

# CMPT 280

## Topic 13: AVL Trees

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# References

- Textbook, Chapter 13

# AVL Trees

- AVL trees are ordered binary trees with a guarantee on how they are balanced.
- Basic insertion/deletion algorithms remain unchanged.
- Insertion/deletion algorithms are augmented to maintain the balance guarantee.
- This makes AVL trees another example of **searchable dispensers**.

# AVL Property

## AVL Property

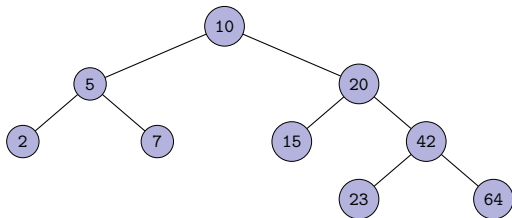
An AVL tree is an ordered binary tree in which the maximum imbalance is 1.

This property, if maintained, keeps the tree "almost" balanced.

# Types of Imbalance

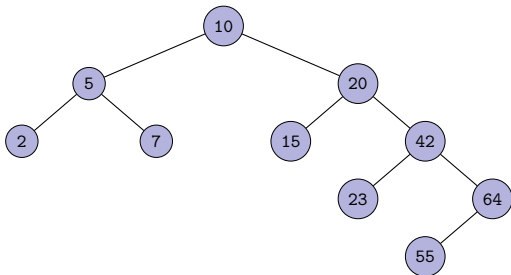
- If we start with an AVL tree, insertion of an element may cause an imbalance.
- Four possible imbalance situations: LL, LR, RL, RR.
- Critical node: node with an imbalance of 2, and all descendants have imbalance  $\leq 1$ .

## Exercise 1



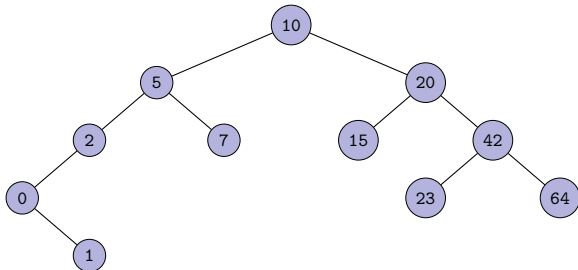
- Is it an AVL tree?
- If not, where is the critical node and what type of imbalance is exhibited?

## Exercise 2



- Insert element 55 starting with tree in Ex. 1.
- Is it an AVL tree?
- If not, where is the critical node and what type of imbalance is exhibited?

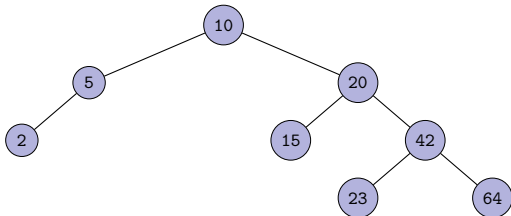
## Exercise 3



- Insert elements 0 (which doesn't break the AVL property) and then 1 starting with tree in Ex. 1.
- Is it an AVL tree?
- If not, where is the critical node and what type of imbalance is exhibited?



## Exercise 4

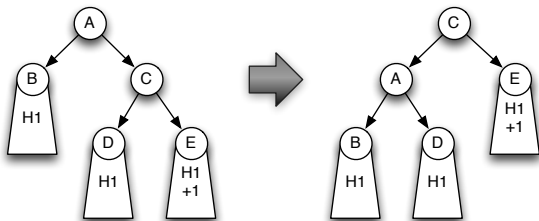


- Starting from the tree in Ex. 1, delete node containing 7.
- Is it an AVL tree?
- If not, where is the critical node and what type of imbalance is exhibited?

## Fixing Imbalance After Insertion/Deletion

- We can repair the imbalance at critical nodes caused by insertion or deletion of a single element to/from an AVL tree.
- It can be done using efficient local adjustments to the trees in the vicinity of the critical node.

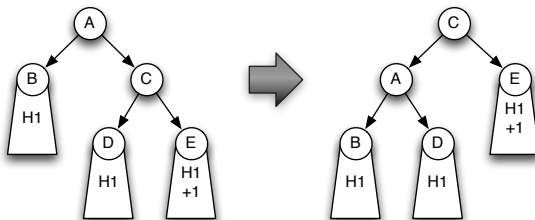
## Left Rotation (aka: RR rotation)



A *left rotation* fixes RR imbalance.  $A$  is the critical node. After the rotation:

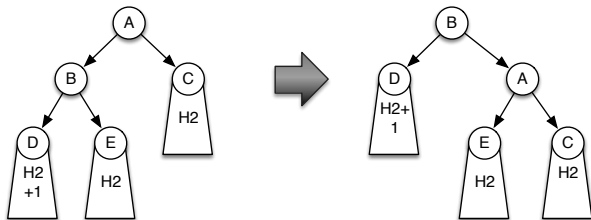
- All nodes in subtrees rooted at  $B$ ,  $D$ , and  $E$  still have the AVL property; they did before, and we did not change them.
- $A$  has the AVL property (both child subtrees have height  $H_1$ ).
- $C$  has the AVL property (both child subtrees have  $H_1 + 1$ ).

