Lecture Note 4

CSCI 6470 Algorithms
(Fall 2024)

Liming Cai

School of Computing UGA

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Chapter 4. Advanced Algorithmic Techniques

Topics to be discussed:

- Dynamic programming
- Greedy algorithms
- Flow networks

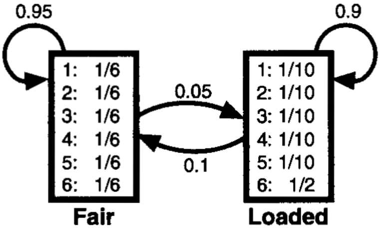
1. Dynamic programming

Introduction to DP with problem:
computing the n^{th} Fibonacci numbers

- naive recursive algorithm (top-down), at least 1.41^n
  (how to describe "at least 1.41^n"?)
- memoized recursive algorithm (top-down, use lookup table) $O(n)$
- iterative algorithm (bottom-up) $O(n)$

1. Dynamic programming

Decoding dishonest dice rollings



A hidden Markov model M

```
O = 1654622316516643254132565442355122126161626   <- observable
S = FFFFFFFFFLLLLLLLFFFFFFFFFFFFFFFFFFFLLLLLLLL   <- hidden dice
```

- decoding question: what are the underlying sequence of dices used?

1. Dynamic programming

A more significant problem:

1. Dynamic programming

A more significant problem:

1. Dynamic programming

Intuitively,

- dynamic programming is an exhaustive search method;
- dynamic programming fills a table(s) with numerical data according to certain order;
- data dependency order in the table implies the desired solution;

1. Dynamic programming

Problem 1: Single-source shortest paths in DAG

- (based on topological sort order), recall how we did it;
- a slightly different order,

for $v = 1$ **to** n (order in a topological sort)
$$dist(v) = \min_{(u,v)\in E}\{dist(u) + l(u,v)\}$$
remember the corresponding `prev`

- how to write this into pseudo code?

1. Dynamic programming

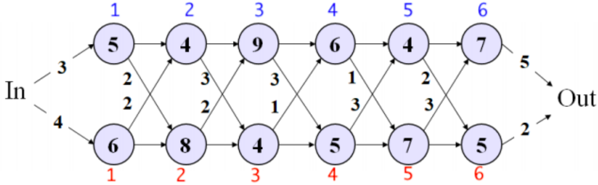
- Fill the table dist in a topological order

```
for v = 1 to n
   dist(v) = infinite;
   prev(v) = nil;
   for all (u, v) in E
     if dist(v) > dist(u) + l(u,v)
        dist(v) = dist(u) + l(u,v);
        prev(v) = u;
```

- Print out all shortest-paths based on dist and prev
  [in class exercise]

1. Dynamic programming

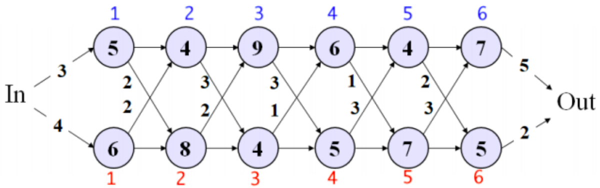
Problem 2: the fastest path through a factory



- $2n$ stations; each station has processing time;
- no time cost for transitions within the same production line;
- there are time costs between two different production lines;
- a path time = sum of all processing and transition times on the path;

1. Dynamic programming

Step 1: analysis of the problem



- the fastest path $\mathtt{In} \rightsquigarrow \mathtt{Out}$ has to be

$$\text{the faster of } \begin{cases} \text{a fastest path } \mathtt{In} \rightsquigarrow 6 \text{ then edge } 6 \to \mathtt{Out}, \\ \text{a fastest path } \mathtt{In} \rightsquigarrow 6 \text{ then edge } 6 \to \mathtt{Out} \end{cases}$$

- the fastest path $\mathtt{In} \rightsquigarrow 4$ has to be

$$\text{the faster of } \begin{cases} \text{a fastest path } \mathtt{In} \rightsquigarrow 3 \text{ then edge } 3 \to 4, \\ \text{a fastest path } \mathtt{In} \rightsquigarrow 3 \text{ then edge } 3 \to 4 \end{cases}$$

1. Dynamic programming

In general,

- for every $k = 2, 3, \ldots, n$,
  the fastest path $\texttt{In} \rightsquigarrow k$ has to be

$$\text{the faster of} \begin{cases} \text{a fastest path } \texttt{In} \rightsquigarrow k-1 \text{ then } k-1 \to k, \\ \text{a fastest path } \texttt{In} \rightsquigarrow k-1 \text{ then } k-1 \to k \end{cases}$$

- for every $k = 2, 3, \ldots, n$,
  the fastest path $\texttt{In} \rightsquigarrow k$ has to be

$$\text{the faster of} \begin{cases} \text{a fastest path } \texttt{In} \rightsquigarrow k-1 \text{ then edge } k-1 \to k, \\ \text{a fastest path } \texttt{In} \rightsquigarrow k-1 \text{ then } k-1 \to k \end{cases}$$

- what about $k = 1$?

  the fastest path $\texttt{In} \rightsquigarrow 1$ is $\texttt{In} \to 1$

  the fastest path $\texttt{In} \rightsquigarrow 1$ is $\texttt{In} \to 1$

1. Dynamic programming

Now what?

