Unit IV Posets and Lattice

Relation: Let A be a non-empty set. A relation R on the set A is the subset of $A \times A$.

eg ① If $A = \{1, 2, 3\}$, then

$$A \times A = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\}$$

Now (i) $R_1 = \{(1,1), (2,2), (3,3)\}$

(ii) $R_2 = \{(1,2), (2,1), (1,1)\}$

(iii) $R_3 = \{(3,4), (2,1), (1,3)\}$

Here, R_1 and R_2 are the relations on the set A but R_3 is not a relation on the set A because $(3,4) \notin A \times A$.

* If $(a,b) \in R$; we say that "a is related to b" and is written as aRb.

* If $(a,b) \notin R$; we say that "a is not related to b" and is written as $a \not{R} b$.

* The domain of a relation R is the set of all first elements of the ordered pairs which belong to R and the range is the set of the second elements of the ordered pair. ie

if $(a,b) \in R$, then

(i) Domain $= \{a : (a,b) \in R \ \forall\ a \in A\}$

(ii) Range $= \{b : (a,b) \in R \ \forall\ b \in A\}$

* **Types of Relation:**

(i) **Reflexive Relations:** A relation R on a set A is reflexive if aRa for every $a \in A$ i.e.
if $(a, a) \in R$ $\forall$ $a \in A$.

→ Thus R is not reflexive if there exists $a \in A$ such that $(a, a) \notin R$.

(ii) **Symmetric Relation:-** A relation R on a set A is symmetric if aRb then bRa $\forall$ $(a, b) \in R$ ie

if $(a, b) \in R$ then $(b, a) \in R$ $\forall$ $(a, b) \in R$

Thus R is not symmetric if there exists $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

(iii) **Antisymmetric Relation:-** A relation R on a set A is antisymmetric if whenever aRb and bRa then $a = b$ ie,

if $(a, b) \in R$ and $(b, a) \in R$ $\Rightarrow$ $a = b$

Thus R is not antisymmetric if aRb and bRa both exists but $a \neq b$.

(iv) **Transitive Relation:** A relation R on a set A is transitive if whenever aRb and bRc then aRc ie
whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$.

Thus R is not transitive if there exists $a, b, c \in A$ such that $(a, b), (b, c) \in R$ but $(a, c) \notin R$.

Q.1: Determine which of the relations in eq ① are reflexive, symmetric, antisymmetric and transitive.

Solution: (i) R_1 – reflexive, symmetric, antisymmetric transitive

(ii) R_2 – symmetric and transitive

Q2: Check whether the relation $\leq$ (less than or equal) on the set of integers are reflexive, symmetric, antisymmetric and transitive.

Solution: $\because$ $x \leq x$ $\forall$ $x \in Z$, so $\leq$ is reflexive

$\because$ $1 \leq 2$ but $2 \nleq 1$ for $1, 2 \in Z$, so $\leq$ is not symmetric.

$\leq$ is antisymmetric since $a \leq a, a \leq a \Rightarrow a = a$ $\forall$ $a \in Z$. If $a \leq b$, then $b \leq a$ must be hold to check the final result $a = b$. That is possible only when both the elements are same. So there not exists a single pair of distinct elements which hold the first two conditions ie aRb and bRa.

$\because$ $a \leq b$, $b \leq c$ $\Rightarrow$ $a \leq c$

so $\leq$ is transitive.

Partial Ordering Relations:- A relation R on a set A is called a partial ordering or partial order of A if

(i) R is reflexive

(ii) R is antisymmetric, and

(iii) R is transitive

~~A set~~ Example (i) "$\leq$" is a partial order on the set **Z**

(ii) "$\subseteq$" is a partial order on the power set

POSET: A set S together with a partial order relation R is called a partial ordered set or poset and is denoted by (S, R)

Ex (i) The relation $\subseteq$ of set inclusion is a partial ordering on any collection of sets since set inclusion has the three properties:

(i) $A \subseteq A$ for any set A

(ii) If $A \subseteq B$ and $B \subseteq A$, then $A = B$

(iii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(2) The relation $\leq$ on the set $\mathbb{R}$ of real numbers is reflexive, antisymmetric and transitive. Thus $\leq$ is a partial ordering on $\mathbb{R}$.

