MATH 302: Homework 5

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March 18, 2024

1 Section 4.5

Problems: 14, 21, 24, 38

14. "If today is Tuesday, what day of the week will it be 1,000 days from today?"

    Since there are 7 days in a week, we calculate the remainder of 1000/7 or $1000 \mod 7 = 6$ Thus 6 days after Tuesday is *Monday*

21. "Suppose b is any integer. If $b \mod 12 = 55$, what is $8b \mod 12$? In other words, if division of b by 12 gives a remainder of 5, what is the remainder when $8b$ is divided by 12? Your solution should show that you obtain the same answer no matter what integer you start with."

    We can rewrite b as $b = 12 * k + 55$ where k is any non-negative integer. Thus if we multiply b by 8, we get $8b = 8(12 * k + 55) = 96k + 440 = 96k + 432 + 8 = 12(8k + 36) + 8$. We can denote $8k + 36$ as an arbitrary integer, q. Thus, $8b = 12 * q + 8$ and $8b \mod 12 = 8$.

24. "Prove that for all integers m and n, if $m \mod 5 = 2$ and $n \mod 5 = 1$ then $mn \mod 5 = 2$."

    We begin by rewriting m and n as $m = 5j + 2, n = 5k + 1$ where j and k are arbitrary, non-negative integers. Thus we can rewrite mn as $mn = (5j+2)(5k+1) = 5jk+5j+5k+2 = 5(jk+j+k)+2$. Since j and k are arbitrary integers, we can simplify this to $mn = 5q + 2$. Where q is a non-negative integer. By definition, this can be rewritten and proves that $mn \mod 5 = 2$.

38. **Prove:** "For every integer m, $m^2 = 5k$, or $m^2 = 5k + 1$, or $m^2 = 5k + 4$ for some integer k."

    We can rewrite m as $m = 5q + r$ where q is any integer and $0 \leq r < 5$. Thus, we can calculate $m^2 = 25q^2 + 10qr + r^2 = 5(5q^2 + 2qr) + r^2$ We can rewrite this as $m^2 = 5s + r^2$ where $s = 5q^2 + 2qr =$ any positive integer. The last step is to figure out possible values for r. Since $r \in \mathbb{Z}, 0 \leq r < 5$, the only options are $r = 0, 1, 2, 3, 4$

If $r = 0, 1, 2$, then $m^2 = 5q, 5q + 1, 5q + 4$, respectively. (Case 1)

If $r = 3$, then $m^2 = 5q + 9 = 5q + 5 + 4 = 5(q + 1) + 1 \equiv 5k + 1$. (Case 2)

Lastly, if $r = 4$, then $m^2 = 5q + 16 = 5q + 15 + 1 = 5(q + 3) + 1 \equiv 5k + 1$. (Case 3).

Therefore, for every integer m, $m^2 = 5k$, or $m^2 = 5k + 1$, or $m^2 = 5k + 4$ for some integer k.

2 Section 4.6

Problems: 7, 19, 20

7. Given that k is an integer, $\lceil k + \frac{1}{2} \rceil = k + 1$ as $k < k + \frac{1}{2} < k + 1$

19. **Prove or Disprove:** "For every real number x, $\lceil x - 1 \rceil = \lceil x \rceil - 1$." We can divide this into two cases: $r \in \mathbb{Z}$ and $r \notin \mathbb{Z}$

    If $r \in \mathbb{Z}$, then $\lceil r - 1 \rceil = r - 1 \equiv \lceil r \rceil - 1 = r - 1$

    If $r \notin \mathbb{Z}$, then r must have some non-integer component. As such, $\lceil r \rceil = r + 1$ Thus, $\lceil r - 1 \rceil = r$. With these two equivalencies, we can state that $\lceil r - 1 \rceil = r \equiv \lceil r \rceil - 1 = (r + 1) - 1 = r$

20. **Prove or Disprove:** $\forall x, y \in \mathbb{R}, \lceil xy \rceil = \lceil x \rceil \lfloor y \rfloor$

    This can be disproved in the case that $x = 1, y = 1.1$ as $1 * 1.1 = 1.1$ and $\lceil 1.1 \rceil = 2$, but $\lceil 1 \rceil = \lfloor 1.1 \rfloor = 1$ and $1 \neq 2$

3 Section 4.7

Problems: 4, 6, 17

4. "Use proof by contradiction to show that for every integer m, $7m + 4$ is not divisible by 7" Suppose not and suppose that $7m + 4$ is divisible by 7. Thus, $7m + 4$ would have to be written in a way such that there is no remainder. This is not possible. As such, $7m + 4$ is not divisible by 7. $\lightning$

6. "There is no greatest negative real number." $\rightarrow$ There is a greatest negative real number.

    Proof by Contradiction: Suppose there is a greatest negative real number, n, such that $\forall x < 0, x < n < 0$. We can divide n by two and know that $n < \frac{n}{2} < 0$ As such, n is not the greatest negative number. $\lightning$

17. "For all prime numbers a, b, and c, $a^2 + b^2 \neq c^2$"

    Proof by Contradiction: Let's assume that $a^2 + b^2 = c^2$ for all prime numbers a, b, and c. Then, $c^2 - b^2 = a^2 = (c - b)(c + b) = a^2$. Let $b = 3, c = 2$, thus $c - b < 0$ but a^2 must be positive. As such, this is not possible. $\lightning$

4 Section 4.8

Problems: 14, 23

14. **Prove or Disprove:** The sum of any two positive irrational numbers is irrational.

    This can be easily disproved by counter-example. $\sqrt{2}$ and $1-\sqrt{2}$ are both irrational. The sum of these two numbers is $\sqrt{2}+(1-\sqrt{2}) = 1$. Thus the sum of two positive, irrational numbers can be rational and this statement is disproved.

23. Prove that for any integer, a, $9 \not| (a^2 - 3)$

    By the quotient remainder rule, we can state that $9 \mid (a^2-3)$ if $9k = a^2-3$ where k is any integer. We cannot rewrite this in such a way where 9 is a common divisor of 9. As such, 9 cannot divide $a^2 - 3$.

5 Section 4.10

Problems: 15, 16

15. 832 and 10,933

    (a) $10933 \mod 832 = 117$
    (b) $832 \mod 117 = 13$
    (c) $117 \mod 13 = 0$

    Therefore, 13 is the GCD.

16. 4,131 and 2,431

    (a) $4131 \mod 2431 = 1700$
    (b) $2431 \mod 1700 = 731$
    (c) $1700 \mod 731 = 238$
    (d) $731 \mod 238 = 17$
    (e) $238 \mod 17 = 0$

    Therefore, the GCD is 17.

6 Section 5.1

Problems: 6, 60

6. $f_n = \lfloor \frac{n}{4} \rfloor * 4, n \geq 1$. The first four terms are $0, 0, 0, 4$

60.

$$2 * \sum_{k=1}^{n} (3k^2 + 4) + 5 * \sum_{k=1}^{n} (2k^2 - 1) \tag{1}$$

$$= \sum_{k=1}^{n} (2(3k^2 + 4) + 5 * (2k^2 - 1)) \tag{2}$$