5.5 Bessel's Eq^n

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0 \qquad \nu \geq 0, \text{ real.}$$

Assume $y = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$\Rightarrow \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r} + \sum_{0}^{\infty} (m+r) x^{m+r} a_m + \sum_{0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{0}^{\infty} a_m x^{m+r} = 0$$

$m=0$: $r(r-1)a_0 + r a_0 \cancel{+ a_0} - \nu^2 a_0 = 0$

$$\Rightarrow a_0 \left[r(r-1) + r - \nu^2 \right] = 0$$

$$\boxed{r = \pm \nu}$$

for $r=\nu$: $\left[\nu(\nu+1)-\nu^2\right]a_1=0$

for $m=1$ $\Rightarrow (2\nu+1)a_1=0 \Rightarrow \boxed{a_1=0}$

$m=2,3,\ldots$: $(m+r)(m+r-1)a_m+(m+r)a_m+a_{m-2}-\nu^2 a_m=0$

$r=\nu$ $\quad m=2s \Rightarrow a_{2s}=-\dfrac{a_{2s-2}}{2^2 s(\nu+s)} \longrightarrow$

$$a_2=\frac{-a_0}{2^2(1)(\nu+1)}$$

$$a_4=\frac{-a_2}{2^2\,2\,(\nu+2)}=\frac{+a_0}{2^4\,2!\,(\nu+1)(\nu+2)}$$

$$\boxed{a_{2s}=\frac{(-1)^s a_0\,\nu!}{2^{2s}(s!)(\nu+1)(\nu+2)(\nu+3)\ldots(\nu+s)\nu}}\quad s=1,2,3,$$

Further assume

$r = \nu = \text{Integer}$

$= n$

Choose $a_0 = \text{const.} = \dfrac{1}{2^n (n!)}$

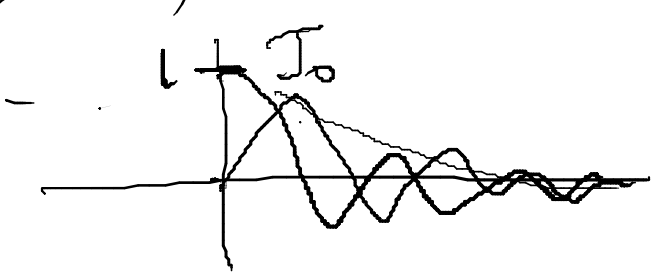
$$a_{2m} = \frac{(-1)^m}{2^{2m+n}\,(n+m)!\;m!}$$

$$y = \sum_{m=2,4,6\ldots} a_m x^{m+r} = \sum_{m=0}^{\infty} a_{2m} x^{2m+n}$$

Suppose $n=0$

Bessel fns of 1st kind $\displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m}(m!)^2} x^{2m}$

$$J_0(x) = 1 - \frac{x^2}{2^2 (1!)^2} + \cdots$$



$$J_n(x) = \sum a_{2m} x^{2m+n}$$

$$J_n \overset{x\to\infty}{\sim} \sqrt{\frac{2}{\pi x}} \cos\left[x - \frac{n\pi}{2} - \frac{\pi}{4}\right]$$

~~$J_\nu(x)$~~ Bessel fn. for any $\nu \geq 0$ (1st)

Gamma fn. : $\Gamma(\nu) = \int_0^\infty e^{-t}\, t^{\nu-1}\, dt$

~~$\Gamma(\nu+1)$~~

$$\Gamma(\nu+1) = \int_0^\infty \underset{\uparrow 2}{e^{-t}}\, \underset{\uparrow 1}{t^{\nu}}\, dt$$

$$= \cancel{-e^{-t} t^{\nu} \Big|_0^\infty} + \nu \int_0^\infty e^{-t}\, t^{\nu-1}\, dt$$

0

$= \nu\, \Gamma(\nu) = \nu(\nu-1)\Gamma(\nu-2)$

$= \nu(\nu-1) \cdot \cancel{\Gamma(1)}$

1

n: Int

Chk: $\Gamma(n+1) = n\Gamma(n) = n!$

$$J_\nu(x) = x^{\nu} \sum \frac{(-1)^m\, x^{2m}}{2^{2m+\nu}\, (m!)\, \Gamma(\nu+m+1)}$$