(3) The relation "a divides b" written as $a \mid b$, is a partial order relation on the set $\mathbb{N}$ of natural numbers.

(4) However, "a divides b" is not a partial order relation on the set of integers $\mathbb{Z}$ since $a \mid b$ is not antisymmetric, for example, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$ ie 1 divides -1 and -1 divides 1 but 1 and -1, both are not equal means distinct elements so the relation "a divides b" is not antisymmetric on $\mathbb{Z}$.

* Examples of Posets are (i) $(A, \subseteq)$

(ii) $(\mathbb{R}, \leq)$

(iii) $(\mathbb{N}, \mid)$

(iv) $(\mathbb{Z}, \mid)$ is not a poset

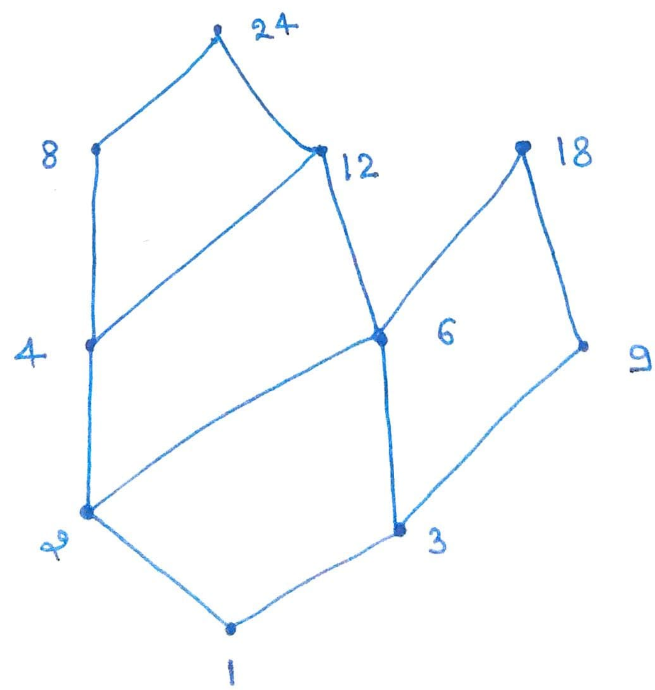
Hasse Diagram

The Hasse diagram of a finite partially ordered set S is the directed graph whose vertices are the element of S and there is a directed edge from a to b whenever $a \ll b$. [Instead of drawing an arrow from a to b, we sometimes place b higher than a and draw a line between them. It is then understood that ~~main~~ movement upwards indicates succession]

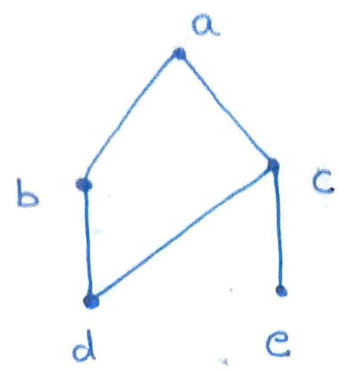
* Let S be a partially ordered set, and suppose $a, b \in S$. We say that a is an immediate predecessor of b, or that b is an immediate successor of a, written as $a \ll b$.

If $a < b$ but no element in S lies between a and b, ie there exists no element c in S such that $a < c < b$.

Ex 1: Let $S = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be an ordered set by the relation "x divides y". Draw the Hasse diagram of $(S, |)$, here $x|y$ stands for "x divides y".



Ex 2: Let $S = \{a, b, c, d, e\}$, if $d \leq b$, $d \leq c$, $b \leq a$, $c \leq a$ and $e \leq c$, then draw the Hasse diagram.



