media house develops in-house country risk scores built upon currency risk, sovereign debt risk, and banking risk.
IV. The World Bank- The World Bank develops country risk scores based on six key indicators. These are corruption, government effectiveness, political stability, regulatory quality, the rule of law, and accountability. The WB’s scores are scaled around zero, with negative

numbers indicating more risk and positive numbers less risk.

Limitations of Risk Services

Risk services greatly help to understand and keep track of country risk, but they are not devoid of limitations. These include:

I. Some of the models used to churn out the final risk score incorporates risks that have very little impact on business. The final score could be appropriate for policy-making or economical reading but otherwise irrelevant for investors.
II. There's no standardization of scores, and each service uses its own protocol and calibration techniques.
III. The scores can be quite misleading when used to measure relative risk. For example, if a country has a PRS score of 80, that doesn't imply it's twice as safe as another country with a score of 40.

Sovereign Default

A sovereign default is the failure or refusal of the government of a sovereign state to pay back its debt in full. Sovereign default risk is, therefore, the risk that a country will default on its debt. Sovereign debt can be denominated in either local or foreign currency.

Foreign Currency Defaults

Foreign currency defaults are defaults that occur on the sovereign currency that's denominated in foreign currency. In most cases, governments find themselves unable to raise the amount of foreign currency required to meet contractual obligations. And the worst thing about foreign currency debt is that governments cannot print foreign currency to make up for the shortfall.

In the last few decades, a majority of sovereign defaults have been based on foreign currency. In addition, countries have been more likely to default on bank debt owed than on sovereign bonds issued. And in dollar value terms, Latin American countries have accounted for much of sovereign defaulted debt in the last 50 years.

Local Currency Defaults

Some countries have previously defaulted on debt denominated in local currency. Examples include Argentina (2002-2004) and Russia (1998-1999). Local currency defaults could be traced down to three main reasons:

I. **Mandatory gold backups:** In the years prior to 1971, the printed currency had to be backed up with gold reserves. Without enough reserves, countries could not print enough cash to pay up debts.

II. **Presence of shared currencies:** When a country shares a currency with other countries, it lacks the freedom to print cash to prevent a sovereign default. This scenario played out in 2015 whereby the Greek government could not print more Euros even in the face of a crippling economic meltdown punctuated by a huge sovereign debt. The country defaulted on a USD1.7 billion IMF payment – becoming the first country to do so since Zimbabwe in 2001.

III. **A reluctance to print cash purely for debt repayment:** Printing cash to meet debt obligations is fraught with dangers, including reputation risk, political instability, and the very real possibility of an economic recession. The local currency can dramatically lose value, forcing investors to shun financial investments in favor of real assets such as real estate and precious stones.

Consequences of Sovereign Default

I. **Reputation loss:** Sovereign default can lead to a dramatic deterioration of both diplomatic and economic ties between states. The defaulting government suffers a loss of trust, making it incredibly difficult to negotiate debt in the future.

II. **Political instability:** Following a sovereign default event, the populace may lose its confidence in their leadership, leading to wave after wave of demos, unrest, as well as coups in extreme cases.

III. **Real output declines:?** As investors increasingly shun financial assets in favor of real assets, domestic consumption may also decrease. This, in turn, may lead to a drop in production.

IV. **Capital markets are thrust into a state of chaos and turmoil:** Following a default event, investors increasingly grow reluctant to commit their funds in long-term investments. The demand for long-term securities such as bonds declines, making it difficult for firms to raise funds for expansion and other operations.

Factors that Influence the Level of Sovereign Default Risk

I. **Degree of indebtedness:** The larger the debt a sovereign state has, the more likely it is to default on the part of the debt.

II. **The size of revenue:** A high amount of revenue reduces the chances of a sovereign default event occurring. If a government has regular cash inflows in the form of taxes and other revenue streams such as income earned from the sale of state-owned natural resources, the less likely it is to default on a payment. The opposite is true.

III. **Stability of revenue:** Countries with more stable revenue streams have less default risk. Normally, stability in the revenue collected increases as a country's economic activities become more diversified. Countries that depend too much on a specific source of income have higher exposure to default risk.

IV. **Political risk:** If the leadership of a country is somewhat immune from public pressure – something common in autocracies like Iran and other Middle East nations, default events can easily occur. In such situations, the leadership enjoys the stability of tenure, giving them a free hand to dictate the nation's financial priorities. Democracies, on the other hand, are less likely to default because their leadership is constantly under pressure to deliver.

V. **The level of backing/support a country enjoys courtesy of other sovereign states:** Countries that form part of a regional economic block usually offer each other some implicit backing on matters of debt. For example, an EU member state is unlikely to default on a sovereign debt because other EU countries are likely to chip in. In fact, when countries like Spain and Portugal joined the EU, renowned credit rating agencies reduced their assessment of default risk in these countries. However, such support is not guaranteed. That's why investors don't read too much from this kind of backing.

How Rating Agencies Measure Sovereign Default Risks

Rating agencies offer opinions on a firm or country's ability to incur and/or pay back the debt. While assessing sovereign risk, agencies consider:

- Political and social risks – including issues like public participation in political decision making, respect for the rule of law, transparency in government, and long-term stability of political institutions.
- International security – including the safety and security of a country's borders.
- Regime legitimacy
- Power and governance structures in place
- Existing government obligations –including both local and foreign debt

Shortcomings of Ratings and Rating Agencies

I. **Ratings are upward biased:** In the aftermath of the 2007/2008 financial crisis, rating agencies were widely criticized for having awarded unduly high credit ratings to banks and other financial institutions. In general, rating agencies have been accused of being far too optimistic in their assessment of sovereign ratings. An upward bias on corporate ratings could be explained by the fact that the same corporates double up as the credit agencies' remunerators. This argument, however, does not hold when it comes to sovereign ratings because individual governments are not required to pay the rating agencies.

II. **They are at times reactive rather than proactive:** Rather than updating their ratings before an actual credit event occurs, rating agencies sometimes downgrade countries after a problem has become evident. This does little to protect investors.

III. **Too much interdependence:** Although rating agencies claim to work independently albeit using similar risk indicators, they have been accused of exhibiting herd behavior so that an upgrade/downgrade by one agency is soon replicated by other agencies. As such, independence is lost.

IV. **Vicious cycle:** Sometimes rating agencies have been accused of worsening a crisis by unduly downgrading ratings – painting a situation as being worse than it actually is.

Sovereign Default Spread

Sovereign default spread describes the difference between the rate of interest on a sovereign bond denominated in a foreign currency and the rate of interest on a riskless investment in that currency. For example, suppose that country A has a 10-year dollar-denominated bond with a market interest rate of 8%. At the same time, the 10-year U.S. Treasury bond is trading at 2%. This implies that the sovereign default spread for country A is 6%. The sovereign default spread represents the market's assessment of a government's credit risk.

Here is a table representing the sovereign default spread in some specific countries as compared to the 10-years US Treasury bonds:

Country	Moody's rating	$10-Year Bond Rate	U.S. T. Bond Rate	Default Spread
Argentina	B3	6.96%	2.85%	4.11%
Turkey	Baa2	6.15%	2.85%	3.30%
Peru	3.55%	3.55%	2.85%	0.70%
Chile	3.35%	3.35%	2.85%	0.50%

Advantages of Using the Sovereign Default Spread as a Predictor of Defaults

- Compared to rating agencies, the market differentiation for risk is more granular. In other words, the market has an even more refined understanding of country risk compared to credit ratings. For example, countries A and B could have the same Moody's rating, but the sovereign spread for A may be greater than the spread for B. That could mean the market sees more default risk in A than in B.

- Market-based spreads reflect changes in real-time, unlike credit ratings which can be reactive rather than proactive. As such, market-based spreads can be more effective at predicting imminent danger.

Disadvantages of Using the Sovereign Default Spread as a Predictor of Defaults

Market-based spreads tend to be far more volatile than credit ratings and can be affected by variables with little or no correlation to default. For example, liquidity and market forces of supply and demand can trigger shifts in spreads that have nothing to do with default.

Conclusion

Both credit ratings and market spreads are useful measures of default. Sovereign bond markets usually price bonds guided by credit ratings. In addition, credit rating agencies also leverage market data to make changes in ratings.

Questions

Question 1

In the context of sovereign default spread, which of the following statements is *least* accurate?

A. It describes the difference between the rate of interest on a sovereign bond denominated in a foreign currency and the rate of interest on a riskless investment in the local currency

B. A higher spread represents greater sovereign risk

C. Market-based spreads tend to be far more volatile than credit ratings

D. There is a strong correlation between sovereign ratings and market default spreads

The correct answer is **A**.

Sovereign default spread describes the difference between the rate of interest on a sovereign bond denominated in a foreign currency and the rate of interest on a riskless investment in that currency. For example, suppose that country A has a 20-year dollar-denominated bond with a market interest rate of 8%. At the same time, the 20-year U.S. Treasury bond is trading at 2%. This implies that the sovereign default spread for country A is 6%.

Question 2

Which of the following statements is *most* accurate in reference to the effect of a country's position in the economic cycle on its risk exposure?

A. Mature growth countries are insured from economic shocks

B. Young, growth companies are more exposed to risk partly because they have limited resources to overcome setbacks

C. In markets, a shock to global markets will travel across the world, but mature market equities will often show much greater reactions, both positive and negative, to the same news

D. A country that is still in the early stages of economic growth will generally have less exposure than a mature country, provided it is well-governed and has a solid legal system.

The correct answer is **B**.

Young, growth companies are more exposed to risk partly because they have limited resources to overcome setbacks and also because they rely too much on a stable macroeconomic environment so as to register success in terms of GDP.

Choice **A** is inaccurate: Mature growth countries are not insured from economic shocks. They are still exposed to risk but on a lower scale compared to young, emerging markets.

Choice **C** is inaccurate: In markets, a shock to global markets will travel across the world, but **early growth** market equities will often show much greater reactions, both positive and negative to the same news

Choice **D** is also inaccurate: A country that is still in the early stages of economic growth will generally have more risk exposure than a mature country, even if it has good governance backed up by a solid legal system.

Reading 50: Measuring Credit Risk

After completing this reading, you should be able to:

- Explain the distinctions between economic capital and regulatory capital and describe how economic capital is derived.
- Describe the degree of dependence typically observed among the loan defaults in a bank's loan portfolio, and explain the implications for the portfolio's default rate.
- Define and calculate expected loss (EL).
- Define and explain unexpected loss (UL).
- Estimate the mean and standard deviation of credit losses assuming a binomial distribution.
- Describe the Gaussian copula model and its application.
- Describe and apply the Vasicek model to estimate default rate and credit risk capital for a bank.
- Describe the CreditMetrics model and explain how it is applied in estimating economic capital.
- Describe and use Euler's theorem to determine the contribution of a loan to the overall risk of a portfolio.
- Explain why it is more difficult to calculate credit risk capital for derivatives than for loans.
- Describe challenges to quantifying credit risk.

Economic Capital, Regulatory Capital, and Debt Capital

Economic capital of a bank is the approximate amount of capital that the bank requires to absorb the losses from the loan portfolios. In other words, it is the cushion that a bank estimates it will need in order to remain solvent.

Regulatory capital is the amount of capital the regulators require the banks to maintain. For instance, the Global bank requirements are determined by the Basel Committee on Banking Supervision (BCBS) based in Switzerland. Supervisors then implement the BCBS requirements in each country.

Banks also have what is called **debt capital**, funded by bondholders. In case the incurred losses deplete the equity capital, the debt holders should incur losses before the depositors.

The equity capital is described as “going concern capital” because the bank is solvent if its capital is positive. On the other hand, the debt capital is described as the “gone concern capital” because it acts as a cushion to the depositors when the bank becomes insolvent (no longer a going concern).

Banks face many risks through their transactions, which need to be quantified. Credit risk is a primary risk that the banks have concentrated since the inception of the banking industry. It is majorly explained by the fact that the banks’ activities mainly involve taking deposits and making loans. The loans made are predisposed to some level of default of risk and hence some level of credit loss (credit risk).

Credit risk is quantified using different models. This chapter discusses three models:

i. The mean and standard deviation of the loss from a loan portfolio are determined from the features of individual loans.
ii. The Vasicek model – used by the bank regulators to approximate the extreme percentile of the loss distribution.
iii. CreditMetrics – used by banks to estimate economic capital.

Degree of Dependence Typically Observed among the Loan Defaults in a Bank's Loan Portfolio

The defaults among bank borrowers are not independent because if this would have been the case, then we would expect a similar default rate each year.

In reality, loans do not default independently of one another. Instead, there are good and bad years for defaults.

Among the reasons why companies do not default independently is the economy. Risks associated with the economy are usually systematic and non-diversifiable, which cannot be diversified away by banks and bondholders. Good economic conditions immediately before and during the year lower the probability of default for all companies. On the other hand, bad economic conditions immediately before and during the year raise the probability of default for all companies during the year.

Another reason why companies do not default independently of each other is the credit contagion. Credit contagion is the process by which a problem in one company spreads to other companies as well. To illustrate this, consider three companies X, Y, and Z. Suppose that company X purchases goods from company Y and that in turn, Company Y buys goods from Company Z. Now, if Company X goes bankrupt, Company Y will suffer and it may even go bankrupt depending on the volume of transactions between the two companies. This could lead to Company Z failing even though it had no direct exposure to Company X.

The importance of credit contagion to non-financial companies is still a matter of discussion. The banking regulators are concerned, however, about the potential for credit contagion. Such risks are called systemic risks. In the event that Bank X fails, bank Y will suffer a huge loss as a result of the transactions it has with bank X. Bank Y, may thus default. Bank Z will be affected as well if it has outstanding transactions with Bank Y.

Calculation of the Expected Loss

The expected loss, EL, is the average credit loss that we would expect from an exposure or a portfolio over a given period. It is the anticipated deterioration in the value of a risky asset. In mathematical terms,

$$\text{EL} = \text{EA} \times \text{PD} \times \text{LGD}$$

Where:

EA = exposure amount also known as exposure at default (EAD).

PD = probability of default.

LGD = loss given default also known as loss rate.

Credit loss levels are not constant but rather fluctuate from year to year. The expected loss represents the anticipated average loss that can be statistically determined. Businesses will typically have a budget for the EL and try to bear the losses as part of the standard operating cash flows.

Exam tip: *The expected loss of a portfolio is equal to the summation of expected losses of personal losses.*

$$EL_P = \sum EA_i \times PD_i \times LGD_i$$

Example: Calculating Expected Loss (EL) and Unexpected Loss (UL)

A Canadian bank recently disbursed a CAD 2 million loan, of which CAD 1.6 million is currently outstanding. According to the bank's internal rating model, the beneficiary has a 1% chance of defaulting over the next year. In case that happens, the estimated loss rate is 30%. The probability of default and the loss rate have standard deviations of 6% and 20%, respectively. Determine the expected loss figures for the bank.

$$EL = EA \times PD \times LR$$

Where:

EA = CAD 1,600,000

PD = 1%

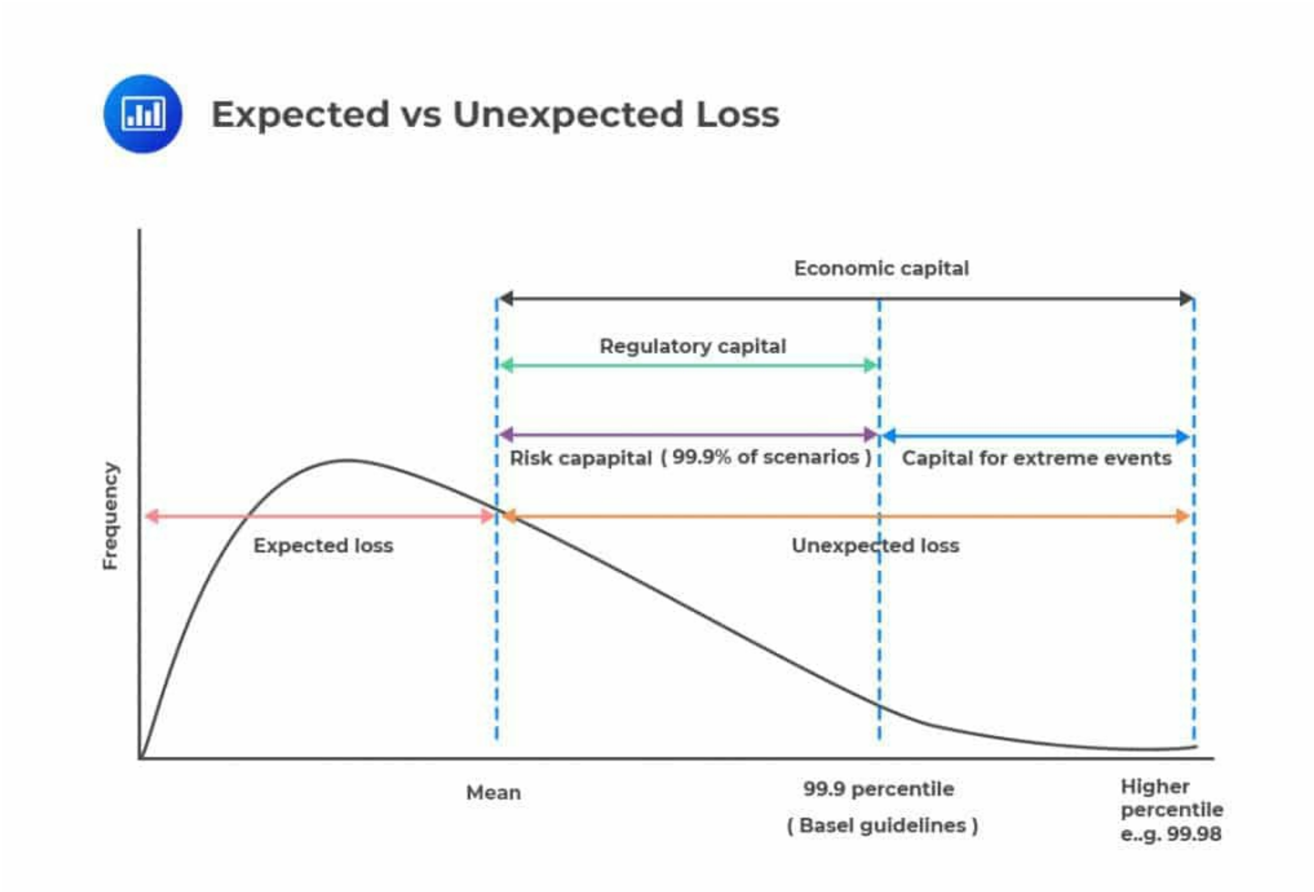
LR = 30%

Thus,

$$EL = 1,600,000 \times 0.01 \times 0.3 = \text{CAD } 4,800$$

Unexpected Loss

Unexpected loss is the average total loss over and above the expected loss.



It is the variation in the expected loss. It is calculated as the standard deviation from the mean at a certain confidence level.

Let UH_L denote the unexpected loss at the horizon for asset value V_H. Then,

$$UH_L \equiv \sqrt{(\text{var}(VH))}$$

You will usually apply the following formula to determine the value of the unexpected loss:

$$UL = EA \times \sqrt{PD \times \sigma_{LR}^2 + LR^2 \times \sigma_{PD}^2}$$

Where

$$\sigma_{PD}^2 = PD \times (1 - PD)$$

The Mean and Standard Deviation of Credit Losses

Given that a bank has *n* loans, define the following quantities:

L_i – the amount borrowed in the ith assumed to be constant throughout the year.

p_i – the probability of the default for the ith loan

R_i – the recovery rate in the event of the default by the ith loan in the event of the default by the ith loan

p_{ij} – the correlation between losses on the ith and jth loan

σ_i – the standard deviation of loss on the ith and jth loan

σ_p – the standard deviation of loss from the portfolio

a – the standard deviation of portfolio loss as a fraction of the size of the portfolio

If the ith loan defaults, the loss is given by:

$$L_i(1 - R_i)$$

Intuitively, the probability distribution for the loss from the *i*th loan is made of a probability p_i that there will be a loss of this amount and the probability $1 - p_i$ that there is no loss, which is typically a binomial distribution. Therefore, we can present the mean and the standard deviation of the loss.

The mean of the loss is given by:

$$p_i \times L_i(1 - R_i) + (1 - p_i) \times 0 = p_i L_i(1 - R_i)$$

Now, recall that for a random variable X, the variance is defined as:

$$\sigma_x = E(x^2) - [E(x)]^2$$

Where E denotes the expectation. Intuitively,

$$\sigma_i^2 = E(\text{Loss}^2) - [E(\text{Loss})]^2$$

This follows immediately that:

$$\sigma_i^2 = p_i[L_i(1-R_i)]^2 - [p_i L_i(1-R_i)]^2 = (p_i - p_i^2)[L_i(1-R_i)]^2$$

So that the standard deviation of the loss is given by:

$$\sigma_i = \sqrt{p_i - p_i^2}\,[L_i(1-R_i)]$$

Note that we can also calculate the standard deviation of the loan portfolio from the losses on the individual loans. It is given by:

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}\sigma_i\sigma_j \ldots\ldots\ldots\ldots\ldots \text{Eq1}$$

Now, the standard deviation expressed as the percentage of the size of the portfolio is

$$\alpha = \frac{\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}\sigma_i\sigma_j}}{\sum_{i=1}^{n} L_i} \ldots\ldots\ldots\ldots\ldots \text{Eq2}$$

Assume that all loans have the same principal L, all recovery rate R are equal, and all default probabilities are equal and denoted by P, and the correlation coefficient is defined as:

$$\rho_{ij} = \begin{cases} 1 & \text{when} \quad i = j \\ \rho & \text{when} \quad i \neq j \end{cases}$$

Where ρ is constant.

Therefore, the standard deviation of the loss from loan *i* is the same for all *i* so that the common standard deviation denoted by σ is given by:

$$\sigma = \sqrt{p - p^2}\,[L(1-R)]$$

Therefore, Eq1 reduces to:

$$\sigma_P^2 = n\sigma^2 + n(n-1)\rho\sigma^2$$

Also, the standard deviation of the loss from the loan portfolio as a percentage of its size (Eq2) reduces to:

$$\alpha = \frac{\sigma_P}{nL} = \frac{\sigma\sqrt{1 + (n-1)\rho}}{L\sqrt{n}}$$

Example: Loss From a Loan Portfolio as a Percentage of its Size

The bank of Baroda has a portfolio consisting of 10,000 loans, each loan amounting to $2 million and has a 1% probability of default over the following year. The recovery rate is 40%, and the correlation coefficient is 0.1. Calculate α, the standard deviation of the loss from the loan portfolio as a percentage of its size?

Solution

The α parameter is given by:

$$\alpha = \frac{\sigma\sqrt{1 + (n-1)\rho}}{L\sqrt{n}}$$

For this case,

L = $2 million

P=0.01

ρ = 0.1

n = 10,000

R = 0.4

$$\begin{aligned}\sigma &= \sqrt{p - p^2}\,[L(1 - R)] \\ &= \sqrt{0.01 - 0.01^2}\,[2(1 - 0.4)] = 0.1194\end{aligned}$$

Therefore,

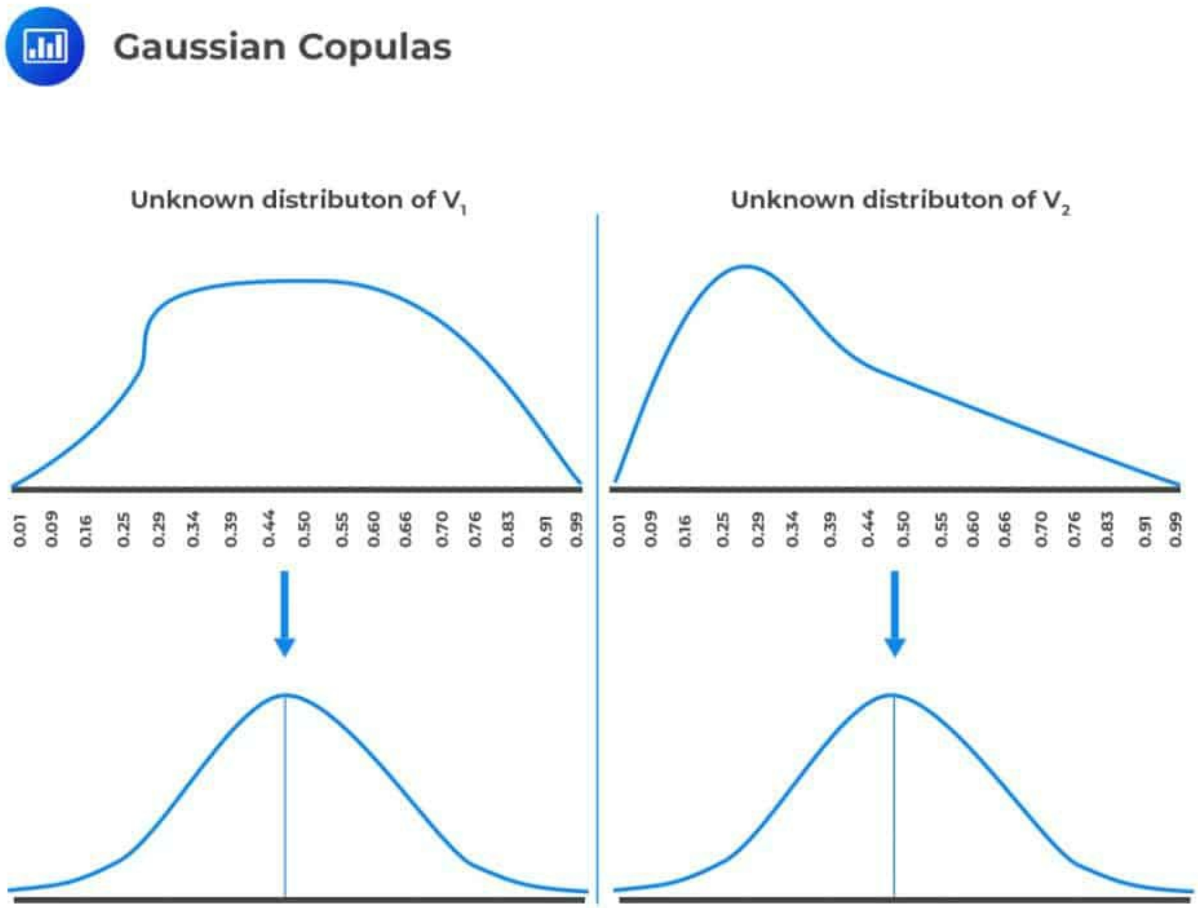
$$\alpha = \frac{0.1194\sqrt{1 + (10,000 - 1) \times 0.1}}{2 \times \sqrt{10,000}} = 0.01889$$

The Gaussian Copula Model

A **Gaussian copula** maps the marginal distribution of each variable to the **standard normal distribution,** which, by definition, has a mean of zero and a standard deviation of one. Copula correlation models create a joint probability distribution for two or more variables while still preserving their marginal distributions. The joint probability of the variables of interest is implicitly defined by mapping them to other variables whose distribution properties are known.

We can demonstrate how Gaussian copulas work using a simple example:

Let us define two variables V_1 and V_2 that have unknown distributions and unique marginal distributions. V_1 and V_2 are mapped into new variables U_1 and U_2, respectively, that have standard normal distributions. The mapping is done on a percentile-to-percentile basis to create a Gaussian copula.



For example, the one-percentile point of the V_1 distribution is mapped to the one-percentile point of the U_1 distribution. Similarly, the 50-percentile point of the V_1 distribution is mapped to the 50-percentile point of the U_1 distribution.

Before mapping variables V_1 and V_2 to the normal distribution, it is very difficult to define a relationship between them since their marginal distributions are unknown and are pretty much incomprehensible structures. Once they have been mapped to the standard normal distribution as new variables U_1 and U_2, respectively, we can now define a relationship between them since the standard normal distribution has a known structure. The Gaussian copula, therefore, helps us to define a correlation between variables when it is not possible to define a correlation directly.

A One-Factor Correlation Model

Assume now that we have many variables V_i for all i=1,2,…,n for which each of the V_i can be mapped to a standard normal distribution U_i. The main challenge that remains is to define the correlation between U_i distributions because the presence of many unique distributions means that we have to specify numerous correlation parameters. To address this issue, we use the one-factor model.

The one-factor model is defined as:

$$U_i = a_i F + \sqrt{1 - a_i^2} Z_i$$

Where F is a common factor for all U_i and Z_i is a component of U_i that is unrelated to the factor F and uncorrelated to each other. The a_i are the parameter values that lie between -1 and +1, that is $a_i \in [-1, +1]$.

The variables F and Z_i have the standard normal distributions, that is, $F \sim N(0,1)$ and $Z_i \sim N(0,1)$. Therefore, U_i is a sum of two independent normal distributions, and it is, therefore, a normal variable with a mean of 0 and a standard deviation of 1. The variance of U_i is 1 since F and Z_i are uncorrelated so that:

$$\begin{aligned}
\text{Var}(U_i) &= \text{Var}[a_i F + \sqrt{1 - a_i^2} Z_i] \\
&= a_i^2 \text{Var}(F) + (\sqrt{1 - a_i^2})^2 \text{Var}(Z_i) \\
&= a_i^2 .1 + (\sqrt{1 - a_i^2})^2 .1 = a_i^2 + 1 - a_i^2 = 1
\end{aligned}$$

Note that we are summing the variances of two uncorrelated variables.

So in a nutshell, the one-factor model takes one standard normally distributed variable U_i and defines it in relation to two other variables, which are both standard normally distributed. The factor F affects across all U_i but factor Z_i affects only U_i.

The coefficient of correlation between U_i and U_j comes in as a result of their dependence on the common factor F and thus is $a_i\ a_j$. The correlation coefficient between U_i and U_j is defined from the basic statistics as

$$\rho_{U_i, U_j} = \frac{E(U_i U_j) - E(U_i) E(U_j)}{\sigma_{U_i} \sigma_{U_j}}$$

But $E(U_i)=E(U_j)=0$ and $\sigma_{U_i} = \sigma_{U_j} = 1$ and thus the above equation reduces to

$$\rho_{U_i,U_j} = \frac{E(U_iU_j) - 0}{1} = E(U_iU_j)$$

Now,

$$E(U_iU_j) = E[(a_iF + \sqrt{1-a_i^2}Z_i)(a_jF + \sqrt{1-a_j^2}Z_j)]$$

But F is uncorrelated with all Z_i and Z_i is uncorrelated with Z_j so that

$$E(FZ_i) = E(FZ_j) = E(Z_iZ_j) = 0$$

And the above equation reduces to:

$$E(U_iU_j) = E(a_i, a_jF^2) = a_i, a_jE(F^2) = a_i, a_j.1 = a_i, a_j$$
$$\therefore \rho_{U_i,U_j} = a_ia_j$$

Note that the immediate result stems from the fact that $E(F^2) = 1$ because F is a standard normal variable, $F \sim N(0,1)$ so that:

$$E(F^2) - [E(F)]^2 = Var(F) = 1$$
$$\Rightarrow E(F^2) - [0]^2 = 1$$
$$\therefore E(F^2) = 1$$

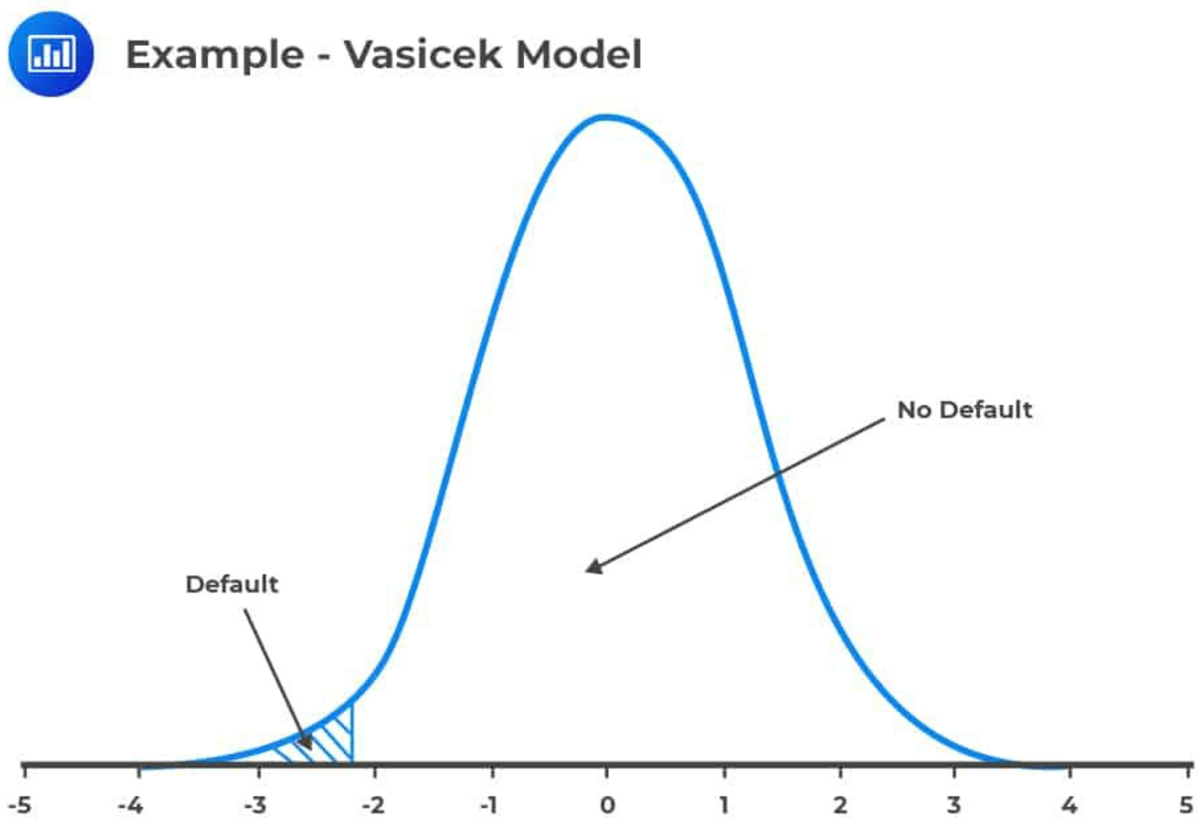
A notable example of a one-factor model is the capital asset pricing model (CAPM). In CAPM, the correlation coefficient between two assets is assumed to arise from the dependence on the common factor – the return from the market index. However, CAPM is significantly palatable since it specifies the correlation between the returns of the different assets.

The Vasicek Model

The Vasicek model uses the Gaussian copula model to define the correlation between the defaults when determining the capital for loan portfolios. The Vasicek model has an advantage over the standard deviation of the loss from the loan portfolio as a percentage of its size, α in that the unexpected loss can be determined analytically.

Now, assume that the probability of default (PD) is the same for all firms in a huge portfolio so that the PD for a company *i* for one year is mapped to a standard normal U_i as described earlier.

Consider the following figure below:



The values at the further left (shaded region) tail of this standard normal distribution is the distribution of the default, while the rest is the distribution of no default.

Mathematically, a firm *i* defaults if:

$$U_i \leq N^{-1}(PD)$$

Where N^{-1} is the inverse of the inverse cumulative normal distribution. That is if the probability of default for the loan portfolio of a bank is 1%, then the bank defaults if:

$$U_i \leq N^{-1}(0.01) = -2.326$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

This can be stated as a bank defaults if:

$$U_i \in (-\infty, -2.326)$$

And no default if:

$$U_i \in (-2.326, \infty)$$

Recall that the one-factor model is given by:

$$U_i = a_i F + \sqrt{1 - a_i^2} Z_i$$

To make the model more palatable, a_i's are assumed to be equal for all i so that $a_i = a$ and thus, the one-factor model is rewritten as:

$$U_i = aF + \sqrt{1 - a^2} Z_i$$

Thus the correlation between each pair of U_i's is:

$$\rho_{U_i, U_j} = a_i a_j = a.a = a^2$$

Now, the factor F can be taken as an index of the recent economic health. That is, if the F is high the doing economy is healthy implying that U_i is high, making the default relatively low. Otherwise, if F is relatively low, then U_i is also relatively low, and thus making default most likely.

For a large portfolio, the default rate defined as the probability that:

$$U_i \leq N^{-1}(PD)$$

which was discussed earlier. Now using the properties of a normal distribution:

$$\text{Default Rate as a Function of F} = N\left(\frac{N^{-1}(PD) - aF}{\sqrt{1 - a^2}}\right)$$

We anticipate that the default rate is not exceeded with a 99.9% likelihood and thus given by the low value of F. Moreover, we require that the probability of the true value of F will be worse than other F' and it will be 0.1%. Now, recall that F is normally distributed so that:

$$F' = N^{-1}(0.001)$$

and thus,

$$99.9 \text{ percentile for default rate} = N\left(\frac{N^{-1}(PD) - aN^{-1}(0.001)}{\sqrt{1-a^2}}\right)$$

Now, ρ is defined as the correlation between each pair of U_i and is given by:

$$\rho = a^2 \Rightarrow a = \sqrt{\rho}$$

Therefore,

$$99.9 \text{ percentile for default rate} = N\left(\frac{N^{-1}(PD) - \sqrt{\rho}N^{-1}(0.001)}{\sqrt{1-\rho}}\right)$$

By the property of the normal distribution, we know that:

$$N(0.001) = -N(0.999)$$

The formula above can be rewritten as:

$$99.9 \text{ percentile for default rate} = N\left(\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}}\right)$$

Now consider a loan portfolio with the same default probability. The last equation allows us to convert the average default rate (PD) into a portfolio default rate, which is interestingly only once in 1000 years!

