IF223 - Algorithmique Distribuée

2021-2022

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Graph Theory : Reminder

Graph G = (V, E)

V = **set of nodes** (finite, non-empty)

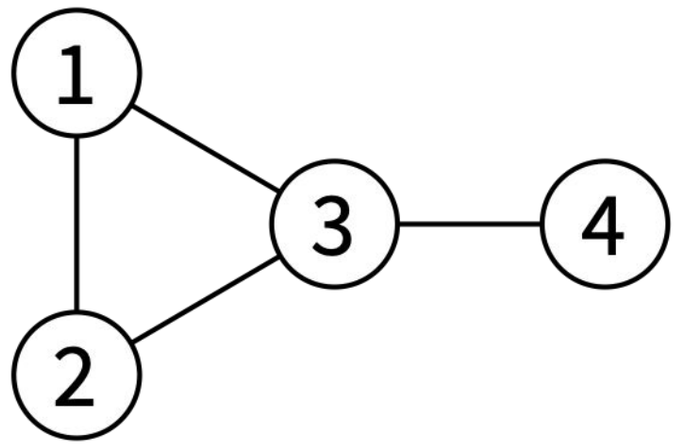
E = **set of edges** (unordered pairs of nodes)

$G = (V, E)$

$V = \{1, 2, 3, 4\}$

$E = \{ \{1,2\}, \{1,3\}, \{2,3\}, \{3,4\} \}$

G:



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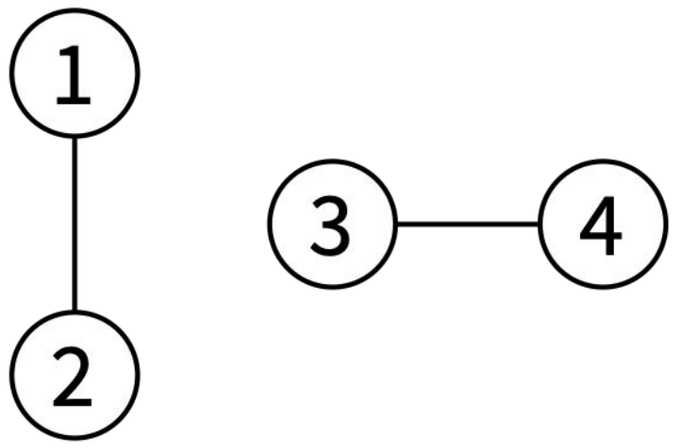

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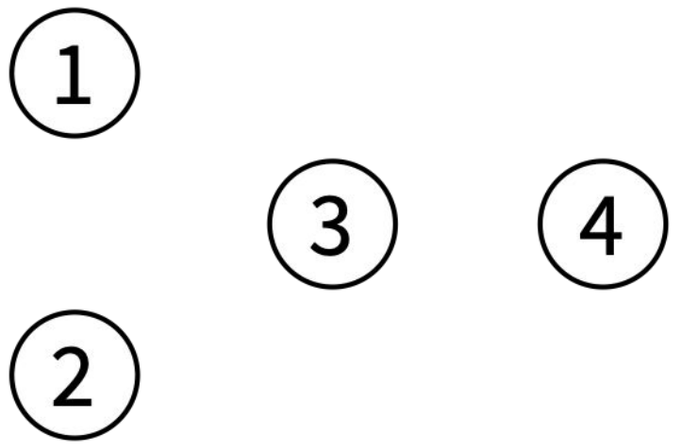


Graph G = (V, E)

V = **set of nodes** (a.k.a. "vertices")
E = **set of edges**

Usually nodes are denoted with *u*, *v*
(if more nodes needed: *s*, *t*, *u*, *v*, *u'*, *v'*, v_1, v_2, etc.),
edges are denoted with e, e', e_1, e_2, etc.

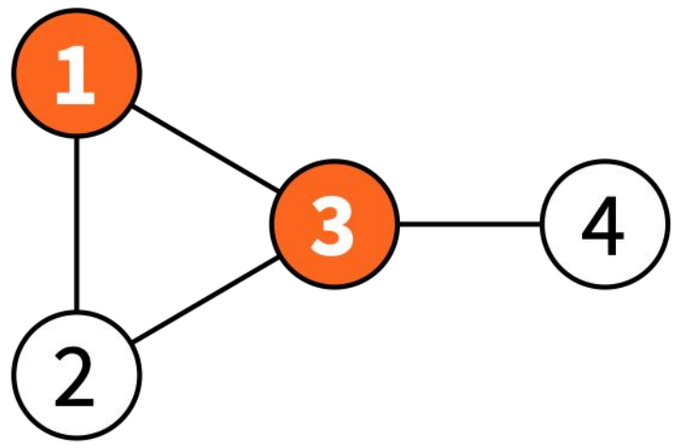
Convention: $n = |V|$, $m = |E|$

Graph G = (V, E)

u and *v* are **adjacent nodes**
= **nodes** *u* and *v* are **neighbours**
= there is an **edge** {*u*, *v*}

$G = (V, E)$
$V = \{\mathbf{1}, 2, \mathbf{3}, 4\}$
$E = \{ \{1,2\}, \mathbf{\{1,3\}}, \{2,3\}, \{3,4\} \}$

G:



Graph G = (V, E)

e_1 and e_2 are "adjacent edges"
= they share an endpoint
= their intersection is non-empty

$G = (V, E)$

$V = \{1, 2, 3, 4\}$

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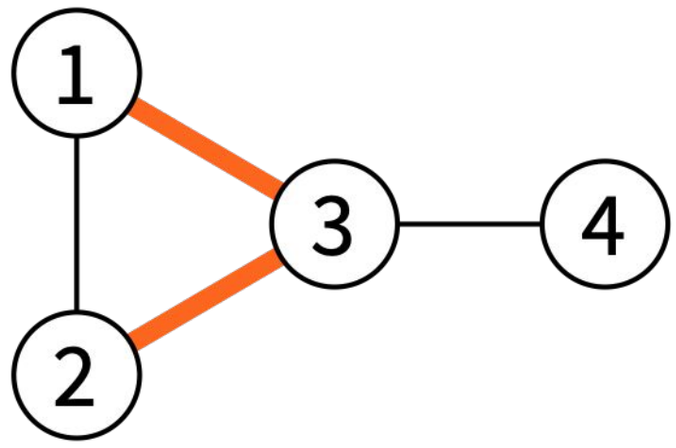
G:



Graph G = (V, E)

Node *v* is incident to edge *e*
= *v* is an endpoint of *e*
= *v* is a member of *e*

$G = (V, E)$
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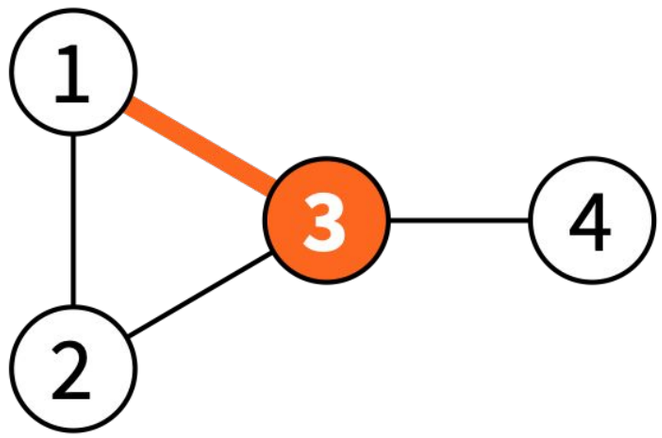
G:



Graph G = (V, E)

Node of degree *k*
= # node adjacent to *k* nodes
= # node incident to *k* edges

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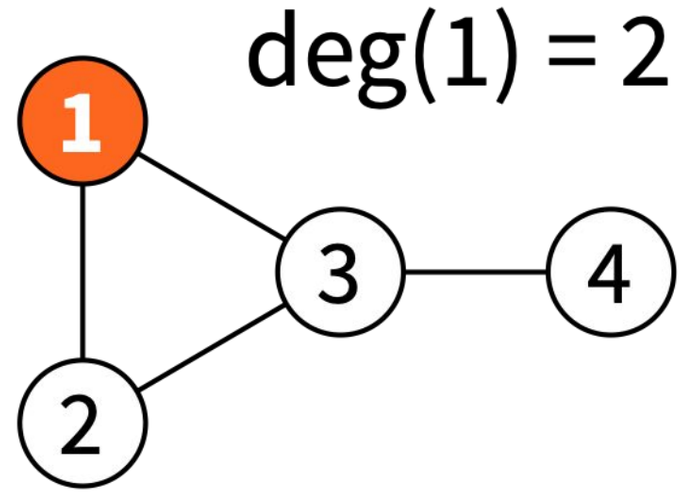

Graph G = (V, E)

***k*-regular graph**

= all nodes have degree *k*

= all nodes have *k* neighbours

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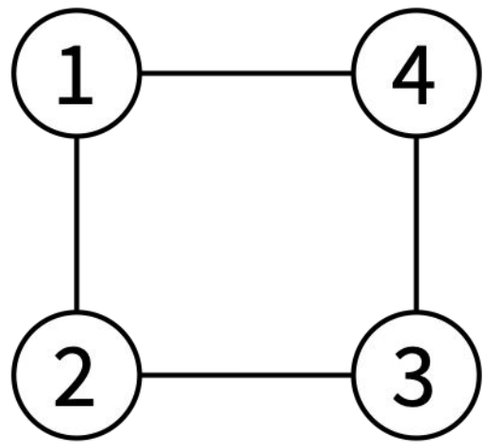
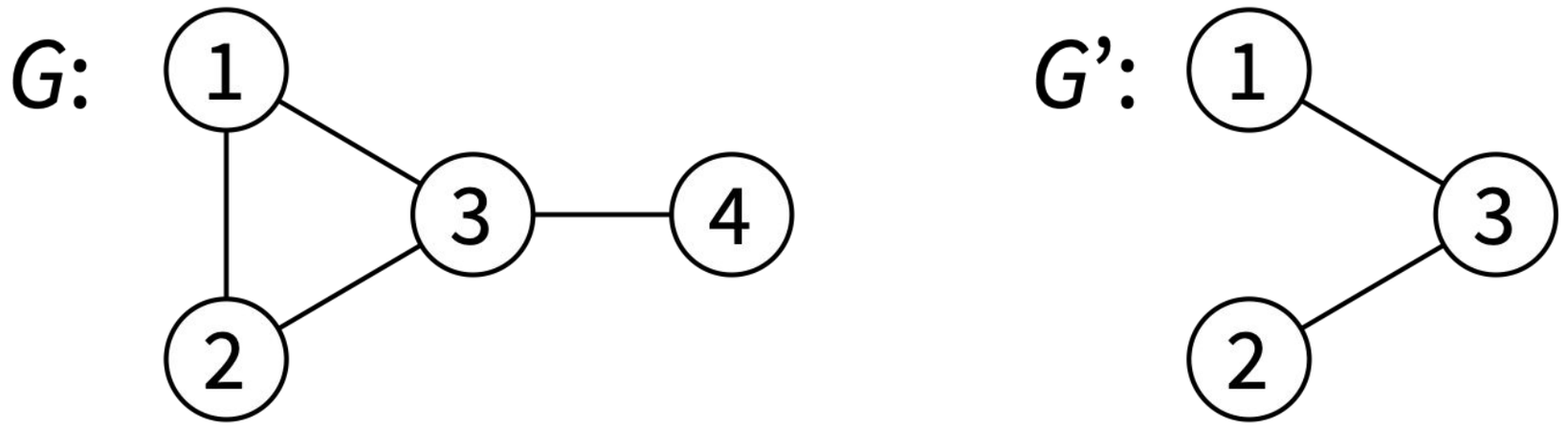
G:



Subgraph

Graph $G' = (V', E')$ is a subgraph of $G = (V, E)$:
$V' \subseteq V$ and $E' \subseteq E$



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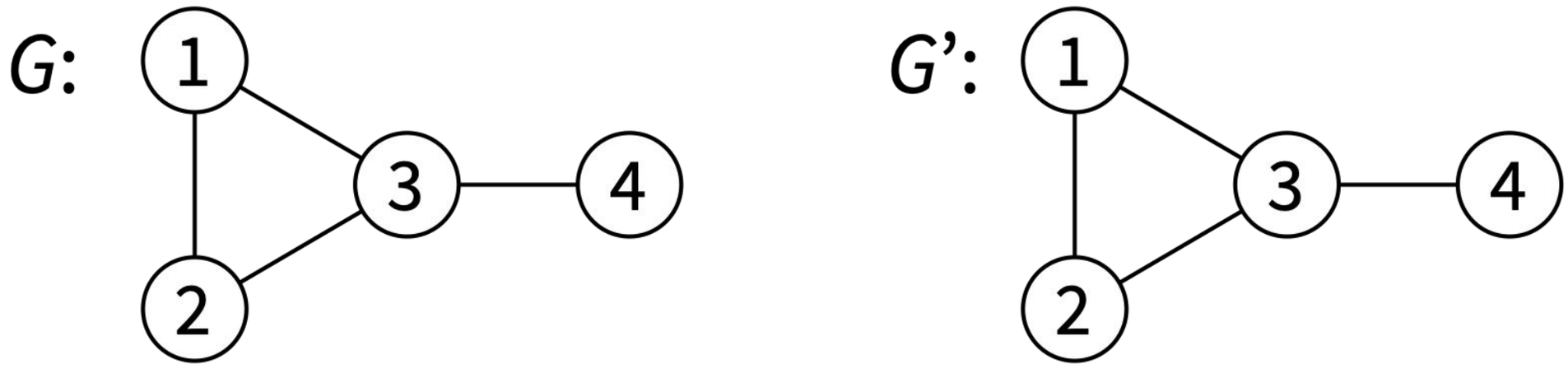

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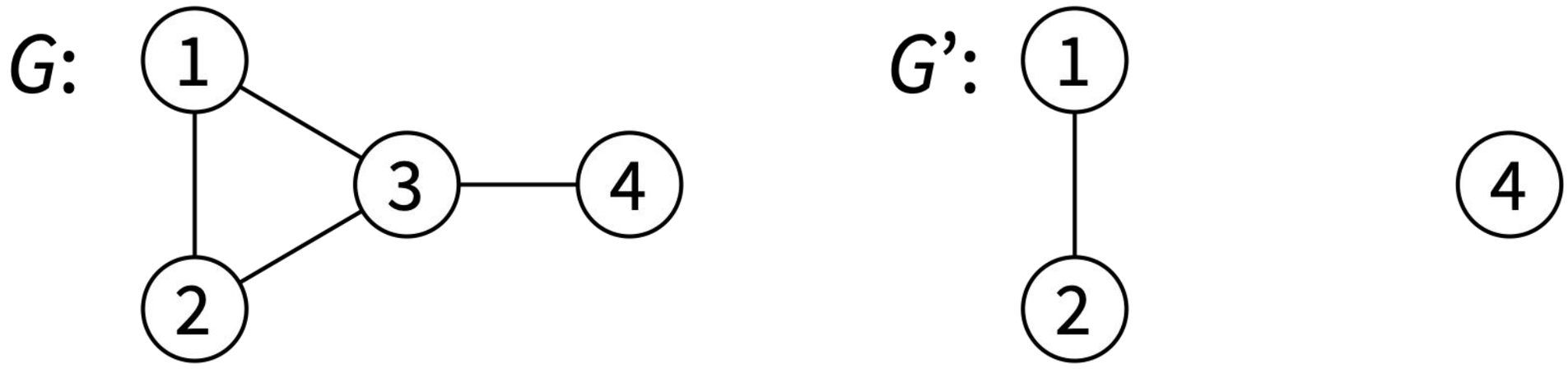

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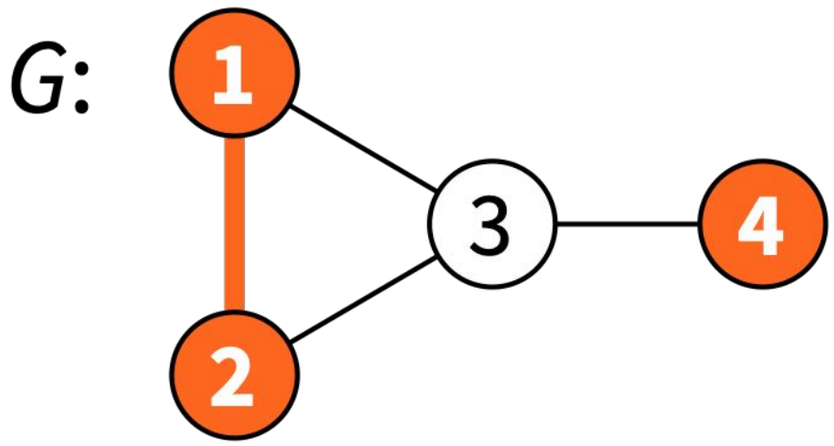
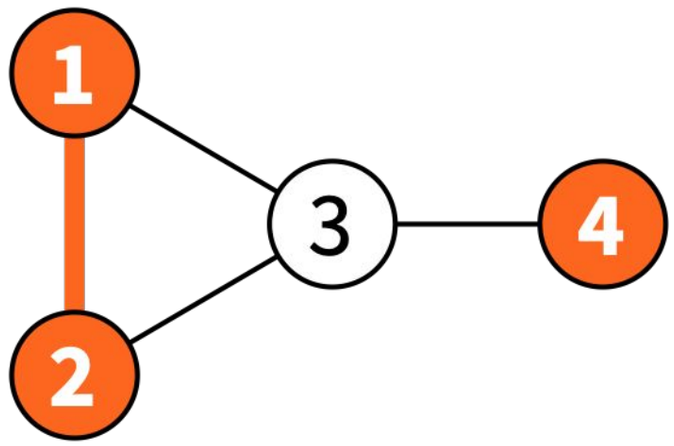


Induced Subgraph

Subgraph induced by nodes *V'*
= all nodes of *V'* and all edges that connect them

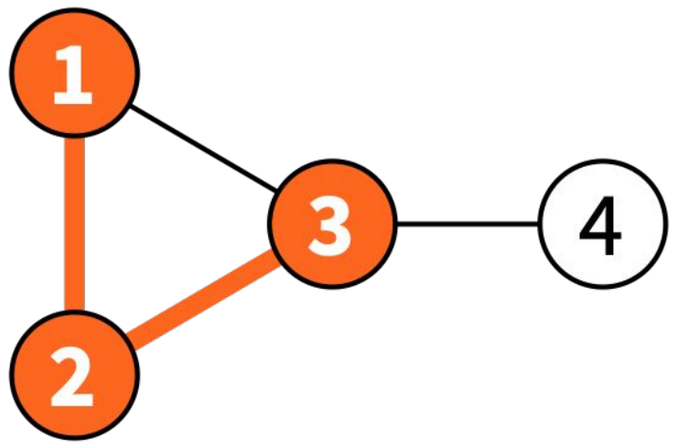
G:



Induced Subgraph

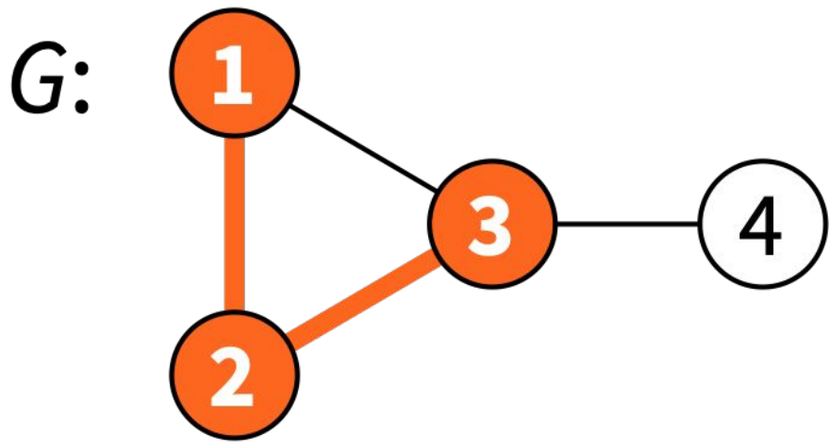
Subgraph induced by edges *E'*
= all edges of *E'* and all of their endpoints

G:



Induced Subgraph

This is *not* a subgraph induced by any set of nodes — why?