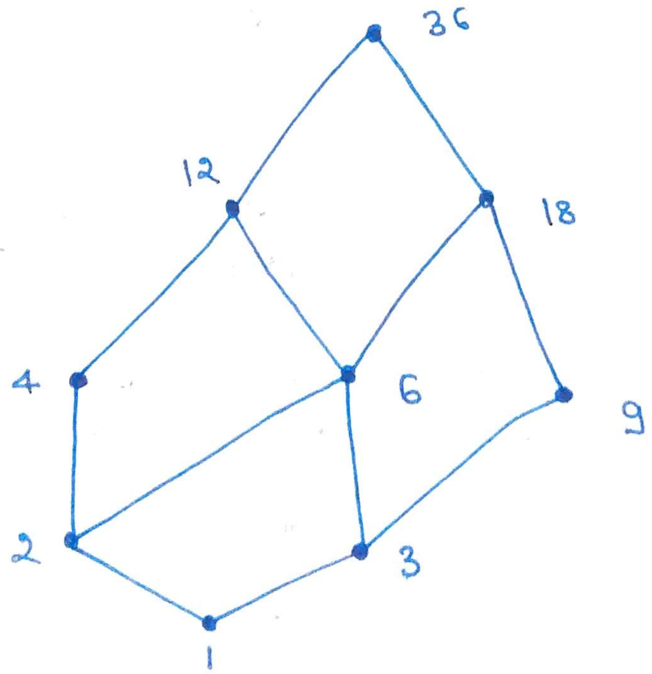
Ex 3: For any positive integer m, we will let Dm denote the set of divisors of m ordered by divisibility. Draw the Hasse diagram of D_{36}.

Solution: D_{36} = set of divisors of 36

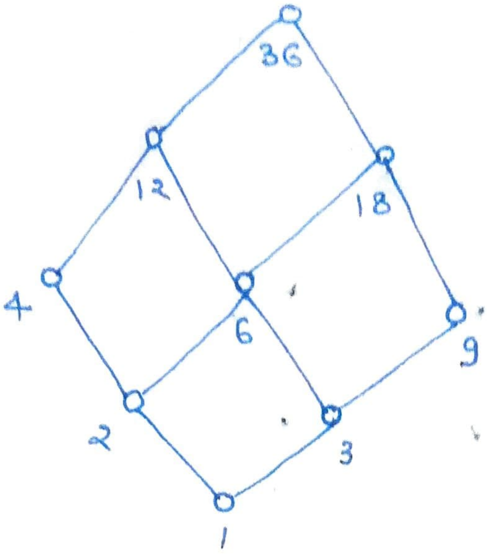
$D_{36} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$

$R = |$ ie $a|b$ ie a divides b"
relation

poset $= (D_{36}, |)$



POSES $= (D_{36}, |)$

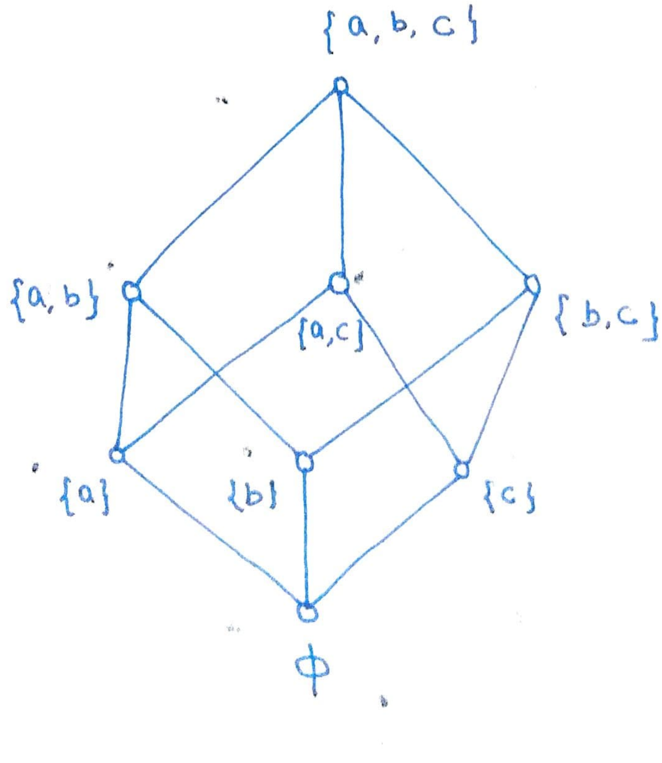


Ex 4: Let $S = \{a, b, c\}$, partial order relation $(R) = \subseteq$ then draw the Hasse diagram of $(P(S), \subseteq)$, where $P(S)$ is the power set of S.

