## 5.24 OUTPUT

```
Enter the range -
Lower Limit a - 0
Upper Limit b - 6
Enter number of subintervals - 6
Value of the integral is 1.3662
Press Enter to Exit
```

## EXAMPLES

**Example 1.** *Use Trapezoidal rule to evaluate* $\int_0^1 x^3\,dx$ *considering five sub-intervals.*

**Sol.** Dividing the interval (0, 1) into 5 equal parts, each of width $h = \frac{1-0}{5} = 0.2$, the values of $f(x) = x^3$ are given below:

| $x$: | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|---|
| $f(x)$: | 0 | 0.008 | 0.064 | 0.216 | 0.512 | 1.000 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ |

By Trapezoidal rule, we have

$$\int_0^1 x^3\,dx = \frac{h}{2}\,[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$

$$= \frac{0.2}{2}\,[(0 + 1) + 2(0.008 + 0.064 + 0.216 + 0.512)]$$

$$= 0.1 \times 2.6 = 0.26.$$

**Example 2.** *Evaluate* $\int_0^1 \frac{dx}{1+x^2}$ *using*

(*i*) *Simpson's* $\frac{1}{3}$ *rule taking* $h = \frac{1}{4}$

(*ii*) *Simpson's* $\frac{3}{8}$ *rule taking* $h = \frac{1}{6}$

(*iii*) *Weddle's rule taking* $h = \frac{1}{6}$

*Hence compute an approximate value of* $\pi$ *in each case.*

**Sol.** (*i*) The values of $f(x) = \dfrac{1}{1+x^2}$ at $x = 0, \dfrac{1}{4}, \dfrac{2}{4}, \dfrac{3}{4}, 1$ are given below:

| | | | | | |
|---|---|---|---|---|---|
| $x$: | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
| $f(x)$: | 1 | $\frac{16}{17}$ | 0.8 | 0.64 | 0.5 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

By Simpson's $\dfrac{1}{3}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}\,[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{1}{12}\left[(1+0.5) + 4\left\{\frac{16}{17} + .64\right\} + 2(0.8)\right] = 0.785392156$$

Also $$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore \qquad \frac{\pi}{4} \simeq 0.785392156 \quad \Rightarrow \quad \pi \simeq 3.1415686$$

(*ii*) The values of $f(x) = \dfrac{1}{1+x^2}$ at $x = 0, \dfrac{1}{6}, \dfrac{2}{6}, \dfrac{3}{6}, \dfrac{4}{6}, \dfrac{5}{6}, 1$ are given below:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$: | 0 | $\frac{1}{6}$ | $\frac{2}{6}$ | $\frac{3}{6}$ | $\frac{4}{6}$ | $\frac{5}{6}$ | 1 |
| $f(x)$: | 1 | $\frac{36}{37}$ | $\frac{9}{10}$ | $\frac{4}{5}$ | $\frac{9}{13}$ | $\frac{36}{61}$ | $\frac{1}{2}$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

By Simpson's $\dfrac{3}{8}$ rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}\,[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[\left(1+\frac{1}{2}\right)+3\left\{\frac{36}{37}+\frac{9}{10}+\frac{9}{13}+\frac{36}{61}\right\}+2\left(\frac{4}{5}\right)\right]$$

$$= 0.785395862$$

Also, $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

$\therefore \quad \frac{\pi}{4} = 0.785395862$

$\Rightarrow \quad \pi = 3.141583$

(*iii*) By Weddle's rule, using the values as in (*ii*),

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{10}(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6)$$

$$= \frac{3\left(\frac{1}{6}\right)}{10}\left\{1+5\left(\frac{36}{37}\right)+\frac{9}{10}+6\left(\frac{4}{5}\right)+\frac{9}{13}+5\left(\frac{36}{61}\right)+\frac{1}{2}\right\}$$

$$= 0.785399611$$

Since $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

$\therefore \quad \frac{\pi}{4} = 0.785399611$

$\Rightarrow \quad \pi = 3.141598.$

**Example 3.** *Evaluate*

$$\int_0^6 \frac{dx}{1+x^2} \textit{ by using}$$

(*i*) *Simpson's one-third rule*

(*ii*) *Simpson's three-eighth rule*

(*iii*) *Trapezoidal rule*

(*iv*) *Weddle's rule.*

**Sol.** Divide the interval (0, 6) into six parts each of width $h = 1$.

The values of $f(x) = \frac{1}{1+x^2}$ are given below:

| $x$: | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f(x)$: | 1 | 0.5 | 0.2 | 0.1 | $\frac{1}{17}$ | $\frac{1}{26}$ | $\frac{1}{37}$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