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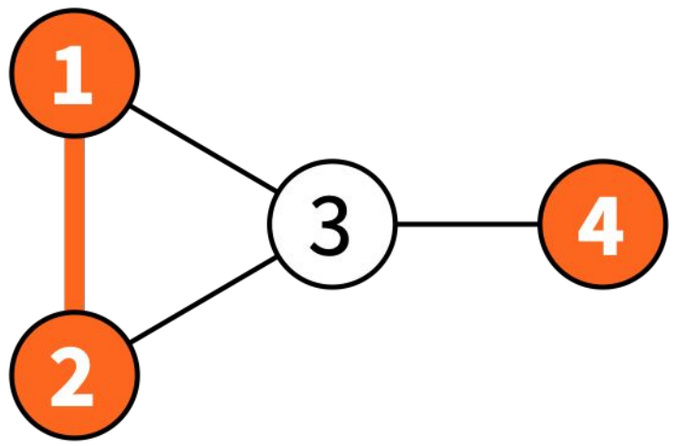
G:

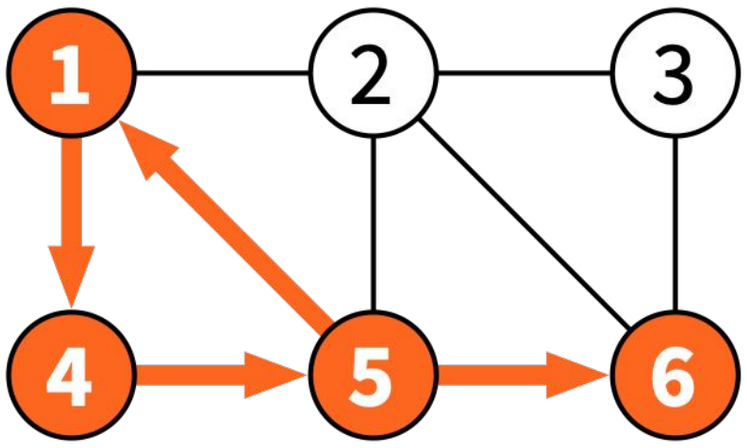


Walks, paths and connectivity

Walk = alternating sequence of incident nodes and edges

$w = (5, \{5,1\}, 1, \{1,4\}, 4, \{4,5\}, 5, \{5,6\}, 6)$

G:



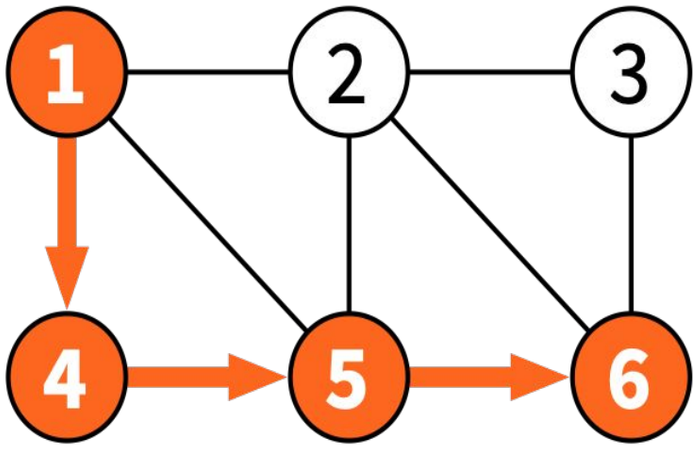
Walks, paths and connectivity

Path = walk visiting each node at most once

Length of a path = number of *edges*

$w = (1, \{1,4\}, 4, \{4,5\}, 5, \{5,6\}, 6)$

G:

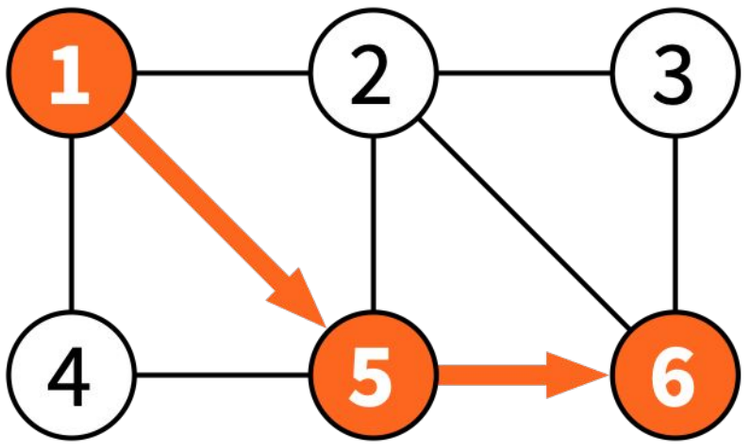


Walks, paths and connectivity

Distance = length of a ***shortest path***

$w = (1, \{1,5\}, 5, \{5,6\}, 6)$

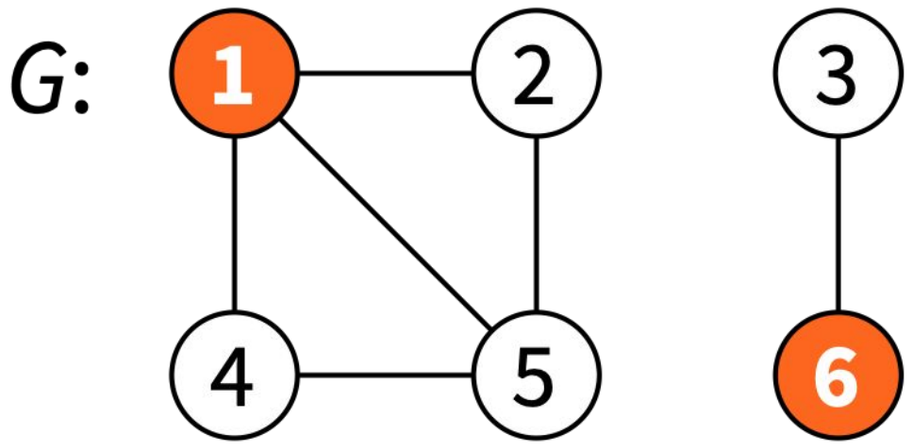
G:



dist(1, 6) = 2

Walks, paths and connectivity

Distance = length of a *shortest path*
(infinite if no such path exists)



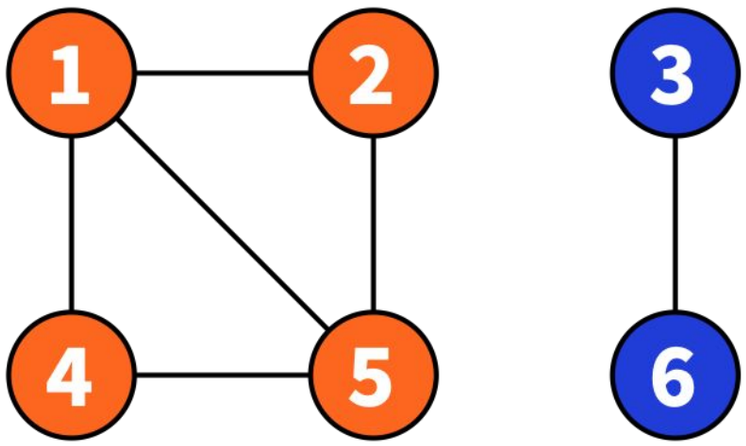
dist(1, 6) = ∞

Walks, paths and connectivity

Connected component C**:**

there is a path between any two nodes of C

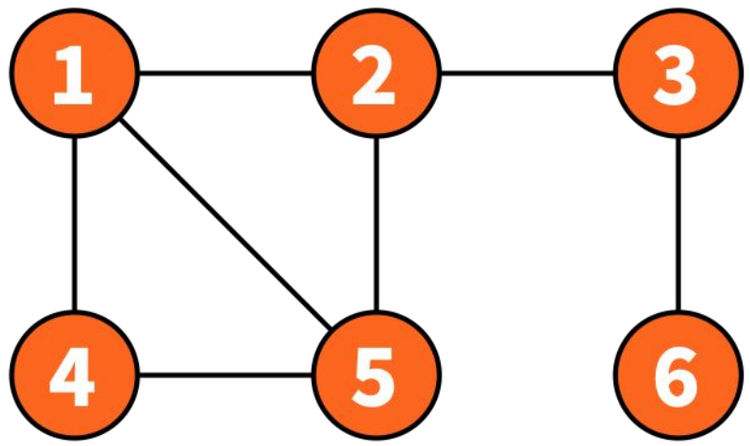
G:



Walks, paths and connectivity

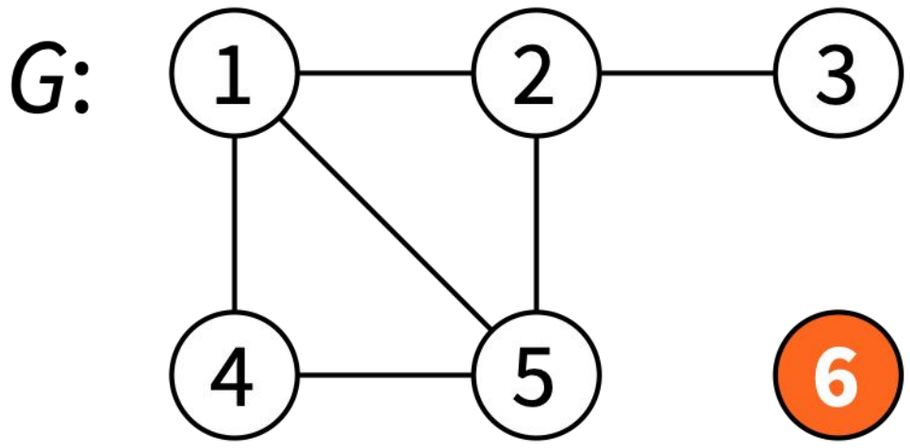
Graph is connected if only 1 connected component

G:



Walks, paths and connectivity

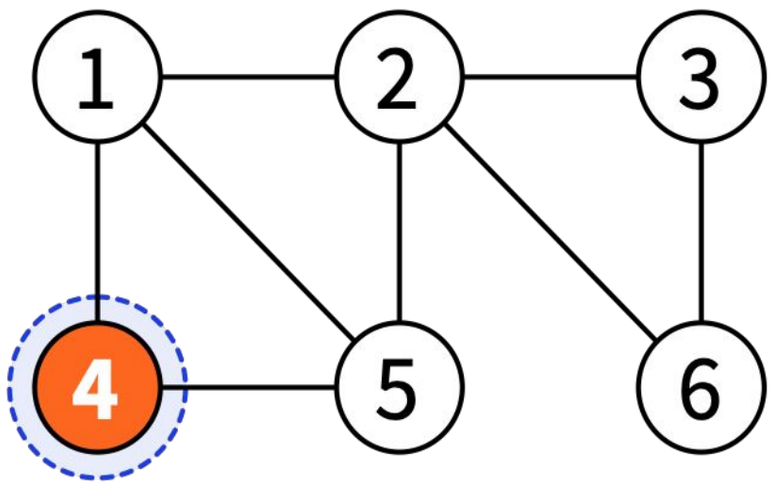
Isolated node = node of degree 0



Walks, paths and connectivity

ball(v, r) = "radius-*r* neighbourhood of *v*"
= nodes at distance at most *r* from node *v*

G:



ball(4, 0) = {4}
ball(4, 1) = {4, 1, 5}
ball(4, 2) = {4, 1, 5, 2}
ball(4, 3) = *V*

Walks, paths and connectivity

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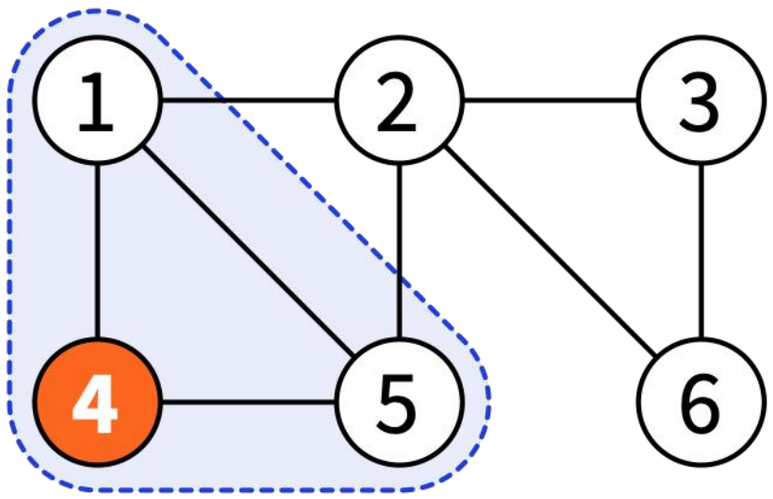

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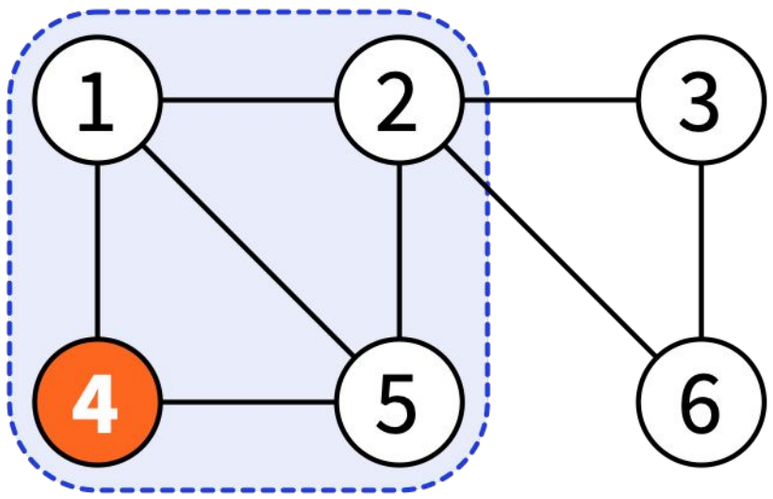

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Walks, paths and connectivity

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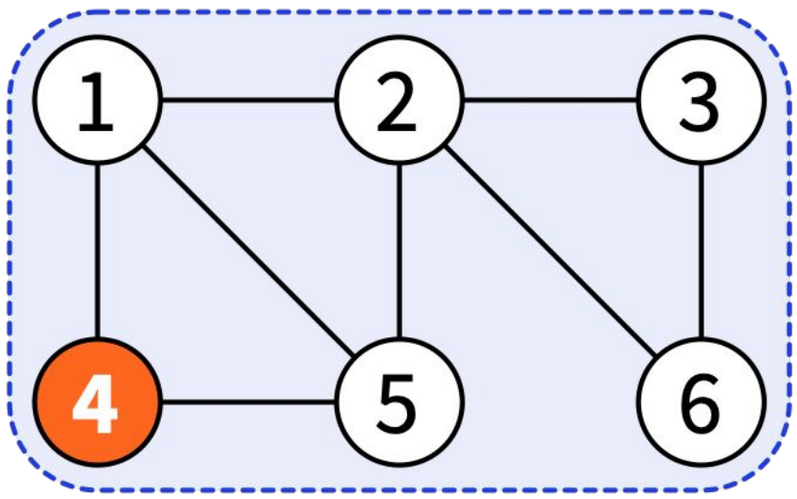

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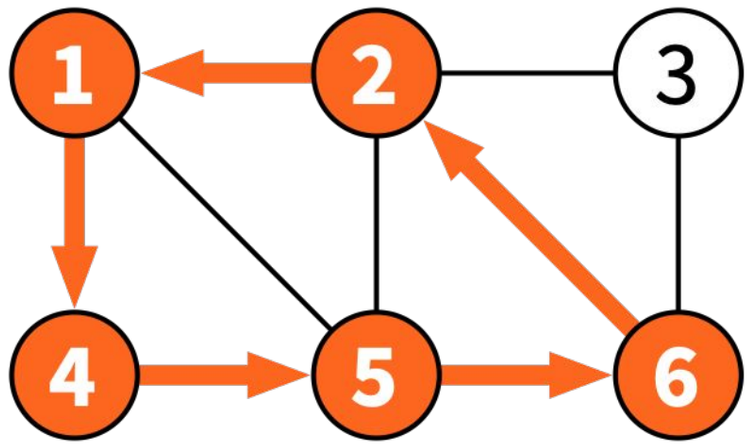
Walks, paths and connectivity

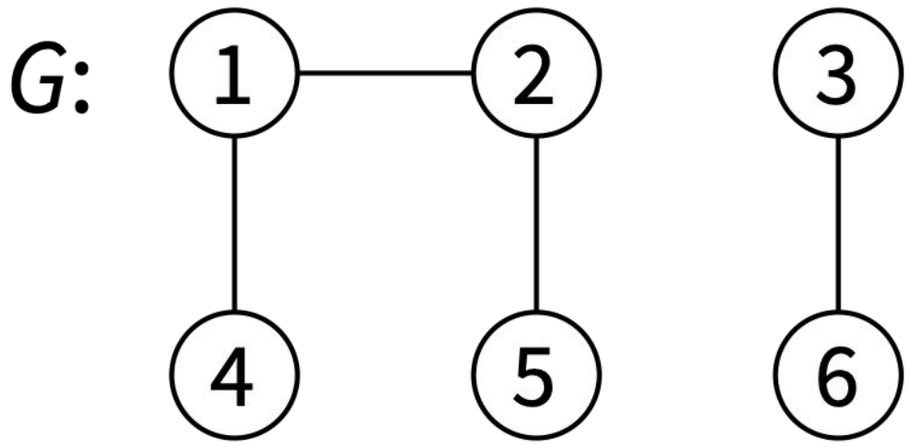
Cycle = closed walk that visits each node and each edge at most once (length ≥ 3)

G:



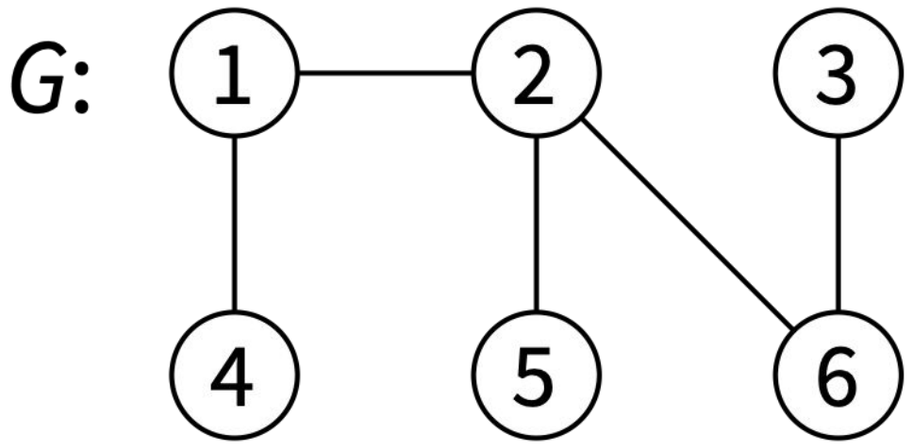
Walks, paths and connectivity

Acyclic graph = graph without any cycles



Walks, paths and connectivity

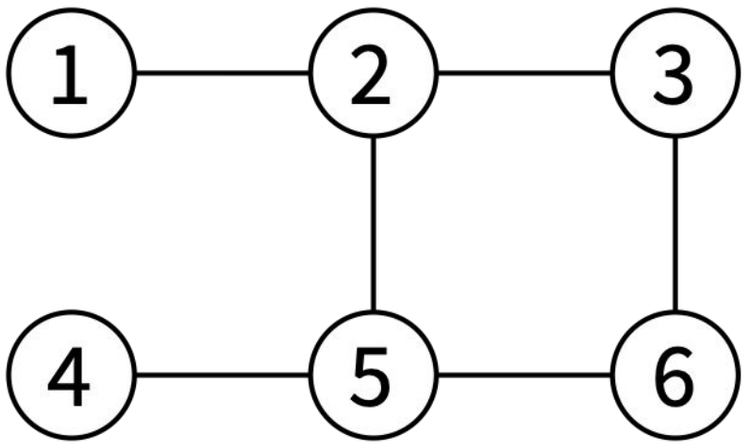
Tree = connected acyclic graph
Forest = acyclic graph



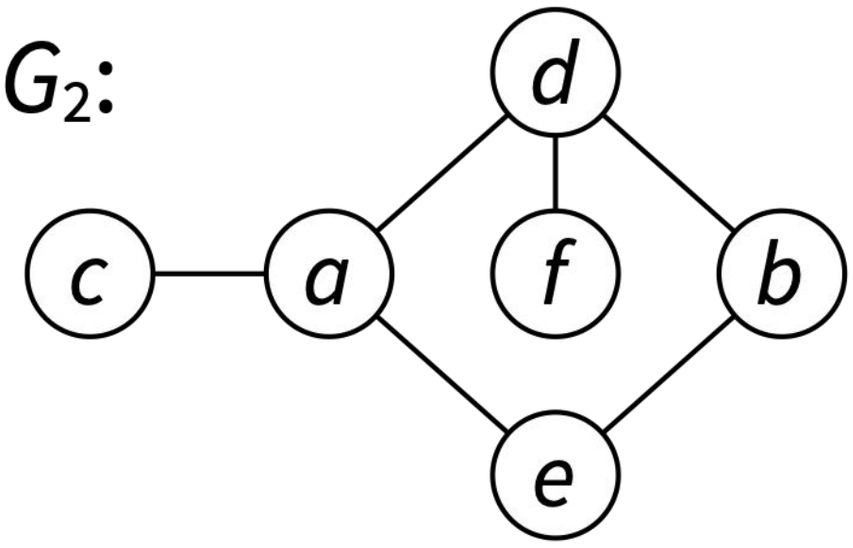
Isomorphism

Isomorphism from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$: bijection f: $V_1 \rightarrow V_2$ that preserves adjacency

G_1:



G_2:



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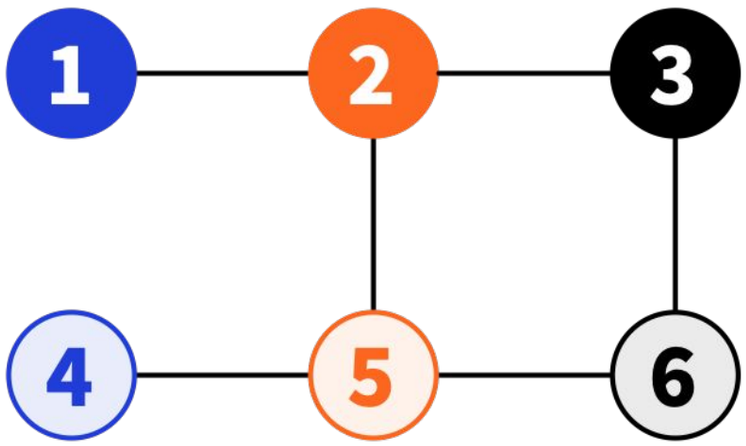
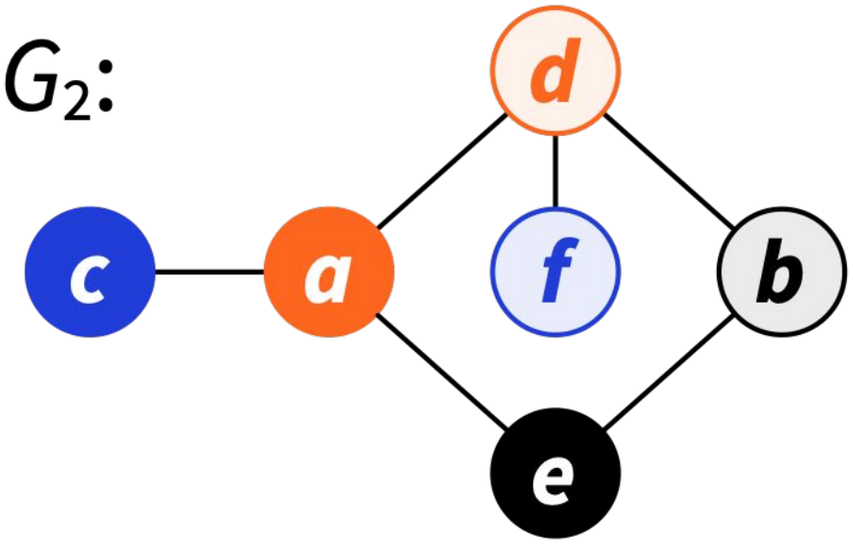


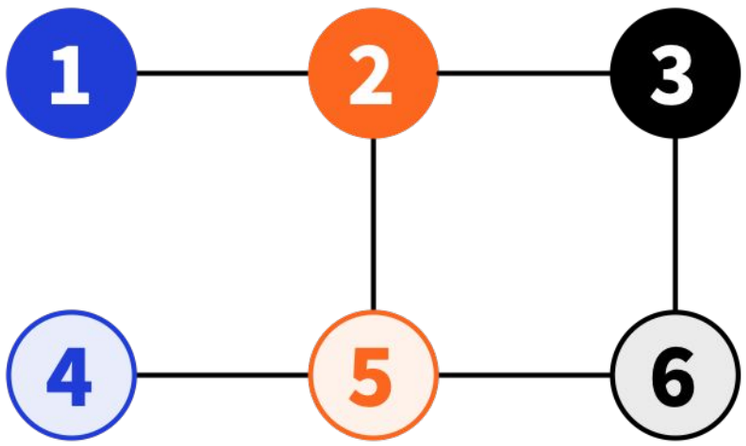
G_2:



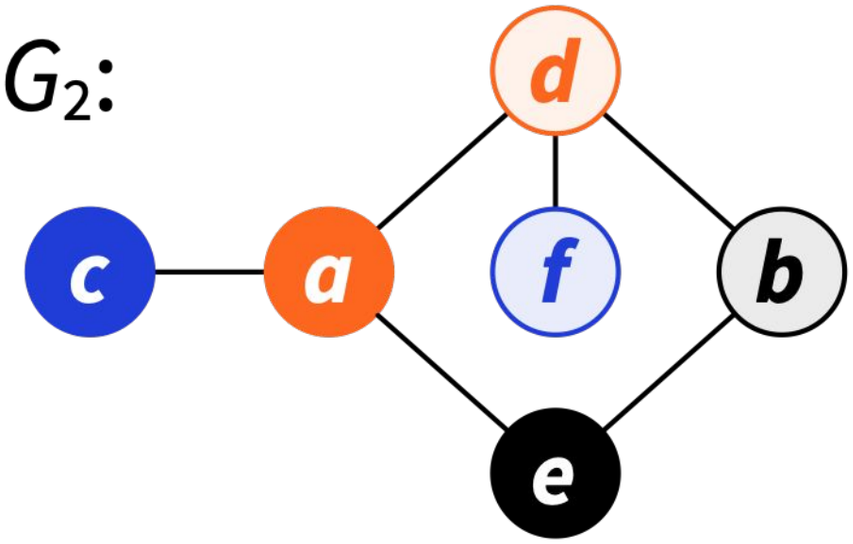
Isomorphism

Graphs are isomorphic if there exists an isomorphism from one to another

G_1:



G_2:



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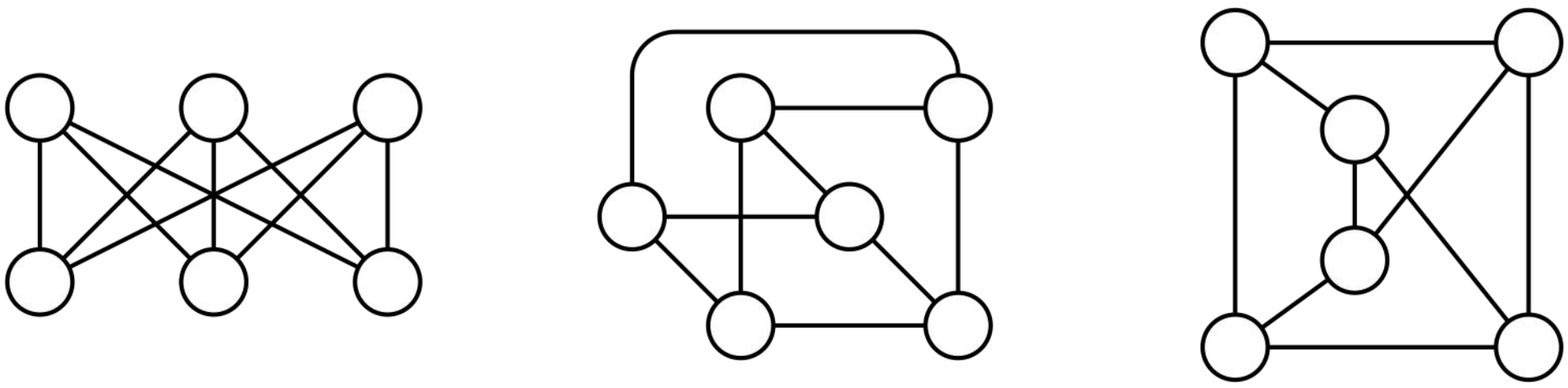
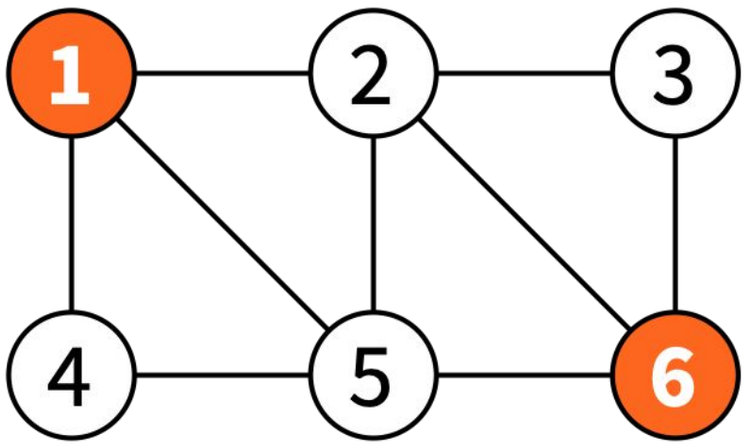


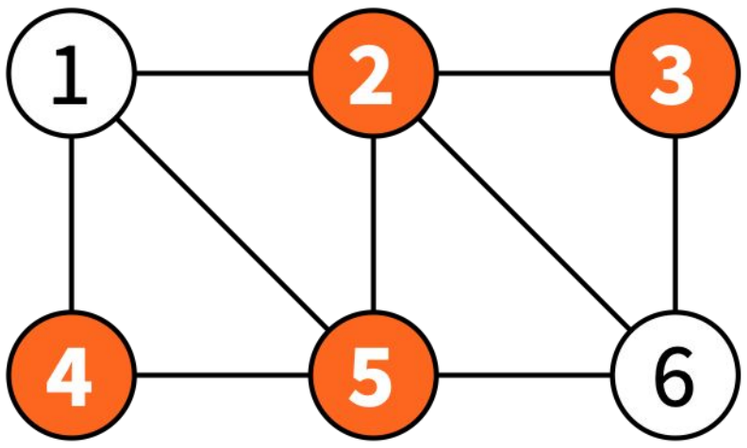
Graph problems

Independent set: non-adjacent nodes



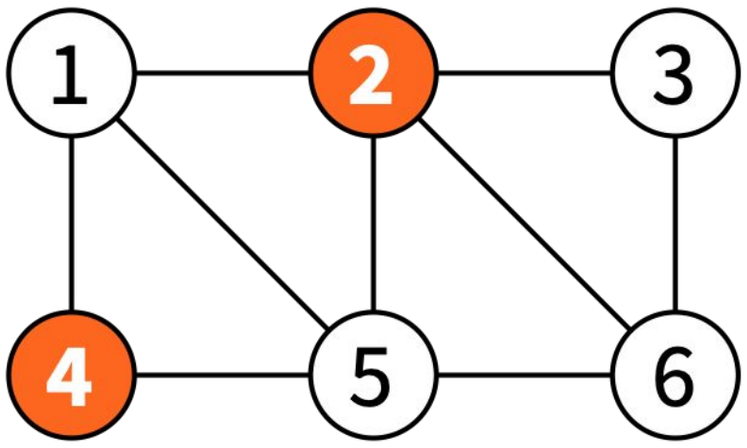
Graph problems

Vertex cover: at least one endpoint of each edge (all edges are "covered" with these nodes)



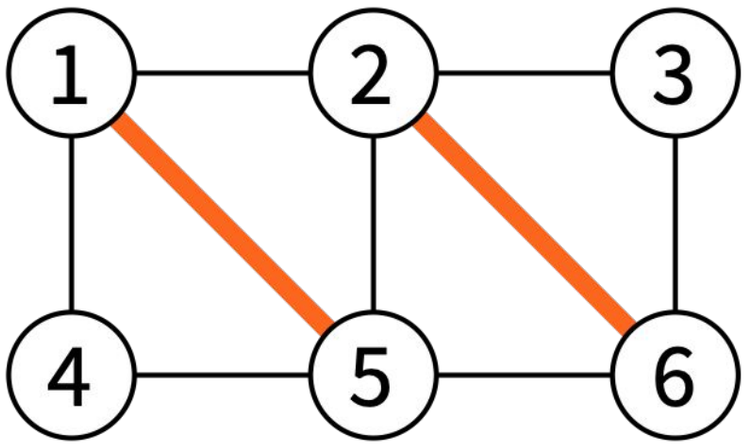
Graph problems

Dominating set: all other nodes have a neighbour in this set



Graph problems

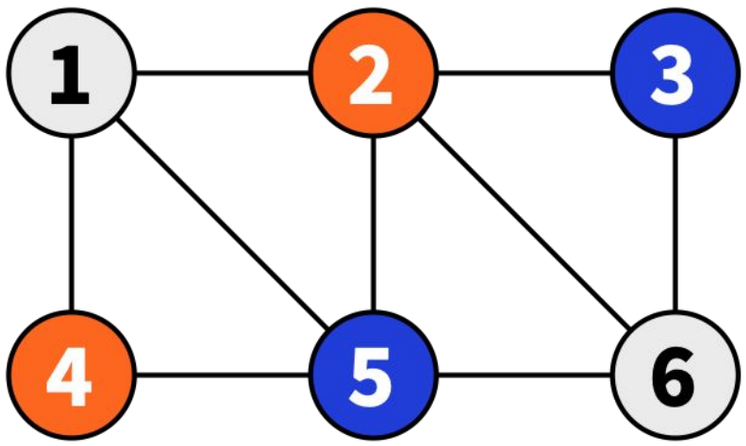
Matching: non-adjacent edges



Graph problems

Vertex coloring:

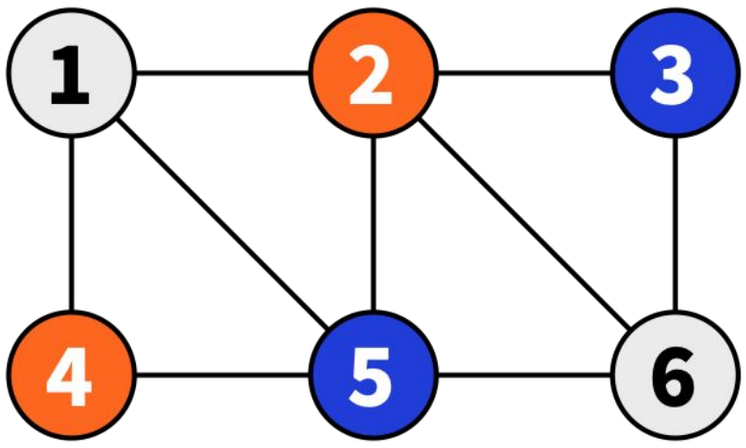
adjacent nodes have different colours



Graph problems

Vertex coloring:

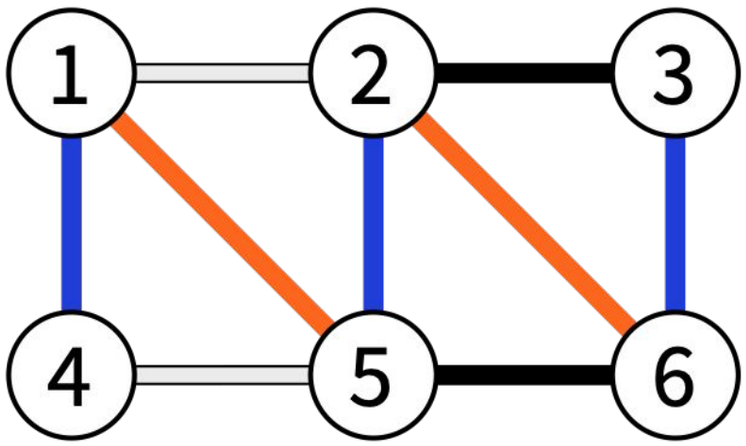
each colour class is an independent set



Graph problems

Edge coloring:

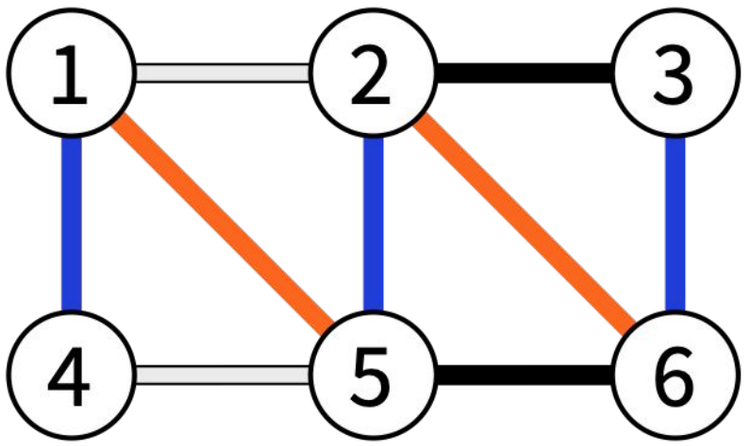
adjacent edges have different colours



Graph problems

Edge coloring:

each color class is a matching

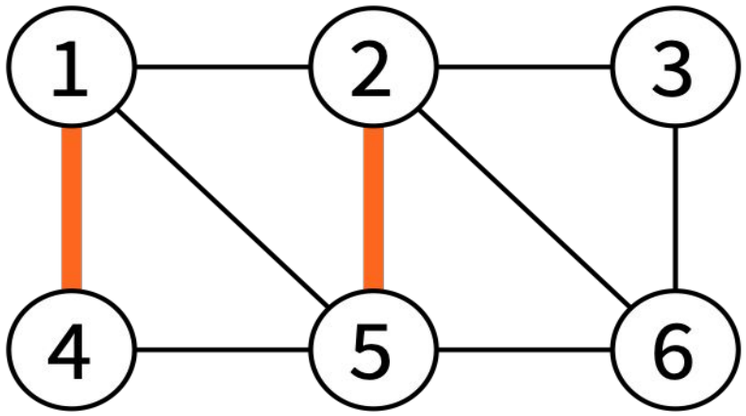


Maximisation problems

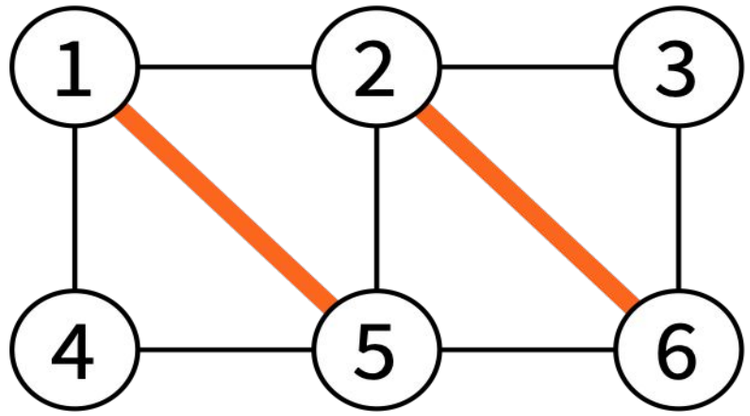
- maximal = **cannot add anything**
- maximum = **largest possible size**
- x-approximation = **at least $1/x$ times maximum**