Moreover, when $\rho = 0$,

$$\begin{aligned} 99.9 \text{ percentile for default rate} &= N\left(\frac{N^{-1}(PD) + \sqrt{0}N^{-1}(0.999)}{\sqrt{1-0}}\right) \\ &= N(N^{-1}(PD)) = PD \end{aligned}$$

This makes much sense because if the firms default independently, the "law of large numbers" makes sure that the default rate in a large portfolio is always the same.

Example: Calculating the 99.9 Percentile for Default Rate Under the

Vasicek Model

The Bank of Baroda has a loan portfolio that has a default rate of 2% and a correlation coefficient of 0.3. Under the Vasicek Model, what is the 99.9 percentile for the portfolio default rate?

Solution

The 99.9 percentile under the Vasicek model is given by:

$$\text{99.9 percentile for default rate} = N\left(\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}}\right)$$

So for the information provided in the question,

$$= N\left(\frac{N^{-1}(0.02) + \sqrt{0.3}N^{-1}(0.999)}{\sqrt{1-0.3}}\right) = N(-0.42) = 34\%$$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

If a loan has the same PD, the same ρ and the same loss given default (LGD) and the same principle, the Basel II capital requirement for banks under the IRB approach is given by:

$$(\text{WCDR} - \text{PD}) \times \text{LGD} \times \text{EAD}$$

Where WCDR is defined as the worst-case default rate, and it is 99.9 percentile of the default rate distribution defined as in Vasicek model. LGD is the loss given the default, which is defined as one minus the recovery rate. EAD is the total exposure at default, which is the sum of the principal of all the loans.

The Basel II equation provides a way of calculating the unexpected loss with a 99.9% confidence level. This can be seen by the fact that WCDR×LGD is the percentile point of loss rate distribution, while WCDR×LGD×EAD is the loss at the 99.9 percentile. Therefore, the expected loss is given by:

$$PD \times LGD \times EAD$$

For a non-homogeneous loan portfolio, the one-factor model equation can be rewritten so that for loan *i*:

$$\sum_{i} (WCDR_i - PD_i) \times LGD_i \times EAD_i$$

Where the variables are as defined earlier for each loan i.

The last equation gives a way of calculating the capital for each loan separately, and then the results are added. Moreover, the equation can be adjusted to include the maturity adjustment factor.

Basel II defines correlation ρ in that the banks must assume different conditions. Based on the IRB approach, the banks estimate PD while EAD and LGD estimates are approximated in accordance with Basel II rules or using the bank's internal model but subject to circumstances and regulatory requirements.

CreditMetrics Model

The CreditMetrics model is used by the banks to calculate economic capital where each borrower is given an external or internal credit rating. A one-year transition table is then utilized to define the changes in the credit ratings.

The loan portfolio of the bank is determined at the beginning of one year, and the Monte Carlo simulation is used to define how the rating changes over the year. In each of the simulation experiments, the ratings of all borrowers are determined at the end of the year, which allows the bank to reevaluate the portfolio.

Reevaluation of the portfolio involves calculating the credit loss of the portfolio, which is defined as the value of the portfolio at the beginning of the year, less the value of the portfolio at the end of the year. Through numerous simulation trials, complete credit loss distribution is produced.

Example: Applications of the CreditMetrics Model

To demonstrate the CreditMetrics model, consider a bank X whose credit ratings are A, B, C, and default. The probabilities of the rating transition are as follows: a B-rated can transition to rating A with a probability of 5%, 85% chance of staying in B, 14% of transitioning to C, and 3% chance of defaulting as shown in the table below.

Rating Transition Movement	Probability of Transition
B to A	6%
Remains in B	85%
B to C	14%
Defaults	3%

In each Monte Carlo simulation experiment, a number is sampled from a standard normal distribution to ascertain what will happen to a borrower. Now given that $N^{-1}(0.06) = -1.555, N^{-1}(0.90) = 1.282, N^{-1}(0.97) = 1.881$. From these values, the corresponding "rule table" is as shown below:

Standard Normal Sample	Rating Transition
Less than -1.555	B to A
Between -1.555 and 1.282	Remains in B
Between 1.282 and 1.881	B to C
Greater than 1.881	Defaults

However, since the bank borrowers do not default independently, sampling should be done in a way that reflects the correlation between the samples. As such, a factor model (such as the one-factor model) is used to capture the relationship between the normal distributions. In other words, the Monte Carlo simulation should be based on the Gaussian model such that probabilities of the rating transitions for each borrower are converted to a normal distribution, and the correlations are defined on those distributions and not rating transitions themselves.

Notably, the correlations between the traded equities are usually used in CreditMetrics, which can be justified by the Monte Carlo simulation, where the company defaults if its market value becomes less than the book value of its debt.

Lastly, as can be seen in the CreditMetrics model, as opposed to the Vasicek model, it includes both the effects of rating variations and defaults.

Euler's Theorem on Risk Allocation

Leonhard Euler developed a model that can be used to divide risk measures. The Euler Theorem is based on the homogenous functions F of a set of variables $x_1, x_2, \ldots, x_n$ in which a feature defined as

$$F(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda F(x_1, x_2, \ldots, x_n)$$

Where λ is a constant.

Now, define:

$$Q_i = x_i \frac{\Delta F_i}{\Delta x_i} = \frac{\frac{\Delta F_i}{\Delta x_i}}{x_i}$$

Where:

Δx_i = small change in x_i

ΔF_i = small change in F_i

Q_i = ratio of ΔF_i to a proportional change $\frac{\Delta x_i}{x_i}$ in x_i

Euler showed that as Δx_i tends to 0,

$$F = \sum_{i=1}^{n} Q_i$$

So, many risk measures are homogeneous functions, which is a property of a coherent risk measure.

Now, if a portfolio is adjusted such that each position is multiplied by some constant λ, a risk measure is typically multiplied by λ. So the Euler's theorem provides a way to allocate a risk measure F, which is defined as a function of many trades into its components.

As an application of Euler's theorem on the credit risk, we can determine the contribution of each loan in a portfolio to the overall risk measure. Consider the following example.

Example: Application of the Euler's Theorem

The Basley bank has three loans, A, B, and C. Losses from the loans are 1.0, 1.2, and 1.3, respectively. The correlations between the losses are as shown in the table below:

	Loan A	Loan B	Loan C
Loan A	1	0	0
Loan B	0	1	0.8
Loan C	0	0.8	1

The standard deviation of the total loss is given by:

$$\sigma_P = \sqrt{\sigma_A^2 + \sigma_B^2 + \sigma_C^2 + 2\rho_{AB}\sigma_A\sigma_B + 2\rho_{AC}\sigma_A\sigma_C + 2\rho_{BC}\sigma_B\sigma_C}$$
$$= \sqrt{1.0^2 + 1.2^2 + 1.3^2 + 0 + 0 + (2 \times 0.8 \times 1.2 \times 1.3)} \approx 2.57$$

Now assume that the size of the loan A is increased by 1% so that its new standard deviation is now $1 \times 1.01 = 1.01$. The increase in the standard deviation of the loan portfolio is:

$$= \left(\sqrt{1.01^2 + 1.2^2 + 1.3^2 + (2 \times 0.8 \times 1.2 \times 1.3)}\right) - \left(\sqrt{1.0^2 + 1.2^2 + 1.3^2 + (2 \times 0.8 \times 1.2 \times 1.3)}\right) = 0.0039$$

Now using the equation:

$$Q_i = x_i \frac{\Delta F_i}{\Delta x_i} = \frac{\frac{\Delta F_i}{\Delta x_i}}{x_i}$$

It implies that,

$$Q_A = \frac{0.0039}{0.01} = 0.39$$

Now if the size of loan B is also increased by 1%, so that its standard deviation of loss is now $1.2 \times 1.01 = 1.2120$ and thus the increase in the loan portfolio is given by:

$$= (\sqrt{1.0^2 + 1.2120^2 + 1.3^2 + 2 \times 0.8 \times 1.2120 \times 1.3}) - (\sqrt{1.0^2 + 1.2^2 + 1.3^2 + 2 \times 0.8 \times 1.2 \times 1.3}) = 0.01($$

And thus,

$$Q_B = \frac{0.01115}{0.01} = 1.045$$

Again if the loan size of loan C is increased by 1% then,

$$Q_C = 1.142$$

So,

$$Q_A + Q_B + Q_C = 0.39 + 1.045 + 1.142 \approx 2.477$$

As per the context of Euler's theorem, we have divided the total loss of 2.477 into the loan contributions from loans A, B, and C. It is easy to see that that the contribution of loan A is low because it is uncorrelated with loans B and C, hence contributes less risk to the entire portfolio.

Credit Risk Capital for Derivatives

Similar to loans, derivatives (such as options and swaps) generate credit risk. For instance, if company X purchases an option from company Y, the credit risk to company X arises in the sense that company Y might default and thus fail to honor its obligation. In the case of a swap, if company X enters into an interest rate swap with company Y, the credit risk to company X occurs if company Y defaults when the value of the swap is positive to company X.

The credit risk capital for derivatives can be similarly calculated using the equation:

$$\sum_i (WCDR_i - PD_i) \times LGD_i \times EAD_i$$

However, it is challenging to compute EAD for the derivatives transactions. Note that in the case of loans, EAD is the amount that has been advanced or expected to be advanced to the borrower. Obviously, in derivatives, exposures vary with the value of the derivative. The solution to this challenge is addressed by the Basel Committee's rules of computing EAD – setting the exposure at default for the derivatives equal the current exposure plus add-on amount. The current exposure is the maximum amount of capital that might be lost in case of a default today. The add-on amount is an additional amount for the possibility of the exposure worsening by the time a default occurs.

An additional challenge for the derivatives involves netting agreements so that all the outstanding derivatives with a given counterparty may be considered a single derivative in case of a default. As such, the equation above cannot be utilized on transaction-by-transaction grounds, yet it must be implemented on a counterparty-by-counterparty basis.

Challenges to Quantifying Credit Risk

The analysis of the credit risk needs many estimates, such as PD. Just like through-the-cycle and point-in-time for credit ratings, we can differentiate between the through-the-cycle PD (mean of the PD over the economic cycle) and point-in-time PD (indicates the current economic times). The banks are required to estimate the through-the-cycle PD for regulatory reasons, but point-in-time PD is estimated for internal uses.

The accounting standards, such as IFRS 9, require that the loans should be valued for accounting reasons. As such, the expected losses over one year or loan's life must be computed and subtracted from the loan amount, at which a point-in-time estimate is required. Therefore, the bank is faced with the difficulty of estimating both through-the-cycle estimates for regulators and point-time estimators to satisfy the auditors.

The recovery rate or loss given default is usually negatively correlated with default rate so that in adverse economic conditions contribute to credit risk is twofold: the default rate increases while recovery decreases.

The exposure to default (the amount the borrower owes at the time of default) is such that in case of an overdraft facility or line of credit, it can be conservatively approximated as the borrowing limit given to the customer. Usually, for a term loan, the exposure to default is the expected principal during a given year. However, for a portfolio of derivatives, the exposure to default is complex to be calculated during a year, which might lead to wrong-way risk. Wrong-way risk occurs where a counterparty to an institution is most likely to default if the value of the derivative is negative to the bank and positive to the counterparty.

The correlations are challenging to approximate. Despite the fact the Gaussian model is easy to use, there is no surety that it will reflect how bad the amount lost by a loan in one-in-thousand days occurrence or the case of economic capital, which is an extreme event.

Apart from the credit risk, a bank must also get concerned about other risks it faces. Such risks include market, operational, liquidity, and strategic risks. Typically, these risks are assigned to different arms in a bank but not necessarily independent of each other. These risks interact and consequently impacting both economic and regulatory capital requirements of the bank.

Question

Big Data Inc., a U.S. based cloud technology and computing firm, has been offered a USD 10 million term loan fully repayable in exactly two years. The bank behind the offer estimates that it will be able to recover 65% of its exposure if the borrower defaults, and the probability of that happening is 0.8%. The bank's expected loss one year from today is *closest* to:

A. USD 52,000

B. USD 26,000

C. USD 14,000

D. USD 28,000

The correct answer is **D**.

$$\text{EL} = \text{EA} \times \text{PD} \times \text{LR}$$

$\text{EA} = \text{USD} \quad 10,000,000$

$\text{PD} = 0.8\%$

$\text{LR} = 35\%$

$$\text{EL} = 10,000,000 \times 0.008 \times 0.35 = \text{USD } 28,000$$

Maturity is irrelevant since the loan is fully repayable in two years.

Option A is incorrect: The loss given default is taken to be 65%. Note that $\text{LGD} = 1 - \text{Recovery rate}$.

Option B is incorrect. The loss given default is taken to be 65% and the final result dividend by two.

Option C is incorrect. The final result is incorrectly divided by 2.

Reading 51: Operational Risk

After completing this reading, you should be able to:

- Describe the different categories of operational risk and explain how each type of risk can arise.
- Compare the basic indicator approach, the standardized approach, and the advanced measurement approach for calculating operational risk regulatory capital.
- Describe the standardized measurement approach and explain the reasons for its introduction by the Basel committee.
- Explain how a loss distribution is derived from an appropriate loss frequency distribution and loss severity distribution using Monte Carlo simulations.
- Describe the common data issues that can introduce inaccuracies and biases in the estimation of loss frequency and severity distributions.
- Describe how to use scenario analysis in instances when data is scarce.
- Describe how to identify causal relationships and how to use Risk and Control Self-Assessment (RCSA) and Key Risk Indicators (KRIs) to measure and manage operational risks.
- Describe the allocation of operational risk capital to business units.
- Explain how to use the power-law to measure operational risk.
- Explain the risks of moral hazard and adverse selection when using insurance to mitigate operational risks.

According to the Basel Committee, operational risk is *"the risk of direct and indirect loss resulting from inadequate or failed internal processes, people, and systems or from external events."*

The International Association of Insurance Supervisors describes the operational risk as to the risk of adverse change in the value of the capital resource as a result of the operational occurrences such as inadequacy or failure of internal systems, personnel, procedures, and controls as well as the external events.

Operational risk emanates from internal functions or processes, systems, infrastructural flaws, human factors, and outside events. It includes legal risk but leaves out reputational and strategic risks in part because they can be difficult to measure quantitatively.

This chapter primarily discusses the methods of computing the regulatory and economic capital for operational risk and how the firms can reduce the likelihood of adverse occurrence and severity.

The Basel Committee's Seven Categories of Operational Risk

1. **Internal fraud:** Internal fraud encompasses acts committed internally that diverge from a firm's interests. These include forgery, bribes, tax non-compliance, mismanagement of assets, and theft.
2. **External fraud:** External fraud encompasses acts committed by third parties. Commonly encountered practices include theft, cheque fraud, hacking, and unauthorized access to information.
3. **Clients, products, and business practices:** This category has much to do with intentional and unintentional practices that fail to meet a professional obligation to clients. This includes issues such as fiduciary breaches, improper trading, misuse of confidential client data, and money laundering.
4. **Employment practices and work safety:** These are acts that go against laws put in place to safeguard the health, safety, and general well-being of both employees and customers. Issues covered include unethical termination, discrimination, and the coerced use of defective protective material.
5. **Damage to physical assets:** These are losses incurred to either natural phenomena like earthquakes or human-made events like terrorism and vandalism
6. **Business disruption and system failures:** This included supply-chain disruptions and system failures like power outages, software crashes, and hardware malfunctions.
7. **Execution, delivery, and process management:** This describes the failure to execute transactions and manage processes correctly. Issues such as data entry errors and unfinished

legal documents can cause unprecedented losses.

Large Operational Risks

The three large operational risks faced by financial institutions include cyber risk, compliance risks, and rogue trader risk.

Cyber Risk

The banking industry has developed technologically. This development is evident through online banking, mobile banking, credit and debit cards, and many other advanced banking technologies. Technological advancement is beneficial to both the banks and their clients, but it can also be an opportunity for cybercriminals. Cybercriminals can either be individual hackers, organized crime, nation-states, or insiders.

The cyber-attack can lead to the destruction of data, theft of money, intellectual property and personal and financial data, embezzlement, and many other effects. Therefore, financial institutions have developed defenses mechanisms such as account controls and cryptography. However, financial institutions should be aware that they are vulnerable to attacks in the future; thus, they should have a plan that can be executed on short notice upon the attack.

Compliance Risks

Compliance risks occur when an institution incurs fines due to knowingly or unknowingly ignoring the industry's set of rules and regulations, internal policies, or best practices. Some examples of compliance risks include money laundering, financing terrorism activities, and helping clients to evade taxes.

Compliance risks not only lead to hefty fines but also reputational damage. Therefore, financial institutions should put in place structures to ensure that the applicable laws and regulations are adhered to. For example, some banks have developed a system where suspicious activities are detected as early as possible.

Rogue Trader Risk

Rogue trader risk occurs when an employee engages in unauthorized activities that consequently lead to large losses for the institutions. For instance, an employee can trade in highly risky assets while hiding losses from the relevant authorities.

To protect itself from rogue trader risk, a bank should make the front office and back office independent of each other. The front office is the one that is responsible for trading, and the back office is the one responsible for record-keeping and the verifications of transactions.

Moreover, the treatment of the rogue trader upon discovery matters. Typically, if unauthorized trading occurs and leads to losses, the trader will most likely be disciplined (such as lawful prosecution). On the other hand, if the trader makes a profit from an unauthorized trading, this violation should not be ignored because it breeds a culture of risk ignorance, which can lead to adverse financial drawbacks.

Comparing the Three Approaches for Calculating Regulatory Capital

The Basel Committee on Banking Supervision develops the global regulations which are instituted by the supervisors of each of the banks in each country. Basel II, which was drafted in 1999, revised the methods of computing the credit risk capital. Basel II regulation includes the approaches to determine the operational risk capital.

The Basel Committee recommends three approaches that could be adopted by firms to build a capital buffer that can protect against operational risk losses. These are:

I. Basic indicator approach
II. Standardized approach
III. Advanced measurement approach (AMA)

Basic Indicator Approach

Under the **basic indicator approach,** the amount of capital required to protect against operational risk losses is set equal to **15%** of annual gross income over the previous three years. Gross income is defined as:

$$\text{Gross income} = \text{Interest earned} - \text{Interest paid} + \text{Noninterest income}$$

Standardized Approach

To determine the total capital required under the **standardized approach** is similar to the primary indicator method, but a bank's activities are classified into eight distinct business lines, with each of the lines having a beta factor. The average gross income for each business line is then multiplied by the line's beta factor. After that, the capital results from all eight business lines are summed up. In other words, the percentage applied to gross income varies in all business lines.

Below are the eight business lines and their beta factors:

Business Line	Beta Factor
Corporate Finance	18%
Retail Banking	12%
Trading and Sales	18%
Commercial Banking	15%
Agency Services	15%
Retail Brokerage	12%
Asset Management	12%
Payment and Settlement	18%

To use the standardized approach, a bank has to satisfy several requirements. The bank must:

I. Have an operational risk management function tasked with the identification, assessment, monitoring, and control of operational risk

II. Consistently keep records of losses incurred in each business line.

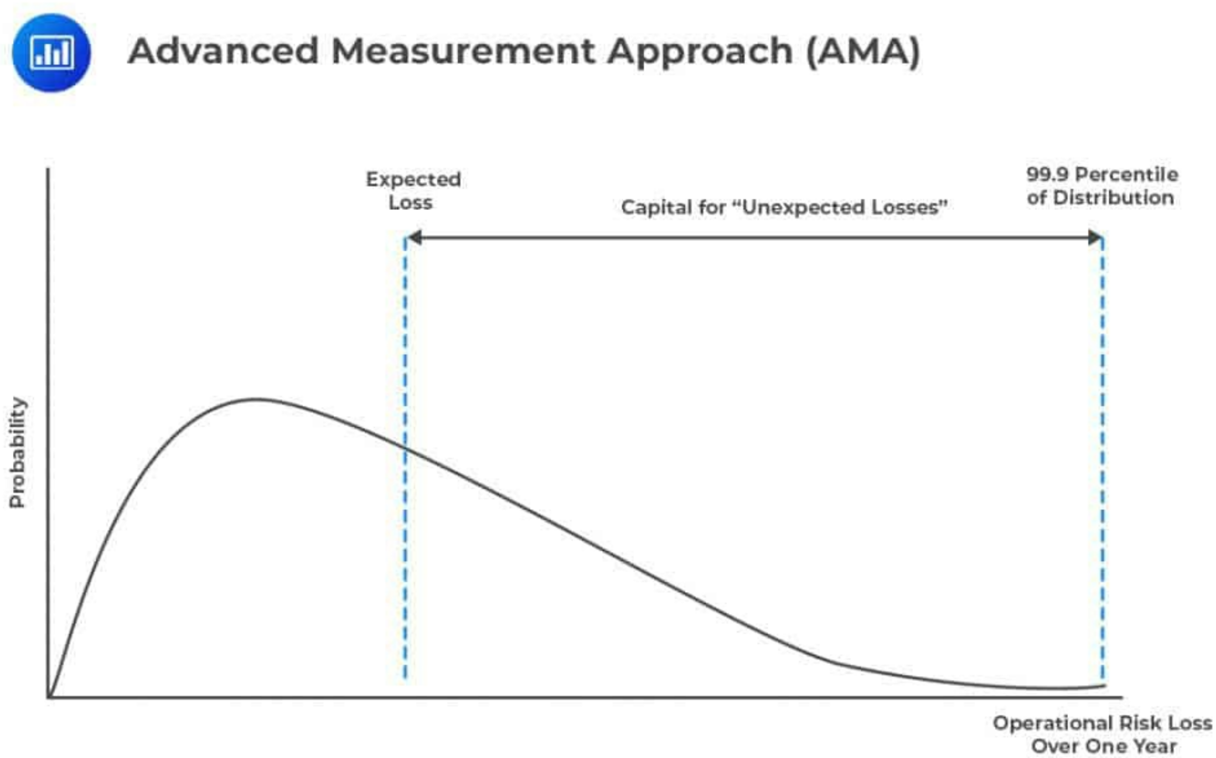
III. Regularly report operational risk losses incurred in all business lines.

IV. Install an operational risk management system that's well documented.

V. Regularly subject its operational risk management processes to independent reviews by both internal and external auditors.

Advanced Measurement Approach (AMA)

The AMA approach is much more complicated compared to other approaches. Under this method, the banks are required to treat operational risk as credit risk and set the capital equal to the 99.9 percentile of the loss distribution less than the expected operational loss, as shown by the figure below.



Moreover, under the AMA approach, banks are required to take into consideration every combination of the eight business lines mentioned in the standardized approach. Combining the seven categories of operational risk with the eight business lines gives a total of ($7 \times 8 =$) 56 potential sources of operational risk. The bank must then estimate the 99.9 percentile of one-year loss for each combination and then aggregate each combination together to determine the total capital requirement.

To use the AMA method, a bank has to satisfy all the requirements under the standardized approach, but the bank must also:

I. Be able to estimate unexpected losses, guided by the use of both external and internal data.

II. Have a system capable of allocating economic capital for operational risk across all business lines in a way that creates incentives for these business lines to manage operational risk better.

Currently, the Basel Committee has replaced the three approaches with a new standardized measurement (SMA) approach. Despite being abandoned by the Basel Committee, some aspects of the AMA approach are still being used by some banks to determine economic capital.

The AMA approach opened the eyes of risk managers to operational risk. However, bank regulators found flaws in the AMA approach in that there is a considerable level of variation in the calculation done by different banks. In other words, if different banks are provided with the same data, there is a high chance that each will come up with different capital requirements under the AMA.

Standardized Measurement Approach (SMA)

The Basel Committee announced in March 2016 to substitute all three approaches for determining operational risk capital with a new approach called the standardized measurement approach (SMA).

The SMA approach first defines a quantity termed as Business Indicator (BI). BI is similar to gross income, but it is structured to reflect the size of a bank. For instance, trading losses and operating expenses are treated separately so that they lead to an increase in BI.

The BI Component for a bank is computed from the BI employing a piecewise linear relationship. The loss component is calculated as:

$$7X + 7Y + 5Z$$

Where X, Y, and Z are the approximations of the average losses from the operational risk over the past ten years defined as:

- X – all losses
- Y – losses greater than EUR 10 million
- Z – losses greater than EUR 100 million

The computations are structured so that the losses component and the BI component are equal for a given bank. The Basel provides the formula used to calculate the required capital for the loss and BI components.

Determination of Operational Risk Loss Distribution

Computations of the economic capital require the distributions for various categories of operational risk losses and the aggregated results. The operational risk distribution is determined by **the average loss frequency and loss severity.**

Average Loss Frequency

The term **"average loss frequency"** is defined as the average number of times that large losses occur in a given year. The loss frequency distribution shows just how the losses are distributed over one year, specifying the mean and variance.

If the average losses in a given year are λ, then the probability of n losses in a given year is given by Poisson distribution defined as

$$\Pr(n) = \frac{e^{-\lambda}\lambda^{n}}{n!}$$

Example: Calculating the Probability of Loss

The average number of losses of a given bank is 6. What is the probability that there are precisely 15 losses in a year?

Solution

From the question, $\lambda = 6$. Therefore,

$$\Pr(15) = \frac{e^{-6}6^{15}}{15!} \approx 0.001$$

Loss Severity

Loss severity is defined as a probability distribution of the size of each loss. The mean and variance of the loss severity are modeled using the lognormal distribution. That is, suppose the standard deviation of the loss size is σ, and the mean is μ, then the mean of the logarithm of the loss size is given by:

$$\ln\left(\frac{\mu}{\sqrt{1+w}}\right)$$

The variance is given by:

$$\ln(1+w)$$

Where:

$$w = \left(\frac{\sigma}{\mu}\right)^2$$

Example: Calculating the Logarithm of Mean and Variance Loss Severity.

The estimated mean and standard deviation of the loss size is 50 and 20, respectively. What is the mean and standard deviation of the logarithm of the loss size?

Solution

We start by calculating w,

$$w = \left(\frac{20}{50}\right)^2 = 0.16$$

So the mean of the logarithm of the loss size is given by:

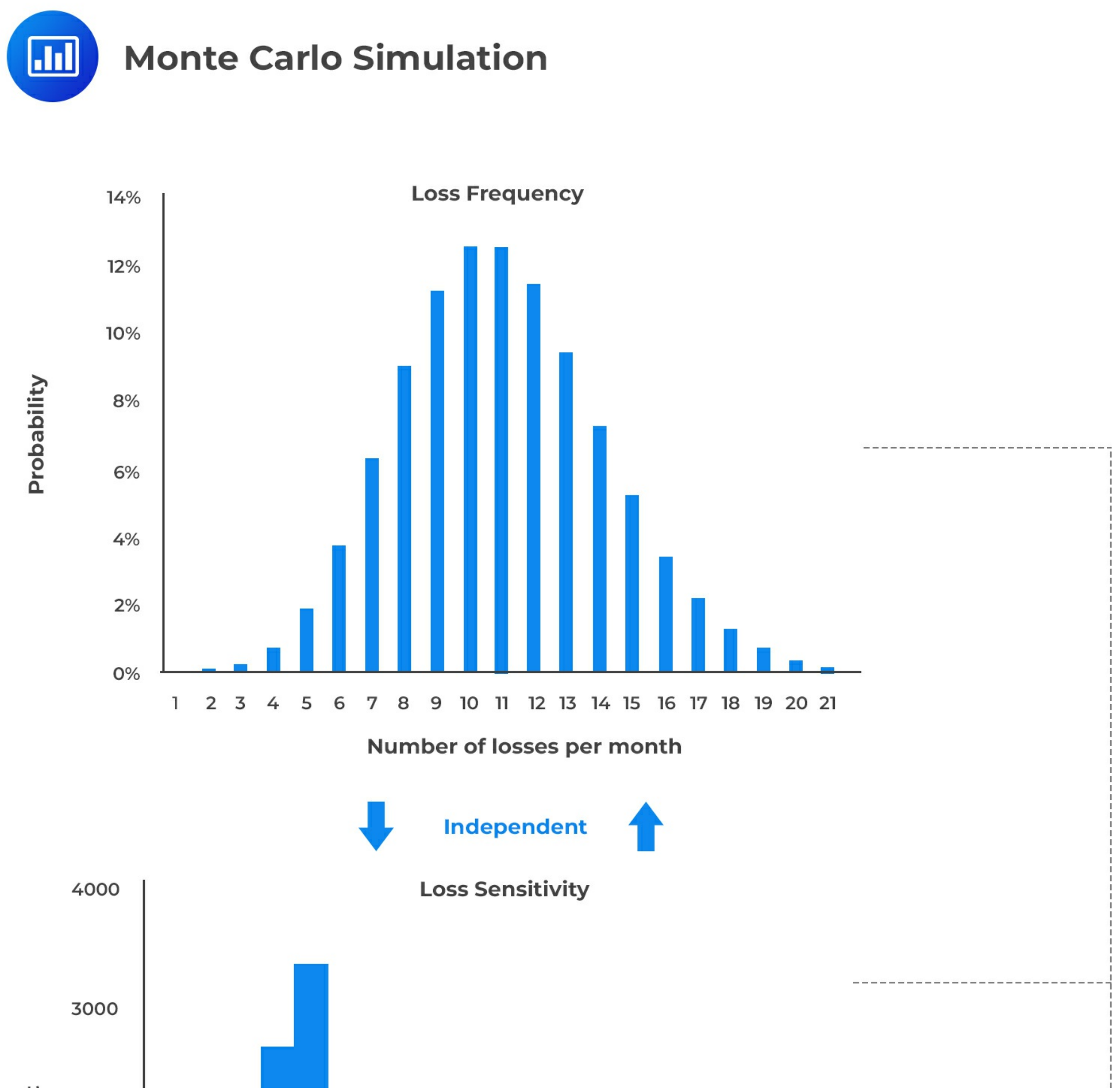
$$\ln\left(\frac{\mu}{\sqrt{1+w}}\right) = \ln\left(\frac{50}{\sqrt{1.16}}\right) = 3.8378$$

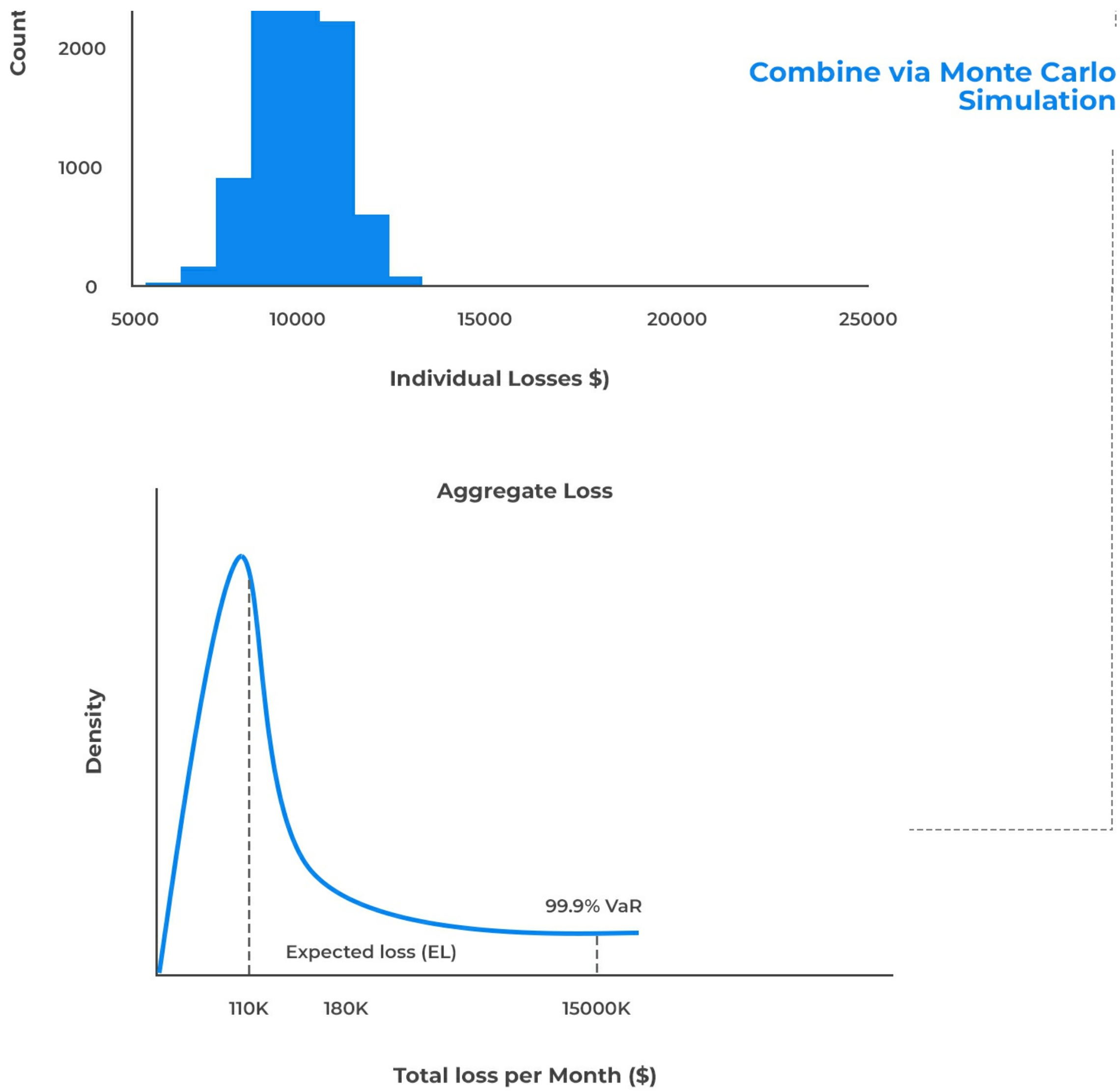
The variance of the logarithm of the log size is given by:

$$\ln(1+w) = \ln(1.16) = 0.1484$$

Monte Carlo Simulation for Operational Risk

After estimating λ, μ, and σ, Monte Carlo simulation can be utilized to determine the probability distribution of the loss as illustrated below:





The necessary steps of the Monte Carlo Simulations are as follows:

I. Sampling is done from the Poisson distribution to determine the number of loss events (=n) in a year. For instance, we can sample the percentile of Poisson distribution as a random number between 0 and 1

II. Sample n times from the lognormal distribution of the loss size for each of the n loss occurrences.

III. Sum the n loss sizes to determine the total loss.

IV. Repeat the process (steps I to III) many times.

Example: Demonstration of the Monte Carlo Simulation

Step 1: Sampling is done from the Poisson distribution

Assume the average loss frequency is six, and a number sampled in step I is 0.29. Therefore, 0.29 corresponds to three loss events in a year because using the formula.

$$\Pr(n) = \frac{e^{-\lambda}\lambda^{n}}{n!}$$

$$\Pr(0) + \Pr(1) + \Pr(2) = 0.284 \text{ and } \Pr(0) + \Pr(1) + \Pr(2) + \Pr(3) = 0.4660$$

Therefore 0.29 lies in between two cumulative probabilities.

Also, assume that the loss size has a mean of 60 and a standard deviation of 5 using the formulas $\ln\left(\frac{\mu}{\sqrt{1+w}}\right)$ and $\ln(1+w)$.

Step 2: Sample n times from the lognormal distribution

In step II we will sample three times from the lognormal distribution using the mean

$$\ln\left(\frac{60}{\sqrt{1+\left(\frac{5}{60}\right)^{2}}}\right) = 4.0909 \text{ and standard deviation } \ln\left(1+\left(\frac{5}{60}\right)^{2}\right) = 0.0069$$

Now, assume the sampled numbers are 4.12, 4.70, and 5.5. Note that the lognormal distribution gives the logarithm of the loss size. Therefore we need to exponentiate the sampled numbers to get the actual losses. As such, the three losses are $e^{4.12}$=61.56, $e^{4.70}$=109.95 and $e^{5.5}$=244.69.

Step 3: Sum the n loss sizes to determine the total loss

This gives the total loss of 416.20 (61.56+109.95+244.69) in the trial herein.

Step 4: Repeat the process (steps I to III) many times

Step 4 requires that the same process be repeated many times to generate the probability distribution for the total loss, from which the desired percentile can be computed.

Estimation Procedures for Loss Frequency and Loss Severity

The estimation of the loss frequency and loss severity involves the use of data and subjective judgment. Loss frequency is estimated from the banks' data or subjectively estimated by operational risk professionals after considering the controls in place.

In the case that the loss severity cannot be approximated from the bank's data, the loss incurred by other financial institutions may be used as a guide. The methods by which banks share information have been laid out. Moreover, there exist data vendor services (such as Factiva), which are useful at supplying data on publicly reported losses incurred by other banks.

Common Data Issues in the Estimation of Loss Frequency and Severity Distributions

I. **Inadequate historical records:** The data available for operational risk losses – including loss frequency and loss amounts – is grossly inadequate, especially when compared to credit risk data. This inadequacy creates problems when trying to model the loss distribution of expected losses.

II. **Inflation:** When modeling the loss distribution using both external and internal data, an adjustment must be made for inflation. The purchasing power of money keeps on changing so that a $10,000 loss recorded today would not have the same effect as a similar loss recorded, say, ten years ago.

III. **Firm-specific adjustments:** No two firms are the same in terms of size, financial structure, and operational risk management. As such, when using external data, it is essential to make adjustments to the data in cognizance of the different characteristics of the source and your bank. A simple proportional adjustment can either underestimate or overestimate the potential loss.

The generally accepted scale adjustment for firm size is as follows:

$$\text{Estimated Loss for Bank A} = \text{Observed Loss for Bank B} \times \left(\frac{\text{Bank A Revenue}}{\text{Bank B Revenue}}\right)^{0.23}$$

Example: Calculating the Estimated Loss based on its Size

Suppose that Bank A has revenues of USD 20 billion and incurs a loss of USD 500 million. Another bank B has revenues of USD 30 billion and experiences a loss of USD 300 million. What is the estimated loss for Bank A?