(*i*) By Simpson's one-third rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\,[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{1}{3}\left[\left(1+\frac{1}{37}\right)+4\left(0.5+0.1+\frac{1}{26}\right)+2\left(0.2+\frac{1}{17}\right)\right]$$

$$= 1.366173413.$$

(*ii*) By Simpson's three-eighth rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\,[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8}\left[\left(1+\frac{1}{37}\right)+3\left(.5+.2+\frac{1}{17}+\frac{1}{26}\right)+2(.1)\right]$$

$$= 1.357080836.$$

(*iii*) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\,[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

$$= \frac{1}{2}\left[\left(1+\frac{1}{37}\right)+2\left(.5+.2+.1+\frac{1}{17}+\frac{1}{26}\right)\right]$$

$$= 1.410798581.$$

(*iv*) By Weddle's rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{10}\,[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3}{10}\left[1 + 5(.5) + .2 + 6(.1) + \frac{1}{17} + 5\left(\frac{1}{26}\right) + \frac{1}{37}\right]$$

$$= 1.373447475.$$

**Example 4.** *The speed, v meters per second, of a car, t seconds after it starts, is shown in the following table:*

| $t$ | 0 | 12 | 24 | 36 | 48 | 60 | 72 | 84 | 96 | 108 | 120 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $v$ | 0 | 3.60 | 10.08 | 18.90 | 21.60 | 18.54 | 10.26 | 5.40 | 4.50 | 5.40 | 9.00 |

*Using Simpson's rule, find the distance travelled by the car in 2 minutes.*

**Sol.** If $s$ meters is the distance covered in $t$ seconds, then

$$\frac{ds}{dt} = v$$

$$\therefore \qquad \Big[s\Big]_{t=0}^{t=120} = \int_0^{120} v\, dt$$

since the number of sub-intervals is **10 (even).** Hence, by using Simpson's $\frac{1}{3}$rd rule,

$$\int_0^{120} v\, dt = \frac{h}{3}\, [(v_0 + v_{10}) + 4(v_1 + v_3 + v_5 + v_7 + v_9) + 2(v_2 + v_4 + v_6 + v_8)]$$

$$= \frac{12}{3}\, [(0 + 9) + 4(3.6 + 18.9 + 18.54 + 5.4 + 5.4)$$

$$+ 2(10.08 + 21.6 + 10.26 + 4.5)]$$

$$= 1236.96 \text{ meters.}$$

Hence, the distance travelled by car in 2 minutes is 1236.96 meters.

**Example 5.** *Evaluate* $\int_{0.6}^{2} y\, dx$, *where y is given by the following table:*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| *x:* | 0.6 | 0.8 | 1.0 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
| *y:* | 1.23 | 1.58 | 2.03 | 4.32 | 6.25 | 8.36 | 10.23 | 12.45. |

**Sol.** Here the number of subintervals is 7, which is neither even nor a multiple of 3. Also, this number is neither a multiple of 4 nor a multiple of 6, hence using Trapezoidal rule, we get

$$\int_{0.6}^{2} y\,dx = \frac{h}{2}\ [(y_0 + y_7) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6)]$$

$$= \frac{0.2}{2}\ [(1.23 + 12.45) + 2(1.58 + 2.03 + 4.32 + 6.25 + 8.36 + 10.23)]$$

| Here $h = 0.2$

$$= 7.922.$$

**Example 6.** *Find* $\int_{1}^{11} f(x)\,dx$, *where f(x) is given by the following table, using a suitable integration formula.*

| $x$: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(x)$: | 543 | 512 | 501 | 489 | 453 | 400 | 352 | 310 | 250 | 172 | 95 |

**Sol.** Since the number of subintervals is 10 (even) hence we shall use Simpson's $\frac{1}{3}$rd rule.

$$\int_{1}^{11} f(x)\,dx = \frac{h}{3}\ [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{1}{3}\ [(543 + 95) + 4(512 + 489 + 400 + 310 + 172) + 2(501 + 453 + 352 + 250)]$$

$$= \frac{1}{3}\ [638 + 7532 + 3112] = 3760.67.$$

**Example 7.** *Evaluate* $\int_{0}^{1} \frac{dx}{1+x}$ *by dividing the interval of integration into 8 equal parts. Hence find* $\log_e 2$ *approximately.*

**Sol.** Since the interval of integration is divided into an even number of subintervals, we shall use Simpson's one-third rule.

Here, $$y = \frac{1}{1+x} = f(x)$$

$$y_0 = f(0) = \frac{1}{1+0} = 1, \quad y_1 = f\left(\frac{1}{8}\right) = \frac{1}{1+\frac{1}{8}} = \frac{8}{9}, \quad y_2 = f\left(\frac{2}{8}\right) = \frac{4}{5}$$

$$y_3 = f\left(\frac{3}{8}\right) = \frac{8}{11}, \quad y_4 = f\left(\frac{4}{8}\right) = \frac{2}{3}, \quad y_5 = f\left(\frac{5}{8}\right) = \frac{8}{13}$$

$$y_6 = f\left(\frac{6}{8}\right) = \frac{4}{7}, \qquad y_7 = f\left(\frac{7}{8}\right) = \frac{8}{15} \qquad \text{and} \quad y_8 = f(1) = \frac{1}{2}$$

