The Tower of Hanoi

20cp

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This project complies with the guidelines stated in the Project Handbook [1].

Acknowledgement of Sources

For all ideas taken from other sources (books, articles, internet), the source of the ideas is mentioned in the main text and fully referenced at the end of the report.

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Date 04/28/2024

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1 Introduction

The Tower of Hanoi is a well known puzzle that has captivated the interest of mathematicians, computer scientists, and the like for many years. Despite its deceptively simple appearance, the puzzle contains many concealed ideas and scope for complication, leading to a wealth of unproven conjectures in the field. The standard variant contains 3 columns and has a trivial solution, but the optimal solution for the 4 column variant was only discovered in 2014, and still no satisfactory proofs for solutions to 5 columns or more exist. The simplicity of the rules, combined with the elegance of the solutions discovered thus far, renders the Tower of Hanoi puzzles particularly captivating to those engaged.

The puzzle was invented by a French mathematician by the name of Édouard Lucas in 1883, and was published posthumously as *"La Tour d'Hanoï"* in one of four of his *Récréations mathématiques* [2]. Unsurprisingly, Lucas was well regarded for his work in number theory and recreational maths, fields that encompass the problems within The Tower of Hanoi. An interesting aside, Lucas marketed the puzzle under the pseudonym N. Claus de Siam, an anagram of Lucas d'Amiens, referring to his hometown. The puzzle has appeared in mainstream media such as *Doctor Who* [3] and *Star Wars* [4], and is often used as an educational tool for inquisitive individuals [5].

In addition to its appearances in popular culture, there exist many myths surrounding the Tower of Hanoi; which although irrelevant to the maths, may provide motivation to many. The myths appear in many forms; a summary of the main variations would approximate to the following [6]:

There exists a temple, inside which one can find 3 brass columns. Upon one of the columns lies 64 golden discs, stacked in order of decreasing radius such that the largest disc lies at the bottom and the smallest at the top (see Figure 1). The priests who happen to inhabit the temple are able to move the discs, so long as:

1. Only one disc is moved at a time
2. A disc is only be moved from the top of one column to the top of a different column
3. A larger disc is never be placed on a smaller disc.

These are the rules to the problem, which will be referred to simply as "the rules". We also note that an "authorised configuration", which we will also refer to as a "configuration", is any arrangement of discs such that no larger disc is found atop a smaller disc.

The puzzle is complete when the priests are able to recreate the stack of discs on a column different to that which they started on, without violating the rules. According to legend, completion of the puzzle will result in the end of the world [2].

This lays the groundwork for our standard problem, which we will now formally define:

Remark 1.1 (The Standard Tower of Hanoi Problem)**.** Let there be N discs each of different radius in our puzzle. Let there exist 3 columns, such that one of the columns contains the N discs stacked in order of decreasing radius with the largest at the bottom and the smallest at the top, as in Figure 1. Without deviating from the rules, what is the minimum number of moves required to recreate the stack on a different column?

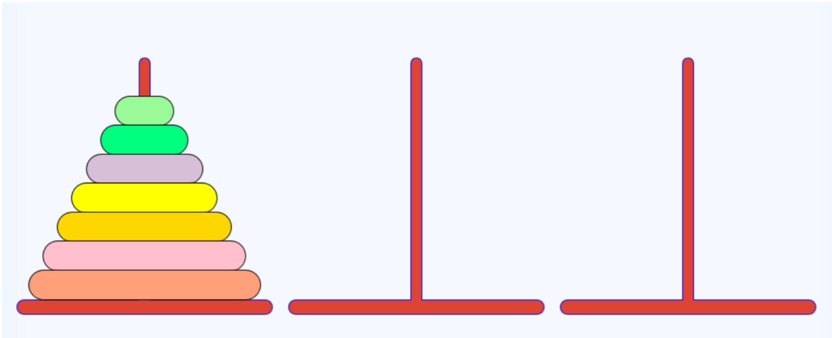


Figure 1: Starting configuration in the Standard Tower of Hanoi Problem for $N = 7$ [7].

Despite the standard problem's underlying complexity, the setup of the problem and the rules pave the way for a magnitude of variants and further problems. For example, what is the average number of moves to get from any authorised configuration to another? What do diagrams of paths look like? What if discs can only move to the column adjacent to their current location, or if we change/add other rules? What happens as the number of columns p grows larger than 3? Excluding the last, these questions have all been shown to have solutions, and demonstrate the possibility of variants as well as the intricacy of the problem.

The first prominent statement of a variant involving more than 3 columns was presented by Henry Dudeney in his 1907 publication *The Canterbury Puzzles* [8].The problem, which he named *The Reve's Puzzle*, operates under the same rules as our standard problem, but has an additional fourth column. As well as stating the problem, Dudeney proposes a solution which shall be discussed later.

Regarding the general problem of $p \geqslant 4$, a proof of optimality for a $p \geqslant 4$ column solution has not yet been found. Despite this, there exists a well known conjecture that claims to do so: the Frame-Stewart Conjecture [9]. The conjecture has been verified for $p = 4$, and is widely believed to be true for $p = 5$ by extensive computer calculations [10], but has not been rigorously demonstrated to be true despite several recent attempts.

The Tower of Hanoi demonstrates recursive properties relative to the number of discs, leading to the emergence of interesting patterns and also facilitating the application of inductive and recursive solutions. As a result, the problem is of great interest to computer scientists due to the potential for designing complex algorithms, though further investigation is beyond the mathematical scope of this paper.

The aim of this paper is to highlight several interesting results arising from the Tower of Hanoi, and provide a detailed exposition of Bousch's remarkable 2014 paper [11]. We will first analyse the solution to the trivial 3 column case and discuss some compelling results following, specifically those related to the famous *Hanoi Graphs*. We will then discuss the Frame-Stewart Conjecture, though much of the focus will be directed towards expositing Thierry Bousch's demonstration of the conjecture for *the Reve's Puzzle*, also referred to as *the Dudeney Problem*.

2 The Standard Tower of Hanoi Problem with three columns

We start our investigation with the standard and quintessential 3 column problem. Although the solution is somewhat trivial, it lays a useful foundation for our approach to further problems; the result also being essential to Bousch's 4 column solution. Note that although extensive notation will not be required for the 3 column problem, we will prescribe detailed notation to configurations, paths, and moments when discussing the 4 column variant. For now, let the 3 columns be called 0, 1, and 2.

2.1 Solution to The Standard Tower of Hanoi Problem with three columns

Let the problem be as stated in Remark 1.1, with all discs on column 0 at the start. Assume we are required to move all discs to column 2, whilst respecting the rules. It should be noted that by symmetry, the solution is the same for any start and end column, since there are no restrictions as to which column a disc can move based on its current location.

A recurring idea through all problems in the Tower of Hanoi is to identify critical configurations in the path from the start configuration to the end. In the standard problem, for any number of discs N, we must reach a configuration such that:

- The largest disc is on column 0 (the start column).
- All $N-1$ other (smaller) discs are on column 1.
- Column 2 is empty.

This configuration is depicted in the following diagram:

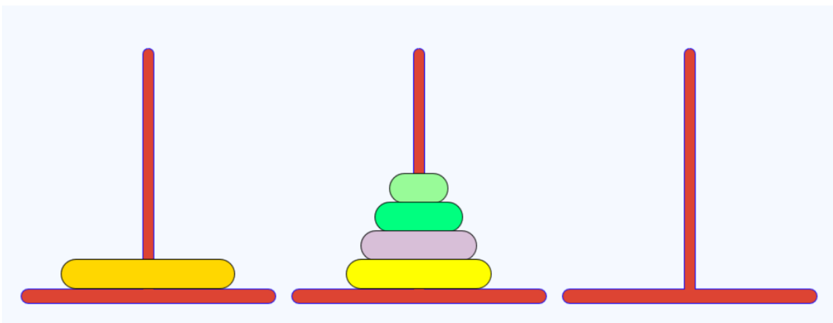


Figure 2: Critical configuration for $N = 5$ in the Standard Tower of Hanoi [7].

This configuration is critical for two reasons: we will reach a corresponding configuration for any N number of discs, and we are able to solve recursively about this position. Let T_N denote the minimal number of moves required to solve the standard problem for N discs.

If we have N discs in our problem, it will intuitively take T_{N-1} moves to such a state, as by symmetry it is the same problem but for $N-1$ discs. Once in this state, it is clear that we must move the largest disc to column 2, and then repeat the T_{N-1} movements of the discs on column 1 to column 2. It will therefore take

$$T_N = 2 \cdot T_{N-1} + 1$$

moves to solve the problem. We set our base case $T_1 = 1$ from the straightforward fact that it would take 1 move to move 1 disc to another column. We can solve this recursion to obtain our solution:

$$T_N = 2^N - 1. \tag{1}$$

Referring back to the myth, this demonstrates that 64 discs would require

$$2^{64} - 1 = 18446744073709551615$$

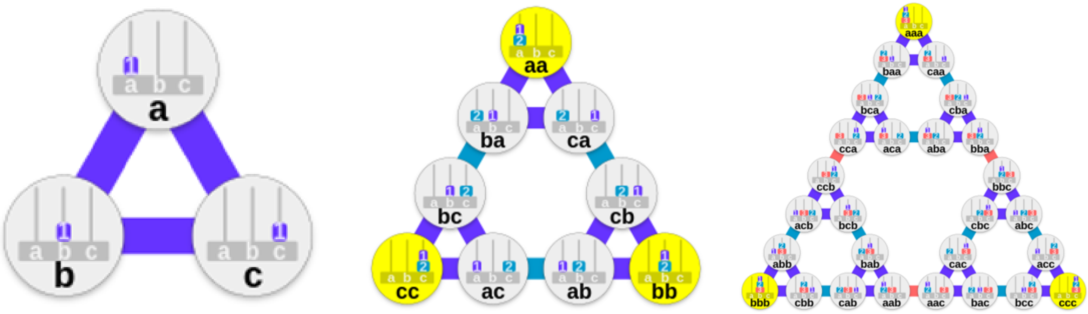
moves, such that if one move was made every second it would take 585 billion years to solve the puzzle (42 times the current age of the universe).

2.2 Other Results in The Standard Tower of Hanoi

There are many other interesting results which arise from the setup of the problem. In the following section, we will examine the outcomes of mapping the puzzle's configurations onto a graph, and uncover subsequent questions that arise as a result.

2.2.1 Tower of Hanoi Graphs

We can make several interesting observations by representing configurations of the Tower of Hanoi diagrammatically. By representing each configuration of the puzzle as a node and joining those which are one move away from each other by an edge, we observe some satisfying patterns. It is worth noting that each diagram will have 3^N nodes, since every disc has a choice of 3 columns where the order is fixed. As an aside, all such graphs will have $\frac{3(3^N-1)}{2}$ edges, the proof for which can be found in [12]. The diagrams, often referred to as *Hanoi Graphs*, appear as follows:



(a) N=1 (b) N=2 (c) N=3

Figure 3: Tower of Hanoi graphs for $N = 1, 2, 3$ [13].

As N grows larger, a pattern grows clearer - the *Hanoi Graphs* tend towards the shape of the Sierpiński triangle, which is more evident on the graph for $N = 7$:

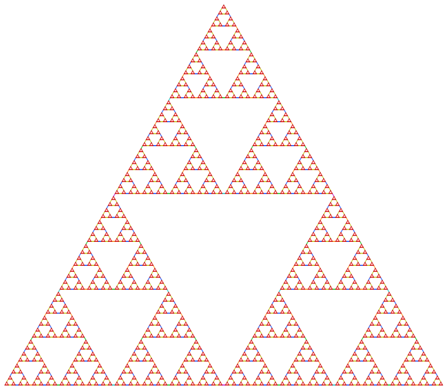


Figure 4: Tower of Hanoi graph for $N = 7$ [14].

We can make various deductions from the shape of such graphs, which shall be stated without proof. Should the reader wish to verify these, the proofs can be found in [6]. First, we see that from any arbitrary position to an end position, there is a unique shortest route. We also notice, however, that from any arbitrary position to another arbitrary position, there is exactly one or two shortest routes.

2.2.2 Average Distance from an Arbitrary State to Another

In 1990, Hinz and Schief made a particularly notable discovery; they were able to compute the average distance between any two arbitrary states [15]. The proof is generally based on the fact that the largest disc will move either exactly once or exactly twice. The result for N discs, which shall be stated without proof, is as follows:

$$\frac{466}{885}\cdot 2^N - \frac{1}{3} - \frac{3}{5}\cdot\left(\frac{1}{3}\right)^N + \left(\frac{12}{59} + \frac{18}{1003}\sqrt{17}\right)\left(\frac{5+\sqrt{17}}{18}\right)^N + \left(\frac{12}{59} - \frac{18}{1003}\sqrt{17}\right)\left(\frac{5-\sqrt{17}}{18}\right)^N.$$

Should the reader wish to verify this result, the proof can be found in [15].