Solution: $S = \{a, b, c\}$

$P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

POSET $= (P(S), \subseteq)$



Comparable Elements: The elements a and b of a poset (S,R) are called comparable if either aRb or bRa ie. either a is related to b or b is related to a by the relation R.

When a and b are elements of S such that neither a is related to b nor b is related to a by the relation R ie. $a \not R b$ nor $b \not R a$, then a and b are called non-comparable.

Ex. ① Poset $(Z^+, |)$, In this element 2 and 6 are comparable since 2 divides 6 ie $2|6$. But 2 and 3 are not comparable since neither 2 divides 3 nor 3 divides 2 ie $2 \nmid 3$ and $3 \nmid 2$.

Total Order Relation: When every two elements in the set are comparable, the relation is called a total ordering for the Poset (S,R).

eg ① $S = \{1, 2, 3, 4, 5\}$, $R = \leq$ (less than or equal)

so here every two elements of set S are comparable so the relation "$\leq$" is a total ordering.

Totally ordered set (or linearly ordered set):

If (S, R) is a poset and every two elements of S are comparable then S is called a totally ordered or linearly ordered set. and the relation R is called a total order or linear order. A totally ordered set is also called a chain. Ex: $S = \{1, 2, 3, 4, 5, 6\}$

$R = \leq$

(2) The poset $(Z, \leq)$ is totally ordered.

(3) The poset $(Z^+, |)$ is not totally ordered because every two elements are not comparable eg 3 and 5, 7 and 9 are not comparable.

Well Ordered Set: ~~poset~~ (S, R) is a well-ordered set if it is a poset such that the relation R is a totally ordering and every non empty subset of S has a ~~least~~ (first) element.

Minimal, maximal, First and Last Elements.

Let S be a partially ordered set with the relation R. An element a in S is called a minimal element if no other element of S strictly precedes (is less than) a.

Similarly, an element b in S is called a maximal element if no element of S strictly succeds (is larger than) b.

Geometrically, a is a minimal element of S if a is at the lowest ~~level~~ level in Hasse diagram and b is the maximal element of S if b is at the highest level in Hasse diagram. Note that S can have more than one minimal and maximal element. If S is infinite, then S may have no minimal and no maximal element.

Ex (1) The poset $(Z, \leq)$, where Z is the set of all integers and the relation $\leq$ is usual less than or equal, has no minimal and maximal elements.

If S is finite, then S must have at least one minimal and maximal element.

An element a in S is called a first element if a precedes every other element in S ie if $a \in S$ and for every element x in S

$$a \precsim x \quad (\text{ie } a \text{ precedes } x)$$

Similarly, an element b in S is called a Last element if for every element y in S, b succedes y ie

if $b \in S$ and for every element y in S

$$y \precsim b \quad (\text{ie } b \text{ succedes } y)$$

Note that S can have at most one first element, which must be a minimal element, and S can have at most one last element, which must be a maximal element. Generally, S may have neither a first nor a last element, even when S is finite.

Ques 1: Determine the first, last, minimal and maximal elements in the following posets.

(a) $(D_{36}, |)$ (b) $(\mathbb{Z}, \leq)$ (c) $(\mathbb{N}, |)$ (d) $(\mathbb{Z}^+, \leq)$