Solution

$$\text{Estimated Losss for Bank A} = \text{Observed Loss for Bank B} \times \left(\frac{\text{Bank A Revenue}}{\text{Bank B Revenue}}\right)^{0.23}$$

$$= 300 \times \left(\frac{20}{30}\right)^{0.23} = \text{USD } 273.29 \text{ million.}$$

Scenario Analysis in Instances when Data is Scarce

Scenario analysis aims at estimating how a firm would get along in a range of scenarios, some of which have not occurred in the past. It's particularly essential when modeling low-frequency high-severity losses, which are essential to determine the extreme tails of the loss distribution.

The objective of the scenario analysis is to list events and create a scenario for each one. Scenarios considered come from:

- The firms' own experience;
- The experience of other firms;
- Market analysts and consultants; or
- The risk management unit in liaison with senior management.

For each scenario, loss frequency and loss severity are approximated. Monte Carlo simulations are used to determine a probability distribution for total loss across diverse types of losses. The loss frequency estimates should capture the existing controls at the financial institution and the type of business.

Estimation of the probability of rare events is challenging. One method is to state several categories and ask operational risk experts to put each loss into a category. For instance, some of the categories might be a scenario that happens once every 1,000 years on average, which is equivalent to $\lambda = 0.001$. The bank could also use a scenario happening once every 100 years on average, which is equivalent to $\lambda = 0.01$, and so on.

Operational risk experts estimate the loss severity, but rather than in the form of the mean and standard deviation, it is more suitable to estimate the 1 percentile to 99 percentile range of the loss distribution. These estimated percentiles can be fitted with the lognormal distribution. That is, if 1 percentile and 99 percentiles of the loss are 50 and 100 respectively, then 3.91 (ln(50)) and 4.61 (ln(100)) are 1 and 99 percentiles for the logarithm of the loss distribution, respectively.

The concluding point in scenario analysis is that it takes into consideration the losses that have never been incurred by a financial institution but can occur in the future. Managerial judgment is used to analyze the loss frequency and loss severity which can give hints on how such loss events may appear, which in turn assist the firms in setting up plans to respond to loss events or reduce the likelihood of it happening.

Allocation of Operational Risk Capital to Business Units

Typically, economic capital is allocated to business units, after which the return on capital is computed. Similar to credit risk, the same principles are used in the allocation of operational risk capital. The provision of operational risk capital to business units acts as an incentive to the business unit manager to reduce the operational risk because if the manager reduces the loss frequency and severity, less operational capital will be allocated. Consequently, the profit from the business unit will improve.

In a nutshell, the allocation of operational risk capital should sensitize the manager on the benefits of operational risk. Operational risk reduction does not necessarily reach an optimal point because there exists operational risk in a firm that cannot be avoided. Therefore, cost-benefit analysis is carried out when operational risk is reduced by increasing the operational cost.

Use of the Power Law to Measure Operational Risk

The power-law states that, if v is a random variable and that x is another random variable which has a higher value than v, i.e., (v>x), then it is true that:

$$\Pr(v > x) = Kx^{-\alpha}$$

Where K and α are the parameters.

The power law holds for some probability distributions, and it describes the fatness of the right tail of the probability distribution of v. K is a scale factor, and α depends on the fatness of the right tail of the distribution. That is, the fatness of the right tail increases with a decrease in α.

According to the mathematician G.V Gnedenko, the power for many distributions increases as x tends to infinity. Practically, the power law is usually taken to be true for the values of x at the top 5% of the distribution. Some of the distributions in which the power law holds to be true are the magnitude of earthquakes, trading volume of the stocks, income of individuals, and the sizes of the corporations.

Generally, the power law holds for the probability distributions of random variables resulting from aggregating numerous independent random variables. Adding up the independent variables, we usually get a normal distribution, and fat tails arise when the distribution is a result of many multiplicative effects.

According to Fontnouvelle (2003), the power law holds for the operational risk losses, which turns to be crucial.

Example: Measuring Operational Risk Using the Power Law

A risk manager has established that there is a 95% probability that losses over the next year will not exceed $50 million. Given that the power law parameter is 0.7, calculate the probability of the loss exceeding (a) 20 million, (b) 70 million, and (c) 100 million.

Solution

According to the power law

$$\Pr(v > x) = Kx^{-\alpha}$$

This implies from the question that,

$$0.05 = K50^{-0.7} \Rightarrow K = 0.7731$$

Thus,

$$P(v > x) = 0.7731V^{-0.7}$$

Now,

when x=20

$$p(v > 20) = 0.7731\,(20)^{-0.7} = 0.09495$$

when x=70

$$p(v > 20) = 0.7731\,(70)^{-0.7} = 0.03951$$

when x=100

$$p(v > 100) = 0.7731\,(100)^{-0.7} = 0.03071$$

Managing Operational Risk

It is crucial to measure operational risk and to compute the required amount of operational risk capital. However, it is also imperative to reduce the likelihood of significant losses and severity in case an event occurs. More often, financial institutions learn from each other. That is, if significant losses are incurred in one of the financial institutions, risk managers of other financial institutions will try and study what happened so that they can make necessary plans to avoid a similar event.

Some of the methods of reducing operational risk include: reducing the cause of losses, risk control, and self-assessment, identifying key risk indicators (KRI's), and employee education.

Causal Relationship

Causal relationships describe the search for a correlation between firm actions and operational risk losses. It is an attempt to identify firm-specific practices that can be linked to both past and future operational risk losses. For example, if the use of new computer software coincides with losses, it is only wise to investigate the matter in a bid to establish whether the two events are linked in any way.

Once a causal relationship has been identified, the firm should then decide whether or not to act on it. This should be done by conducting a cost-benefit analysis of such a move.

Risk and Control Self Assessment (RCSA)

Risk and control self-assessment (RCSA) involves asking departmental heads and managers to single out the operational risks in their jurisdiction. The underlying argument is that unit managers are the focal point of the flow of information and correspondence within a unit. As such, they are the persons best placed to understand the risks pertinent to their operations.

Some of the approaches RCSA methods include:

I. Analyzing the historical incidences with line managers
II. Requesting line managers to complete a risk questionnaire
III. Carrying out interviews with line managers and their staff
IV. Utilizing the suggestion boxes and intranet reporting portals
V. Executing brainstorming in a workshop environment
VI. Analyzing the reports from third parties such as auditors and regulators

RCSA should be done periodically, such as yearly. The problem with this approach is that managers may not divulge information freely if they feel they are culpable or the risk is out of control. Also, a manager's perception of risk and its potential rewards may not conform to the firm-wide assessment. For these reasons, there is a need for independent review.

Key Risk Indicators (KRIs)

Key risk indicators seek to identify firm-specific conditions that could expose the firm to operational risk. KRIs are meant to provide firms with a system capable of predicting losses, giving the firm ample time to make the necessary adjustments. Examples of KRIs include:

- Staff turnover
- Number of vacant positions
- Number of failed transactions over a specified time period
- Percentage of employees that take up the maximum leave days on offer

The hope is that key risk indicators can identify potential problems and allow remedial action to be taken before losses are incurred.

Education

It is essential to educate the employees on the prohibited business practices and breeding risk culture where such unacceptable practices might be executed. Moreover, the legal branch of a financial institution educates the employees to be cautious when writing emails and answering phone calls. Essentially, employees should be mindful that their emails and recorded calls could become public knowledge.

Moral Hazard and Adverse Selection when Using Insurance to Mitigate Operational Risks

Earlier in the reading, we saw that a bank using the AMA approach could reduce its capital charge, subject to extensive investment in operational risk management. One of the ways through which a bank can achieve this is by taking an insurance cover. That way, the firm is eligible for compensation if it suffers a loss emanating from a covered risk.

For all its advantages, taking an insurance policy comes with two problems:

1. **Moral Hazard:** Moral hazard describes the observation that an insured firm is likely to act differently in the presence of an insurance cover. In particular, traders might increasingly

take high-risk positions in the knowledge that they are well protected from heavy losses. Without such an insurance policy, the traders would be a bit more cautionary and restricted in their trading behavior.

In a bid to tame the moral hazard problem, insurers use a range of tactics. These may include deductibles, coinsurance, and policy limits. Stiff penalties may also be imposed in case there is indisputable evidence of reckless, unrestricted behavior.

A firm can intentionally keep insurance cover private. This ensures that its traders do not take unduly high-risk positions.

2. **Adverse Selection:** Adverse selection describes a situation where the risk seller has more information than the buyer about a product, putting the buyer at a disadvantage. For example, a company providing life assurance may unknowingly attract heavy smokers, or even individuals suffering from terminal illnesses. If this happens, the company effectively takes on many high-risk persons but very few low-risk individuals. This may result in a claim experience that's worse than initially anticipated.

   On matters trading, firms with poor internal controls are more likely to take up insurance policies compared to firms with robust risk management frameworks. To combat adverse selection, an insurer has to go to great lengths toward understanding a firm's internal risk controls. The premium payable can then be adjusted to reflect the risk of the policy.

Question

Question 1

Melissa Roberts, FRM, has observed 12 losses in her portfolio over the last four years. She believes the frequency of losses follows a Poisson distribution with a parameter λ. The probability that she will observe a total of 4 losses over the next year is *closest* to:

A. 17%

B. 16%

C. 20%

D. 0.53%

The correct answer is **A.**

We need to find the parameter λ, which from the questions we have,

$$\lambda = \frac{12 \text{ losses}}{4 \text{ years}} = 3 \text{ losses per year}$$

The probability of n losses in a given year is given by Poisson distribution, defined as

$$\Pr(n) = \frac{e^{-\lambda}\lambda^{n}}{n!}$$

$$\Pr(n = 4) = \frac{e^{-3}3^{4}}{4!} = 0.168$$

Question 2

According to the Basel Committee, a bank has to satisfy certain qualitative standards to be allowed to use the advanced measurement approach when computing the economic capital required. Which of the following options is NOT one of the standards?

A. The bank must have a system capable of allocating economic capital for operational risk across all business lines in a way that creates incentives for these business lines to manage operational risk better.

B. Internal and external auditors must regularly and independently review all operational risk management processes. The review must include the policy development process and independent scrutiny of the risk management function.

C. The bank's operational risk measurement system should only make use of internally generated data to avoid the bias associated with external data.

D. The bank must have an operational risk management function tasked with identification, assessment, monitoring, and control of operational risk.

The correct answer is **C**.

The Basel committee does not rule out the use of external data by banks. In fact, the committee recommends the use of a combination of both external and internal data to estimate the unexpected loss. External data may not conform to a particular firm, but firms are allowed to scale the data to fit their profiles. In some cases, internal data may be either insufficient or entirely unavailable, forcing the firm to look elsewhere.

Reading 52: Stress Testing

After completing this reading, you should be able to:

- Describe the rationale for the use of stress testing as a risk management tool.
- Explain key considerations and challenges related to stress testing, including choice of scenarios, regulatory specifications, model building, and reverse stress testing.
- Describe the relationship between stress testing and other risk measures, particularly in enterprise-wide stress testing.
- Describe stressed VaR and stressed ES, including their advantages and disadvantages, and compare the process of determining stressed VaR and ES to that of traditional VaR and ES.
- Describe the responsibilities of the board of directors, senior management, and the internal audit function in stress testing governance.
- Describe the role of policies and procedures, validation, and independent review in stress testing governance.
- Describe the Basel stress testing principles for banks regarding the implementation of stress testing.

Stress testing is a risk management tool that involves analyzing the impacts of the extreme scenarios that are unlikely but feasible. The main question for financial institutions is whether they have adequate capital and liquid assets to survive stressful times. Stress testing is done for regulatory purposes or for internal risk management by financial institutions. Stress testing can be combined with measurement of the risk such as the Value-at-Risk (VaR) and the Expected Shortfall (ES) to give a detailed picture of the risks facing a financial institution.

This chapter deals with the internally generated stress testing scenarios, regulatory requirements of stress testing, governance issues of stress testing, and the Basel stress testing principles.

Rationale for the Use of Stress Testing as a Risk Management Tool

- Stress testing serves to warn a firm's management of potential adverse events arising from the firm's risk exposure and goes further to give estimates of the amount of capital needed to absorb losses that may result from such events.

- Stress tests help to avoid any form of complacency that may creep in after an extended period of stability and profitability. It serves to remind management that losses could still occur, and adequate plans have to be put in place in readiness for every eventuality. This way, a firm is able to avoid issues like underpricing of products, something that could prove financially fatal.

- Stress testing is a key risk management tool during periods of expansion when a firm introduces new products into the market. There may be very limited loss data or none at all, for such products, and hypothetical stress testing helps to come up with reliable loss estimates.

- Under pillar 1 of Basel II, stress testing is a requirement of all banks using the Internal Models Approach (IMA) to model market risk and the internal ratings-based approach to model credit risk. These banks have to employ stress testing to determine the level of capital they are required to have.

- Stress testing supplements other risk management tools, helping banks to mitigate risks through measures such as hedging and insurance. By itself, stress testing cannot address all risk management weaknesses, nor can it provide a one-stop solution.

Comparison between Stress Testing and the VaR and ES

Recall that the VaR and ES are estimated from a loss distribution. VaR enables a financial institution to conclude with X% likelihood that the losses will not exceed the VaR level during time T. On the other hand, ES enables the financial institutions to conclude whether the losses exceed the VaR level during a given time T and hence the expected loss will be the ES amount.

VaR and ES are backward-looking. That is, they assume that the future and the past are the same. This is actually one disadvantage of VaR and ES. On the other hand, stress testing is forward-looking.

It asks the question, “what if?”.

While stress testing largely does not involve probabilities, VaR, and ES models are founded on probability theory. For example, a 99.9% VaR can be viewed as a 1-in-1,000 event.

The backward-looking ES and VaR consider a wide range of scenarios that are potentially good or bad to the organization. However, stress testing considers a relatively small number of scenarios that are all bad for the organization.

Specifically, for the market risk, VaR/ES analysis often takes a short period of time, such as a day, while stress testing takes relatively long periods, such as a decade.

The primary objective of stress testing is to capture the enterprise view of the risks impacting a financial institution. The scenarios used in the stress testing are often defined based on the macroeconomic variables such as the unemployment rates and GDP growth rates. The effect of these variables should be considered in all parts of an institution while considering interactions between diverse areas of an institution.

Stressed VaR and Stressed ES

Conventional VaR and ES are calculated from data spanning from one to five years, where a daily variation of the risk factors during this period is used to compute the potential future movements.

However, in the case of the stressed VaR and stressed ES, the data is obtained from specifically stressed periods (12-month stressed period on current portfolios according to Basel rules). In other words, stressed VaR and stressed ES generates conditional distributions and conditional risk measures. As such, they are conditioned to a recurrence of a given stressed period and thus can be taken as a historical stress testing.

Though stressed VaR and stressed ES might be objectively similar, they are different. Typically the time horizon for the stressed VaR/ES is short (one to ten days), while for the stress testing, it considers relatively longer periods.

For instance, assume that a stressed period is the year 2007. The stressed VaR would conclude that if there was a repeat of 2007, then there is an X% likelihood that losses over a period of T days will not surpass the stressed VaR level. On the other hand, stressed ES would conclude that if the losses over T days do not exceed the stressed VaR level, then the expected loss is the stressed ES.

However, stress testing would ask the questions “if the following year (2008) is the same as in 2007, will the financial institution survive?” Alternatively, what if the conditions of the next year are twice as adverse as that of 2007, will the financial institution survive? Therefore, stress testing does not consider the occurrence of the worst days of 2008 but rather the impact of the whole year.

There is also a difference between conventional VaR and the stressed VaR. Conventional VaR can be back-tested while stressed VaR cannot. That is, if we can compute one-day VaR with 95% confidence, we can go back and determine how effective it would have worked in the past. We are not able to back-test the stressed VaR output and its results because it only considers the adverse conditions which are generally infrequent.

Types of Scenarios in Stress Testing

The basis of choosing a stress testing scenario is the selection of a time horizon. The time horizon should be long enough to accommodate the full analysis of the impacts of scenarios. Long time horizons are required in some situations. One-day to one-week scenarios can be considered, but three months to two-year scenarios are typically preferred.

The regulators recommend some scenarios, but in this section, we will discuss internally chosen scenarios. They include using historical scenarios, stressing key variables, and developing ad hoc scenarios that capture the current conditions of the business.

Historical Scenarios

Historical scenarios are generated by the use of historical data whose all relevant variables will behave in the same manner as in the past. For instance, variables such as interest rates and credit rate spreads are known to repeat past changes. As such, actual changes in the stressed period will be assumed to repeat themselves while proportional variations will be assumed for others. A good example of a historical scenario is the 2007-2008 US housing recession, which affected a lot of financial institutions.

In some cases, a moderately adverse scenario is made worse by multiplying variations of all risk factors by a certain amount. For instance, we could multiply what happened in the loss-making one-month period and increase the frequency of movement of all relevant risk movements by ten. As a result, the scenario becomes more severe to financial institutions. However, this approach assumes linear relationships between the movements in risk factors, which is not always the case due to correlations between the risk factors.

Other historical scenarios are based on one-day or one-week occurrences of all market risk factors. Such events include terrorist attacks (such as 9/11 terrorist attacks) and one-day massive movement of interest rates (such as on April 10, 1992, when ten-year bond yields changed by 8.7 standard deviations).

Stressing Key Variables

A scenario could be built by assuming that a significant change occurs in one or more key variables. Such changes include:

- A 2% decline in the GDP
- A 25% decrease in equity prices
- A 100% increase in all volatilities
- A 4% increase in the unemployment rate
- A 200-basis point increase in all interest rates

Some other significant variations could occur in factors such as money exchange rates, prices of commodities, and default rates.

In the case of the market risk, small changes in measured using the Greek letters (such as delta and gamma). The Greek letters cannot be used in stress testing because the changes are usually large. Moreover, Greeks are used to measure risk from a unit market variable over a short period of time, while stress testing incorporates the interaction of the different market variables over a long period of time.

Ad Hoc Stress Tests

The stress testing scenarios we have been discussing above are performed regularly, after which the results are used to test the stability of the financial structure of a financial institution in case of extreme conditions. However, the financial institutions need to develop ad hoc scenarios that capture the current economic conditions, specific exposures facing the firm, and update analysis of potential future extreme events. The firms either generate new scenarios or modify the existing scenarios based on previous data.

An example of an event that will prompt the firms to develop an ad hoc scenario is the change in the government policy on an important aspect that impacts the financial institutions or change in Basel regulation that requires increment of the capital within short periods of time.

The boards, senior management, and economic experts use their knowledge in markets, global politics, and current global instabilities to come with adverse scenarios. The senior management carries out a brain-storming event, after which they recommend necessary actions to avoid unabsorbable risks.

Using the Stress Testing Results

While stress testing, it is vital to involve the senior management for it to be taken seriously and thus used for decision making. The stress-testing results are not only used to satisfy the “what if” question, but also the Board and management should analyze the results and decide whether a certain class of risk mitigation is necessary. Stress testing makes sure that the senior management and the Board do not base their decision-making on what is most likely to happen, but also consider other alternatives less likely to happen that could have a dramatic result on the firm.

Model Building

It is possible to see how the majority of the relevant risk factors behave in a stressed period while building a scenario, after which the impact of the scenario on the firm is analyzed in an almost direct manner. However, scenarios generated by stressing key variables and ad hoc scenarios capture the variations of a few key risk factors or economic variables. Therefore, in order to exhaust the scenarios, it is necessary to build a model to determine how the “left out” variables are expected to behave in a stressed market condition. The variables stated in the context of the stress testing are termed as **core variables,** while the remaining variables are termed as **peripheral variables.**

One method is performing analysis, such as regression analysis, to relate the peripheral variables to the core variables. Note that the variables are based on the stressed economic conditions. Using the data of the past stressed periods is most efficient in determining appropriate relationships.

For example, in case of the credit risk losses, data from the rating agencies, such as default rates, can be linked to an economic variable such as GDP growth rate. Afterward, general default rates expected in various stressed periods are determined. The results can be modified (scaled up or down) to determine the default rate for different loans or financial institutions. Note that the same analysis can be done to the recovery rates to determine loss rates.

The Knock-On Effects

Apart from the immediate impacts of a scenario, there are also knock-on effects that reflect how financial institutions respond to extreme scenarios. In its response, a financial institution can make decisions that can further worsen already extreme conditions.

For instance, during the 2005-2006 US housing price bubble, banks were concerned with the credit quality of other banks and were not ready to engage in interbank lending, which made funding costs for banks rise.

Reverse Stress Testing

Recall that stress testing involves generating scenarios and then analyzing their effects. Reverse stress testing, as the name suggests, takes the opposite direction by trying to identify combinations of circumstances that might lead financial institutions to fail.

By using historical scenarios, a financial institution identifies past extreme conditions. Then, the bank determines the level at which the scenario has to be worse than the historical observation to cause the financial institution to fail. For instance, a financial institution might conclude that twice the 2005-2006 US housing bubble will make the financial institution to fail. However, this kind of reverse stress testing is an approximation. Typically, a financial institution will use complicated models that take into consideration correlations between different variables to make the market conditions more stressed.

Finding an appropriate combination of risk factors that lead the financial institution to fail is a challenging feat. However, an effective method is to identify some of the critical factors such as GDP growth rate, unemployment rates, and interest rate variations, then build a model that relates all other appropriate variables to these key variables. After that, possible factor combinations that can lead to failure are searched iteratively.

Regulatory Stress Testing

US, UK, and EU regulators require banks and insurance companies to perform specified stress tests. In the United States, the Federal Reserve performs stress tests of all the banks whose consolidated assets are over USD 50 billion. This type of stress test is termed as Comprehensive Capital Analysis and Review (CCAR). Under CCAR, the banks are required to consider four scenarios:

I. Baseline Scenario
II. Adverse Scenario
III. Scenario
IV. An internal Scenario

The baseline scenario is based on the average projections from the surveys of the economic predictors but does not represent the projection of the Federal Reserve.

The adverse and the severely adverse scenarios describe hypothetical sets of events which are structured to test the strength of banking organizations and their resilience. Each of the above scenarios consists of the 28 variables (such as the unemployment rate, stock market prices, and interest rates) which captures domestic and international economic activity accompanied by the Board explanation on the overall economic conditions and variations in the scenarios from the past year.

Banks are required to submit a capital plan, justification of the models used, and the outcomes of their stress testing. If a bank fails to stress test due to insufficient capital, the bank is required to raise more capital while restricting the dividend payment until the capital has been raised.

Banks with consolidated assets between USD 10 million and USD 50 million are under the Dodd-Fank Act Stress Test (DFAST). The scenarios in the DFAST are similar to those in the CCAR. However, in the DFAST, banks are not required to produce a capital plan.

Therefore, through stress tests, regulators can consistently evaluate the banks to determine their ability to extreme economic conditions. However, they recommend that banks develop their scenarios.

Responsibilities of the Board of Directors, Senior Management and the Internal Audit Function in Stress Testing Activities

For effective operation of stress testing, the Board of directors and senior management should have distinct responsibilities. What's more, there should be some shared responsibilities, although a few roles can be set aside exclusively for one of the two groups.

Responsibilities of the Board of Directors

1. **The buck stops with the Board:** The Board of directors is "ultimately" responsible for a firm's stress tests. Even if board members do not immerse themselves in the technical details of stress tests, they should ensure that they stay sufficiently knowledgeable about stress-testing procedures and interpretation of results.
2. **Continuous involvement:** Board members should regularly receive summary information on stress tests, including results from every scenario. Members should then evaluate these results to ensure they take into account the firm's risk appetite and overall strategy.
3. **Continuous review:** Board members should regularly review stress testing reports with a view to not just critic key assumptions but also supplement the information with their views that better reflect the overall goals of the firm.
4. **Integrating stress testing results in decision making:** The Board should make key decisions on investment, capital, and liquidity based on stress test results along with other information. While doing this, the Board should proceed with a certain level of caution in

cognizance of the fact that stress tests are subject to assumptions and a host of limitations.

5. **Formulating stress-testing guidelines:** It's the responsibility of the Board to come up with guidelines on stress testing, such as the risk tolerance level (risk appetite).

Responsibilities of Senior Management

1. **Implementation oversight:** Senior management has the mandate to ensure that stress testing guidelines authorized by the Board are implemented to the letter. This involves establishing policies and procedures that help to implement the Board's guidelines.
2. **Regularly reporting to the Board:** Senior management should keep the Board up-to-date on all matters to do with stress testing, including test designs, emerging issues, and compliance with stress-testing policies.
3. **Coordinating and Integrating stress testing across the firm:** Members of senior management are responsible for propagating widespread knowledge on stress tests across the firm, making sure that all departments understand its importance.
4. **Identifying grey areas:** Senior management should seek to identify inconsistencies, contradictions, and possible gaps in stress tests to make improvements to the whole process.
5. **Ensuring stress tests have a sufficient range:** In consultations with the Board of directors, senior management has to ensure that stress testing activities are sufficiently severe to gauge the firm's preparation for all possible scenarios, including low-frequency high-impact events.
6. **Using stress tests to assess the effectiveness of risk mitigation strategies:** Stress tests should help the management to assess just how effective risk mitigation strategies are. If such strategies are effective, significantly severe events will not cause significant financial strain. If the tests predict significant financial turmoil, it could be that the hedging strategies adopted are ineffective.
7. **Updating stress tests to reflect emerging risks:** As time goes, an institution will gradually gain exposure to new risks, either as a result of market-wide trends or its investment activities. It is the responsibility of senior management to develop new stress-testing techniques that reflect the institution's new risk profile.

Role of the Internal Audit

Internal audit should:

- Independently evaluate the performance, integrity, and reliability of stress-testing activities;
- Ensure that stress tests across the organization are conducted in a sound manner and remain relevant in terms of the scenarios tested;
- Assess the skills and expertise of the staff involved in stress-testing activities;
- Check that approved changes to stress-testing policies and procedures are implemented and appropriately documented;
- Evaluate the independent review and validation exercises;

To accomplish all the above, internal audit staff must be well qualified. They should be well-grounded in stress-testing techniques and technical expertise to be able to differentiate between excellent and inappropriate practices.

The Role of Policies and Procedures, Validation, and Independent Review in Stress Testing Governance

Policies and Procedures

A financial institution should set out clearly stated and understandable policies and procedures governing stress testing, which must be adhered to. The policies and procedures ensure that the stress testing of parts of a financial institution converges to the same point.

The policies and procedures should be able to:

- Explain the purpose of stress testing;
- Describe the procedures of stress testing;
- State the frequency at which the stress testing can be done;

- Describe the roles and responsibilities of the parties involved in stress testing;
- Provide an explanation of the procedures to be followed while choosing the scenarios;
- Describe how the independent reviews of the stress testing will be done;
- Give clear documentation on stress testing to third parties (e.g., regulators, external auditors, and rating agencies);
- Explain how the results of the stress testing will be used and by whom;
- They were amended as the stress testing practices changes as the market conditions change;
- Accommodate tracking of the stress test results as they change through time; and
- Document the activities of models and the software acquired from the vendors or other third parties.

Validation and Independent Review Governance

The stress testing governance covers the independent review procedures, which are expected to be unbiased and provide assurance to the board that stress testing is carried out while following the firm's policies and procedures. Financial institutions use diverse models that are subject to independent review to make sure that they serve the intended purpose.

Validation and independent review should involve the following:

- Ensuring that validation and independent review are conducted on an ongoing basis;
- Ensuring that subjective or qualitative aspects of a stress test are also validated and reviewed, even if they cannot be tested in quantitative terms;
- Acknowledging limitations in stress testing;
- Ensuring that stress-testing standards are upheld;
- Acknowledging data weaknesses or limitations, if any;

- Ensuring that there is sufficient independence in both validation and review of stress tests;
- Ensuring that third-party models used in stress-testing activities are validated and reviewed to determine if they are fit for the purpose at hand;
- Ensuring that stress tests results are implemented rigorously, and verifying that any departure from the recommended actions is backed up by solid reasons.

Basel Stress-Testing Principles

The Basel Committee emphasizes that stress testing is a crucial aspect by requiring that the market risk calculations are based on the internal VaR and the Expected Shortfall (ES) models, which should be accompanied by " rigorous and comprehensive" stress testing. Moreover, banks that use the internal rating approach of the Basel II to calculate the credit risk capital should perform a stress test to evaluate the strength of their assumptions.

Influenced by the 2007-2008 financial crisis, the Basel Committee published the principles of stress-testing for the banks and corresponding supervisors. The overarching emphasis of the Basel committee was the importance of stress testing in determining the amount of capital that will cushion banks against losses due to large shocks.

Therefore, the Basel committee recognized the importance of stress testing in:

- Giving a forward-looking perspective on the evaluation of risk;
- Overcoming the demerits of modes and historical data;
- Facilitating the development of risk mitigation, or any other plans to reduce risks in different stressed conditions;
- Assisting internal and external communications;
- Supporting the capital and liquidity planning procedures; and
- Notifying and setting of risk tolerance.

When the Basel committee considered the stress tests done before 2007-2008, they concluded that:

- It is crucial to involve the Board and the senior management in stress testing. The Board and the senior management should be involved in stress testing aspects such as choosing scenarios, setting stress testing objectives, analysis of the stress testing results, determining the potential actions, and strategic decision making. During the crisis, banks that had senior management interested in developing a stress test, which eventually affected their decision-making, performed fairly well.

- The approaches of the stress-testing did not give room for the aggregation of different exposures in different parts of a bank. That is, experts from different parts of the bank did not cooperate to produce an enterprise-wide risk view.

- The scenarios chosen in the stress tests were too moderate and were based on a short period of time. The possible correlations between different risk types, products, and markets were ignored. As such, the stress test relied on the historical scenarios and left out risks from new products and positions taken by the banks.

- Some of the risks were not considered comprehensively in the chosen scenarios. For example, counterparty credit risk, risks related to structured products, and product awaiting securitizations were partially considered. Moreover, the effect of the stressed scenario on liquidity was underrated.

Basel Committee Stress Testing Principles

According to the Basel Committee on Banking Supervision's "Stress Testing Principles" published in December 2017:

1. **Stress testing frameworks should incorporate an effective governance structure.**

   The stress testing frameworks should involve a governance structure that is clear, documented, and comprehensive. The roles and responsibilities of senior management, oversight bodies, and those concerned with stress testing operations should be clearly stated.

The stress testing framework should incorporate a collaboration of all required stakeholders and the appropriate communication to stakeholders of the stress testing methodologies, assumptions, scenarios, and results.

2. **Stress testing frameworks should have clearly articulated and formally adopted objectives.**

The stress testing frameworks should satisfy the objectives that are documented and approved by the Board of an organization or any other senior governance. The objective should be able to meet the requirements and expectations of the framework of the bank and its general governance structure. The staff mandated to carry out stress testing should know the stress testing framework's objectives.

3. **Stress testing frameworks should capture material and relevant risks and apply sufficiently severe stresses.**

Stress testing should reflect the material and relevant risk determined by a robust risk identification process and key variables within each scenario that is internally consistent. A narrative should be developed explaining a scenario that captures risks, and those risks that are excluded by the scenario should be described clearly and well documented.

4. **Stress testing should be utilized as a risk management tool and to convey business decisions.**

Stress testing is typically a forward-looking risk management tool that potentially helps a bank in identifying and monitoring risk. Therefore, stress testing plays a role in the formulation and implementation of strategic and policy objectives. When using stress testing results, banks and authorities should comprehend crucial assumptions and limitations such as the relevance of the scenario, model risks, and risk coverage. Lastly, stress testing as a risk management tool should be done regularly in accordance with a well-developed schedule (except ad hoc stress tests). The frequency of a stress test depends on:

- The objective of the stress testing framework;
- The size and complexity of the financial institution; and
- Changes in the macroeconomic environment.

5. **Resources and organizational structures should be adequate to meet the objectives of the stress testing framework.**

   Stress testing frameworks should have adequate organizational structures that meet the objectives of the stress test. The governance processes should ensure that the resources for stress testing are adequate, such that these resources have relevant skill sets to implement the framework.

6. **Stress tests should be supported by accurate and sufficiently granular data and robust IT systems.**

   Stress tests identify risks and produce reliable results if the data used is accurate and complete, and available at an adequately granular level and on time. Banks and authorities should establish a sound data infrastructure which is capable of retrieving, processing, and reporting of information used in stress tests. The data infrastructure should be able to provide adequate quality information to satisfy and objectives of the stress testing framework. Moreover, structures should be put in place to cover any material information deficiencies.

7. **Models and methodologies to assess the impacts of scenarios and sensitivities should be fit for purpose.**

   The models and methodologies utilized in stress testing should serve the intended purpose. Therefore,

   - There should be an adequate definition of coverage, segmentation, and granularity of the data and the types of risks based on the objectives of the stress test framework. All is done at the modeling stage;

- The complexity of the models should be relevant to both the objectives of the stress testing and target portfolios being assessed using the models; and
- The models and the methodologies in a stress test should be adequately justified and documented.

The model building should be a collaborative task between the different experts. As such, the model builders engage with stakeholders to gain knowledge on the type of risks being modeled and understand the business goals, business catalysts, risk factors, and other business information relevant to the objectives of the stress testing framework.

8. **Stress testing models, results, and frameworks should be subject to challenge and regular review.**

Periodic review and challenge of stress testing for the financial institutions and the authorities is important in improving the reliability of the stress testing results, understanding of results' limitations, identifying the areas that need improvement and ensuring that the results are utilized in accordance with the objectives of the stress testing framework.

9. **Stress testing practices and findings should be communicated within and across jurisdictions.**

Communicating the stress testing results to appropriate internal and external stakeholders provides essential perspectives on risks that would be unavailable to an individual institution or authority. Furthermore, disclosure of the stress test results by banks or authorities improves the market discipline and motivates the resilience of the banking sector towards identified stress.

Banks and authorities who choose to disclose stress testing results should ensure that the method of delivery should make the results understandable while including the limitations and assumptions on which the stress test is based. Clear conveyance of stress test results prevents inappropriate conclusions on the resilience of the banks with different results.

Question 1

Hardik and Simriti compare and contrast stress testing with economic capital and value at risk measures. Which of the following statements regarding differences between the two types of risk measures is most accurate?

A. Stress tests tend to calculate losses from the perspective of the market, while EC/VaR methods compute losses based on an accounting point of view

B. While stress tests focus on unconditional scenarios, EC/VaR methods focus on conditional scenarios

C. While stress tests examine a long period, typically spanning several years, EC models focus on losses at a given point in time, say, the loss in value at the end of year t.

D. Stress tests tend to use cardinal probabilities while EC/VaR methods use ordinal arrangements

The correct answer is **C.**

Option A is inaccurate: Stress tests tend to calculate losses from the perspective of accounting, while EC/VaR methods compute losses based on a market point of view.

Option B is inaccurate: While stress tests focus on conditional scenarios, EC/VaR methods focus on unconditional scenarios.

Option D is also inaccurate: Stress tests do not focus on probabilities. Instead, they focus on ordinal arrangements like “severe,” “more severe,” and “extremely severe.” EC/VaR methods, on the other hand, focus on cardinal probabilities. For instance, a 95% VaR loss could be interpreted as 5-in-100 events.

Question 2

One of the approaches used to incorporate stress testing in VaR involves the use of stressed inputs. Which of the following statements most accurately represents a genuine disadvantage of relying on risk metrics that incorporate stressed inputs?

A. The metrics are usually more conservative (less aggressive)

B. The metrics are usually less conservative (more aggressive)

C. The capital set aside, as informed by the risk metrics, is likely to be insufficient

D. The risk metrics primarily depend on portfolio composition and are not responsive to emerging risks or current market conditions.

The correct answer is **D.**

The most common disadvantage of using stressed risk metrics is that they do not respond to current issues in the market. As such, significant shocks in the market can "catch the firm unaware" and result in extensive financial turmoil.

Question 3

Sarah Wayne, FRM, works at Capital Bank, based in the U.S. The bank owns a portfolio of corporate bonds and also has significant equity stakes in several medium-size companies across the United States. She was recently requested to head a risk management department subcommittee tasked with stress testing. The aim is to establish how well prepared the bank is for destabilizing events. Which of the following scenario analysis options would be the best for the purpose at hand?

A. Hypothetical scenario analysis

B. Historical scenario analysis

C. Forward-looking hypothetical scenario analysis and historical scenario analysis

D. Cannot tell based on the given information

The correct answer is **C.**

Scenario analyses should be dynamic and forward-looking. This implies that historical scenario analysis and forward-looking hypothetical scenario analysis should be combined. Pure historical scenarios can give valuable insights into impact but can underestimate the confluence of events that are yet to occur. What's more, historical scenario analyses are backward-looking and hence neglect recent developments (risk exposures) and current vulnerabilities of an institution. As such, scenario design should take into account both specific and systematic changes in the present and near future.

Question 4

Senior management should be responsible for which of the following tasks?

I. Ensuring that stress testing policies and procedures are followed to the letter
II. Assessing the skills and expertise of the staff involved in stress-testing activities
III. Evaluating the independent review and validation exercises
IV. Making key decisions on investment, capital, and liquidity based on stress test results along with any other information available.
V. Propagating widespread knowledge on stress tests across the firm, and making sure that all departments understand its importance

A. I, II, and IV

B. I and V

C. III and IV

D. V only

The correct answer is **B.**

Roles II and III belong to internal audit. Role IV belongs to the board of directors.