We can see that as $N \to \infty$, we are left with

$$\frac{466}{855}\cdot 2^N - \frac{1}{3} + o(1),$$

where $o(1)$ denotes powers of fractions smaller than 1 that become negligible compared to the exponentially growing term 2^N.

This result yields an interesting property - we can interpret the fraction $\frac{466}{885} \approx 52.6\%$ to be the fraction of the full solution required.

2.2.3 Open Problems and Conjectures

Such results provide valuable insight into the nature of the problem, and it is hoped that they provide further motivation to investigate other topics within the Tower of Hanoi. It is worth mentioning that several open conjectures, which may be found in [6] and [16], exist; however, the detailed statement of all such conjectures falls beyond the scope of this paper.

Nevertheless, we will discuss the most notable of these results, the Frame-Stewart Conjecture [9].

3 The Frame-Stewart Conjecture

The Frame-Stewart Conjecture is arguably the most exciting result within the field; a proposed solution to the optimal number of moves for $p \geqslant 3$ pegs with N discs. The conjecture remains very difficult to prove: Donald Knuth, a renowned computer scientist, said "I doubt if anyone will ever resolve the conjecture; it is truly difficult" [17]. The complexity arises from the rapid growth in critical configurations as p increases; the addition of pegs significantly increases the number of possible paths from start to end. The conjecture was proven in 2014 by Bousch [11] for $p = 4$. Defined recursively, the Frame-Stewart Conjecture is as follows.

Remark 3.1 (Frame-Stewart Conjecture)**.** Let $FS(N, p)$ denote the optimal number of moves to solve the Tower of Hanoi with N discs and $p \geqslant 3$ columns. We have

$$FS(N,p) = \min_{1 \leqslant M < N} \Big[2FS(M,p) + FS(N-M, p-1) \Big],$$

where

$$FS(N,3) = 2^N - 1 \text{ and } FS(1,p) = 1.$$

The algorithm states that in order to move all discs to a new column, we must first move M discs to an intermediate column, move the remaining $N - M$ discs to the destination column, after which we again move the same M discs to the destination column. During the move of the $N - M$ discs to the destination column, a column is taken up by the M discs, meaning that we have $p - 1$ columns to do so. We must also determine in each case the optimal M.

In [18], Szegedy shows

$$FS(N,p) \geq 2^{(1 \pm o(1)) c_p N^{1/(p-2)}}$$

where $c_p = \frac{1}{2}\left(\frac{12}{p(p-1)}\right)^{1/(p-2)}$.

This is an important result, as it is the first attempt at such a strategy for computing the lower bound of the solution. We can demonstrate the weakness of the expression by comparing it to our known result when $p = 4$.

Using Bousch's results [11], we have as an approximation of the solution for the $p = 4$ problem with large N:

$$\log_2 FS(N,4) \approx \sqrt{2N}.$$

However, applying $p = 4$ to Szegedy's lower bound, we obtain

$$\log_2 FS(N, 4) \geqslant (\mathrm{o}(1)) \cdot \frac{1}{2} N^{\frac{1}{2}} \sim \frac{1}{2}\sqrt{N}.$$

Comparing the results, our approximation of Bousch's solution for $FS(N, 4)$ demonstrates a dominating exponent of $\sqrt{2N}$, whereas Szegedy's lower bound predicts $\frac{1}{2}\sqrt{N}$. The two expressions exhibit a discrepancy in their growth rates, differing approximately by a factor of $2\sqrt{2}$ in the exponent, showing a substantial incorrectness in Szegedy's expression.

Chen and Shen [19] later sharpened the expression, showing that the c_p constant can be increased such that

$$\log_2 FS(N, p) \geqslant (1 \pm \mathrm{o}(1))(N \cdot (p-2)!)^{\frac{1}{p-2}},$$

which for $p = 4$ has,

$$\log_2 FS(N, 4) \geqslant (o(1))(2N)^{\frac{1}{2}} \sim \sqrt{2N}.$$

This is a far stronger bound for $p = 4$; in fact, it matches asymptotically the exponent present in the Frame-Stewart conjecture for all values of p.

3.1 Bousch's Proof & the Frame-Stewart Conjecture for *The Reve's Puzzle*

In 2014 Bousch made a significant contribution - he proved that the Frame-Stewart conjecture holds for the 4 column problem (*The Reve's Puzzle*) [11]. In [8], Dudeney shows (for certain values of N) that the puzzle can be solved in $\Phi(N)$ moves, where

$$\Phi(N) = 2^{\nabla 0} + 2^{\nabla 1} + \ldots + 2^{\nabla (N-1)},$$

and ∇n is the largest p such that $\frac{p(p+1)}{2} \leqslant n$.

We will later show that this is equivalent to

$$\Phi(N) = \min_{0 \leqslant M < N} 2\Phi(M) + 2^{N-M} - 1,$$

which satisfies the Frame-Stewart Conjecture.

Therefore, for $N > 0$ (assume solved for lower values), fixing M such that $M < N$, we have

$$\Phi(N) = 2\Phi(M) + 2^{N-M} - 1.$$

We therefore know that it is possible to transfer discs from one column to another in $\Phi(N)$ moves, though we have not yet shown that it is optimal. If we are able to show that this solution is optimal, we have then verified the Frame-Stewart Conjecture in the case $p = 4$. We will display these results in the following section.

4 Thierry Bousch's Solution to *the Reve's Puzzle* [11]

4.1 Overview

In this section, we will prove that $\Phi(N)$ is the optimal solution to *the Reve's Puzzle* by exposition of Bousch's 2014 paper [11]. To do so, we will define an intermediate function Ψ, and show for suitably separated cases that the number of moves required in each case is greater than our Ψ function, and therefore greater than $\Phi(N)$. By showing true for each case, we can be certain that there is no further optimal solution.

Remark 4.1 (*The Reve's Puzzle*)**.** Let there be N discs each of different radius in our puzzle. Let there exist 4 columns, such that one of the columns contains the N discs stacked in order of decreasing radius with the largest at the bottom and the smallest at the top, as in Figure 5. Without deviating from the rules, what is the minimum number of moves required to recreate the stack on a different column?

The proof will take the following shape:

- In Section 4.2.1, we will define suitable notation for the problem.
- In Section 4.2.2 we will prove any lemmas required for the proof, after which we state the main result we are trying to prove in Section 4.2.4.
- In Section 4.4, we show that the Theorem is true for all cases of the problem. The resulting lower bound is then stated in Section 4.5.

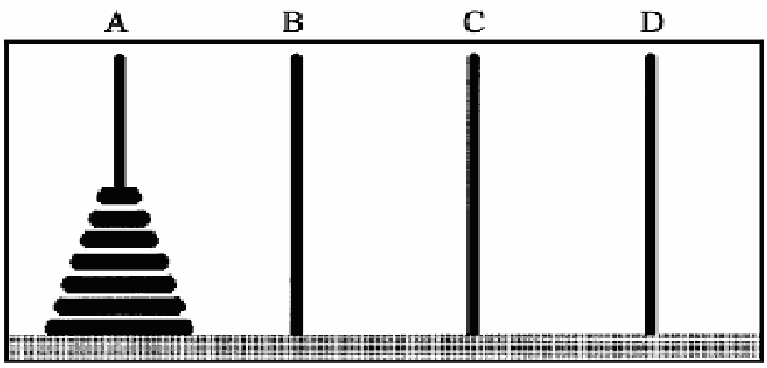


Figure 5: Starting Position for $N = 7$ in *the Reve's Puzzle.* [20]

4.2 Definitions, Lemmas, and the Recurrence Hypothesis

4.2.1 Definitions

- Let $\mathbb{N}$ be the set $\{0, 1, 2, 3, \ldots\}$.
- Let $[n]$ denote the set $\{0, 1, 2, \ldots, n-1\}$. Note $[0] = \emptyset$.
- Let $\Delta n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.
- For $n \in \mathbb{N}$, we denote ∇n as the triangular root, that is, the largest $p \in \mathbb{N}$ such that $\Delta p \leq n$.
- To demonstrate:

n	0	1	2	3	4	5	6	7	8	9	10
Δn	0	1	3	6	10	15	21	28	36	45	55
∇n	0	1	1	2	2	2	3	3	3	3	4

- The Δ and ∇ functions have several properties which we will use:
  - Both Δ and ∇ are increasing functions in $\mathbb{N}$
  - $\nabla(\Delta n) = n$
  - $\Delta(\nabla n) \leqslant n$
  - $M > N \implies \nabla M \geqslant \nabla N$
  - $M > N \implies \Delta M > \Delta N$.

Definition 4.2. Let $\Phi : \mathbb{N} \to \mathbb{N}$ such that:

$$\begin{aligned}\Phi(N) &= 2^{\nabla 0} + 2^{\nabla 1} + 2^{\nabla 2} + \ldots + 2^{\nabla(N-1)} \\ &= \sum_{i=0}^{N-1} 2^{\nabla i}.\end{aligned}$$

This can also be expressed as:

$$\Phi(N) = \min_{0 \leqslant M < N} 2\Phi(M) + 2^{(N-M)} - 1 \quad (N \geqslant 1).$$

It follows that:

$$\forall a, b \in \mathbb{N},\ \Phi(a+b) \leq 2\Phi(a) + 2^b - 1.$$

Throughout the proof we will refer back to several results and lemmas. One such result is the optimal solution to the 3 column problem with N discs, which we now know from (1) to be $2^N - 1$.

4.2.2 Lemmas and the Ψ Function

We now define several lemmas and functions which we will find to be useful throughout the proof.

Lemma 4.3. *For $n, p \in \mathbb{N}$ where $p \leqslant n+1$, we have*

$$\Phi(\Delta n + p) = 1 + (n + p - 1)2^n \tag{2}$$

Proof. We have by definition $\Phi(N) = \sum_{M=0}^{N-1} 2^{\nabla M}$. Writing N as $\Delta n + p$, M as $\nabla m + r$, and grouping terms with common m, we obtain:

$$\Phi(\Delta n + p) = \sum_{m=0}^{n-1} \left[(m+1)2^m\right] + p2^n.$$

To compute this sum, we can say

$$\sum_{m=0}^{n-1} [(m+1)x^m] = \frac{d}{dx} \sum_{m=0}^{n-1} x^{m+1} = \frac{d}{dx} \sum_{m=1}^{n} x^m,$$

and therefore,

$$\begin{aligned} \sum_{m=0}^{n-1} [(m+1)x^m] &= \frac{d}{dx} \left(\frac{1 - x^{n+1}}{1 - x} \right) \\ &= \frac{-nx^n - x^n + nx^{n+1} + 1}{(1-x)^2}. \end{aligned}$$

By substituting $x = 2$ and simplifying, we arrive at

$$\sum_{m=0}^{n-1} [(m+1)x^m] = 1 + 2^n(n-1),$$

and so

$$\begin{aligned} \Phi(\Delta n + p) &= 1 + 2^n(n-1) + p2^n \\ &= 1 + (n + p - 1)2^n. \end{aligned}$$

$\square$

We now define the Ψ function, which will be integral to the proof.

Definition 4.4. Let Ψ be a function such that

$$\Psi : \mathcal{P}_{\text{fin}}(\mathbb{N}) \to \mathbb{N},$$

where $\mathcal{P}_{\text{fin}}(\mathbb{N})$ denotes the set of finite subsets of $\mathbb{N}$.

We define for all $L \in \mathbb{N}$,

$$\Psi_L(E) = (1-L)2^L - 1 + \sum_{n \in E} 2^{\min(\nabla n, L)},$$

where $E \subset \mathbb{N}$.

We also define

$$\Psi(E) = \operatorname*{Sup}_{L \in \mathbb{N}} \Psi_L(E).$$

The Ψ function has several useful properties:

- Since $\Psi_L(E) \to -\infty$ as $L \to \infty$, we can see that the upper bound is a maximum.
- $\Psi(E)$ is an integer for all $E \subset \mathbb{N}$.
- $\Psi(E)$ is definitely bounded below by $\Psi(\emptyset) = 0$.
- $\Psi(E)$ is an increasing function w.r.t inclusion.