## Left Rotation



Keeping in mind that  $A$  might be a subtree of an even larger tree, if implementing this rotation, which nodes have references that need to be changed?

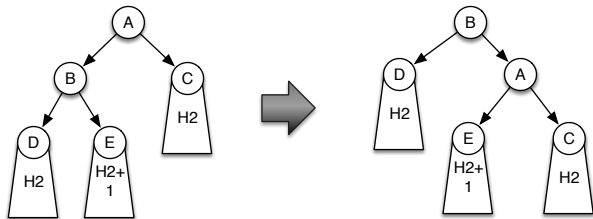
## Right Rotation



A *right rotation* fixes LL imbalance.  $A$  is a critical node. After the rotation:

- All nodes in subtrees rooted at  $C$ ,  $D$ , and  $E$  still have the AVL property; they did before, and we did not change them.
- $A$  has the AVL property (both child subtrees have height  $H_2$ ).
- $B$  has the AVL property (both child subtrees have  $H_2 + 1$ ).

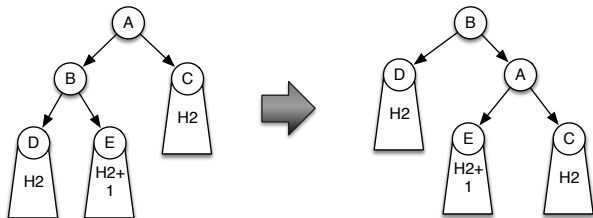
## Fixing LR Imbalance



A *right rotation* **fails** to repair LR imbalance! After the rotation:

- All nodes in subtrees rooted at  $C$ ,  $D$ , and  $E$  still have the AVL property; they did before, and we did not change them.
- $A$  has the AVL property (child subtrees have height  $H_2 + 1$  and  $H_2$ ).
- $B$  **does not have the AVL property** (left subtree has height  $H_2$ , right subtree has height  $H_2 + 2$ ).

## Fixing LR Imbalance



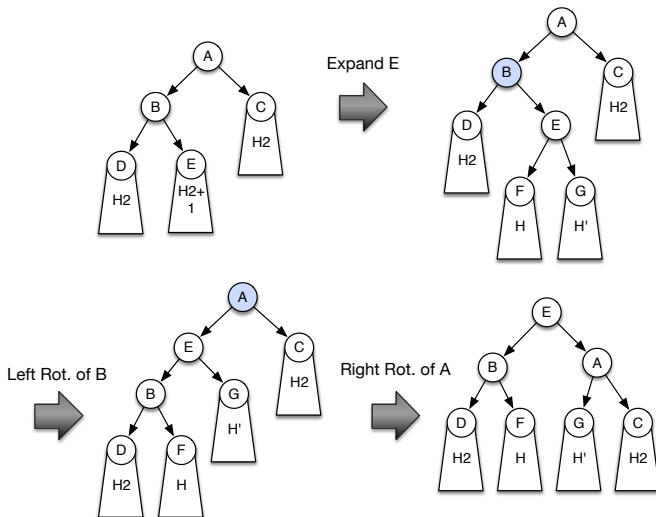
- The problem:  $B$  was right heavy before the rotation.
- The solution: make subtree rooted at  $B$  left heavy (but still an AVL tree) because we know that doesn't cause problems with right rotations.
- How?

## Double Right Rotation

A *double right rotation* (or LR rotation) fixes LR imbalance. It consists of a left rotation **of the left subtree of the critical node**, followed by a right rotation of the critical node.



# Double Right Rotation

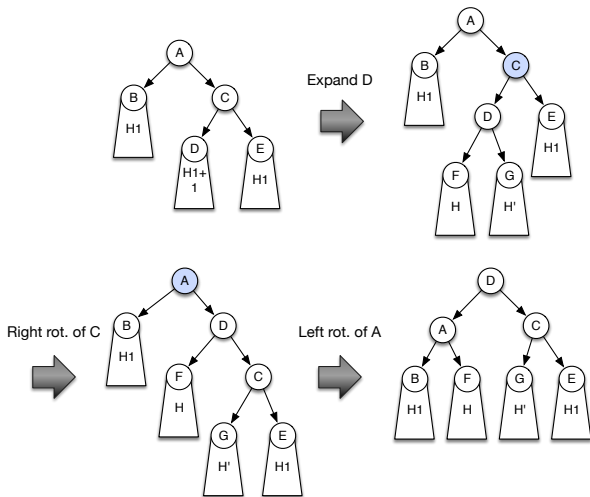


Note:  $\max\{H, H'\} = H_2, |H - H'| \leq 1$

## Double Left Rotation

A *double left rotation* fixes RL imbalance. It consists of a right rotation **of the right subtree of the critical node**, followed by a left rotation of the critical node.