Reading 53: Pricing Conventions, Discounting, and Arbitrage

After completing this reading, you should be able to:

- Define discount factor and use a discount function to compute present and future values.
- Define the "law of one price," explain it using an arbitrage argument, and describe how it can be applied to bond pricing.
- Identify arbitrage opportunities for fixed income securities with certain cash flows.
- Identify the components of a US Treasury coupon bond, and compare and contrast the structure to Treasury STRIPS, including the difference between P-STRIPS and C-STRIPS.
- Construct a replicating portfolio using multiple fixed income securities to match the cash flows of a given fixed-income security.
- Differentiate between "clean" and "dirty" bond pricing and explain the implications of accrued interest with respect to bond pricing.
- Describe the common day-count conventions used in bond pricing.

Discount Factors and Their Use to Compute Present and Future Values

A discount factor for a particular term gives the value today of one unit of currency due at the end of that term. It's essentially a **discount rate.** The discount factor for t years is denoted as $d(t)$. For example, if $d(1) = 0.85$, then the present value of, say, \$1 to be received a year from today is given by $d(t) \times \$1 = \$0.85$. Discount factors can easily be extracted from Treasury bond prices. The discount factor $d(t)$ is the factor which, when multiplied by the total amount of money to be received (principal + interest), gives the price (present value) of the bond. However, when performing these calculations, it's important to note that cash flows with different timings have different discount factors, in line with the time value of money. For example, the discount factor that applies to interest due in six months will be different from the discount factor for interest due in a year, i.e., $d(0.5) \neq d(1)$, and $d(1) < d(0.5)$.

We can obtain discount factors from coupon-bearing bonds.

To illustrate this, let's look at an example.

Example: Discount Factors

Suppose that three bonds with semi-annual coupon payments have the following cashflows:

	Maturity	Coupon Rate	Dirty Price
Bond 1	1/1/2019	2%	100.2535
Bond 2	1/7/2019	3%	100.3240
Bond 3	1/1/2020	4%	100.1020

Assuming $100 face value, what is the value of d(1.5)?

Solution

The cashflows provided by the bonds are as follows:

	1/1/2019	1/7/2019	1/1/2020
Bond 1	101	−	−
Bond 2	1.5	101.5	−
Bond 3	2	2	102

The price of each bond can be found by discounting the cashflows of that particular bond,

Thus,

$$\begin{aligned} 101 * d(0.5) &= 100.2535 \\ \Rightarrow d(0.5) &= \frac{100.2535}{101} = 0.992609 \end{aligned}$$

Similarly,

$$\begin{aligned} 1.5 * d(0.5) + 101.5 * d(1) &= 100.3240 \\ \Rightarrow d(1) &= \frac{100.3240 - 1.5(0.992609)}{101.5} \\ &= 0.973745 \end{aligned}$$

$$
\begin{aligned}
2 * d(0.5) + 2 * d(1) + 102 * d(1.5) &= 100.1020 \\
\Rightarrow d(1.5) &= \frac{100.1020 - 2(0.992609) - 2(0.973745}{102} \\
&= 0.942836
\end{aligned}
$$

Bond Price Quotations

A bond quote is the last price at which a bond is traded, expressed as a percentage of par value (100). Those bonds sold at a discount are priced at less than 100, and another group, although fewer, are sold at a premium and are priced at more than 100.

US T-bonds are quoted in dollars and fractions of a dollar – paving the way for the so-called "32nds" convention. And as the wording suggests, 32 portions of a dollar are considered. For example, if we have a T-bond quoted at 98–16, this means 98 "full" dollars plus 16/32 of a dollar, i.e., 0.5 dollars. Hence, the quote represents a price of $98.50.

Corporate or municipal bonds, on the other hand, use dollars and eight fractions of a dollar.

A "+" sign at the end of a quote represents half a tick. For example,

$$
98 - 16 + \text{implies } 98 + \frac{16.5}{32}
$$

Treasury Bills

Treasury bills are short term debt obligation issued by the government, which usually last for one year or less and do not pay coupons. They are usually quoted at a discount to the face value of 100. The cash price is the face value minus the quoted discount rate.

Let Q be the quoted price of a T-bill and C be the cash price. If the cash price is 97 and there are 90 days until the maturity of the Treasury bill, then:

$$
\begin{aligned}
Q &= \frac{360}{n}(100 - C) \\
&= \frac{360}{90}(100 - 97) \\
&= 12
\end{aligned}
$$

This means that we would pay 97 today to get the face value of 100 in 90 days. Our discount is 12 for every 100 of face value, which means our annual discount rate is around $\frac{12}{100} = 12\%$.

Treasury Bonds

Treasury bonds last for more than one year and usually pay coupons. The accrued interest is the amount of coupon payment accrued between two coupon dates. When we are talking of treasury bonds,

$$\text{Cash price} = \text{Quoted price} + \text{Accrued interest}$$

The Law of One Price and How it Applies to Bond Pricing

The Law of one price states that the price of a security, commodity, or asset should be the same in two different markets, say, A and B. In other words, if two securities have the same cash flows, they must have the same price. Otherwise, a trader can generate a risk-free profit by buying on market A and selling on market B in one risk-free move. Such a possibility is called an arbitrage opportunity.

The Law of one price describes security price quite well because, in case of an arbitrage opportunity, traders rush en masse to take advantage of it. Within no time, market forces of supply and demand adjust the price to eliminate any deviations.

Let’s look at an example.

Consider a 1-year maturity bond with a face value of \$100, a coupon rate of 10%, paying coupons semi-annually. Assume that the borrowing (bank) interest rate is 5% per annum.

The present value of the cash flows from this bond is:

$$\text{PV} = \frac{5}{1.025} + \frac{105}{1.025^2} = \$104.82$$

If the bond has a price of \$100, an investor can borrow \$100 from a bank and buy the bond. After six months, they will be able to repay \$5 after receiving the first coupon. At this point, the debt outstanding will be equal to:

$$\text{debt} = \$100 + \$100 \times 2.5\% - \$5 = \$97.5$$

At the end of the year, the investor will pocket the principal ($100) as well as the second coupon of $5, making a total of $105.

$$\text{Debt at this point} = \$97.5(1.025) = \$99.94$$

Thus, after fully repaying the debt, they will be left with $\$105 - \$99.94 = \$5.06$, which would effectively be a risk-free profit.

To exploit this situation, eagle-eyed investors in an efficient market would attempt to buy this bond by borrowing funds from banks. Increased demand would drive the price up so that at the end of the day, there would be no arbitrage opportunity.

Liquidity

Liquidity refers to how easily an asset converts to cash. It affects the price of a bond since it determines how easily the bond could be sold in the future.

Liquidity issues have, at times, causing a violation of the Law of one price, and therefore we can not always conclude that arbitrage opportunities do not exist at all.

Components of US Treasury Coupon Bond and the Structure of Treasury Strips

STRIP stands for Separate Trading of Registered Interest and Principal of Securities. A US Treasury coupon can have strips in two distinct securities: The principal security, also known as the P-STRIP, and the detached coupons, also called C-STRIPS. The two types of securities can then be traded separately via a broker.

For instance, suppose we have a 10-year bond with a $100,000 face value and a 10% annual interest rate. Assuming it initially pays coupons semi-annually, 21 zero-coupon bonds can be created. That's the 20 C-STRIPS plus the principal strip (P-STRIP). Each of the C-STRIPS has a $5,000 face value, which is the amount of each coupon. The P-STRIP to be received at maturity has a face value of $100,000.

Why are STRIPS so popular?

- They have a very high credit quality because US Treasury securities back them.
- Because they are sold at a discount, investors do not need a large stash of money to purchase them
- The payout is known in advance as long as the investor holds them to maturity
- They offer a range of maturity dates and can, therefore, be used to match liabilities due at specific points in the future.
- STRIPS are eligible for inclusion in tax-deferred retirement plans and non-taxable accounts such as pension funds, in which their value would grow tax-free until your retirement.

Disadvantages of STRIPS

- Shorter-term STRIPS tend to trade rich while longer-term STRIPS tend to trade cheap
- Sometimes they can be quite illiquid
- They typically trade very close to the fair value, thus potential profits are small

How to Construct a Replicating Portfolio Using Multiple Fixed Income Securities to Match the Cash Flows of a Given Fixed Income Security

Here's an example of how a replicating portfolio can be created from multiple fixed income securities:

Assume we have a 2-year fixed-income security with $100 face value and a 20% coupon rate, paid on a semi-annual basis. Assume further that the security has a yield to maturity of 5%.

The present value of the security would be:

$$PB_{B_1} = \frac{10}{1.025^1} + \frac{10}{1.025^2} + \frac{10}{1.025^3} + \frac{110}{1.025^4} = \$128.21$$

If this bond is determined to be trading cheap, a trader can carry out an arbitrage trade by purchasing the undervalued bond and shorting a portfolio that mimics (replicates) the bond's cash flows. Assume that in addition to our bond above, which we shall call bond 1, we have four fixed income securities with the following characteristics:

Bond	Coupon	PV	FV	Time to maturity
Bond 2	14%	$106.35	$100	Six months
Bond 3	24%	$122.58	$100	12 months
Bond 4	10%	$113.07	$100	18 months
Bond 5	12%	$120.94	$100	24 months

Note that these bonds also pay semi-annual coupons.

Using the above multiple fixed-income securities, we can create a replicating portfolio. However, we must first determine the percentage face amounts of each bond to purchase, F_i, where i=1,2,3,4, which match bond 1 cash flows in every semi-annual period.

$$\text{Bond1} \quad CF_t = F_2 \times \frac{14\%}{2} + F_3 \times \frac{24\%}{2} + F_4 \times \frac{10\%}{2} + F_5 \times \frac{12\%}{2}$$

In these types of calculations, the most straightforward approach to obtaining the values of F_i involves starting from the end and then working backward. The logic here is simple. At 24 months, only bond 5 makes a payment. Hence at this point, all other values are equal to zero.

$$\$110 = F_2 \times 0 + F_3 \times 0 + F_4 \times 0 + F_5 \times (100 + \frac{12}{2})\%$$

Solving this gives

$$F_5 = \frac{110}{106\%} = 103.77\%$$

Thus, we have to purchase 103.77%×100=$103.77 face value of bond 5

At 18 months, only bonds 4 and 5 make a payment. We can, therefore, obtain the value of F_4 as follows:

$$\$10 = F_2 \times 0 + F_3 \times 0 + F_4 \times (100 + \frac{10}{2})\,\% + 103.77 \times \frac{12\%}{2}$$

Solving this gives

$$F_4 = \frac{10 - 103.77 \times 0.06}{1.05} = 3.59\%$$

Thus, we have to purchase $3.59 face value of bond 4

To solve for F_3,

$$\$10 = F_2 \times 0 + F_3 \times (100 + \frac{24}{2})\,\% + 3.59 \times \frac{10\%}{2} + 103.77 \times \frac{12\%}{2}$$
$$F_3 = \frac{10 - 0.18 - 6.23}{1.12} = 3.21\%$$

Similarly,

$$\$10 = F_2 \times (100 + \frac{14}{2})\,\% + 3.21 \times \frac{24\%}{2} + 3.59 \times \frac{10\%}{2} + 103.77 \times \frac{12\%}{2}$$
$$F_2 = \frac{10 - 0.39 - 0.18 - 6.23}{1.07} = 2.99915 \approx 3.00\%$$

We can create Cash flows from the replicating portfolio as the product of each bond's initial cash flows and the face amount percentage. For example, the cash flow from bond five at 12 months is equal to:

$$\frac{(12\%)}{2} \times \$100 \times 103.77 = \$6.23$$

Similarly, the cash flow from bond two at six months

$$\frac{(14\%)}{2} \times \$100 \times 3.00\% = \$0.21$$

Bond	Coupon	Face Amount	CF(t=6)	CF(t=12)	CF(t=18)	CF(t=24)
Bond 2	14%	2.99%	3.2			
Bond 3	24%	3.21%	0.39	3.59		
Bond 4	10%	3.59%	0.18	0.18	3.77	110
Bond 5	12%	103.77%	6.23	6.23	6.23	
Total cashflows			10	10	10	110
Bond 1 cashflows			10	10	10	110

Where t= time in months

As can be seen above, the cash flows from the four bonds replicate bond one cash flows.

Clean vs. Dirty Bond Pricing

The dirty price of a bond is a bond pricing quote that's equal to the present value of all future cash flows, including interest accruing on the next coupon payment date. Bonds do trade in the secondary-market before paying any coupon, or after clearing several coupons. In other words, the day a trader buys or sells the bond could be in between coupon payment dates.

In line with the principle of the time value of money, it's only fair to compensate the seller of a bond for the number of days they have held the bond between coupon payment dates. We call this compensation of the accrued interest – the interest earned in between any two coupon dates.

$$\text{Accrued interest} = c\left(\frac{(\text{Number of days that have elapsed since the last coupon was paid}}{\text{Number of days in the coupon period})}\right)$$

For example, suppose a $1,000 par value bond pays semi-annual coupons at a rate of 20%, and we've had 120 days since the last coupon was paid. Assuming that there are 30 days in a month,

$$\text{Accrued interest} = \frac{120}{180} \times \$100 = \$66.70$$

The seller would be compensated to the tune of $67.70, while the buyer would see out the coupon period and receive the remaining $33.30.

The **clean price** of a bond is the price that doesn't include any coupon payments.

The dirty and clean prices are also known as the full and quoted prices, respectively.

Day-Count Conventions

When computing the accrued interest, we use one of several day-count conventions. These include:

- Actual/actual
- Actual/360
- Actual/365
- 30/360
- 30E/360 (E stands for Europe)

Interpretation of these conventions is relatively straightforward. For example, the actual/actual convention considers the actual number of days between two coupon dates. The 30/360 convention assumes there are 30 days in any given month and 360 days in a year.

For purposes of the exam, note the following:

I. We use the actual/actual convention for US government bonds
II. US corporate and municipal bonds use the 30/360 convention
III. The actual/360 convention is common in money markets

Exam tips:

- If coupons are paid semi-annually, the denominator should be 180 in both actual/360 and 30/360 conventions. Similarly, the denominator would be 90 for quarterly coupons.
- Almost all US Treasury trades settle T + 1, which means that the exchange of bonds for cash happens one business day after the trade date.

- Clean price = dirty price – accrued interest

Dirty Price Formula

$$\text{Price} = \frac{C}{(1+y)^{k}} + \frac{C}{(1+y)^{k+1}} + \frac{C}{(1+y)^{k+2}} + \cdots \frac{C+F}{(1+y)^{(k+n-1)}}$$

Where:

P = price

C = semi-annual coupon

k = number of days until the next coupon payment divided by the number of days in the coupon period, determined as per the relevant day-count convention.

y = periodic required yield

n = number of periods remaining, including the present one.

F = face value (par value) of the bond

Question 1

A \$1,000 par value US corporate bond pays coupons semi-annually on January 1 and July 1 at the rate of 20% per year. Mike Brian, FRM, purchases the bond on March 1, 2018, intending to keep it until maturity. The bond is scheduled to mature on July 1, 2021. Compute the dirty price of the bond, given that the required annual yield is 10%.

A. \$1,310.25

B. \$502.50

C. \$400.25

D. \$1,100

The correct answer is A.

As a US corporate issue, this bond is valued based on the 30/360 day-count convention. Under this convention, the number of days between the settlement date (March 1, 2018) and the next coupon date (July 1, 2018) is 120 (= 4 months at 30 days per month).

Each coupon payment is valued at $\frac{20\%}{2} \times \$1,000 = \$100$

$$\text{Price} = \frac{C}{(1+y)^k} + \frac{C}{(1+y)^{k+1}} + \frac{C}{(1+y)^{k+2}} + \cdots \frac{C+F}{(1+y)^{(k+n-1)}}$$

Where:

P = price

C = semi-annual coupon

k = number of days until the next coupon payment divided by the number of days in the coupon period, determined as per the relevant day-count convention.

y = periodic required yield

n = number of periods remaining, including the present one.

F = face value (par value) of the bond

In this case, n=7

$$\text{Price} = \frac{100}{(1.05)^{0.67}} + \frac{100}{(1.05)^{1.67}} + \frac{100}{(1.05)^{2.67}} + \frac{100}{(1.05)^{3.67}} + \frac{100}{(1.05)^{4.67}} + \frac{100}{(1.05)^{5.67}} + \frac{100+1000}{(1.05)^{6.67}}$$
$$= 96.78 + 92.18 + 87.79 + 83.61 + 79.62 + 75.83 + 794.44 = 1,310.25$$

Question 2

An analyst has been asked to check for arbitrage opportunities in the Treasury bond market by comparing the cash flows of selected bonds with the cash flows of combinations of other bonds. If a 1-year zero-coupon bond is priced at USD 97.25 and a 1-year bond paying a 20% coupon semi-annually, is priced at USD 114.50, what should be the price of a 1-year Treasury bond that pays a coupon of 10% semi-annually?

A. $105.88

B. $100

C. $103.35

D. $105

The correct answer is **A.**

The secret here is to replicate the 1-year 10% bond using the other two treasury bonds whose price we already know. To do this, you could solve a system of equations to determine the weight factors, F1 and F2, which correspond to the proportion of the zero and the 20% bond to be held, respectively.

At every coupon date, the cash flow from the 10% bond should match cash flows from the zero-bond and the 20% bond.

At t=1, the 10% coupon bond pays 105, and both the zero-bond and the 20% also have got payouts of 100 and 110, respectively

$$105 = F1 \times (100) + F2 \times 110 \ldots\ldots\ldots\ldots \text{equation1}$$

At t=0.5, the 10% coupon bond pays 5, the zero bond pays 0, and the 20% bond pays 10

$$5 = F1 \times 0 + F2 \times 10 \ldots\ldots\ldots\ldots \text{equation2}$$

Solving equation 2,

$$F2 = \frac{5}{10} = 0.5$$

Solving equation 1,

$$\begin{aligned} 105 &= 100F1 + 0.5 \times 110 \\ 50 &= 100F1 \\ F1 &= 0.5 \end{aligned}$$

Thus, the price of the

$$\begin{aligned} 10\%\text{coupon bond} &= 0.5 \times \text{price of zero bond} + 0.5 \times \text{price of 20\% bond} \\ &= 0.5 \times 97.25 + 0.5 \times 114.5 \qquad = \$105.88 \end{aligned}$$

Note: You should assume the prices are given as per $100 face value

Reading 54: Interest Rates

>After completing this reading, you should be able to:

- Calculate and interpret the impact of different compounding frequencies on a bond's value.
- Define spot rate and compute spot rates given discount factors.
- Interpret the forward rate and compute forward rates given spot rates.
- Define the par rate and describe the equation for the par rate of a bond.
- Interpret the relationship between spot, forward, and par rates.
- Assess the impact of maturity on the price of a bond and the returns generated by bonds.
- Define the "flattening" and "steepening" of rate curves and describe a trade to reflect expectations that a curve will flatten or steepen.
- Describe a swap transaction and explain how a swap market defines par rates.
- Describe overnight indexed swap (OIS) and distinguish OIS rates from LIBOR swap rates.

Different Compounding Frequencies and Their Effect on Bond Value

Besides annual interest payments, most securities on today's market have much shorter accrual periods. For Example, interest may be payable monthly, quarterly (every three months), or semi-annually resulting in different present values or future values depending on the frequency of compounding employed. Here's how to calculate the PV and the FV of an investment with multiple compounding periods per year.

The most important thing is to ensure that the interest rate used corresponds to the number of compounding periods present per year.

Future Value

$$FV = PV\{(1 + \frac{r_q}{m})\}^{m \times n}$$

Where:

r_q is the quoted annual rate,

m represents the number of compounding periods (per year)

Lastly, n is the number of years

Present Value

If we make PV the subject of the formula above, we have:

$$PV = FV\{(1 + \frac{r_q}{m})\}^{-m*n}$$

Example: Simple Time Value of Money Calculation

Suppose you wish to have $20,000 in your savings account at the end of the next four years. Assume that the account offers a return of 10 percent per year, compounded monthly. How much would you need to invest now to have the specified amount after the four years?

Solution

First, we write down the formula to use,

$$PV = FV\{(1 + \frac{r_q}{m})\}^{-m*n}$$

Second, we establish the components that we already have:

r_q=0.10

m=12

n=4 years

Then, we factor everything into the equation to find our PV.

$$PV = 20000\{1 + \frac{0.1}{12}\}^{-12\times 4} = \$13,429$$

Therefore, you will need to invest at least $13,429 in your account to ensure that you have $20,000 after three years.

We can do this with the financial calculator with the following inputs:

$$N = 12 * 4 = 48; \frac{I}{Y} = \frac{10}{12} = 0.833; PMT = 0; FV = 20,000;$$
$$CPT \Rightarrow PV = -13,429$$

Here, we can see that the PV on the financial calculator is a negative value since it's a cash outflow. The FV has a positive sign since it's a cash inflow.

We can also rearrange the future value formula to obtain the holding period return (HPR) as follows:

$$r_q = m\left[(\frac{FV}{PV})^{\frac{1}{mn}} - 1\right]$$

Calculating Discount Factors Given Interest Rate Swap Rates

If we have a series of interest swap rates, it is possible to derive discount factors. The notional amount, which is technically never exchanged between counterparties, determines the size of both fixed and floating leg payments.

If we could exchange the notional amount, the fixed leg of the swap would resemble a fixed coupon-paying bond, with fixed leg payments acting like semi-annual, fixed coupons, and the notional amount acting like the principal payment. Floating rate payments would act like coupon payments of the floating-rate bond.

We denote the discount factor for t-years as d(t). The methodology used to come up with discount factors when dealing with interest rate swaps is similar to that used to find discount factors when dealing with bonds.

Example: Calculating Discount Factors

Compute the discount factors for maturities ranging from six months to two years, given a notional swap amount of $100 and the following swap rates:

Maturity (years)	Swap rates
0.5	0.75
1.0	0.85
1.5	0.98
2.0	1.20

The four discount factors d(0.5), d(1.0), d(1.5) and d(2.0) can be calculated as follows:

$$\left(100+\frac{0.75}{2}\right)d(0.5)=100$$

$$d(0.5)=\frac{100}{100.375}=0.9963$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{0.85}{2}d(0.5)+\left(100+\frac{0.85}{2}\right)d(1.0)=100$$

$$0.425\times 0.9963+100.425\ d(1.0)=100$$

$$d(1.0)=\frac{100-0.4234}{100.425}=0.9916$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{0.98}{2}d(0.5)+\frac{0.98}{2}d(1.0)+\left(100+\frac{0.98}{2}\right)d(1.5)=100$$

$$0.49\times 0.9963+0.49\times 0.9916+100.49\ d(1.5)=100$$

$$d(1.5)=\frac{100-0.4882-0.4859}{100.49}=0.9854$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\frac{1.20}{2}d(0.5)+\frac{1.20}{2}d(1.0)+\frac{1.20}{2}d(1.5)(100+\frac{1.20}{2})\,d(2.0)=100$$

$$d(2.0)=\frac{100-0.5978-0.5950-0.5912}{100.6}=0.9763$$

Maturity (years)	Discount factor
0.5	0.9963
1.0	0.9916
1.5	0.9854
2.0	0.9763

Computing Spot Rates Given Discount Factors

A t-period spot rate is a yield to maturity on a zero-coupon bond that matures in t years, assuming semi-annual compounding. We denote the t-periodic spot rate as z(t).

Spot rates and discount factors are related as shown in the following formula, assuming semi-annual coupons:

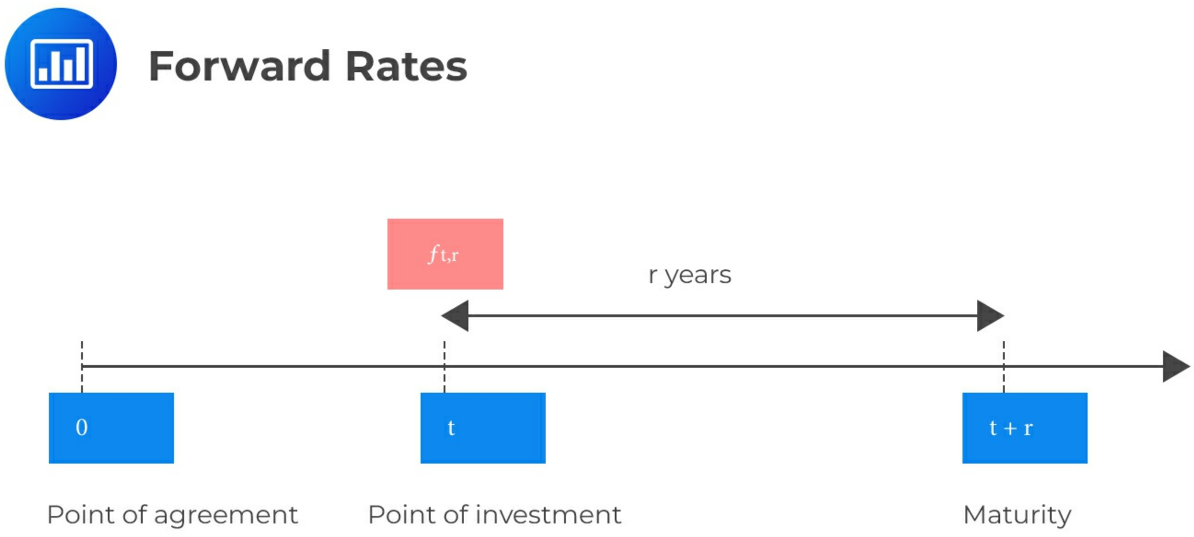
$$z(t)=2\left[\left(\frac{1}{d(t)}\right)^{\frac{1}{2t}}-1\right]$$

Spot Rate vs. Forward Rates

A spot interest rate gives you the price of a financial contract on the spot date. The spot date is the day when the funds involved in a business transaction are transferred between the parties involved. It could be two days after a trade, or even on the same day, we complete the deal. A spot rate of 5% is the agreed-upon market price of the transaction based on current buyer and seller action.

In theory, forward rates are prices of financial transactions that might take place at some future point. The spot rate tells you "how much it would cost to execute a financial transaction today". The forward rate, on the other hand, tells you “how much would it cost to execute a financial transaction at a future date X”.

We agree on spot and forward rates in the present. The only difference comes in the timing of execution.



Example: Converting Spot Rates into Forward Rates

Compute the six-month forward rate in six months, given the following spot rates:

Z (0.5) =1.6%

Z (1.0) =2.2%

Solution

The six-month forward rate, f(1.0), on an investment that matures in one year, must solve the following equation:

$$(1+\frac{0.022}{2})^2 = (1+\frac{0.016}{2})^1 \times 1+\frac{f(1.0)}{2}^1$$

$$1.0221 = 1.008 \times (1+\frac{f(1.0)}{2})^1$$

$$1.01399 - 1 = \frac{f(1.0)}{2}$$

$$f\,(1.0) = 0.02797 = 2.8\%$$

Par Rate

The par rate is the rate at which the present value of a bond equals its par value. It's the rate you'd use to discount of all a bond's cash flows so that the price of the bond is 100 (par). For a 100-par value, the two-year bond that pays semi-annual coupons, and we can easily calculate the 2-year par rate provided we have the discount factor for each period

$$\frac{\text{Par Rate}}{2}[d(0.5)+d(1.0)+d(1.5)+d(2.0)]+100\,d(2.0)=100\ldots\ldots\ldots\text{(i)}$$

In general, for any maturity T, and assuming a par value of \$1,

$$\frac{C_T}{2}\sum_{t=1}^{2T}d\left(\frac{t}{2}\right)+d(T)=1$$

The sum of the discount factors is called the annuity factor, A_T, and is given by:

$$\sum_{t=1}^{2T}d\left(\frac{t}{2}\right)$$

Therefore, equation (i) above, becomes

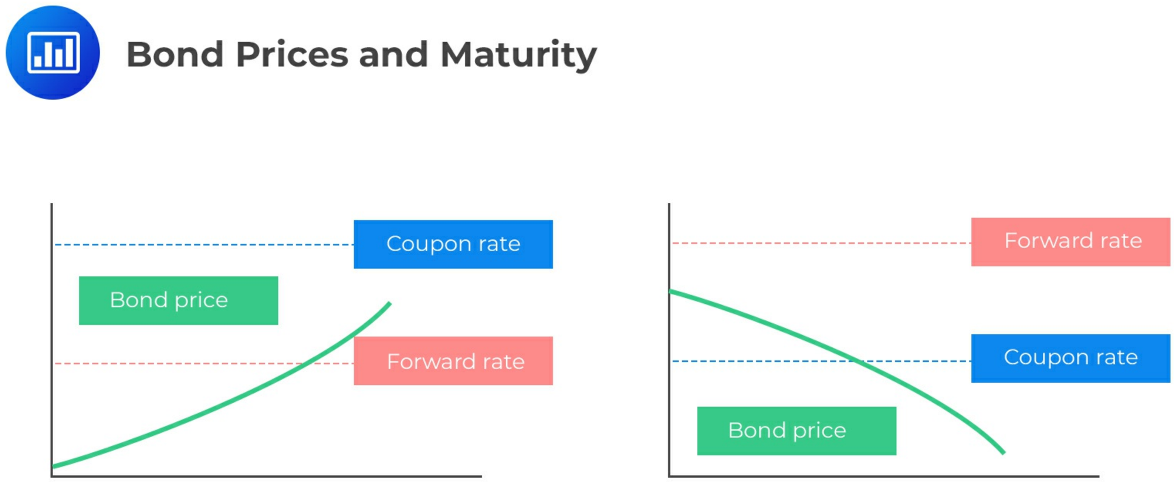
$$\frac{\text{Par Rate}}{2}[A_T]+100\,d(T)=100$$

Thus,

$$\text{Par Rate}=\frac{2\times100\times[1-d(T)]}{A_T}$$

Impact of Maturity on the Price of a Bond and the Returns Generated by Bonds

In general terms, bond prices will tend to increase with maturity whenever the coupon rate is above the forward rate throughout the maturity extension. The opposite holds: bond prices will tend to decrease with maturity whenever the coupon rate is below the forward rate for a maturity extension.



To help you understand just how this happens, assume we have two investors with the opportunity to invest in either STRIPS or a five-year bond. The investor who opts for the 5-year bond (which utilizes forward rates) will have a simple task: they will invest at the onset and then wait to receive regularly scheduled coupon payments, plus the principal amount at maturity. The investor who opts for STRIPS (which utilizes spot rates) will roll them over as they mature throughout the five years. Rolling over implies that when one STRIP expires, the investor will use the proceeds to invest in the next six-month contract, and so on for five years.

In market conditions where short-term rates are above the forward rates utilized by bond prices, the investor who rolls over the STRIPS will tend to outperform the investor in the 5-year bond. The opposite is exact: In market conditions where short-term rates are below the forward rates utilized by bond prices, the investors who roll over the STRIPS will tend to underperform the investor in the 5-year bond.

Properties of Spot, Forward and Par Rates

- If the term structure is constant, and that all spot rates are equal, then all par rates and all forward rates equal the spot rate.
- If the term structure is increasing, the par rate for a specific maturity is below the spot

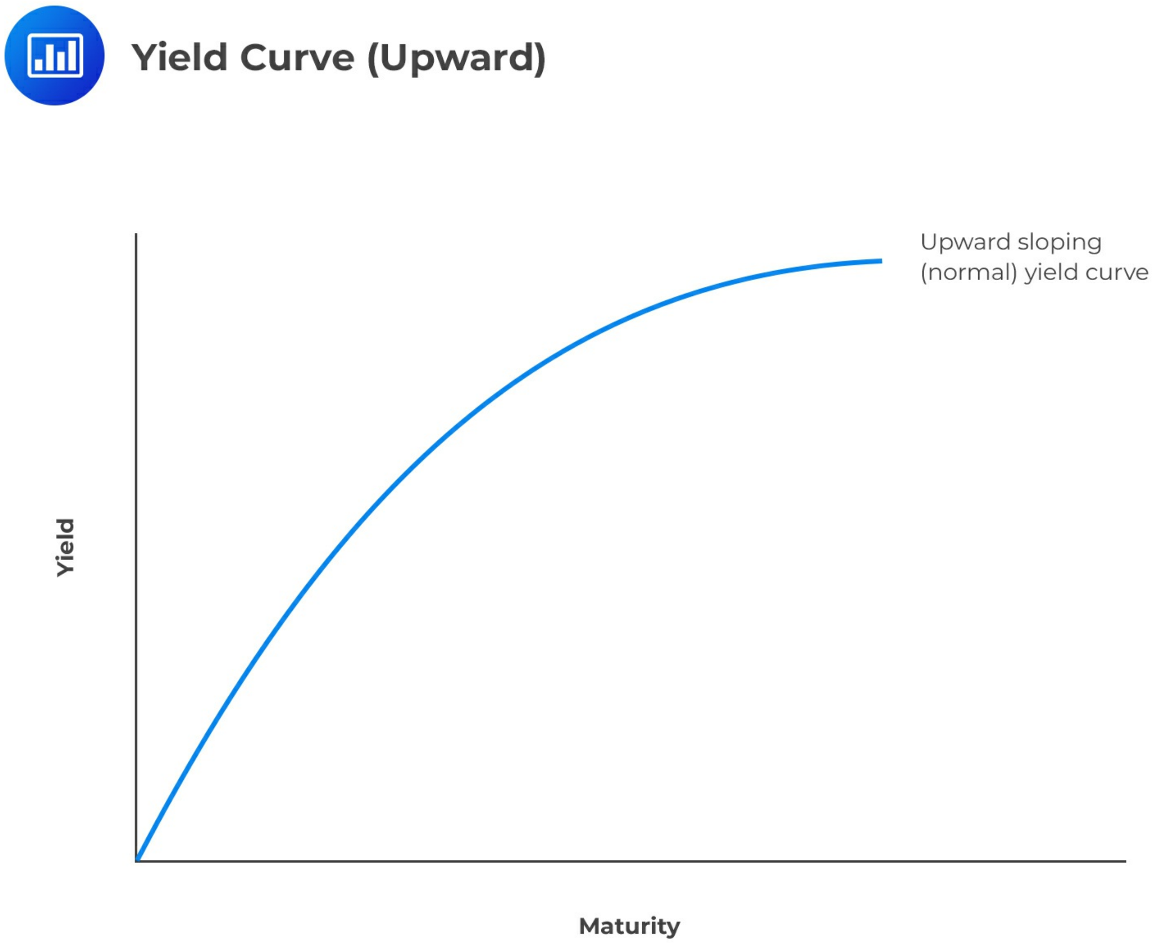
rate for the same timeline.

- If the term structure is decreasing, the par rate for a specific maturity is above the spot rate for the same maturity.
- If the term structure is increasing, forward rates for a period starting at time T are higher than the spot rate for maturity T.

Flattening and Steepening of Rate Curves

A yield curve represents the yield of each bond along a maturity spectrum that's plotted on a graph. The most and widely accepted yield curves pit the three-month versus two-year T-bonds or the five-year versus ten-year T-bonds. On occasion, we may use the Federal Funds Rate versus the 10-year Treasury note.

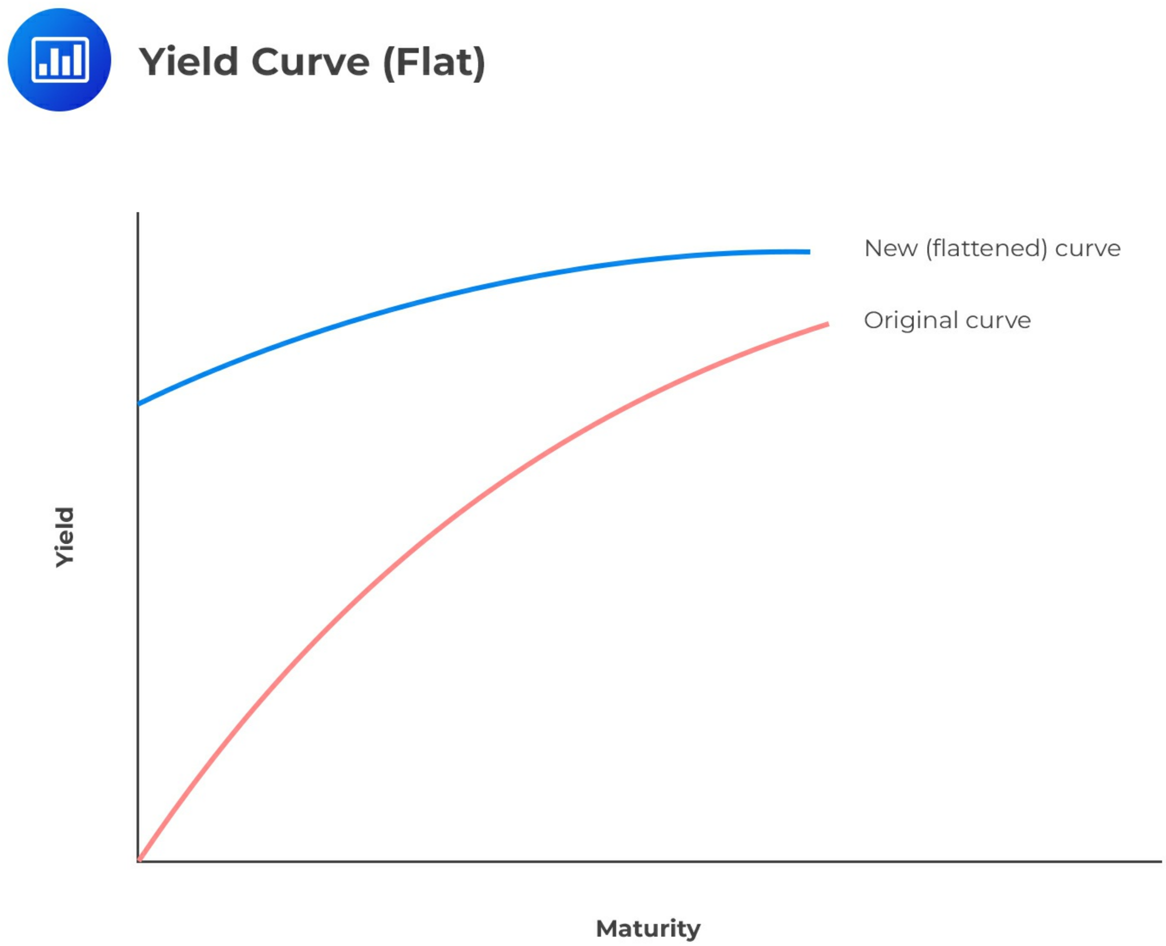
The yield curve typically **slopes upwards,** indicating that the interest rate on long-term bonds is higher than the rate on short-term bonds, reflecting the investors' demands to be compensated for taking on more risk by investing in long-term bonds. Such a curve is said to be normal.



