DS 203 - Assignment 1

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Exercise 1

1) $P(A) = 0.8$ $A \rightarrow$ Manufactured in A

$P(B) = 0.2$ $B \rightarrow$ Manufactured in B

$P(D/A) = 0.3$ $D \rightarrow$ Defective

$P(D/B) = 0.1$

$\therefore P(D) = P(A \cap D) + P(B \cap D)$

as A & B partition the set

$$P(D) = P(A) \cdot P(D/A) + P(B) \cdot P(D/B)$$

$$= (0.8)(0.3) + 0.2(0.1)$$

$$P(D) = 0.24 + 0.02 = 0.26$$

2) $P(A/D) = ?$

$$P(A/D) = \frac{P(A \cap D)}{P(D)} = \frac{P(A) \cdot P(D/A)}{P(D)}$$

$$= \frac{(0.8)(0.3)}{0.26} = \frac{24}{26} = \frac{12}{13}.$$

Exercise 2

$E_1 \rightarrow$ web server is working.

$A_1 \rightarrow$ First attempt succeeds

$A_2 \rightarrow$ Second attempt succeeds

$A_K \rightarrow K^{th}$ attempt succeeds

$P(E_1) = 0.8$

$P(A_K | E_1) = 0.9$

$A_K | E_1$ and A_j / E_1 $K \neq j$ are independent

a) P(first attempt fails)

$$P(\bar{A}_1) = P(\bar{E}_1) + P(E_1) P(\bar{A}_1 | E_1)$$
$$= 0.2 + 0.8(0.1)$$
$$= 0.28$$

b) P(server works | first attempt fails)

$$P(E_1 | \bar{A}_1) = \frac{P(E_1 \cap \bar{A}_1)}{P(\bar{A}_1)} = \frac{P(E_1) \cdot P(\bar{A}_1 | E_1)}{P(\bar{A}_1)}$$

$$= \frac{(0.8) \cdot (0.1)}{0.2 + 0.8(0.1)} = \frac{2}{7}$$

$$= 0.286$$

c) P (second attempt fails | first attempt fails)

$$P(\bar{A}_2 | \bar{A}_1) = \frac{P(\bar{A}_1 \cap \bar{A}_2)}{P(\bar{A}_1)}$$

$$= \frac{0.2 + 0.8 \underset{\bar{A}_1/E_1}{(0.1)} \underset{\bar{A}_2/E_1}{(0.1)}}{0.2 + 0.8(0.1)}$$

$$= \frac{0.208}{0.28} = 0.742$$

d) P(server works | First and second attempts fail)

$$P(E_1 | \bar{A}_1 \cap \bar{A}_2) = \frac{P(E_1 \cap (\bar{A}_1 \cap \bar{A}_2))}{P(\bar{A}_1 \cap \bar{A}_2)} = \frac{P(E_1) \cdot P(\bar{A}_1 \cap \bar{A}_2 / E_1)}{P(E_1) P(\bar{A}_1 \cap \bar{A}_2 | E_1) + P(\bar{E}_1) \cdot P(\bar{A}_1 \cap \bar{A}_2 / \bar{E}_1)}$$

$$= \frac{0.8(0.1)(0.1)}{0.8(0.1)(0.1) + 0.2} = 0.03846$$

Exercise 3

$E_1 \rightarrow$ Probability that atleast one roll of two dice rolls is six

$$E_1 = \left\{ \begin{matrix} (6,1), (6,2) \ldots (6,5) \\ (1,6), (2,6) \ldots (5,6) \end{matrix}, (6,6) \right\}$$

$$P(E_1) = \frac{|E_1|}{|S|} = \frac{11}{36} = 0.3056$$

$E_2 \rightarrow$ Two faces are different

$$E_2 = \{ (i,j), i \neq j \quad 1 \le i,j \le 6 \}$$

$$n(E_2) = 6 \times 5 = 30$$

$$E_1 \cap E_2 = \left\{ \begin{matrix} (1,6), (2,6) \ldots (5,6) \\ (6,1), (6,2) \ldots (6,5) \end{matrix} \right\}$$

