EE 325: Probability and Random Processes

Tutorial 12 and Assignment 4

Assignment Due: 1159pm on Tuesday, 02 October 2021

1. For $k = 1, \dots, n$, Y_k and Z_k are independent, zero-mean Gaussian random variables with variance σ_k^2, and a_k are real constants. Consider the random process

$$\mathsf{X}(t) := \sum_{k=1}^{n} \mathsf{Y}_k \cos a_k t + \mathsf{Z}_k \sin a_k t.$$

Obtain the mean $\mu_{\mathsf{X}}(t)$ and autocovariance $C_{\mathsf{XX}}(t_1, t_2)$ of this random process.

   (a) Is $\mathsf{X}(t)$ a WSS process?

   (b) Is $\mathsf{X}(t)$ an ergodic process?

   (c) Is $\mathsf{X}(t)$ a Gaussian process for $n = 1$?

   (d) Is $\mathsf{X}(t)$ a Gaussian process for $n > 1$?

2. $\mathsf{N}(t)$ is a Poisson counting process of rate λ. Determine the mean $\mu_{\mathsf{N}}(t)$ and the autocovariance $C_{\mathsf{NN}}(t_1, t_2)$ of this process. Next consider the process

$$\mathsf{X}(t) = \mathsf{N}(t+1) - \mathsf{N}(t)$$

Determine the mean $\mu_{\mathsf{X}}(t)$ and the autocovariance $C_{\mathsf{XX}}(t_1, t_2)$ of this process.

3. Let X_n be a sequence of zero-mean, uncorrelated random variables with variance σ^2. Let w_i, $0 < w_i < 1$, for $i = 0, \dots, k$, be a set of constants. The sequence

$$\mathsf{Y}_n = \sum_{i=0}^{k} w_i \mathsf{X}_{n-i},$$

is called the weighted moving window average of X_n. Obtain $C_{\mathsf{YY}}(n_1, n_2)$, the autovariance of the sequence Y_n.

4. $\mathsf{X}(t)$ is a WSS Gaussian process with mean $\mu_{\mathsf{X}}(t) = 3$ and autocovariance $C_{\mathsf{XX}}(\tau) = e^{-0.2|\tau|}$. Find (i) $\mathsf{Pr}(\mathsf{X}(5) \le 2)$ and (ii) $\mathsf{Pr}(|\mathsf{X}(5) - \mathsf{X}(8)| \le 1)$.

5. For $k = 1, \dots, n$, R_k and Θ_k are independent random variables. The pdf of R_k are

$$f_{\mathsf{R}_k}(r) = \begin{cases} \frac{r}{a_k^2} e^{-r^2/(2a_k^2)} & \text{for } r > 0 \\ 0 & \text{for } r \le 0 \end{cases}$$

for positive constants a_k. Θ_k are uniformly distributed in $(0, 2\pi)$. b_k are arbitrary positive constants. Define $\mathsf{X}(t)$ as

$$\mathsf{X}(t) = \sum_{k=1}^{n} \mathsf{R}_k \cos(b_k t + \Theta_k).$$

Determine the mean $\mu_{\mathsf{X}}(t)$ and autocovariance $C_{\mathsf{XX}}(t_1, t_2)$.

6. White noise with unit power spectral density is input to an ideal low pass filter of bandwidth B. If $\mathsf{Y}(t)$ is the output, determine the autocovariance function of the output and hence determine $\mathsf{Pr}(Y(1) > 1)$.

Assignment

1. $\mathsf{X}(n)$ is a zero-mean WSS sequence with autocovariance $C_{\mathsf{XX}}(\tau)$ for $\tau = \ldots, -2, -1, 0, 1, 2, \ldots$. Let $\hat{\mathsf{X}}(n+1) = a\mathsf{X}(n)$ be the linear estimate for $\mathsf{X}(n+1)$. Find the minimum mean square error linear estimate for X_{n+1}, i.e., determine the a that minimizes

$$\mathsf{E}\Big((\mathsf{X}(n+1) - a\mathsf{X}(n))^2 \Big).$$

2. $\mathsf{X}(t)$ is a real valued, bandpass random signal with power spectral density for positive frequencies given by

$$S_{\mathsf{XX}}(\omega) = 3\,(U(\omega - 9000) - U(\omega - 11000)) + 400\delta(x - 10000)$$

Here $U(\omega)$ is the unit step function. Determine $\mathsf{E}(\mathsf{X}(t))$, the mean value of the signal, and $\mathsf{E}(\mathsf{X}^2(t))$, the mean power in the random signal.

3. Consider a random sequence that changes as follows.

$$\mathsf{Q}_n = \begin{cases} Q_{n-1} + 1 & \text{with probability } \lambda(1-\mu) \\ \max\{Q_{n-1} - 1,\ 0\} & \text{with probability } (1-\lambda)\mu \\ Q_{n-1} & \text{with probability } \mu\lambda + (1-\lambda)(1-\mu) \end{cases}$$

where $0 < \lambda, \mu \leq 1$. Write a program to simulate this sequence, and obtain the time average and ensemble average for a given λ and μ. Using the simulation program, determine which of the following values corresponds to a stationary Q_n and to an ergodic Q_n.

(a) $\lambda = 0.7, \mu = 0.9$.

(b) $\lambda = 0.9, \mu = 0.7$.

(c) $\lambda = 0.7, \mu = 0.7$.

Provide some intuitive explanation for your finding and comment on a possible generalization. Experiment with different initialisations Q_0 and comment on the dependence of your answer on this dependence.

4. X_n is a random sequence that takes values in the sets $\{1, 2, 3, 4, 5, 6\}$. It evolves as follows

$$\mathsf{Pr}(\mathsf{X}_n = j | \mathsf{X}_{n-1} - i) = p_{i,j}$$

Write a simulation program to simulate the sequence X_n when the p_{ij} are given. For each of the following cases, use the results of your simulation to determine the stationarity and ergodicity of X_n. Explain your answers.

(a) $p_{ii} = 0.3$ for $i = 1, \ldots, 6$, $p_{12} = p_{34} = p_{43} = p_{56} = p_{65} = 0.7$, $p_{21} = 0.3, p_{24} = p_{25} = 0.2$, and $p_{ij} = 0$ for all other i, j.

(b) Same as above except that $p_{52} = 0.2$ $p_{56} = 0.5$ $p_{56} = 0.5$

(c) $p_{13} = p_{24} = p_{35} = p_{46} = p_{51} = p_{62} = 0.4$, $p_{14} = p_{23} = p_{36} = p_{45} = p_{52} = p_{61} = 0.6$, and $p_{ij} = 0$ for all other i, j.

(d) Same as above except $p_{61} = 0.3$ and $p_{66} = 0.3$.