Other than a standard yield curve, we could have a flattening yield curve or a steepening yield curve.

Flattening Yield Curve

A flat yield curve indicates that little difference exists between short-term and long-term rates for similarly rated bonds. It may manifest as a result of long-term interest rates falling more than short-term interest rates or short-term rates increasing more than long-term rates.



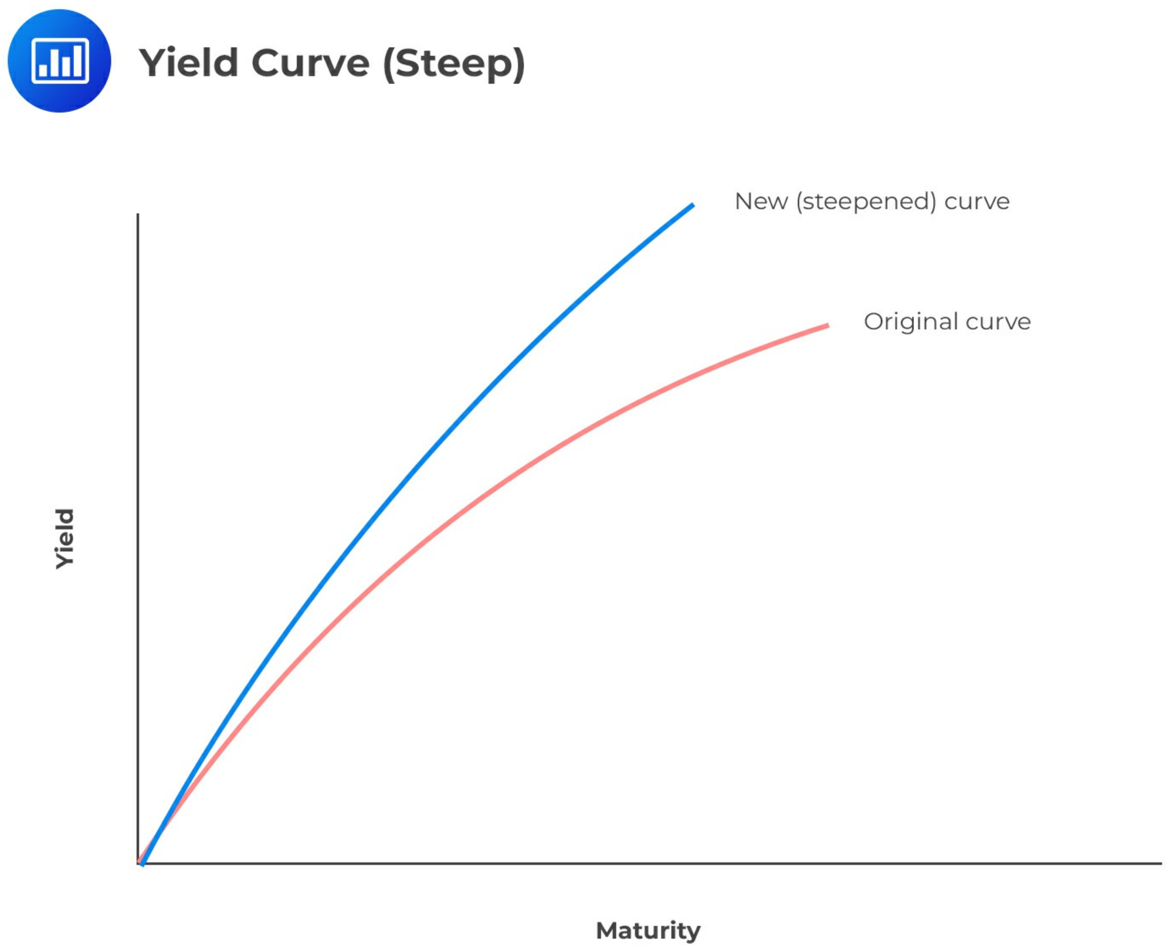
A flattening curve reflects the expectations of investors about the macroeconomic outlook. It may be the result of the following events:

- A decrease in expected inflation, causing the premium loaded on long-term rates to decrease; or

- An impending increase in the Federal Funds Rate. (An increase in the FFR could cause an increase in short term rates while long-term rates remain relatively stable, causing the curve to flatten.)A trader who anticipates a flattening of the yield curve can buy a long-term bond and sell a short-term bond because they expect bond prices to rise in the long-term.

Steepening Yield Curve

A steepening yield curve indicates a widening gap between the yields on short-term bonds and long-term bonds. A steepening curve could occur when long-term rates rise faster than short-term rates. Sometimes, short-term rates can also show some defiance by decreasing even as long-term rates rise.



For example, assume that a two-year note was at 2.3% on July 15, and the 10-year was at 3.3%. By August 30, the two-year note could have risen to 2.38% and the 10-year to 3.5%. The difference would effectively go from 1 percentage point to 1.12 percentage points, resulting in a steeper yield curve.

A steepening curve reflects the expectations of investors about the macroeconomic outlook. It may be the result of the following events:

- an increase in expected inflation, causing the premium loaded on long-term rates by investors to increase; or
- an impending decrease in the Federal Funds Rate.

A trader who anticipates a steepening of the yield curve can sell a long-term bond and buy a short-term bond because they expect bond prices to fall in the long-term (bond prices fall as rates increase).

LIBOR and Overnight Rates

LIBOR (London Interbank Offered Rate) is a rate of interest at which banks across the world borrow from each other. LIBOR is derived from average estimates of different borrowing rates provided by different AA-rated banks globally. These rates are quoted for different currencies, and the borrowing periods are between one day and one year.

Although we use LIBOR rates as a benchmark borrowing rate, we should note that the estimates used to come up with the average borrowing rate are subject to manipulation by the different banks. In efforts to curb these issues, the market has come up with other benchmarks, and an excellent example of this is the overnight rate.

Overnight Rate

While the borrowing period for LIBOR ranges from one day to one year, overnight rates apply for a day only. Banks with insufficient funds may borrow from those banks with surplus funds to meet their financial needs overnight.

Brokers always accompany these transactions, and different countries have systems that monitor the activities of these brokers. For example, we have the Federal Reserve in the US, the Sterling Overnight Index Average (SONIA) in the UK, and the Euro Overnight Index Average (EONIA) in the Eurozone.

Swaps

A swap is a contract in which two parties agree to exchange a sequence of cash flows for a given period. A contract in which one party agrees to swap a fixed rate for Libor is the most frequently exercised type of swap.

A swap has two legs: a fixed leg in which the interest rate is fixed and a floating one in which the rates are varying, and the interest is compound. Each day, banks will come up with a new Overnight Index; hence the rates keep on fluctuating, i.e., swaps are associated with a fixed interest rate and a variable interest rate.

Overnight Indexed Swap (OIS)

An OIS is a swap in which an investment compounded daily for three months is exchanged with a fixed rate of interest for five years.

The floating rate per three-month is given by:

$$(1 + r_1 d_1)(1 + r_2 2) \ldots . (1 + r_i d_i) \ldots (1 + r_n d_n) - 1$$

Where:

r is the overnight rate per day on a day i.

d is the number of calendar days where r applies and,

n is the number of trading days in the three months.

OIS is the fixed rate in the overnight indexed swaps.

While Libor swaps define par bond that can be used to determine the Libor term structure, OIS rates describe par bonds that can be used to determine an OIS term structure.

Question 1

Given the following spot rates, compute the 6-month forward rate in 1 year.

$z(1.0) = 3.25\%$

$z(1.5) = 3.60\%$

A. 4.2%

B. 1.5%

C. 4.3%

D. 5.1%

The correct answer is **C**.

Solution:

The 6-month forward rate on an investment that matures in 1.5 years must solve the following equation:

$$\left(1+\frac{0.036}{2}\right)^3 = \left(1+\frac{0.0325}{2}\right)^2 \times \left(1+\frac{f(1.5)}{2}\right)^1$$

$$1.05498 = 1.03276 \times \left(1+\frac{f(1.5)}{2}\right)^1$$

$$1.02152 - 1 = \frac{f(1.5)}{2}$$

$$f(1.5) = 0.04303 = 4.3\%$$

Question 2

Consider a bond with a par value of USD 1,000 and maturity in four years. The bond pays a coupon of 4% annually. The spot rate curve is as follows:

n-year	Spot rate
1	5%
2	6%
3	7%
4	8%

The value of the bond is closest to:

A. $1,160

B. $500

C. $870.78

D. $850

The correct answer is **C**.

The value of the bond is the present value of all future cash flows (coupons plus principal), discounted at the various spot rates.

$$\text{Each} \quad \text{coupon} = 4\% \times 1,000 = \$40$$

$$\begin{aligned} PV &= \frac{40}{(1+0.05)^1} + \frac{40}{(1+0.06)^2} + \frac{40}{(1+0.07)^3} + \frac{40+1000}{(1+0.08)^4} \\ &= 38.10 + 35.60 + 32.65 + 764.43 = \$870.78 \end{aligned}$$

Reading 55: Bond Yields and Return Calculations

After completing this reading, you should be able to:

- Distinguish between gross, and net realized returns and calculate the realized return for a bond over a holding period, including reinvestments.
- Define and interpret the spread of a bond and explain how to derive a spread from a bond price and a term structure of rates.
- Define, interpret, and apply a bond's yield-to-maturity (YTM) to bond pricing.
- Compute a bond's YTM, given a bond structure and price.
- Calculate the price of an annuity and perpetuity.
- Explain the relationship between spot rates and YTM.
- Define the coupon effect and explain the relationship between the coupon rate, YTM, and bond prices.
- Explain the decomposition of P&L for a bond into separate factors, including carry roll-down, rate change, and spread change effects.
- Explain the following four common assumptions in carry roll-down scenarios: realized forwards, unchanged term structure, consistent yields, and realized expectations of short-term rates; and calculate carry roll down under these assumptions.

Gross vs. Net Realized Returns

The gross realized return on investment has two components: Any increase in the price of the asset plus income received while holding the investment. When dealing with bonds,

$$\text{Gross realized return}_{t-1,t} = \frac{\text{Ending value} + \text{Coupon - Beginning value}}{\text{Beginning value}}$$

Example: A Bond’s Gross Realized Return over Six Months

What is the gross realized return for a bond that is currently selling for $1,060 if it was purchased exactly six-months ago for $1,000 and paid a $20 coupon today?

Solution

$$\text{Gross realized return} = \frac{\text{Ending value} + \text{Coupon-Beginning value}}{\text{Beginning value}}$$
$$= \frac{1,060 + 20 - 1000}{1000} = 8\%$$

When calculating the gross realized return for multiple periods, it's essential to consider whether coupons received are reinvested. If the coupons are reinvested, they will earn some interest at a given rate.

Example: Gross Realized Return over One Year With Reinvested Coupons

A bond purchased exactly six months ago for $1,000 paid a $20 coupon today. Suppose the coupon is reinvested at an annual rate of 4.4% for the next six months and that the bond is worth $1,080 after one year. What is the realized return on the bond over the one-year period?

Solution

$$\text{Gross realized return} = \frac{\text{Ending value} + \text{Coupon} + \text{Coupon Investment} - \text{Beginning value}}{\text{Beginning value}}$$
$$= \frac{1,080 + 20 + 20 \times 1.022 - 1,000}{1000} = 12.04\%$$

In addition to reinvestment income, we can also consider borrowing costs. If the investor buys the bond using borrowed funds, they will be expected to pay some interest at the end of the investment period. In these circumstances, the net realized return is calculated as follows:

$$\text{Net realised return} = \frac{\text{Ending value} + \text{Coupon} - \text{Beginning value} - \text{Financing costs}}{\text{Beginning value}}$$

Example: Example: Gross Realized Return over One Year With Reivested Coupons and Financing Costs

An investor purchased a bond exactly six months ago at $980 (per $1,000 nominal value). The purchase was entirely financed at an annual rate of 2%. Today, the bond is worth $995.

Given that the bond paid a coupon of $20 today, determine the net realized return

Solution

$ $\text{Net realised return}=\cfrac{\text{Ending value} + \text{Coupon}-\text{Beginning value} - \text{Financing costs}}{ \text{Beginning value} } $

$$\text{Net realised return} = \frac{995 + 20 - 980 - 9.8}{980} = 2.57\%$$

where $9.8 = \frac{2\%}{2} \times 980$

Bond Spread

The spread of a bond is the difference between its market price and the price computed according to spot rates or forward rates – the term structure of interest rates.

As a relative measure, a bond's spread helps us determine whether the bond is trading cheap or rich relative to the yield curve. We incorporate spread in the bond price formula as follows:

Recall that given a 2-year bond with a face value of P, paying annual coupons each of amount C, its price is given by:

$$\text{Market bond price} = \frac{C}{(1 + f(1.0))} + \frac{C+P}{(1 + f(1.0)) \times (1 + f(2.0))}$$

To incorporate the spread s, we assume that the bond is trading at a premium or discount to this computed price. We can find the bond’s spread using the following formula:

$$= \frac{C}{(1 + f(1.0) + s)} + \frac{C+P}{(1 + f(1.0) + s) \times (1 + f(2.0) + s)}$$

Yield to Maturity

Yield to maturity (YTM) of fixed income security is the total return anticipated if we hold the security until it matures. Yield to maturity is considered a long-term bond yield, but we express it as an annual rate. In other words, it's the security's internal rate of return as long as the investor holds it up to maturity. To compute a bond's yield to maturity, we use the following formula:

$$p = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} \cdots + \frac{F + C_N}{(1+y)^N}$$

Where:

P = price of the bond

C_t = annual cash flow in year t

N = time to maturity in years

y = annual yield (YTM to maturity)

F = face value

Example: Yield to Maturity

Suppose a two-year bond with a coupon of 5% sells for USD 106. What is the yield to maturity expressed with semi-annual compounding?

Solution

We can find the yield y (expressed with semi-annual compounding) by solving the equation:

$$106 = \frac{2.5}{1 + y/2} + \frac{2.5}{(1 + y/2)^2} + \frac{2.5}{(1 + y/2)^3} + \frac{102.5}{(1 + y/2)^4}$$

We can solve this by trial and error, to get y=1.93%

When cash flows are received multiple times every year, we can slightly modify the above formula such that:

$$p = \frac{C_1}{(1+y)^1} + \frac{C_2}{(1+y)^2} + \frac{C_3}{(1+y)^3} \cdots + \frac{F + C_n}{(1+y)^n}$$

Where:

P = price of the bond

C_t = periodic cash flow in period t

n = N × m = number of periods (= years × number of periods per year)

F = face value

Provided all cash flows received are reinvested at the YTM; the yield to maturity is equal to the bond's realized return.

For zero-coupon bonds that are not accompanied by recurring coupon payments, the yield to maturity is equal to the normal rate of return of the bond. We use the formula below to determine YTM for zero-coupon bond:

$$\text{Yield to maturity} = \left(\frac{\text{Face value}}{\text{Current price of bond}}\right)^{\frac{1}{\text{year to maturity}}} - 1$$

Exam tip: The yield to maturity assumes cash flows will be reinvested at the YTM and assumes that we hold the bond until maturity.

Prices of Annuities and Perpetuities

An annuity is a series of annual payments of PMT until the final time T. The value of an ordinary annuity is given by:

$$PV_{annuity} = PMT\frac{1-(1+r)^{-T}}{r}$$

Where:

r = discount rate

Perpetuity is a type of annuity whose cash flows continue for an **infinite amount of time.** The present value of a perpetuity is given by:

$$PV_{perpetuity} = \frac{PMT}{r}$$

Suppose we receive a semi-annual coupon at the rate of USD 3 per annum forever. Suppose further that the yield to maturity is 6%. Then, the present value of the perpetuity is

$$\frac{3}{0.06} = USD\ 50$$

The Relationship between Spot Rates and YTM

We can use both the spot rate and the yield to maturity to determine the fair market price of a bond. However, while the yield to maturity is constant, the spot rate varies from one period to the next to reflect interest rate expectations as time goes.

The spot rate is a more accurate measure of the fair market price when interest rates are believed to rise and fall over the coming years.

Given a bond's cash flows and the applicable spot rates, you can easily calculate the price of a bond. You can then determine the bond's YTM by equating the price to the present values of cash flows discounted at the YTM.

The Coupon Effect

The coupon effect describes the fact that reasonably priced bonds of the same maturity but different coupons have different yields to maturity, which implies that yield is not a reliable measure of relative value. Even if fixed-income security A has a higher yield than fixed security B, A is necessarily not a better investment.

It also follows that if two bonds have identical features save for the coupon, the bond with the smaller coupon is more sensitive to interest rate changes. In other words, given a change in yield, the lower coupon bond will experience a higher percentage change in price compared to the bond with larger coupons. The most sensitive bonds are zero-coupon bonds, which do not make any coupon payments.

Exam tips:

- The lower the coupon rate, the higher the interest-rate risk. The greater the coupon rate, the lower the interest rate risk.
- If coupon rate > YTM, the bond will sell for more than par value or at a premium.
- If the coupon rate < YTM, the bond will sell for less than par value, or at a discount.
- If coupon rate= YTM, the bond will sell for par value.

Over time, the price of premium bonds will gradually fall until they trade at par value at maturity. Similarly, the price of discount bonds will gradually rise to par value as maturity gets closer. This phenomenon is known as "pulling to par."

Components of a Bond’s P&L

We generate the bond's profitability or loss through price appreciation and explicit cash flows. The total price appreciates as follows:

$$P_{t+1}(R_{t+1}s_{t+1}) - P_t(R_t s_t)$$

There are three components of price appreciation:

I. **Carry-roll-down component**: The carry-roll-down component comprises of price changes that emanate from a deviation of term structure from the original structure to an expected term structure, denoted as R^e, as maturity approaches. It does not account for spread changes

$$\text{Carry roll down component} = P_{t+1}(R^e_{t+1}s_t) - P_t(R_t s_t)$$

II. **Rate changes component**: The rate changes component accounts for price changes due to interest rate movements from an expected term structure to the term structure that exists at time t+1.

$$\text{Rate changes component} = P_{t+1}(R_{t+1}s_t) - P_t(R^e_{t+1}s_t)$$

This component also doesn't account for spread changes.

III. **Spread change component**: As the words suggest, the spread change component accounts for price changes emanating from changes in the bond's spread from time t to t+1.

$$\text{Spread change component} = P_{t+1}(R_{t+1}s_{t+1}) - P_{t+1}(R_{t+1}s_t)$$

Assumptions in Carry-Roll-Down Scenarios

I. Realized forwards: The return to a bond held to maturity is the same as rolling the investment one period at a time at the forward rates. However, in reality, some forwards are realized above or below the initial forwards.

II. Unchanged term structure

III. Unchanged yields

Practice Questions

Question 1

A bond currently selling for $1,060 was purchased exactly 12 months ago for $1,000 and paid a $20 coupon six months ago. Today, the bond paid a $20 coupon. The coupon received six months ago was reinvested at an annual rate of 2%. Given that the purchase price was entirely financed at a yearly rate of 1%, the net realized return of the bond is closest to:

A. 9%

B. 8%

C. 10%

D. 11%

The correct answer is **A.**

$$\begin{aligned}\text{Netrealised return}_{t-1,t} &= \frac{BV_t + C_t - BV_{t-1}}{BV_{t-1}} - \text{\% Financing Costs} \\ &= \frac{1,060 + 20 + 20(1 + \frac{2\%}{2}) - 1000}{1000} - 1.0\% \\ &= 10.02\% - 1\% = 9.02\% \approx 9\%\end{aligned}$$

Alternatively, Financial Costs $= 0.01 \times 1000 = 10$

So that,

$$\begin{aligned}\text{Netrealised return}_{t-1,t} &= \frac{BV_t + C_t - BV_{t-1} - \text{Financing Costs}}{BV_{t-1}} \\ &= \frac{1,060 + 20 + 20(1 + \frac{2\%}{2}) - 1000 - 10}{1000} \\ &= 9.02\% \approx 9\%\end{aligned}$$

Question 2

On Jan 1 2017, Commercial Bank of India issued a six-year bond paying an annual coupon of 6% at a price reflecting a yield to maturity of 4%. As of Dec 31, 2017, interest rates remain unchanged. Holding all other factors constant, and assuming a flat term structure of interest rates, how was the bond's price affected? The price:

A. Remained constant
B. Decreased
C. Increased
D. Increased, but only in the second half of the year

The correct answer is **B.**

From the data given, it's clear that the bond's coupon is higher than the yield. As such, the bond must have traded at a premium – implying the price must have been higher than the face value. Provided the yield doesn't change; a bond's price will always converge to its face value. Since the price starts higher, it must decrease. This phenomenon is called 'pulling to par."

Reading 56: Applying Duration, Convexity, and DV01

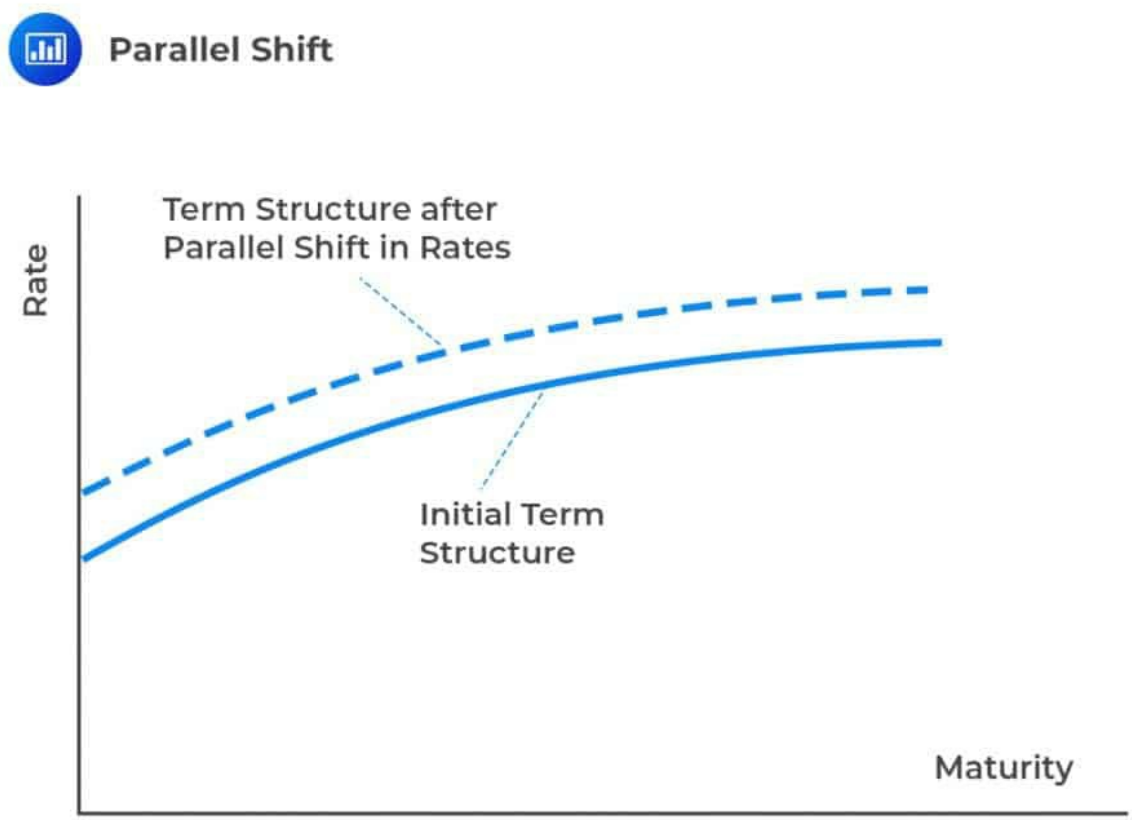
After completing this reading, you should be able to:

- Describe a one-factor interest rate model and identify common examples of interest rate factors.
- Define and compute the DV01 of fixed income security, given a change in yield and the resulting change in price.
- Calculate the face amount of bonds required to hedge an option position given the DV01 of each.
- Define, compute, and interpret the effective duration of fixed income security given a change in yield and the resulting change in price.
- Compare and contrast DV01 and effective duration as measures of price sensitivity.
- Define, compute, and interpret the convexity of fixed income security, given a change in yield and the resulting change in price.
- Explain the process of calculating the effective duration and convexity of a portfolio of fixed income securities.
- Describe an example of hedging based on effective duration and convexity.
- Construct a barbell portfolio to match the cost and duration of a given bullet investment, and explain the advantages and disadvantages of a bullet versus barbell portfolios.

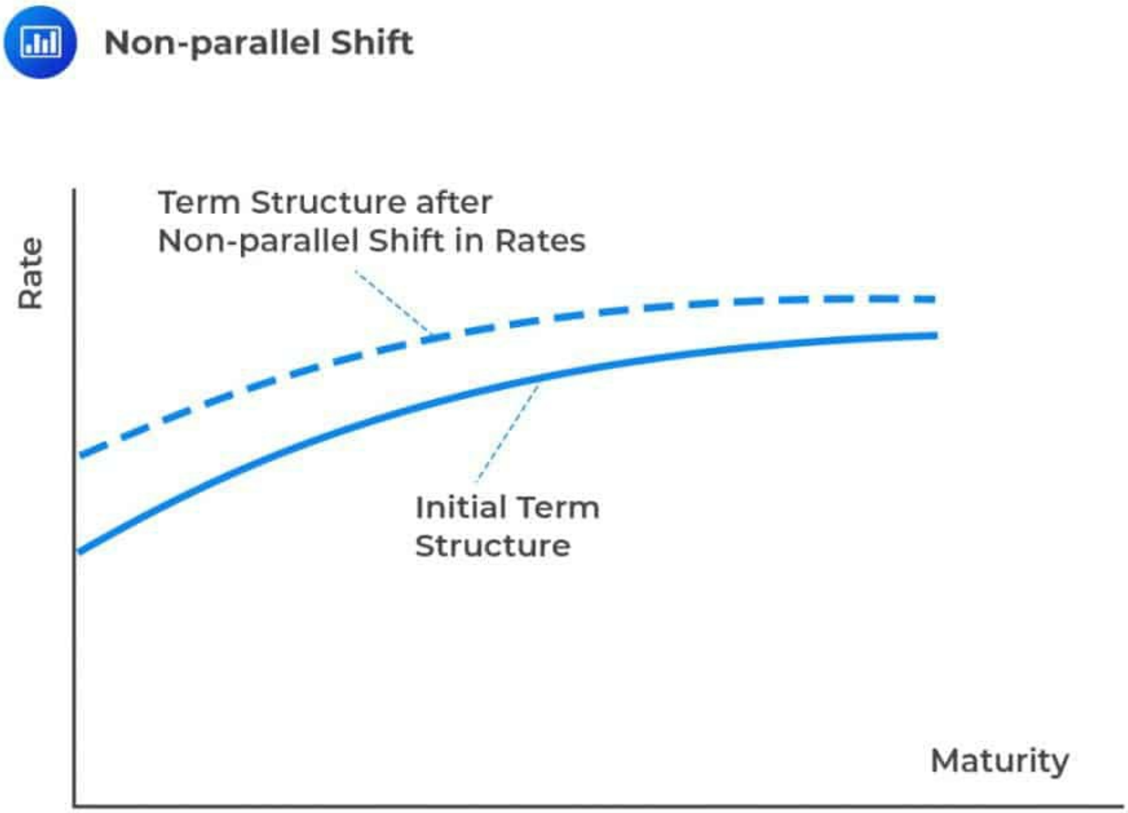
This chapter discusses one-factor risk metrics, which include DV01, duration, and convexity, as used in the analysis of fixed-income portfolios. We consider these measures to quantify the consequence of a parallel shift in the interest rate term structure: DV01 and duration consider a small parallel shift in the term structure while convexity extends the duration to accommodate larger parallel shifts. Hedging can, in accordance with these risk metrics, be efficient for a parallel shift of term structure as compared to non-parallel shifts.

One-factor Interests Rate Models

The one-factor assumption states that when the rates are driven by one factor, the change of one interest rate can be used to determine the change in all other interest rates over a short period of time. For instance, a one-factor model assumes that all interest rates changes by the same amount. As such, the shape of the term structure never changes. That is, if a one-year spot rate increases by two basis points, all other spot rates increase by two basis points. DV01, duration, and convexity are examples of such models.



It is worth noting that shifts in the term structure are not always parallel. For instance, a one-factor model might predict that if the one-year spot rate increases by five basis points over a short period, then the two-year increases by three basis points, and the ten-year rate increases by one basis point.



DVO1

DV01 reflects the effect of a one-basis movement in interest rates on the value of a portfolio. DV01 is defined as:

$$\text{DV01} = -\frac{\Delta P}{\Delta r}$$

Where

Δr = the size of a parallel shift in the interest rate term structure measured in basis points

ΔP = resultant change in the value of the position being considered

Note that for a long position in bonds, the DV01 is positive due to a negative correlation between the bond's price and interest rate changes.

DV01 is defined in three different ways:

i. **Year-based DV01:** defined as the change in the price from a one-basis point increase in the yield of a bond.
ii. **DVDZ or DPDZ:** defined as the change in price from a one-point increase in all spot rates.
iii. **DVDF or DPDF:** defined as a change in price from a one-basis point increase in the forward rate.

Example: Calculating DV01 from the One-Basis Point Change in Spot Rates

A two-year treasury bond has a face value of USD 100,000, with an annual coupon rate of 8% paid semiannually. The spot rates, as shown in the table below:

Maturity Period (Years)	Spot Rate (%)
0.5	6.0
1.0	6.5
1.5	7.0
2.0	7.5

What is the value of the DV01 if:

i. the spot rates increase by five basis points?
ii. the spot rates decrease by five basis points?

Solution

i. The price of a bond with no spread is USD 101,003.01 calculated using the formula:

$$P = \frac{C}{2}\sum_{i=1}^{2T}\left(\frac{1}{1+\frac{y}{2}}\right)^{i} + \frac{100,000}{(1+\frac{y}{2})^{2T}}$$

In this case,

$$c = 100,000 \times 0.08 = 8,000, T = 2$$

$$\Rightarrow p = \frac{8000}{2}\left(\frac{1}{1+\frac{0.06}{2}} + \frac{1}{\left(1+\frac{0.065}{2}\right)^2} + \frac{1}{\left(1+\frac{0.07}{2}\right)^3} + \frac{1}{\left(1+\frac{0.075}{2}\right)^4}\right) + \frac{100,000}{\left(1+\frac{0.075}{2}\right)^4}$$
$$= 4000\left(\frac{1}{1.03} + \frac{1}{(1.0325)^2} + \frac{1}{(1.035)^3} + \frac{1}{(1.0375)^4}\right) + \frac{100,000}{(1.0375)^4}$$
$$= 101,003.01$$

If the spot rates are each increased by 5 basis points (0.05%) so that the six-month spot rate is 6.05%, the one-year spot rate is 6.55%, and so on, the price of the bond is USD 100,911.18, calculated as:

$$= \frac{8000}{2}\left(\frac{1}{1.03025} + \frac{1}{(1.03275)^2} + \frac{1}{(1.03525)^3} + \frac{1}{(1.03775)^4}\right) + \frac{100,000}{(1.03775)^4}$$
$$= 100,911.18$$

We know that:

$$DV01 = -\frac{\Delta P}{\Delta r} = -\frac{100,911.18 - 101,003.01}{5}$$
$$= \frac{91.83}{5} = 18.366$$

Note that the rise of the spot rates by 5 basis points decreases the bond price by 91.83 (=100,911.18 − 101,003.01), and the DV01, in this case, measures the decline of a bond price for each one-basis point increase in spot rates.

ii. Assume now that the spot rates decrease by 5 basis points so that the six-month spot rate is 5.95% (=6 − 0.05), the one-year spot rate is 6.45% (=6.5 − 0.05) and so on. Under decreased spot rates, the price of the bond is USD 101,094.96 calculated as

$$= 4,000\left(\frac{1}{1.02975} + \frac{1}{(1.03225)^2} + \frac{1}{(1.03475)^3} + \frac{1}{(1.03725)^4}\right) + \frac{100,000}{(1.03725)^4} = 101,094.96$$

And thus,

$$\text{DV01} = -\frac{\Delta P}{\Delta r} = -\frac{101,094.96 - 101,003.01}{-5}$$
$$= \frac{91.95}{-5} = 18.39$$

As such, an increase of spot rates by five basis points causes the price of the bond to increases by 91.95 (=101,094.96 − 101,003.01), and thus DV01, in this case, measures the increase of bond price for each one-basis point decrease in spot rates.

It is worth noting that the DV01 for the decrease and increases of the basis points are slightly different because the bond price is not a linear function of interest rates. We estimate the DV01 by averaging the estimates above:

$$\text{DV01} = \frac{18.366 + 18.39}{2} = 18.38$$

Example: Calculating DV01 from the one-basis point change in bond yield

Assume that now bond yield increases by one basis point. Consider the spot rates, as in the example above.

Maturity Period (Years)	Spot Rate (%)
0.5	6.0
1.0	6.5
1.5	7.0
2.0	7.5

A two-year treasury bond has a face value of USD 100,000, with an annual coupon rate of 8% paid semiannually. The spot rates, as shown in the table above.

What is the value of DV01 if the bond yield

i. Increases by 5 basis points?

ii. Decreases by the 5 basis points?

Solution

Recall that, using the spot rates, we calculated the bond price to be 101,003.01. To find the yield to maturity of the bond (bond yield), we solve the equation:

$$\frac{4,000}{1+\frac{y}{2}}+\frac{4,000}{(1+\frac{y}{2})^2}+\frac{4,000}{(1+\frac{y}{2})^3}+\frac{104,000}{(1+\frac{y}{2})^4}=101,003.01$$

Using a financial calculator with the variables N=4, PMT=4000, PV=−101,003.01, we get:

$$\frac{y}{2}=3.7255\% \Rightarrow y=7.45\%$$

i. Now, if the bond yield increases by 5 basis point to 7.50% (=7.45 + 0.05), the price of the bond is 100,912.846 (using the financial calculator) and thus for an increase of 5 basis points in bond yield, the bond price decreases by 90.164 (=100,912.85 − 101,003.01). We can, therefore, calculate the DV01:

$$\begin{aligned}\text{DV01} &= -\frac{\Delta P}{\Delta r} = -\frac{100,912.85-101,003.01}{5}\\ &= \frac{-90.164}{5}=18.033\end{aligned}$$

ii. Similarly, if the bond yield decreases by 5 basis points to 7.40% (=7.45 − 0.05), the bond price is 101,096.71. A decrease of bond yield by 5 basis points increases the bond price by 93.70 (=101,096.71 − 101,003.01). Thus,

Similarly, if the bond yield decreases by 5 basis points to 7.40% (=7.45 − 0.05), the bond price is 101,096.71. A decrease of bond yield by 5 basis points increases the bond price by 93.702 (=101,096.71 − 101,003.01). Thus,

$$
\begin{aligned}
\text{DV01} &= -\frac{\Delta \text{P}}{\Delta \text{r}} = -\frac{101,096.71 - 101,003.01}{-5} \\
&= \frac{-93.164}{5} = 18.632
\end{aligned}
$$

The average of the estimates is 18.33 given as:

$$
\frac{18.0328 + 18.632}{2} \approx 18.33
$$

In the cases of the forward rates, the analogy is similar to that of the spot rates. However, this chapter will primarily emphasize on DV01 computed from a one-basis-point parallel shift in the term structure of **spot rates.**

Hedging Using DV01 Metric

In any position that depends on the interest, DV01 can be computed efficiently. DV01 can, therefore, be used to hedge a position. For example, assume that a bank has a position whose DV01 is −40. By the definition of DV01, the banks will gain from their position if interest rates increases and will undoubtedly lose value if interest rates decrease. More specifically, if all the interest increases by the one-basis point, the value of the bank's position increases by USD 40. On the contrary, if all interest rates decrease by one basis point, the value of the bank's position decreases by USD 40.

Now assume that this bank wants to hedge its position with a coupon bond that pays an annual coupon rate of 8% payable semiannually, has a face value of USD 100,000, and a bond yield of 7.50%. If we can determine a position in the coupon bond that is exactly 40, the bank's position would have been hedged.

A hedge ratio determines the amount of par of the hedge position that needs to be bought or sold for every $1 par value of the original position. The goal of hedging is to lock in the value of a position even in the face of small changes in yield. The hedge ratio is given by:

$$
\text{HR} = \frac{\text{DV01}_{\text{Initial Position}}}{\text{DV01}_{\text{Hedge Position}}}
$$

Assume that DV01 of the coupon bond is 18.33. We use the hedge ratio and the bond's face value to increase the DV01 of the bond to 40. In this case, we need to increase the value of the position by:

$$100,000 \times \frac{40}{18.33} = \$218,221.50$$

Thus, adding the USD 218,221.50 of coupon bond's position to the bank's portfolio protects against small changes in the term structure. In other words, a downward (upward) movement of the term structure will result in a gain (loss) on the existing position, which will be offset by a loss (gain) on the position in the 8% coupon bond.