Hence the table of values is

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$: | 0 | $\frac{1}{8}$ | $\frac{2}{8}$ | $\frac{3}{8}$ | $\frac{4}{8}$ | $\frac{5}{8}$ | $\frac{6}{8}$ | $\frac{7}{8}$ | 1 |
| $y$: | 1 | $\frac{8}{9}$ | $\frac{4}{5}$ | $\frac{8}{11}$ | $\frac{2}{3}$ | $\frac{8}{13}$ | $\frac{4}{7}$ | $\frac{8}{15}$ | $\frac{1}{2}$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ | $y_8$ |

By Simpson's $\frac{1}{3}$rd rule,

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{3}\,[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{1}{24}\left[\left(1+\frac{1}{2}\right) + 4\left(\frac{8}{9}+\frac{8}{11}+\frac{8}{13}+\frac{8}{15}\right) + 2\left(\frac{4}{5}+\frac{2}{3}+\frac{4}{7}\right)\right]$$

| Here $h = 1/8$

$$= 0.69315453$$

Since, $$\int_0^1 \frac{dx}{1+x} = \Big[\log_e (1+x)\Big]_0^1 = \log_e 2$$

$\therefore$ $$\log_e 2 = 0.69315453.$$

**Example 8.** *Find, from the following table, the area bounded by the curve and the x-axis from x = 7.47 to x = 7.52.*

| | | | | | | |
|---|---|---|---|---|---|---|
| *x:* | *7.47* | *7.48* | *7.49* | *7.50* | *7.51* | *7.52* |
| *f(x):* | *1.93* | *1.95* | *1.98* | *2.01* | *2.03* | *2.06.* |

**Sol.** We know that

$$\text{Area} = \int_{7.47}^{7.52} f(x)\, dx$$

with $h = 0.01$, the trapezoidal rule gives,

$$\text{Area} = \frac{.01}{2}\,[(1.93 + 2.06) + 2(1.95 + 1.98 + 2.01 + 2.03)]$$

$$= 0.09965.$$

**Example 9.** *Use Simpson's rule for evaluating*

$$\int_{-0.6}^{0.3} f(x)\,dx$$

*from the table given below:*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *x:* | – 0.6 | – 0.5 | – 0.4 | – 0.3 | – 0.2 | – 0.1 | 0 | .1 | .2 | .3 |
| *f(x):* | 4 | 2 | 5 | 3 | – 2 | 1 | 6 | 4 | 2 | 8 |

**Sol.** Since the number of subintervals is 9(a multiple of 3), we will use Simpson's 3/8th rule here.

$$\therefore \quad \int_{-0.6}^{0.3} f(x)\,dx = \frac{3(.1)}{8}\,[(4+8)+3\{2+5+(-2)+1+4+2\}+2(3+6)]$$

$$= 2.475.$$

**Example 10.** *Evaluate* $\int_1^2 e^{-\frac{1}{2}x}\,dx$ *using four intervals.*

**Sol.** The table of values is:

| | | | | | |
|---|---|---|---|---|---|
| $x$: | 1 | 1.25 | 1.5 | 1.75 | 2 |
| $y = e^{-x/2}$: | .60653 | .53526 | .47237 | .41686 | .36788 |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

Since we have four (even) subintervals here, we will use Simpson's $\frac{1}{3}$rd rule.

$$\therefore \quad \int_1^2 e^{-\frac{1}{2}x}\,dx = \frac{h}{3}\,[(y_0+y_4)+4(y_1+y_3)+2y_2]$$

$$= \frac{.25}{3}\,[(.60653+.36788)+4(.53526)+.41686)+2(.47237)]$$

$$= 0.4773025.$$

**Example 11.** *Find* $\int_0^6 \frac{e^x}{1+x}\,dx$ *approximately using Simpson's* $\frac{3}{8}$*th rule on integration.*

**Sol.** Divide the given integral of integration into 6 equal subintervals, the arguments are 0, 1, 2, 3, 4, 5, 6; $h = 1$.

$$f(x) = \frac{e^x}{1+x};\; y_0 = f(0) = 1$$

$$y_1 = f(1) = \frac{e}{2}, \qquad y_2 = f(2) = \frac{e^2}{3}, \qquad y_3 = f(3) = \frac{e^3}{4},$$

$$y_4 = f(4) = \frac{e^4}{5}, \qquad y_5 = f(5) = \frac{e^5}{6}, \qquad y_6 = f(6) = \frac{e^6}{7}$$

The table is as below:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$: | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
| $y$: | 1 | $\frac{e}{2}$ | $\frac{e^2}{3}$ | $\frac{e^3}{4}$ | $\frac{e^4}{5}$ | $\frac{e^5}{6}$ | $\frac{e^6}{7}$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ |

Applying Simpson's three-eighth rule, we have

$$\int_0^6 \frac{e^x}{1+x}\,dx = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3\right]$$

$$= \frac{3}{8}\left[\left(1+\frac{e^6}{7}\right) + 3\left(\frac{e}{2}+\frac{e^2}{3}+\frac{e^4}{5}+\frac{e^5}{6}\right) + 2\,\frac{e^3}{4}\right]$$

$$= \frac{3}{8}\left[(1 + 57.6327) + 3(1.3591 + 2.463 + 10.9196 + 24.7355) + 2(5.0214)\right]$$

$$= 70.1652.$$

NOTE *It is not possible to evaluate* $\int_0^6 \frac{e^x}{1+x}\,dx$ *by using usual calculus method. Numerical integration comes to our rescue in such situations.*

**Example 12.** *A train is moving at the speed of 30 m/sec. Suddenly brakes are applied. The speed of the train per second after t seconds is given by*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| *Time (t):* | 0 | 5 | 10 | 15 | 20 | 25 | 30 | 35 | 40 | 45 |
| *Speed (v):* | 30 | 24 | 19 | 16 | 13 | 11 | 10 | 8 | 7 | 5 |

*Apply Simpson's three-eighth rule to determine the distance moved by the train in 45 seconds.*

**Sol.** If $s$ meters is the distance covered in $t$ seconds, then

$$\frac{ds}{dt} = v \qquad \Rightarrow \qquad \Big[s\Big]_{t=0}^{t=45} = \int_0^{45} v\,dt$$

Since the number of subintervals is **9 (a multiple of 3)** hence by using Simpson's $\left(\frac{3}{8}\right)^{\text{th}}$ rule,

$$\int_0^{45} v\, dt = \frac{3h}{8}\,[(v_0 + v_9) + 3(v_1 + v_2 + v_4 + v_5 + v_7 + v_8) + 2(v_3 + v_6)]$$

$$= \frac{15}{8}\,[(30 + 5) + 3(24 + 19 + 13 + 11 + 8 + 7) + 2(16 + 10)]$$

$$= 624.375 \text{ meters.}$$

Hence the distance moved by the train in 45 seconds is **624.375** meters.

**Example 13.** *Evaluate* $\int_0^4 \frac{dx}{1+x^2}$ *using Boole's rule taking*

(*i*) $h = 1$ (*ii*) $h = 0.5$

*Compare the results with the actual value and indicate the error in both.*

**Sol.** (*i*) Dividing the given interval into 4 equal subintervals (*i.e.*, $h = 1$), the table is as follows:

| | | | | | |
|---|---|---|---|---|---|
| $x$: | 0 | 1 | 2 | 3 | 4 |
| $y$: | 1 | $\frac{1}{2}$ | $\frac{1}{5}$ | $\frac{1}{10}$ | $\frac{1}{17}$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ |

using Boole's rule,

$$\int_0^4 y\, dx = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

$$= \frac{2(1)}{45}\left[7(1) + 32\left(\frac{1}{2}\right) + 12\left(\frac{1}{5}\right) + 32\left(\frac{1}{10}\right) + 7\left(\frac{1}{17}\right)\right]$$

$$= 1.289412 \text{ (approx.)}$$

$$\therefore \qquad \int_0^4 \frac{dx}{1+x^2} = 1.289412.$$

(*ii*) Dividing the given interval into 8 equal subintervals (*i.e.*, $h = 0.5$), the table is as follows:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$: | 0 | .5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| $y$: | 1 | 0.8 | 0.5 | $\frac{4}{13}$ | .2 | $\frac{4}{29}$ | .1 | $\frac{4}{53}$ | $\frac{1}{17}$ |
| | $y_0$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ | $y_8$ |

using Boole's rule,

$$\int_0^4 y dx = \frac{2h}{45}[7(y_0)+32(y_1)+12(y_2)+32(y_3)+7(y_4)$$

$$+\ 7(y_4)+32(y_5)+12(y_6)+32(y_7)+7(y_8)]$$

$$= \frac{1}{45}\left[7(1)+32(.8)+12(.5)+32\left(\frac{4}{13}\right)+7(.2)+7(.2)\right.$$

$$\left.+32\left(\frac{4}{29}\right)+12(.1)+32\left(\frac{4}{53}\right)+7\left(\frac{1}{17}\right)\right]$$

$$= 1.326373$$

$$\therefore \qquad \int_0^4 \frac{dx}{1+x^2} = 1.326373$$

But the actual value is

$$\int_0^4 \frac{dx}{1+x^2} = \left(\tan^{-1} x\right)_0^4 = \tan^{-1}(4) = 1.325818$$

$$\text{Error in result I} = \left(\frac{1.325818 - 1.289412}{1.325818}\right) \times 100 = 2.746\%$$

$$\text{Error in result II} = \left(\frac{1.325818 - 1.326373}{1.325818}\right) \times 100 = -0.0419\%.$$

**Example 14.** *A river is 80 m wide. The depth 'y' of the river at a distance 'x' from one bank is given by the following table:*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| *x:* | *0* | *10* | *20* | *30* | *40* | *50* | *60* | *70* | *80* |
| *y:* | *0* | *4* | *7* | *9* | *12* | *15* | *14* | *8* | *3* |

*Find the approximate area of cross-section of the river using*

(*i*) *Boole's rule.*

(*ii*) *Simpson's* $\frac{1}{3}$*rd rule.*

**Sol.** The required area of the cross-section of the river

$$= \int_0^{80} y\,dx$$

Here the number of sub intervals is 8.