Lemma 4.5. *For all $n \in \mathbb{N}$, we have:*

$$\Psi[n] = \frac{\Phi(n+1) - 1}{2} = \frac{1}{2}(2^{\nabla 1} + 2^{\nabla 2} + \ldots + 2^{\nabla n}).$$

Proof. We have already shown that $\Psi[0] = 0$. Assume $n \geq 1$, and write $n = \Delta m + p$ with $m = \nabla n$, from which it follows that $m \geqslant 1$ and $0 \leq p \leq m$.
From Lemma 4.3, we have

$$\Phi(\Delta m) = 1 + (m-1)2^m$$

and

$$\Phi(n+1) = 1 + (m+p)2^m.$$

We start by computing a discrete derivative defined for all $L \in \mathbb{N}$:

$$\Psi_{L+1}[n] - \Psi_L[n] = -(L+1)2^L + \sum_{k \in [n]} 2^{\min(\Delta k, L+1)} - 2^{\min(\Delta k, L)}.$$

Since $2^{L+1} - 2^L = 2^L$, and counting the number of $\Delta k \geq L+1$, we can see that this is equivalent to

$$\Psi_{L+1}[n] - \Psi_L[n] = 2^L \left[\#\{k \in [n] : k \geq \Delta(L+1)\} - (L+1)\right];$$

and therefore,

$$\Psi_{L+1}[n] - \Psi_L[n] = 2^L \left[[n - \Delta(L+1)]^+ - (L+1)\right].$$

The expression is strictly positive for $n \geq \Delta(L+1) + L + 2 = \Delta(L+2)$, which is equivalent to $n \geq \Delta(L+2)$. Since $n = \Delta m + p$, we have that the expression reaches a maximum for L at:

$$\begin{aligned} n &= \Delta(L+1) \\ \nabla n &= \nabla(\Delta(L+1)) \\ m &= L+1. \end{aligned}$$

As such, we have by definition

$$\begin{aligned} \Psi[n] = \Psi_{m-1}[n] &= (2-m)2^{m-1} - 1 + \sum_{0 \leqslant k < \Delta m} 2^{\nabla k} + \sum_{\Delta m \leqslant k < n} 2^{m-1} \\ &= (2-m)2^{m-1} - 1 + \Phi(\Delta m) + (n - \Delta m)2^{m-1} \\ &= (2-m)2^{m-1} + (m-1)2^m + p2^{m-1} \\ &= (m+p)2^{m-1} = \frac{\Phi(n+1)-1}{2}, \end{aligned}$$

which concludes the proof. □

Consequently, we obtain the following inequality for any $a, b \in \mathbb{N}$:

$$\Psi[a+b] \leqslant 2\Psi[a] + 2^{b-1}. \tag{3}$$

Lemma 4.6. *For all $n \in \mathbb{N}$, we have that $\Psi[n+2] \geqslant 2^{(\nabla n)+1}$.*

Proof. Let $s = \nabla n$. We know that $\Psi[\cdot]$ is an increasing function, so it suffices to show that $\Psi[\Delta s + 2] \geqslant 2^{s+1}$, since $\Delta s \leqslant n$.

Since $\Psi[2] = 2$ and $\Psi[3] = 4$, we can see that the inequality holds (with equality) for $s = 0$ and $s = 1$.

For $s \geqslant 2$, we have from Lemmas 3.1 and 3.3:

$$\begin{aligned} \Psi[\Delta s + 2] &= \frac{\Phi(\Delta s + 3) - 1}{2} \\ &= \frac{1 + (s + 3 - 1)2^s - 1}{2} \\ &= \frac{(s+2)2^s}{2} = (s+2)2^{s-1} \geqslant 2^{s-1} = 2^{\nabla n + 1}. \end{aligned}$$

This concludes the proof. □

Lemma 4.7. *For all subsets E of $\mathbb{N}$, we have $n \leqslant \Psi[n] \leqslant \Psi(E) \leqslant 2^n - 1$, where $n = ||E||$.*

Proof. We will prove each inequality separately.

1. $\underline{n \leqslant \Psi[n]}$
   Since we know that the Ψ_L function is bounded below by $L = 0$, we can say that $\Psi[n] \geqslant \Psi_0[n] = n$ which proves the inequality.
2. $\underline{\Psi[n] \leqslant \Psi(E)}$
   Let $e_0 < e_1 < \ldots < e_{n-1}$ be the n elements of E. It is clear that $e_k \geqslant k \ \forall \ k \leqslant n - 1$. Evaluating $\Psi_L(E)$ we obtain

$$\begin{aligned} \Psi_L(E) &= (1-L)2^L - 1 + \sum_{0 \leqslant k < n} 2^{\min(\nabla e_k, L)} \\ &\geqslant (1-L)2^L - 1 + \sum_{0 \leqslant k < n} 2^{\min(\nabla k, L)} = \Psi_L[n]. \end{aligned}$$

3. $\underline{\Psi(E) \leqslant 2^n - 1}$
   We have

$$\begin{aligned} \Psi_L(E) &= (1-L)2^L - 1 + \sum_{k \in E} 2^{\min(\nabla k, L)} \\ &\leqslant (1-L)2^L - 1 + \sum_{k \in E} 2^L = (1+n-L)2^L - 1. \end{aligned}$$

   We know that for any $s \in \mathbb{N}, 2^s \geqslant 1 + s$. Let $s = n - L$. We now have

$$\begin{aligned} (1+n-L)2^L - 1 = (1+s)2^L - 1 &\leqslant 2^s 2L - 1 \\ &= 2^{s+L} - 1 = 2^n - 1, \end{aligned}$$

   so,

$$\begin{aligned} \Psi_L(E) \leqslant (1+n-L)2^L - 1 &\leqslant 2^n - 1 \\ \implies \Psi(E) &\leqslant 2^n - 1. \end{aligned}$$

Since all 3 inequalities hold, the lemma is proved. □

Lemma 4.8. *Let A and B be finite subsets of $\mathbb{N}$. We have $\Psi(A) - \Psi(B) \leqslant \sum_{k \in A-B} 2^{\nabla k}$.*

Proof. We define $L \in \mathbb{N}$ such that $\Psi_L(A) = \Psi(A)$. We can say

$$\begin{aligned} \Psi(A) - \Psi(B) \leqslant \Psi_L(A) - \Psi_L(B) &\leqslant \Psi_L(A) - \Psi_L(A \ \cap \ B) \\ &= \sum_{k \in A-B} 2^{\min(\nabla k, L)} \leqslant \sum_{k \in A-B} 2^{\nabla k} \\ \implies \Psi(A) - \Psi(B) &\leqslant \sum_{k \in A-B} 2^{\nabla k}. \end{aligned}$$

The lemma is proved. □

Lemma 4.9. *Let A be a subset of $\mathbb{N}$, and s an element of $\mathbb{N}$ such that $A-[\Delta s]$ contains at most s elements. We have $\Psi(A)-\Psi(A-\{a\}) \leqslant 2^{s-1}\ \forall\ a \in A$.*

Proof. Suppose that $A \neq \emptyset$ and that $s \geqslant 1$, then we can say that for all $s-1 \leqslant L$ we have

$$\begin{aligned}\Psi_{L+1}(A)-\Psi_L(A) &= 2^L[\#\{n \in A : n \geqslant \Delta(L+1)\}-(L+1)] \\ &\leqslant 2^L[\#\{n \in A : n \geqslant \Delta s\}-s] \leqslant 0.\end{aligned}$$

This follows from the fact that the set $A-[\Delta s]$ is equivalent to $\{n \in A : n \geqslant \Delta\ s\}$, which we know has at most s elements. From this, we can see that the function is decreasing as L increases from $s-1$. Therefore, there exists an $L \leqslant s-1$ such that $\Psi_L(A)=\Psi(A)$. As such it is true that,

$$\Psi(A)-\Psi(A-\{a\}) \leqslant \Psi_L(A)-\Psi_L(A-\{a\}).$$

Computing the RHS, we see from the definition of $\Psi_L(E)$ that:

$$\begin{aligned}\Psi_L(A)-\Psi_L(A-\{a\}) &= \Big((1-L)2^L-1+\sum_{n\in A}2^{\min(\nabla n,L)}\Big)-\Big((1-L)2^L-1+\sum_{n\in A-\{a\}}2^{\min(\nabla n,L)}\Big) \\ &= \sum_{n\in A}2^{\min(\nabla n,L)}-\sum_{n\in A-\{a\}}2^{\min(\nabla n,L)} \\ &= 2^{\min(\nabla a,L)} \leqslant 2^L \leqslant 2^{s-1}\end{aligned}$$

$$\implies \Psi(A)-\Psi(A-\{a\}) \leqslant 2^{s-1}.$$

The lemma is proved. □

Lemma 4.10. *Let n and s be natural numbers such that $s \geqslant 1$ and $n \geqslant \Delta(s-1)$. Let A be a subset of $[n]$. We have $\Psi(A\cup\{b_1,b_2,\dots b_s\})-\Psi(A) \leqslant \Psi[n+s]-\Psi[n]$, where $b_1,b_2,\dots,b_s \in \mathbb{N}$.*

Proof. Let $A_t = A\cup\{b_1,\dots,b_t\}$ for $0 \leqslant t \leqslant s$.

To prove the lemma, it is sufficient to prove

$$\Psi(A_t)-\Psi(A_{t-1}) \leqslant \Psi[n+t]-\Psi[n+t-1]. \tag{4}$$

Computing the RHS of (4) we see

$$\begin{aligned}\Psi[n+t]-\Psi[n+t-1] &= \frac{\Phi(n+t+1)-1}{2}-\frac{\Phi(n+t)-1}{2} \\ &= \frac{\Phi(n+t+1)-\Phi(n+t)}{2} \\ &= \frac{2^{\nabla(n+t)}}{2} \\ &= 2^{\nabla(n+t)-1}.\end{aligned}$$

We can now rewrite (4) as

$$\Psi(A_t)-\Psi(A_{t-1}) \leqslant 2^{\sigma-1},$$

where $\sigma = \nabla(n+t)$.

From the previous lemma, we know that (4) holds if the cardinality of $A - [\Delta\sigma]$ is at most σ.

Note that $\Delta\sigma \leqslant n+t$ and $\Delta(\sigma+1) > n+t$, so $\Delta\sigma + \sigma \geqslant n+t$.

From our definition of A_t, we can provide an upper bound to the limit:

$$\#\left(A_t - [\Delta\sigma]\right) \leqslant t + \#([n] - [\Delta\sigma]) = t + (n - \Delta\sigma)^+.$$

We only take values $n - \Delta\sigma \geqslant 0$, since we are referring to exclusion from a set. Therefore, this can be rewritten as

$$t + \#([n] - [\Delta\sigma]) = t + (n-\Delta\sigma)^+ = \max(t, t+n-\Delta\sigma).$$

Since $\Delta(\sigma+1) > n+t$, it follows that

$$\max(t, t+n-\Delta\sigma) \leqslant \max(t,\sigma).$$

Consequently, proving $t \leqslant \sigma$ will prove the lemma. We have:

$$\begin{aligned}
\Delta t - t = \Delta(t-1) &\leqslant \Delta(s-1) \leqslant n \\
\Delta t &\leqslant n+t \\
t = \nabla(\Delta t) &\leqslant \nabla(n+t) = \sigma.
\end{aligned}$$

The lemma is proved. ∎

Lemma 4.11. *Let A and B be subsets of $\mathbb{N}$. We have*

$$\begin{aligned}
\Psi(A) + \Psi(B) \geqslant \frac{\Phi(n+3)-5}{4} &= \frac{1}{2}\Psi[n+2] - 1 \\
&= \frac{1}{4}(2^{\nabla 3} + 2^{\nabla 4} + \ldots + 2^{\nabla(n+2)}),
\end{aligned}$$

where n is the cardinality of $A \cup B$.

Proof. Let $E = A \cup B$, and L be any natural number. We have from Lemma 4.7 that $\Psi_L(E) \geqslant \Psi_L[n]$, where n is the cardinality of E. As such, we can say

$$\Psi(A) + \Psi(B) \geqslant \Psi_L(A) + \Psi_L(B) = \Psi_L(A\cap B) + \Psi_L(A \cup B)$$

from the inclusion-exclusion principle.

