EE 325: Probability and Random Processes

Assignment 2

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1. **[10 Marks]** Consider the digital binary symmetric communication channel that works as follows. The transmitter transmits a sequence of binary symbols. For this problem, consider the transmission of just one symbol $X \in \{x_0, x_1\}$. x_0 is transmitted with probability $p_0 = 0.2$ and x_1 is transmitted with probability $p_2 = 0.8$. The channel can introduce an error by 'flipping the symbol', i.e., x_0 will be received as symbol y_0 with probaility 0.6 and as y_1 with probability 0.4. Similarly, x_1 will be received as symbol y_1 with probaility 0.6 and as y_0 with probability 0.4. This is represented in the Figure. 1

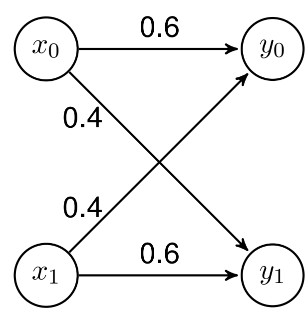

Figure 1: Binary Symmetric Channel

Find the following

   (a) $\mathbb{P}(y_0)$, $\mathbb{P}(y_1)$ i.e., probabilities of receiving y_0, y_1 at receiver

   (b) $\mathbb{P}(x_0|y_0)$, $\mathbb{P}(x_0|y_1)$, $\mathbb{P}(x_1|y_0)$, $\mathbb{P}(x_1|y_1)$.

2. **[20 Marks]** A person possesses five coins, two of which are are heads on both sides, one has a tail on both sides two are normal fair coins. With eyes shut, the person picks a coin at random and tosses it.

   (a) What is the probability that the lower face (the face that is not seen) of the coin is a head?

   (b) The person opens eyes and sees that the coin is showing heads on top. What is the probability that the lower face of the coin is a head?

   (c) The person shuts the eyes and tosses the same coin again. What is the probability that the lower face of the coin is a head?

   (d) The eyes are opened, and the person sees that the coin is showing heads. What is the probability that the lower face of the coin is a head?

3. **[15 marks]** There are R brown balls and B black balls in an urn. Balls are drawn at random without replacement. Let A_k be the event that a brown ball is drawn for the first time on the k-th draw. Find p_k, the probability of A_k. Now consider the case when B and R are increased to ∞ while keeping $\alpha = R/(B + R)$. Find p_k as $B + R \to \infty$.

4. **[15 marks]** There are n urns of which the r-th urn contains $r - 1$ brown balls and $n - r$ black balls. You pick an urn at random and pick two balls at random without replacement. What is the probability that the second ball is black. What is conditional probability that the second ball is black given that the first ball is black.

5. **[20 marks]** 10% of the surface area of a sphere is white and the rest is black. There are no assumptions on how this white part is distributed on the surface. Prove that it is always possible to inscribe a cube with all its vertices black. Think of a randomly inscribed cube. Let A_i be the probability a random vertex is white. Now obtain an upper bound on the probability that at least one of the vertices is white. Show that this strictly less than one. This proves that there is at least one cube with all black vertices.

6. **[40 marks]**

   (a) You want to estimate the fish population in Powai lake. You do the following: Catch 100 fish, mark them and release them back into the lake. Allow the fish to mix well and then you catch 100 fish again. Of these 10 are those that were marked before. Assume that the fish population is n and has not changed between the catches. Find the probability of the outcome of your experiment.

   (b) Now consider a generalisation. Catch m fish, mark them and release them back into the lake. Allow the fish to mix well and then you catch m fish. Of these p are those that were marked before. Assume that the actual fish population in the lakes is n and has not changed between the catches. Let $P_{m,p}(n)$ be the probability of the event (for a fixed p recatches out of m) coming from n fish in the lake. Generate a plot for $P_{m,p}(n)$ as a function of n for the following values of m and $p : m = 100$ and $p = 10, 20, 50, 75$. For each of these p, use the plots to estimate (educated guess) the actual value of n. Call these four estimates $\hat{n}_1, \ldots, \hat{n}_4$.

   (c) Now for each of the four $\hat{n}_i$, simulate the previous problem. One simulation will for case i will do the following. Take $\hat{n}_i$ fish in the pond, mark $m = 100$ of these, mix up the fish and catch $m = 100$ random fish. Repeat the experiment 500 times for each i, calculate the sample average from the 500 experiments and compare with the corresponding p from the previous question. Comment on the comparison between them.