(a) First element – 1
Last element – 36
minimal element – 1
maximal element – 36

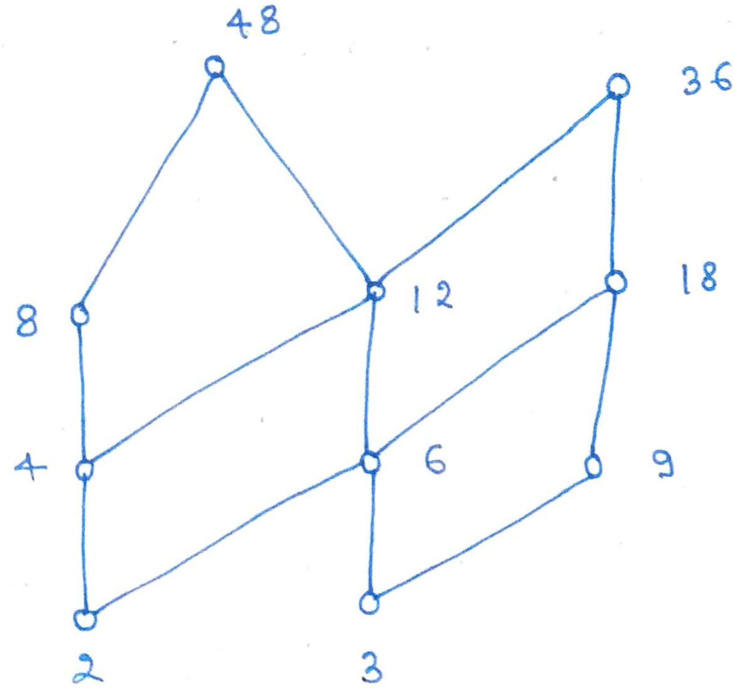
(b) First – does not exists
Last – does not exists
minimal – does not exists
maximal – does not exists

(c) Minimal – 1
maximal – does not exists
First – 1
Last – does not exists

(d) minimal – 1
maximal – not exists
First – 1
Last – not exists

Ques 2: Let $S = \{2, 3, 4, 6, 8, 9, 12, 18, 36, 48\}$ and $aRb \equiv a|b$ is "a divides b". Determine the minimal, maximal, first and last elements if they exists in the poset $(S, |)$.

Solution:



minimal elements — 2 and 3

maximal elements — 36 and 48

First element — does not exists

Last element — does not exists

* First element is also known as least element.

* Last element is also known as greatest element.

Supremum AND Infimum :-

Let A be a subset of a partially ordered set S. An element M in S is called an upper bound of A if M succeeds every element of A, i.e if for every x in A, we have

$$x \precsim M \qquad (\preceq \text{ stands for comparable})$$

If an upper bound of A precedes every other upper bound of A, then it is called the supremum of A and is denoted by sup (A). There can be at most one sup(A). However, sup (A) may not exist.

An element m in a poset S is called a lower bound of a subset A of S if m precedes every element of A. i.e. if for every y in A, we have

$$m \precsim y \qquad (\precsim \text{ stands for comparable})$$

If a lower bound of A succeeds every other lower bound of A, then it is called the infimum of A and is denoted by inf (A). There can be at most one inf(A) although inf (A) may not exists.

Least upper bound = Supremum

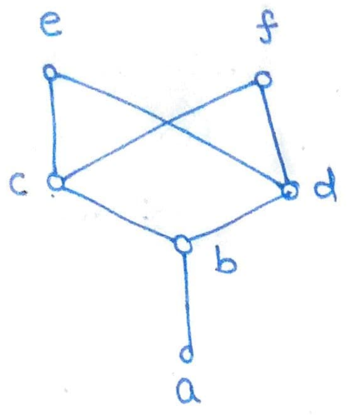
lub (A) = sup (A)

Greatest lower bound = Infimum

glb (A) = Inf (A)

If A has an upper bound we say A is bounded above, and if A has a lower bound we say A is bounded below. In particular, A is bounded if A has an upper and lower bound.

Ex 1. Let $S = \{a, b, c, d, e, f\}$ be ordered as show below and let $A = \{b, c, d\}$. Determine upper bound, lower bound, supremum and infimum of A.