Further, we have

$$\Psi_L(A\cap B) + \Psi_L(A\cup B) \geqslant \Psi_L(\varnothing) + \Psi_L(E) \geqslant \Psi_L[0] + \Psi_L[n].$$

We now write $n+3 = \Delta m + p$, where $m = \nabla(n+3)$, such that $m \geqslant 2$ (this follows from

$n+3 \geqslant 3 \; \forall \; n \in \mathbb{N}$), and $0 \leqslant p \leqslant m$. Since $m \geqslant 2$ and $n = \Delta m + p - 3$, we observe:

$$\begin{aligned} 2m + p &\geqslant 4 \\ 4m + 2p &\geqslant 8 \\ m + 2p - 6 &\geqslant -3m + 2 \\ m^2 + m + 2p - 6 &\geqslant m^2 - 3m + 2 \\ 2\Big[\frac{m(m+1)}{2} + p - 3\Big] &\geqslant 2\Big[\frac{(m-2)(m-1)}{2}\Big] \\ \Delta m + p - 3 &\geqslant \Delta(m-2) \\ \implies n &\geqslant \Delta(m-2). \end{aligned}$$

We also note directly from Lemma 4.3:

$$\begin{aligned} \Phi(n+3) &= 1 + (m+p-1)2^m \\ \Phi(\Delta(m-2)) &= 1 + (m-3)2^{m-2}. \end{aligned}$$

We now let $L = m-2$. From our definition of Ψ_L,

$$\Psi_L[0] + \Psi_L[n] = (1-L)2^{L+1} - 2 + \sum_{0 \leqslant k < n} 2^{\min(\nabla k, L)},$$

and substituting $L = m-2$ we reach

$$= (3-m)2^{m-1} - 2 + \sum_{0 \leqslant k \leqslant n} 2^{\min(\nabla k, m-2)}.$$

The point at which ∇k grows larger than $m-2$ is $k = \Delta(m-2)$. By separating these cases, we obtain

$$= (3-m)2^{m-1} - 2 + \sum_{0 \leqslant k < \Delta(m-2)} 2^{\nabla k} + \sum_{\Delta(m-2) \leqslant k < n} 2^{m-2}.$$

After inserting our definition of Φ and counting the elements in the second sum, we have

$$= (3-m)2^{m-1} - 2 + \Phi\big(\Delta(m-2)\big) + \big(n - \Delta(m-2)\big)2^{m-2}. \quad (*)$$

From Lemma 4.3, we observe $\Phi\big(\Delta(m-2)\big) = 1 + (m-1)2^{m-2}$.

We also note that $n = \Delta m + p - 3$, so

$$\begin{aligned} (n - \Delta(m-2))2^{m-2} &= (\Delta m - \Delta(m-2) + p - 3)2^{m-2} \\ &= (m + (m-1) + p - 3)2^{m-2} \\ &= 2m + p - 4. \end{aligned}$$

Substituting these relations into $(*)$ and simplifying, we arrive at

$$\begin{aligned} &= (3-m)2^{m-1} - 1 + (m-3)2^{m-2} + (p + 2m - 4)2^{m-2} \\ &= (m+p-1)2^{m-2} - 1 = \frac{\Phi(n+3) - 5}{4}. \end{aligned}$$

The lemma is proved. □

4.2.3 Defining Notation in *the Reve's Puzzle*

We will now assign notation to *the Reve's puzzle* with 4 columns.

- Let $\mathcal{C}$ be the set of columns. In the 4 column case, we have $\mathcal{C} = \{0, 1, 2, 3\}$.
- Let the number of discs be N, with the discs labelled 0 to $N-1$ in order of increasing radius (0 is the smallest disc, $N-1$ the largest). We will refer to the set of discs in the problem as $[N] = \{0, 1, \ldots, N-1\}$.
- We can describe a stack, or configuration, as a function from $[N] \to \mathcal{C}$. For example, the starting configuration for the 4 column problem with 5 discs would be $\{0, 0, 0, 0, 0\}$.
- We denote the distance (number of moves) between two stacks $\mathbf{u}$, $\mathbf{v} : [N] \to \mathcal{C}$, as $d(\mathbf{u}, \mathbf{v})$. Note that $d(\mathbf{m}, \mathbf{n}) = d(\mathbf{n}, \mathbf{m})$ for any configurations $\mathbf{m}$ and $\mathbf{n}$ since we can perform the same sequence of moves in reverse.

4.2.4 Recurrence Hypothesis

The main result of Bousch's paper is as follows.

Theorem 4.12. *Let $\mathcal{C} = \{0, 1, 2, 3\}$. Let N be a natural number, and $\mathbf{u}$, $\mathbf{v} : [N] \to \mathcal{C}$ be 2 authorised configurations of the discs. Assume that in stack $\mathbf{v}$, columns 0 and 1 are empty, that is to say $\mathbf{v}[N] \subseteq \{2, 3\}$. We then have*

$$d(\mathbf{u}, \mathbf{v}) \geqslant \Psi\{k \in [N] : \mathbf{u}(k) = 0\}. \tag{5}$$

The remainder of the proof will prove this theorem.

4.3 Preliminaries to the Proof

4.3.1 Critical Configurations

We fix the number N, and assume the result is true for any strictly smaller number of discs. Let $\mathbf{u}$ and $\mathbf{v}$ be two stacks of N discs, with $\mathbf{v}[N] \subseteq \{2,3\}$, meaning that in the state $\mathbf{v}$ all discs are on the final two columns. We define E as the set of discs on the 0^{th}column in the starting configuration $\mathbf{u}$, that is, $E = \{k \in [N] : \mathbf{u}(k) = 0\}$. We choose our N to be 1, which means we carry out the proof for $N \geqslant 1$ ($N = 0$ has a trivial solution of 0 moves).

We define $\mathbf{u}'$, $\mathbf{v}' : [N-1] \to \mathcal{C}$ as configurations equivalent to $\mathbf{u}$ and $\mathbf{v}$ but with the largest disc removed (they therefore have $N-1$ total discs). It is clear that the distance between the states $\mathbf{u}'$ and $\mathbf{v}'$ is less than that between $\mathbf{u}$ and $\mathbf{v}$, and by applying the recurrence hypothesis we see that

$$\begin{aligned} d(\mathbf{u},\mathbf{v}) \geqslant d(\mathbf{u}',\mathbf{v}') &\geqslant \Psi\{k \in [N-1] : \mathbf{u}'(k) = 0\} \\ &= \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} \\ &= \Psi(E - \{N-1\}). \end{aligned} \tag{6}$$

Note that disc $N-1$ is the largest and that $[N-1]$ refers to all discs up to, but not including, $N-1$.

The substitution of $\mathbf{u}$ for $\mathbf{u}'$ comes from the fact that Ψ is acting on $[N-1]$, which does not include $N-1$ itself, so $\mathbf{u}$ and $\mathbf{u}'$ are equivalent in this case. We then insert our definition of E.

If E does not contain disc $N-1$, it follows from (6) that $d(\mathbf{u},\mathbf{v}) \geqslant \Psi(E)$, which is the result we are trying to prove. The case $N-1 \notin E$ is therefore settled.

Now that we have settled the trivial case of the largest disc $N-1$ not being on the starting column, we will assume throughout the remainder of the proof that E contains $N-1$, that is, the column 0 contains the largest disc in the starting configuration.

It is now given that disc $N-1$ will start in column 0 (in configuration $\mathbf{u}$) and end in column 2 or 3 (in configuration $\mathbf{v}$). We will henceforth without loss of generality assume that in configuration $\mathbf{v}$ disc $N-1$ is on column 2, or $\mathbf{u}(N-1) = 0$ and $\mathbf{v}(N-1) = 2$.

In order to divide the puzzle into convenient sections, it is useful to assign notation to distances and their relative configurations.

Let $D = d(\mathbf{u},\mathbf{v})$ be the distance between $\mathbf{u}$ and $\mathbf{v}$. Let $\gamma : [D+1] \to \mathcal{C}^{[N]}$ be a geodesic path between $\mathbf{u}$ and $\mathbf{v}$ in the graph $\mathcal{C}^{[N]}$. This can be visualised by imagining a graph of every configuration, with each configurations that are one move apart joined by an edge: γ is the shortest path from $\mathbf{u}$ to $\mathbf{v}$. $\gamma(i)$ represents the configuration after move i, for example $\gamma(0) = \mathbf{u}$ and $\gamma(D) = \mathbf{v}$. As such, the distance between any two configurations on the geodesic is $d(\gamma(i),\gamma(j)) = |i-j| \ \forall\ i,j \in [D+1]$.

We will also use the notation $\gamma_i(k)$ to mean the column on which disc k is located after move i.

Let $E' = \{k \in E : \exists t \in [D+1] \gamma_t(k) = 3\}$ be the set of discs in E (that start on column 0 in $\mathbf{u}$) that enter column 3 at any time.

Consider the case where E' is empty. This means that the discs in E never pass through column 3, but still all end on column 2. In this case, we can treat the discs in E as following the rules of the 3 column problem, which tells us that $D \geqslant 2^{\#E} - 1$. However, from Lemma 4.7, we know that $\Psi E \leqslant 2^{\#E} - 1$, so it is clear that the inequality $D \geqslant \Psi E$ is satisfied. This case is now settled.

Since we have settled the case where E' is empty, we can now throughout the proof assume that E' is not empty, and we name its largest element T. We also define the set $E'' = \{k \in E : k > T\}$ as all the elements in E that are larger than T, and denote its cardinality K. Since T is the largest element of E, it is clear that any greater element - and therefore any element of E'' - will never enter column 3. We define the elements of E'' as $b_1, b_2, \ldots, b_k$. We see that:

$$E \subseteq \underbrace{[T] \sqcup \{T\}}_{\text{contains } E'} \sqcup \underbrace{\{b_1, \ldots, b_K\}}_{E''} \subseteq [N], \tag{7}$$

from which, by counting the elements in the above relation, it follows that:

$$T + K + 1 \leqslant N. \tag{8}$$

This is a result of the facts:

- $\#[T] \leqslant T$
- $\#\{T\} = 1$
- $\#\{b_1, \ldots, b_K\} = K$.

Note that $\sqcup$ indicates a disjoint union.

4.3.2 Critical Moments

We define critical configurations based on the knowledge that:

- Both discs T and $N-1$ will start in column 0.
- T will at some point leave column 0 (t_0).
- By definition, T will at some point move to column 3 (t_1).
- $N-1$ will at some point leave column 0 (t_2).
- $N-1$ will at some point enter column 2, and not leave until $\mathbf{v}$ is reached (t_3+1).

We now formally define these moments in the path γ, which shall be useful throughout the proof. We start by defining the movements of T. Note that $\mathbf{x_i}$ refers to the movement of T to or from column i, and $\mathbf{z_i}$ refers to the movement of $N-1$ to or from column i. All moments t_i that are defined will be in the set $[D+1]$.

Let $t_0 = \min\{i : \gamma_i(T) \neq 0\}$ be the first instant that T leaves column 0. Since it is the first moment that T moves, we see that T must be in column 0 just before, or equivalently $\gamma_{t_0-1}(T) = 0$.

Let us call the configuration just before T moves for the first time $\mathbf{x_0}$, where $\mathbf{x_0} = \gamma(t_0-1)$. Note that at $\mathbf{x_0}$ we have T on column 0; and that both column 0 and another unspecified column contain no discs $< T$ - otherwise moving T would not be possible.

Let $t_1 = \min\{i : \gamma_i(T) = 3\}$ be the first moment disc T is found in column 3, so $\gamma_{t_1}(T) = 3$. We call the corresponding configuration $\mathbf{x_3} = \gamma(t_1)$, which will have T in column 3 but no disc $< T$, and another unspecified column contains no disc $< T$.

It is therefore true that

$$1 \leqslant t_0 \leqslant t_1 \leqslant D,$$

which comes from the fact that T definitely starts in column 0, so it cannot enter column 3 before it leaves column 0.

Let $t_2 = \min\{i : \gamma_{t_i}(N-1) \neq 0\}$ be the first moment disc $N-1$ is not in column 0. We define $\mathbf{z_0} = \gamma(t_2-1)$ as the moment just before $N-1$ leaves column 0; in this configuration we have $N-1$ on column 0 and another unspecified column empty - otherwise disc $N-1$ cannot move.

Let $t_3 = \max\{i : \gamma_{t_i}(N-1) \neq 2\}$ be the last instant where disc $N-1$ is not in column 2 (it will definitely enter at some point). We define $\mathbf{z_2} = \gamma(t_3+1)$ to be the configuration just *after* $N-1$ moves into column 2, in which there is one unspecified column empty, and the largest disc (only) in column 2.

As before, we let $\mathbf{x_i}'$ and $\mathbf{z_j}'$ denote $\mathbf{x_i}$ and $\mathbf{z_j}$ with disc $N-1$ removed. We also let $\mathbf{u}'', \mathbf{v}'', \mathbf{x_i}'', \mathbf{z_j}''$ be equivalent to $\mathbf{u}, \mathbf{v}, \mathbf{x_i}, \mathbf{z_j}$ respectively, but with all discs $\geqslant T$ removed, or equivalently only including discs in the set $[T]$.

