

MATH 304: Homework 6

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1 Section 5.2

Problems: 3, 7, 12, 29

3. For each positive integer, n , let $P(n)$ be the formula:

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(a) $P(1) : 1 = \frac{1 \cdot 2 \cdot 3}{6} = 1$, $P(1)$ is true.

(b) $P(k) : 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$

(c) $P(k+1) : 1^2 + 2^2 + \dots + (k+1)^2$. We want to show this equals $\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

(d) First, we show that $P(2)$ is true. $1^2 + 2^2 = 5$ and $\frac{2 \cdot (3) \cdot (5)}{6} = 5$. Thus, $P(2)$ is true.

We assume that $P(k)$ is true and $1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Next, we want to prove that $P(k+1)$ is true to prove this by mathematical induction. $P(k+1) = 1^2 + 2^2 + \dots + (k+1)^2 = 1^2 + 2^2 + \dots + k^2 + (k+1)^2$

Since we know $P(k)$ is true, we can substitute that in.

Thus, $P(k+1) = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$.

And $P(k+1) = (k+1) \left[\frac{k(2k+1)}{6} + (k+1) \right] = (k+1) \left[\frac{2k^2 + k + 6k + 6}{6} \right] = (k+1) \left[\frac{(2k+3)(k+2)}{6} \right]$

Looking back on what we want to prove, we want to show that $P(k+1) = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} = (k+1) \frac{(k+2)(2k+3)}{6}$

Thus, we have shown that

$P(k+1) : 1^2 + 2^2 + \dots + (k+1)^2 = (k+1) \frac{(k+2)(2k+3)}{6}$.

And by mathematical induction, $\forall n \in \mathbb{Z}, n \geq 2, P(n) = \frac{n(n+1)(2n+1)}{6}$

7. **Prove:** $\forall n \in \mathbb{Z}, n \geq 1, 1 + 6 + 11 + 16 + \dots + (5n - 4) = \frac{n(5n-3)}{2}$

First, we prove that for $n=1$, $P(n)$ is true. $P(1) : 1 = \frac{1(5-3)}{2}$ is true.

Thus, we assume that for some $k \geq 1$, $P(k)$ is true. In other words, $1 + 6 + 11 + 16 + \dots + (5k - 4) = \frac{k(5k-3)}{2}$ for any $k \geq 1$

We want to show that this is also true for $P(k+1)$ and show that $P(k+1) = \frac{(k+1)(5(k+1)-3)}{2} = (k+1)\left[\frac{5k+2}{2}\right]$.

We begin by writing $P(k+1) = 1 + 6 + 11 + 16 + \dots + (5(k+1) - 4)$
 $= 1 + 6 + 11 + 16 + \dots + (5k - 4) + (5(k+1) - 4)$
 $= 1 + 6 + 11 + 15\dots + (5k - 4) + (5k + 1)$

We can use our assumption to rewrite this as $P(k+1) = \frac{k(5k-3)}{2} + (5k+1)$
 $= \frac{5k^2-3k}{2} + \frac{10k+2}{2} = \frac{5k^2+7k+2}{2} = \frac{(5k+2)(k+1)}{2} = (k+1)\left[\frac{5k+2}{2}\right]$

Thus, we find that $P(k+1) = (k+1)\left[\frac{5k+2}{2}\right]$, which is what we sought to prove. As such, this is proven by mathematical induction.

12. **Prove:** $\forall n \in \mathbb{Z}, n \geq 1, \frac{1}{1*2} + \frac{1}{2*3} + \dots + \frac{n}{n(n+1)} = \frac{n}{n+1}$

First, we must prove that $P(1)$ is true. $P(1) = \frac{1}{1*2} = \frac{1}{2}$ is a true statement.

Next, we assume that for some $k \in \mathbb{Z}, k \geq 1, P(k)$ is true. We write this as:

$$\frac{1}{1*2} + \frac{1}{2*3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

We now must prove that this holds true for $P(k+1) = \frac{k+1}{k+2}$

We write $P(k+1) = \frac{1}{1*2} + \frac{1}{2*3} + \dots + \frac{1}{k(k+1)} + \frac{(1)}{(k+1)((k+1)+1)}$
 $= \frac{1}{1*2} + \frac{1}{2*3} + \dots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$

We rewrite this using $P(k)$ as $P(k+1) = \frac{k}{k+1} + \frac{1}{(k+2)(k+1)}$
 $= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+2)(k+1)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$.

This is what we expected to solve as $P(k+1)$, and this is thus proven by mathematical induction.