(*i*) *By Boole's rule,*

$$\int_0^{80} y\,dx = \frac{2h}{45}\,[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4 + 7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8]$$

$$= \frac{2\,(10)}{45}\,[7(0) + 32(4) + 12(7) + 32(9) + 7(12) + 7(12) + 32(15) + 12(14) + 32(8) + 7(3)]$$

$$= 708$$

Hence the required area of the cross-section of the river = 708 sq. m.

(*ii*) *By Simpson's* $\frac{1}{3}$*rd rule*

$$\int_0^{80} y\,dx = \frac{h}{3}\,[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)]$$

$$= \frac{10}{3}\,[(0 + 3) + 4(4 + 9 + 15 + 8) + 2(7 + 12 + 14)]$$

$$= 710$$

Hence the required area of the cross-section of the river = 710 sq. m.

**Example 15.** *Evaluate* $\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,dx$ *approximately using Weddle's rule correct to 4 decimals.*

**Sol.** Let $f(x) = \sin x - \log x + e^x$. Divide the given interval of integration into 12 equal parts so that the arguments are: 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4.

The corresponding entries are

$y_0 = f(0.2) = 3.0295$, $y_1 = f(0.3) = 2.8494$, $y_2 = f(0.4) = 2.7975$,

$y_3 = f(0.5) = 2.8213$, $y_4 = f(0.6) = 2.8976$, $y_5 = f(0.7) = 3.0147$

$y_6 = f(0.8) = 3.1661$, $y_7 = f(0.9) = 3.3483$, $y_8 = f(1) = 3.5598$,

$y_9 = f(1.1) = 3.8001$, $y_{10} = f(1.2) = 4.0698$, $y_{11} = f(1.3) = 4.3705$

$y_{12} = f(1.4) = 4.7042$

Now, by Weddle's rule,

$$\int_{0.2}^{1.4} f(x)\, dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}]$$

$$= \frac{3}{10}(0.1)[3.0295 + 14.2470 + 2.7975 + 16.9278 + 2.8976 + 15.0735 + 3.1661 + 3.1661 + 16.7415 + 3.5598 + 22.8006 + 4.0698 + 21.8525 + 4.7042]$$

$$= (0.03)[135.0335] = 4.051.$$

**Example 16.** *A solid of revolution is formed by rotating about x-axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following coordinates.*

| | | | | | |
|---|---|---|---|---|---|
| *x:* | *0* | *0.25* | *0.5* | *0.75 1* | |
| *y:* | *1* | *0.9896* | *0.9589* | *0.9089* | *0.8415* |

*Estimate the volume of the solid formed using Simpson's rule.*

**Sol.** If V is the volume of the solid formed then we know that

$$V = \pi \int_0^1 y^2\, dx$$

Hence we need the values of $y^2$ and these are tabulated below correct to four decimal places

| $x$ | 0 | .25 | .5 | .75 | 1 |
|---|---|---|---|---|---|
| $y^2$ | 1 | .9793 | .9195 | .8261 | .7081 |

with $h = 0.25$, Simpson's rule gives

$$V = \pi \frac{(0.25)}{3}[(1 + .7081) + 4(.9793 + .8261) + 2(.9195)]$$

$$= 2.8192.$$

**Example 17.** *A tank is discharging water through an orifice at a depth of x meter below the surface of the water whose area is A $m^2$. Following are the values of x for the corresponding values of A.*

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *A:* | *1.257* | *1.39* | *1.52* | *1.65* | *1.809* | *1.962* | *2.123* | *2.295* | *2.462* | *2.650* | *2.827* |
| *x:* | *1.5* | *1.65* | *1.8* | *1.95* | *2.1* | *2.25* | *2.4* | *2.55* | *2.7* | *2.85* | *3* |

*Using the formula* $(0.018)\ T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}}\, dx$, *calculate T, the time (in seconds) for the level of the water to drop from 3.0 m to 1.5 m above the orifice.*

**Sol.** Here $h = 0.15$

The table of values of $x$ and the corresponding values of $\frac{A}{\sqrt{x}}$ is

| $x$ | 1.5 | 1.65 | 1.8 | 1.95 | 2.1 | 2.25 | 2.4 | 2.55 | 2.7 | 2.85 | 3 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y = \frac{A}{\sqrt{x}}$ | 1.025 | 1.081 | 1.132 | 1.182 | 1.249 | 1.308 | 1.375 | 1.438 | 1.498 | 1.571 | 1.632 |