Two observations:

- the problem is to find a shortest path from station `In`;
  every path is associated with a time (`dist`);
- shortest paths are recursively defined;
  so fastest times can be recursively defined;

1. Dynamic programming

Step 2: define numerical objective function

For $k = 1, 2, \ldots, n$, $i = 1, 2$:

- Label with $(1,1), \ldots, (1,n)$ for stations in production line 1; and with $(2,1), \ldots, (2,n)$ for production line 2;
- Let $pt_i(k)$ be the processing time on station (i,k);
- Let $tt_i(k-1)$ be the transfer time from station $(i, k-1)$ to station $(\widetilde{i}, k)$, where $\widetilde{i}$ is the opposite production line of i;
- Define **function** $ft_i(k)$ to be the fastest time of a path from station `In` to station (i,k);

  Then

$$ft_i(k) = \min \begin{cases} ft_i(k-1) + pt_i(k) \\ ft_{\widetilde{i}}(k-1) + tt_{\widetilde{i}}(k-1) + pt_i(k) \end{cases} \qquad k \geq 2$$

$$ft_i(1) = \text{the known time from } \mathtt{I} \text{ to station } (i,1) + pt_i(1)$$

1. Dynamic programming

Step 3: Establish and fill DP tables

- establish a table $F_{2\times n}$ to store values of function $ft_i(k)$, where $i = 1, 2$ and $k = 1, 2, \ldots, n$;
- establish a table $\texttt{prev}_{2\times n}$ to store previous stations
- fill the tables using the recursive formulas for $ft_i(k)$, with an iterative program;
- write the pseudo code for table filling (in-class exercise)

1. Dynamic programming

Step 4: Trace back the fastest path

- table `prev` should contain enough information about the fastest path
- but wait, what is the fastest time through the factory?
- from the fastest time, we know the last station of which production line is on the fast path before station `Out`;
- traceback can start from that station, and recursively;
- write pseudo code for traceback (in-class exercise)

1. Dynamic programming

Complexity of a DP algorithm

- essentially the time to fill tables
  $=$ table size $\times$ cell filling time
- plus the time to trace back solution(s) (how much is it?)

1. Dynamic programming

Characteristics of problems that can be solved with DP:

(1) **Optimal substructures**

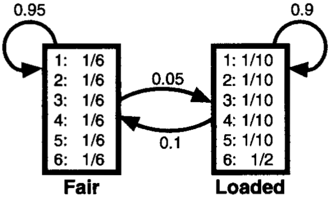
- the solution to the problem can be recursively constructed from solutions to some subproblems;
- solutions to subproblems should also be optimal;

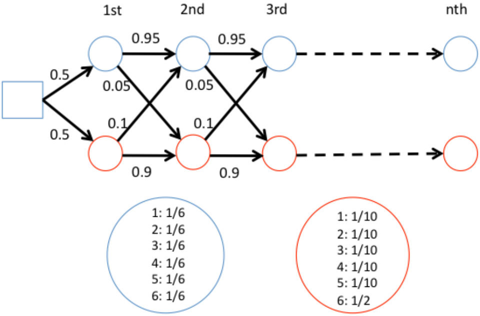
(2) **Overlapping subproblems**

- one subproblem solution is shared by more than one other problem to construct their solutions

1. Dynamic programming

Problem 3: Decoding dishonest dice rolls





$O = o_1 o_2 \ldots o_n$ observed dice roll outcomes;
$S = d_1 d_2 \ldots d_n$ the sequence of dice **with highest probability**

1. Dynamic programming

Probability of dice rollings:

- emission probability $e_F(k) = \frac{1}{6}$ for all $k = 1, 2, \ldots, 6$;
- transition probability

$$t_{FF} = 0.95, t_{FL} = 0.05, t_{LL} = 0.9, t_{LF} = 0.1$$

- computing probability of rolling 2466 with dice $\texttt{FFLL}$

$$0.5 \times e_F(2) \times t_{FF} \times e_F(4) \times t_{FL} \times e_L(6) \times t_{LL} \times e_L(6) = ?$$

is it different from with dice $\texttt{FFFF}$? (in-class exercise)

1. Dynamic programming

Step 1: problem analysis

Analog between decoding dice and finding the fastest path through factory

- a sequence of dice consists of either `F` or `L` dices in each position;

  a path through factory consists of stations either in production line 1 or line 2;

- the most like sequence is one with the highest probability;

  the fastest path is one with smallest time;

1. Dynamic programming

- the most likely sequence ends at either `Fair` or `Loaded` die;
- for $k \geq 1$,
  the most likely sequence of length k ending at `Fair` die is
  (1) either the most likely sequence of length $k - 1$ ending at `Fair` die followed by `Fair` die,
  (2) or the most likely sequence of length $k - 1$ end at `Loaded` die followed by `Fair`,

  whichever has higher probability

1. Dynamic programming

Step 2: definition of objective function

Define $m(k, F)$ to be the highest probability of a sequence of k dice ending at `Fair` die to emit the first k observed numbers. Then

Recursively,

$$m(k, F) = \max \begin{cases} m(k-1, F) \times t_{FF} \times e_F(o_k); \\ m(k-1, L) \times t_{LF} \times e_F(o_k); \end{cases}$$

$m(k, L) =?$ (in-class exercise)

base cases:

$m(1, F) = 0.5 \times e_F(o_1)$

$m(1, L) = 0.5 \times e_L(o_1)$

1. Dynamic programming

Step 3: fill DP tables

- what tables are needed?
- how to fill the tables?
- pseudo code for the table filling process (in-class exercise)

1. Dynamic programming

Step 4: trace back solutions

- what solutions?
- how to get the solutions?
- pseudo code for traceback (in-class exrecise)

1. Dynamic programming

The **Decoding dishonest dice** problem has the characteristics

- optimal substructure, what is it in the problem?
- overlapping subproblems, what are they in the problem?

1. Dynamic programming

Problem 4: Knapsack problem

1. Dynamic programming

Problem 4: Knapsack problem

- input: n items, of size/weight s_i and value v_i, $i = 1, \ldots, n$, and a knapsack of volume W;

  output: a subset of items $A \subseteq \{1, 2, \ldots, n\}$, such that

$$\sum_{i \in A} v_i \text{ is maximized, subject to } \sum_{i \in A} s_i \leq W$$

- there is a recursive solution to this problem.

1. Dynamic programming

Step 1: problem analysis

In the previous three problems, subproblems are "prefixes";
do we have "prefix subproblems" for Knapsack?

how to select some items from the first k items into a space of ? volume X, $X \leq W$.

