MATH 302: Homework 6

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1 Section 5.2

Problems: 3, 7, 12, 29

3. For each positive integer, n, let $P(n)$ be the formula:
$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

   (a) $P(1) : 1 = \frac{1*2*3}{6} = 1, P(1)$ is true.

   (b) $P(k) : 1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$

   (c) $P(k+1) : 1^2 + 2^2 + ... + (k+1)^2$. We want to show this equals $\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

   (d) First, we show that $P(2)$ is true. $1^2 + 2^2 = 5$ and $\frac{2*(3)*(5)}{6} = 5$. Thus, $P(2)$ is true.
   We assume that $P(k)$ is true and $1^2 + 2^2 + ... + k^2 = \frac{k(k+1)(2k+1)}{6}$.
   Next, we want to prove that $P(k+1)$ is true to prove this by mathematical induction. $P(k+1) = 1^2 + 2^2 + ... + (k+1)^2 = 1^2 + 2^2 + ... + k^2 + (k+1)^2$
   Since we know $P(k)$ is true, we can substitute that in.
   Thus, $P(k+1) = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$.
   And $P(k+1) = (k+1)[\frac{k(2k+1)}{6} + (k+1)] = (k+1)[\frac{2k^2+k+6k+6}{6}] = (k+1)[\frac{(2k+3)(k+2)}{6}]$
   Looking back on what we want to prove, we want to show that $P(k+1) = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = (k+1)\frac{(k+2)(2k+3)}{6}$
   Thus, we have shown that
   $P(k+1) : 1^2 +^2 +... + (k+1)^2 = (k+1)\frac{(k+2)(2k+3)}{6}$.
   And by mathematical induction, $\forall n \in \mathbb{Z}, n \geq 2, P(n) = \frac{n(n+1)(2n+1)}{6}$

7. **Prove:** $\forall n \in \mathbb{Z}, n \geq 1, 1+6+11+16+...+(5n-4) = \frac{n(5n-3)}{2}$
First, we prove that for n=1, $P(n)$ is true. $P(n) : 1 = \frac{1(5-3)}{2}$ is true.
Thus, we assume that for some $k \geq 1, P(k)$ is true. In other words, $1+6+11+16+...+(5k-4) = \frac{k(5k-3)}{2}$ for any $k \geq 1$
We want to show that this is also true for $P(k+1)$ and show that $P(k+1) = \frac{(k+1)(5(k+1)-3)}{2} = (k+1)[\frac{5k+2}{2}]$.
We begin by writing $P(k+1) = 1+6+11+16+...+(5(k+1)-4)$
$= 1+6+11+16+...+(5k-4)+(5(k+1)-4)$
$= 1+6+11+15...+(5k-4)+(5k+1)$
We can use our assumption to rewrite this as $P(k+1) = \frac{k(5k-3)}{2} + (5k+1)$
$= \frac{5k^2-3k}{2} + \frac{10k+2}{2} = \frac{5k^2+7k+2}{2} = \frac{(5k+2)(k+1)}{2} = (k+1)[\frac{5k+2}{2}]$
Thus, we find that $P(k+1) = (k+1)[\frac{5k+2}{2}]$, which is what we sought to prove. As such, this is proven by mathematical induction.

12. **Prove:** $\forall n \in \mathbb{Z}, n \geq 1, \frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{n}{n(n+1)} = \frac{n}{n+1}$
First, we must prove that $P(1)$ is true. $P(1) = \frac{1}{1*2} = \frac{1}{2}$ is a true statement.
Next, we assume that for some $k \in \mathbb{Z}, k \geq 1, P(k)$ is true. We write this as:
$\frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$
We now must prove that this holds true for $P(k+1) = \frac{k+1}{k+2}$
We write $P(k+1) = \frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{k(k+1)} + \frac{(1)}{(k+1)((k+1)+1)}$
$= \frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$
We rewrite this using $P(k)$ as $P(k+1) = \frac{k}{k+1} + \frac{1}{(k+2)(k+1)}$
$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+2)(k+1)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$.
This is what we expected to solve as $P(k+1)$, and this is thus proven by mathematical induction.