29. Write the sequence in closed form: $1 - 2 + 2^2 - 2^3 + \dots + (-1)^n * 2^n$
 $= \frac{(-1)^{k+1}-1}{-2} * \frac{2^{k+1}-1}{1}$

2 Section 5.3

Problems: 4, 7, 15

4. For each positive integer n , let $P(n)$ be the sentence that describes the following divisibility property:
 $5^n - 1$ is divisible by 4.

(a) $P(0) = 5^0 - 1 = 4, 4 \mid 4$. Thus, $P(0)$ is true

(b) $P(k) = 5^k - 1$, we will assume that $P(k)$ is true. In other words, we will assume $4 \mid 5^n - 1$

(c) $P(k+1) = 5^{k+1} - 1$ We seek to prove that $4 \mid P(k)$.

- (d) The inductive step must assume that $P(k)$ is true and prove that $4 \mid 5^{k+1} - 1$ for any $k \geq 0$. That proves that this predicate is true for all values of $n \in \mathbb{Z}, n \geq 0$
7. For each positive integer n , let $P(n)$ be the sentence:
In any round-robin tournament involving n teams, the teams can be labeled $T_1, T_2, T_3, \dots, T_n$, so that T_i beats T_{i+1} for every $i = 1, 2, \dots, n$.
- (a) $P(2)$ is a case with teams T_1, T_2 and T_1 beats T_2 .
- (b) $P(k)$ is a tournament in which there are teams T_1, T_2, \dots, T_k . We assume that the teams are ordered in such a way that the T_1 is in first place.
- (c) $P(k+1) = T_1, T_2, \dots, T_k, T_{k+1}$.
- (d) In the inductive step, you assume that there is a value $k, k \geq 2$ and you must prove that $P(k) \implies P(k+1)$. In other words, if $P(k)$ is true, you must prove that $P(k+1)$ is also true.
15. **Prove by Mathematical Induction:** $n(n^2 + 5)$ is divisible by 6, for each integer $n \geq 0$
First, verify that $P(0)$ is true: $P(0) = 0(0 + 5) = 0, 6 \mid 0$. Therefore, $P(0)$ is true.
Next, assume that $P(k)$ is true such that $P(k) = k(k^2 + 5), 6 \mid k(k^2 + 5)$
We must prove that $6 \mid P(k+1)$. We write $P(k+1) = (k+1)((k+1)^2 + 5)$
 $= (k+1)(k^2 + 2k + 6) = (k^3 + 2k^2 + 6k + k^2 + 2k + 6) = (k^3 + 3k^2 + 8k + 6)$
 $= (k^3 + 5k) + (3k^2 + 3k + 6)$ Since we know that $k^3 + 5k$ is divisible by 6 (as made in our assumption), we can rewrite this as $6r$ where r is any integer. We also rewrite the other portion as
 $6(\frac{3n(n+1)}{2} + 1)$ Thus, we can rewrite everything as
 $P(k+1) = 6(r + (\frac{3n(n+1)}{2} + 1))$. We also know that in any case, the numerator in our fraction will be divisible by two since $n+1$ or n MUST be positive. As such, we can rewrite $P(k+1)$ as $6(r+k)$ where r and k are arbitrary integers. Thus, we know that $6 \mid P(k+1)$. Thus, by mathematical induction, $\forall n \geq 0, 6 \mid P(n)$.

3 Section 5.4

Problems: 10, 13, 16, 17

10. "The introductory example solved with ordinary mathematical induction in Section 5.3 can also be solved using strong mathematical induction. Let $P(n)$ be "any n¢ can be obtained using a combination of 3¢ and 5¢ coins." Use strong mathematical induction to prove that $P(n)$ is true for every integer $n \geq 8$."
First, we test that $P(8)$ is valid. We can verify it is true as $1 * 5¢ + 1 * 3¢ = 8¢$. Next, we assume that $P(k)$ and the set $\{P(a), P(a+1), \dots, P(k)\}$ is

all true, where $a = 8$.

We now seek to represent $P(k + 1)$ in cents. We know that any integer $k \geq a$ exists in this format. Every number can also be rewritten as a sum of two numbers smaller than it. For instance, we can rewrite $P(k + 1)$ as $P(k - 2) + 3\text{¢}$. Since $P(k - 2)$ is defined in our assumption set, we know it must exist. As such, we can prove that any integer $k \geq 8$ can be represented with 3¢ and 5¢ coins.

13. "Use strong mathematical induction to prove the existence part of the unique factorization of integers theorem (Theorem 4.4.5). In other words, prove that every integer greater than 1 is either a prime number or a product of prime numbers." First, we prove that $P(a) = P(2)$ is true. We know that $2 = 1 * 2$, so 2 can be rewritten as a product of prime number. Next, we assume that $P(k)$ can be defined as a product of prime numbers and thus the set of $\{P(a), P(a + 1), \dots, P(k)\}$ is defined as true. We seek to show that $P(k + 1)$ is true. We split this into two cases: $P(k + 1)$ is prime or not prime.