$$P(E_1 | E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{(10/36)}{(30/36)} = \frac{1}{3}$$

$$= 0.33$$

Exercise 4

$E_1 \rightarrow$ colorblind

$M \rightarrow$ male

$\bar{M} \rightarrow$ female

$P(E_1/M) = 0.05$

$P(E_1/\bar{M}) = 0.01$

$P(M/E_1) = ?$

$$P(M/E_1) = \frac{P(M \cap E_1)}{P(E_1)} = \frac{P(M) \cdot P(E_1/M)}{P(M) \cdot P(E_1/M) + P(\bar{M}) \cdot P(E_1/\bar{M})}$$

Since equal no of males and females

$P(M) = P(\bar{M}) = 0.5.$

$$P(M/E_1) = \frac{0.5(0.05)}{0.5(0.05) + 0.5(0.01)} = 5/6 \,//.$$

Exercise 5

a) Claim: E is independent of itself

$\Rightarrow P(E) = 0$ or $P(E) = 1$

Proof:

$P(E|E) = P(E)$

$\frac{P(E \cap E)}{P(E)} = P(E) \Rightarrow P(E) = (P(E))^2$

$P(E)[P(E) - 1] = 0 \Rightarrow P(E) = 0$ or $P(E) = 1$

Hence Proved!

b) $P(A) = 0.3$, $P(B) = 0.4$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

given : A & B are independent events

$$P(A \cap B) = P(A) \cdot P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$$

$$= 0.3 + 0.4 - (0.3)(0.4)$$

$$= 0.7 - 0.12$$

$$P(A \cup B) = 0.58$$

given: A and B are mutually exclusive $\Rightarrow$ $P(A \cap B) = 0$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.3 + 0.4 - 0$$

$$P(A \cup B) = 0.7$$

(c) $P(A) = 0.6$ and $P(B) = 0.8$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.6 + 0.8 - P(A \cap B) \leq 1$$

$$\Rightarrow 1.4 - 1 \leq P(A \cap B)$$

$$\Rightarrow P(A \cap B) \geq 0.4.$$

$\therefore\ P(A\cap B) \geq 0.4$

$\Rightarrow$ If A & B were independent then

$P(A\cap B) = P(A)\cdot P(B) = (0.6)(0.8) = 0.48$

$= 0.48 \geq 0.4$

Hence, A & B can be independent.

$\Rightarrow$ If A and B are mutually exclusive then $P(A\cap B) = 0 \ngeq 0.4$

Hence, A & B cannot be mutually exclusive.

Exercise 6

1) $$F(x) = \begin{cases} e^{-x^2/4}, & \text{if } x < 0 \\ 1 - e^{x^2/4}, & \text{if } x \geq 0 \end{cases}$$

$\lim_{x\to+\infty} F(x) = 1$

$\lim_{x\to-\infty} F(x) = 0$

$\int_0^x f(x)\,dx = F(x)$

where f is the P.D.F

$F'(x) = f(x)$

$$f(x) = F'(x) = \begin{cases} \dfrac{-xe^{-x^2}}{2}, & x < 0 \\ \dfrac{xe^{-x^2}}{2}, & x \geq 0 \end{cases}$$

$F'(x) \geq 0 \quad \forall \quad x \in \mathbb{R} \quad \rightarrow$

$F(x)$ in non decreasing in x

$F(x)$ is right continuous $\forall$ $x \in \mathbb{R}$.