$r = -\nu$

$$J_{-\nu}(x) = x^{-\nu} \sum \frac{(-1)^m\, x^{2m}}{2^{2m-\nu}\, m!\, \Gamma(m-\nu+1)}$$

General sol: $y = A\, J_\nu(x) + B\, J_{-\nu}(x)$

Ckk: LI

ν: Integer: $J_{-n}(x) = (-1)^n J_n(x)$

$$J_{-n}(x) = \sum \frac{(-1)^m x^{2m-n}}{2^{2m-n}\, \underbrace{\Gamma(m-n+1)}_{(m-n)!}\, (m!)}$$

$m = s+n$

$s = m-n$

$2m-n = 2(s+n) - n$

$$= \sum \frac{(-1)^{s+n} x^{2s+n}}{2^{2s+n}\, (s+n)!\, s!}$$

$$= (-1)^n \sum \frac{(-1)^s x^{2s+n}}{\quad} = (-1)^n J_n(x)$$

(b) $(x^{\nu} J_{\nu})' = x^{\nu} J_{\nu-1}$

LHS $\quad x^{\nu} J_{\nu} = x^{\nu} \sum \frac{(-1)^m \, x^{2m+\nu}}{2^{2m+\nu} (m!) \, \Gamma(m+\nu+1)}$ $\quad 2m+2\nu$

$$(x^{\nu} J_{\nu})' = \sum \frac{(-1)^m \cdot (2m+2\nu) \, x^{2m+2\nu-1}}{2^{2m+\nu} (m!) \, \Gamma(m+\nu+1)}$$

$$= x^{\nu} \sum \frac{x^{2m+\nu-1} (-1)^m \, \cancel{2}\cancel{(m+\nu)}}{2^{2m+\nu-1} (m!) \, \cancel{(\nu+m)} \, \Gamma(\nu+m)}$$

$$= x^{\nu} \sum \frac{(-1)^m \, x^{2m+\nu-1}}{2^{2m+\nu-1} (m!) \, \Gamma(m+(\nu-1)+1)}$$

$$= x^{\nu} J_{\nu-1}$$

(c) $\left(x^{-\nu} J_\nu\right)' = -x^{-\nu} J_{\nu+1}$

d) $J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu$

e) $J_{\nu-1} - J_{\nu+1} = 2J_\nu'$

Thm. Bessel fn. for Half int. of order ν!

$\left\{ \Gamma(\nu+m) \; \Gamma(\nu+m-1) \; a \right)$ $\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x \quad / \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

Proof. $J_{1/2}(x) = x^{1/2} \sum \frac{(-1)^m x^{2m}}{2^{2m+1/2} (m!) \Gamma(m+3/2)}$

$\Gamma(m+3/2) = (m+3/2)\Gamma(m+1/2)$

$\nu = -1/2$

$\Gamma(\nu+1) = \int_0^\infty e^{-t} t^\nu \, dt$ $\quad (m+1/2)(m-1/2) \cdots \frac{\Gamma(1/2)}{\sqrt{\pi}} (m+1/2)\Gamma(m+1/2)$

$$2^{m+1}\,\Gamma\left(m+\tfrac{3}{2}\right) = \underbrace{\left(m+\tfrac{1}{2}\right)\left(m-\tfrac{1}{2}\right)\cdots}_{(m+1)\text{ factors}}\Gamma\left(\tfrac{1}{2}\right)$$

$$= \underbrace{(2m+1)(2m-1)(2m-3)\cdots}_{(2m+1)!}$$

$$\underbrace{2m}_{2(m)} \qquad \underbrace{(2m-2)}_{2(m-1)} \qquad \underbrace{}_{2(m-2)}$$

$$2^{2m+1}(m!)\,\Gamma\left(m+\tfrac{3}{2}\right)$$

$$= (2m+1)!\,\sqrt{\pi}\;2^{m}(m!)$$

$$J_{1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}}\sum\frac{(-1)^m x^{2m+1}}{(2m+1)!\,\sqrt{\pi}} = \sqrt{\frac{2}{\pi x}}\,\sin x$$

P.P. (5.5) 5, 10, 15, 25, 26.