Maximisation problems

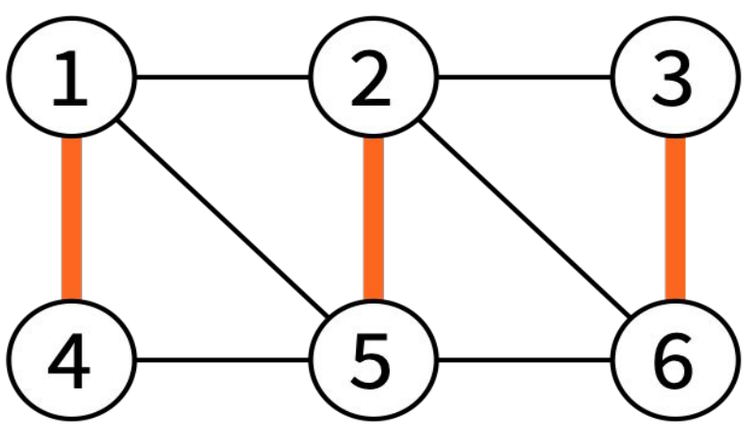
Matching



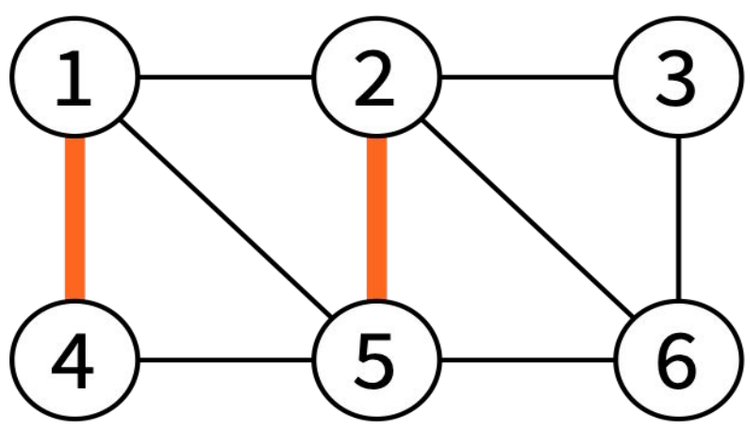
Maximal matching



Maximum matching

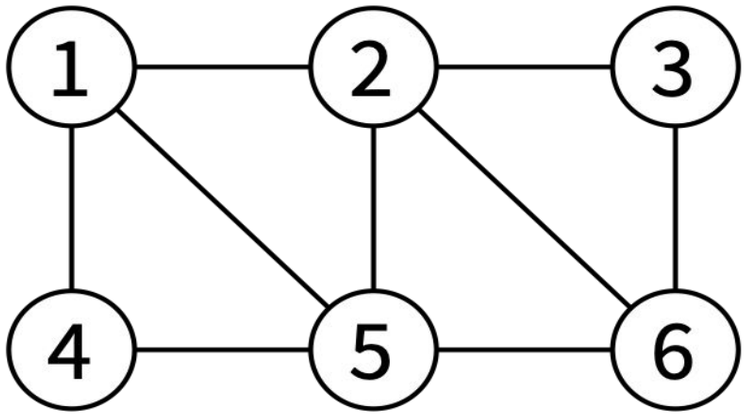


2-approximation

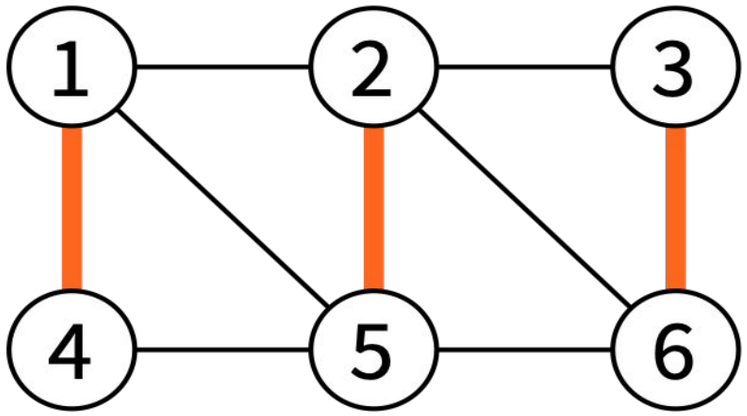


Maximisation problems

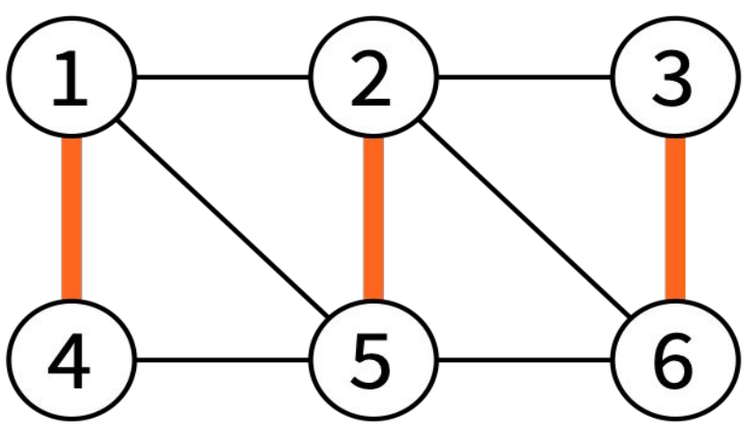
Matching



Maximal matching



Maximum matching



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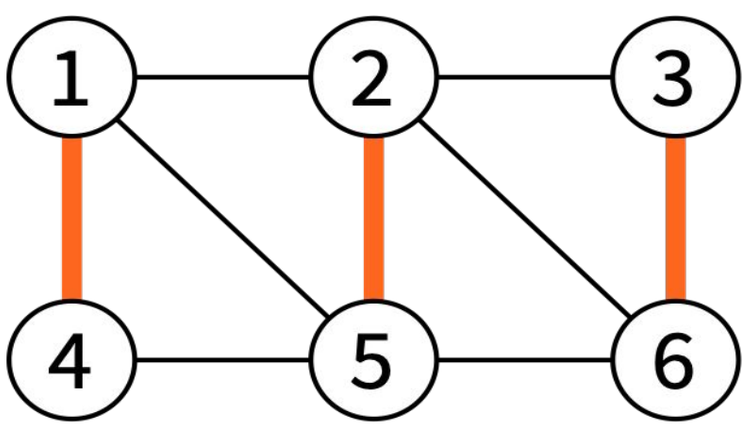
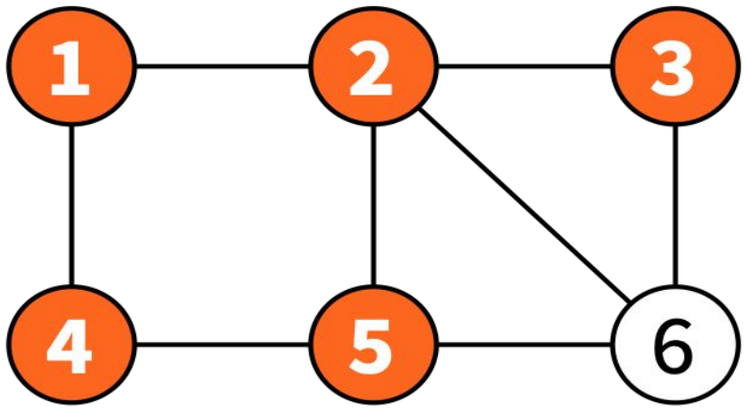


Minimisation problems

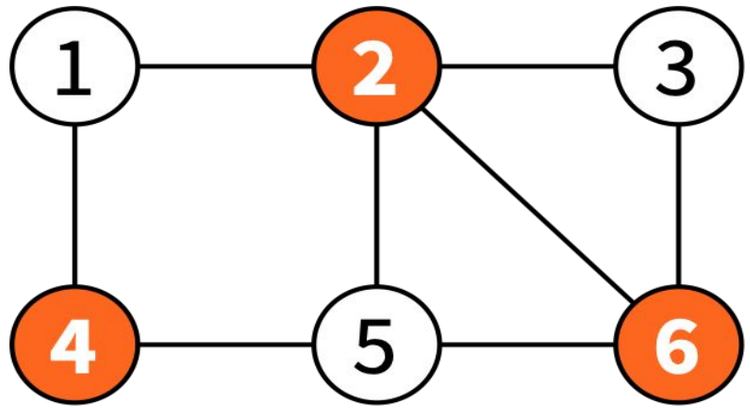
- minimal = **cannot remove anything**
- minimum = **smallest possible size**
- *x*-approximation =
  at most *x* times minimum

Minimisation problems

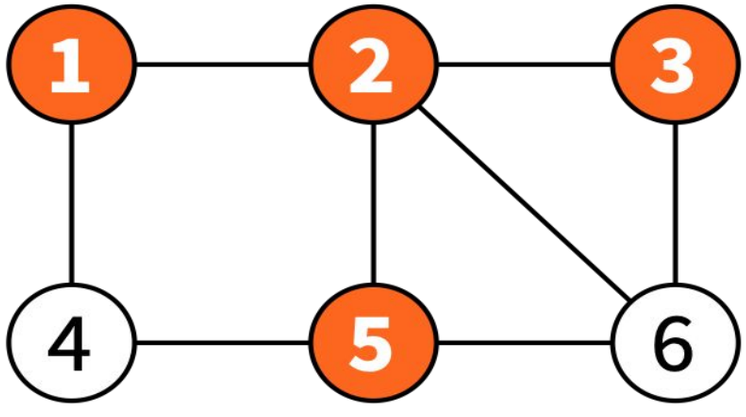
Vertex cover (VC)



Minimum VC



Minimal VC



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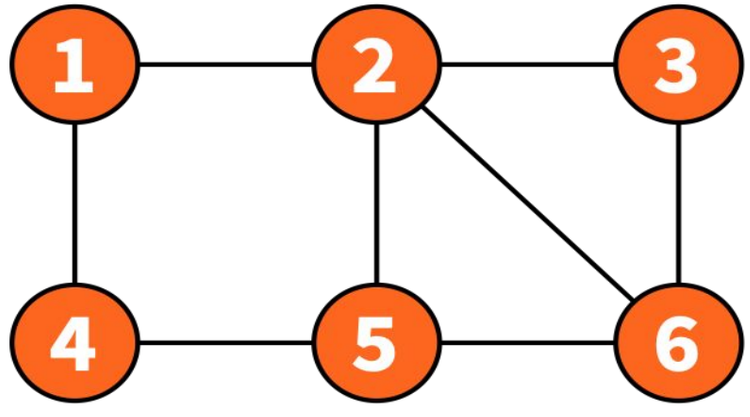

Approximation

- **Approximations are always feasible solutions!**
- **“2-approximation of minimum vertex cover”**
  - vertex cover
  - ≤ 2 times as large as minimum vertex cover

Optimization problem

An **optimization problem** has the following characteristics:

- A solution is a subset of one of the data in the problem;
- There are in general several admissible solutions;
- Each (admissible) solution has an associated value (usually a cost or a gain).

The **optimization problem** consists not only in finding an admissible solution, but also in finding a admissible solution, but to find a solution of minimal value or maximum value.

Example : giving change

- Problem : we have coins of values v1, v2, .., vn ;
- Solution : a sequence of (values of) coins with total S ;
- Best solution : the one using the least number of coins.

Greedy Algorithms

It is a particular strategy to solve an optimization problem.

A greedy algorithm is an algorithm that builds a solution :

- element by element without ever going back ;

- based on local considerations.

Example of greedy algorithm

Greedy algorithm

1. As long as S > 0 ;

2. Choose the piece with the largest value v less than S ;

3. Repeat with S - v.

Example of greedy algorithm

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Correct example

S =16 with pieces of 10,5,2, 1:

- 16 - 10 = 6
- 6 - 5 = 1
- 1 - 1 = 0

3 pieces

Example of greedy algorithm

Greedy algorithm

1. As long as S > 0 ;

2. Choose the piece with the largest value v less than S ;

3. Repeat with S - v.

Correct example

S =16 with pieces of 10,5,2, 1:

- 16 - 10 = 6
- 6 - 5 = 1
- 1 - 1 = 0

3 pieces

Incorrect example

S=16 with pieces of 9,8,1:

- 16 - 9 = 7
- 7 - 1 = 6
- …

8 pieces (best case is 8 + 8 = 16)

Coloration of a graph with a greedy algorithm

Idea of the algorithm

We have a graph G = (V , E). For any v ∈ V :

- We look at the set of colors already assigned to the neighbors of v.
- We deduce the smallest natural number that does not belong to this set.
- We assign this color to v.

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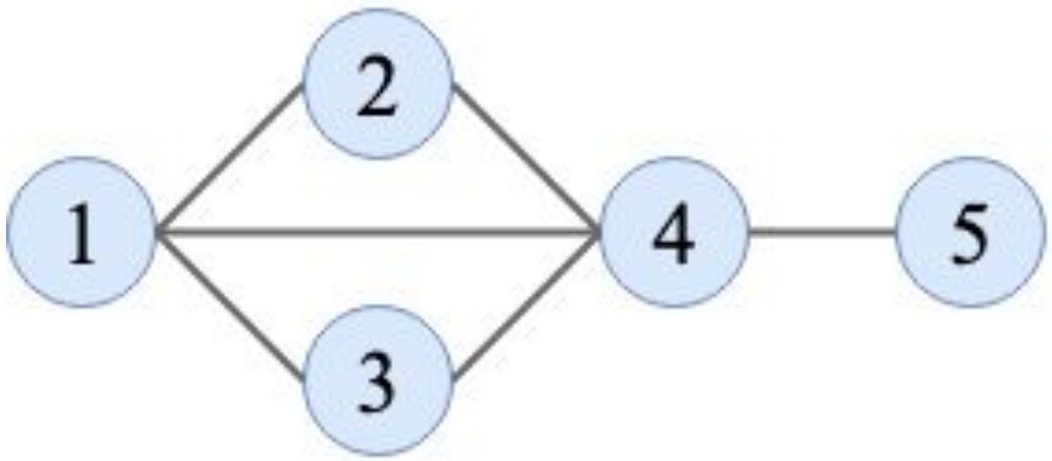

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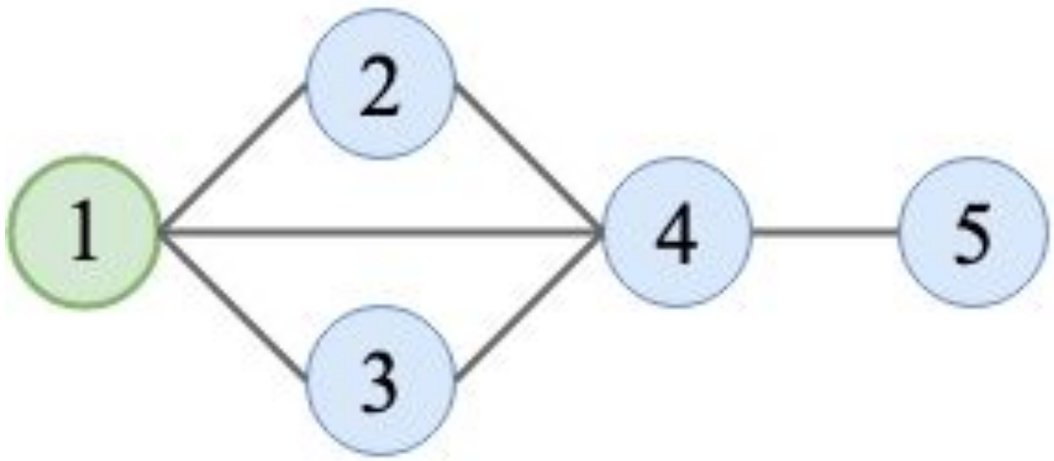

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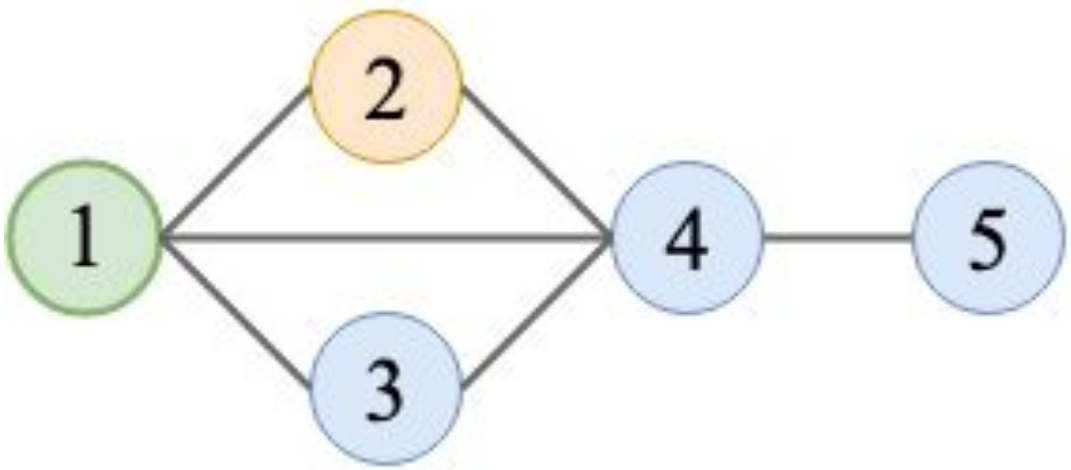

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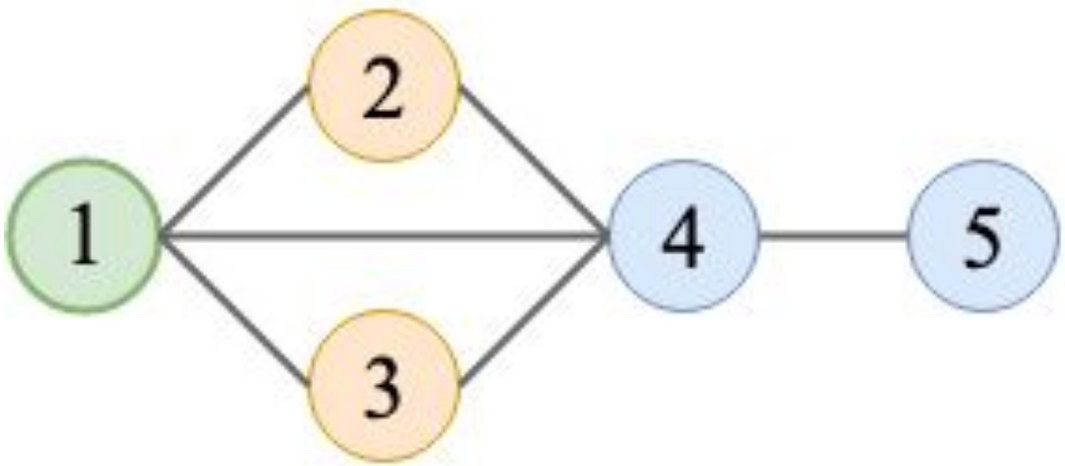

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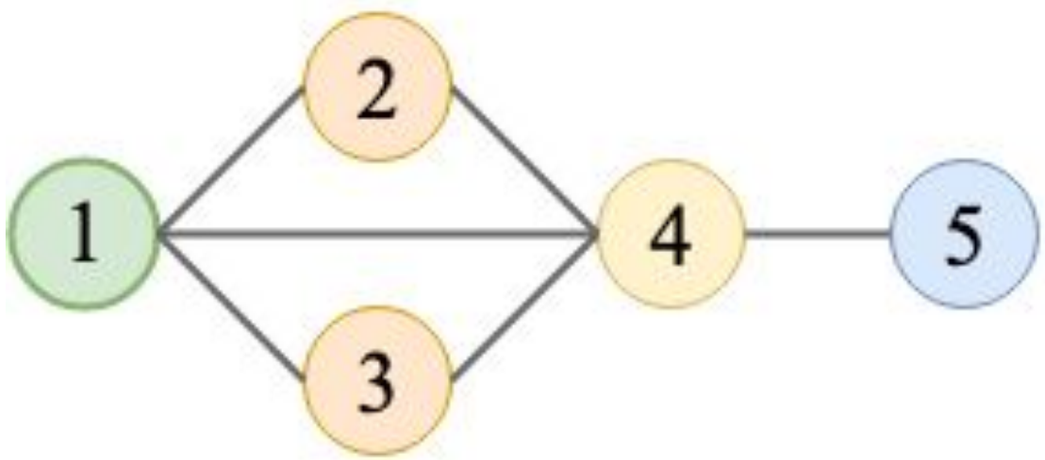

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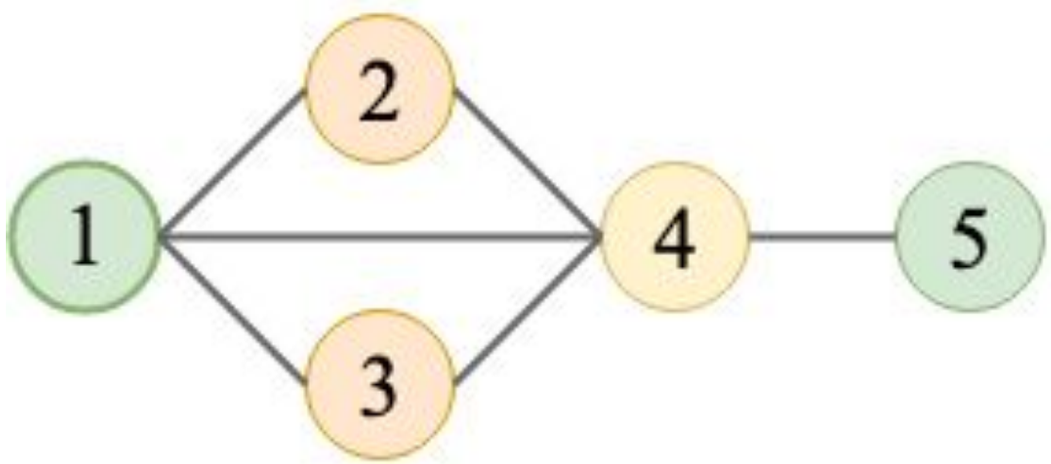

Reminders about tree

a tree is a connected graph without cycle.
Be careful, the notion is different from the trees usually used in algo:
- No arity constraint ;
- No notion of root

Properties

- Between two given vertices of a tree, there is always exactly an (elementary) chain;
- A tree with n vertices has n - 1 edges.

Spanning Tree

A subgraph of G is said to be covering if it contains all vertices of G.

Be careful, a covering subgraph is not necessarily connected.