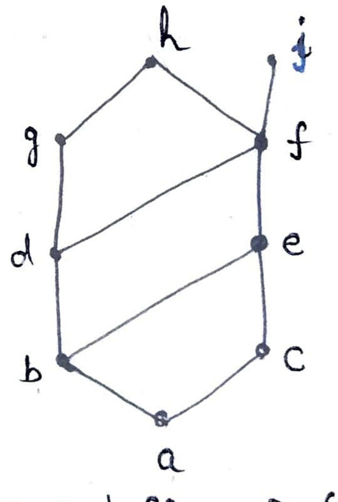
Solution: Upper bound of $A = \{e, f\}$

Lower bound of $A = \{b, a\}$

$\text{Sup}(A)$ = does not exist because e and f are not comparable

$\text{Inf}(A) = b$ since b and a both are lower bound and b succeeds a

Ex 2: Determine the lower, upper bound, supremum and infimum of the subsets $\{a, b, c\}$, $\{j, h\}$, $\{b, d, g\}$ and $\{a, c, d, f\}$ in the poset with the Hasse diagram as shown in Fig.



Solution: for $\{a, b, c\} = A$ (let)

upper bound $= \{e, f, h, j\}$

lower bound $= \{a\}$

$\text{Sup}(A) = e$, $\text{Inf}(A) = a$

for $\{j, h\} = B$ (let)

upper bound $= \{\ \} =$ no upper bound

lower bound $= \{f, d, e, b, c, a\}$

$\text{Sup}(B)$ = does not exist

$\text{Inf}(B) = f$

for $\{b, d, g\} = C$ (let)

upper bound $= \{g, h\}$

lower bound $= \{b, a\}$

$\text{Sup}(C) = g$, $\text{Inf}(C) = b$

for $\{a, c, d, f\} = D$ (let)

upper bound $= \{f, j, h\}$

lower bound $= \{a\}$

$\text{Sup}(D) = \text{lub}(D) = f$

$\text{Inf}(D) = a$

Solution : $S = \{ \{1\}, \{2\}, \{4\}, \{1,2\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,3,4\}, \{2,3,4\} \}$, relation $R = \subseteq$

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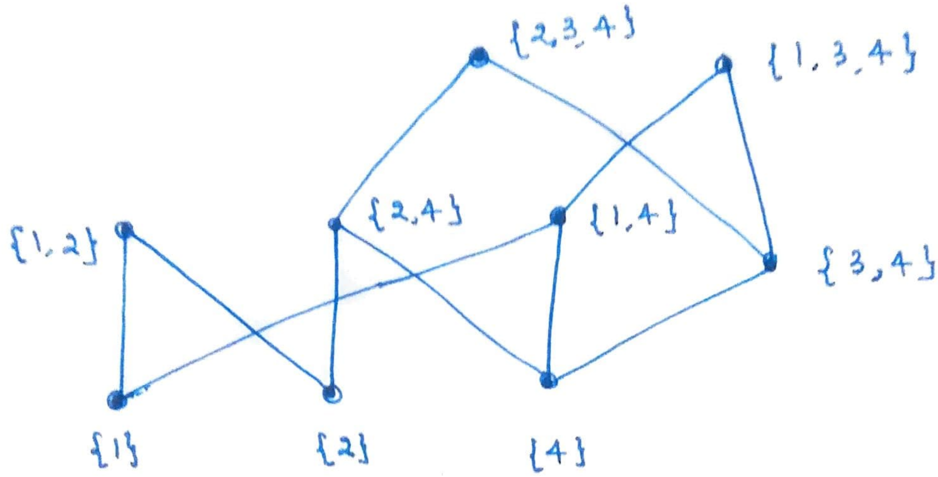

(a) maximal Elements $= \{1,2\}, \{2,3,4\}, \{1,3,4\}$

(b) minimal Elements $= \{1\}, \{2\}, \{4\}$

(c) No, there is no greatest element

(d) No, there is no least element

(e) upper bounds of $\{\{2\}, \{4\}\} = \{\{2,4\}, \{2,3,4\}\}$

(f) least upper bound of $\{\{2\}, \{4\}\} = \{2,4\}$

(g) Lower bounds of $\{\{1,3,4\}, \{2,3,4\}\} = \{\{3,4\}, \{4\}\}$

(h) Greatest lower bound of $\{\{1,3,4\}, \{2,3,4\}\} = \{3,4\}$