29. Write the sequence in closed form: $1-2+2^2-2^3+...+(-1)^n * 2^n$
$= \frac{(-1)^{k+1}-1}{-2} * \frac{2^{k+1}-1}{1}$

2 Section 5.3

Problems: 4, 7, 15

4. For each positive integer n, let $P(n)$ be the sentence that describes the following divisibility property:
$5^n - 1$ is divisible by 4.
   (a) $P(0) = 5^0 - 1 = 4, 4 \mid 4$. Thus, $P(0)$ is true
   (b) $P(k) = 5^k - 1$, we will assume that $P(k)$ is true. In other words, we will assume $4 \mid 5^n - 1$
   (c) $P(k+1) = 5^{k+1} - 1$ We seek to prove that $4 \mid P(k)$.

(d) The inductive step must assume that $P(k)$ is true and prove that $4 \mid 5^{k+1} - 1$ for any $k \geq 0$. That proves that this predicate is true for all values of $n \in \mathbb{Z}, n \geq 0$

7. For each positive integer n, let $P(n)$ be the sentence:
In any round-robin tournament involving n teams, the teams can be labeled $T_1, T_2, T_3, ..., T_n$, so that T_i beats T_{i+1} for every $i = 1, 2, ..., n$.

(a) $P(2)$ is a case with teams T_1, T_2 and T_1 beats T_2.

(b) $P(k)$ is a tournament in which there are teams $T_1, T_2, ..., T_k$. We assume that the teams are ordered in such a way that the T_1 is in first place.

(c) $P(k+1) = T_1, T_2, ..., T_k, T_{k+1}$.

(d) In the inductive step, you assume that there is a value $k, k \geq 2$ and you must prove that $P(k) \implies P(k+1)$. In other words, if $P(k)$ is true, you must prove that $P(k+1)$ is also true.

15. **Prove by Mathematical Induction:** $n(n^2 + 5)$ is divisible by 6, for each integer $n \geq 0$
First, verify that $P(0)$ is true: $P(0) = 0(0+5) = 0, 6 \mid 0$. Therefore, $P(0)$ is true.
Next, assume that $P(k)$ is true such that $P(k) = k(k^2+5), 6 \mid k(k^2+5)$ We must prove that $6 \mid P(k+1)$. We write $P(k+1) = (k+1)((k+1)^2+5)$ $= (k+1)(k^2+2k+6) = (k^3+2k^2+6k+k^2+2k+6) = (k^3+3k^2+8k+6)$ $= (k^3+5k) + (3k^2+3k+6)$ Since we know that k^3+5k is divisible by 6 (as made in our assumption), we can rewrite this as $6r$ where r is any integer. We also rewrite the other portion as
$6(\frac{3n(n+1)}{2} + 1)$ Thus, we can rewrite everything as
$P(k+1) = 6(r + (\frac{3n(n+1)}{2} + 1)$. We also know that in any case, the numerator in our fraction will be divisible by two since $n+1$ or n MUST be positive. As such, we can rewrite $P(k+1) as 6(r+k)$ where r and k are arbitrary integers. Thus, we know that $6 \mid P(k+1)$. Thus, by mathematical induction, $\forall n \geq 0, 6 \mid P(n)$.

3 Section 5.4

Problems: 10, 13, 16, 17

10. "The introductory example solved with ordinary mathematical induction in Section 5.3 can also be solved using strong mathematical induction. Let $P(n)$ be "any n¢ can be obtained using a combination of 3¢ and 5¢ coins." Use strong mathematical induction to prove that $P(n)$ is true for every integer $n \geq 8$."
First, we test that $P(8)$ is valid. We can verify it is true as $1 * 5¢ + 1 * 3¢ =$ 8¢. Next, we assume that $P(k)$ and the set $\{P(a), P(a+1), ..., P(k)\}$ is

all true, where $a = 8$.
We now seek to represent $P(k+1)$ in cents. We know that any integer $k \geq a$ exists in this format. Every number can also be rewritten as a sum of two numbers smaller than it. For instance,we can rewrite $P(k+1)$ as $P(k-2) + 3¢$. Since $P(k-2)$ is defined in our assumption set, we know it must exist. As such, we can prove that any integer $k \geq 8$ can be represented with 3¢ and 5¢ coins.