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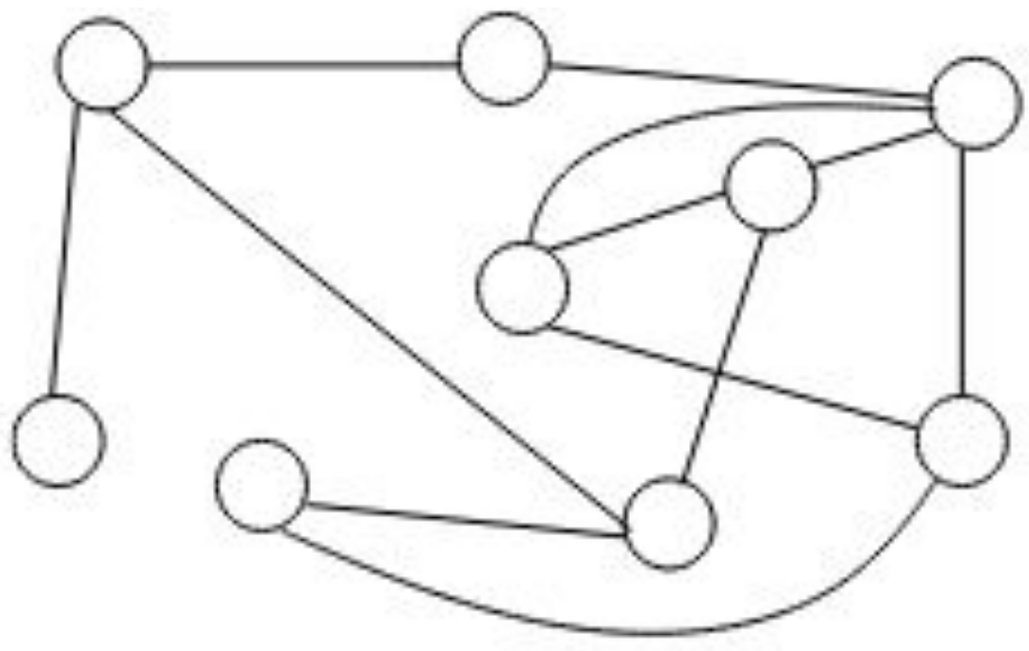

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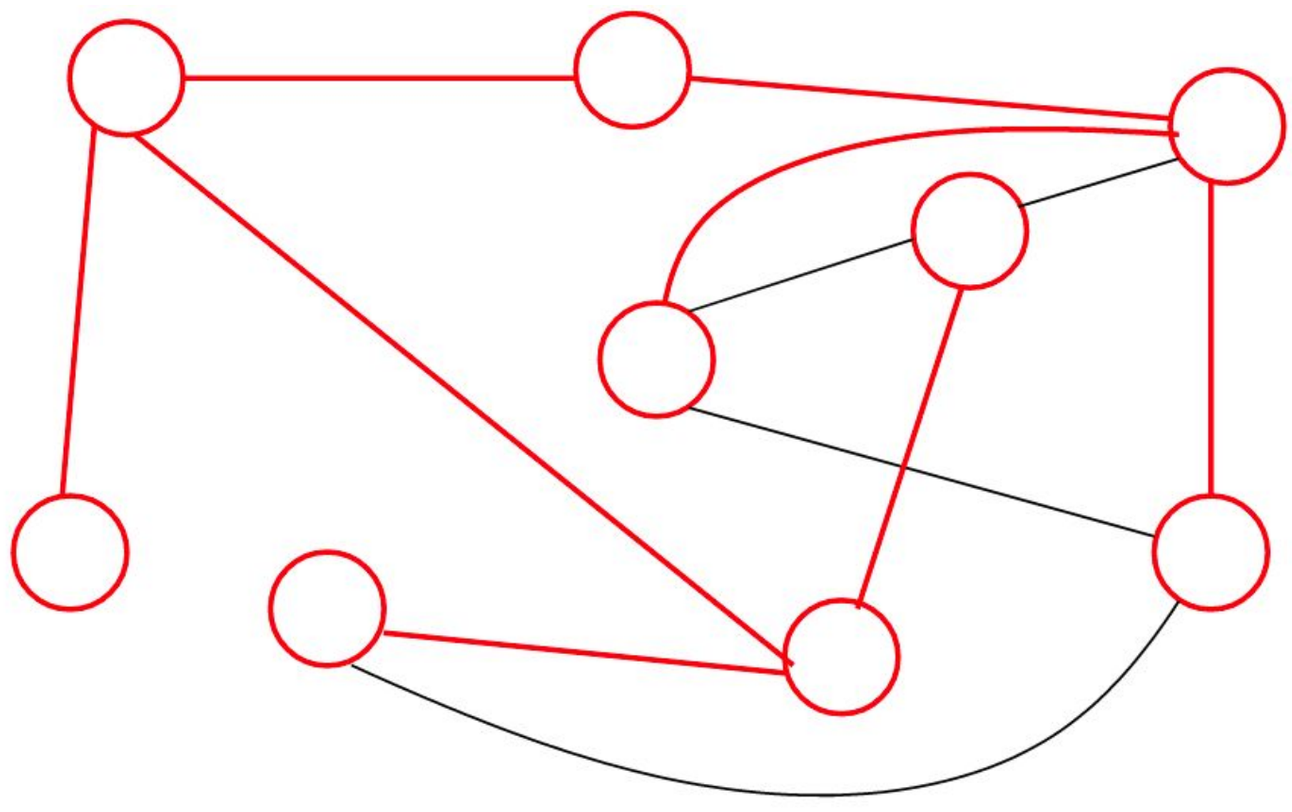


Spanning Tree

If A is a **spanning subgraph** of a graph G with V vertices the following characterizations are equivalent:

- A is a spanning tree of G ;

- A is cycle-free and has V - 1 edges;

- A is connected

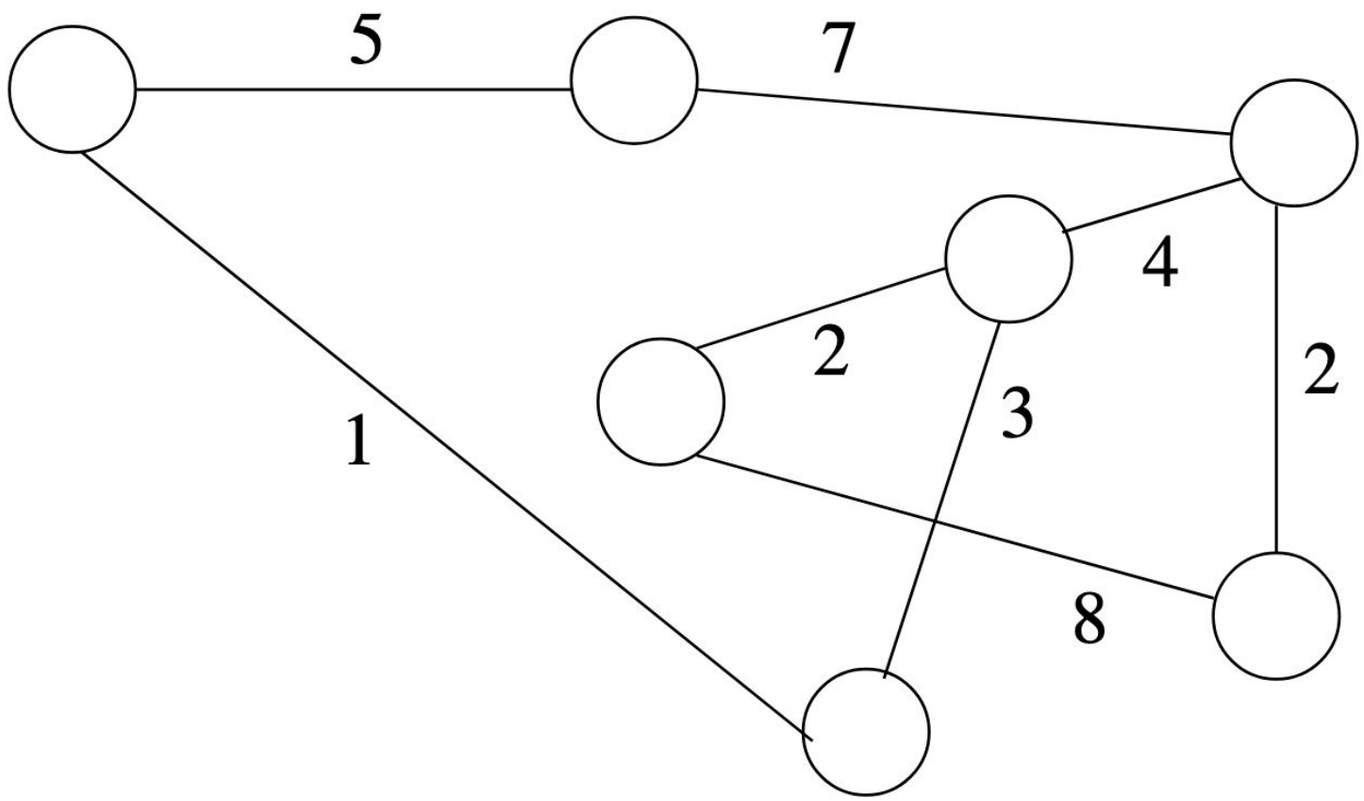
- We cannot add an edge to A without creating a cycle;

- We cannot remove an edge from A without breaking its connectedness

Spanning Tree in a weighted graph

Minimum weight spanning tree

The weight of a subgraph is the sum of the weights of the edges. We are looking for a minimal weight spanning tree (MST).



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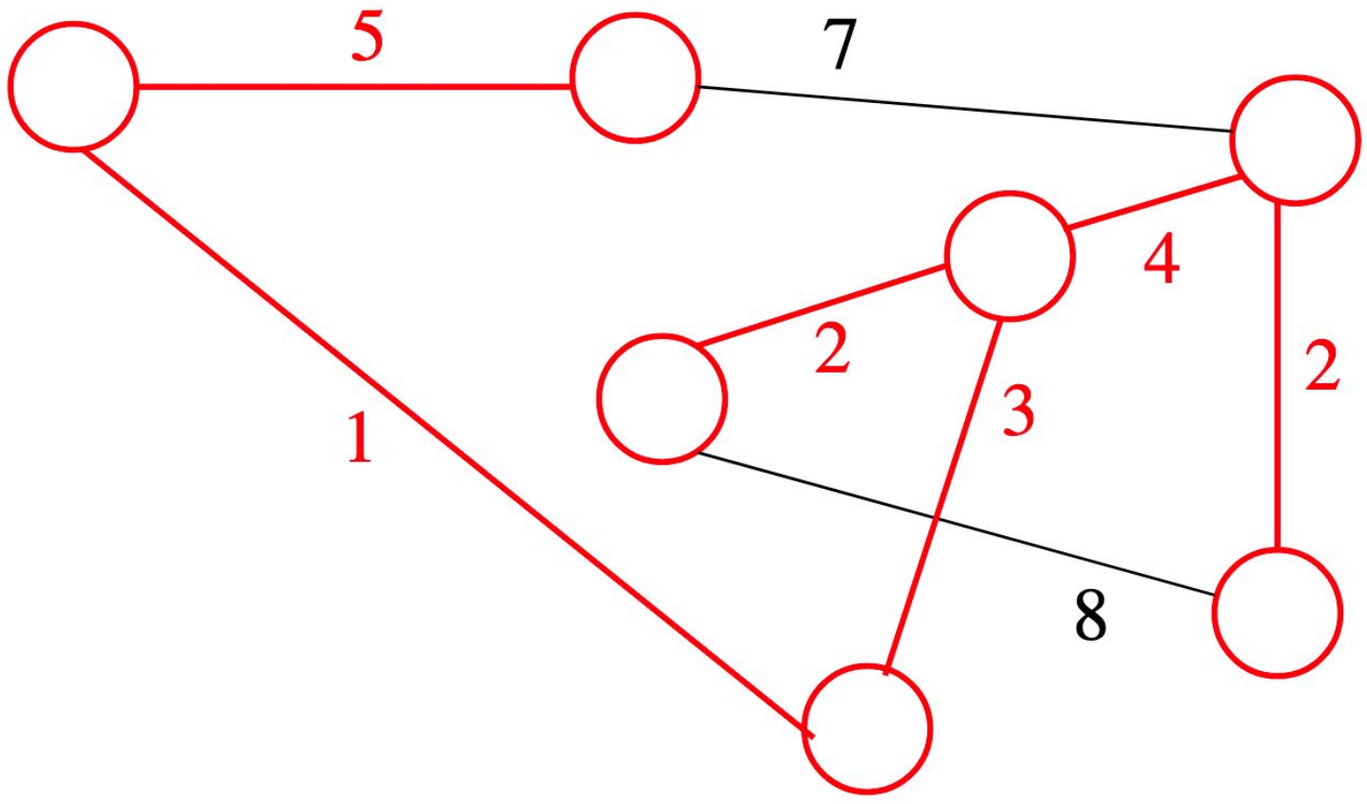
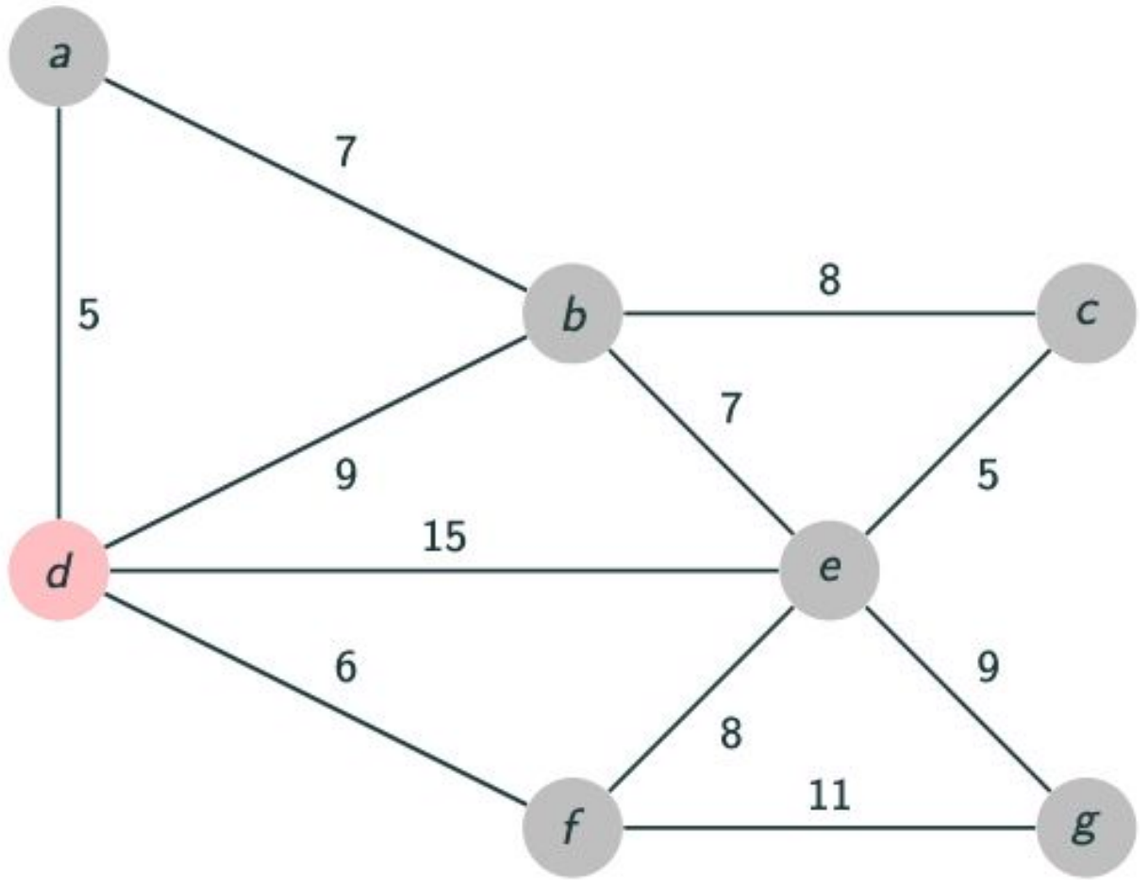


Prim's Algorithm

```
A := (∅, ∅)
ajouterSommet( choisirSommet(G), A )
while nbSommets(A) < V
    // Recherche de l'arête min incidente à A
    m := +∞
    foreach x ∈ A do
        foreach y ∈ ensVoisins(x) do
            if y ∉ ensSommets(A) et poids(x, y) < m
                m := poids(x, y)
                ymin := y
    ajouterSommet( ymin, A )
    ajouterArete( (x, ymin), A )
return A
```

Prim's Algorithm on an example



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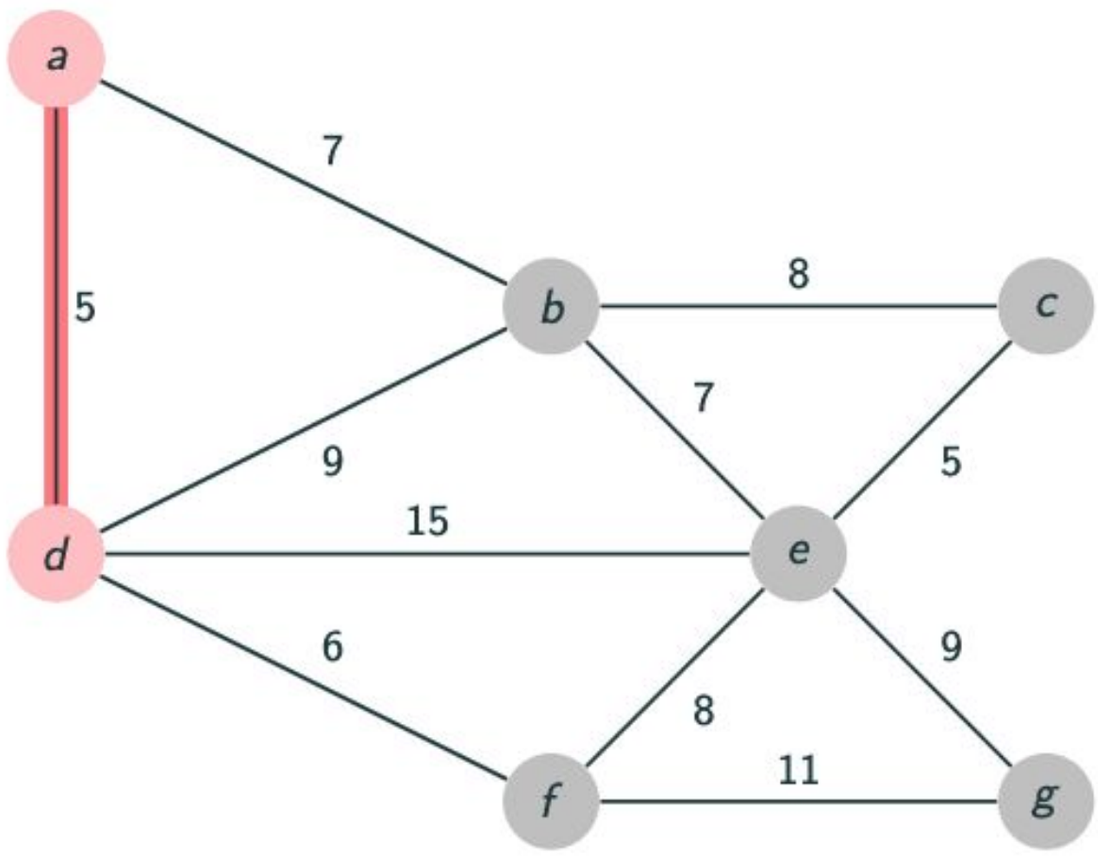

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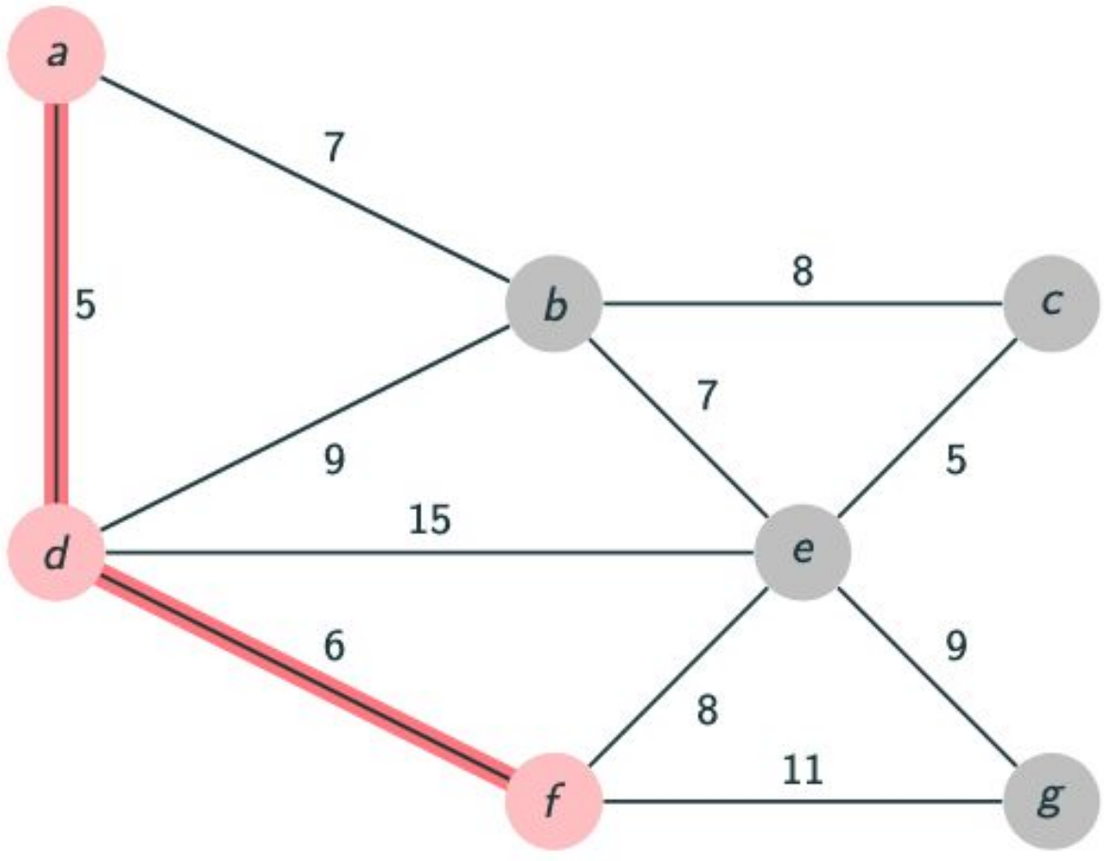