- either item k is selected, with gain of value v_k but decrease of available space to $X - s_k$;
- or discard item k, with no change in value and no change in available space

1. Dynamic programming

Step 2: define objective function

- associated with a solution is the total value of selected items;
- define objective function $V(k, X)$ to be the maximum value of items selected from $\{1, 2, \ldots k\}$. Then

$$V(k, X) = \max \begin{cases} V(k-1, X - s_k) + v_k & X \geq s_k \\ V(k-1, X) \end{cases}$$

base cases

$$V(0, X) = 0; \; X = 0, 1, 2 \ldots, W$$

$$V(k, X) = 0; \; k = 0, 1, 2, \ldots, n$$

1. Dynamic programming

Step 3: Fill DP tables

- dimensions of tables: $(n+1) \times (W+1)$
- data dependence, `prev` info;
- fill the table, pseudo code (taken-home exercise)

	X = 0	1	2	3	4	5	6	7	8
k=0	0	0	0	0	0	0	0	0	0
k=1	0	0	1	1	1	1	1	1	1
k=2	0	0	1	2	2				
k=3	0								
k=4	0								

item	1	2	3	4
value	1	2	5	6
size	2	3	4	5

W=8

base cases

V(0, X) = 0,
V(k, 0) = 0

recursive cases

$$V(1,1) = \max \begin{cases} V(0, 1-s1) + v1 & \text{not possible} \\ V(0, 1) = 0 \end{cases}$$

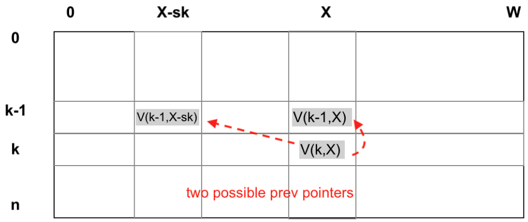
$$V(2,3) = \max \begin{cases} V(1, 3-s2) + v2 = V(1, 0) + 2 = 2 \\ V(1, 3) = 1 \end{cases}$$

1. Dynamic programming

Step 4. Trace back optimal packing

- How to identify `prev` info? recall

$$V(k,X) = \max \begin{cases} V(k-1, X-s_k) + v_k & X \geq s_k \\ V(k-1, X) \end{cases}$$



- pseudo code for traceback of optimal solution from DP tables

Taken-home exercises IV(A)

1. Run Bellman-Ford algorithm on a directed graph of at least 5 vertices to find shortest paths from the source vertex to all others.

   (1)After the shortest paths are found, choose one path consisting of the most edges (say, 4 edges) to show that every one of these edges was actually relaxed in one edge relaxation phase.

   (2) Modify your graph slightly so it contains a negative cycle. Show how the algorithm is able to detect the negative cycle.

2. Find an example for the **Fastest Path Through Factory** problem; fill in dynamic programming tables `time` and `prev`. Also write the recursive trace back algorithm to output a fastest path based on the data in table `prev`.

3. Consider an alternative dynamic programming solution to the **Fastest Path Through Factory** problem. Instead of computing the fastest path from station `In` to every station (i, k), we compute the fastest path from every station (i, k) to station `Out`.

   (1) Explain why the idea should work.

   (2) Design a numerical function associated with new fastest paths including recursive and base cases.

Taken-home exercises IV(A) (con't)

4 Find an example for the **0-1 Knapsack** problem; fill in dynamic programming tables `value` and `prev`. Also write the recursive trace back algorithm to output an optimal packing of items based on the data in table `prev`.

5 Consider a slightly modified version of the **0-1 Knapsack** problem. Instead of finding an optimal packing of items to maximize the total value of packed items, this version is, given a threshold for value, to minimize the total space of the packed items whose total value should meet or exceed the given threshold.

(1) What should be an output (solution) of this new problem?

(2) Identify a recursive pattern in the solution.

(3) Design a numerical function associated with solution including recursive and base cases.

1. Dynamic programming

Problem 5: Edit Distance problem

measuring distance between two input strings, based on how many

(1) matches;
(2) insertions;
(3) deletions;
(4) mismatches;

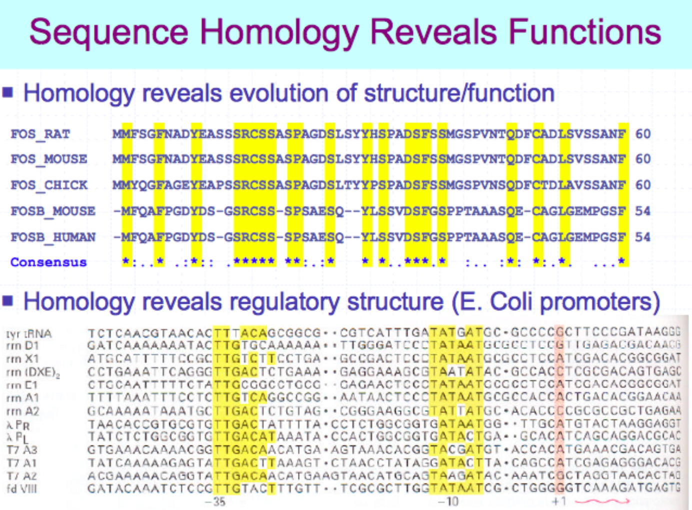
```
E V O L V I N G            edited    _ E V O L V _ I _ N G
R E V O L U T I O N          ==>     R E V O L U T I O N _
```

- scores is a part of input;
  e.g., match 0, insertion/deletion 1, mismatch 2.
  the above edit gives 6 points. Is there a lower score edit?
- **the goal of the problem is to find a lowest score edit.**

1. Dynamic programming

Problem 5: Edit Distance problem

A significant application: biological sequence alignment



1. Dynamic programming

Step 1 identify optimal substructure

Handle the problem recursively:

```
E V O L V I N [G]          3 possible    G    _    G
R E V O L U T I O [N]      scenarios     N    N    _
```

3 subproblems: to find lowest score edits for

```
(1)  E V O L V I N           G
     R E V O L U T I O       N

(2)  E V O L V I N G         _
     R E V O L U T I O       N

(3)  E V O L V I N           G
     R E V O L U T I O N     _
```

Lowest score edit is chosen over the 3 subproblems.

1. Dynamic programming

Step 2 define objective function

- input two strings $x[1..m]$ and $y[1..n]$;
- define $E(i,j)$ be the smallest distance (lowest score) between prefixes $x[1..i]$ and $y[1..j]$;
- then the recursive formula for $E(i,j)$:

$$E(i,j) = \min \begin{cases} E(i-1, j-1) + \texttt{diff}(i,j) \\ E(i, j-1) + 1 \\ E(i-1, j) + 1 \end{cases}$$

where

$$\texttt{diff}(i,j) = \begin{cases} 0 & x[i] = y[j] \\ 2 & x[i] \neq y[j] \end{cases}$$

$\texttt{diff}$ and all scores can be redefined for other problems!