$F(x)$ in a valid CDF

$$P(x^2 > 5) = P(x > \sqrt{5}) + P(x < -\sqrt{5})$$

$$= \int_{-\infty}^{-\sqrt{5}} f(x)\,dx + \int_{\sqrt{5}}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{-\sqrt{5}} \frac{-x}{2} e^{-x^2} dx + \int_{\sqrt{5}}^{\infty} \frac{x e^{-x^2/2}}{2} dx$$

$$= \frac{1}{4}\int_{-\infty}^{-5} e^{t}\,dt - \frac{1}{4}\int_{-5}^{-\infty} e^{t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{-5} e^{t}\,dt = \frac{1}{2}\left[e^{-5}\right] = \frac{1}{2e^5}$$

2) $$F(x) = \begin{cases} 0, & x < 0 \\ 0.5 + e^{-x}, & 0 \leq x < 3 \\ 1, & x \geq 3 \end{cases}$$

$$\lim_{x \to \infty} F(x) = 1 \qquad \lim_{x \to -\infty} F(x) = 0$$

$F(x)$ in right continuous $\forall x \in \mathbb{R}$

$F'(x)$ denote the first derivative

$$F'(x) = \begin{cases} 0 & x < 0 \\ -e^{-x} & 0 < x < 3 \\ 0 & x > 0 \end{cases}$$

$F(0) = 1.5 \Rightarrow \Leftarrow$ As $0 \leq F(x) \leq 1$

$F(x)$ is not non-decreasing $\forall x \in R \Rightarrow F$ is not a valid CDF.

3)

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.5 + x/20, & 0 \leq x \leq 10 \\ 1, & x \geq 10 \end{cases}$$

$\lim_{x \to \infty} F(x) = 1 \quad \lim_{x \to -\infty} F(x) = 0$

$F(x)$ is right continuous $\forall x \in \mathbb{R}$

$$F'(x) = \begin{cases} 0, & x < 0 \\ 1/20, & 0 < x < 10 \\ 0, & x > 10 \end{cases}$$

$F'(x) \geq 0 \quad \forall x \in R \Rightarrow F$ is non decreasing $\forall x \in \mathbb{R}$

$\therefore$ PDF $f(x) = F'(x)$

as $F(x) = \int_{-\infty}^{x} f(x)\,dx$

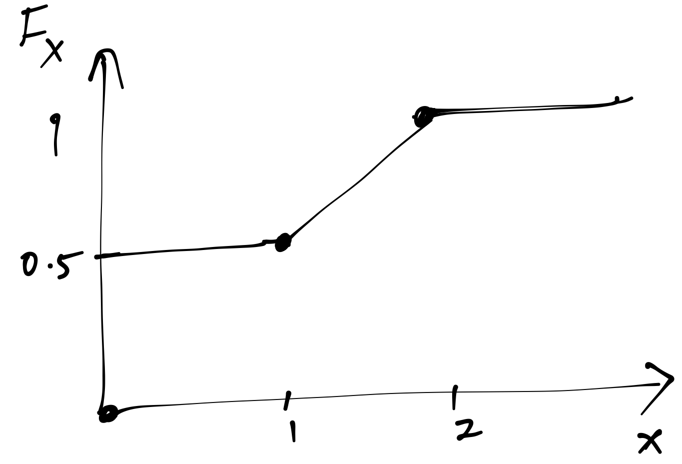
$\therefore P(x^2 > 5) = P(x < -\sqrt{5}) + P(x > \sqrt{5})$

$$P(x^2>5) = \int_{-\infty}^{-\sqrt{5}} 0\,dx + \int_{\sqrt{5}}^{10} \frac{1}{20}\,dx$$

$$P(x^2>5) = \frac{1}{20}(10-\sqrt{5})$$

$$\therefore P(x^2>5) = \frac{1}{20}(10-\sqrt{5}) = 0.388$$

Exercise 7



1) $F_X(x) = P(X \le x)$

$P(X \le 0.8) = F_X(0.8)$

$= 0.5$

$$P(X \le x) = F(x) = \int_{-\infty}^{x} f(x)\,dx$$

f(x) is the probability density function

now using FTC $F'(x) = f(x)$

$\delta(x)$ denote dirac delta function

$$\therefore \quad f(x) = F'(x) = \begin{cases} \frac{1}{2}\delta(x), & x < 1 \\ \frac{1}{2} & 1 < x < 2 \\ 0 & x > 2 \end{cases}$$