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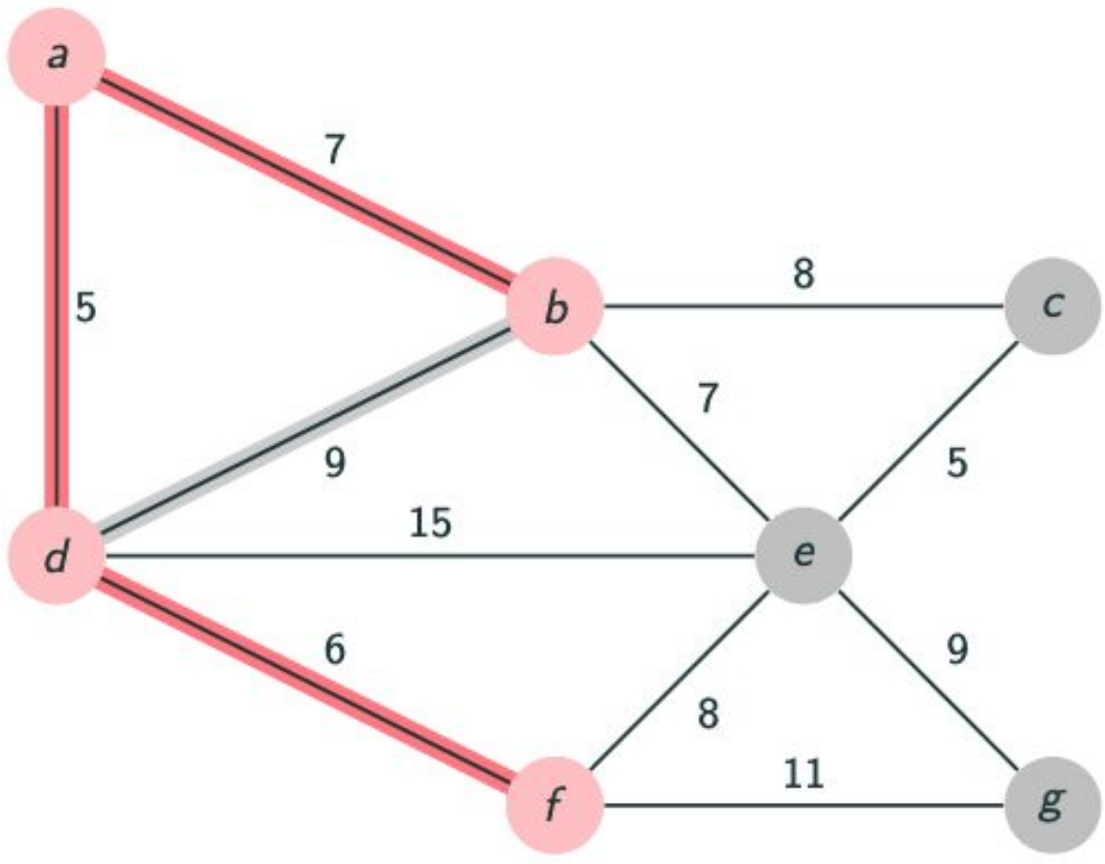

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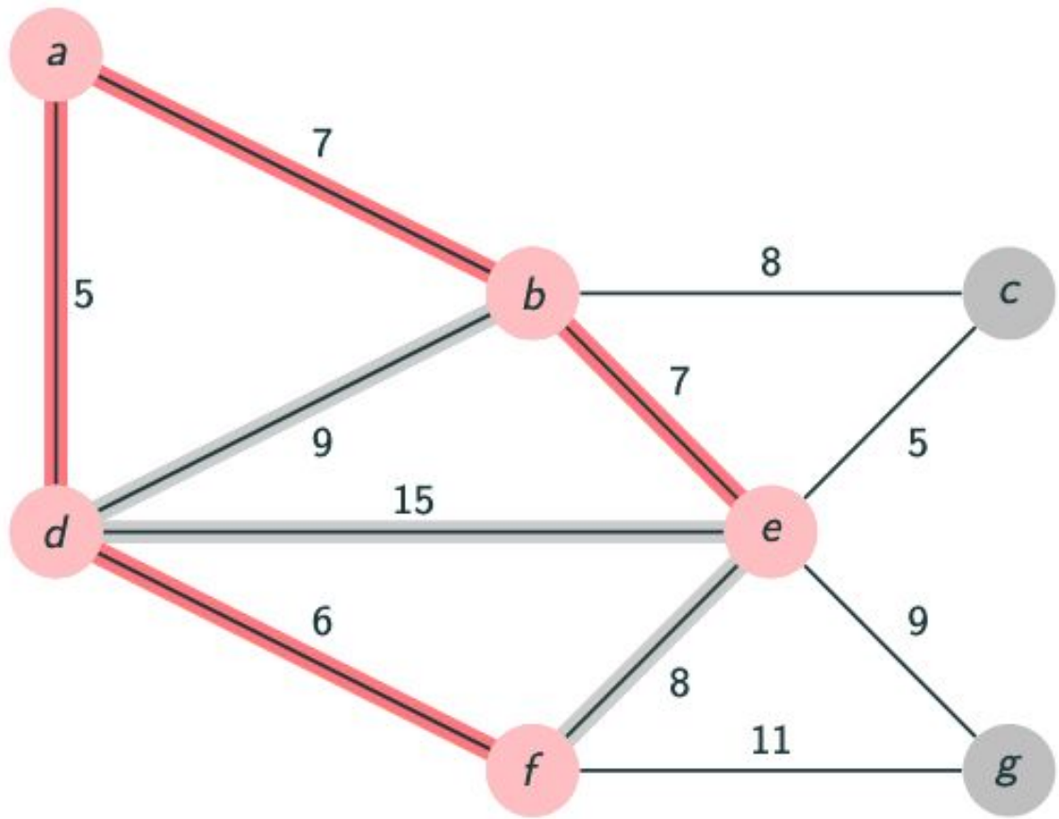

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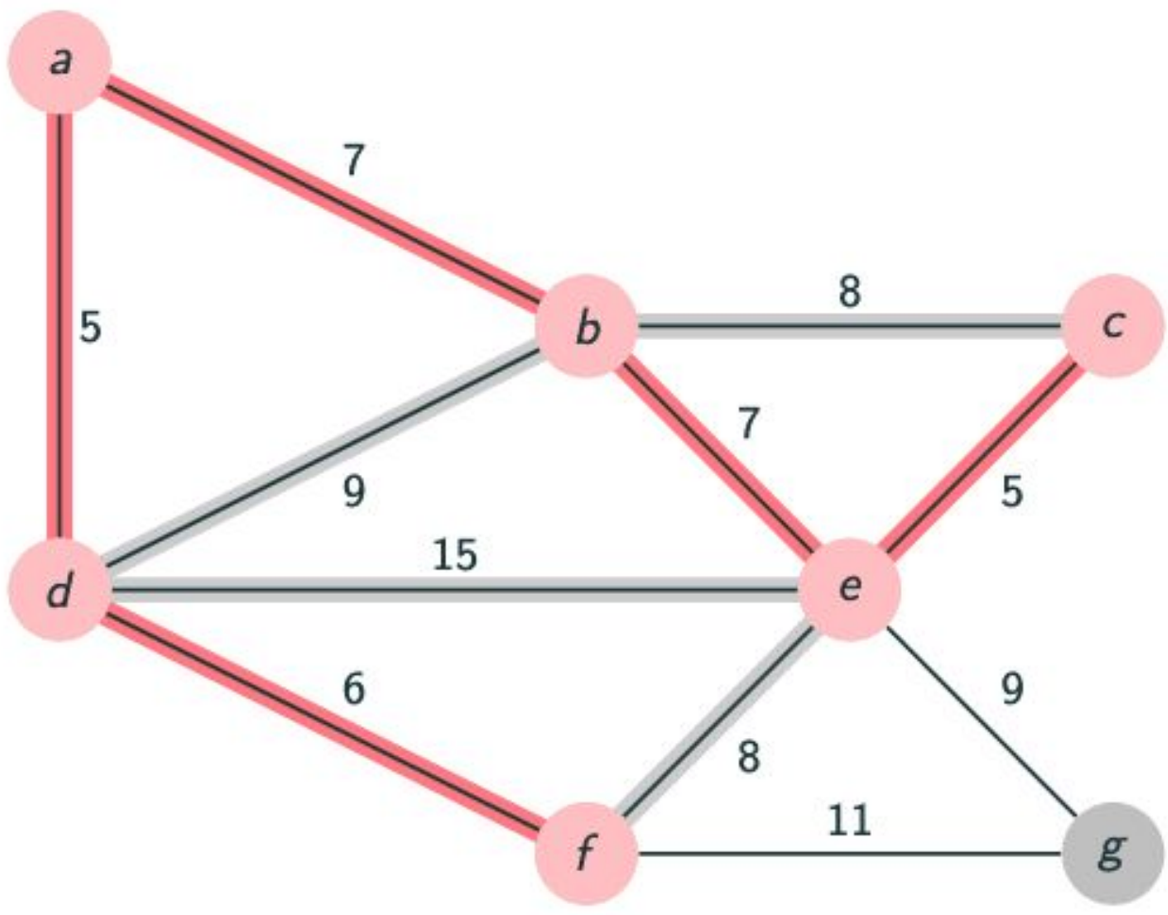

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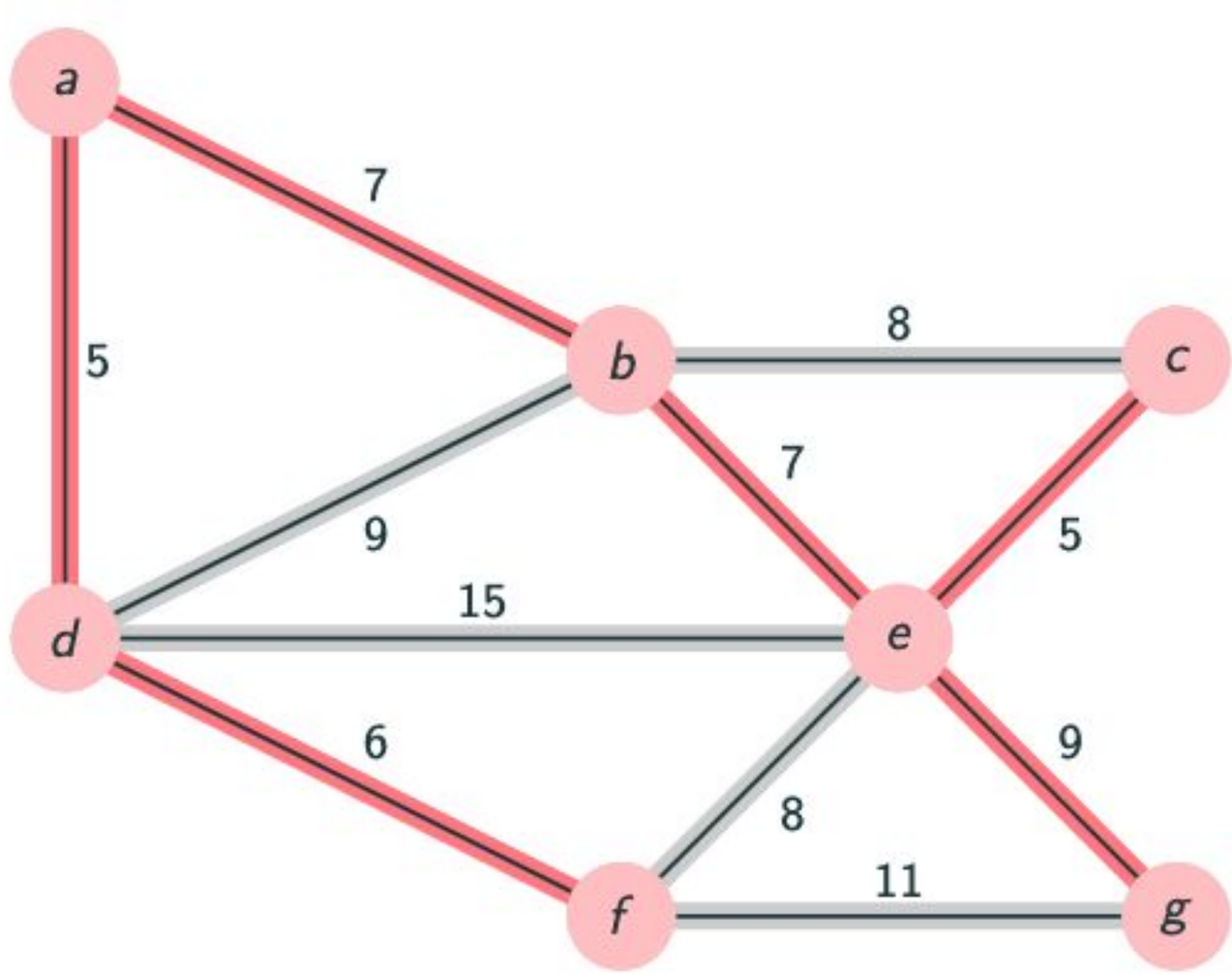


Is it a greedy algorithm ?

We find the characteristics of a greedy algorithm:

- The constructed solution is a set (of edges) ;
- The admissible solutions are covering trees;
- We are looking for a solution of minimal total weight;
- Prim's algorithm proceeds only by adding edges (and vertices) in A;
- he choice of the next edge is based only on its weight.
-

The algorithm therefore has a cost in $O(V \times C)$ where C is the cost of choosing the next edge;

Graph theory and distributed algorithms

- **Network ≈ graph:** node ≈ computer, edge ≈ link
- **Graph theory used to:**
  - define: model of computing,
    what we want to solve, what we assume …
  - prove: correctness of algorithms,
    time complexity, impossibility results …

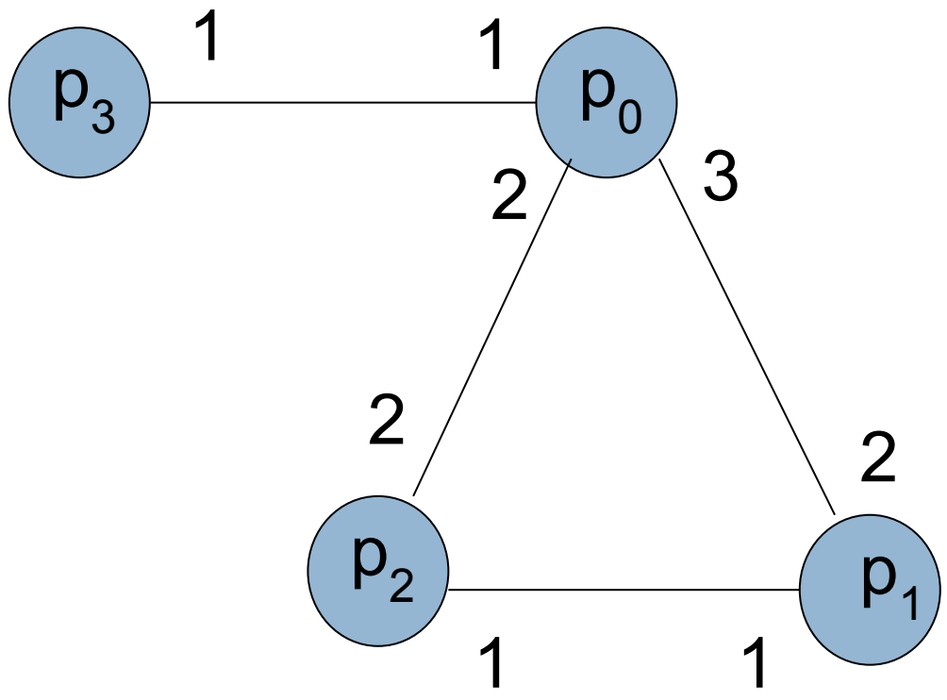
Distributed Algorithm : Message Passing Model

Message-Passing Model

- In a message-passing system, processors communicate by sending messages over communication channels, where each channel provides a bidirectional connection between two specific processors.
- The pattern of connections provided by the channels describes the **topology** of the system.
- The topology is represented by an undirected graph in which
  - each node represents a processor and
  - an edge exists between two nodes if and only if there is a communication channel between the corresponding processors.

Message-Passing Model

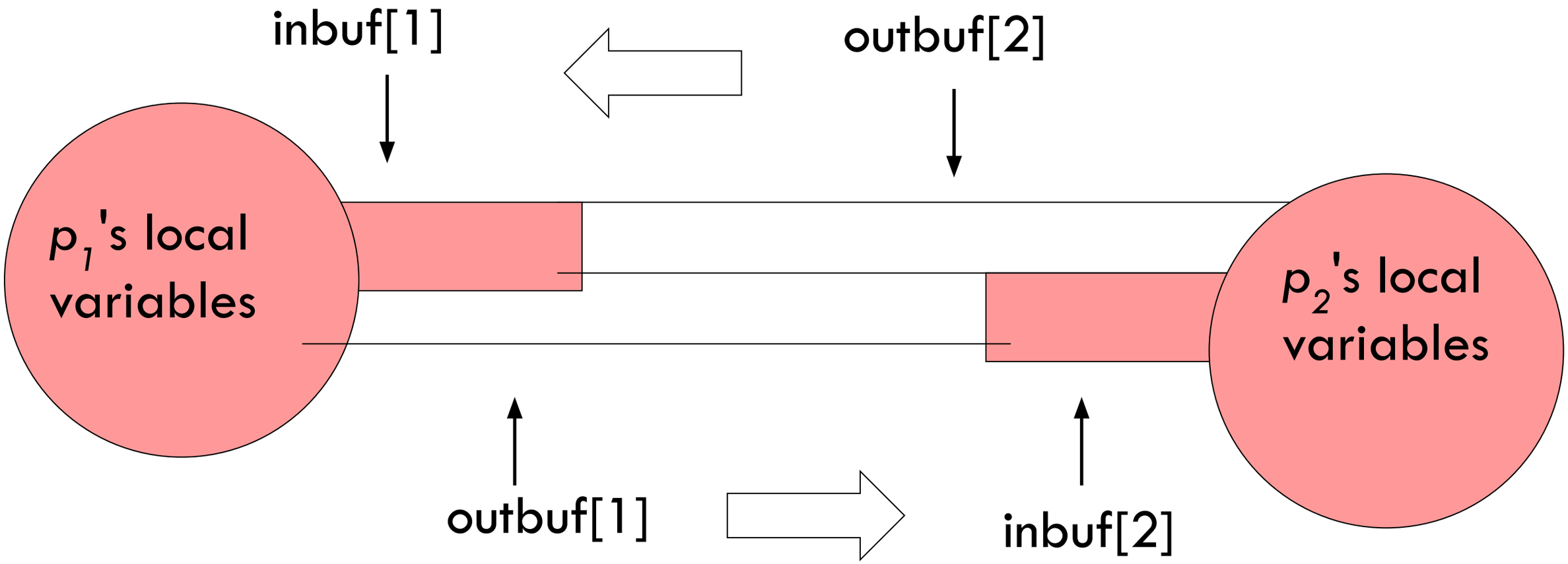
- processors are p_0, p_1, …, p_{n-1} (nodes of graph)
- each processor p_i labels its incident channels with the integers 1, 2, …r where r is the degree of p_i
  - 1, 2, 3 for p_0 in the exemple



Modeling Processors and Channels

- Processor is a state machine including
  - local state of the processor
  - mechanisms for modeling channels
- Channel directed from processor p_i to processor p_j is modeled in two pieces:
  - *outbuf* variable of p_i : msgs p_i has sent to p_j *and not yet delivered to* p_j
  - *inbuf* variable of p_j : msgs *delivered by* p_i *and not yet processed by* p_j
- *Outbuf* corresponds to physical channel, *inbuf* to incoming message queue

Modeling Processors and Channels



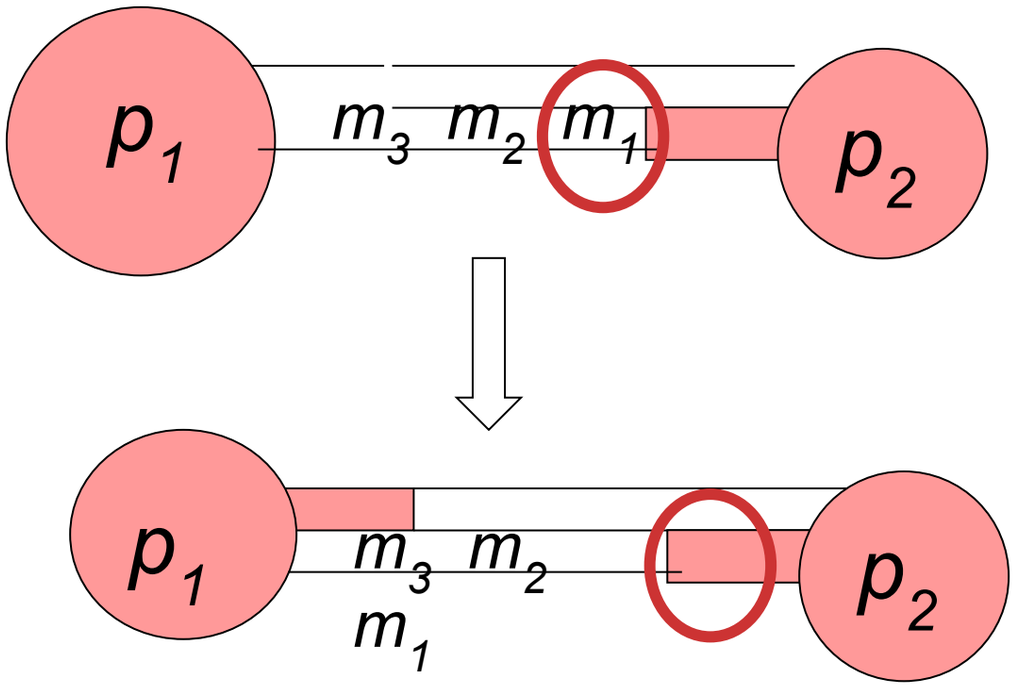
Pink area (**local vars + inbuf**) is ***accessible* state for a processor.**