Note that for any states $\mathbf{m}$ and $\mathbf{n}$,

$$d(\mathbf{m}'', \mathbf{n}'') \leq d(\mathbf{m}', \mathbf{n}') \leq d(\mathbf{m}, \mathbf{n});$$

an important property which shall be frequently used.
To summarise:

Notation	**Definition**	**Explanation**
t_0	$\min\{t : \gamma_t(T) \neq 0\}$	First moment disc T leaves column 0.
t_1	$\min\{t : \gamma_t(T) = 3\}$	First moment disc T is found in column 3.
t_2	$\min\{t : \gamma_t(N-1) \neq 0\}$	First moment disc $N-1$ is not in column 0.
t_3	$\max\{t : \gamma_t(N-1) \neq 2\}$	Last moment where disc $N-1$ is not in column 2.
$\mathbf{x_0}$	$\gamma(t_0 - 1)$	Configuration just before T first leaves column 0.
$\mathbf{x_3}$	$\gamma(t_1)$	Configuration when T is first found in column 3.
$\mathbf{z_0}$	$\gamma(t_2 - 1)$	Configuration just before $N-1$ first leaves column 0.
$\mathbf{z_2}$	$\gamma(t_3 + 1)$	Configuration just after $N-1$ finally moves into column 2.
$\mathbf{u}'', \mathbf{v}''$	Equivalent to $\mathbf{u}, \mathbf{v}$ including only discs in the set $[T]$.	
$\mathbf{x_i}'', \mathbf{z_j}''$	Equivalent to $\mathbf{x_i}, \mathbf{z_j}$ including only discs in the set $[T]$.	

Table 1: Moments and Configurations in *the Reve's Puzzle*

Because such positions are generally defined and can take multiple appearances, diagrams will not be provided in order to prevent confusion.

Notation	Definition	Explanation
$\mathcal{C}$	Set of columns	$\mathcal{C}$ is defined as $\{0, 1, 2, 3\}$, for the 4 column problem.
N	Number of discs	Discs are labelled from 0 to $N-1$, where 0 is the smallest and $N-1$ the largest.
$\mathbf{u}$	Starting configuration	A function $\mathbf{u} : [N] \to \mathcal{C}$ mapping each disc to its starting column.
$\mathbf{v}$	Final configuration	A function $\mathbf{v} : [N] \to \mathcal{C}$ mapping each disc to its ending column, with all discs ending in columns 2 or 3.
$\mathbf{u}'$	$u : [N-1] \to \mathcal{C}$	Similar to $\mathbf{u}$, but with the largest disc $(N-1)$ removed from the configuration.
$\mathbf{v}'$	$v : [N-1] \to \mathcal{C}$	Similar to $\mathbf{v}$, but with the largest disc $(N-1)$ removed from the configuration.
D	$d(\mathbf{u}, \mathbf{v})$	The minimum number of moves required to transform configuration $\mathbf{u}$ into $\mathbf{v}$.
γ	$\gamma : [D+1] \to \mathcal{C}^{[N]}$	The geodesic path from $\mathbf{u}$ to $\mathbf{v}$. $\gamma(k)$ returns the configuration after the k^{th} move.
$\gamma_k(j)$	Position of disc j after k^{th} move	For example, $\gamma_0(N-1) = 0$ indicates that before any moves, the largest disc is in column 0.
E	$\{k \in [N] : \mathbf{u}(k) = 0\}$	The set of discs that start in column 0.
E'	$\{k \in E : \exists t \in [D+1], \gamma_t(k) = 3\}$	The set of discs starting in column 0 that enter column 3 at some point.
T	$\max(E')$	The largest disc that starts on column 0 and enters column 3 at some point.
E''	$\{k \in E : k > T\}$	The set of discs larger than T. Denoted by K elements: $b_1 < b_2 < \ldots < b_K$.

Table 2: Definitions and Notations for *the Reve's Puzzle*

Now that we have defined these moments, we can apply our recurrence hypothesis from (5) in Theorem 4.12:

$$\begin{aligned} d\left(\mathbf{u}, \mathbf{z}_0\right) \geqslant d\left(\mathbf{u}', \mathbf{z}_0'\right) &\geqslant \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} \\ &= \Psi(E - \{N-1\}), \end{aligned} \tag{9}$$

and similarly,

$$d(\mathbf{u}, \mathbf{x}_0) \geqslant d(\mathbf{u}'', \mathbf{x}_0'') \geqslant \Psi(E \cap [T]), \tag{10}$$

following the same logic as equation (6). The inequalities (9) and (10) are equivalent for the case $T = N-1$, and hold for all cases throughout the proof.

4.4 Proof of Theorem 4.12

4.4.1 Separation of Problem into Cases, "Sub-Cases", and "Sub-Sub-Cases"

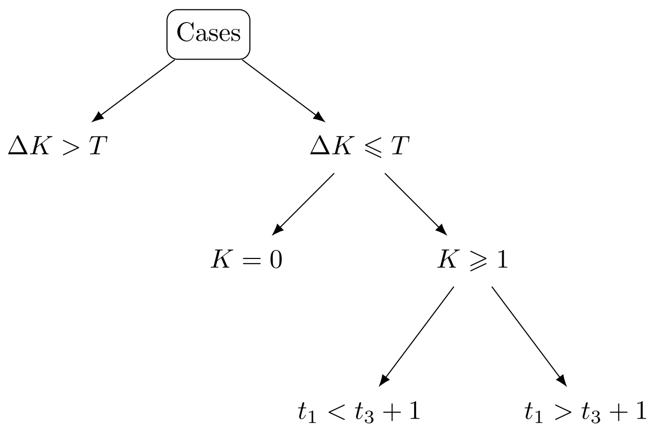
We have already proved that the theorem holds for the following cases:

- The case where the largest disc is not on the starting column ($N-1 \notin E$)
- The case where no disc enters the final column ($E' = \emptyset$).

As a result, we make the assumptions that:

- The largest disc $N-1$ starts on column 0: $\gamma_0(N-1) = 0$
- E' is not empty, and therefore there exists a T which is the largest element of E'.

We will now split the problem into multiple cases and show that Theorem 4.12 holds for each. The overarching distinction to make is where our T lies in the starting configuration, and we will see that it is convenient to separate these cases into subsequent sub-cases:



We need to show that $D = d(\mathbf{u}, \mathbf{v}) \geqslant \Psi\{k \in [N] : \mathbf{u}(k) = 0\} = \Psi E$ for all such cases. We will do so in the following order:

1. $\Delta K > T$
2. $\Delta K \leqslant T, K = 0$
3. $\Delta K \leqslant T, K \geqslant 1, t_1 > t_3 + 1$
4. $\Delta K \leqslant T, K \geqslant 1, t_1 < t_3 + 1$.

These cases account for all possible paths γ from $\mathbf{u}$ to $\mathbf{v}$.

4.4.2 Case 1: $\Delta K > T$

We first examine the simplest case, $\Delta K > T$.

It is always true that $t_2 \leqslant t_3 + 1$ ($N-1$ cannot be in column 2 before it has left column 0), so we can say that the solution must be reached in this order:

$$\mathbf{u} \longrightarrow \mathbf{z_0} \longrightarrow \mathbf{z_2} \longrightarrow \mathbf{v}$$

$$\boxed{\begin{array}{c}\mathbf{u}\\ \text{Start}\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{z_0}\\ N-1 \text{ leaves column } 0\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{z_2}\\ N-1 \text{ enters column } 2\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{v}\\ \text{End}\end{array}}$$

It should be noted that this is the expected result; it seems counterintuitive to move the largest disc into a column that it will not end on, though we cannot assume that this is optimal without sufficient proof.

$\underline{d(\mathbf{u}, \mathbf{z_0})}$

Since $T \geqslant 0$ always, it follows by the definition of the case that $K \geqslant 1$, meaning that E'' always has at least one element. As such, we know that b_K always exists in this case, and $T < b_1 < \ldots < b_K$ by definition. Therefore $b_K = N - 1$; the largest disc never passes through the third column.

Since $\Delta K > T$, we can say that

$$\{x \in E : x \geqslant \Delta K\} \subseteq \{x \in E : x > T\} = E''. \tag{11}$$

It follows from our definition of E'' that the cardinality of $\{x \in E : x \geqslant \Delta K\}$ is at most K. Recall Lemma 4.9 which states $\Psi(A) - \Psi(A - \{a\}) \leqslant 2^{s-1}$, where A is a subset of $\mathbb{N}$, a is any element in A, and s an element of A such that $A - [\Delta s]$ contains at most s elements. We have an equivalent, where A corresponds to E and s corresponds to K, that is,

$$\Psi(E) - \Psi(E - \{N-1\}) \leqslant 2^{K-1}$$

or

$$\Psi(E) - 2^{K-1} \leqslant \Psi(E - \{N-1\}). \tag{12}$$

This is true as the set $E - [\Delta K]$ is equivalent to $\{x \in E : x \geqslant \Delta K\}$, which we know has less than K elements from (11). We choose $N-1$ to be our equivalent to a, as the lemma states we are able to choose any element of E.

Inserting this into (9), we obtain

$$\begin{aligned} d(\mathbf{u}, \mathbf{z_0}) &\geqslant \Psi(E - \{N-1\}) \geqslant \Psi(E) - 2^{K-1} \\ d(\mathbf{u}, \mathbf{z_0}) &\geqslant \Psi(E) - 2^{K-1}. \end{aligned} \tag{13}$$

<u>$d(\mathbf{z_0}, \mathbf{z_2})$</u>

It is trivial that

$$d(\mathbf{z_0}, \mathbf{z_2}) \geqslant 1, \tag{14}$$

as we are simply moving disc $N-1$ from column 0 to column 2.

<u>$d(\mathbf{z_2}, \mathbf{v})$</u>

In the configuration $\mathbf{z_2}$ there will be two columns that contain no discs less than $N-1$: column 2 and column $c = \gamma_{t_3}(N-1)$ which is the column $N-1$ moved from. As we know in this case, $N-1$ never enters column 3 (since $K \geqslant 1$), so c must be either 0 or 1. This tells us that the discs in E'', $b_1, b_2, \ldots, b_{K-1}$, must all be in column $1-c$:

- They cannot be in column 2 as there cannot be any discs $< b_k = N-1$.
- They cannot be in column 3 by the definition of the set E".
- They cannot be in column c as we have defined it to be the column which $N-1$ moves from.
- If $N-1$ moved from column 0 they will be in column 1 (and vice versa), meaning that they must be in column $1-c$.

We also know that in the final configuration $\mathbf{v}$, all of the discs are on column 2. This recreates the 3 column problem for the discs $b_1, b_2, \ldots, b_{K-1}$ as they must go from column $1-c$ to column 2 without entering column 3. That will take at least 2^{K-1} moves, so

$$d(\mathbf{z_2}, \mathbf{v}) \geq 2^{K-1} - 1. \tag{15}$$

$\underline{d(\mathbf{u}, \mathbf{v})}$

We can now show that Theorem 4.12 holds for the case $\Delta K > T$. Combining (13), (14), and (15), we get

$$\begin{aligned} D = d(\mathbf{u}, \mathbf{v}) &= d(\mathbf{u}, \mathbf{z_0}) + d(\mathbf{z_0}, \mathbf{z_2}) + d(\mathbf{z_2}, \mathbf{v}) \\ &\geq (\Psi E - 2^{K-1}) + 1 + (2^{K-1} - 1) = \Psi E. \end{aligned}$$

This proves that Theorem 4.12 holds for the case $\Delta K > T$.

4.4.3 Case 2: $\Delta K \leqslant T$

We now examine the case $\Delta K \leqslant T$. In this case, inequality (12) no longer holds, but we can suggest an alternative.

Proposition 4.13. *We have*

$$\Psi E - \Psi(E - \{N-1\}) \leqslant 2^{\nabla(T+K+1)-1}. \tag{16}$$

Proof. To demonstrate this, let $s = \nabla(T+K+1)$. If we can show that $E - [\Delta s]$ contains at most s elements, the inequality must be true by Lemma 4.9. From (7), we know that E is contained within $[T+1] \sqcup E''$, which has at most $T+K+1$ elements.

If we assume that the subtraction of the set $[\Delta s]$ from E only removes elements from $[T+1]$ and not E'' (sufficient, but not necessarily true), we can set an upper bound to the cardinality of $E - [\Delta s]$ as

$$(T+1-\Delta s)^+ + K,$$

where $(T+1-\Delta s)^+$ is not considered for values $\leqslant 0$.

We can rewrite this as a pair of inequalities:

$$\begin{cases} K \leqslant s & \text{if } T+1-\Delta s \leqslant 0, \\ T+1-\Delta s + K \leqslant s & \text{if } T+1-\Delta s > 0. \end{cases} \tag{17}$$

If we can show that both of these inequalities hold, Proposition 16 is proven.