1. Dynamic programming

Step 3 DP table filling (taken-home exercise)

Step 4 Solution trace back (taken-home exercise)

1. Dynamic programming

Pseudocode for DP table filling and solution trace back

- DP table filling:

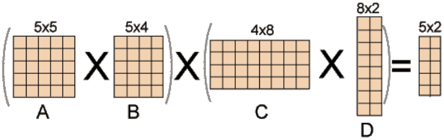
  (1) design of tables,
  (2) order of cells to fill,
  (3) iterative (can it be recursive?)

- Trace back:

  repetitive process following `prev` information (can it be recursive?)

1. Dynamic programming

Problem 6: Matrix Chain Multiplication (MCM)



- $((A \times B) \times (C \times D))$: $5 \times 5 \times 4 + 4 \times 8 \times 2 + 5 \times 4 \times 2 = 204$ ops
- $(((A \times B) \times C) \times D)$: $5 \times 5 \times 4 + 5 \times 4 \times 8 + 5 \times 8 \times 2 = 340$ ops

Problem MCM:

Input: dimensions $p_0 \times p_1, p_1 \times p_2, \ldots, p_{n-1} \times p_n$ of n matrices $A_1, A_2, \ldots, A_n$

Output: a grouping order (parenthesization) for the matrices to minimize the total number of scalar multiplication ops.

1. Dynamic programming

Work on the following steps in the classroom:

- Identify subproblems and relationships of their solutions;

  subproblem: $A_i, A_{i+1}, \ldots, A_j$

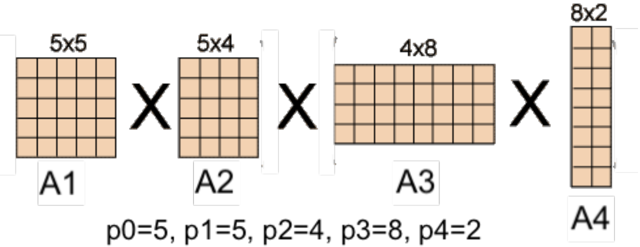
  how to break it into "smaller" subproblems?

- Formulation of objective function and its recurrent relations

  what should the objection function be associated with any subproblem?

- What may the DP tables look like?
- How to traceback for a solution (parenthesization)

1. Dynamic programming



	1	2	3	4
1	0	5x5x4	?	?
2		0	5x4x8	?
3			0	4x8x2
4				0

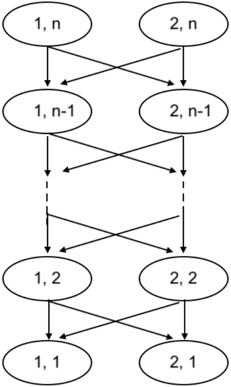
table for m(i, j)

1. Dynamic programming

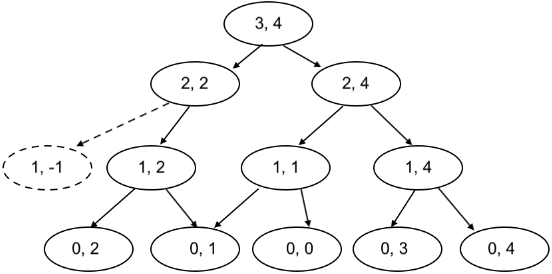
Exploitation of problem structure

- problem decomposition (always top-down, recursive structure)
- top-down implementation (mostly recursive)
- bottom-up implementation (mostly iterative)
- where does DP fit?
- memoization (using more space to gain time speed up)
- memoized DP

1. Dynamic programming



Assembly Line

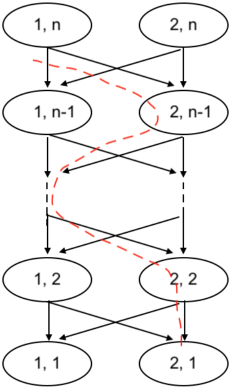


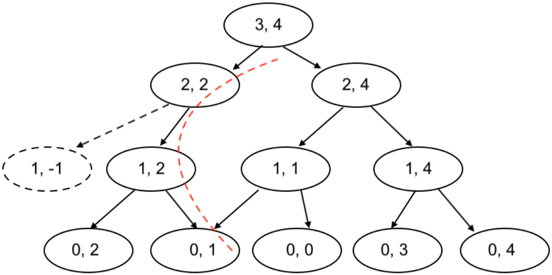
item	size	value
1	1	20
2	3	40
3	2	50

W = 4

0-1 Knapsack

1. Dynamic programming





item	size	value
1	1	20
2	3	40
3	2	50

W = 4

Assembly Line

0-1 Knapsack

Taken-home exercises IV(B)

1. Solve the **Edit Distance** problem on two input sequences of length at least 4. Fill out the DP tables (`dist` and `prev` tables) and trace back the optimal editing.
2. Let `prev` be DP table used for solving problem **Edit Distance**. If in cell `prev`$[i, j]$, the information is "$\uparrow$", what does it mean? What if the information is "$\leftarrow$" (and "$\nwarrow$") instead?
3. We may solve the **Edit Distance** using the "backward" style of dynamic programming. Explain the technical details regarding how to design an objective function that has recurrence relation for the DP.

Taken-home exercises IV(B)

4. The **Longest Common Subsequence** (LCS) problem is a special case of **Editing Distance** problem, where a match of letters is rewarded with 1 point and all other penalties have 0 point. The LCS problem is a maximization problem instead of minimization.

   Give the objective function for LCS and its recurrence relation tailored from Editing Distance.

5. Solve the **Matrix Chain Multiplication** on a chain of 4 matrices. Fill in the DP tables.

2. Greedy algorithms

Problem 1: Fractional Knapsack problem

- input: n items, of value v_i and size s_i, and knapsack size W;
- output: $f_1, f_2, \dots, f_n$, $0 \le f_i \le 1$, such that

$$\sum_{i=1}^{n} f_i v_i \text{ is maximized}$$

subject to $\sum_{i=1}^{n} f_i s_i \le W$

Not only options of items and but also options of fractions!