Processor state and configuration

- The *state* of a processor p_i contains 2r components where r is the number of neighbors of p_i . In particular for every neighbor j
  - *outbuf[j]* : is the set of msgs p_i has sent to p_j *and not yet delivered to* p_j
  - *inbuf[j]* variable of p_i : msgs from p_j *delivered by* p_i *and not yet processed by* p_i
- A *configuration* is a vector $C=(q_0,q_1,\ldots,q_{n-1})$ where q_i is a state of process p_i
  - An initial configuration is a vector $(q_0,q_1,\ldots,q_{n-1})$ such that q_i is the initial state of process p_i

A Deliver Event

- Moves a message from sender's outbuf to receiver's inbuf; message will be available next time receiver takes a step



A Computation Event

- Occurs at one processor and represents a computation step where the process applies its transition function :
  - Start with old accessible state (local vars + incoming messages)
  - Apply transition function of processor's state machine; handles all incoming messages
  - End with new accessible state with empty inbufs, and new outgoing messages

Execution

- The behavior of the system over time is modeled as an execution : a sequence of configurations alternating with events :
  - config, event, config, event, config, …
- in first config: each processor is in initial state and all inbufs are empty
- for each consecutive (config, event, config), new config is same as old config except:
  - if delivery event: specified msg is transferred from sender's outbuf to receiver's inbuf
  - if computation event: specified processor's state (including outbufs) change according to transition function

Asynchronous Systems

- A system is said to be asynchronous if there is no fixed upper bound on how it takes for a message to be delivered or how much time elapses between two consecutive steps of a processor (ex. Internet)
- Observation : there are usually upper bounds on message delays and processors step times, but sometimes they are very large, infrequently reached and they can vary over time :

  ☞ It is often desirable to write an algorithm independent of any timing parameters

Asynchronous Executions

- An execution is *admissible for the asynchronous model* if:
  - every message in an outbuf is eventually delivered
  - every processor takes an infinite number of steps
- No constraints on when these events take place: arbitrary message delays and relative processor speeds are not ruled out
- Models reliable system (no message is lost and no processor stops working)

Synchronous Message Passing Systems

- An execution is *admissible* for the synchronous model if it is an infinite sequence of "rounds"
- A"round" is a sequence of deliver events that move all messages in transit into inbuf's, followed by a sequence of computation events, one for each processor

Synchronous Message Passing Systems

- The new definition of admissible captures lockstep unison feature of synchronous model.
- This definition also implies
  - every message sent is delivered
  - every processor takes an infinite number of steps.
- **Time** is measured as number of rounds until termination.

Message Complexity Measure

- **Message complexity:** maximum number of messages sent in any admissible execution

- This is a worst-case measure

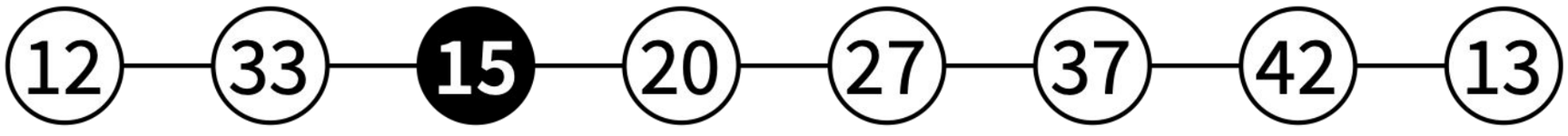
Distributed Algorithm : Locality

Locality

- **Output of a node can only depend on what it knows**
- **After *T* time steps, a node can only know about things up to distance *T***

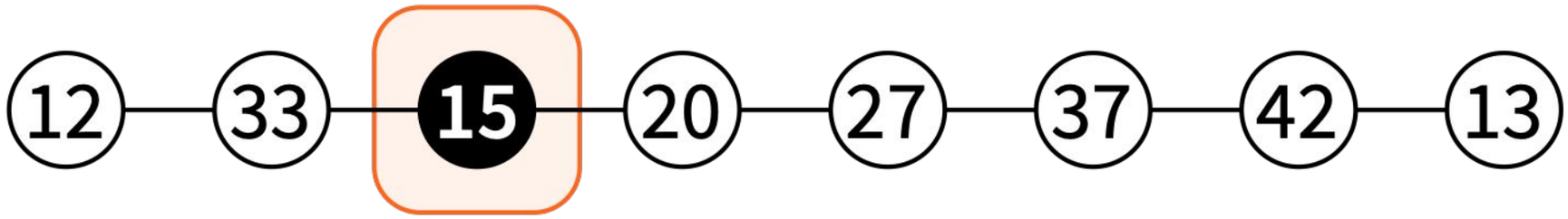
Locality

- **Who knows that node 15 exists?**
  - initially, only node 15
  - everyone else has to learn it by exchanging messages



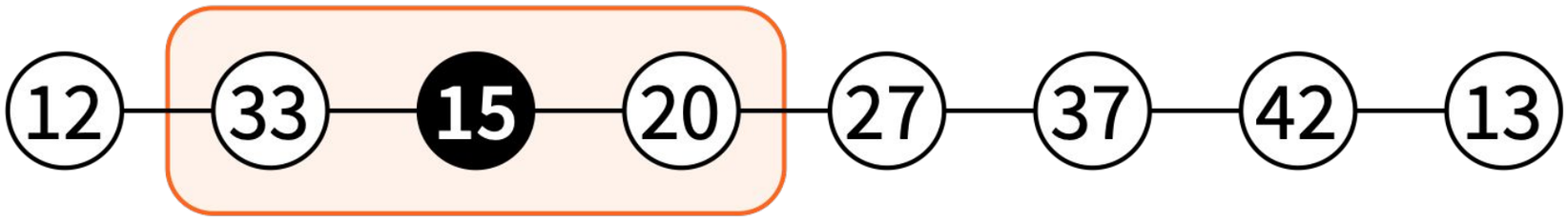
Locality

- **Who knows about node 15 at time $T = 0$?**
  - initial state, before we exchange any messages



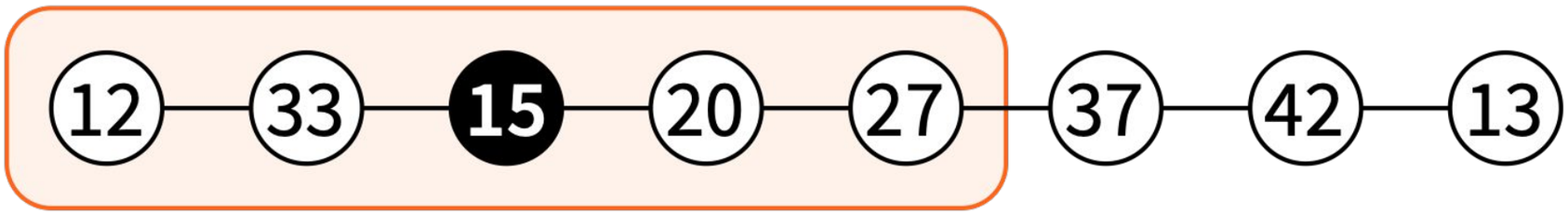
Locality

- **Who knows about node 15 at time $T = 1$?**
  - after 1 communication round



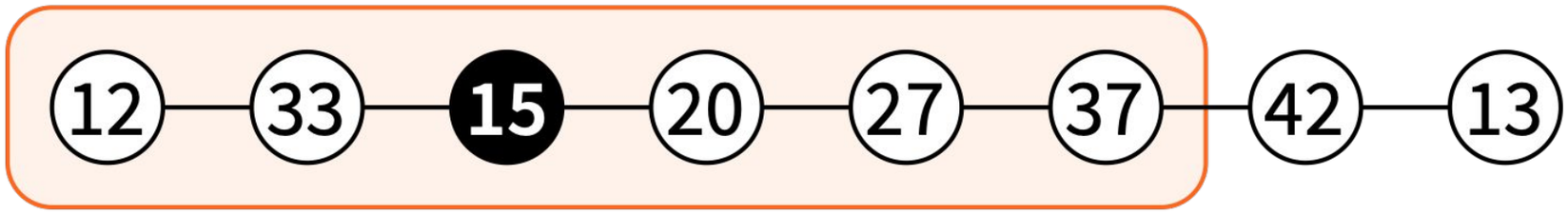
Locality

- **Who knows about node 15 at time $T = 2$?**
  - after 2 communication rounds



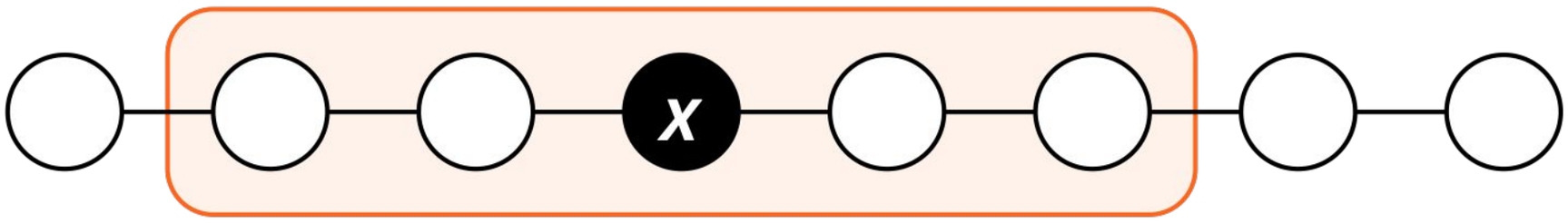
Locality

- **Who knows about node 15 at time $T = 3$?**
  - after 3 communication rounds



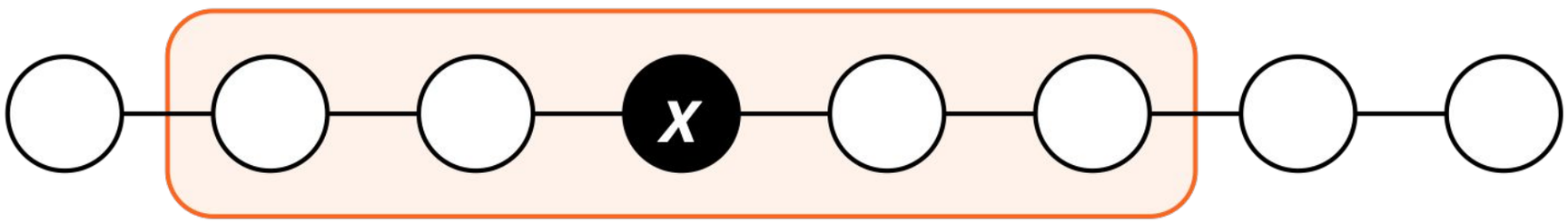
Locality

- **After *T* communication rounds, only nodes up to distance *T* from node *x* can know anything about node *x***
  - distance = “number of hops”



Locality

- **After** *T* communication rounds**,**
  node *x* **can only know about other nodes that are within** distance *T* **from it**
  - distance = “number of hops”



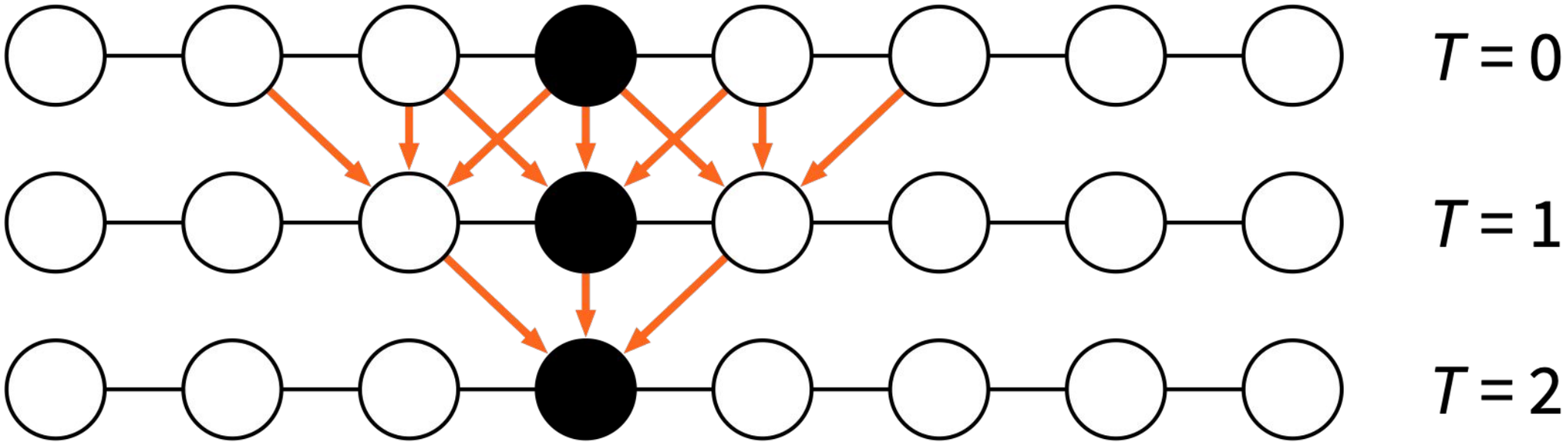
Locality

- **My state at time T only depends on:**
  - my state at time $T - 1$, and
  - messages that I received on round T,

  which only depend on:
    - the state of my neighbours at time $T - 1$

Locality

- **State at time *T* only depends on initial information within distance *T***

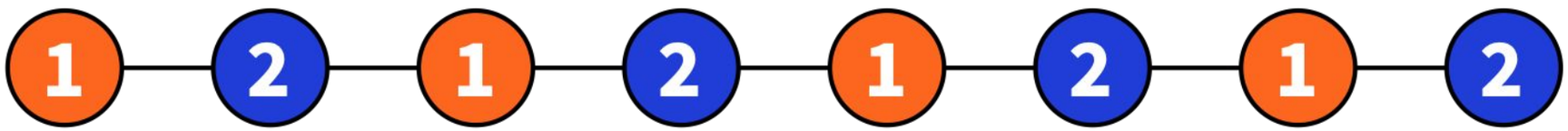


Locality

- **Time = distance**
- **Fast algorithm = “local” algorithm**
  - outputs only depend on local neighbourhoods

Using locality to prove lower bounds

- **Example: 2-colouring of a path**
- **Upper bound: possible in time $O(n)$**
- **Lower bound: not possible in time $o(n)$**



Algorithm for 2-coloring

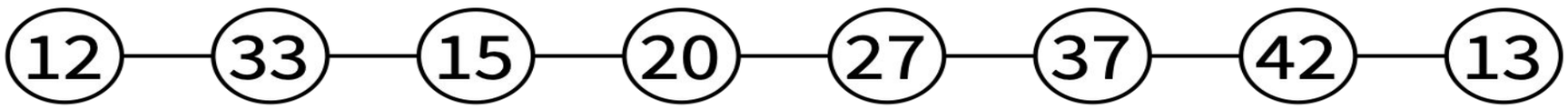
- **Assumption: path, unique identifiers**

Algorithm for 2-coloring

- **Assumption: path, unique identifiers**
- **Two phases:**
  - find the endpoint with smaller identifier
  - starting from this end, assign colours 1, 2, 1, 2, …

Algorithm for 2-coloring

- **Messages:**
  - “**ID *x***” = there is an endpoint with identifier *x*
  - “**colour *c***” = my colour is *c*



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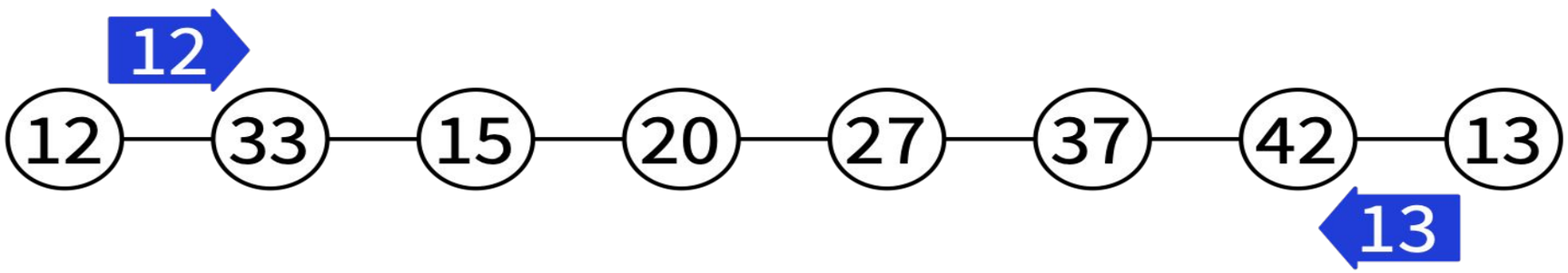

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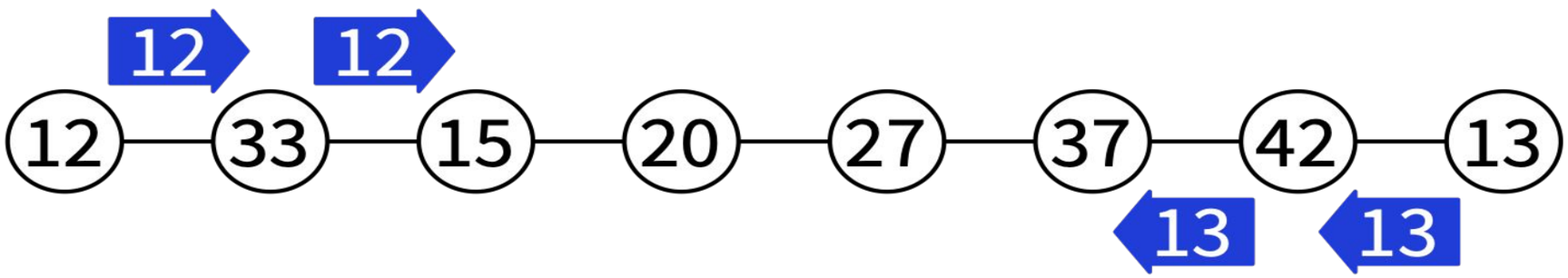

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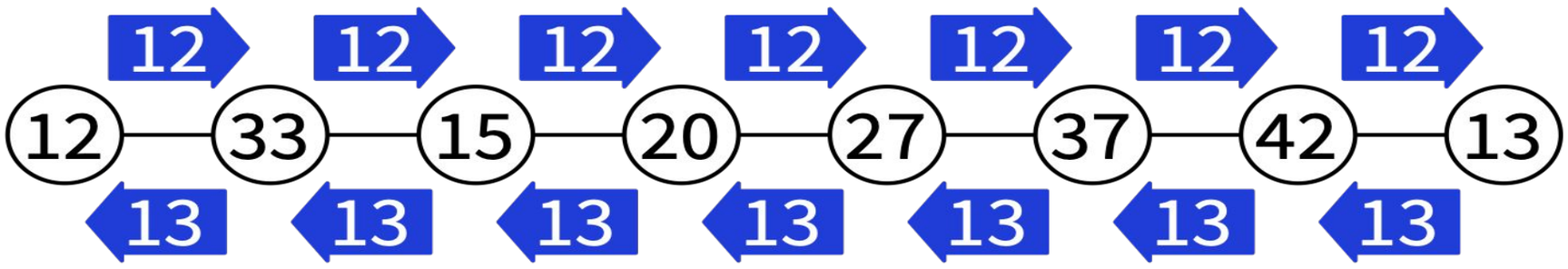

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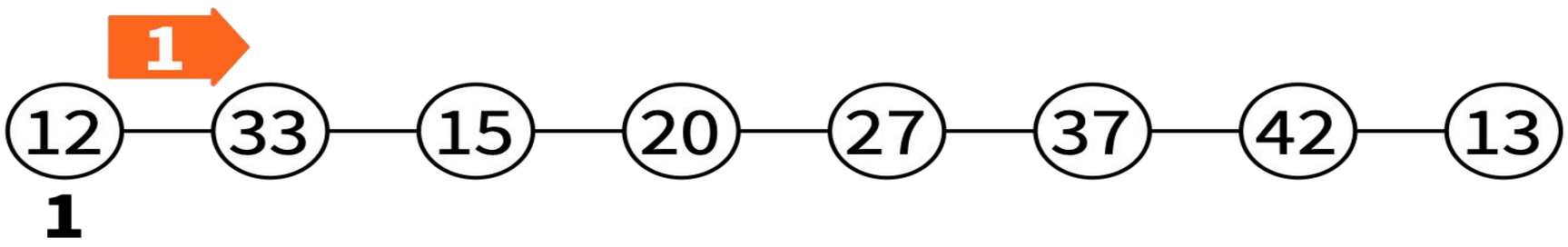