To prove the first inequality, we use the fact that in this case $\Delta K \leqslant T$, so $K \leqslant \nabla T$. We also have by definition $\nabla T \leqslant s$, from which it follows that $K \leqslant \nabla T \leqslant s$, proving the first inequality.

From our definition of s as a triangular root, we have that $\Delta s \leqslant T+K+1 < \Delta(s+1)$. To prove the second inequality, we use the fact that $\Delta(s+1) = \Delta s + (s+1)$, meaning we can rearrange as follows:

$$\begin{aligned} T+K+1 &< \Delta(s+1) \\ T+K+1 &\leqslant \Delta(s+1) - 1 \\ T+K+1 &\leqslant \Delta s + s \\ T+1-\Delta s + K &\leqslant s. \end{aligned}$$

The second inequality is proven.

Since we have proven the two inequalities, (16) is proven. $\square$

Inserting (16) into (9), we arrive at:

$$d(\mathbf{u}, \mathbf{z_0}) \geqslant d(\mathbf{u}', \mathbf{z_0}') \geqslant \Psi E - 2^{\nabla(T+K+1)-1} \tag{18}$$

which is a crucial result to this case. Note that this result holds for the entirety of the $\Delta K \leqslant T$ case, since the separations into sub-cases only occur after the route γ reaches the state $\mathbf{z_0}$.

4.4.3.1 i) $K = 0$

We now address the case where $K = 0$. This implies that there are no discs larger than T, so $T = N-1$. Consequently, $N-1$ enters column 3 before finally arriving column 2, or equivalently $t_1 \leqslant t_3$. Note that because $T = N-1$, we see that $\mathbf{x_0}$ and $\mathbf{z_0}$ are equivalent configurations. In this case, the solution must therefore follow the route:

$$\mathbf{u} \longrightarrow \mathbf{x_0} = \mathbf{z_0} \longrightarrow \mathbf{x_3} \longrightarrow \mathbf{z_2} \longrightarrow \mathbf{v}$$

$$\boxed{\begin{matrix}\mathbf{u}\\ \text{Start}\end{matrix}} \longrightarrow \boxed{\begin{matrix}\mathbf{x}_0 = \mathbf{z_0}\\ T \text{ leaves column } 0\end{matrix}} \longrightarrow \boxed{\begin{matrix}\mathbf{x}_3\\ T \text{ enters column } 3\end{matrix}} \longrightarrow \boxed{\begin{matrix}\mathbf{z}_2\\ N-1 \text{ enters column } 2\end{matrix}} \longrightarrow \boxed{\begin{matrix}\mathbf{v}\\ \text{End}\end{matrix}}$$

$\underline{d(\mathbf{u}, \mathbf{z_0} = \mathbf{x_0})}$

We have

$$d(\mathbf{u}, \mathbf{z_0}) \geqslant d(\mathbf{u}', \mathbf{z_0}') \geqslant \Psi E - 2^{\nabla(T+K+1)-1}$$

from (18).

$\underline{d(\mathbf{x_0} = \mathbf{z_0}, \mathbf{z_2})}$

As we will see, we can compute a sufficient lower bound for the distance $d(\mathbf{x_0} = \mathbf{z_0}, \mathbf{z_2})$ without explicitly computing intermediate distances $d(\mathbf{x_0} = \mathbf{z_0}, \mathbf{x_3})$ and $d(\mathbf{x_3}, \mathbf{z_2})$.

Let $c = \gamma_{t_3}(N-1)$ be the column from which $N-1$ finally moves to column 2 (so $c \neq 2$). In the configuration $\mathbf{z_2}'$, we have that columns 2 and c are empty, so all of the discs must be in the other two columns. We name these two columns a and b, such that:

$$\{0, 1, 2, 3\} - \{2, c\} = \{a, b\}.$$

We will see that it is convenient to choose $a \in \{0, 3\}$ and $b \in \{0, 1\}$, which forces the combinations:

c	0	1	3
a	3	3	0
b	1	0	1

We now define sets A and B as the sets containing the discs on columns a and b respectively in configuration $\mathbf{z_2}$:

$$A = \{k \in [N-1] : \mathbf{z_2}(k) = a\}$$
$$B = \{k \in [N-1] : \mathbf{z_2}(k) = b\}.$$

For any a, b and c we have $A \sqcup B = [N-1]$; in all cases every disc except $N-1$ is on column a or b. The cardinality of $A \cup B$ is $N-1$, so therefore by Lemma 4.11,

$$\Psi A + \Psi B \geqslant \frac{1}{2}\Psi[N+1] - 1,$$

which, from Lemma 4.5,

$$= \frac{1}{4}(2^{\nabla(N+1)} + 2^{\nabla N}) + \frac{1}{2}\Psi[N-1] - 1. \tag{19}$$

As a side note, from inserting (8) into (19), we see

$$\begin{aligned}
\Psi A + \Psi B &\geqslant \frac{1}{4}(2^{\nabla(N+1)} + 2^{\nabla N}) + \frac{1}{2}\Psi[N-1] - 1 \\
&\geqslant \frac{1}{4}(2^{\nabla(T+K+2)} + 2^{\nabla(T+K+1)}) + \frac{1}{2}\Psi[N-1] - 1 \\
&\geqslant \frac{1}{4}(2^{\nabla(T+K+1)} + 2^{\nabla(T+K+1)}) + \frac{1}{2}\Psi[N-1] - 1 \\
&= \frac{1}{4}(2 \cdot 2^{\nabla(T+K+1)}) + \frac{1}{2}\Psi[N-1] - 1 \\
&= \frac{1}{2} \cdot 2^{\nabla(T+K+1)} + \frac{1}{2}\Psi[N-1] - 1 \\
&= 2^{\nabla(T+K+1)-1} + \frac{1}{2}\Psi[N-1] - 1.
\end{aligned} \tag{20}$$

Consider the configurations $\mathbf{x_a}'$ and $\mathbf{z_2}'$. In both possible configurations $\mathbf{x_a}'$ we have column a empty as well as one other column. Similarly, in $\mathbf{z_2}'$, we have column 2 empty as well as one other. This means to get from $\mathbf{z_2}'$ to either configuration $\mathbf{x_a}'$, we have to move all of the discs on column a to another column, which will take at least ΨA moves by our recurrence hypothesis, or

$$d(\mathbf{z_2}', \mathbf{x_a}') \geqslant \Psi A. \tag{21}$$

Between the configurations $\mathbf{z_0}$ and $\mathbf{z_2}$, we know that $N-1$ must go from column 0 to 3 before finally moving to column 2, and will therefore make at least 2 moves.

By our defined route, we also know that between $\mathbf{z_0}$ and $\mathbf{z_2}$ we will have to move all discs $< N-1$ at least $d(\mathbf{x_a}, \mathbf{z_2})$ times, which from (21) we know to be ΨA, and therefore

$$d(\mathbf{z_0}, \mathbf{z_2}) \geqslant d(\mathbf{x_a}', \mathbf{z_2}') \geqslant \Psi A + 2. \tag{22}$$

$\underline{d(\mathbf{z_2}, \mathbf{v})}$

Similar to (21), we know that columns b and $1-b$ will both be empty in $\mathbf{v}'$, but in $\mathbf{z_2}$ there is guaranteed to be a disc on one of these columns. Therefore, to get from $\mathbf{z_2}'$ to $\mathbf{v}'$ we will have to move discs to column b or $1-b$, which will take at least ΨB moves, and so

$$d(\mathbf{z_2}, \mathbf{v}) \geqslant d(\mathbf{z_2}', \mathbf{v}') \geqslant \Psi B. \tag{23}$$

$\underline{d(\mathbf{u},\mathbf{v})}$

Finally, combining (18), (22), and (21) we have:

$$\begin{aligned} D = d(\mathbf{u},\mathbf{v}) &= d(\mathbf{u},\mathbf{z_0}) + d(\mathbf{z_0},\mathbf{z_2}) + d(\mathbf{z_2},\mathbf{v}) \\ &\geqslant \left(\Psi E - 2^{\nabla(T+K+1)-1}\right) + (\Psi A + 2) + (\Psi B). \end{aligned}$$

Rearranging, and inserting (20), we arrive at

$$\begin{aligned} &= \Psi E + \left[\Psi A + \Psi B\right] - 2^{\nabla(T+K+1)-1} + 2 \\ &\geqslant \Psi E + \left[(2^{\nabla(T+K+1)-1} + \frac{1}{2}\Psi[N-1] - 1)\right] - 2^{\nabla(T+K+1)-1} + 2 \\ &= \Psi E + \frac{1}{2}\Psi[N-1] + 1 \geqslant \Psi E. \end{aligned}$$

We have shown that Theorem 4.12 holds in the case $K = 0$ for $\Delta K \leqslant T$.

4.4.3.2 ii) $K > 0$

Having settled the case where $K = 0$, we now move onto the case where $K \geqslant 0$ (still with $\Delta K \leqslant T$). $K \geqslant 1$ implies that there exist one or many discs larger than T, and therefore $T < b_K = N - 1$; the largest disc does not enter column 3.

As such, the moments t_1 (with configuration $\mathbf{x_3}$) and $t_3 + 1$ (with configuration $\mathbf{z_2}$) are now separate and unrelated. We cannot have $t_1 = t_3 + 1$: t_1 is reached by moving T whilst $t_3 + 1$ is reached by moving $N - 1$, but we know $T < N - 1$. Therefore, we are required to split this final "sub-case" into further "sub-sub-cases":

- a) $t_1 > t_3 + 1$
- b) $t_1 < t_3 + 1$.

$\underline{\textbf{a) } \mathbf{t_1 > t_3 + 1}}$

We start by supposing that $t_1 > t_3 + 1$, meaning that $N - 1$ moves finally to column 2 before T first enters column 2. As such, the solution in this case must follow the route:

$$\mathbf{u} \longrightarrow \mathbf{z_0} \longrightarrow \mathbf{z_2} \longrightarrow \mathbf{x_3} \longrightarrow \mathbf{v}$$

$$\boxed{\begin{array}{c}\mathbf{u}\\ \text{Start}\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{z_0}\\ N-1 \text{ leaves column } 0\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{z_2}\\ N-1 \text{ enters column } 2\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{x_3}\\ T \text{ enters column } 3\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{v}\\ \text{End}\end{array}}$$

$\underline{d(\mathbf{u},\mathbf{z_0})}$

From (18), we have $d(\mathbf{u},\mathbf{z_0}) \geqslant d(\mathbf{u}',\mathbf{z_0}') \geqslant \Psi E - 2^{\nabla(T+K+1)-1}$.

$\underline{d(\mathbf{z_0}, \mathbf{z_2})}$

It is clear that

$$d(\mathbf{z_0}, \mathbf{z_2}) \geqslant 1, \tag{24}$$

as we are simply moving $N-1$ from column 0 to column 2.

$\underline{d(\mathbf{z_2}, \mathbf{x_3})}$

Consider the configuration $\mathbf{x_3}$, where T is found for the first time in column 3. We will have two columns which contain no discs $< T$: column 3 and column $d = \gamma_{t_1-1}(T)$, the column from which T moved. All discs $< T$ must therefore be split across the two other columns, which we call a and b (notation unrelated to previous cases); formally,

$$\{0,1,2,3\} - \{3,d\} = \{a,b\}.$$

We also know that in the configuration $\mathbf{x_3}$, $N-1$ has moved for the final time into column 2. As before, we denote by $c = \gamma_{t_3}(N-1)$ the column from which $N-1$ finally moved into column 2. It must be true that $c \in \{0,1\}$ since $N-1$ is moving into column 2 and never enters column 3 in this case.

Notation	**Definition**	**Explanation**	**Range**
d	$\gamma_{t_1-1}(T)$	The column from which T moves to column 3.	$\{0,1,2\}$
$\{a,b\}$	$\{0,1,2,3\} - \{3,d\}$	The two columns containing discs $< T$ in $\mathbf{x_3}$.	$a,b \in \{0,1,2\}, a \neq b$
c	$\gamma_{t_3}(N-1)$	The column from which $N-1$ finally moves to column 2.	$\{0,1\}$

Table 3: Explanation of Notations

We choose $a \in \{2,c\}$ and $b \in \{0,1\}$ (we will later see why this is useful):

- If $d = 0$ or 1 (equivalent by symmetry), we know columns $1-d$ and 2 contain discs $< T$, so $a = 2$ and $b = 1-d$.
- If $d = 2$, then columns 0 and 1 contain discs $< T$. By definition, columns 0 and 1 are equivalent to c and $1-c$, so we can say $a = c$ and $b = 1-c$ which satisfies our choice of a and b.