Effective Duration (D)

Effective duration measures the percentage change in the price of a bond (or other instruments) caused by small changes in all rates. Note that effective duration is different from DV01 because DV01 measures actual price changes against small changes in all rates.

Effective duration is defined as:

$$D = -\frac{\frac{\Delta P}{P}}{\Delta r} = -\frac{\Delta P}{P \times \Delta r}$$

The effective duration can also be rewritten as:

$$\Delta P = -D \times P \times \Delta r$$

When the change in all rates is measured basis points, the effective duration is equivalent to **DV01 divided by the bond price.**

Consider the DV01 example on the spot rates where we had calculated the price of the bond USD 101,003.01 and the DV01 of 18.38 so that the duration is:

$$\frac{18.33}{101003.01} = 0.000182 = 0.0182\%$$

Effective duration gives the proportional change in the price of an instrument corresponding to a one-basis point change in all interest rates.

Typically, the effective duration is stated as a percentage change in the price of an instrument for a 100-basis-change in all rates by multiplying the effect of one-basis-point change by 100. Therefore, our example above would be stated as 1.82% per 100 basis points.

However, for the sake of clarity, the duration in this chapter will be reported as a decimal. In the decimal reporting system, one basis point is equivalent to 0.0001, and thus, we measure duration per 10,000 basis points. Therefore, the duration calculated above will be:

$$0.000182 \times 10,000 = 1.82$$

Callable and Puttable Bonds

A callable bond is one that an issuing party has the right to purchase back the bond at a pre-determined price at a particular time in the future before the maturity period of the bond. The call feature of the bond should not be ignored as it reduces the duration of the bond.

A practical approach to address the callable feature of a bond while calculating duration is outlined as follows:

- Calculate the value of the bond today;
- Compute the bond value if all interest rates increase by one basis point while incorporating the effect of the one-basis point increase on the bond's callable probability of the bond; and
- Compute the effective duration from the percentage change in the price.

A puttable bond is one that a holder has a right to ask for early repayment. Calculating the effective duration of a puttable bond is similar to that of a callable bond. That is, the probability of the put options increases with the increase in the interest rates.

Comparing DV01 and Effective Duration

DV01 is useful in measuring the effect of all rate changes on the value of a position. DV01 is also appropriate in measuring the changes in **swaps** and **interest rate futures**.

Effective duration is appropriate in measuring the effect of rate changes on the value of a position as a percentage. Effective duration is an appropriate measure in **bond valuation**.

DV01 increases with an increase in the position size, but effective duration does not. In other words, if the value of a position is doubled, DV01 doubles, but effective duration does not.

Effective Convexity

Convexity measures the sensitivity of duration measure to movement in the interest rates. Denote convexity by C and value of a position by P, convexity is defined as:

$$C = \frac{1}{P}\left[\frac{P^{+} + P^{-} - 2P}{(\Delta r)^{2}}\right]$$

Where:

P^{+} = the value of the position when all rates increase by Δr

P^{-} = the value of the position corresponding to the decrease of all rates by Δr

Note that Δr is given in decimal.

Example: Calculating Convexity

A two-year treasury bond has a face value of USD 100,000, with an annual coupon rate of 8% paid semiannually. The spot rates, as shown in the table below:

Maturity Period (Years)	Spot Rate (%)
0.5	6.0
1.0	6.5
1.5	7.0
2.0	7.5

What is the value of the convexity if all spot rates change by 5 basis points?

Solution

As computed earlier, the price of a bond with no interest spread is USD 101,003.01 calculated as:

$$\begin{aligned}
p &= \frac{8000}{2}\left(\frac{1}{1+\frac{0.06}{2}}+\frac{1}{(1+\frac{0.065}{2})^2}+\frac{1}{(1+\frac{0.07}{2})^3}+\frac{1}{(1+\frac{0.075}{2})^4}\right)+\frac{100,000}{(1+\frac{0.075}{2})^4}\\
&= 101,003.01
\end{aligned}$$

If the spot rates are each increased by 5 basis points (0.05%) so that the six-month spot rate is 6.05%, the one-year spot rate is 6.55%, and so on, the price of the bond is USD 100,911.18 calculated as:

$$= 4000\left(\frac{1}{1.02975}+\frac{1}{(1.03225)^2}+\frac{1}{(1.03475)^3}+\frac{1}{(1.03725)^4}\right)+\frac{100,000}{(1.03725)^4} = 101,094.96$$

So,

$$\begin{aligned}
C &= \frac{1}{P}\left[\frac{P^{+}+P^{-}-2P}{(\Delta r)^2}\right]\\
&= \frac{1}{101,003.01}\left[\frac{100,911.18+101,094.96-2\times 101,003.01}{(0.0005)^2}\right]\\
&= 4.752
\end{aligned}$$

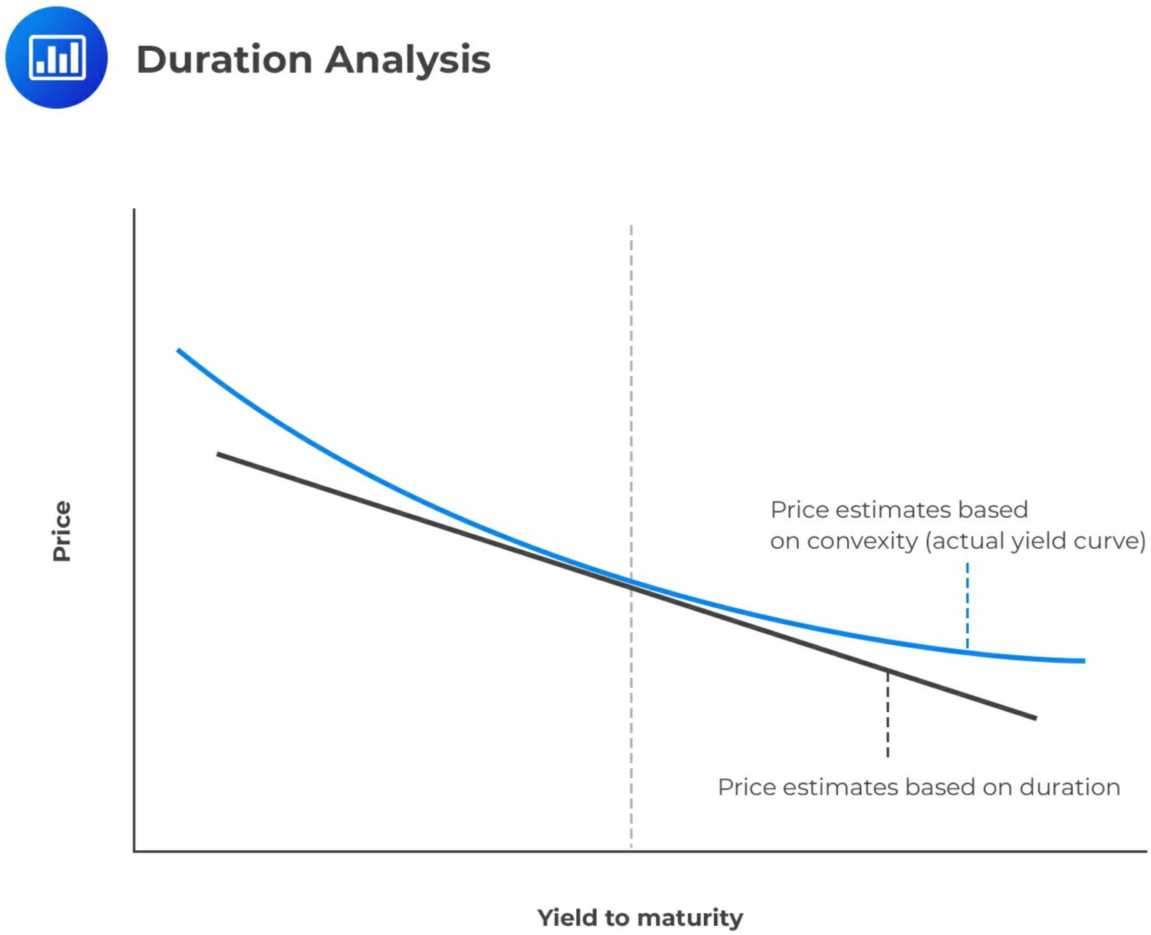
The Impact of Parallel Shifts

Recall that the effective duration can be written as:

$$\Delta P = -D \times P \times \Delta r$$

The duration of a bond is a linear relationship between the bond price and interest rates where, as interest rates increase, bond price decreases. It is a good measure of sensitivity to interest rates for small/sudden changes in yield. However, for much larger changes in yield, the duration is not so desirable since the relationship between price and interest rates tends to be non-linear.

Consider the following diagram:



The relationship between $\frac{\Delta P}{P}$ and Δr is non-linear. However, effective duration approximates this as a linear relationship because it is the gradient of the curve linking $\frac{\Delta P}{P}$ and Δr. As seen on the graph, the linear approximation by effective duration becomes less efficient as the rate changes become large.

The combination of the duration and convexity is a quadratic function which provides a better estimation. As such the price change is given by

$$\Delta P = -DP\Delta r + \frac{1}{2}CP(\Delta r)^2$$

Where Δr is given in decimal.

By combining the duration and the convexity, we can reach a convenient estimate, even for large changes. Consider the example in the DV01 section.

Example: Price Change Approximation Using Effective Duration and Convexity

A two-year treasury bond has a face value of USD 100,000, with an annual coupon rate of 8% paid semiannually. The spot rates, as shown in the table below:

Maturity Period (Years)	Spot Rate (%)
0.5	6.0
1.0	6.5
1.5	7.0
2.0	7.5

Note that we had calculated the DV01 for the 5 basis point movement of all the spot rates to be 18.38, the price of the bond with no spread is USD 101,003.01, the duration to be 1.82 and convexity to be 4.752.

Now we want to:

i. Estimate actual bond price change due to an increase in all the spot rates by 30 basis points;
ii. Estimate bond price change using effective duration; and
iii. Estimate bond price change using a combination of effective duration and convexity.

Solutions

i. Actual Change due to a change in all rates

Because all the spot rates increase by 30 basis points (0.3%), the six-month spot rate is 6.3%, the one-year spot rate is 6.8%, and so on. So the price due to new rates is given by the formula:

$$P = \frac{C}{2}\sum_{i=1}^{2T}\left(\frac{1}{1+\frac{y}{2}}\right)^{i} + \frac{100,000}{(1+\frac{y}{2})^{2T}}$$

In this case,

$$c = 100,000 \times 0.08 = 8,000, T = 2$$

$$\begin{aligned}
&= \frac{8000}{2}\left(\frac{1}{1+\frac{0.063}{2}} + \frac{1}{(1+\frac{0.068}{2})^2} + \frac{1}{(1+\frac{0.073}{2})^3} + \frac{1}{(1+\frac{0.078}{2})^4}\right) + \frac{100,000}{(1+\frac{0.078}{2})^4} \\
&= 4000\left(\frac{1}{1.0315} + \frac{1}{(1.034)^2} + \frac{1}{(1.0365)^3} + \frac{1}{(1.039)^4}\right) + \frac{100,000}{(1.039)^4} \\
&= 100,453.63 \\
\Rightarrow \Delta P &= 100,453.63 - 101,003.01 = -549.38
\end{aligned}$$

Therefore the price of the bond decreases by **549.38** (=100,453.63 − 101,003.01).

ii. Estimated price change using effective duration

The estimated price change using effective duration is given by:

$$\begin{aligned}
\Delta P &= -DP\Delta r \\
&= -1.82 \times 101,003.01 \times 0.003 = -551.48
\end{aligned}$$

iii. Estimated price change using a combination of duration and convexity

The estimated price change provided by the duration is a good approximation of the price change, but we can improve it by combining duration and convexity.

The price change estimation due to a combination of duration and convexity is given by:

$$\begin{aligned}\Delta P &= -DP\Delta r + \frac{1}{2}CP(\Delta r)^2 \\ &= -1.82 \times 101,003.01 \times 0.003 + \frac{1}{2} \times 4.752 \times 101,003.01 \times (0.003)^2 \\ &= -551.48 + 2.1598 = -549.32\end{aligned}$$

Hedging Based on Effective Duration and Convexity

Hedging Based on Duration only

Hedging using effective duration analogous to that of DV01. Now denote the duration of investment by D_V, and its investment value by V. On the other hand, denote the effective duration of a bond by D_B and its value by P. By the definition of effective duration,

$$\Delta V = -V D_V \Delta r$$

and

$$\Delta V = -VD_B \Delta r$$

Where Δr is the size of a small parallel shift in the term structure. These small parallel shifts are hedged against if:

$$\begin{aligned}-VD_V\Delta r - PD_B\Delta r &= 0 \\ \Rightarrow \Delta r\,(-VD_V - PD_B) &= 0 \\ \therefore -VD_V - PD_B &= 0\end{aligned}$$

If we make P the subject, the position required in the bond is:

$$P = \frac{VD_V}{D_B}$$

Example: Hedging Based on Effective Duration

A bank has a position of USD 12 million with an effective duration of 5 and a bond with an effective duration of 6. How will the bank hedge against its position?

Solution

The position required is given by:

$$\begin{aligned} P &= -\frac{VD_V}{D_B} \\ &= -\frac{5 \times 12}{6} = -10 \end{aligned}$$

Therefore, the bank should short bonds worth 10 million to hedge against its position. This is true because:

$$\Delta V = -12 \times 5 \times \Delta r = -60\Delta r$$

and

$$\begin{aligned} \Delta P &= -10 \times 6 \times \Delta r = 60\Delta r \\ \Rightarrow \Delta V + \Delta P &= -60\Delta r + 60\Delta r = 0 \end{aligned}$$

Hedging Based on Duration and Convexity

Hedging based on both duration and convexity is a complex undertaking where we are trying to reduce both the effective duration and convexity to zero. We need two bonds. Define:

P_1 = value of the first bond

D_1 = duration of the first bond

C_1 = convexity of the first bond

P_2 = value of the second bond

D_2 = duration of the second bond

C_2 = convexity of the second bond

Also, define

V = value of the position to be hedged

P_V = duration of the position to be hedged

C_V = convexity of the position to be hedged

Now by the definition of approximating the price change using a combination of the duration and convexity we have:

$$\Delta V = -VD_V\Delta r + \frac{1}{2}VC_V(\Delta r)^2$$
$$\Delta P_1 = -P_1D_1\Delta r + \frac{1}{2}P_1C_1(\Delta r)^2$$

and

$$\Delta P_2 = -P_2D_2\Delta r + \frac{1}{2}P_2C_2(\Delta r)^2$$

If we make both the duration and convexity to be zero, we have:

$$-VD_V - P_1D_1 - P_2D_2 = 0$$

Also,

$$VC_V - P_1C_1 - P_2C_2 = 0$$

These equations must be satisfied for a position to be hedged.

Example: Hedging Based on Convexity and Duration

A bank has a position of USD 12 million with an effective duration of 5 and a convexity of 9. The bank wishes to hedge its position with two bonds where the first bond has an effective duration of 6 and a convexity of 9. The second bond has a duration of 4 and a convexity of 7. How will the bank hedge against its position?

Solution

We know that a position is hedged against if the following equations are satisfied:

$$VD_V - P_1D_1 - P_2D_2 = 0$$
$$VC_V - P_1C_1 - P_2C_2 = 0$$

Therefore, we have $V = 12, D_V = 5,$ and $C_V = 9$. For the bonds we have $C_1 = 9, D_1 = 6$ also $C_2 = 7, D_2 = 4$ so that:

$$60 - 6P_1 - 4P_2 = 0$$
$$108 - 9P_1 - 7P_2 = 0$$

Solving the above equation simultaneously, we get:

$$P_1 = -2 \text{ and } P_2 = 18$$

Therefore, for the bank to hedge its position, it must take a short position of USD 2 in the first bond and a long position of USD 18 million in the second bond. In other words, by combining these positions in bonds, there is no duration or convexity exposure, and thus the bank is hedged against large parallel shifts in the term structure though it will be exposed to non-parallel shifts.

Yield-Based Duration and Convexity

Yield-Based Duration

Consider a bond whose price is P and yield is y. If the cash flows from the bond are $c_1, c_2, \ldots, c_n$ at times $t_1, t_2, \ldots, t_n$ respectively, the relationship between P and y (continuously compounded) is:

$$P = \sum_{i=1}^{n} C_i e^{-yt_i}$$

If we differentiate with respect to y, we have:

$$\frac{dp}{dy} = -\sum_{i=1}^{n} C_i e^{-yt_i}$$
$$\Rightarrow \frac{\Delta P}{\Delta y} = -\sum_{i=1}^{n} t_i C_i e^{-yt_i}$$

Now, the yield-based duration is defined as the proportional change in the bond price due to a small change in the yield. More precisely,

$$D = -\frac{1}{P}\frac{\Delta P}{\Delta y} = -\frac{1}{P}\left(-\sum_{i=1}^{n} t_i C_i e^{-yt_i}\right)$$
$$\Rightarrow D = -\sum_{i=1}^{n} t_i \frac{C_i e^{-yt_i}}{P}$$

The expression $\frac{C_i e^{-yt_i}}{P}$ in the formula denotes the proportion of the value of the bond received at the time t_i. Therefore, the duration can be computed by taking the average of the times when the bond's cash flows were received weighted by the bond's value at each time. As such, the duration will measure the time an investor must wait to receive cash flows.

Typically, the yield on the bond is given as a semi-annual compounding rate rather than continuous compounding. As such, the expression for the duration is divided by $\frac{1}{1+\frac{y}{2}}$ to get:

$$D = \left(\frac{1}{1+\frac{y}{2}}\right)\sum_{i=1}^{n} t_i \frac{C_i e^{-yt_i}}{P}$$

The resulting equation gives the formula for the **modified duration** while the expression without dividing by $\frac{1}{1+\frac{y}{2}}$ is termed as **Macaulay Duration**. Modified duration is slightly different from an effective duration.

Yield-Based Convexity

Recall convexity measures the sensitivity of duration measure to movement in the interest rates. In the calculus context, yield based convexity is the derivative of the duration. Therefore,

$$C = \frac{1}{P}\sum_{i=1}^{n} t_i^2 C_i e^{-yt_i} = \sum_{i=1}^{n} t_i^2 \frac{C_i e^{-yt_i}}{P}$$

Intuitively, convexity can be defined as the weighted (by bond's value) average of the squared time to maturity. When yields are given as semi-annual compounding, the convexity expression is multiplied by $\frac{1}{(1+\frac{y}{2})^2}$ so that:

$$C = \frac{1}{(1+\frac{y}{2})^2}\sum_{i=1}^{n} t_i^2 \frac{C_i e^{-yt_i}}{P}$$

The last expression is termed as **Modified convexity,** and it is slightly different from effective convexity.

Portfolio Interest-Rate Sensitivity Measures

We now need to consider a group of instruments (such bonds) where we need to compute the interest-rate sensitivity measures we have been discussing.

DV01 for a Portfolio

The DV01 of a portfolio is simply the sum of individual DV01 of instruments in the portfolio.

For instance, if a bank has four positions whose DV01s are 9, 10, 12, and 13 (in USD millions), then the DV01 for the portfolio is 44 (=9+10+12+13).

Duration for a Portfolio

Portfolio duration can be calculated as the **weighted sum** of the individual durations. The weight attached to each security is equal to its value as a percentage of total portfolio value.

$$\text{Portfolio duration} = \sum_{j=1}^{k} w_j D_j$$

Where:

D_j = duration of the bond j

w_j = market value of the bond j divided by total portfolio market value

k = number of bonds in the portfolio

For instance, consider a portfolio consisting of four instruments with values (in USD millions) of 8, 13, 15 and 18, and respective effective durations are eight, seven, six, and four. The effective duration for the portfolio is given by:

$$D_{Eff} = \frac{8}{8+13+15+18}\times 8 + \frac{13}{8+13+15+18}\times 7 + \frac{15}{8+13+15+18}\times 6 + \frac{18}{8+13+15+18}\times 4$$
$$= 1.1852 + 1.6852 + 1.6667 + 1.3333 = 5.87$$

The effective portfolio duration is used to determine the impact of the small parallel shift of the term structure of the interest rates. Similar computations can be used in yield-based durations.

Convexity for a Portfolio

Portfolio convexity is computed using a similar approach. It's defined as the value-weighted average of the individual bond convexities making up the portfolio.

$$\text{Portfolio convexity} = \sum_{j=1}^{k} w_j C_j$$

Barbell Portfolio vs. Bullet Portfolio

A **barbell** is an investment strategy applicable to fixed-income securities whereby half the portfolio is made up of long-term bonds and the other half of short-term bonds. On the other hand, a bullet strategy is an investment strategy where the investor buys bonds concentrated only in the intermediate maturity range.

Given the prices, coupon rates, maturities, yields, durations, and convexities of a set of bonds, it is possible to construct a barbell portfolio with the same cost and duration as the bullet portfolio. This involves determining the proportion of each security in the Barbell that should be bought, such that their total value equals that of the bullet.

Example: Demonstration of Barbell and Bullet Portfolio

Consider the following three bonds:

Bonds	Bond Value	Effective Duration	Effective Convexity
2-year, 3% coupon	88.90	4.567	21.23
5-year, 5% coupon	99.80	7.767	78.90
10-year, 7% coupon	120.45	13.43	180.56

Assume that the term structure of the interest rates is flat at 5%, compounded semiannually. Note that this means that the yield on all instruments is 5%.

Now, assume that an investor wants a portfolio with a duration of 7.767. The investor can either buy a 5-year, 5% coupon (bullet investment) or build a portfolio of other two bonds to have the desired duration of 7.767 (barbell investment).

If the investor wishes to construct a bond from the other two bonds, denote:

- The proportion of the 2-year, 3% coupon bond by α, and
- The proportion of the 10-year, 7% coupon bond by $1\text{-}\alpha$.

Thus, the duration of the bond is

$$4.567\alpha + 13.43(1-\alpha) = 7.767$$
$$\Rightarrow \alpha = 0.6389$$

Therefore, the investor can create a bond with a duration of 7.767 by either investing all his money in a 5-year coupon bond, or invest 63.89% of his funds into the 2-coupon bond, and 36.11% of his funds in the 10-year coupon bond.

Note that the portfolios have equal duration but different convexities. The convexity of the 5-year, 5% coupon bond is 78.90. For the portfolio consisting of 2-year, 3% coupon, and 10-year, 7% coupon bonds, the convexity is given by 78.8088 ($=0.6389\times21.3+0.3611\times180.56$) which is better than investing in one bond only.

An arbitrage opportunity can occur if one invests an amount in the barbell portfolio and short the same amount of the bullet portfolio. The arbitrage opportunity is possible if the movement in the term structure is parallel (which is not always true). Typically, the bullet investment is profitable for many non-parallel movements of the term structure as compared to the barbell investment. Most of the models are constructed in such a way there are no arbitrage opportunities to the investors.

Advantages and Disadvantages of Barbell Portfolios

Advantages:

- It may be more liquid compared to a barbell portfolio;
- Spreading out bond purchases ensures higher yields when rates are rising; and
- The investor need not have a "war chest" at the onset since the portfolio is built gradually.

Disadvantages:

- When the yield curve flattens (short rates go up; long rates go down), the Barbell **outperforms** the bullet.

Note: Sometimes, the bullet and Barbell have the same duration, but they will have different convexities.

Practice Question

The price of a 3-year bond is USD 10,000. The DV01 of the bond is 50. What is the estimated bond price change if all rates increase by five basis points?

A. 25

B. 5

C. 250

D. 50

The correct answer is **C**.

Recall that when the change in all rates is measured in basis points, the effective duration is equivalent to DV01 divided by the bond price. Therefore, in this case, the effective duration is given by:

$$D = -\frac{50}{10,000} = -0.005 = -50$$

(Note: We measure the effective duration in decimal, i.e., per 10,000 basis points.)

The estimated price change using effective duration is given by:

$$\begin{aligned}\Delta P &= -DP\Delta r\\ &= -50 \times 10,000 \times -0.0005 = \text{USD } 250\end{aligned}$$

(Note: 5 *basis points* = 0.05% = 0.0005.)

Reading 57: Modeling and Hedging Non-Parallel Term Structure Shifts

After completing this reading you should be able to:

- Describe the principal components analysis and explain its use in understanding term structure movements.
- Define key rate exposures and know the characteristics of key rate exposure factors including partial ‘01s and forward-bucket ‘01s.
- Describe key-rate shift analysis.
- Define, calculate, and interpret key rate ‘01 and key rate duration.
- Compute the positions in hedging instruments necessary to hedge the key rate risks of a portfolio.
- Relate key rates, partial ‘01s, and forward-bucket ‘01s, and calculate the forward-bucket ‘01 for a shift in rates in one or more buckets.
- Apply key rate and multi-factor analysis to estimating portfolio volatility.

Principal Components Analysis

This is a statistical technique that can be used to explain movements in term structure in historical data. Daily movements in the rates of various maturities are observed and certain factors (term structure movements) are identified. Term structure movements have the following properties:

- The daily term structure movements observed are a linear combination of the factors (e.g., a combination of 2 units of the first factor, one unit of the second factor, and 4 units of the third factor);
- The factors are uncorrelated; and
- The first two or three factors account for most of the observed daily movements.

The importance of a factor can be measured by the standard deviation of its factor scores.

The following are the three most important factors driving Treasury rates:

- Factor 1 is a shift in the term structure where all rates move in the same direction by roughly (but not exactly) the same amount.
- Factor 2 is a shift where short-term rates move in one direction and long-term rates move in the other direction. It corresponds to steepening or flattening of the term structure.
- Factor 3 is a bowing of the term structure where relatively short-term and relatively long-term rates move in one direction while intermediate rates move in the other direction.

In chapter 12, we looked at just one factor (a parallel shift in the term structure).

The main weakness attributable to single-factor approaches to portfolio hedging has much to do with the assumption that movements in the entire term structure can be exhaustively described by one interest rate factor. In other words, the single-factor approach erroneously assumes that all rate changes within the term structure of interest rates are driven by a single factor.

From a practical point of view, rates in different regions of the term structure are not always correlated. As an example, the single-factor approach tells us that the 6-month rate can perfectly predict the change in the 30-year rate. This in turn informs the hedging of the 30-year bond with a 6-month bill. Such a move is unlikely to hedge the total risk inherent in the 30-year bond.

Predicted changes in the 30-year rate based purely on changes in the 6-month rate can be quite misleading. That's because rates in different regions of the term structure (yield curve) are not always correlated. The risk of such non-parallel shifts along the yield curve is known as yield curve risk.

Using the **principal components analysis**, multiple factors are identified and assessed in relation to their relative importance in describing movements in the term structure.

This chapter discusses how the metrics introduced in Chapter 12 can be extended to multi-factor models.

Key Rate Exposures

Key rate exposures help to describe the risk distribution along the term structure given a bond portfolio. They help describe how to execute the perfect hedge using highly liquid benchmark bonds. Bonds used for this purpose are normally government bonds issued in the recent past, which means they are likely to be trading at or near par.

Partial '01s are used to measure and hedge the risk of portfolios of swaps or portfolios that combine both bonds and swaps in terms of the most liquid money market and swap instruments.

Forward bucket '01s are also used to measure and to hedge the risk of portfolios of swaps/bond combinations, but the difference here is that instead of measuring risk based on other comparable securities on the market, they measure risk based on changes in the shape of the yield curve. Forward bucket '01s present an intuitive way to understand the yield curve risk of a portfolio, but they are otherwise not efficient at recommending the perfect hedges to neutralize such risks. To compute forward '01s, the yield curve is divided into several defined regions.

A Description of the Key-rate Shift Analysis

The assumption behind the key rate shift analysis is that the entire spectrum of rates can be considered as a function of a few select rates at specified points along the yield curve. Thus, to measure risk and predict interest rate movements, a small number of key rates are used, usually those of highly liquid government bonds.

The rates most commonly used are the U.S. Treasury 2-, 5-, 10-, and 30-year par yields. As the words suggest, a "key rate shift" occurs when any of these rates shifts by one basis point. The key rate technique indicates that changes in each key rate will affect rates from the term of the previous key rate to the term of the subsequent key rate.

The key rate shift approach enables analysts to estimate changes in all rates based on a few select rates.

Definition, Calculation, and Interpretation of Key Rate ’01 and Key Rate Duration

By definition, a Key rate '01 (key rate DV01) is the effect of a dollar change of a one basis point shift around each key rate on the value of the security.

The Key rate '01 is computed using the same logic as the DV01 formula used in the single-factor approach.

$$\text{Key rate '01} = -\frac{\Delta \text{BV}}{10,000 \times \Delta \text{y}}$$

Where:

ΔBV = change in bond value

Δy = change in yield (0.01%)

Note that yield here implies the yield to maturity.

The change in bond value here is **measured in reference to the initial** bond value.

Example: Key Rate DV01s and Durations of the May 15, 2045, C-STRIP as of May 28, 2015

In the table below, column (1) gives the initial price of a C-STRIP and its present value after the application of key rate one basis point shifts.

	Value	Key Rate ‘01	Calculation
Initial value	26.11485		
2-year shift	26.11582	−0.001	$-\frac{26.11582-26.11485}{10{,}000*0.01\%}$
5-year shift	26.11885	−0.040	$-\frac{26.13885-26.11885}{10{,}000*0.01\%}$
10-year shift	26.13885	−0.024	$-\frac{26.13885-26.11485}{10{,}000*0.01\%}$
30-year shift	26.01192	0.103	$-\frac{26.01192-26.11485}{10{,}000*0.01\%}$

The key rate ’01 with respect to the 10-year shift is calculated as:

$$\text{Key rate '01} = -\frac{\Delta BV}{10,000 \times \Delta y} \qquad = -\frac{26.13885 - 26.11485}{10,000 \times 0.01\%} = -0.024$$

How do we interpret the key rate?

A key rate of -0.024 implies that the C-STRIP increases in price by 0.024 per $100 face value for a one basis point 10-year shift.

Exam tip: Just like the DV01, a negative key rate ’01 implies an increase in value after a given shift, relative to the initial value. A positive key rate ’01 implies a decrease in value after a given shift, relative to the initial value.

Key Rate Duration

The effective duration calculates expected changes in price for a bond or portfolio of bonds given a basis point change in yield, but it is only valid for parallel shifts in the yield curve. The key rate duration presents an improvement to the effective duration because it gives the expected changes in price when the yield curve shifts in a manner that is not perfectly parallel.

The key rate duration is actually analogous to duration so that:

$$\text{Key rate duration} = -\frac{1}{P}\left(\frac{\partial P}{\partial y}\right)$$

Thus, the key rate duration with respect to the 10-year shift is calculated as:

$$\begin{aligned}\text{Key rate duration} &= -\left(\frac{1}{26.11485}\right) \times \left(\frac{26.13885 - 26.11485}{0.01\%}\right)\\ &= -9.19\end{aligned}$$

Alternatively, recall that:

$$\text{DV01} = \text{Duration} \times 0.0001 \times \text{Bond value}$$

Thus,

$$\begin{aligned}\text{Duration} &= \frac{\text{DV01}}{0.0001 \times \text{ Bond value}}\\ &= \frac{-0.024}{0.0001 \times 26.11485}\\ &= -9.19\end{aligned}$$

How do we interpret the key rate duration?

Interpreting each key rate duration in isolation can be quite difficult. That's because, in practice, it's highly unlikely that a single point on the yield curve will exhibit an upwards or downwards shift while all other points remain constant. For this reason, analysts tend to compare key rate durations across the curve.

Exam tip: The sum of all the key rate durations along a portfolio yield curve is equal to the effective duration of the portfolio.

The Key Rate Exposure Technique in Multi-factor Hedging Applications

Before looking at some examples, let's try to map out a typical problem investors often find themselves in.

Let's say an investor holds a (long) bond position and is afraid that the bond will lose some value in the future. We call such a position the "underlying exposure." How exactly can the position lose value? Suppose the position has a 5-year key rate exposure of $0.05. This implies that the position will drop in value by $0.05 if there's a one basis point shock to the 5-year key rate. If the bond is trading at, say, $94 per $100 face value, then the new price will be $93.95 per $100 face value following a one basis point increase in the 5-year key rate (Remember that bond prices fall when interest rates rise). To avoid the loss, the investor must identify and sell short another bond whose key rate exposure matches that of the underlying exposure.

Selling short is all about selling high and buying low. The investor will borrow the bond and sell it, anticipating an increase in the 5-year key rate which will result in a decrease in the bond's price. The investor will still be obligated to "return" the bond to its owner but he will buy it at a lower price and get to keep the difference, which will offset the loss incurred on the long position (underlying exposure). This is the argument behind key rate exposure hedging.

Note that the hedging security need not have a 5-year key rate exposure of exactly $0.05. It could have an exposure of, say, 0.045, implying that the investor will not sell short exactly $100 face value of the bond; the amount will be slightly more.

In key rate exposure hedging, therefore, the secret lies in determining the face amount "F" that's needed to **neutralize** the key rate exposure of the underlying position.

Example 1 on Multi-factor Hedging

An underlying exposure (bond position) has a ten-year key-rate '01 of +$880. If this key rate exposure can be hedged by trading a ten-year bond that itself has a 10-year KR01 of $0.0520 per 100 face amount, what is the hedge trade?

Solution

A positive key rate '01 implies a decrease in value after a given shift, relative to the initial value. Thus, a ten-year key rate '01 of +$880 implies that the bond position stands to lose $880 if there happens to be a one basis point shock to the ten-year key rate. To avoid this scenario, we must determine the face amount F(10) of the ten-year bond that must be sold short to neutralize the key rate exposure. We proceed as follows:

$$\frac{0.0520}{100} \times F(10) = 880$$

$$F(10) = \frac{880}{0.00052} = \$1,692,308$$

Note that since the key rate 01s are reported per 100 face value, they need to be divided by 100 in the hedging equation. However, the key rate 01 of the initial bond position (underlying exposure) is reported for the face amount to be hedged, so it stands as it is.

The hedge trade requires us to short the $1.692 million face amount of the ten-year bond so as to neutralize the exposure to the ten-year key rate. If there's a one basis point shock to the ten-year key rate, the long position will lose $880, while the short position will gain approximately $880 (=

0.052/100 × 1,692,000).

Important note:

Before looking at the second example, it is important to understand exactly what key rates stand for. When we say that, for example, the 5-year key rate changes, what we mean is that if the 5-year par rate changes; all other par rates are unchanged. It is easy to think of the 5-year key rate as the 5-year spot rate, but it is not; it's the par rate. Key rates are not spot rates. (*Par rate denotes the coupon rate for which the price of a bond is equal to its nominal value (or par value).*

This leads us to a very important observation: a bond priced **at par** (i.e., purchase price = par value = $100) **only has** price sensitivity to key rates at the same tenor as its maturity. For instance, a 5-year coupon-paying **par bond** has **zero sensitivity** to a change in the 2-year key rate. However, a 5-year **premium/discount** bond will have some sensitivity to the 2-year key rate.

The reason, as we have seen above is that the 5-year par rate doesn't change. We compute the price of a bond by discounting all its cash flows by its YTM. If the 5-year par rate doesn't change, then the YTM on a 5-year par bond doesn't change, and therefore the price of a 5-year ***par*** bond doesn't change.

Example 2 on Multi-factor Hedging

Let's use an example to illustrate how to pull off the perfect hedge under multi-factor hedging:

Suppose we have a 30-year option-free bond paying semi-annual coupons of $5,000 in a flat rate environment of 5% across all maturities. Using the concepts learnt in the preceding learning outcome statements, we can compute the following key rate '01s and key rate durations, assuming a one-basis point shift in the key rates used:

	Value	Key Rate '01	Key Rate Duration
Initial value	145, 066.45		
2-year shift	145, 061.23	5.22	0.36
5-year shift	145, 050.68	15.77	1.09
10-year shift	144, 989.02	77.43	5.34
30-year shift	145, 000.95	65.50	4.52
Total		163.92	11.31

For example,

The key rate '01 with respect to the 5-year shift is calculated as:

$$\begin{aligned}\text{Key rate '01} &= -\frac{\Delta BV}{10,000 \times \Delta y} \\ &= -\frac{145,050.68 - 145066.45}{10,000 \times 0.01\%} = 15.77\end{aligned}$$

And the corresponding key rate duration is:

$$\begin{aligned}\text{duration} &= \frac{\text{DV01}}{0.0001 \times \text{bond value}} \\ &= \frac{15.77}{0.0001 \times 145066.45} \\ &= 1.09\end{aligned}$$

A key rate '01 of 15.77 implies that the bond decreases in value by $15.77 for a one basis point shock to the 5-year key rate.

We can easily come up with the other key rate 01's and key rate durations by performing similar calculations.

Now to illustrate how hedging is carried out in this scenario, assume we have **four other** different securities, each with the following key rate exposures:

To illustrate how hedging is carried out based on key rates, assume we have **four other** different securities, each with the following key rate exposures:

Security	Exposure	(per 100	face value)	
	2-year key rate	5-year key rate	7-year key rate	30-year key rate
2-year security	0.001			
5-year security	0.0015	0.045		
10-year security	0.002	0.001	0.1	
30-year security				0.20

Note: In the table above, we assume that the 2-year bond and the 30-year bond are trading at par, in which case they are only exposed to the key rate corresponding to their maturity dates (2 years and

30 years, respectively). On the other hand, the 5-year and 10-year securities are trading at a premium.