2. Greedy algorithms

There are **greedy algorithms** for this problem.

(1) Idea:

- compute "value density" $d_i = \frac{v_i}{s_i}$;
- sort items according to d_i, non-decreasingly;
- choose items in this order;
  pack the current item (whole or a part, space allowed)

(2) Efficient: only linear time is required + sorting time;

(3) unlike DP which guarantees an optimal solution,
a greedy algorithm may not;

- we need to prove the greedy strategy leads to optimal solutions.

2. Greedy algorithms

A **greedy-choice property** for Fractional Knapsack:

The item with the maximum value density is in some optimal solution.

Proof:

- Assume items are ordered 1 to n based on density d_i;
- Let $S = \{f_1, f_2, \dots, f_n\}$ be any solution.
- If $f_1 < 1$, going through all $f_i, i = 2, \dots, n$, to update

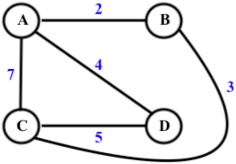
$$f_1 = f_1 + \min\{1 - f_1, \frac{f_i s_i}{s_1}\}, \text{ and reduce } f_i \text{ accordingly}$$

  until $f_1 = 1$ or $f_1 \times s_1 = W$.
- This yields solution S', which contains item 1 and is optimal

2. Greedy algorithms

Problem 2: Minimum spanning tree (MST)

- what is a spanning tree of a graph G?



- significance of spanning tree and MST

2. Greedy algorithms

- Again **greedy-choice property**

  A problem has a **greedy choice property** if its optimal solution is computed from only one specific choice.

- MST problem has a greedy choice property.

2. Greedy algorithms

Intuitively,

- an edge with the smallest weight should be in some m.s.t.
  why?
- is this alway true as the idea being used repetitively?
  when does it not work?
- more precise terms are needed.

2. Greedy algorithms

Some terminologies:

- a **cut** in a graph $G = (V, E)$ is a partition of set V into two:

$$(S, V - S), \text{ where } S \subset V, (S \neq \emptyset)$$

- an edge (u, v) **crosses** cut $(S, V - S)$ if $u \in S, v \in V - S$;
- an edge is a **light edge** crossing a cut it is of the smallest weight among all edges that cross the cut.

2. Greedy algorithms

Theorem: A greedy-choice property for MST problem:

> *Let G be a given graph. Then any light edge crossing any cut of the graph is in some minimum spanning tree of the graph.*

Proof: (using the Exchange method)

- let T be an m.s.t. for G and (u, v) is a light edge crossing some cut $(S, V - S)$;
- if (u, v) is in T, then the theorem is proved;
- otherwise, let edge (x, y) in T that crosses the cut $(S, V - S)$;
- then $T \cup \{(u, v)\}$ contains a cycle; why?
- let $T' = T \cup \{(u, v)\} - \{(x, y)\}$. Then T' is a spanning tree; why
- because (x, y) and (u, v) cross $(S, V - S)$ and (u, v) is a light edge, T' is also m.s.t. for G. why?
- Because T' contains (u, v), the theorem is proved.

2. Greedy algorithms

Based on the greedy-choice property, if we can identify a light edge crossing some cut (any cut), then we can safely add the edge into partially constructed m.s.t.

- the process repeats, adding one edge at a time;
- what cut should we identify? and identify another cut after adding an edge;

2. Greedy algorithms

```
function grow-tree(V,E);
     A = empty_set;
     while |A| < |V|-1 do
        if (u,v) not in A
             & is a light edge cross some new cut
             & A U {(u,v)} does not form a cycle
        then A = A U {(u,v)};
     return A;
```

- how to identify a cut (then a light edge)?
- how to check cyclicality?

2. Greedy algorithms

This leads to two different MST algorithms: Prim's and Kruskal's

Prim's:

- start from any single vertex a, let $S = \{a\}$; $T = \emptyset$;
- find a light edge (u, v) crossing the cut $(S, V - S)$; then $T = T \cup \{(u, v)\}$; $S = S \cup \{v\}$;
- if $|T| < n - 1$, repeat the above step;

Technically, how to identify every light edge (efficiently)?

- a new cut evolves from an old cut; a light edge crossing the new cut may be identified with little effort;

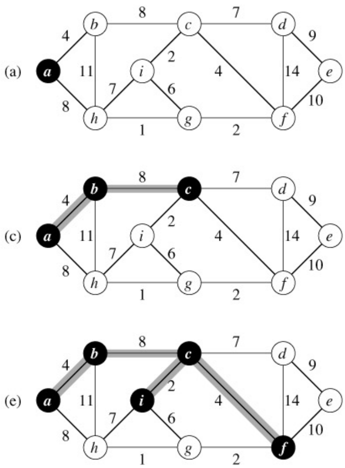
2. Greedy algorithms

```
function prim (G, w)
for all u in V
  cost(u) = infinity;
  prev(u) = nil;
pick an arbitrary vertex s
cost(s) = 0;
T = empty_set;
H = makequeue(V);
while H is not empty
  u = dequeue(H);
  T = T U {(prev(u), u)};
   for every (u, v) in E
    if cost(v) > w(u, v)
       cost(v) = w(u, v);
       prev(v) = u;
return (T, prev)
```

2. Greedy algorithms

- what does the list `prev` look like?
- does it work on directed graphs?
- what would happen if the graph is not connected?
- how to implement a priority queue?
- time complexity? $O(|E| + |V| \log |V|)$

2. Greedy algorithms



dynamic changes of the priority queue.

But wait, are we sure algorithm `prim` finds an m.s.t.?

We need to prove `T` generated by `prim` is an m.s.t.

We prove a more general **claim**:

At every iteration of the while loop, `T` is contained in some m.s.t.
called the **loop-invariant** for the while loop.

We prove the claim by induction on k, of the k^{th} iteration.

2. Greedy algorithms

Claim: At every iteration of the while loop in algorithm `prim`, set `T` is contained in some m.s.t.