By considering columns 0 and 1 to be equivalent by symmetry, these conditions force the combinations:

d	0 or 1	2
a	2	c
b	$1-d$	$1-c$

As in previous sections, we define sets A and B to be the stacks on columns a and b respectively in $\mathbf{x_3}$:

$$A = \{k \in [T] : \mathbf{x_3}(k) = a\}$$
$$B = \{k \in [T] : \mathbf{x_3}(k) = b\}$$

where $A \cup B = T$ and $A \cap B = \emptyset$ (A and B are complementary in $[T]$).

Now consider the states $\mathbf{x_3}''$ and $\mathbf{z_2}''$. In $\mathbf{z_2}''$ we have columns 2 and c empty, meaning to get from $\mathbf{x_3}''$ to $\mathbf{z_2}''$ we must at least move all discs in $[T]$ from these columns to another. Since these columns are represented by column a, we therefore have:

$$d(\mathbf{z_2}, \mathbf{x_3}) = d(\mathbf{x_3}, \mathbf{z_2}) \geq d(\mathbf{x_3}'', \mathbf{z_2}'') \geq \Psi A. \tag{25}$$

<u>$d(\mathbf{x_3}, \mathbf{v})$</u>

Consider the states $\mathbf{x_3}''$ and $\mathbf{v}''$. In the state $\mathbf{v}''$ we know all columns except 2 are empty. Since columns $1-d$ and $1-c$ can never be column 2, when moving from $\mathbf{x_3}''$ to $\mathbf{v}''$ we must at least move all discs in [T] from these columns. Since these columns are represented by column b, we therefore have:

$$d(\mathbf{x_3}, \mathbf{v}) \leq d(\mathbf{x_3}'', \mathbf{v}'') \leq \Psi B. \tag{26}$$

<u>$d(\mathbf{u}, \mathbf{v})$</u>

Therefore, combining (18), (24), (25) and (26), we arrive at:

$$\begin{aligned} D = d(\mathbf{u}, \mathbf{v}) &= d(\mathbf{u}, \mathbf{z_0}) + d(\mathbf{z_0}, \mathbf{z_2}) + d(\mathbf{z_2}, \mathbf{x_3}) + d(\mathbf{z_2}, \mathbf{x_3}) + d(\mathbf{x_3}, \mathbf{v}) \\ &\geqslant \Psi E - 2^{\nabla(T+K+1)-1} + 1 + \Psi A + \Psi B, \\ &\geqslant \Psi E - 2^{\nabla(T+K+1)} + \frac{1}{2}\Psi\big[T+2\big]. \end{aligned} \tag{27}$$

The last line here comes from applying Lemma 4.11 and rearranging, as we know that the cardinality of $A \cup B$ is $T-1$ because A and B are complementary in $[T]$.

<u>Proof that the theorem holds</u>

If we can show that the expression in (27) is greater than or equal to ΨE, we have proven that the theorem holds for this "sub-sub-case". It is therefore sufficient to show that:

$$\begin{aligned} -2^{\nabla(T+K+1)} + \frac{1}{2}\Psi\big[T+2\big] &\geqslant 0 \\ \implies \frac{1}{2}\Psi\big[T+2\big] &\geqslant 2^{\nabla(T+K+1)}, \end{aligned} \tag{28}$$

In order to prove this, we must first prove some preliminary propositions.

Let $s = \nabla(T + K + 1)$.

Proposition 4.14.

$$s \geqslant K + 1 \tag{29}$$

Proof. Using the fact that throughout this case $\Delta K \leqslant T$, we have

$$\begin{aligned} \Delta K &\leqslant T \\ \Delta K + K + 1 &\leqslant T + K + 1, \end{aligned}$$

which, by the definition of the Δ function, is equivalent to

$$\Delta(K + 1) \leqslant T + K + 1,$$

and therefore,

$$K + 1 \leqslant \nabla(T + K + 1) = s.$$

□

Proposition 4.15.

$$T + K + 1 \geqslant \Delta s$$

Proof. Taking the definition of s and applying known properties of the Δ function,

$$\begin{aligned} \nabla(T + K + 1) &= s \\ T + K + 1 &\leqslant \Delta s. \end{aligned}$$

□

Proposition 4.16.

$$\nabla(T + 2) \geqslant \nabla T \geqslant s - 1 \tag{30}$$

Proof. Rewriting T and inserting Propositions 4.14 and 4.15, we get:

$$\begin{aligned} T + 2 > T &= (T + K + 1) - (K + 1), \\ &\geqslant \Delta s - s \\ &= \Delta(s - 1). \end{aligned}$$

Now that we have $T + 2 > T \geqslant \Delta(s - 1)$, it follows that $\nabla(T + 2) \geqslant \nabla T \geqslant s - 1$ from the properties of the ∇ function. □

In order to prove (28), recall Lemma 4.6. By Lemma 4.6 we have that:

$$\Psi[T + 2] \geqslant 2^{\nabla((T+2)+1)}.$$

Inserting Proposition 4.16, it therefore follows that

$$\begin{aligned}\Psi[T+2] &\geqslant 2^{(s-1)+1}\\ &= 2^{s}\\ &= 2^{\nabla(T+K+1)}.\end{aligned}$$

Therefore, it is true that

$$\frac{1}{2}\Psi[T+2] \geqslant 2^{\nabla(T+K+1)-1}.$$

This proves (28), and therefore proves this "sub-sub-case", since:

$$\begin{aligned}
& & \frac{1}{2}\Psi\big[T+2\big] &\geqslant 2^{\nabla(T+K+1)}\\
&\implies & -2^{\nabla(T+K+1)} + \frac{1}{2}\Psi\big[T+2\big] &\geqslant 0\\
&\implies & \Psi E - 2^{\nabla(T+K+1)} + \frac{1}{2}\Psi\big[T+2\big] &\geqslant \Psi E\\
&\implies & D = d(\mathbf{u},\mathbf{v}) &\geqslant \Psi E.
\end{aligned}$$

We have proved the theorem holds for the "sub-sub-case" $t_1 < t_3 + 1$.

b) $\mathbf{t_1 < t_3 + 1}$

We now move to the case where $t_1 < t_3 + 1$, meaning T moves to column 3 before $N-1$ finally moves to column 2. As such, the solution in this case must follow the route:

$$\mathbf{u} \longrightarrow \mathbf{x_0} \longrightarrow \mathbf{x_3}, \mathbf{z_0} \longrightarrow \mathbf{z_2} \longrightarrow \mathbf{v}$$

$$\boxed{\begin{array}{c}\mathbf{u}\\ \text{Start}\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{x_0}\\ T \text{ leaves column } 0\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{x_3}, \mathbf{z_0}\\ T \text{ enters column } 3,\\ N-1 \text{ leaves column } 0\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{z_2}\\ N-1 \text{ enters column } 2\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{v}\\ \text{End}\end{array}}$$

To compute a sufficient lower bound for D we compute the distances $d(\mathbf{u}, \mathbf{x_0})$ and $d(\mathbf{u}, \mathbf{z_0})$ separately, but consider in each case only $[T]$ and only $b_1, \ldots, b_{K-1}$ respectively, allowing us to sum them separately. It is possible that these configurations are equivalent, but reached at different times.

$d(\mathbf{u}, \mathbf{x_0})$ for $[T]$

We first consider the movements of discs in $[T]$ from the start configuration.

Between $\mathbf{u}$ and $\mathbf{x_0}$ (the configuration just before T moves), we will have to move all discs $< T$ out of column 0. From our recurrence hypothesis, this will take at least $\Psi(E \cap [T])$ moves. That is to say, moving discs only in $[T]$:

$$d(\mathbf{u}, \mathbf{x_0}) \geqslant \Psi(E \cap [T]). \tag{31}$$

$d(\mathbf{u}, \mathbf{z_0})$ for $T \cap \{b_1, \ldots, b_{K-1}\}$

We now consider movements of $b_1, .., b_{K-1}$ from the starting configuration $\mathbf{u}$. We have that $T \cap \{b_1, \ldots, b_{K-1}\} = \emptyset$, so we know that no movements counted in our $d(\mathbf{u}, \mathbf{x_0})$ for $[T]$ will be counted in this computation.

Consider the state $\mathbf{z_0}$, where $N-1$ leaves column 0. We know that we have two columns with no discs $< N-1$: column 0 and another column which cannot be column 3, because $T < N-1 = b_K$ implies $N-1$ never enters column 3. Therefore, in the state $\mathbf{z_0}$, discs $b_1, \ldots, b_{K-1}$ must all be in either column 1 or all be in column 2. They cannot be shared across columns 1 and 2, since that would force $N-1$ to move to column 3 which is impossible in this case.

We now have for the set $\{b_1, \dots, b_{K-1}\}$:

- There are $K-1$ discs in the set.
- In $\mathbf{u}$ all $K-1$ discs are in column 0.
- In $\mathbf{z_0}$ all $K-1$ discs are in column 1 or 2 exclusively.
- No discs may enter column 3 at any time.

These conditions recreate a 3 column problem for $K-1$ discs, for which the solution is bounded below by $2^{K-1}-1$. Equivalently, for discs in $\{b_1, \dots, b_{K-1}\}$ we have

$$d(\mathbf{u}, \mathbf{z_0}) \geqslant 2^{K-1} - 1. \tag{32}$$

$\underline{d(\mathbf{z_2}, \mathbf{x_3})}$

Consider the state $\mathbf{z_2}'$, where $N-1$ has entered column 2 (ignoring $N-1$). There will be two empty columns: column 2 and column $c = \gamma_{t_3}(N-1)$, the column from which $N-1$ moved into column 2. We know that c cannot be 3, as $T < b_k = N-1$, so $N-1$ never enters column 3. Since all the discs must be split across the non-empty columns, we can say that all discs are therefore on columns $b = 1-c$ and 3. Given that this is true, it must follow that all discs $b_1, \dots, b_{K-1}$ are in column b, as they cannot be in column 3 by definition.

Now consider the state $\mathbf{x_3}''$, where T enters column 3 (regarding only discs in $[T]$). We know that column 3 will be empty, as well as one other unspecified column.

Therefore, to get from $\mathbf{z_2}''$ to $\mathbf{x_3}''$, we must move all discs $< T$ from column 3 to another column. Let $A = \{k \in [T] : \mathbf{z_2}(k) = 3\}$ be the set of all discs $< T$ in column 3 in the configuration $\mathbf{z_2}$. We have

$$d(\mathbf{z_2}, \mathbf{x_3}) \geqslant d(\mathbf{z_2}'', \mathbf{x_3}'') \geqslant \Psi A. \tag{33}$$

$\underline{d(\mathbf{z_0}, \mathbf{z_2})}$

It is trivial that at least one move is made between $\mathbf{z_0}$ and $\mathbf{z_2}$, as we are simply moving $N-1$ from column 0 to column 2, giving us

$$d(\mathbf{z_0}, \mathbf{z_2}) \geqslant 1. \tag{34}$$

$\underline{d(\mathbf{x_0}, \mathbf{x_3})}$

It is trivial that at least one move is made between $\mathbf{x_0}$ and $\mathbf{x_3}$, as we are simply moving T from column 0 to column 3, from which it follows

$$d(\mathbf{x_0}, \mathbf{z_2}) \geqslant \mathbf{x_3}. \tag{35}$$

$\underline{d(\mathbf{z_2}, \mathbf{v})}$

Similarly, in the state $\mathbf{v}'$, we know that columns b and $1-b$ ($b, 1-b \in \{0,1\}$) must be empty since all discs are on column 3. Therefore, to get to $\mathbf{v}'$ from $\mathbf{z_2}'$, we must move all the discs from column b to another column. Let $B = \{k \in [N-1] : \mathbf{z_2}(k) = b\}$ be the set of all discs $< N-1$ in column b in the configuration $\mathbf{z_2}$. We have

$$d(\mathbf{z_2}, \mathbf{v}) \geqslant d(\mathbf{z_2}', \mathbf{v}') \geqslant \Psi B. \tag{36}$$

$\underline{d(\mathbf{u}, \mathbf{z_2})}$

We now combine these results to produce a lower bound for $d(\mathbf{u}, \mathbf{z_2})$.

