ME 581
Final Exam

Problem 1 (20 pts)	
Problem 2 (40 pts)	
Problem 3 (40 pts)	

1) The exam will be available at 3:00PM-EST on Dec 9, 2022. You have 72 hours to finish the exam starting at the time you open the file in Gradescope.
2) The latest you can submit the exam is Dec 15, 2022 at 3:00PM. Therefore, you should plan start working on the exam before Dec 12, 2022 at 3:00PM.
3) The exam should be done on your own and should not be discussed with any other person. If you have questions, please email Prof Koslowski (marisol@purdue.edu)
4) The exam will be solved using a Jupyter notebook and submitted as a single pdf file in Gradescope.
5) Notes, books, homework, laptops, and calculators are allowed.
6) You can use all the functions that you wrote for previous homework assignments, so make sure that they are working properly.
7) You do not need a proctor.

Problem 1 (20 points)

A piston is compressing a gas in a closed tube with a piston moving at constant velocity v [m/s]. At $t = 0$ [s], the mass density is $\rho_0 = 1.23$ [kg/m^3]. As the gas is compressed the velocity of the gas inside the tube, $u(x, t)$ is

$$u(x, t) = -v\frac{x}{(L-vt)} \text{ m/s}$$

where $L = 120$ [mm] is the piston location at $t = 0$ [s], see Figure 1.

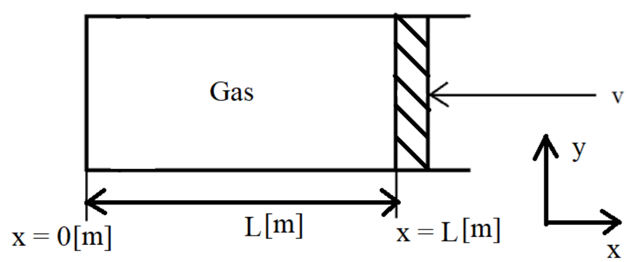

Figure 1. Schematic diagram of the system described in Problem 1

We will use the one-dimensional continuity equation to find the mass density, $\rho(t)$, as a function of time.

$$\frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} = 0$$

(a) Write the discretized form of the continuity equation.
(b) Implement Euler's method to solve the mass density from $t = 0$ s to $t = 0.0045$ s. Plot $\rho(t)$ vs t for $v = 20$ m/s, $v = 23$ m/s, and $v = 26$ m/s.

Problem 2 (40 points)

The following equation represents the velocity of a viscous fluid in 1D in the domain $[0, L]$ m.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \vartheta\frac{\partial^2 u}{\partial x^2}$$

where u is the fluid velocity in m/s, ϑ is the kinematic viscosity in m^2/s and t is the time seconds. The boundary conditions are,

$$u(0, t) = u(L, t)=0 \text{ for } t \geq 0$$

and the initial condition is, $u(x, 0) = (\sin(\pi x))^{40}$.

Consider, $L = 1$ m, $\vartheta = 0.05$ m^2/s and $\Delta x = 0.001$ m.

(a) Write the discretized partial differential equation using explicit forward time and central space scheme.
(b) Determine an appropriate time-step size (Δt) to get a stable solution.
(c) Plot $u(x, t)$ vs x for $t = 0$ s, $t = 0.125$ s, $t = 0.25$ s, $t = 0.375$ s, and $t = 1$ s in the same plot.

Problem 3 (40 points)

The concentration of ink in a shallow plate satisfies the Laplace equation:

$$\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} = 0 \; on \; R = \{(x, y) |\, 0 < x < 2\; cm\, , 0 < y < 2\; cm\}$$

The boundary conditions are shown in the figure.

Use the second order central finite difference method to approximate the concentration.

a) Use dx=dy=2/3 cm. Write the system of equations as $\boldsymbol{A} \cdot c = b$, where $\underline{c}$ is a vector that contains the concentration at the grid points. Write $\boldsymbol{A}$, $\underline{c}$, and $\underline{b}$.
b) Use dx=dy=0.1 cm and plot
   1. The concentration vs x along $y = 0.5\; cm, 1cm, 1.5\; cm$
   2. The concentration vs y along $x = 0.5\; cm, 1cm, 1.5\; cm$
   3. Contour plot of the concentration $c(x, y)$.