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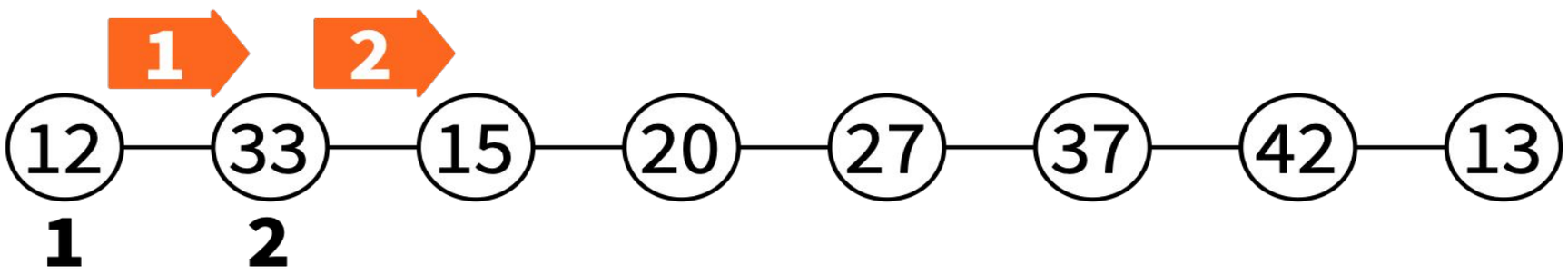

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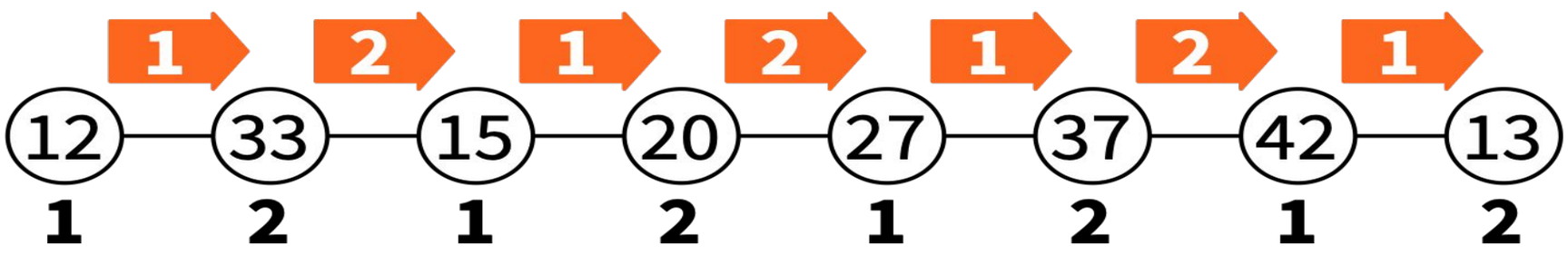


BASIC DISTRIBUTED ALGORITHMS IN MESSAGE-PASSING SYSTEMS

Broadcast on a spanning tree

Broadcast over a Rooted Spanning Tree

- Problem : (Single message) Broadcast a distinguished processor p_r has a message M it wishes to send to all other processors
- Suppose processors already have information about a rooted spanning tree of the communication topology
  - tree: connected graph with no cycles
  - spanning tree: contains all processors
  - rooted: there is a unique root node p_r

Broadcast Over a Rooted Spanning Tree

- root initially sends *M* to its children
- when a processor receives *M* from its parent
  - sends *M* to its children
  - terminates (sets a local boolean to true)
- Implemented via *parent* and *children* local variables at each processor
  - indicate which incident channels lead to parent and children in the rooted spanning tree

Broadcast Over a Rooted Spanning Tree : pseudo-code

Algorithm 1 Spanning tree broadcast algorithm.

Initially ⟨M⟩ is in transit from p_r to all its children in the spanning tree.

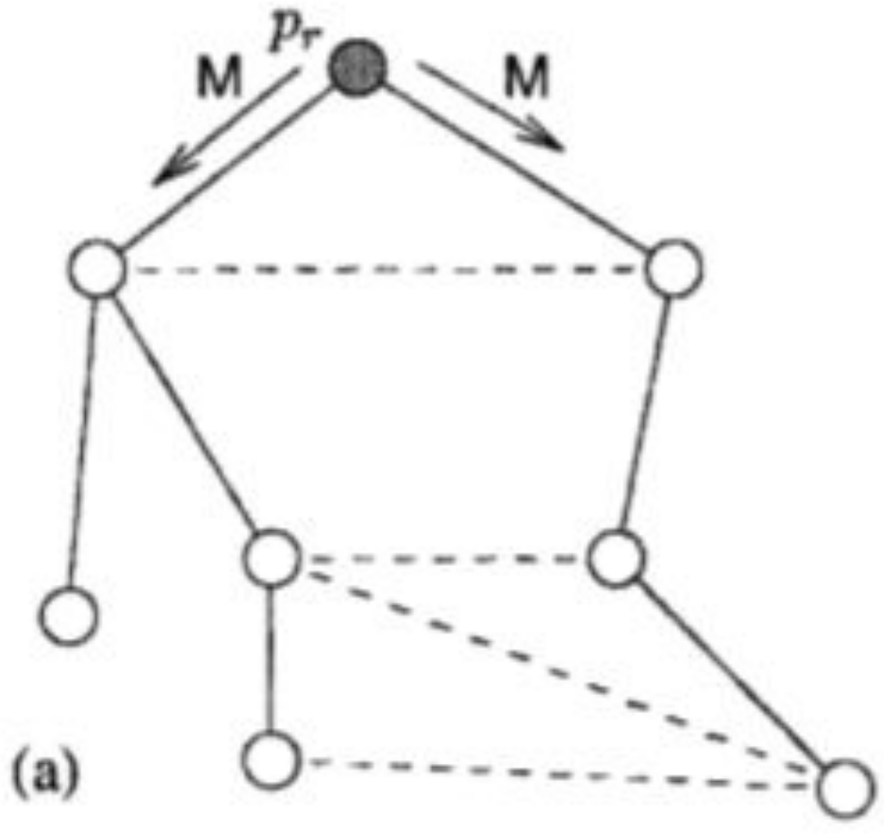
Code for p_r:

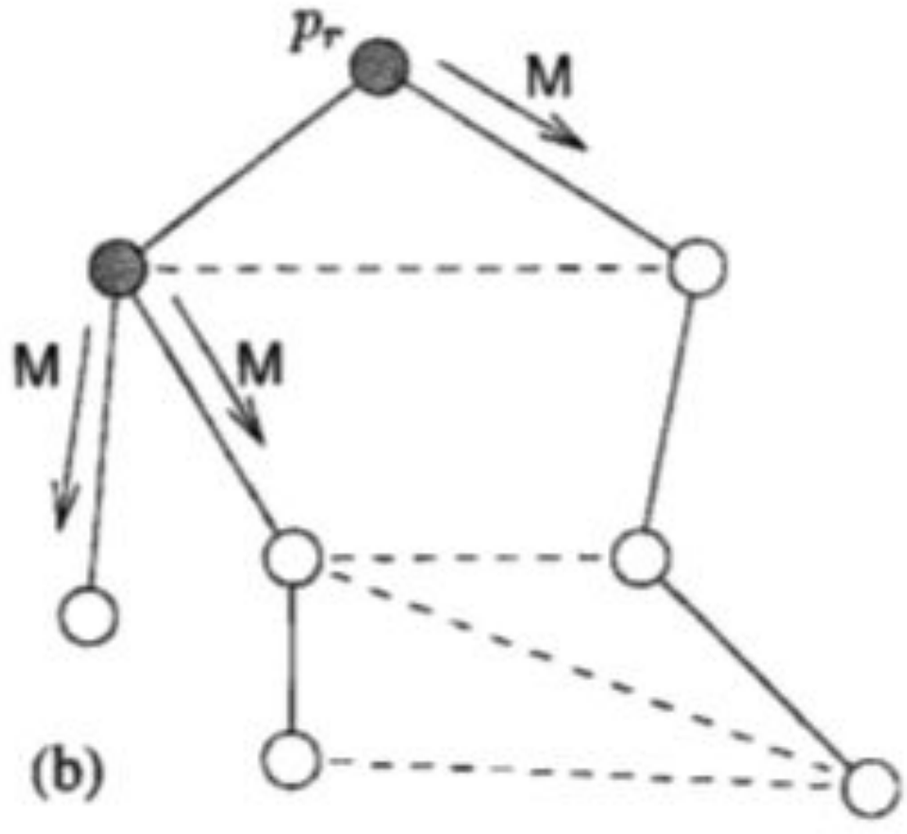
1: upon receiving no message: // first computation event by p_r
2: terminate

Code for p_i, $0 \leq i \leq n-1, i \neq r$:

3: upon receiving ⟨M⟩ from parent:
4: send ⟨M⟩ to all children
5: terminate

Broadcast Over a Rooted Spanning Tree : two steps in an asynchronous execution





Message Complexity Analysis

- Number of messages is $n - 1$, since
  - one message is sent over each spanning tree edge
  - A spanning tree of n nodes has exactly n-1 edges
- Same message complexity for synchronous and asynchronous model

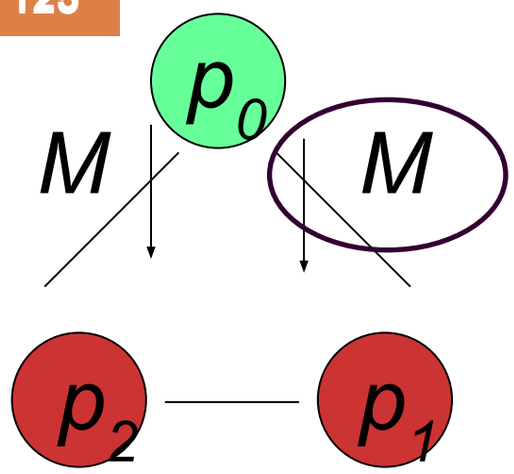
Broadcast without a spanning tree

- The algorithm starts from a given processor p_r which sends the message M to all its neighbors
- When processor p_i receives M for the first time, from some neighboring processor p_j
  - It sends M to its neighbors except p_j
- Message complexity : a processor will not send <M> more than once on any communication channel
  - <M> is sent at most twice on each communication channel (once by each processor using the channel)

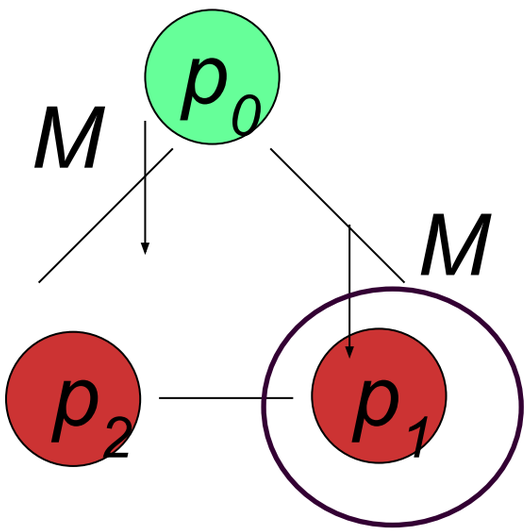
A simple algorithm to solve broadcast : called Flooding

- Each processor's local state consists of variable *color*, either red or green
- Initially:
  - p_0: *color* = green, all outbufs contain *M*
  - others: *color* = red, all outbufs empty
- Transition: If *M* is in an inbuf[j] and *color* = red, then change *color* to green and send *M* on all outbufs excluding outbuf[j]

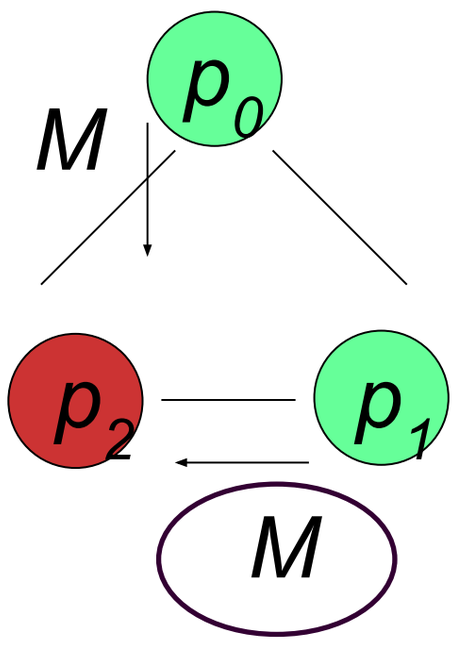
Example: (Asynchronous) Flooding



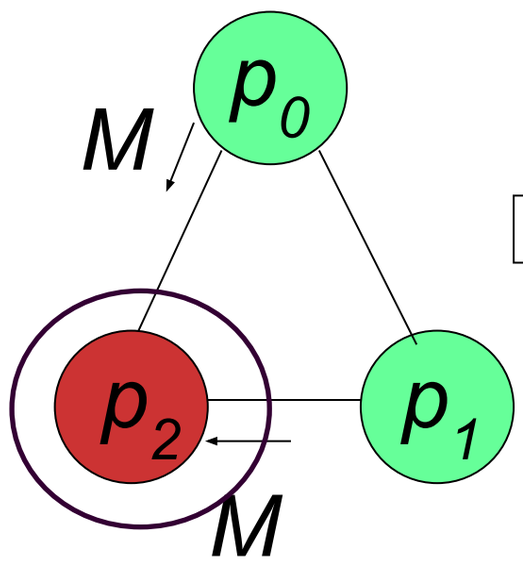
deliver event at p_1 from p_0



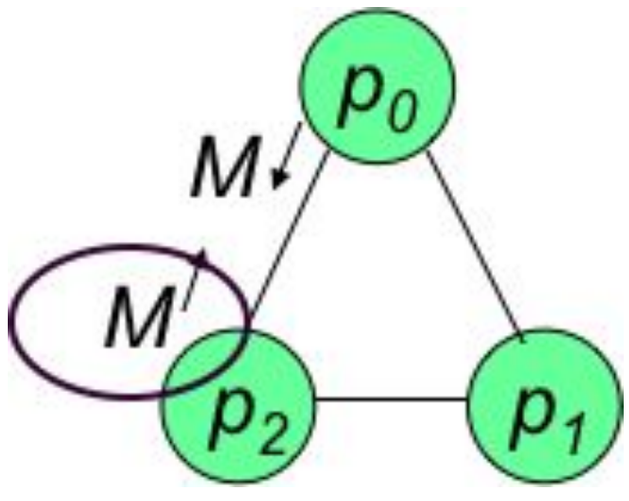
computation event by p_1



deliver event at p_2 from p_1



computation event by p_2



etc. to deliver all messages

Nondeterminism

- The previous execution is not the only admissible execution of the Flooding algorithm on that triangle.
- There are several, depending on the order in which messages are delivered.
- For instance, the message from p_0 could arrive at p_2 before the message from p_1 does.

Termination

- For technical reasons, admissible executions are defined as infinite.
- But often algorithms terminate.
- To model algorithm termination, identify *terminated* states of processors: states which, once entered, are never left
- Execution has *terminated* when all processors are terminated and no messages are in transit (in inbufs or outbufs)

Termination of Flooding Algorithm

- Define *terminated* processor states as those in which *color* = green

Message Complexity of Flooding Algorithm

- Message complexity: one message is sent over each edge in each direction. So number is $2m$, where m = number of edges
- m can be n(n-1)/2 where n is the number of processes

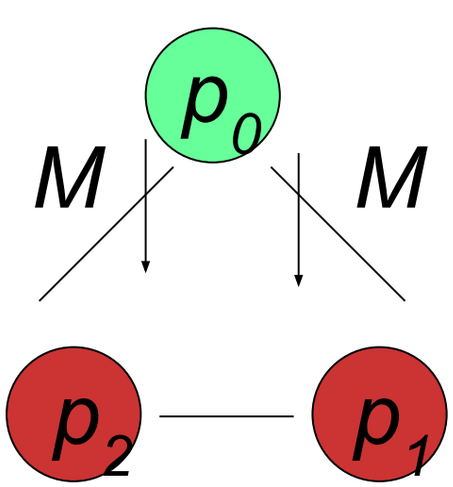
Example of Synchronous Model

- Suppose flooding algorithm is executed in synchronous model on the triangle.
- Round 1:
  - deliver M to p_1 from p_0
  - deliver M to p_2 from p_0
  - p_0 does nothing (as it has no incoming messages)
  - p_1 receives M, turns green and sends M to p_0 and p_1
  - p_2 receives M, turns green and sends M to p_0 and p_1

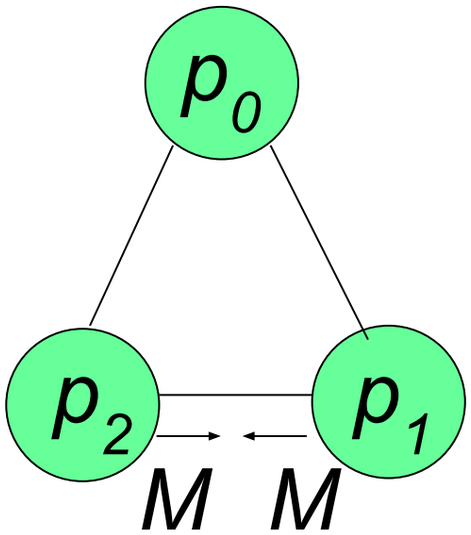
Example of Synchronous Model

- Round 2:
  - deliver M to p_0 from p_1
  - deliver M to p_0 from p_2
  - deliver M to p_1 from p_2
  - deliver M to p_2 from p_1
  - p_0 does nothing since its *color* variable is already green
  - p_1 does nothing since its *color* variable is already green
  - p_2 does nothing since its *color* variable is already green

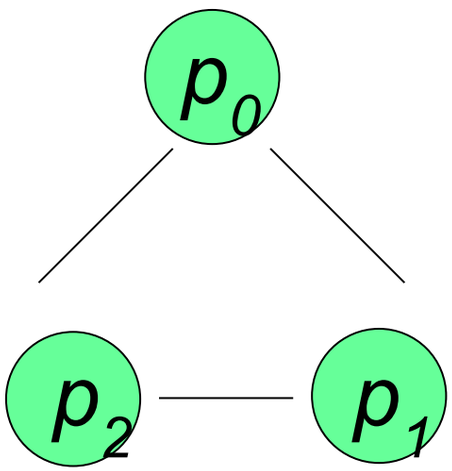
Example of Synchronous Model



round 1 events



round 2 events



Finding a Spanning Tree Given a Root

- Having a spanning tree is very convenient.
- How do you get one?
- Suppose a distinguished processor is known, to serve as the root.

Finding a Spanning Tree Given a Root : Algorithm description

- root sends M to all its neighbors

1. when non-root process *first* gets M
   - set the sender as its parent
   - send "parent" msg to sender
   - send M to all other neighbors (if no other neighbors, then terminate)
2. when get M otherwise
   - send ”already" msg to sender

- Note : it is possible that a process receives M for the first time from several processes (all these messages are in the inbufs). In this case pick one and execute 1. Execute 2. for the other messages.

Finding a Spanning Tree Given a Root : Algorithm description

- Each process pi manages three local variables :
  - *parent* is used to store the process that will be the parent of p_i in the spanning tree. It is initially equal to $\perp$
  - *children* is used to store the processes that will be the children of pi in the spanning tree. Initially it is equal to the empty set.
  - *other* to store the other processes. Initially it is equal to the empty set.
- "terminate" (for example in line 14) indicates that the process pi has entered terminated state.
  - use "parent" and "already" msgs to set children variables and know when to terminate (after hearing from all neighbors)

Finding a Spanning Tree Given a Root : Algorithm pseudo-code for processor p_i


```
1:  upon receiving no message:
2:      if p_i = p_r and parent = ⊥ then          // root has not yet sent ⟨M⟩
3:          send ⟨M⟩ to all neighbors
4:          parent := p_i

5:  upon receiving ⟨M⟩ from neighbor p_j:
6:      if parent = ⊥ then                        // p_i has not received ⟨M⟩ before
7:          parent := p_j
8:          send ⟨parent⟩ to p_j
9:          send ⟨M⟩ to all neighbors except p_j
10:     else send ⟨already⟩ to p_j

11: upon receiving ⟨parent⟩ from neighbor p_j:
12:     add p_j to children
13:     if children ∪ other contains all neighbors except parent then
14:         terminate

15: upon receiving ⟨already⟩ from neighbor p_j:
16:     add p_j to other
17:     if children ∪ other contains all neighbors except parent then
18:         terminate
```