As mentioned, we know all discs $b_1, \ldots, b_{K-1}$ must be in column b and therefore in B. Since all discs $< N-1$ are shared between column 3 and b, we have:

$$\begin{aligned} A &= \{k \in [T] : \mathbf{z_2}(k) = 3\}, \\ B &= \{k \in [N-1] : \mathbf{z_2}(k) = b\} \\ &\supseteq \{k \in [T] : \mathbf{z_2}(k) = b\} \sqcup \{b_1, \ldots, b_{K-1}\}, \end{aligned}$$

$$\implies A \cup B \supseteq [T] \sqcup \{b_1, \ldots, b_{K-1}\}.$$

This shows that the set $A \cup B$ contains $[T]$ (cardinality T) and $\{b_1, \ldots, b_{K-1}\}$ (cardinality $K-1$), so its cardinality is at most $T + K - 1$. Therefore, by Lemma 4.5,

$$\Psi A + \Psi B \geqslant \frac{1}{2}[T + K + 1] - 1. \tag{37}$$

Summarising, we have:

- From (31), discs in $[T]$ move at least $\Psi(E \cap [T])$ times between $\mathbf{u}$ and $\mathbf{x_0}$.
- From (32), discs in $\{b_1, \ldots, b_{K-1}\}$ move at least $2^{K-1} - 1$ times between $\mathbf{u}$ and $\mathbf{z_0}$.
- From (33), between $\mathbf{x_3}$ and $\mathbf{z_2}$ at least ΨA moves are made.
- From (34), between $\mathbf{z_0}$ and $\mathbf{z_2}$ at least 1 move is made.
- From (35), between $\mathbf{x_0}$ and $\mathbf{x_3}$ at least 1 move is made.
- From (36), between $\mathbf{z_2}$ and $\mathbf{v}$ at least ΨB moves are made.
- From (37), $\Psi A + \Psi B \geqslant \frac{1}{2}[T + K + 1] - 1$.

Combining these results, it is clear that $d(\mathbf{u}, \mathbf{z_2})$ satisfies

$$\begin{aligned} d(\mathbf{u}, \mathbf{z_2}) &= d_{[T]}(\mathbf{u}, \mathbf{x_0}) + d_{\{b_1, \ldots, b_{K-1}\}}(\mathbf{u}, \mathbf{z_0}) + d(\mathbf{x_3}, \mathbf{z_2}) + d(\mathbf{z_0}, \mathbf{z_2}) + d(\mathbf{x_0}, \mathbf{x_3}) \\ &\geqslant \Psi(E \cap [T]) + 2^{K-1} - 1 + \Psi A + 2 \\ &= \Psi(E \cap [T]) + 2^{K-1} + \Psi A + 1. \end{aligned}$$

$\underline{d(\mathbf{u}, \mathbf{v})}$

Therefore,

$$\begin{aligned} D = d(\mathbf{u}, \mathbf{v}) &= d(\mathbf{u}, \mathbf{z}_2) + d(\mathbf{z}_2, \mathbf{v}) \\ &\geqslant \Psi(E \cap [T]) + \Psi A + 2^{K-1} + 1 + \Psi B. \end{aligned}$$

Inserting (37), we reach

$$D \geqslant \Psi(E \cap [T]) + \frac{1}{2}\Psi[T + K + 1] + 2^{K-1}. \tag{38}$$

It is true that:

- $E = (E \cap [T]) \sqcup \{T, b_1, \ldots, b_k\}$ by definition of all included sets and elements
- $K + 1 \geqslant 1$ and $T \geqslant \Delta(K + 1 - 1) = \Delta K$ by the definition of our case
- $E \cup [T] \subset [T]$
- The cardinality of $\{T, b_1, \ldots, b_K\}$ is $K + 1$.

We can therefore apply Lemma 4.10, where from the definition,

- s is equivalent to $K + 1$
- n is equivalent to T
- A is equivalent to $E \cup [T]$
- $\{b_1, b_2, ..b_s\}$ is equivalent to $\{T, b_1, \ldots, b_K\}$.

From Lemma 4.10,

$$\begin{aligned} &\Psi\big((E \cup [T]) \cup \{T, b_1, \ldots, b_K\}\big) - \Psi\big(E \cup [T]\big) \leqslant \Psi[T + K + 1] - \Psi[T] \\ &\qquad \implies \Psi E - \Psi\big(E \cup [T]\big) \leqslant \Psi[T + K + 1] - \Psi[T] \\ &\qquad\qquad \implies \Psi E \leqslant \Psi\big(E \cup [T]\big) + \Psi[T + K + 1] - \Psi[T]. \end{aligned}$$

Subtracting this from (38), we arrive at the result

$$\begin{aligned} D - \Psi E &\geqslant \Psi(E \cap [T]) + \frac{1}{2}\Psi[T + K + 1] + 2^{K-1} - \Psi\big(E \cup [T]\big) + \Psi[T + K + 1] - \Psi[T] \\ &= 2^{K-1} - \frac{1}{2}\Psi[T + K + 1] + \Psi[T]. \end{aligned}$$

If we can show that this expression is greater than or equal to 0, it then follows that $D \geqslant \Psi E$, and the "sub-sub-case" is proven.

From (3), we have

$$\begin{aligned}
\Psi[T+1+K] &\leqslant 2\Psi[T+1] + 2^{K-1} \\
\Psi[T+1+K] - 2\Psi[T+1] &\leqslant 2^{K-1} \\
\frac{1}{2}\Psi[T+1+K] - \Psi[T+1] &\leqslant 2^{K-1} \\
\implies 0 &\leqslant 2^{K-1} - \frac{1}{2}\Psi[T+K+1] + \Psi[T].
\end{aligned}$$

The expression is positive, proving that $D \geqslant \Psi E$ for this "sub-sub-case".

We have now proven that $D \geqslant \Psi E$ for all cases and relevant "sub-cases" and "sub-sub-cases".

4.5 Results of The Theorem

We now state the consequences of Theorem 4.12.

Corollary 4.17 (Solution to *the Reve's Puzzle*)**.** *In the 4 column Tower of Hanoi puzzle, it takes at least $\Phi(N)$ moves to transfer N discs from one column to another* [11].

Proof. Suppose $N \geqslant 1$. Let $\mathbf{u}$ be the starting configuration where all discs are on column 0, and $\mathbf{v}$ the final configuration where all discs are on column 2. We know that the path γ must at some point enter the configurations $\mathbf{z_0}$ (configuration when the largest disc leaves column 0) and $\mathbf{z_2}$ (configuration when the largest disc finally enters column 2) as before. The path therefore follows:

$$\mathbf{u} \longrightarrow \mathbf{z_0} \longrightarrow \mathbf{z_2} \longrightarrow \mathbf{v}$$

$$\boxed{\begin{array}{c}\mathbf{u}\\ \text{Start}\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{z_0}\\ N-1 \text{ leaves column } 0\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{z_2}\\ N-1 \text{ enters column } 2\end{array}} \longrightarrow \boxed{\begin{array}{c}\mathbf{v}\\ \text{End}\end{array}}$$

$\underline{d(\mathbf{u}, \mathbf{z_0})}$

In $\mathbf{z_0}'$, we know that column 0 is empty as well as one other. $\mathbf{u}$ has all discs on column 0, so therefore by Theorem 4.12

$$d\left(\mathbf{u}, \mathbf{z}_0\right) \geqslant d\left(\mathbf{u}', \mathbf{z}_0'\right) \geqslant \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} = \Psi[N-1].$$

$\underline{d(\mathbf{z_0}, \mathbf{z_2})}$

It is trivial that $d(\mathbf{z_0}, \mathbf{z_2}) \geqslant 1$, as we are simply moving $N-1$ from column 0 to column 2.

$\underline{d(\mathbf{z_2}, \mathbf{v})}$

Similarly, In $\mathbf{z_2}'$, we know that column 2 is empty as well as one other. $\mathbf{v}$ has all discs on column 2, so therefore by Theorem 4.12

$$d\left(\mathbf{v}, \mathbf{z}_2\right) = d\left(\mathbf{z}_2, \mathbf{v}\right) \geqslant d\left(\mathbf{u}', \mathbf{z}_0'\right) \geqslant \Psi\{k \in [N-1] : \mathbf{u}(k) = 0\} = \Psi[N-1].$$

d($\mathbf{u}$, $\mathbf{v}$)

Combining these and applying Lemma 4.5, we obtain

$$d(\mathbf{u}, \mathbf{v}) = d\left(\mathbf{u}, \mathbf{z}_0\right) + d\left(\mathbf{z}_0, \mathbf{z}_2\right) + d\left(\mathbf{z}_2, \mathbf{v}\right) \geqslant 1 + 2\Psi[N-1] = \Phi(N)$$

which is what we are trying to prove. □

We have shown that since $\Phi(N)$ is less than or equal to the optimal solution $D = d(\mathbf{u}, \mathbf{v})$ in all possible cases, $\Phi(N)$ is the optimal number of moves to solve *the Reve's Puzzle*.

■

References

[1] School of Mathematics. *Undergraduate Project Handbook 2023/24*. School of Mathematics. University of Bristol. Bristol, UK, 2023.

[2] Édouard Lucas. *Récréations mathématiques, Volume 3*. Gauthier-Villars, 1893.

[3] BBC One. *Doctor Who, Season 3, The Celestial Toymaker - The Fourth Dimension*. Accessed: 2024-04-04. BBC. 2023. URL: `https://www.bbc.co.uk/programmes/articles/3DHlYgfSdY2RlXkS2KwjtKK/the-fourth-dimension` (visited on 04/04/2024).

[4] *Tower of Hanoi (video game concept)*. Retrieved on 2024-04-04. Giantbomb.com. URL: `https://www.giantbomb.com/tower-of-hanoi/3015-5744/` (visited on 04/04/2024).

[5] Cynthia Honrales. "Improving Solving Problem Ability with Tower of Hanoi Puzzle". In: July 2022.

[6] Andreas M. Hinz et al. *The Tower of Hanoi – Myths and Maths*. Birkhäuser, 2013.

[7] Rod Pierce. *Play Tower of Hanoi*. Edited by Rod Pierce. July 2023. URL: `http://www.mathsisfun.com/games/towerofhanoi.html` (visited on 04/05/2024).

[8] Henry Ernest Dudeney. *The Canterbury Puzzles: And Other Curious Problems*. Digitized on 19 Mar 2008. E.P. Dutton and Company, 1908, p. 194.

[9] B. M. Stewart and J. S. Frame. "Solution to advanced problem 3819". In: *American Mathematical Monthly* 48.3 (1941). JSTOR 2304268, pp. 216–219. DOI: `10.2307/2304268`.

[10] Jens-P. Bode and Andreas M. Hinz. "Results and open problems on the Tower of Hanoi". In: *Proceedings of the Thirtieth Southeastern International Conference on Combinatorics, Graph Theory, and Computing*. 2014, pp. 113–122.

[11] Thierry Bousch. "La quatrième tour de Hanoï". In: *Bulletin of the Belgian Mathematical Society - Simon Stevin* 21 (2014), pp. 895–912.

[12] R. S. Scorer, P. M. Grundy, and C. A. B. Smith. "Some binary games". In: *The Mathematical Gazette* 28.280 (July 1944), p. 96. DOI: `10.2307/3606393`.

[13] Wikimedia Commons contributors. *File:Tower of Hanoi graph.svg*. Version 785373041. Page name: File:Tower of Hanoi graph.svg. July 21, 2023. URL: `https://commons.`

wikimedia.org/w/index.php?title=File:Tower_of_hanoi_graph.svg&oldid=785373041 (visited on 04/05/2024).

[14] Wikimedia Commons contributors. *File:Hanoi-Graph-7.svg.* Version 805008839. Page name: File:Hanoi-Graph-7.svg. Sept. 26, 2023. URL: https://commons.wikimedia.org/w/index.php?title=File:Hanoi-Graph-7.svg&oldid=805008839 (visited on 04/05/2024).

[15] A.M. Hinz and A. Schief. "The average distance on the Sierpiński gasket". In: *Probability Theory and Related Fields* 87 (1990), pp. 129–138. DOI: 10.1007/BF01217750.

[16] J.-P. Bode and A. M. Hinz. "Results and open problems on the Tower of Hanoi". In: *Congr. Numer.* 139 (1999), pp. 113–122.

[17] W. F. Lunnon. "The Reve's Puzzle". In: *The Computer Journal* 29.5 (1986). URL: https://academic.oup.com/comjnl/article/29/5/478/486278.

[18] Mario Szegedy. "In How Many Steps the k Peg Version of the Towers of Hanoi Game Can Be Solved?" In: *STACS 99, 16th Annual Symposium on Theoretical Aspects of Computer Science.* Trier, Germany: Springer, June 3, 1999. DOI: 10.1007/3-540-49116-3_33.

[19] Xiao Chen and Jian Shen. "On the Frame–Stewart Conjecture about the Towers of Hanoi". In: *SIAM Journal on Computing* 33.3 (2004), pp. 584–589. ISSN: 0097-5397. DOI: 10.1137/S0097539703431019.

[20] Jun Wang et al. "A Non-recursive Algorithm for 4-Peg Hanoi Tower". In: 2007. URL: https://api.semanticscholar.org/CorpusID:62882646.