For the hedge to work, we must neutralize the key rate exposure at each key rate.

Let F_2, F_5, F_{10}, and F_{30} be the face amounts of the bonds in the hedging portfolio to be sold.

2-year key rate exposure: Three bonds, namely the two-year, five-year, and 10-year, have an exposure to the two-year key rate. Therefore, for the two-year key rate exposure of the hedging portfolio to equal that of the underlying position, it must be the case that

$$\text{2-year key rate exposure}: \frac{0.001}{100} \times F_2 + \frac{0.0015}{100} \times F_5 + \frac{0.002}{100} \times F_{10} = 5.22$$

5-year key rate exposure: Only two bonds, namely the five-year and 10-year, have an exposure to the five-year key rate. Therefore, for the five-year key rate exposure of the hedging portfolio to equal that of the underlying position, it must be the case that

$$\text{5-year key rate exposure}: \frac{0.045}{100} \times F_5 + \frac{0.001}{100} \times F_{10} = 15.77$$

10-year key rate exposure: Only the ten-year bond has an exposure to the ten-year key rate. Therefore, for the ten-year key rate exposure of the hedging portfolio to equal that of the underlying position, it must be the case that

$$\text{10-year key rate exposure}: \frac{0.1}{100} \times F_{10} = 77.43$$

30-year key rate exposure: Only the ten-year bond has an exposure to the ten-year key rate. Therefore, for the ten-year key rate exposure of the hedging portfolio to equal that of the underlying position, it must be the case that

$$\text{30-year key rate exposure}: \frac{0.2}{100} \times F_{30} = 65.50$$

Solving equations (1) through (4) simultaneously gives the following solution for the face value of the hedging bonds in the hedging portfolio:

$$F_{30} = 32,750$$
$$F_{10} = 77,430$$
$$F_5 = 33,324$$
$$F_2 = 317,150$$

What then, do these figures imply?

The investor needs to short $317,150 face amount of the 2-year security, short $33,324 face amount of the 5-year security, short $77,430 face amount of the 10-year security, and finally short $32,750 face amount of the 30-year security. Only then would the initial bond position be insured from changes in rates close to the key rates used.

However, such a hedge portfolio is not perfect, and the hedged position is actually only approximately immune due to two main reasons

As is the case whenever derivatives are used for hedging purposes, the quality of hedge deteriorates as the size of the interest rate change increases.

The hedge will work only if the par yields between key rates move as assumed (linearly).

Other reasons why the hedge may not work include:

- Hedging implies more instruments and more transaction costs which may eat up the scooped gains;
- Under the key rate model, the number of key rate durations to be used and the corresponding choice of key rates remain quite arbitrary

Partial '01s and Forward-bucket '01s

Partial '01s

Key rate shifts make use of a few key rates to determine risk exposures and execute hedging strategies. For, example, in this reading, we've used the 2-year, 5-year, 10-year, and 30-year par yields. The hedging strategy must involve all four.

However, when the securities involved contain swaps, we need more to assess the effect of interest rates at more points along the yield curve. There's a need to measure more frequently. This leads us to partial '01s and forward-bucket '01s.

When swaps are taken as the benchmark for interest rates in complex portfolios, risk along the curve is usually measured with Partial '01s or Partial PV01s, rather than with key-rate '01s. Swap market participants fit a swap rate at least once every day from a set of observable par swap rates or futures rates. Using the fitted swap rate curve, the sensitivity of a portfolio can be measured in terms of changes in the rates of the fitting securities.

It follows that by definition, **partial '01** (PV01) **is the change in the value of a portfolio after** a one-basis-point decline in that fitted rate and refitting of the curve. All other fitted rates are **unchanged.** With partial '01s, yield curve shifts are able to be fitted more precisely because we are constantly fitting securities.

Example of Partial '01s

If a curve-fitting algorithm fits the three-month London Interbank Offered Rate (LIBOR) rate and par rates at 2-, 5-, 10-, and 30-year maturities, then, the two-year partial '01 would be the change in the value of a portfolio for a one-basis-point decline in the two-year par rate and refitting of the curve, where the three-month LIBOR and the par 5-, 10-, and 30-year rates are kept the same.

Forward-bucket '01s

While key rates and partial '01s do a fantastic job expressing the exposures of a position in terms of hedging securities, forward-bucket '01s present a far more direct and intuitive way to convey the exposures of a position to different parts of the curve.

A bucket is a jargon for a region of the term structure of interest rates.

Forward-bucket '01s are computed by shifting the forward rate over each of several defined regions of the term structure on the region at a time. They, however, aren't the quickest way to determine the hedges required to pull off that perfect "immunization".

The first step under this methodology is to subdivide the term structure into buckets. The 5 most common buckets are 0-2 years, 2-5 years, 5- 10 years, 10-15 years, and 20-30 years. After that, each forward-bucket '01 is computed by shifting the forward rates in that bucket by one basis point. In so doing, the analyst may have to shift all of a bucket's semiannual forward rates, quarterly forward rates, or even shorter-term rates.

Example: Computation of Forward-Bucket '01s of a 5-year Swap Given a 0-2 year Bucket and a 2-5 year Bucket.

The table below lists the cash flows of the fixed side of the 100 notional amount of a swap, the current forward rates (marked "current") as of the pricing date, and the three shifted forward curves.

		Forward Rates			
Term	Cash flow	Current	0-2 shift	2-5 shift	shift-all
0.5	1.06	1.012	1.022	1.012	1.022
1	1.06	1.248	1.258	1.248	1.258
1.5	1.06	1.412	1.422	1.412	1.422
2	1.06	1.652	1.662	1.652	1.662
2.5	1.06	1.945	1.945	1.955	1.955
3	1.06	2.288	2.288	2.298	2.298
3.5	1.06	2.614	2.614	2.624	2.624
4	1.06	2.846	2.846	2.856	2.856
4.5	1.06	3.121	3.121	3.131	3.131
5	101.06	3.321	3.321	3.331	3.331
Present value		99.9955	99.976	99.9679	99.9483
01			0.0195	0.0276	0.0472

Credit: Bruce Tuckman and Angel Serrat, Fixed Income Securities: Tools for Today's Markets, 3rd Edition

- For the "0-2 Shift," forward rates of term 0.5 to 2.0 years are shifted up by one basis point while holding all other forward rates constant.
- For the "2-5 Shift," forward rates of term 2.0 to 5.0 years are shifted up by one basis point while, again, holding all other forward rates constant.
- Lastly, for "Shift All," the forward rates in the curve are shifted

The row labeled "Present Value" gives the present value of the cash flows first under the initial forward rate curve and then under each of the shifted curves.

The forward-bucket '01 for each shift can then be computed as the negative of the difference between the shifted and initial present values, i.e.,

Forward bucket '01 = –(shifted present value – initial present value)
For the 0-2-year shift, for example, the '01 is −(99.976 − 99.9955), or 0.0195

The '01 of the "Shift All" scenario is analogous to a DV01. The forward bucket analysis decomposes this total '01 into 0 .0195 due to the 0-2-year part of the curve and 0.0276 due to the 2-5-year part of the curve.

Hedging across Forward-Bucket Exposures

Referring to the GARP-assigned Tuckman reading, let us say a counterparty enters into a euro 5x10 payer swaption with a strike of 4.044% on May 28, 2010.

This payer swaption gives the buyer the right to pay a fixed rate of 4.044% on a 10-year euro swap in five years. The underlying is a 10-years swap for settlement on May 31, 2015.

The figure below gives the forward-bucket '01s of this swaption for four different buckets, along with other swaps for hedging purposes.

Security	Rate	0-2	2-5	5-10	10-15	All
5×10 payer swaption	4.04%	0.0010	0.0016	−0.0218	−0.0188	-0.0380
5-year swap	2.120%	0.0196	0.0276	0.0000	0.0000	0.0472
10-year swap	2.943%	0.0194	0.0269	0.0394	0.0000	0.0857
15-year swap	3.290%	0.0194	0.0265	0.0383	0.0323	0.1165
5×10 swap	4.044%	0.0000	0.0000	0.0449	0.0366	0.0815

Since the overall forward-bucket '01 of the year swaption is negative (-0.0380), as rates rise, the value of the option to pay a fixed rate of 4.044% in exchange for a floating rate worth par also rises.

The figure below shows forward-bucket exposures of three different ways to hedge this payer swaption (as of May 28, 2010) using securities presented in the previous figure:

Security/Portfolio	0-2	2-5	5-10	10-15	All
5*10 payer swaption	0.001	0.0016	−0.0218	−0.0188	−0.0380
Hedge #1:Long 44.34% of 10-year swaps	0.0086	0.0119	0.0175		0.038
Net position	0.0096	0.0135	−0.0043	−0.0188	0.000
Hedge #2:Long 46.66% of 5*10 swaps			0.0209	0.0171	0.038
Net position	0.001	0.0016	−0.0009	0.0017	0.000
Hedge #3:					
Long 57.55% of 15-year swaps	0.0112	0.0153	0.022	0.0186	0.067
Short 61.55% of 15-year swaps	−0.012	−0.017			−0.029
Net position	0.0002	−0.0001	0.0002	−0.0002	0.000

As is apparent, the third hedge is the best option since this hedge best neutralizes risk in each of the buckets (the lowest net position indicates when risk is best neutralized).

Credit: Bruce Tuckman and Angel Serrat, Fixed Income Securities: Tools for Today's Markets, 3rd Edition

Applying Key Rate and Multi-Factor Analysis to Estimate Portfolio Volatility

Although we have studied at length the term structure of interest rates, we are yet to look at volatility. Just like there is a term structure for interest rates, there is also a term structure for volatility. In fact, the volatility term structure typically slopes downwards when plotted against maturity. This implies that the shorter the maturity of the par-rate, the more volatile it tends to be. The 10-year par rate, for example, is usually more volatile than the 30-year par rate.

In general, portfolios are exposed to interest rates all along the curve but changes in these rates are not perfectly correlated. How can we go about estimating volatilities for the key rates?

Step 1: Estimate the volatility for each key rate as well as the correlation for each pair of key rates.

Step 2: Compute the key-rate 01s of the portfolio

Step 3: Compute the variance and volatility of the portfolio

For example, let's make some assumptions:

- There are two key rates C_1 and C_2
- The key rates of the portfolio are $KR01_1$ and $KR01_2$.
- P gives the value of our portfolio

Then, by the definition of key rates,

$$\Delta P = KR01_1 \times \Delta C_1 + KR01_2 \times \Delta C_2$$

Furthermore, let σ_P^2, σ_1^2 and σ_2^2 denote the variances of the portfolio and of the key rates and let ρ denote the correlation of the key rates. By applying the usual formula for finding the variance of a portfolio, we can estimate portfolio variance:

$$\sigma_P^2 = \sigma_1^2 KR01_1^2 + \sigma_2^2 KR01_2^2 + 2\rho_{1,2}\sigma_1\sigma_2 KR01_1 KR01_2$$

Note that this methodology can be applied equally well to partial '01s or forward-bucket '01s.

Questions

Question 1

A risk manager at an Indian bank helps manage a portfolio of investment-grade option-free bonds. After a lengthy market analysis, the manager strongly recommends portfolio hedging using the key rates of 2-year, 5-year, 7-year, and 20-year exposures. According to the manager, the 2-year rate has increased by 10 basis points in the recent past.

How will the increase affect the 20-year rate?

A. It will increase by 10 basis points

B. It will decrease by 10 basis points

C. It will increase by 20 basis points

D. It will increase by zero basis points

The correct answer is **D**.

The key rate technique indicates that changes in each key rate will affect rates from the term of the previous key rate to the term of the subsequent key rate. In this case, the 2-year key rate will affect all rates from 0 to 5 years; the 5-year key rate affects all rates from 2 to 7 years; the 7-year key rate affects all rates from 5 to 20 years; and the 20-year key rate affects all rates from 7 years to the end of the curve.

Question 2

Suppose we have a 30-year option-free bond paying semiannual coupons of $4,000 in a flat rate environment of 5% across all maturities. The following table provides the initial price of the bond and its present value after application of a one basis point shift in four key rates:

	Value	Key Rate '01
Initial value	138, 200.55	
2-year shift	138, 195.23	5.32
5-year shift	138, 187.33	13.22
7-year shift	138, 172.91	27.64
30-year shift	138, 180.25	20.30
Total		66.48

Suppose further that there are four other different bonds with the following key rate exposures:

Security	Exposure	(per 100	face value)	
	2-year key rate	5-year key rate	7-year key rate	30-year key rate
2-year security	0.001			
5-year security	0.0015	0.045		
7-year security	0.002	0.001	0.1	
30-year security				0.20

If we wish to fully hedge our initial position using these four securities, determine the face amount of the 5-year security we need to short (assume that the 2-year bond and the 30-year bond are trading at par):

A. 27,640

B. 10,150

C. 28,764

D. 30,000

The correct answer is **C**.

If we assume that the 2-year bond and the 30-year bond are trading at par, they are only exposed to the key rate corresponding to their maturity dates (2 years and 30 years, respectively). The face amount we need for each security is given by F_i.

$$2\text{-year key rate exposure:} \frac{0.001}{100} \times F_2 + \frac{0.0015}{100} \times F_5 + \frac{0.002}{100} \times F_7 = 5.32$$

$$5\text{-year key rate exposure:} \frac{0.045}{100} \times F_5 + \frac{0.001}{100} \times F_7 = 13.22$$

$$7-\text{year key rate exposure :} \frac{0.1}{100} \times F_7 = 27.64$$

$$30\text{-year key rate exposure:} \frac{0.2}{100} \times F_{30} = 20.30$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$F_{30} = 10,150$$
$$F_7 = 27,640$$
$$F_5 = \frac{13.22 - 0.2764}{0.00045} = 28,764$$

Reading 58: Binomial Trees

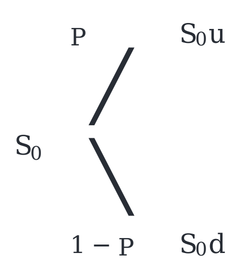
After completing this reading you should be able to:

- Calculate the value of an American and a European call or put option using a one-step and two-step binomial model.
- Describe how volatility is captured in the binomial model.
- Describe how the value calculated using a binomial model converges as time periods are added.
- Define and calculate the delta of a stock option.
- Explain how the binomial model can be altered to price options on stocks with dividends, stock indices, currencies, and futures.

Pricing Options Using the Binomial Model

The binomial option pricing model is a simple approximation of returns which, upon refining, converges to the analytic pricing formula for vanilla options. The model is also useful for valuing American options that can be exercised before expiry.

The model can be represented as:



The notation used is as follows:

S_0=stock price today

P =probability of a price rise

u=The factor by which the price rises

d=The factor by which the price falls

Over a small time interval Δt, the price today rises or falls to one of only two potential future values: S_0u, and S_0d.

The underlying price is assumed to follow a random walk.

Risk-neutral Valuation

The following formula are used to price options in the binomial model:

u=size of the up move factor=$e^{\sigma\sqrt{t}}$, and

d=size of the down move factor=$e^{-\sigma\sqrt{t}} = \frac{1}{e^{\sigma\sqrt{t}}} = \frac{1}{u}$

σ is the annual volatility of the underlying asset's returns and t is the length of the step in the binomial model.

π_u =probability of an up move=$\frac{e^{rt}-d}{u-d}$

π_d=probability of a down move=$1-\pi_u$

Let f_u be the value of an option when the price goes up and f_d the value when the price goes down.

The value,f of the option, for one step-binomial is given by:

$$f = e^{-rt}(\pi f_u + (1-\pi)f_d)$$

Where,

$$\pi = \frac{e^{rt} - d}{u - d}$$

Example: Risk-neutral Valuation

The price of an exchange-quoted zero-dividend share is $30. Over the past year, the stock has exhibited a standard deviation of 17%. The continuously compounded risk-free rate is 5% per annum.

Compute the value of a 1-year European call option with a strike price of $30 using a one-period binomial model:

The up- and down-move factors are:

$$u = e^{0.17 \times \sqrt{1}} = 1.1853$$
$$d = \frac{1}{1.1853} = 0.8437$$

The risk-neutral probabilities of an up- and down-move are:

$$\pi_u = \frac{(e^{0.05 \times 1}) - D}{U - D} = \frac{1.0513 - 0.8437}{1.1853 - 0.8437} = 0.61$$
$$\pi_d = 1 - 0.61 = 0.39$$

Exhibit 1: Binomial Tree – Stock

$$1.1853 \times \$30 = \$35.60$$

$$\$30$$

$$0.8437 \times \$30 = \$25.30$$

Exhibit 2: Binomial Tree – Option

$$\max(0, \$35.6 - \$30) = \$5.6$$

$$c_0$$

$$\max(0, \$25.3 - \$30) = \$0$$

The expected value of the option in one year is given by:

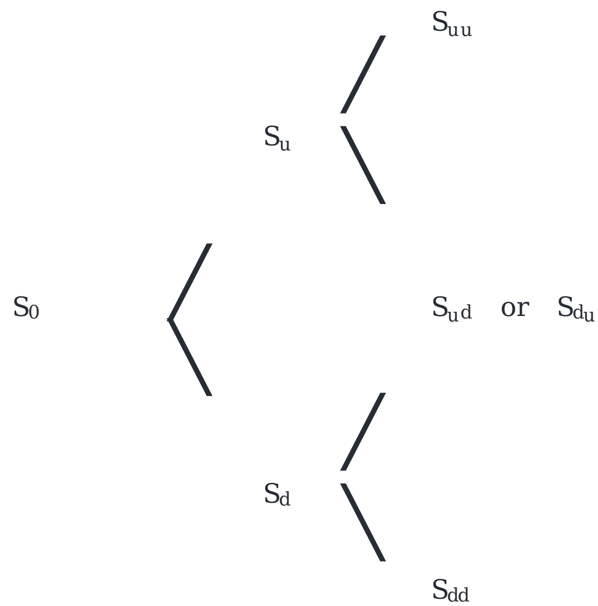
$$c_u \times \pi_u + c_d \times \pi_d = \$5.6 \times 0.61 + \$0 \times 0.39 = \$3.42$$

The expected value of the option at present is given by:

$$c_0 = \$3.42e^{(-0.05\times 1)} = \$3.25$$

Two-step Binomial model

In the two-period model, the tree is expanded to create room for a greater number of potential outcomes. Exhibit 3 below presents the two-period stock price tree:



The two-step model uses the same formulae used in the one-step version to calculate the value of an option.However, here, we replace t with Δt, which is the length of one-step. If we have say, an option that matures in one year period, then for a two-step binomial model, $\Delta t = 1/2 = 0.5$

Thus, the value of an option is given by:

$$f = e^{-r\Delta t}\left(\pi f_u + (1-\pi) f_d\right)$$

and

$$\pi = \frac{e^{r\Delta t} - d}{u - d}$$

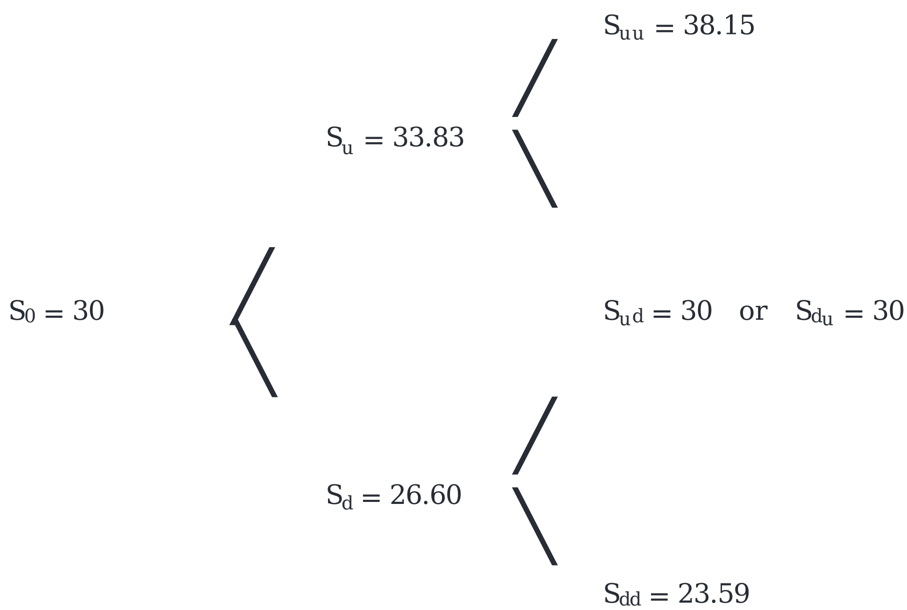
Example: Two-Step Binomial

The price of an exchange-quoted zero-dividend share is \$30. Over the past year, the stock has exhibited a standard deviation of 17%. The continuously compounded risk-free rate is 5% per annum. Compute the value of a 1-year European call option with a strike price of $30 using a two-step binomial model

The up- and down-move factors are:

$$u = e^{0.17 \times \sqrt{0.5}} = 1.1277$$
$$d = \frac{1}{1.1277} = 0.8867$$

So that,



The option values are

$S_{uu} = \$38.15$	$f_{uu} = \max(\$38.15 - \$30, 0)$	$f_{uu} = \$8.15$
$S_{ud} = \$30$	$f_{ud} = \max(\$30 - \$30, 0)$	$f_{ud} = \$0$
$S_{du} = \$30$	$f_{du} = \max(\$30 - \$30, 0)$	$f_{du} = \$0$
$S_{dd} = \$23.59$	$f_{dd} = \max(\$23.59 - \$30, 0)$	$f_{dd} = \$0$

The risk-neutral probability is:

$$\pi = \frac{(e^{0.05 \times 0.5}) - d}{u - d} = \frac{1.0253 - 0.8867}{1.1277 - 0.8867} = 0.58$$

Thus,

$$f_u = e^{-r\Delta t}(\pi f_{uu} + (1-\pi)f_{ud}) = e^{-0.05\times 0.5}(0.58 \times 8.15 + 0.42 \times 0) = 4.6103$$

and

$$f_d = e^{-r\Delta t}(\pi f_{ud} + (1-\pi)f_{dd}) = e^{-0.05\times 0.5}(0.58 \times 0 + 0.42 \times 0) = 0$$

Thus, the value of the option is

$$f = e^{-r\Delta t}(\pi f_u + (1-\pi)f_d) = e^{-0.05\times 0.5}(0.58 \times 4.6102 + 0.42 \times 0) = 2.6079$$

Note: The value of a put can be calculated once the value of the call has been determined, using the put-call parity relationship.

$$\text{Call Price} + \text{PV of Strike Price} = \text{Put Price} + \text{Stock Price}$$

Increasing the Number of Steps in the Binomial Model

Binomial models with one or two steps are unrealistically simple. Assuming only one or two steps would yield a very rough approximation of the option price. In practice, the life of an option is divided into 30 or more time steps. In each step, there is a binomial stock price movement.

As the number of time steps is increased, the binomial tree model makes the same assumptions about stock price behavior as the Black– Scholes–Merton model. When the binomial tree is used to price a European option, the price **converges to the Black-Scholes-Merton** price as the number of time steps is increased.

Delta

The delta, Δ, of a stock option, is the **ratio of the change in the price of the stock option to the change in the price of the underlying stock**. It is the number of units of the stock an investor/trader should hold for each option shorted in order to create a riskless portfolio. This process is called **delta-hedging**.

The delta of a call option is always between 0 and 1 because as the underlying asset increases in price, call options increase in price. The delta of a put option, on the other hand, is always between -1 and 0 because as the underlying security increases, the value of put options decrease.

For instance, suppose that when the price of a stock change from \$20 to \$22, the call option price changes from \$1 to \$2. We can calculate the value of delta of the call as:

$$\frac{2-1}{22-20} = 0.5$$

This means that if the underlying stock increases in price by \$1 per share, the option on it will rise by \$0.5 per share, all else being equal.

Suppose that an investor is long one call option on the stock above (with a delta of 0.5, or 50 since options have a multiplier of 100). The investor could delta hedge the call option by **shorting** 50 shares of the underlying stock. Conversely, if the investor is long one put on the stock (with a delta of -0.5, or -50), they would maintain a delta neutral position by **purchasing** 50 shares of the underlying stock.

Generally,

$$\Delta = \frac{f_u - f_d}{S_u - S_d}$$

How Volatility is Captured in the Binomial Model

As the standard deviation increases, so does the divide (dispersion) between stock prices in up and down states (S_u and S_d, respectively). Suppose there was no deviation at all. Would we have a binomial tree in the first place? The answer is no.

With zero standard deviation, (S_u would be equal to S_d, and instead of a tree, we would have a straight line. But provided there's some deviation, the gap between stock prices in the upstate and stock prices in the downstate increasingly widens as the deviation increases.

To capture volatility, therefore, it would be paramount to evaluate stock prices at each time period present in the tree.

How the Modified Binomial Model can be Altered to Price Options on Non-zero Dividend Stocks, Stock Indices, Currencies, and Futures

Given a stock that pays a continuous dividend yield q, the following formula can be used to price the resulting option:

$$\text{Probability of an up move} = \pi_u = \frac{e^{(r-q)t} - d}{u - d}$$
$$\text{Probability of a down move} = 1 - \pi_u$$

u=size of the up move factor=$e^{\sigma\sqrt{t}}$, and

d=size of the down move factor=$e^{-\sigma\sqrt{t}} = \frac{1}{e^{\sigma\sqrt{t}}} = \frac{1}{u}$

Note: The sizes of the up move factor and down move factor are the same as in the zero-dividend model.

Sometimes it may also be necessary to price options constructed with a stock index as the underlying, for instance, an option on the S&P 500 index. Such an option would be valued in a manner similar to that of the dividend-paying stock. It’s assumed that the stocks forming part of the index pay a dividend yield equal to q.

The binomial model can also be modified to incorporate the unique characteristics of options on futures. Of note is the fact that futures contracts are largely considered cost-free to initiate, and therefore in a risk-neutral environment, they are zero-growth instruments. The only formula that changes is that of the probability of an up move, where:

$$\pi_u = \frac{1 - d}{u - d}$$

When dealing with options on currencies, a plausible assumption is that the return earned on a foreign currency asset is equal to the foreign risk-free rate of interest. As such, the probability of an up move is given by:

$$\pi_u = \frac{e^{(r_{DC} - r_{FC})t} - d}{u - d}$$

American Options

To value an American option, we check for early exercise at each node. If the value of the option is greater when exercised, we assign that value to the node. If that's not the case, we assign the value of the option unexercised. We then work backward through the tree as usual.

The Binomial Model When Time is Continuous

The binomial model is essentially a discrete-time model where we evaluate option values at discrete times, say, intervals of one year, intervals of six months, intervals of three months, etc.

However, if we were to shrink the length of time intervals to arbitrarily small values, we'd end up with a continuous-time model where the price can move at non-discrete times. The binomial model converges to the continuous-time model when time periods are made arbitrarily small.

Questions

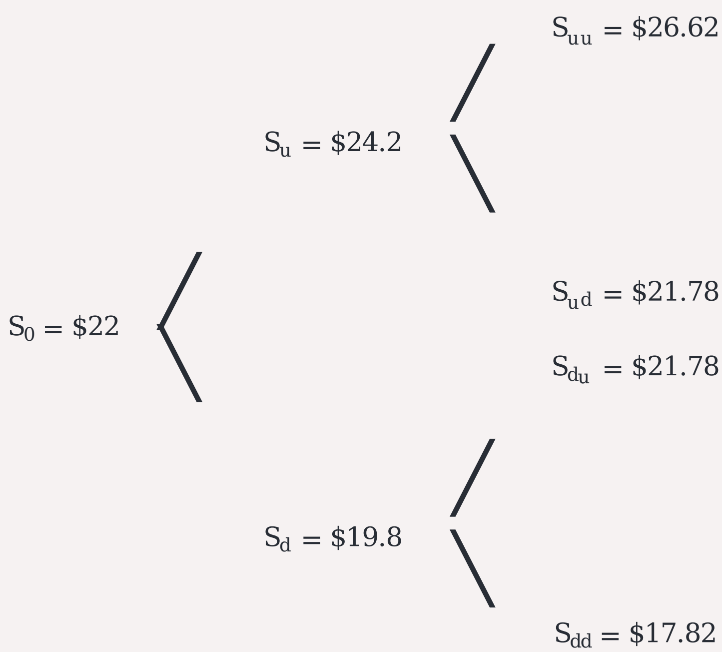
Question 1

Suppose we have a 6-month European call option with K = \$23. Suppose the stock price is currently \$22 and in two-time steps of three months, the stock can go up or down by 10%. The up move factor is 1.1 while the down move factor is 0.9. The risk-free rate of interest is 12%.

Compute the value of the call today.

A. \$2
B. \$1.54
C. \$1.45
D. \$0

The correct answer is **C**.



$S_u = 22 \times 1.1 = 24.2$, $S_{uu} = 22 \times 1.1 \times 1.1$ Other values at other nodes are calculated using the relevant up/down factors.

$$\pi_u = \frac{e^{rt} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523,$$
$$\pi_d = 1 - 0.6523 = 0.3477$$

Let f represent the value of the call:

$S_{uu} = 26.62$	$f_{uu} = \max(\$26.62 - \$23, 0)$	$f_{uu} = \$3.62$
$S_{ud} = \$21.78$	$f_{ud} = \max(\$21.78 - \$23, 0)$	$f_{ud} = \$0$
$S_{du} = \$21.78$	$f_{du} = \max(\$21.78 - \$23, 0)$	$f_{du} = \$0$
$S_{dd} = \$17.82$	$f_{dd} = \max(\$17.82 - \$23, 0)$	$f_{dd} = \$0$

The expected value of the call six months from now is given by:

$$\begin{aligned} &0.6523 \times 0.6523 \times \$3.62 + 0.6523 \times 0.3477 \times \$0 \\ &+ 0.3477 \times 0.6523 \times \$0 + 0.3477 \times 0.3477 \times \$0 \\ &= \$1.54 \end{aligned}$$

$$\text{Value of the call today} = \frac{\$1.54}{e^{0.12 \times 0.5}} = \$1.45$$

Question 2

A 1-year $50 strike European call option exists on ABC stock currently trading at $49. ABC pays a continuous dividend of 3% and the current continuously compounded risk-free rate is 4%. Assuming an annual standard deviation of 3%, compute the value of the call today.

A. $0.31

B. $0.30

C. $0.47

D. $0

The correct answer is **B**.

$$u = e^{\sigma\sqrt{t}} = e^{0.03 \times 1} = 1.03$$
$$d = \frac{1}{1.03} = 0.97$$

Note that the stock is dividend-paying, and therefore the formula for the probability of an up move is given by:

Probability of an up move=$\pi_u = \frac{e^{(r-q)t} - d}{u-d} = \frac{e^{(0.04-0.03)1} - 0.97}{1.03-0.97} = 0.67$

Probability of a down move $= 1 - 0.67 = 0.33$

Let S represent the price of the stock and f represent the value of the call. This is a one-step binomial process.

$S_u = \$49 \times 1.03 = \$50.47$	$f_u = \max(\$50.47 - \$50, 0) = \$0.47$
$S_d = \$49 \times 0.97 = \$47.53$	$f_d = \max(\$47.53 - \$50, 0) = \$0$

$$\text{Value of the call option one year from today} = (\$0.47 \times 0.67 + \$0 \times 0.33) = \$0.31$$
$$\text{Value of the call today} = \frac{\$0.31}{e^{0.04}} = \$0.30$$

Reading 59: The Black-Scholes-Merton Model

After completing this reading you should be able to:

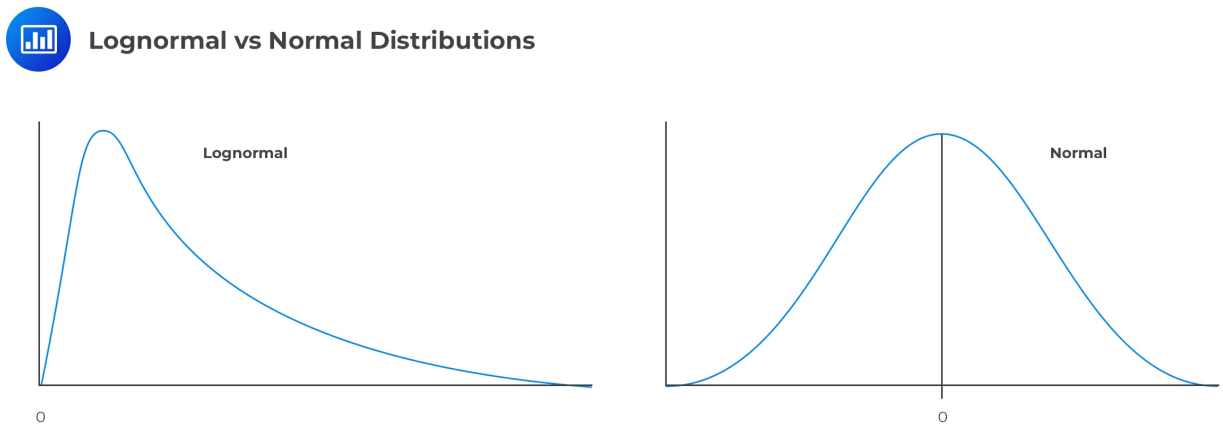
- Explain the lognormal property of stock prices, the distribution of rates of return, and the calculation of expected return.
- Compute the realized return and historical volatility of a stock.
- Describe the assumptions underlying the Black-Scholes-Merton option pricing model.
- Compute the value of a European option using the Black-Scholes-Merton model on a non-dividend-paying stock.
- Define implied volatilities and describe how to compute implied volatilities from market prices of options using the Black-Scholes-Merton model.
- Explain how dividends affect the decision to exercise early for American call and put options.
- Compute the value of a European option using the Black-Scholes-Merton model on a dividend-paying stock.
- Describe warrants, calculate the value of a warrant, and calculate the dilution cost of the warrant to existing shareholders.

Suppose we have a random variable X. This variable will have a lognormal distribution if its natural log (ln X) is normally distributed. In other words, when the natural logarithm of a random variable is normally distributed, then the variable itself will have a lognormal distribution.

The two most essential characteristics of the lognormal distribution are as follows:

- It has a lower bound of zero, i.e., a lognormal variable cannot take on negative values.
- The distribution is skewed to the right, i.e., it has a long right tail.

These characteristics are in direct contrast to those of the normal distribution, which is symmetrical (zero-skew) and can take on both negative and positive values. As a result, the normal distribution cannot be used to model stock prices because stock prices cannot fall below zero. The lognormal distribution is also used to value options.



The Lognormal Property of Stock Prices

A crucial part of the BSM model is that it assumes stock prices are log-normally distributed. Precisely,

$$\ln S_T \sim N\left(\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right)$$

Where:

S_T=stock price at time T

S_0=stock price at time 0

μ=expected return on stock per year

σ=annual volatility of the stock price

Note: The above relationship holds because mathematically, if the natural logarithm of a random variable x, ln(x) is normally distributed, then x has a lognormal distribution. It's also imperative to note that the BSM model assumes stock prices are lognormally distributed, with stock returns being normally distributed. Specifically, continuously compounded annual returns are normally distributed with:

$$\text{a mean of } [\mu - \frac{\sigma^2}{2}] \text{ and a variance of } \frac{\sigma^2}{T}$$

Example: Mean and standard deviation given a lognormal distribution

ABC stock has an initial price of $60, an expected annual return of 10%, and annual volatility of 15%. Calculate the mean and the standard deviation of the distribution of the stock price in six months.

$$\begin{aligned}
&\ln S_T \sim N\left(\ln S_0 + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\right) \\
=&N\left[\ln 60 + (0.10 - \frac{0.15^2}{2})0.5, 0.15^2 \times 0.5\right] \\
&\ln S_T \sim N[4.139, 0.011] \\
&(\text{In this case,we have } \sigma = \sqrt{0.011} = 0.1049)
\end{aligned}$$

Sometimes, the Global Association of Risk Professionals (GARP) may want to test your understanding of the lognormal concept by involving confidence intervals. Since $\ln S_T$ is log-normally distributed, 95% of values will fall within 1.96 standard deviations of the mean. Similarly, 99% of the values will fall within 2.58 standard deviations of the mean. For example, to obtain the 99% confidence interval for stock prices using the above data, we will proceed as follows:

$$\begin{gathered}
\ln S_T \sim N[4.139, 0.011] \\
\ln S_T = \mu \pm z_\alpha \times \sigma \\
(\text{In this case,we have } \mu = \sqrt{0.011} = 0.1049) \\
4.139 - z_\alpha \times \sigma < \ln S_T < 4.139 + z_\alpha \times \sigma \\
e^{4.139 - z_\alpha \times \sigma} < S_T < e^{4.139 + z_\alpha \times \sigma} \\
47.86 < S_T < 82.24 \\
[e^{(\ln x)} = x]
\end{gathered}$$

Expected Stock Price

Using the properties of a lognormal distribution, we can show that the expected value of S_T, $E(S_T)$, is:

$$E(S_T) = S_0 e^{\mu T}$$

μ=expected rate of return

Example: Expected value of a stock

The current price of a stock is $40, with an expected annual return of 15%. What is the expected value of the stock in six months?

Solution

$$\text{Expected stock price} = \$40e^{0.15\times 0.5} = \$43.11$$

Realized Portfolio Return

$$R_{pr} = [r_1 \times r_2 \times r_3 \times \cdots \times r_n]^{\frac{1}{n}} - 1$$

r_i=portfolio return at the time i

The continuously compounded return realized over some time of length T is given by:

$$\frac{1}{T}\ln\left(\frac{S_T}{S_0}\right)$$

Example: Realized return

The realized return of a stock initially priced at $50 growing, with volatility, to $87 over five periods would simply be:

$$\text{Realized return } = \frac{1}{5}\ln\left(\frac{\$87}{\$50}\right) = 11.08\%$$

Estimating Historical Volatility

We can calculate historical volatility from daily price data of stock. We simply need to calculate continuously compounded returns per day and then determine the standard deviation.