Using Simpson's $\frac{1}{3}$rd rule, we get

$$\int_{1.5}^{3} \frac{A}{\sqrt{x}}\, dx = \frac{.15}{3}[(1.025+1.632) + 4(1.081 + 1.182 + 1.308 + 1.438 + 1.571) + 2(1.132 + 1.249 + 1.375 + 1.498)]$$

$$= 1.9743$$

Using the formula

$$(0.018)\mathrm{T} = \int_{1.5}^{3} \frac{A}{\sqrt{x}}\, dx$$

We get $0.018\mathrm{T} = 1.9743 \Rightarrow \mathrm{T} = 110$ sec. (approximately).

**Example 18.** *Using the following table of values, approximate by Simpson's rule, the arc length of the graph* $y = \frac{1}{x}$ *between the points (1, 1) and* $\left(5, \frac{1}{5}\right)$

| | | | | | |
|---|---|---|---|---|---|
| *x:* | *1* | *2* | *3* | *4* | *5* |
| $\sqrt{\frac{1+x^4}{x^4}}$: | *1.414* | *1.031* | *1.007* | *1.002* | *1.001.* |

**Sol.** The given curve is

$$y = \frac{1}{x}$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{x^2}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{x^4}} = \sqrt{\frac{1+x^4}{x^4}}$$

$\therefore$ The arc length of the curve between the points (1, 1) and $\left(5, \frac{1}{5}\right)$

$$= \int_1^5 \sqrt{\frac{1+x^4}{x^4}}\, dx$$

$$= \frac{h}{3}[(1.414 + 1.001) + 4(1.031 + 1.002) + 2(1.007)]$$

$$= \frac{1}{3}(2.415 + 8.132 + 2.014) = 4.187$$

**Example 19.** *From the following values of* $y = f(x)$ *in the given range of values of* $x$*, find the position of the centroid of the area under the curve and the x-axis*

| | | | | | |
|---|---|---|---|---|---|
| $x$: | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
| $y$: | 1 | 4 | 8 | 4 | 1 |

*Also find*

(*i*) *the volume of solid obtained by revolving the above area about x-axis.*

(*ii*) *the moment of inertia of the area about x-axis.*

**Sol.** Centroid of the plane area under the curve $y = f(x)$ is given by $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{\int_0^1 xy\, dx}{\int_0^1 y\, dx}$$

and

$$\bar{y} = \frac{\int_0^1 \frac{y}{2}\cdot y\, dx}{\int_0^1 y\, dx} = \frac{\int_0^1 \frac{y^2}{2}\, dx}{\int_0^1 y\, dx} \qquad (50)$$

From the given data, we obtain

| | | | | | |
|---|---|---|---|---|---|
| $x$: | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
| $y$: | 1 | 4 | 8 | 4 | 1 |
| $xy$: | 0 | 1 | 4 | 3 | 1 |
| $\frac{y^2}{2}$: | $\frac{1}{2}$ | 8 | 32 | 8 | $\frac{1}{2}$ |

$\therefore$ By Simpson's rule,

$$\int_0^1 xy\,dx = \frac{(1/4)}{3}[(0 + 1) + 4(1 + 3) + 2(4)] = \frac{25}{12}$$

$$\int_0^1 \frac{y^2}{2}\,dx = \frac{1}{12}\left[\left(\frac{1}{2}+\frac{1}{2}\right) + 4(8+8) + 2(32)\right] = \frac{129}{12}$$

$$\int_0^1 y\,dx = \frac{1}{12}[(1 + 1) + 4(4 + 4) + 2(8)] = \frac{50}{12}$$

From (50), $$\bar{x} = \frac{25/12}{50/12} = \frac{1}{2} = 0.5$$

$$\bar{y} = \frac{129/12}{50/12} = \frac{129}{50} = 2.58$$

$\therefore$ Centroid is the point (0.5, 2.58).

(*i*) We know that

$$\text{V} = \text{Volume} = \pi\int_0^1 y^2\,dx$$

$\therefore$ Required volume $= \pi.2\int_0^1 \frac{y^2}{2}dx = 2\pi\times\frac{129}{12} = 67.5442$

(*ii*) We know that moment of inertia of the area about the $x$-axis is given by

$$\text{M.I.} = \frac{1}{3}\rho\int_a^b y^3\,dx$$

where $\rho$ is the mass per unit area.