Proof:

- base case: $k = 0$,
  the algorithm has yet to enter the while loop. Then $T = \emptyset$, therefore, it is contained in every m.s.t..
- assumption: at iteration k, $T \subseteq \mathcal{T}$ for some m.s.t., $\mathcal{T}$.
- induction: at iteration $k+1$, $T' = T \cup \{(u, v)\}$, where edge (u, v) is a light edge cross cut $(S, V - S)$, and S is exactly the set of those vertices in T.

  (1) if $(u, v) \in \mathcal{T}$, then $T' \subseteq \mathcal{T}$, we prove the claim.

  (2) otherwise, $\mathcal{T}$ has to contain a different edge (x, y) crossing the cut $(S, V - S)$. (why?)
- let $\mathcal{T}' = \mathcal{T} \cup \{(u, v)\} - \{(x, y)\}$. $\mathcal{T}'$ is also an m.s.t. (why?)
- $T' \subseteq \mathcal{T}'$ (why?), we prove the claim.

2. Greedy algorithms

```
function Kruskal (G=(V, E), w)

1. Sort edges by weight in the nondecreasing order;
2. forest F = emptyset;
3. for every edge (u, v) in the sorted order;
4.   if u and v not belonging to the same tree in F
5.      F = F U {(u, v)};
6.      update forest F;
```

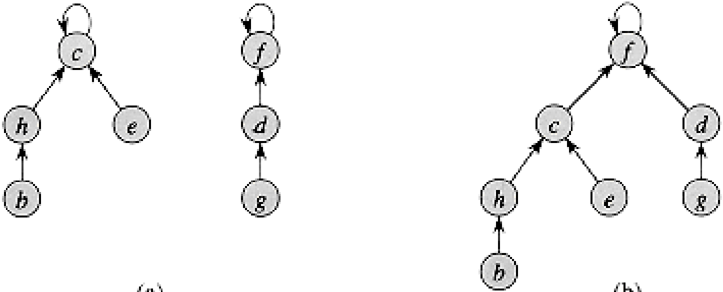
- complexity depends on how to implement steps 2, 4, and 5
- use set to store a tree in F, with operations

  make-setu, *find*(u), *union*(u, v).

2. Greedy algorithms

Disjoint-set

- $\textsc{Make Set}(x)$: create a set of single element x;
  $\textsc{Find Set}(x)$: identify the set that contains element x;
  $\textsc{Union}(x, y)$: union the two sets containing x and y into one;



(a) (b)

2. Greedy algorithms

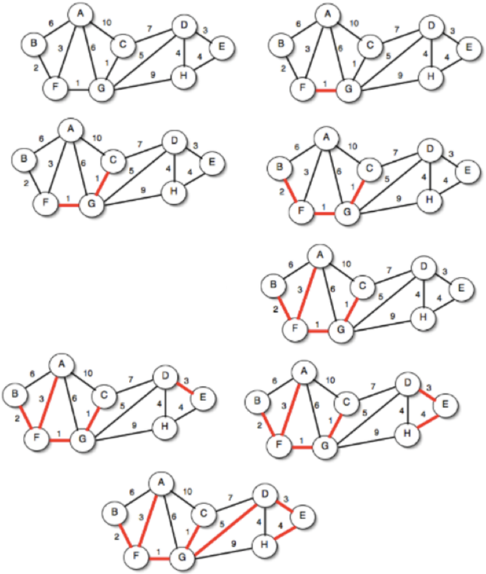
```
function Kruskal (G=(V, E), w)

Sort edges by weight in the nondecreasing order;
for every u in V,
   make_set(u);
for every edge (u, v) in the sorted order;
  if find(u) not = find(v)
     F = F U {(u, v)};
     union(u, v);
```

Time complexity:

2. Greedy algorithms

Execution of `Kruskal`'s:



disjoint sets

[A] [B] [C] [D] [E] [F] [G] [H]

[A] [B] [C] [D] [E] [F] [G] [H]

[A] [B] [C] [D] [E] [F, G] [H]

[A] [B] [D] [E] [C, F, G] [H]

[A] [D] [E] [B, C, F, G] [H]

[D] [E] [A, B, C, F, G] [H]

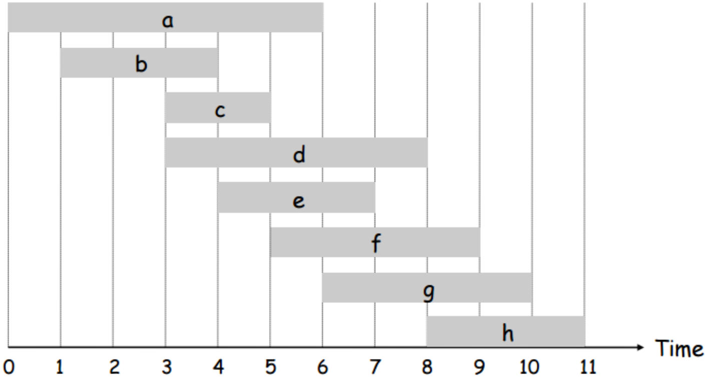
[D, E] [A, B, C, F, G] [H]

[D, E, H] [A, B, C, F, G]

[A, B, C, D, E, F, G, H]

2. Greedy algorithms

<u>Problem 3</u>: **Activity Scheduling**
Input: n activities, each with start time s_i and finish time f_i;
Output: max number of activities allowed to use a venue exclusively;



2. Greedy algorithms

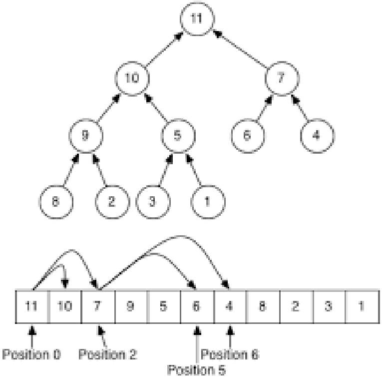
Greedy-choice property for **Activity Scheduling**:

> The activity with the earliest finish time is contained in some optimal scheduling

Proof: (in-classroom exercise)

3. Some data structures and implementations

heap implementation of priority queue



- heap: a complete binary tree, in which every node u satisfies:

  for max heap $\texttt{key}(u) \geq \texttt{key}(lc(u))$ and $\texttt{key}(u) \geq \texttt{key}(rc(u))$

- storage: array $A[0..n-1]$, $A[k]$'s children: $A[2k+1]$, $A[2k+2]$;