The continuously compounded return for the day i is calculated as:

$$\ln\left(\frac{S_i}{S_{i-1}}\right)$$

The volatility of short periods can be scaled to give the volatility of more extended periods.

For example,

$$\text{Annual volatility} = \text{daily volatility} \times \sqrt{(\text{no.of trading days in a year})}$$

Note that this formula is useful throughout the whole FRM part 1 and FRM part 2 exams in estimating volatility.

Conversely,

$$\text{daily volatility} = \frac{\text{annual volatility}}{\sqrt{\text{no.of trading days in a year}}}$$

Black-Scholes-Merton Model

The Black-Scholes-Merton model is used to price European options and is undoubtedly the most critical tool for the analysis of derivatives. It is a product of Fischer Black, Myron Scholes, and Robert Merton.

The model takes into account the fact that the investor has the option of investing in an asset earning the risk-free interest rate. The overriding argument is that the option price is purely a function of the volatility of the stock's price (option premium increases as volatility increases).

Assumptions underlying the Black-Scholes-Merton Option Pricing Model

i. There is no arbitrage.

ii. The price of the underlying asset follows a lognormal distribution.

iii. The continuous risk-free rate of interest is constant and known with certainty.

iv. The volatility of the underlying asset is constant and known.

v. The underlying asset has no cash flow, such as dividends or interest payments.

vi. Markets are frictionless – no transaction costs, taxes, or restrictions on short sales.

vii. Options can only be exercised at maturity, i.e., they are European-style. The model cannot be used to value American options accurately.

Determining the Value of Zero-dividend European Options using the BSM model

The value of a call option is given by:

$$C_0 = S_0 \times N(d_1) - Ke^{-rT} \times N(d_2)$$

The value of a put option is given by:

$$P_0 = Ke^{-rT} \times N(-d_2) - S_0 \times N(-d_1)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left[r + \left(\frac{\sigma^2}{2}\right)\right]T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - (\sigma\sqrt{T})$$

T =time to maturity,assuming 365 days per year

S_0=asset price

K =exercise price

R_f^c=continuously compounded risk-free rate

σ=volatility of continuously compounded returns on the stock

$N(d_i)$=cumulative distribution function for a standardized normal distribution variable

Example: Valuing a call option using the BSM model

Assume S_0 = \$100, K = \$90, T = 6 months, r = 10%, and σ = 25%.

Calculate the value of a call option.

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{S_0}{K}\right) + \left[r + \left(\frac{\sigma^2}{2}\right)\right]T}{\sigma\sqrt{T}} \\
&= \frac{\ln\left(\frac{100}{90}\right) + \left[0.10 + \left(\frac{0.25^2}{2}\right)\right]0.5}{0.25\sqrt{0.5}} = \frac{0.1053 + 0.0656}{0.1768} = 0.9672 \\
d_2 &= d_1 - (\sigma\sqrt{T}) = 0.9672 - (0.25\sqrt{0.5}) = 0.7904
\end{aligned}
$$

From a standard normal probability table, look up N(0.97) = 0.8333 and N(0.79) = 0.7852.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621

$$
\begin{aligned}
C_0 &= S_0 \times N(d_1) - Ke^{-rT} \times N(d_2) \\
&= 100 \times N(0.8340) - 90e^{-0.10\times 0.50} \times N(0.7852) = \$16.17
\end{aligned}
$$

Note that the intrinsic value of the option is \$10—our answer must be at least that amount.

Exam tips

Tip 1: Given one of either the put value or the call value, you can use the put-call parity to find the other. Precisely,

$$C_0 = P_0 + S_0 - (Ke^{(-R_f^c T)})$$

$$P_0 = C_0 - S_0 + (Ke^{(-R_f^c T)})$$

Tip 2: $N(-d_1) = 1 - N(d_1)$

Tip 3: As S_0 becomes very large, calls (puts) are extremely in-the-money (out-of-the-money)

Tip 4: As S_0 becomes very small, calls (puts) are extremely out-of-the-money (in-the-money)

Tip 5: Although $N(-d_1)$and $N(-d_2)$ can easily be identified from statistical tables; sometimes you'll be asked to compute d_2 and d_2 without the formulas.

The Value of a European Option using the BSM Model on a Dividend-paying Stock

Assume that we have a known dividend d distributed a time $T_1, T1 < T$ where T is the maturity date. To value calls and puts when there are such dividends, we modify the BSM model by replacing S_0 withS, where:

$$S = S_0 - D$$

D is the sum of the PV(discounted at R_f^c) of the dividend payments during the life of the option.

For example,with dividends D_1 and D_2 at times Δt_1 nd Δt_2,

$$S = S_0 - D_1 e^{-(R_f^c)\frac{\Delta t_1}{m}} - D_2 e^{-(R_f^c)\frac{\Delta t_2}{m}}$$

Δt_i represents the amount of time until the ex-dividend date

m a division factor in bringing the Δt to a full year. e.g. $\Delta t = 2\Delta t = 2$ months, m=12 months, so $\frac{\Delta t_1}{m} = \frac{2}{12} = 0.1667$ years .

After this, everything else in the computational formulas remains the same, i.e.,

The value of a call option is given by:

$$C_0 = [S_0 \times N(d_1)] - \left| K \times e^{(-R_f^c \times T)} \times N(d_2) \right|$$

The value of a put option is given by:

$$P_0 = [K \times e^{(-R_f^c \times T)} \times (1 - N(d_2))] - [S \times (1 - N(d_1))]$$

$$d_1 = \frac{\ln(\frac{S}{K}) + [R_f^c + (0.5 \times \sigma^2)]\, T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - (\sigma\sqrt{T})$$

S is simply S_0 adjusted to include dividends payable.

The underlying argument here is that on the ex-dividend dates, the stock prices are expected to reduce by the amounts of the dividend payments.

Exam tip: Sometimes, GARP will give you not a dollar amount "d" of the dividend, but a dividend yield q. For example, you may be told that the dividend yield is 2%, continuously compounded. In such a case, you're still expected to replace S_0 with S where:

$$S = e^{-qT} \times S_0$$

How Dividends affect the Early Exercise for American Calls and Puts

Call option holders have the right but not the obligation to buy shares as per the terms of the contract, but they do not hold shares. As such, they cannot benefit from the rights of shareholders, such as the right to receive dividends – as long as the call options have not been exercised.

When the underlying stock pays dividends, a call option holder will not receive it unless they exercise the contract before the dividend is paid. Whoever owns the stock as of the ex-dividend date receives the cash dividend, so an investor who owns in-the-money call options may exercise early to capture the dividend. In summary, a call option should only be exercised early to take advantage of dividends if:

i. The option is in-the-money
ii. The time value of the option needs to be less than the value of the dividend.

It wouldn't make sense to exercise an out-of-the-money call option and pay an above-market price just to receive a dividend.

Suitable conditions for early exercise of a put option include:

i. The option must be deep in-the-money
ii. High-interest rates
iii. Sufficiently low volatility

Provided these conditions have been met, the holder of an American put option can exercise early, but only after the dividend has been paid. It would make a whole lot more sense to exercise the put option the day after the dividend is paid to collect the dividend, instead of exercising the day before and missing out.

Black's Approximation in Calculating the Value of an American Call Option on a Dividend-paying Stock

Black's approximation sets the value of an American call option as the maximum of two European prices:

I. A European call with the same maturity as the American call being valued, but with the stock price reduced by the present value of the dividend. This implies that S_0 is reduced by the present value of the dividends payable, but all other variables remain the same. For example, if we anticipate two dividends, $S = S_0 - PV$

   Where

$$PV = D_1 e^{-(r)\frac{\Delta t_1}{m}} + D_2 e^{-(r)\frac{\Delta t_2}{m}}$$

   $D_{1,2}$ are the dividends on the ex-dividend dates

   r is the risk-free rate

   Δt_i represent the amount of time until the ex-dividend date

   m is a division factor in bringing the Δt to a full year. If $\Delta t = 2$ months, m=12 months, so

$$\frac{\Delta t}{m} = \frac{2}{12} = 0.1667 \text{ years}$$

Note: All other variables (d_1, d_2, C_0, K, etc.) remain the same.

II. A European option is maturing just before the final ex-dividend date of the American-option. This implies that time to maturity is trimmed down to just before the final dividend is paid. The PV of dividends other than the final one must be deducted from S_0

The largest of the two values (I) and (II) above is the desired Black's approximation for the American call.

Exam tips:

- An American call on a non-dividend-paying stock should never be exercised early.
- An American call on a dividend-paying stock should only be exercised immediately before an ex-dividend date.

Options on Stock Indices, Currencies, and Futures

We can extend the BSM result to valuing other assets such as stock indices, currencies, and futures. For a European option on a stock paying a continuous dividend yield at a rate of q, the value of the call becomes:

$$C_0 = S_0 e^{-qT} \times N(d_1) - Ke^{-rt} \times N(d_2)$$

The value of a put option is given by:

$$P_0 = Ke^{-rt} \times N(-d_2) - S_0 e^{-qT} \times N(-d_1)$$

Where:

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left[r - q + \left(\frac{\sigma^2}{2}\right)\right]T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - (\sigma\sqrt{T})$$

Note that we can also use these formulas to value a European option on a stock index paying dividends at the rate of q when S_0 is the value of the index.

When dealing with an option on foreign currency, we take note that it behaves like a stock paying a dividend yield at the risk-free foreign rate (r_f). We, therefore, set q=(r_f), and we have the following equations of valuation:

$$C_0 = S_0 e^{-r_f T} \times N(d_1) - Ke^{-rT} \times N(d_2)$$
$$P_0 = Ke^{-rT} \times N(-d_2) - S_0 e^{-r_f T} \times N(-d_1)$$
$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left[r - r_f + \left(\frac{\sigma^2}{2}\right)\right]T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - (\sigma\sqrt{T})$$

Example: Valuing an option on foreign currency

Suppose the current exchange rate for a currency is 1.100 and the volatility of the exchange rate is rate is 20%. Calculate the value of a call option to buy 1000 units of the currency in 3 years at an exchange rate of 2.200. The domestic and foreign risk-free interest rates are 2% and 3%, respectively.

Solution

In this case S_0=1.100, K=1.200 , r=0.02, r_f=0.03, σ=0.2 and T=3

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left[r - r_f + \left(\frac{\sigma^2}{2}\right)\right]T}{\sigma\sqrt{T}}$$
$$= \frac{\ln\left(\frac{1.100}{2.200}\right) + \left[0.02 - 0.03 + \left(\frac{0.2^2}{2}\right)\right]3}{0.2\sqrt{3}}$$
$$d_2 = d_1 - \sigma\sqrt{T} = -1.9143 - 0.2\sqrt{3} = -2.2607$$

From standard normal tables,

$$N(d_1) = N(-1.91) = 1 - 0.9719 = 0.0281$$

$$N(d_2) = N(-2.26) = 1 - 0.9881 = 0.0119$$

The value of the call is therefore given by:

$$\begin{aligned} C_0 &= S_0 e^{-r_f T} \times N(d_1) - Ke^{-rT} \times N(d_2) \\ &= 1.1e^{-0.03\times3} \times 0.0281 – 2.2e^{-0.02\times3} \times 0.0119 = 0.0036 \end{aligned}$$

This is the value of the option to buy one unit of the currency. The value of an option to buy 1000 units is 0.0036×1000=$3.60

When we are considering an option on futures, we realize that the futures price F is typical to a stock paying a dividend yield at the risk-free domestic rate (r). We, therefore, set q=r and $S_0 = F_0$, so that we have the following valuation equations:

$$\begin{aligned} C_0 &= F_0 e^{-rT} \times N(d_1) - Ke^{-rT} \times N(d_2) \\ P_0 &= Ke^{-rT} \times N(-d_2) - F_0 e^{-rT} \times N(-d_1) \\ d_1 &= \frac{\ln\left(\frac{F_0}{K}\right) + \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}} \\ d_2 &= d_1 - \sigma\sqrt{T} \end{aligned}$$

Complications Involving the Valuation of Warrants

Warrants are securities issued by a company on its own stock, which give their owners the right to purchase shares in the company at a specific price at a future date. They are much like options, the only difference being that while options are traded on an exchange, warrants are issued by a company directly to investors in bonds, rights issues, preference shares, and other securities. They are basically used as sweeteners to make offers more attractive.

When warrants are exercised, the company issues more shares, and the warrant holder buys the shares from the company at the strike price. An option traded by an exchange does not change the number of shares issued by the company. However, a warrant allows new shares to be purchased at a price lower than the current market price, which dilutes the value of the existing shares. This is known as dilution.

In an efficient market, the share price reflects the potential dilution from outstanding warrants. We are not necessarily required to consider these when valuing the outstanding warrants. This implies that we can value warrants just like exchange-traded options.

For detachable warrants, their value can be estimated as the difference between the market price of bonds with the warrants and the market price of the bonds without the warrants.

The Black-Scholes-Merton Model can also be used to value warrants using the BSM call/put option formulas, i.e.

$$C_0 = [S_0 \times N(d_1)] - \left|K \times e^{-R_f^c \times T} \times N(d_2)\right|$$

However, the following adjustments must be made:

1. The stock price S_0 s replaced by an "adjusted" stock price. Suppose a company has N outstanding shares worth S_0. This means that the value of the company's equity is NS_0. Further, assume that the company has decided to issue M number of warrants with each warrant giving the holder the right to buy one share for K. If the stock prices change to S_T at time T, the (adjusted) stock price which accounts for the dilution effect of the issued warrants, is:

$$S_{\text{adjusted}} = \frac{(NS_0 + MK)}{N + M}$$

2. The volatility input is calculated on equity (volatility of the value of the shares plus the warrants, not just the shares).
3. A multiplier (haircut) that captures dilution, given by $\frac{N}{N + M}$.

Implied Volatility

The volatility of the stock price is the only unobservable parameter in the BSM pricing formula. The implied volatility of an option is the volatility for which the BSM option price equals the market price.

Implied volatility represents the expected volatility of a stock over the life of the option. It is influenced by market expectations of the share price as well as by supply and demand of the underlying options. As expectations rise, and the demand for options increases, the implied volatility increases. The opposite is true.

If we use the observable parameters in the BSM formula (S_0, K, r, and T) and set the BSM formula equal to the market price, then it's possible to solve for volatility that satisfies the equation. However, there is no closed-form solution for the volatility, and the only way to find it is through iteration.

Practice Questions

Question 1

ABC stock is currently trading at $70 per share. Dividends of $1 are expected with ex-dividend dates in three months and six months. An American option on ABC stock has a strike price of $65 and 8 months to maturity. Given that the risk-free rate is 10% and the volatility is 32%, compute the price of the option:

A. $9.85

B. $12.5

C. $10

D. $10.94

The correct answer is **D**.

The current price of the share must be adjusted to take into account the expected dividends.

The present value of the dividends is

$$e^{-0.25\times 0.1} + e^{-0.50\times 0.1} = 1.9265$$
$$S_0 = 70 - 1.9265 = 68.0735$$

Next, calculate the variables required,

S_0=68.0735

K=65

σ=0.32

r=0.1

T=0.6667

$$d_1 = \frac{\ln\left(\frac{68.0735}{65}\right) + [0.1 + (\frac{0.32^2}{2})]0.6667}{0.32\sqrt{0.6667}} = 0.5626$$

$$d_2 = d_1 - 0.32\sqrt{0.6667} = 0.3013$$

$N(d_1)=0.7131$

$N(d_2)=0.6184$

The call price is

$$68.0735 \times 0.7131 - 65e^{-0.1\times 0.6667} \times 0.6184 = 10.94$$

Question 2

A two year stock is currently priced at $100. Assume that the expected return from the stock is 35% per annum, and its volatility is 20% per annum. Calculate the mean and standard deviation of the distribution, and determine the 95% confidence interval for the stock price

	Mean	Standard deviation	95% CI
A	150	30	$110 < S_T < 330$
B	201.38	58.12	$112.30 < S_T < 336.57$
C	5.27	0.28	$112.3 < S_T < 336.57$
D	0.35	0.2	$112.30 < S_T < 336.57$

The correct answer is **C**.

In this case,

$S_0 = 100$

$\mu = 0.35$ and,

$\sigma = 0.20$

The mean and standard deviation of the logarithm of the stock price at the end of two years is given by:

$$\ln S_T \sim \{\ln 100 + (0.35 - \frac{0.2^2}{2})\,2, 0.2^2 \times 2\} \sim (5.27, 0.28^2)$$

Because the logarithm of the stock price is normally distributed, we know the 95% confidence interval for the logarithm of the stock price is

$$5.27 - 1.96 \times 0.28 < \ln S_T < 5.27 + 1.96 \times 0.28$$

$$4.7212 < \ln S_T < 5.8188$$

$$e^{4.7212} < S_T < e^{5.8188}$$

$$112.30 < S_T < 336.57$$

<u>**Reading 60: Option Sensitivity Measures: The “Greeks”**</u>

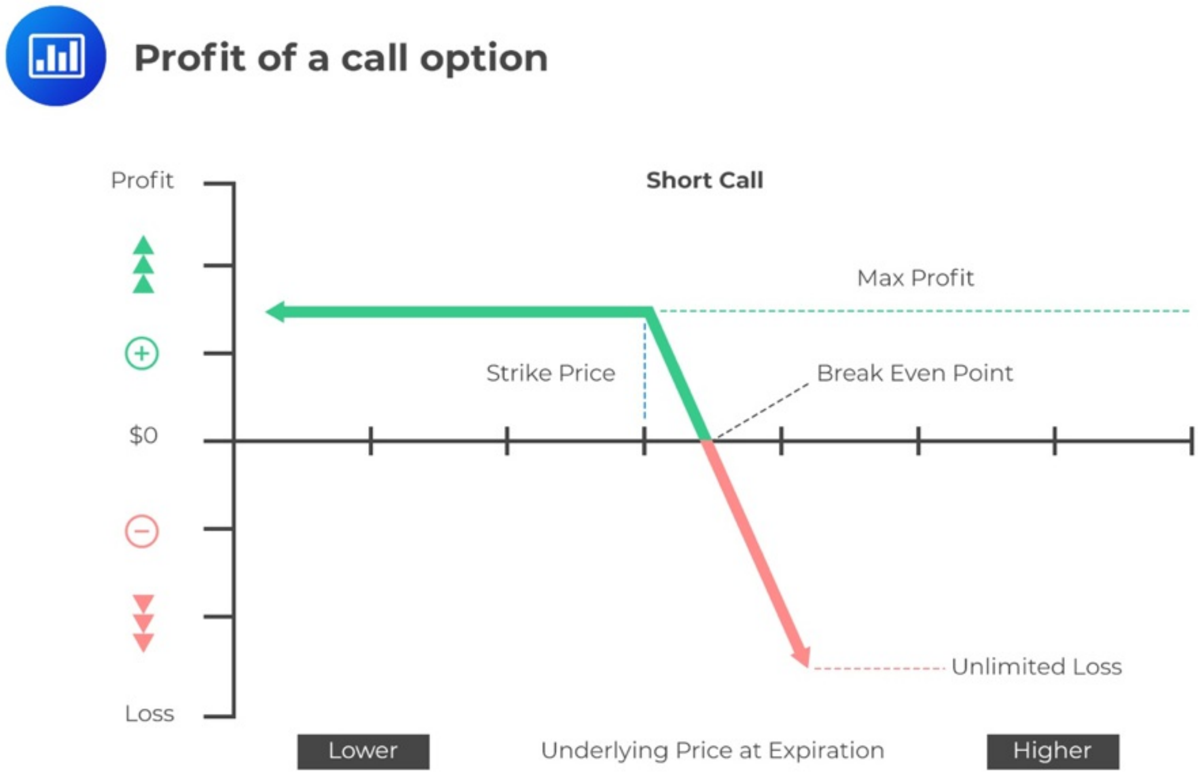
After completing this reading you should be able to:

- Describe and assess the risks associated with naked and covered option positions.
- Describe the use of a stop-loss hedging strategy, including its advantages and disadvantages, and explain how this strategy can generate naked and covered option positions.
- Describe delta hedging for an option, forward, and futures contracts.
- Compute the delta of an option.
- Describe the dynamic aspects of delta hedging and distinguish between dynamic hedging and hedge-and-forget strategy.
- Define and calculate the delta of a portfolio.
- Define and describe theta, gamma, vega, and rho for options positions and calculate the gamma and vega for a portfolio.
- Explain how to implement and maintain a delta-neutral and a gamma-neutral position.
- Describe the relationship between delta, theta, gamma, and vega.
- Describe how portfolio insurance can be created through option instruments and stock index futures.

Risks Associated with Naked and Covered Option Positions

A naked option position occurs when a trader sells a call option without insurance in the form of a holding of the underlying shares. If the option position is backed by ownership of the underlying shares, the position is said to be covered (clothed).

Selling naked call options is laden with risks on the part of the trader. If the market price is below the pre-agreed strike price on the expiration date, the seller makes a gain equal to the premium received. However, if the market price soars above the strike price as at expiration, the buyer exercises the option. When that happens, the seller has to deliver the agreed number of shares to the buyer, even if it means buying from the market. Depending on the extent of the price increase, the entire loss arising from a naked option can be absorbed by the premium received. In other cases, the price increase may be so high that the seller is left with a net loss.



On the contrary, the trader may insure themselves by selling covered options. This may be a safer strategy but one that's also laden with downside risk. If the stock falls, the seller will get to keep the entire premium, but the shares under their ownership will now be worthless. Sometimes the price fall may be too high such that the total value lost in the long position exceeds the premium received.

Example:

Suppose a firm sells 10,000 naked call options on a stock on a stock currently going for $30 a share. The strike price is $33 and the option premium is $4

Scenario 1: Price at expiry = $29

The buyer will not exercise the premium. Total income generated $= 10,000 \times \$4 = \$40,000$

Scenario 2: Price at expiry = $37

The buyer will exercise the options, and the seller is obliged to honor the contract. As such, the seller buys 10,000 shares from the market at a cost of $37 per share and hands them to the buyer.

$$\begin{aligned}\text{Net loss} &= \text{premium received} + \text{contract proceeds} - \text{cost of shares delivered}\\ &= \$4 \times 10,000 + \$33 \times 10,000 - \$37 \times 10,000 = \$0\end{aligned}$$

In this scenario, the loss is absorbed by the premium, resulting in a net loss of $0

Scenario 3: Price at expiry = $38

Again, the buyer exercises the options.

$$\begin{aligned}\text{Net loss} &= \text{premium received} + \text{contract proceeds} - \text{cost of shares delivered}\\ &= \$4 \times 10,000 + \$33 \times 10,000 - \$38 \times 10,000 = \$10,000\end{aligned}$$

How Naked and Covered Option Positions Generate a Stop Loss Trading Strategy

A stop-loss trading strategy is a strategy where the trader initially gets into a naked option position but later on seeks cover when the option moves in-the-money. In other words, protection is sought only when market conditions are such that the call writer stands to lose.

With a naked call position, this strategy requires the purchase of the underlying asset immediately the market price rises above the option's strike price. But as soon as the market price returns to a position that's below the strike price, the trader sells the underlying asset.

Although this sounds like a simple, executable plan on paper, it's a lot more complicated in practice thanks to transaction costs and price uncertainty. In practice, buy/sell costs increase as fluctuations in the strike price increase. It also becomes even more difficult to predict whether the option will be in-the-money or out-of-the-money at expiration.

Delta of an Option

Delta is a measure of the degree to which an option is exposed to changes in the price of the underlying asset. It's the ratio of the change in the price of the call option to the change in the price of the underlying.

$$\text{Delta} = \Delta = \frac{\text{change in the call option price}}{\text{change in the price of the underlying}}$$

For example, if we have a delta value of 0.5, it means that when the price of the underlying moves by a point, the price of the corresponding call option will change by half a point. If delta = 0.5, a $1 increase in the underlying's price triggers a $0.5 increase in the price of the call option.

How to Compute Delta of an Option

Call Option

Delta of a Call option is closely related to N (d_1) in the Black-Scholes Pricing model. Precisely,

$$\Delta_c = e^{-qT} N\,(d_1)$$

Where

$$d_1 = \frac{\ln \frac{S}{K} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$$

q is the dividend yield(%)

S=price of the underlying

K=strike price of the option

r=risk-free interest rate

σ=volatility of the underlying

T =time to option's expiry

Exam tips:

I. Delta of a call option is always positive (between 0 and 1). The delta of an at-the-money call option is close to 0.5. Delta moves to 1 as the call goes deep in-the-money (ITM). It moves to zero as the call goes deep out-of-the-money (OTM).

II. If the underlying does not pay dividends the delta of a call option simplifies to:

$$\Delta_c = e^{-qT} N(d_1) = e^0 N(d_1) = N(d_1)$$

(q is zero in this case, and any number raised to power zero is equal to 1)

Put option

The delta of a put option is

$$\Delta_p = e^{-qT} [N(d_1) - 1]$$

It behaves similar to the call delta, except for the sign (between 0 and -1). As with the call delta, if there are no dividends,

$$\Delta_p = e^0 [N(d_1) - 1] = [N(d_1) - 1]$$

Exam tip: The delta of an at-the-money put option is close to -0.5. Delta moves to -1 as the put goes deep in-the-money. It moves to zero as the put goes deep out-of-the-money.

Delta of a Forward

The delta of a forward contract is given by:

$$\Delta_f = e^{-qT}$$

Where q is the dividend yield and T is time to expiry.

By definition, all forward positions have a delta of approximately 1. What does that imply?

It means the underlying asset and the corresponding forward contract have a one-to-one relationship. As a result, a forward sale position can always be perfectly hedged by buying the same number of securities at the spot price.

Delta of a Futures Contract

Unlike forward contracts, the delta value of a futures contract is not ordinarily equal to 1. This is because futures and spot prices move in lockstep, but are not exactly identical.

For a futures position on a stock that does not pay dividends,

$$\Delta_{futures} = e^{rT}$$

Where r is the risk-free rate and T is the time to maturity.

For a futures position on a stock that pays a dividend,

$$\Delta_{futures} = e^{(r-q)T}$$

Where q is the dividend yield.

The Dynamic Aspects of Delta Hedging

Delta hedging is an attempt to reduce (hedge) the risk associated with price movements in the underlying, by offsetting long and short positions. For instance, a long call position could be offset by shorting the underlying stock. Since delta is actually a function of the price of the underlying asset, it continually changes as the underlying's price changes.

When delta changes, the initially option-hedged position is, again, thrust into a state of imbalance. In other words, the number of stocks is no longer matched with the right number of options, exposing the trader to possible loss.

The overall goal of delta-hedging (a delta-neutral position) is to combine a position in the underlying with another position in an option such that the value of the portfolio remains fixed even in the face of constant changes in the value of the underlying asset.

Delta Hedging with Stock

An options position can be hedged using shares of the underlying. A share of the underlying has a delta equal to 1 because of the value changes by $1 for a $1 change in the stock. For instance, suppose an investor is long one call option on a stock whose delta is 0.6. Because options are usually held in multiples of 100, we could say that the delta is 60. In such a scenario, the investor could delta hedge the call option by shorting 60 shares of the underlying. The converse is true: If the investor is long one put option, he would delta hedge the position by going long 60 shares of the underlying.

Delta Hedging with Options

Sometimes an options position can be delta hedged using another options position that has a delta that's opposite to that of the current position. This effectively results in a delta-neutral position. For instance, suppose an investor holds a one call option position with a delta of 0.5. A call with a delta of 0.5 means it is at-the-money. To maintain a delta neutral position, the trader can purchase an at-the-money put option with a delta of -0.5, so that the two cancel out.

Delta of a Portfolio

Suppose we want to determine the delta of a portfolio of options, all on a single underlying. The portfolio delta is equivalent to the weighted average of the deltas of individual options.

$$\text{portfolio delta} = \Delta_{\text{portfolio}} = \sum_{i=1}^{n} w_i \Delta_i$$

w_i represents the weight of each option position while Δ_i represents the delta of each option position.

Portfolio delta gives the change in the overall option position caused by a change in the price of the underlying.

Theta, Gamma, Vega, and Rho for Option Positions

Theta

Theta, θ, tells us how sensitive an option is to a decrease in time to expiration. It gives us the change in the price of an option for a one-day decrease in its time to expiration.

Options lose value as expiration approaches. Theta estimates the value lost per day if all other factors are held constant. Time value erosion is nonlinear, and this has implications on theta. As a matter of fact, the theta of in-the-money, at-the-money, and slightly out-of-the-money options generally increases as expiration nears. On the other hand, the theta of far out-of-the-money options generally decreases as expiration nears.

For a call option,

$$\theta = \frac{\partial c}{\partial t}$$

Where:

∂c=change in call price

∂t=change in time

For European call options that have zero dividends, the Black-Scholes Merton model can be used to calculate theta. Precisely,

$$\theta_{call} = -\frac{S_0 N^{'}(d_1)\,\sigma}{2\sqrt{T}} - rXe^{-rT}N(d_2)$$
$$\theta_{put} = -\frac{S_0 N^{'}(d_1)\,\sigma}{2\sqrt{T}} + rXe^{-rT}N(d_{-2})$$

Where:

$$N^{'}(y) = \frac{1}{\sqrt{2\pi}}e^{-\left(\frac{y^2}{2}\right)},\ y = d_1, d_2$$

In the above equations, the resulting value for theta is measured in years because T is also measured in years. To covert theta into a daily value, divide by 252, assuming 252 trading days in a year.

Gamma

Gamma, Γ, measures the rate of change in an option's Delta per $1 change in the price of the underlying stock. It tells us how much the option's delta should change as the price of the underlying stock or index increases or decreases. Options with the highest gamma are the most responsive to changes in the price of the underlying stock.

Mathematically,

$$\Gamma = \frac{\partial^2 c}{\partial^2 s}$$

Where the numerator and denominator are the partial derivatives of the call and stock prices, respectively.

For European calls and puts on stocks with zero dividends,

$$\Gamma = \frac{N'(d_1)}{S_0 \sigma \sqrt{T}}$$

While delta neutral positions hedge against small changes in stock price, gamma-neutral positions guard against relatively large stock price moves. As such, a delta-neutral position is important, but even more important is one that's also gamma-neutral, because it will be insulated from both small and large price moves.

The number of options that must be added to an existing portfolio to generate a gamma-neutral position is given by:

$$-\left(\frac{\Gamma_p}{\Gamma_T}\right)$$

Where:

Γ_p=gamma of the existing portfolio position

Γ_T =gamma of a traded option that can be added

Example on Delta-Gamma Hedging

A trader has a short option position that's delta-neutral but has a gamma of -800. In the market, there's a tradable option with a delta of 0.8 and a gamma of 2. To maintain the position gamma-neutral and delta-neutral, what would be the trader's strategy?

The number of options that must be added to an existing portfolio to generate a gamma-neutral position is given by:

$$-\frac{\Gamma_p}{\Gamma_T} = -\frac{-800}{2} = 400$$

Buying 400 calls, however, increases delta from zero to 320 (=400×0.8). Therefore, the trader has to sell 320 shares to restore the delta to zero. Positions in shares always have zero gamma.

Relationship among Delta, Theta, and Gamma

The relationship between the three Greeks can best be expressed in the following equation:

$$rP = \theta + rS\Delta + 0.5\sigma^2 S^2 \Gamma$$

Where:

r=risk neutral risk free rate

P =price of the option

θ=option theta

S=price of the underlying stock

Δ=option delta

σ^2=variance of the underlying stock

Γ=option Gamma

If a position is delta-neutral, then $\Delta = 0$, and the above equation narrows down to:

$$rP = \theta + 0.5\sigma^2 S^2 \Gamma$$

Vega

Vega measures the rate of change in an option's price per 1% change in the implied volatility of the underlying stock. And while Vega is not a real Greek letter, it tells us how much an option's price moves in response to a change in volatility of the underlying.

As an example, a Vega of 6 indicates that for a 1% increase in volatility, the option's price will increase by 0.06. For a given exercise price, risk-free rate, and maturity, the Vega of a call equals the Vega of a put.

Mathematically,

$$Vega = \frac{\partial c}{\partial \sigma}$$

Where:

∂c=change in call price

$\partial \sigma$=change in volatility

For European calls and puts on stocks with zero dividends,

$$Vega = S_0 N'(d_1)\sqrt{T}$$

A drop in Vega will typically cause both calls and puts to lose value. An increase in Vega will typically cause both calls and puts to gain value.

Vega decreases with maturity, unlike gamma which increases with maturity. Vega is highest for at-the-money options.

Rho

Rho measures the expected change in an option's price per 1% change in interest rates. It tells us how much the price of an option should fall or rise in response to an increase or decrease in the risk-free rate of interest.

As interest rates increase, the value of call options will generally increase. On the other hand, as interest rates increase, the value of put options will usually decrease. Although rho is not a dominant factor in the price of an option, it takes center stage when interest rates are expected to change significantly.

Long-term options are far more sensitive to changes in interest rates than are short-term options. Furthermore, in-the-money calls and puts are more sensitive to interest rate changes compared to out-of-the-money calls and puts.

Mathematically,

$$\text{rho} = \frac{\partial c}{\partial r}$$

Where:

∂c=change in call price

∂r=change in interest rate

For European calls and puts on stocks that do not pay dividends,

$$\text{rho}_{\text{call}} = XT\,e^{-rT}N\,(d_2)$$
$$\text{rho}_{\text{put}} = -XT\,e^{-rT}N\,(-d_2)$$

Hedging Activities In Practice, And How Scenario Analysis Can Be Used to Formulate Expected Gains and Losses With Option Positions

On paper, attaining neutrality to all the Greeks might appear a straight forward task but in practice, this is hardly the case. Although delta-neutral positions are easy to create and maintain, it's quite difficult to find securities at reasonable prices that can help tame the negative effects of gamma and vega.

Traders usually concentrate on maintaining delta neutrality and then purpose to continuously and closely monitor the other Greeks.

Sometimes traders may use different values of a portfolio value determinant in order to assess how sensitive the portfolio is to that determinant. For example, the traders may work with different values of volatility to estimate the impact on portfolio value. This is called **scenario analysis**.

Under scenario analysis, a single parameter may be varied at a time, but two or more parameters can also be varied simultaneously to estimate their overall effect on the portfolio.

Creating Portfolio Insurance through Option Instruments and Stock Index Futures

Portfolio insurance is the combination of (1) an underlying instrument and (2) either cash or a derivative that generates a minimum value for the portfolio in the event that markets crash and values decline, while still allowing the trader to make a gain in the event that market values rise. The degradation of portfolio value is protected.

The most common insurance strategy involves using put options to lock in the value of an asset. This way, the trader is able to maintain a limit on the portfolio value – even if the underlying's price tumbles, the trader is insulated from prices below the put's strike.

To hedge a portfolio with index options, the trader selects an index with a high correlation to their portfolio. For instance, if the portfolio consists of mainly technology stocks, the Nasdaq Composite Index might be a good fit. If the portfolio is made up of mainly blue-chip companies, then the Dow Jones Industrial Index could be used.

Alternatively, a trader can use stock index futures with a similar end goal. Traders who want to hedge their portfolios need to calculate the amount of capital they want to hedge and find a representative index. Assuming an investor wants to hedge a $500,000 stock portfolio, she would sell $500,000 worth of a specific futures index, such as the S&P 500.

Practice Questions

Question 1

The current stock price of a company is USD 100. A risk manager is monitoring call and put options on the stock with exercise prices of USD 70 and 6 days to maturity. Which of these scenarios is most likely to occur if the stock price falls by USD 1?

Scenario	Call Value	Put Value
A	Decrease by $0.9	Increase by $0.05
B	Decrease by $1	Increase by $1
C	Decrease by $0	Increase by $1
D	Decrease by $0	Increase by $0

The correct answer is **A**.

The call option is deep-in-the-money and therefore must have a delta close to one. The put option is deep out-of-the-money and will, therefore, have a delta close to zero. Therefore, the value of the in-the-money call will decrease by close to USD 1, and the value of the out-of-the-money put will increase by a much smaller amount close to 0. Among the four choices, it's **A** that is closest to satisfying both conditions.

Question 2

XYZ Inc., a non-dividend-paying stock, has a current price of $200 per share. Eric Rich, FRM, has just sold a six-month European call option contract on 200 shares of this stock at a strike price of $202 per share. He wants to implement a dynamic delta hedging scheme to hedge the risk of having sold the option. The option has a delta of 0.50. He believes that the delta would fall to 0.40 if the stock price falls to $195 per share.

Identify what action he should take NOW (i.e., when he has just written the option contract) to make his position delta-neutral

After the option is written, if the stock price falls to $195 per share, identify the action Mr. Rich should take at that time, i.e. LATER, to rebalance his delta-hedged position

A. NOW: buy 200 shares of stock, LATER: buy 100 shares of stock

B. NOW: buy 100 shares of stock, LATER: sell 20 shares of stock

C. NOW: sell 100 shares of stock, LATER: buy 100 shares of stock

D. NOW: sell 100 shares of stock, LATER: buy 20 shares of stock

The correct answer is **B**.

NOW: Eric sold a call on 200 shares, that means he's short delta of 0.50×200, which is delta = -100. To be delta neutral, he must long (i.e. buy) 100 shares of stock.

LATER: As price falls to \$195, the delta moves to $-80 = -0.40 \times 200$. To be delta neutral, Eric's portfolio needs to have 80 shares of stock. He purchased 100 shares at time 0. To rebalance, he must sell 20 shares.