Table for $y^3$ is

| | | | | | |
|---|---|---|---|---|---|
| $x$: | 0 | $\frac{1}{4}$ | $\frac{1}{2}$ | $\frac{3}{4}$ | 1 |
| $y$: | 1 | 4 | 8 | 4 | 1 |
| $y^3$: | 1 | 64 | 512 | 64 | 1 |

$$\int_0^1 y^3\, dx = \frac{1}{12}[(1+1)+4(64+64)+2(512)] = \frac{769}{6}$$

$$\therefore \qquad \text{Reqd. M.I.} = \frac{1}{3}\rho\left(\frac{769}{6}\right) = \frac{769}{18}\rho = 42.7222\,\rho.$$

**Example 20.** *A reservoir discharging water through sluices at a depth h below the water surface, has a surface area A for various values of h as given below:*

| | | | | | |
|---|---|---|---|---|---|
| *h (in meters):* | *10* | *11* | *12* | *13* | *14* |
| *A (in sq. meters):* | *950* | *1070* | *1200* | *1350* | *1530* |

*If t denotes time in minutes, the rate of fall of the surface is given by*

$$\frac{dh}{dt} = -\frac{48}{A}\sqrt{h}$$

*Estimate the time taken for the water level to fall from 14 to 10 m above the sluices.*

**Sol.** From $$\frac{dh}{dt} = -\frac{48}{A}\sqrt{h}, \text{ we have}$$

$$dt = -\frac{A}{48}\frac{dh}{\sqrt{h}}$$

Integration yields,

$$t = -\frac{1}{48}\int_{14}^{10}\frac{A}{\sqrt{h}}\,dh = \frac{1}{48}\int_{10}^{14}\frac{A}{\sqrt{h}}\,dh$$

Here, $y = \frac{A}{\sqrt{h}}$. The table of values is as follows:

| | | | | | |
|---|---|---|---|---|---|
| $h$: | 10 | 11 | 12 | 13 | 14 |
| $A$: | 950 | 1070 | 1200 | 1350 | 1530 |
| $\frac{A}{\sqrt{h}}$: | 300.4164 | 322.6171 | 346.4102 | 374.4226 | 408.9097 |

Applying Simpson's $\frac{1}{3}$rd rule, we have

$$\text{time } t = \frac{1}{48} \cdot \frac{1}{3}[(300.4164 + 408.9097) + 4(322.6171 + 374.4226) + 2(346.4102)]$$

$$= 29.0993 \text{ minutes.}$$

## ASSIGNMENT 5.2

**1.** Evaluate $\int_1^2 \frac{1}{x}\, dx$ by Simpson's $\frac{1}{3}$rd rule with four strips and determine the error by direct integration.

**2.** Evaluate the integral $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ by dividing the interval into 6 parts.

**3.** Evaluate $\int_4^{5.2} \log_e x\, dx$ by Simpson's $\frac{3}{8}$th rule. Also write its programme in 'C' language.

**4.** Evaluate $\int_{30°}^{90°} \log_{10} \sin x\, dx$ by Simpson's $\frac{1}{3}$rd rule by dividing the interval into 6 parts.

**5.** Evaluate $\int_4^{5.2} \log_e x\, dx$ using

(*i*) Trapezoidal rule (*ii*) Weddle's rule.

**6.** Evaluate using Trapezoidal rule

(*i*) $\int_0^{\pi} t \sin t\, dt$ (*ii*) $\int_{-2}^{2} \frac{t\, dt}{5 + 2t}$

**7.** Evaluate $\int_3^7 x^2 \log x\, dx$ taking 4 strips.

**8.** The velocities of a car running on a straight road at intervals of 2 minutes are given below:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| *Time (in minutes)*: | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| *Velocity (in km/hr)*: | 0 | 22 | 30 | 27 | 18 | 7 | 0 |

Apply Simpson's rule to find the distance covered by the car.

**9.** Evaluate $\int_0^1 \cos x\, dx$ using $h = 0.2$.

**10.** Evaluate $\int_0^4 e^x \, dx$ by Simpson's rule, given that $e = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.6$ and compare it with the actual value.

**11.** Find an approximate value of $\log_e 5$ by calculating to 4 decimal places, by Simpson's $\frac{1}{3}$rd rule, $\int_0^5 \frac{dx}{4x+5}$ dividing the range into 10 equal parts.

**12.** Use Simpson's rule, taking five ordinates, to find an approximate value of $\int_1^2 \sqrt{x - \frac{1}{x}} \, dx$ to 2 decimal places.

**13.** Evaluate $\int_0^{\pi/2} \sqrt{\sin x} \, dx$ given that

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$: | 0 | $\pi/12$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $5\pi/12$ | $\pi/2$ |
| $\sqrt{\sin x}$: | 0 | 0.5087 | 0.7071 | 0.8409 | 0.9306 | 0.9878 | 1 |

**14.** The velocity of a train which starts from rest is given by the following table, time being reckoned in minutes from the start and speed in kilometers per hour:

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *Minutes*: | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| *Speed (km/hr)*: | 0 | 10 | 18 | 25 | 29 | 32 | 20 | 11 | 5 | 2 | 0 |

Estimate the total distance in 20 minutes. $\left[\textbf{Hint: } \text{Here step-size } h = \frac{2}{60}\right]$

**15.** A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the following table. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $t(sec)$: | 0 | 10 | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
| $f(cm/sec^2)$: | 30 | 31.63 | 33.34 | 35.47 | 37.75 | 40.33 | 43.25 | 46.69 | 50.67. |

**16.** A curve is drawn to pass through the points given by the following table:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$: | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
| $y$: | 2 | 2.4 | 2.7 | 2.8 | 3 | 2.6 | 2.1 |

Find

(*i*) Center of gravity of the area.