$$\therefore \quad E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{1} x\,\delta(x)\,dx + \int_{1}^{2} x \cdot \frac{1}{2}\,dx = 0 + \left[\frac{x^2}{4}\right]_1^2$$

$$= \frac{3}{4} = 0.75$$

$\therefore \quad E(X) = 0.75$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= 0 + \int_{1}^{2} x^2\left(\frac{1}{2}\right)dx = \left[\frac{x^3}{6}\right]_1^2 = \frac{8-1}{6}$$

$$= 7/6$$

$$\therefore \quad Var(X) = E(x^2) - (E(x))^2$$

$$= 7/6 - (3/4)^2$$

$$= \underline{\underline{0.604}}$$

$$\therefore \quad \underline{\underline{Var(X) = 0.604.}}$$

Exercise 8

$$f(x) = \begin{cases} ce^{-2x}, & 0 \le x < \infty \\ 0, & x < 0 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{\infty} ce^{-2x}\,dx = 0 + \left[\frac{ce^{-2x}}{2}\right]_{\infty}^{0}$$

$$= \frac{c}{2}\left[1 - e^{-\infty}\right] = c/2$$

$$= \frac{c}{2} = 1 \Longrightarrow c = 2.$$

$$P(x > 2) = \int_{2}^{\infty} 2e^{-2x}\,dx$$

$$= \frac{2}{2}\left[e^{-2x}\right]_{\infty}^{2} = \frac{1}{e^4}$$

Exercise 9

for coin, $P(H) = 0.7$, $P(T) = 0.3$

$E_1 \to$ 1 head

$$P(E_1) = P(\{HTT, THT, TTH\})$$

$$= 3(0.7)(0.3)^2 = 0.189$$

$E_2 \rightarrow$ Two heads

$$P(E_2) = P(\{HHT \quad HTH \quad THH\})$$

$$= 3(0.7)^2(0.3) = 0.441.$$

$E_3 \longrightarrow$ three heads

$$P(E_3) = \{HHH\}$$

$$= (0.7)^3 = 0.343$$

$E_0 \longrightarrow$ zero heads

$$P(E_3) = \{TTT\}$$

$$= (0.3)^3 = 0.027$$

$$\therefore \text{PMF } (x) = \begin{cases} 0.027, & x = 0 \\ 0.189, & x = 1 \\ 0.441, & x = 2 \\ 0.343, & x = 3 \end{cases}$$

for no of heeds

Exercise 10

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

To find $P(X \ge 0.4 \mid X \le 0.8)$

$$P(X \geq 0.4 \mid X \leq 0.8) = \frac{P(0.4 \leq X \leq 0.8)}{P(X \leq 0.8)}$$

$$= \frac{\int_{0.4}^{0.8} f_X(x)\,dx}{\int_{0}^{0.8} f_X(x)\,dx} = \frac{\int_{0.4}^{0.8} 2x\,dx}{\int_{0}^{0.8} 2x\,dx} = \frac{(0.8)^2 - (0.4)^2}{(0.8)^2}$$

$$= \underline{\underline{0.75}}$$

Exercise 11

Exponential distributed random variable

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X > a+b \mid X > a) = \frac{P((X > a+b) \cap (X > a))}{P(X > a)}$$

since $a, b > 0 \implies X > a+b \subset X > a$.

$$P(X > a+b \mid X > a) = \frac{P(X > a+b)}{P(X > a)} = \frac{\int_{a+b}^{\infty} \lambda e^{-\lambda x}\,dx}{\int_{a}^{\infty} \lambda e^{-\lambda x}\,dx}$$

$$= \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda a - \lambda b + \lambda a}$$

$$P(X > a+b \mid X > a) = e^{-\lambda b}$$

λ is the parameter.