3. Implementations of priority queue and set

function `build-heap`: to build an initial heap

function `heapify`: adjust nodes to satisfy the heap condition

function `increase-key`: update key for a node in the heap

```
function heapify(A, k, n);    // adjust node from position k
                              // and downward
if k <= n/2
  place in A[k] the largest of A[2k+1], A[2k+2], and A[k]
  if index of largest element is not k
     k = index of the largest
     heapify(A, k, n);
```

3. Some data structures and implementations

usage in prim and Dijkstra's complexity analysis

```
function build-heap(A, n);  // build initial heap

1.   for k = n/2 to 0
2.     heapify(A, k, n)

function increase-key(A, i, key); // update node i's key value

1. if key > A[i]
2      A[i] = key
3. while i > 0 and A[PARENT[i]] < A[i]
4.   exchange A[i] with A[PARENT[i]]
5.   i = PARENT[i]
```

4. Matrix multiplication for graphs

$$\underset{2\times 2}{\begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}} \times \underset{2\times 2}{\begin{bmatrix} 6 & 5 \\ -7 & 1 \end{bmatrix}} = \underset{2\times 2}{\begin{bmatrix} \boxed{-2\times6+1\times-7} & -2\times5+1\times1 \\ 0\times6+4\times-7 & 0\times5+4\times1 \end{bmatrix}}$$

dot product

$$= \begin{bmatrix} \boxed{-19} & -9 \\ -28 & 4 \end{bmatrix}$$

4. Matrix multiplication for graphs

$$c_{11}=a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31}+a_{14}b_{41}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

2 x 4 4 x 3 2 x 3

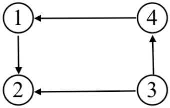
$$c_{22}=a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32}+a_{24}b_{42}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix}$$

4. Matrix multiplication for graphs

Consider an adjacency matrix of a directed graph:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$



$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

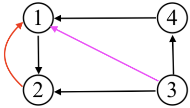
What does A^2 mean? e.g., entry $A^2(3,1)$

$$= A(3,1) \times A(1,1) + A(3,2) \times A(2,1) + A(3,3) \times A(3,1) + A(3,4) \times A(4,1)$$
$$= 0 + 0 + 0 + 1 = 1$$

What does $A^2(3,1) = 1$ mean?

4. Matrix multiplication for graphs

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$



What should A^2 be now?

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

What does $A^2(3,1) = 2$ mean?

4. Matrix multiplication for graphs

Can we conclude?

If $A_{n\times n}$ is a 0-1 adjacency matrix, then A^k contains the information about the number k-step paths $i \rightsquigarrow j$;

- How to get number of paths $i \rightsquigarrow j$, regardless steps?
- What if the given $A(i,i) \neq 0$?
- What if the graph is weighted and shortest paths are desired?

5. All pairs shortest paths

All Pair Shortest Paths Problem

Input: A weighted graph $G = (V, E)$ with edge weight function w;
Output: Shortest paths between every pair of vertices in G.

- If run DIJKSTRA's on every vertex, with total time $O(|V|^2 \log |V| + |V||E|)$, but only on graphs with non-negative edges.
- **Floyd-Warshall** algorithm: $O(|V|^3)$, able to detect negative cycles.

4. Matrix multiplication for graphs

ALL PAIR SHORTEST PATHS

- Can we solve the problem with matrix multiplication?

- Revising dot-product

$$A^2(i,j) = A(i,1) \times A(1,j) + \cdots + A(i,k) \times A(k,j) + \cdots + A(i,n) \times A(n,j)$$

replace $+$ with min;
replace $\times$ with $+$;

- Does the following formulation of shortest distances work?

$$d(i,j) = \min_{1 \le k \le n} \{d(i,k) + d(k,j)\}$$

(Circular data dependencies.)

5. All pairs shortest paths

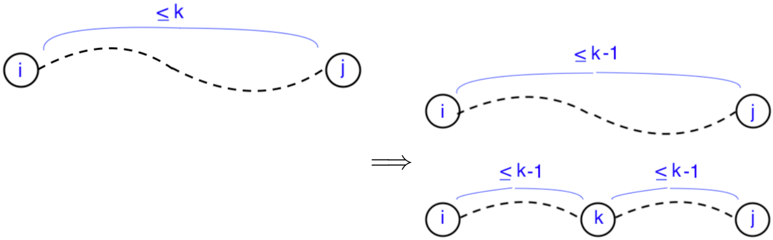
The idea of **Floyd-Warshall** algorithm is to break the "circularity" by computing more refined data.

Define: $D^{(k)}[i,j]$ to be the weight of a shortest path between vertices i and j **on which all intermediate nodes are of indexes** $\leq k$.

Note: the goal is still to compute d_{ij}, which is $D^{(n)}[i,j]$, where $n = |V|$

5. All pairs shortest paths

For $D^{(k)}[i,j]$, we can have recursive formulation, based on two possibilities:

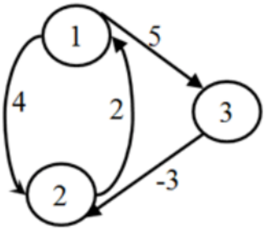


$$D^{(k)}[i,j] = \min \begin{cases} D^{(k-1)}[i,j] & \leftarrow \text{vertex } k \text{ is not on the path} \\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j] & \leftarrow \text{vertex } k \text{ is on the path} \end{cases}$$

base cases: $D^{(0)}[i,j] = w(i,j)$, $D^{(0)} = W$ (there are no intermediate nodes).