# Double Left Rotation



Note:  $\max\{H, H'\} = H_2, |H - H'| \leq 1$

# Computing the Height of a Tree

```
1 Algorithm height(R)
2 R is the root of a binary tree
3
4 if R == null
5     return 0
6 else
7     leftHeight = height(R.left);
8     rightHeight = height(R.right);
9     return 1 + max(leftHeight, rightHeight);
```

Time Complexity?

# Algorithm: Restore AVL Property of Critical Node

```
1  Algorithm signed_imbalance(N)
2  N is a node.
3
4  return height(N.left) - height(N.right)
5
6  Algorithm restoreAVLProperty(R)
7  R is a (possibly critical) node
8
9  imbalanceR = signed_imbalance(R);
10 if abs(imbalanceR) <= 1 return;           // imbalance is 1 or 0
11                                         // node is not critical
12
13 if imbalanceR == 2                       // R is left heavy
14     if signed_imbalance(R.left) >= 0    // R.left is left heavy
15         RightRotate(R)                 // LL imbalance! Do right rotation.
16     else
17         LeftRotate(R.left)              // LR imbalance! Do double right rotation.
18         RightRotate(R)
19 else                                     // (imbalanceR == -2): R is right heavy
20     if signed_imbalance(R.right) <= 0   // R.right is right heavy
21         LeftRotate(R)                   // RR imbalance! Do left rotation.
22     else
23         RightRotate(R.right)            // RL imbalance! Do double-left rotation.
24         LeftRotate(R)
25 }
```

Time Complexity?

*Checks for null omitted for brevity.*

# AVL Tree: Insertion Algorithm

```
1  // Recursively insert data into the tree rooted at R
2  Algorithm insert(data, R)
3  data is the element to be inserted
4  R is the root of the tree in which to insert 'data'
5
6  if data < R.item() and R.left is not null
7      insert (data, R.left)
8  else if data >= R.item() and R.right is not null
9      insert (data, R.right)
10 else {
11     // do normal ordered binary tree
12     // insertion of data as child of R
13 }
14 restoreAVLProperty(R)
```

## AVL Tree: Insertion Algorithm Complexity

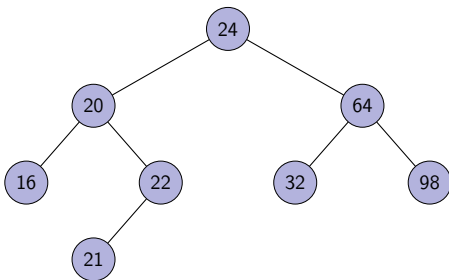
- The tree height is  $\Theta(\log n)$ . Thus, `restoreAVLProperty()` is called on at most  $O(\log n)$  nodes.
- `restoreAVLProperty()` is  $\Theta(\text{signed\_imbalance}()) + \Theta(1) = \Theta(\text{signed\_imbalance}())$
- `signed\_imbalance()` is  $2\Theta(\text{Height}()) = \Theta(\text{Height}())$
- The naive implementation of `Height()` (traversal of entire tree) is  $\Theta(n)$ , but it can be made  $\Theta(1)$  if we store the heights of `N.left` and `N.right` in node `N` (and keep them updated by modifying `insert` & `delete`).
- Thus, insertion into an AVL tree is either  $\Theta(n \log n)$  or  $\Theta(\log n)$ , depending on implementation.

## Exercise 5

Starting with an empty AVL tree, insert the following values into the tree in the order given. Show the resulting tree after each step.

16, 32, 64, 24, 20, 22, 21, 98

If done correctly, you should have the following tree at the end.



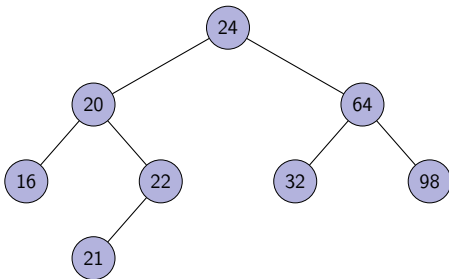


## AVL Tree: Deletion

```
1  Algorithm delete(data, R)
2  data - element to be deleted
3  R - root of tree from which 'data' should be deleted
4
5  if R is null
6      // throw exception, data is not in tree.
7  if data == R.item()
8      if R has zero or one children delete R trivially
9  else
10     replace R.item() with the in-order successor item
11     delete(in-order successor item, R.right)
12 else
13     if data <= R.item()
14         delete(data, R.left);
15     else
16         delete(data, R.right);
17     restoreAVLProperty(R);
18 }
```

Complexity of this deletion algorithm is also identical to that of the AVL tree insertion algorithm.

## Exercise 6



Starting with the AVL tree, above, delete the following values from the tree in the order given.

64, 24, 20, 16, 21

Show the resulting tree after each step.

## Next Class

- Next class reading: Chapter 14: Dictionaries