ME581 Homework 7

Instructions:

- **Solve the following problems in a Jupyter Notebook**
- **Save the notebook as a single PDF, then upload and submit the PDF in Gradescope**
- **Write your own codes to carry out the following PDEs. You can use linear algebra libraries such as linalg.solve(), linalg.inv(), and also the corresponding solvers in matlab if you need.**

Problem 1

Given the following first-order partial differential equation:

$$5\frac{\partial u}{\partial t} + k\frac{\partial u}{\partial x} = q\,x\,,\ \ 0 \leq x < \infty\,, t \geq 0$$

with the initial/boundary conditions:

$$u(x, 0) = sin(2\pi x/L)$$

$$u(0, t) = 0$$

Apply an explicit first-order finite difference method with a spatial discretization of $\Delta x = 0.05$ and a temporal discretization of $\Delta t\ = 0.005$ to approximate the function $u(x, t)$ for $0 < t < 3$ sec. Graph the function $u(x, t)$ at 1 sec intervals from $x = 0$ to $x = 3$m (all in one plot). Use $k =$ 0.5m/s, $q = 2$/m, and $L = 1$m.

Problem 2

Consider a vibrating string fixed at both ends occupying the interval $0 \leq x \leq l$. Suppose the string is plucked in the middle in such a way that its initial displacement $u(x, 0)$ is $2mx/l$ for $0 \leq x \leq \frac{1}{2}l$ and $2m(l - x)/l$ for $\frac{1}{2}l \leq x \leq l$ (so the maximum displacement, at $x = \frac{1}{2}l$, is m), and its initial velocity $u_t(x, 0)$ is zero. The problem can be represented as a wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

subject to the following conditions:

$$u(0, t) = u(l, t) = 0$$

$$u(x, 0) = \begin{cases} 2mx/l, & 0 \leq x \leq \frac{1}{2}l \\ 2m(l - x)/l, & \frac{1}{2}l \leq x \leq l \end{cases}$$

$$u_t(x, 0) = 0$$

Find the displacement $u(x, t)$ in the time interval (0,10) using finite difference method with 500 space and time steps. Plot the displacement at 0, 2.5,5, 7.5, 10s (one plot for each). Use the following constants: $l = 10$m, $c = 1$m/s, $m = 1$m.

Problem 3

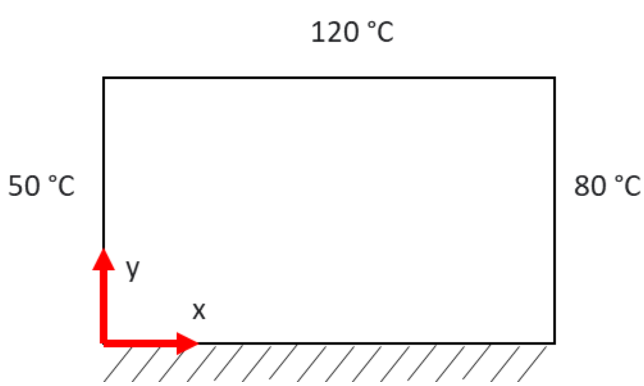
The temperature distribution of a rectangular heated plate satisfies the Laplace equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \text{ } on \text{ } R = \{(x, y) |\, 0 < x < 2\, m\, , 0 < y < 1\, m\}$$

The boundary conditions are shown in the figure. The plate's bottom boundary is insulated ($\frac{\partial T}{\partial y} = 0$ on $y = 0$), and the temperature is fixed in the other boundaries.

Use second order central finite difference method with $dx = dy = 0.05$ m to approximate the temperature distribution. Graph the following:

a) The temperature vs y along $x = 0.5$ m
b) The temperature vs x along $y = 0.2$ m
c) Contour plot of the temperature distribution $T(x, y)$.



Problem 4

Consider the Poisson problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x + y$$

on the region $(1 < x < 2)$, $(0 < y < 1)$ subject to the boundary conditions:

$$\frac{\partial u}{\partial y}(x, 0) = 0$$
$$u(x, 1) = ln(x + 4)$$
$$\frac{\partial u}{\partial x}(1, y) = \frac{2}{y^2 + 1}$$
$$u(2, y) = ln(y^2 + 1)$$

Apply the finite difference method with spacing $\Delta x = \Delta y \ = 0.05$ to calculate an approximation to the solution $u(x, y)$.

a) Plot the approximated solution $u(x = 2, y)$ along the y axis.
b) Plot the approximated solution $u(x, y = 0.5)$ along the x axis.
c) Graph a contour-plot of the approximated solution $u(x, y)$.