(*ii*) Volume of the solid of revolution.

(*iii*) The area bounded by the curve, the $x$-axis and lines $x = 1$, $x = 4$.

**17.** In an experiment, a quantity G was measured as follows:

G(20) = 95.9, G(21) = 96.85, G(22) = 97.77

G(23) = 98.68, G(24) = 99.56, G(25) = 100.41, G(26) = 101.24.

Compute $\int_{20}^{26} G(x) \, dx$ by Simpson's and Weddle's rule, respectively.

**18.** Using the data of the following table, compute the integral $\int_{0.5}^{1.1} xy\, dx$ by Simpson's rule:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $x$: | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 | 1.1 |
| $y$: | 0.4804 | 0.5669 | 0.6490 | 0.7262 | 0.7985 | 0.8658 | 0.9281 |

**19.** Find the value of $\log_e 2$ from $\int_0^1 \frac{x^2}{1+x^3}\, dx$ using Simpson's $\frac{1}{3}$rd rule by dividing the range of integration into four equal parts. Also find the error.

**20.** Use Simpson's rule dividing the range into ten equal parts to show that

$$\int_0^1 \frac{\log(1+x^2)}{1+x^2}\, dx = 0.173$$

**21.** Find by Weddle's rule the value of the integral

$$I = \int_{0.4}^{1.6} \frac{x}{\sinh x}\, dx$$

by taking 12 sub-intervals.

**22.** Evaluate $\int_{0.5}^{0.7} x^{1/2} e^{-x}\, dx$ approximately by using a suitable formula.

**23.** (*i*) Compute the integral

$$I = \sqrt{\frac{2}{\pi}} \int_0^1 e^{-(x^2/2)}\, dx$$

Using Simpson's $\frac{1}{3}$rd rule, taking $h = 0.125$.

(*ii*) Compute the value of I given by

$$I = \int_{0.2}^{1.5} e^{-x^2}\, dx$$

Using Simpson's $\left(\frac{1}{3}\right)$ rule with four subdivisions.

**24.** Using Simpson's $\frac{1}{3}$rd rule, Evaluate the integrals:

(*i*) $\int_{1.0}^{1.8} \frac{e^x + e^{-x}}{2}\, dx$ (taking $h = 0.2$)

(*ii*) $\int_0^{\pi/2} \frac{dx}{\sin^2 x + \frac{1}{4}\cos^2 x}$

**25.** Evaluate $\int_0^1 \sqrt{\sin x + \cos x}\, dx$ correct to two decimal places using seven ordinates.

**26.** Use Simpson's three-eighths rule to obtain an approximate value of

$$\int_0^{0.3} (1 - 8x^3)^{1/2}\, dx$$

**27.** Evaluate $\int_0^{1/2} \frac{dx}{\sqrt{1-x^2}}$ using Weddle's rule.

**28.** Evaluate $\int_0^1 \frac{x^2+2}{x^2+1}\, dx$ using Weddle's rule correct to four places of decimals.

**29.** Using $\frac{3}{8}$th Simpson's rule,

Evaluate: $\int_0^6 \frac{dx}{1+x^4}$.

**30.** Apply Simpson's $\frac{1}{3}$rd rule to evaluate the integral

$$\text{I} = \int_0^1 e^x\, dx \text{ by choosing step size } h = 0.1$$

Show that this step size is sufficient to obtain the result correct to five decimal places.

**31.** (*i*) Obtain the global truncation error term of trapezoidal method of integration.

(*ii*) Compute the approximate value of the integral

$$I = \int (1 + x + x^2)\, dx$$

Using Simpson's rule by taking interval size $h$ as 1. Write a C program to implement.

**32.** The function $f(x)$ is known at one point $x^*$ in the interval $[a, b]$. Using this value, $f(x)$ can be expressed as

$$f(x) = p_0(x) + f'\{\xi(x)\}\,(x - x^*) \quad \text{for} \quad x \in (a, b)$$

where $p_0(x)$ is the zeroth-order interpolating polynomial $p_0(x) = f(x^*)$ and $\xi(x) \in (a, b)$. Integrate this expression from $a$ to $b$ to derive a quadrature rule with error term. Simplify the error term for the case when $x^* = a$.

## 5.25 EULER-MACLAURIN'S FORMULA

This formula is based on the expansion of operators. Suppose $\Delta F(x) = f(x)$, then an operator $\Delta^{-1}$, called inverse operator, is defined as

$$F(x) = \Delta^{-1} f(x) \tag{51}$$