ME581 Homework 5
Due Nov 8 at 11:59PM EST

Instructions:

- **The following problems are to be documented, solved, and presented in Jupyter notebooks.**
- **Save the notebooks as PDFs, then upload and submit the PDFs in Gradescope.**
- **Implement your own code for the IVP (explicit method & implicit method), R-K methods, and numerical instability**

Problem 1

For the initial value problem given by

$$\frac{dy}{dt} = \frac{(siny - e^t)}{\cos t}$$
$$y(0) = 0$$
$$(0 \le t \le 1)$$

(a) Apply the Euler Method with a time step size $h = 0.5, 0.02,\ 0.008, and\ 0.006$. Plot each estimated solution $y(t_i)$ vs. time, and discuss the results for different h regarding the convergence of the solution, stability, and accuracy.

(b) Apply the Backward Euler Method with a time step size $h = 0.02$, and print each estimated solution $y(t_i)$. Solve any nonlinear algebraic equations using Newton's Method. Discuss your results in terms of convergence and accuracy.

(c) Apply the Second Order Runge-Kutta Method with a time step size $h = 0.02$, and print each estimated solution $y(t_i)$.

(d) On a single plot, graph the three solutions with time step h=0.02 and h=0.008 from parts (a)-(c). Label both axes and include a legend. Discuss the differences between these three solutions regarding the accuracy and convergence.

Problem 2

Consider the nonlinear system:

$$\frac{dy_1}{dt} = 3y_1(1 - y_2), y_1(0) = 3$$
$$\frac{dy_2}{dt} = -5y_2(1 - y_1), y_2(0) = 5$$
$$(0 \le t \le 8)$$

of interest in population dynamics,

(a) Determine a time step size h, or the corresponding number N of steps, in the second order Runge-Kutta method that would produce about eight correct decimal digits. Label both axes and include a legend. (Hint: for N=10, 20, 40, 80... compute the solution with N steps and 2N steps and stops as soon as the two solutions agree to within eight decimal places at all grid points common to both solutions). Plot the estimated solution y_{t_i}. Label both axes. Do not print y_{t_i}.

(b) Apply N=610 to the Classical Fourth Order Runge-Kutta Method to obtain the solution. Apply the Classical Fourth Order Runge-Kutta Method with a time step size h=0.1, and (on a new plot) plot the estimated solution y_{t_i}. Label both axes. Do not print y_{t_i}. Repeat for time steps h=0.1, h=0.05, and h=0.01. Use a single plot to graph the estimated solutions y_{t_i}. Label both axes and include a legend. Do not print y_{t_i}.

Problem 3

A study of nonlinear spatial development of a two-dimensional wall jet on curved surfaces, Le Cunff and Zebib (Nonlinear Spatially Developing Görtler Vortices in Curved Wall Jet Flow," Phys. Fluids, 8, pp. 2375-2384, 1996) requires the solution of the initial value problem,

$$
\begin{aligned}
&y''' + \frac{1}{4}(yy'' + 2(y')^2) = 0 \\
&y(0) = 0 \\
&y'(0) = 0 \\
&y''(0) = \frac{2^{5/2}}{9}
\end{aligned}
$$

(a) Convert the differential equation to a system of three first-order differential equations of the form

$$
\begin{aligned}
u_1' &= \cdots \\
u_2' &= \cdots \\
u_3' &= \cdots
\end{aligned}
$$

State this system of equations and the corresponding initial conditions $u_1(0)$, $u_2(0)$, and $u_3(0)$.

(b) Approximate the solution of this system of equations from t = 0 to t = 30 using forward Euler's Method. (*Hint*: through $N = 200$ time steps and a step size of $h = 0.15$. Print each estimated value of $u_1(t_i)$, $u_2(t_i),\ u_3(t_i)$.

(c) Advance the solution of this system of equations from t =0 to t=30 using the Classical Fourth Order Runge-Kutta Method. (*Hint*: through $N = 200$ time steps and a step size of $h = 0.15$. Print each estimated value of $u_1(t_i), u_2(t_i),\ u_3(t_i)$.

(d) On a single plot, graph the two solutions $u_1(t_i)$ from parts (b) and (c). Label both axes and include a legend.

(e) Graph u_1 against u_2 for the solution obtained in part (b). On the same plot, graph u_1 against u_2 for the solution obtained in part (c). Label both axes and include a legend.

Problem 4

For the growth of a population in a closed system, the mathematical model can be described by the second-order initial value problem

$$
\begin{aligned}
&\frac{d^2y}{dt^2} = -\frac{e^y}{\kappa}(1 + 2\frac{dy}{dt}) \\
&y(0) = \ln u_0 \\
&y'(0) = \frac{1 - u_0}{\kappa}.
\end{aligned}
$$

Here, y is the natural logarithm of the population, $u_0\ is$ the initial population, and κ is a dimensionless parameter. For this problem, take $u_0 = 0.5$. Using the Second Order Runge-Kutta Method, perform the following:

(a) Investigate the evolution of the population from $t = 0$ to $t = 10$ by solving the angle $y(t)$ as a function of time for $\kappa = 0.01$. Plot the solution, label both axes, include a legend, and state units. Discuss your choice of time step.

(b) Find numerical solutions for $\kappa = 0.01, 0.05, 0.1\ and\ 0.5$. Plot solutions for $y(t)$ vs t on a single plot. Label both axes, include a legend.

(c) Graph the population y against the velocity y' for each of the solutions in (b). Label both axes, include a legend.

(d) Repeat parts (b) and (c) using an initial population of $u_0 = 0.1$.