If $P(k + 1)$ is prime, then we are done as it can be written as itself*1.

If $P(k + 1)$ is not prime, we know that $P(k + 1)$ must be a product of two numbers that are smaller than it. We will let these be $P(k + 1) = r * s$. Since $(k + 1) > r, s$, we know that r, s must be in the assumption set. As such, $P(k + 1)$ can be rewritten as products of r and s and can be rewritten with their respective factorization multiplied together.

16. "Use strong mathematical induction to prove that for every integer $n \geq 2$, if n is even, then any sum of n odd integers is even, and if n is odd, then any sum of n odd integers is odd. "

We first begin by testing if $P(2)$ is true. We can add 2 odd numbers, $3 + 1 = 4$ and verify that the sum of 2 odd integers is even. We then assume that this is true for $P(k)$, where k is any integer $k \geq 2$. We assume that the entire set from $a = 2$ to k is true, such that $\{P(a), P(a + 1), \dots, P(k)\}$ is true. We can prove $P(k + 1)$ as true by splitting into two cases.

Case 1: k is odd. If k is odd, we know that $k + 1$ must be even. We also know that since $P(k)$ is in the assumption set, $P(k)$ must be odd. Thus, $P(k + 1)$ must add any arbitrary odd integer to this set. We know that the summation of any two odd integers must be even. Thus, $P(k + 1)$ is true when $k + 1$ is even and k is odd.

Case 2: k is even. If k is even, we know that $k + 1$ must be odd. We also know that since $P(k)$ is in the assumption set, $P(k)$ must be even. Thus, $P(k + 1)$ must add any arbitrary odd integer to this set. We know that the summation of any odd and any even numbers must be odd. Thus, $P(k + 1)$ is also true when $k + 1$ is odd and k is even.

17. Compute $4^1, 4^2, 4^3, 4^4, 4^5, 4^6, 4^7, 4^8$ Make a conjecture about the units digit of 4^n where n is a positive integer. Use strong mathematical induction to prove your conjecture.

$$4^1 = 4; 4^2 = 16; 4^3 = 64; 4^4 = 256;$$

$$4^5 = 1024; 4^6 = 4096; 4^7 = 16384; 4^8 = 65536.$$

Conjecture: For any $n \in \mathbb{Z}, n \geq 1$, 4^n is 4 if n is odd and is 6 if n is even.

Base Cases: Let $n = 1, P(n) = 4^1 = 4$. Thus, the units digit is 4.

Let $n = 2, P(n) = 4^2 = 16$. Thus the units digit is 6.

Inductive Step: Let k be any integer, $k \geq 1$. We assume that $P(n)$ is true for all $4 \leq n \leq k$. In other words, we define the set $\{P(a), P(a+1), \dots, P(k)\}$ to be true, where $a = 1$. We now seek to prove $P(k+1) = 4^{k+1}$. We can rewrite this as $4 * 4^k$. We then split this into two cases:

Case 1: k is even. We know by the assumption set that if k is even, the units digit will be 6. As such, we know that we will multiply this number by 4. In any case, $4 * 6 = 24$, which has a units digit of 4.

Case 2: k is odd. Since k is in the assumption set, we know that if k is odd, then $P(k)$ states that the units digit will be 4. As such, we have $4 * 4$, which results in a units place of 6.

4 Section 5.6

Problems: 4, 12

4. $d_k = k(d_{k-1})^2; d_0 = 3$. The next four terms are: $d_1 = 9; d_2 = 162;$
 $d_3 = 52488; d_4 = 5509980288$.

12. Let s_0, s_1, s_2, \dots be defined by the formula $s_n = \frac{(-1)^n}{n!}$ for every integer $n \geq 0$. Show that this sequence satisfies the following recurrence relation for every integer $k \geq 1$: $s_k = \frac{-s_{k-1}}{k}$

We know that s_k can be written as $s_k = \frac{(-1)^k}{k!}$. Thus, we can rewrite

$$s_{k-1} = \frac{(-1)^{k-1}}{(k-1)!}$$

We rewrite the recurrence relation with the definitions provided above as

$$\frac{s_{k-1}}{k} = \frac{-\frac{(-1)^{k-1}}{(k-1)!}}{k} \text{ and simplify:}$$

$$= -\frac{(-1)^{k-1}}{k(k-1)!}$$

$$= \frac{(-1)^k}{k!}$$

We find that this is equivalent to the initial equation provided to us and have thus proven this relation to be true.