ME581 Homework 6

Instructions:

- **Solve the following problems in a Jupyter Notebook**
- **Save the notebook as a single PDF, then upload and submit the PDF in Gradescope**
- **Implement your own code for the BVP, Galerkin method, and heat equation. You can use linear algebra libraries such as linalg.solve(), linalg.inv(), and also the corresponding solvers in matlab if you need.**

Problem 1

Find the approximate solution of the following boundary value problems using the finite central difference method.

$$y'' + \frac{1}{4}xy' + y = x^2, \text{ with } y'^{(0)} = -\frac{4}{3}, \; y(1) = -\frac{2}{3}$$

(a) Write down the discretized version of the PDE.
(b) Plot the approximate solution y(x) vs x in the domain with h=0.1, 0.01, 0.001. Plot the results with both axes and legend.
(c) Discuss your results in terms of accuracy and convergence.

Problem 2

Find the approximate solution of the following boundary value problems using the finite central difference method.

$$u'' = -(x+2)u' + u + (4 + x - x^2)e^{-x} \quad \text{with } u(0) = -2 \text{ and } u(1) = -e^{-1}$$

(a) Write down the discretized version of the PDE.
(b) Plot the approximate solution u(x) vs x in the domain with h=0.2, 0.02, 0.002, 0.0002. Plot the results with both axes and legend.
(c) The exact solution of this BVP is $u(x) = (x-2)e^{-x}$. Confirm the order of accuracy of the numerical method using both the maximum absolute error $(\max_i(|u_i - u_{i_{exact}}|))$. Plot the error vs h.

Problem 3

Find the approximate solution of the following boundary value problems using the Galerkin method, with N=5, 10, 20, 100.

$$u'' = 3x^2 - 4, \text{ with } u(1)=1, u(2)=5$$

(a) Plot the solution u(x) vs x (N=5, 10, 20, 100) in a single plot with both axes and legend. Use Gaussian integration (1 point) to calculate the integrals.
(b) Discuss your results.

Problem 4

1-D heat equation:

$$\frac{\partial h}{\partial t} = k\frac{\partial^2 h}{\partial x^2} 0 \le t < \infty,\ 0 \le x \le 1000.$$

With the boundary condition: $h(0,t) = 0, h(1000,t) = \frac{1}{4}\ (1 - \tanh(t-2))(t)^2 + (1 + \tanh(t-2))e^{-(t-2)}$. Initial condition: $h(x,0) = 0$.

(a) Write down the discretized version of the PDE.
(b) Solve this partial differential equation with k= 500 and determine $h(x,t)$ at different time t=2, 5, 10, 20. Plot the results with both axes and legend.
(c) Plot the solutions at t=2, for k=5, 50, 500 in a single plot.