5. All pairs shortest paths

Example: W is the edge weight matrix;

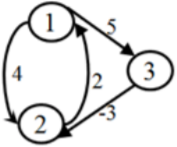


$$W = D^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \infty \\ 3 & \infty & -3 & 0 \end{array}$$

$$P^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}$$

P is the π paths matrix, storing k values

5. All pairs shortest paths



$$D^0 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \infty \\ 3 & \infty & -3 & 0 \end{array}$$

k=1: vertex 1 can be intermediate node

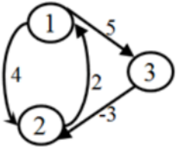
$$D^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 4 & 5 \\ 2 & 2 & 0 & \mathbf{7} \\ 3 & \infty & -3 & 0 \end{array}$$

$$D^1[2,3] = \min(D^0[2,3], \underline{D^0[2,1]+D^0[1,3]})$$
$$= \min(\infty, 7)$$
$$= \mathbf{7}$$

$$P^1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 \end{array}$$

$$D^1[3,2] = \min(D^0[3,2], D^0[3,1]+D^0[1,2])$$
$$= \min(-3, \infty)$$
$$= -3$$

5. All pairs shortest paths



$D^1 =$

	1	2	3
1	0	4	5
2	2	0	7
3	∞	-3	0

k=2: vertices 1, 2 can be intermediate node

$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	**-1**	-3	0

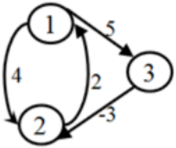
$D^2[1,3] = \min(D^1[1,3], D^1[1,2]+D^1[2,3])$
$= \min (5, 4+7)$
$= 5$

$P^2 =$

	1	2	3
1	0	0	0
2	0	0	1
3	2	0	0

$D^2[3,1] = \min(D^1[3,1], \underline{D^1[3,2]+D^1[2,1]})$
$= \min (\infty, -3+2)$
$= \mathbf{-1}$

5. All pairs shortest paths



$D^2 =$

	1	2	3
1	0	4	5
2	2	0	7
3	-1	-3	0

k=3: vertices 1, 2, 3 can be intermediate node

$D^3 =$

	1	2	3
1	0	**2**	5
2	2	0	7
3	-1	-3	0

$D^3[1,2] = \min(D^2[1,2], \underline{D^2[1,3]+D^2[3,2]})$

$= \min(4, 5+(-3))$

$= \mathbf{2}$

$P^3 =$

	1	2	3
1	0	3	0
2	0	0	1
3	2	0	0

$D^3[2,1] = \min(D^2[2,1], D^2[2,3]+D^2[3,1])$

$= \min(2, 7+(-1))$

$= 2$

5. All pairs shortest paths

Without paths information

FLOYD-WARSHALL(W)

1. $n = rows[W]$
2. $D^{(0)} = W$
3. **for** $k = 1$ **to** n $\leftarrow$ for different layer k
4. **for** $i = 1$ **to** n
5. **for** $j = 1$ **to** n $\swarrow$ compute matrix $D^{(k)}$
6. $D^{(k)}[i,j] = \min \begin{cases} D^{(k-1)}[i,j] \\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j] \end{cases}$
7. **return** $(D^{(n)})$

Time complexity $O(|V|^3)$.

5. All pairs shortest paths

With paths information

initialize path matrices $P = \{P^{(1)}, \ldots, P(^{(n)}\}$ to have zero values

```
FLOYD-WARSHALL(W)
n = rows[W]
D^(0) = W
for k = 1 to n
  for i = 1 to n
    for j = 1 to n
        D^(k)[i,j] = min { D^(k-1)[i,j];
                           D^(k-1)[i,k] + D^(k-1)[k,j]; }
        set P^(k)[i,j] = P^(k-1)[i,j] or P^(k)[i,j] = k, accordingly
return (D^(n), P)
```

$$D^{(k)}[i,j] = \min \begin{cases} D^{(k-1)}[i,j]; \\ D^{(k-1)}[i,k] + D^{(k-1)}[k,j]; \end{cases}$$

Summary of shortest paths algorithms

- A lot of path-related problems can be solved with DFS-like algorithms

  reachability, cycle, path counting, shortest path problems;

  edge relaxation: update distance/path based on edge (u, v);
- **single-source shortest paths on DAG**: DAG-Paths-algorithm, DP;
- **single-target shortest paths on DAG**: DAG-Paths-algorithm, DP
- **single-source shortest paths**: Dijkstra's algorithm;
- **single-source shortest paths**: Bellman-Ford algorithm;
- **all-pairs shortest paths**: Floyd-Warshall algorithm, DP;

6. Linear programming and Max Flow

Example-1: **Knapsack** can be written as

Find $(x_1, x_2, \ldots, x_n)$, such that

$$x_1 v_1 + x_2 v_2 + \cdots + x_n v_n = \sum_{k=1}^{n} x_i v_i \text{ is maximized}$$

subject to

$$x_1 s_1 + \ldots x_n s_n = \sum_{k=1}^{n} x_i s_i \leq B$$

$$x_i \in \{0, 1\}$$

6. Linear programming and Max Flow

Example-2: **MST** can be written as

Find $(e_1, x_2, \ldots, e_m)$, such that

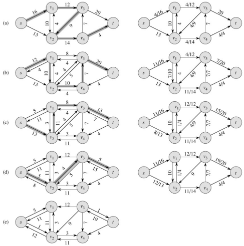
$$e_1 w_1 + e_2 w_2 + \cdots + e_m w_m = \sum_{k=1}^{m} e_i w_i \text{ is minimized}$$

subject to

$$e_1 + \ldots e_m = \sum_{k=1}^{m} e_i = n - 1$$
$$e_i \in \{0, 1\},\ 1 \leq i \leq m$$
$$\sum_{k_i} e_{k_i} \geq 1, \text{ where } e_{k_i} \text{ incident on vertex } k,\ 1 \leq k \leq n$$

6. Linear programming and Max Flow

Example-3 **Max Flow**:



Find $(f_1, f_2, \ldots, f_m)$, such that

$$\sum_j f_{s_j} \text{ is maximized}$$

where e_{s_j} are outgoing edges from source s,

subject to

$f_i \leq w(e_i)$, $1 \leq i \leq m$

$\sum_i f_{i_k} = \sum_j f_{k_j}$, $1 \leq k \leq n$

6. Linear programming and Max Flow

General linear program format:

$$\max \mathbf{c}^T\mathbf{x} \quad \text{or} \quad \min \mathbf{c}^T\mathbf{x}$$

subject to

$$\mathbf{A}\mathbf{x} \leq \mathbf{b} \text{ or } \geq \mathbf{b}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ \dots \\ c_n \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{pmatrix} \quad \mathbf{A} = \begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ a_{2,1} & \dots & a_{2,n} \\ \dots & & \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix} \tag{1}$$