13. "Use strong mathematical induction to prove the existence part of the unique factorization of integers theorem (Theorem 4.4.5). In other words, prove that every integer greater than 1 is either a prime number or a product of prime numbers." First, we prove that $P(a) = P(2)$ is true. We know that $2 = 1 * 2$, so 2 can be rewritten as a product of prime number. Next, we assume that $P(k)$ can be defined as a product of prime numbers and thus the set of $\{P(a), P(a+1), ..., P(k)\}$ is defined as true. We seek to show that $P(k+1)$ is true. We split this into two cases: $P(k+1)$ is prime or not prime.
If $P(k+1)$ is prime, then we are done as it can be written as itself$*1$.
If $P(k+1)$ is not prime, we know that $P(k+1)$ must be a product of two numbers that are smaller than it. We will let these be $P(k+1) = r * s$. Since $(k+1) > r, s$, we know that r, s must be in the assumption set. As such, $P(k+1)$ can be rewritten as products of r and s and can be rewritten with their respective factorization multiplied together.

16. "Use strong mathematical induction to prove that for every integer $n \geq 2$, if n is even, then any sum of n odd integers is even, and if n is odd, then any sum of n odd integers is odd. "

    We first begin by testing if $P(2)$ is true. We can add 2 odd numbers, $3+1 = 4$ and verify that the sum of 2 odd integers is even. We then assume that this is true for $P(k)$, where k is any integer $k \geq 2$. We assume that the entire set from $a = 2$ to k is true, such that $\{P(a), P(a+1), ..., P(k)\}$ is true. We can prove $P(k+1)$ as true by splitting into two cases.
    Case 1: k is odd. If k is odd, we know that $k+1$ must be even. We also know that since $P(k)$ is in the assumption set, $P(k)$ must be odd. Thus, $P(k+1)$ must add any arbitrary odd integer to this set. We know that the summation of any two odd integers must be even. Thus, $P(k+1)$ is true when $k+1$ is even and k is odd.
    Case 2: k is even. If k is even, we know that $k+1$ must be odd. We also know that since $P(k)$ is in the assumption set, $P(k)$ must be even. Thus, $P(k+1)$ must add any arbitrary odd integer to this set. We know that the summation of any odd and any even numbers must be odd. Thus, $P(k+1)$ is also true when $k+1$ is odd and k is even.

17. Compute $4^1, 4^2, 4^3, 4^4, 4^5, 4^6, 4^7, 4^8$ Make a conjecture about the units digit of 4^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.
$4^1 = 4; 4^2 = 16; 4^3 = 64; 4^4 = 256;$

$4^5 = 1024; 4^6 = 4096; 4^7 = 16384; 4^8 = 65536$.
Conjecture: For any $n \in \mathbb{Z}, n \geq 1, 4^n$ is 4 if n is odd and is 6 if n is even.
Base Cases: Let $n = 1, P(n) = 4^1 = 4$. Thus, the units digit is 4.
Let $n = 2, P(n) = 4^2 = 16$. Thus the units digit is 6.
Inductive Step: Let k be any integer, $k \geq 1$. We assume that $P(n)$ is true for all $4 \leq n \leq k$. In other words, we define the set $\{P(a), P(a+1), ..., P(k)\}$ to be true, where $a = 1$. We now seek to prove $P(k+1) = 4^{k+1}$
We can rewrite this as $4 * 4^k$ We then split this into two cases:
Case 1: k is even. We know by the assumption set that if k is even, the units digit will be 6. As such, we know that we will multiply this number by 4. In any case, $4 * 6 = 24$, which has a units digit of 4.
Case 2: k is odd. Since k is in the assumption set, we know that if k is odd, then $P(k)$ states that the units digit will be 4. As such, we have $4 * 4$, which results in a units place of 6.

4 Section 5.6

Problems: 4, 12

4. $d_k = k(d_{k-1})^2; d_0 = 3$. The next four terms are: $d_1 = 9; d_2 = 162$; $d_3 = 52488; d_4 = 5509980288$.

12. Let $s_0, s_1, s_2, ...$ be defined by the formula $s_n = \frac{(-1)^n}{n!}$ for every integer $n \geq 0$. Show that this sequence satisfies the following recurrence relation for every integer $k \geq 1 : s_k = \frac{-s_{k-1}}{k}$
We know that s_k can be written as $s_k = \frac{(-1)^k}{k!}$. Thus, we can rewrite $s_{k-1} = \frac{(-1)^{k-1}}{(k-1)!}$
We rewrite the recurrence relation with the definitions provided above as $\frac{s_{k-1}}{k} = \frac{-\frac{(-1)^{k-1}}{(k-1)!}}{k}$ and simplify:
$= -\frac{(-1)^{k-1}}{k(k-1)!}$
$= \frac{(-1)^k}{k!}$
We find that this is equivalent to the initial equation provided to us and have thus proven this relation to be true.