Exercise 12

$E \rightarrow$ All coins land heads

$$I_E = \begin{cases} 1, & \text{if } E \text{ occurs} \\ 0, & \text{if } E^c \text{ occurs} \end{cases}$$

Ω be the sample space

$I_E = 1$ when E occurs $\Rightarrow$

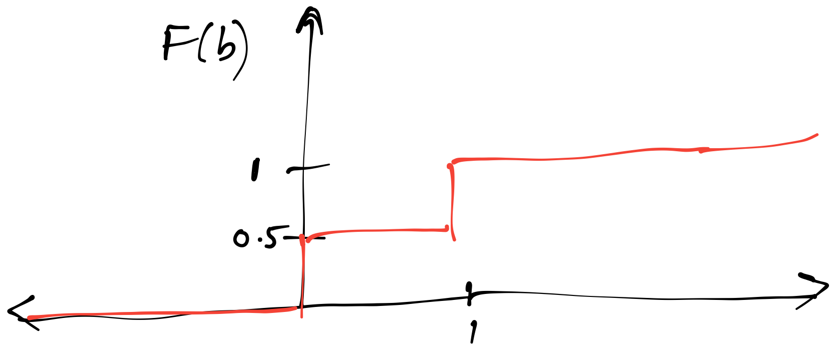
$I_E = 1$ for the outcome $\{HHHHH\} \subset \Omega$

$$P(I_E = 1) = P(\{HHHHH\})$$

$$= \frac{n(\{HHHHH\})}{n(\Omega)} = \frac{1}{32}$$

Exercise 13

$$F(b) = \begin{cases} 0, & b < 0 \\ 1/2, & 0 \le b < 1 \\ 1, & 1 \le b < \infty \end{cases}$$



$$\therefore PMF_X(x) = \begin{cases} 0.5, & x = 0 \\ 0.5, & x = 1 \end{cases}$$

Exercise 14

$E_1 \rightarrow$ First two drawn balls are white of four balls drawn

Since drawn with replacement

$P(W) = \frac{3}{6} = \frac{1}{2}$ $P(B) = 3/6 = 1/2$

Hence getting white and black balls in equiprobable and analogous to a coin toss.

$n(E_1) = \underline{\quad}\,\underline{\quad}\,\underline{\quad}\,\underline{\quad}$ $\{WBWB\}$ in ${}^4C_2 = 6$ ways

$\Omega = \{ \underset{W/B}{\underline{\quad}} \; \underset{W/B}{\underline{\quad}} \; \underset{W/B}{\underline{\quad}} \; \underset{W/B}{\underline{\quad}} \}$ in $2^4 = 16$ ways

$$P(E_1) = \frac{n(E_1)}{n(\Omega)} = \frac{6}{16} = 3/8$$

Exercise 15 :

$X \rightarrow$ The no of tosses needed so that n^{th} toss is the r^{th} head.

$P(\text{head}) = p$ $P(\text{tail}) = 1-p$

$E_1 \rightarrow$ r^{th} head appears on the n^{th} toss

let n be the number of flips needed of which r flips are heads and the last toss landed a head

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \cdots\cdots}_{n-1 \text{ tosses}} \quad \underline{H}$$

$E_2 \rightarrow$ $n-1$ tosses need to be filled by $r-1$ heads and $(n-r)$ tails

$$\therefore P(X=n) = P(E_2) \cdot P(E_1/E_2)$$

$$= P(E_2) \cdot P(E_1) \quad \left\{ \begin{array}{l} \text{Individual coin tosses} \\ \text{are independent events} \end{array} \right\}$$

$$= \left({}^{n-1}C_{r-1}\, p^{r-1} (1-p)^{n-r} \right) \cdot p$$

$$P(X=n) = {}^{n-1}C_{r-1}\, p^{r} (1-p)^{n-r}$$

Hence we are done

Exercise 16:

The no of errors on the page follows poisson distribution with $\lambda = 1$

$$P(X=i) = \frac{e^{-1}}{i!}(1)^{i}$$

$$P(X \geq 1) = P(X=1) + P(X=2) + \cdots \infty$$

$$= \frac{1}{e}\left[\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \cdots \infty \right]$$

$$= \frac{1}{e}\left[e-1 \right] = 1 - \frac{1}{e} = 0.632$$