Recap: Orthogonal fns

$\{y_i\}_{i=1}^{\infty}$ o.g. on $a \le x \le b$.

w.r.t weight $r(x) \Rightarrow$

$$\langle y_n, y_m \rangle \leftarrow \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0$$

O.N: $\sqrt{\langle y_n, y_n \rangle} = \| y_n \| = 1$

Eg1. $y_m(x) = \sin(mx)$ $-\pi \le x \le \pi$ $y'' + m^2 y = 0$

$$\langle y_n, y_m \rangle = \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx \quad p = 1,\ r = 1$$

$m \ne n$ $= 0$

$m \ne n$ $= \pi \Rightarrow \|y_n\| = \sqrt{\pi}$

$$= \int_{-\pi}^{\pi} \frac{\cos(m-n)x}{2}\, dx + \int \frac{\cos(m+n)x}{2}\, dx$$

$$\{y_m\} = \left\{ \frac{\sin mx}{\sqrt{\pi}} \right\}_{m=0}^{\infty}$$

<u>Thm</u>: Suppose p, q, r, p' in S.L. Eqn are real-valued, continuous and $r(x) > 0$ on $a \le x \le b$. Let y_n, y_m be eig-fns. of S.L. problem. $\Rightarrow$ y_n, y_m are orthogonal w.r.t. weight fn. $r(x)$.

$$\Rightarrow \int_a^b r(x)\, y_n\, y_m\, dx = 0$$

Proof: y_n, y_m are sol^n to S.L Eq^n.

$$y_m \Rightarrow \left[(p y_n')' + (q + \lambda_n r) y_n = 0 \right.$$

$$- \quad y_n \Rightarrow \left[(p y_m')' + (q + \lambda_m r) y_m = 0 \right.$$

To show $0 = \int_{x=a}^{b} (\lambda_n - \lambda_m) r \, y_n y_m = \boxed{\int y_m (p y_n')'} - \boxed{\int y_n (p y_m')'}$

1st 2nd

$$= \left[p y_n' y_m - p y_m' y_n \right]_a^b$$

$$+ \int \times$$

$$= p(b) \left[y_n'(b) y_m(b) - y_m'(b) y_n(b) \right]$$

$$- p(a) \left[y_n'(a) y_m(a) - y_m'(a) y_n(a) \right]$$

Case 1: $p(a) = p(b) = 0$ Done

Case 2 $p(a) \neq 0$, $p(b) = 0$

y_n satisfies B.C.

B.C.s $k_1 y_n(a) + k_2 y_n'(a) = 0$

Similarly $k_1 y_m(a) + k_2 y_m'(a) = 0$

?? $(y_n' y_m - y_m' y_n)(a) = 0$

(Assume $k_2 \neq 0$)

Done

Case 3: $p(a) = 0$; $p(b) \neq 0$ Done

Case 4: $p(a) = p(b)$: Assume periodic B.C.s

$y(a) = y(b)$ & $y'(a) = y'(b)$

Eg 1. Legendre Eq^n: $-1 \le x \le 1$

$$\left[(1-x^2)y'\right]' + \lambda y = 0$$

$$p = 1-x^2 \qquad p(a) = p(b) = 0$$

Sol^n: $y_n(x) = P_n(x)$ o.g. w.r.t $r(x) = 1$

$$\Rightarrow \int_{-1}^{1} P_n(x)\, P_m(x)\, dx = 0 \quad n \neq m$$

$$= \frac{2}{2n+1} \quad n = m$$

Eg 2.

$$\cancel{x^2 y'' + x y'}$$

$$\tilde{x} = kx$$

$$(0 \le x \le R) \quad x^2 y'' + x y' + (k^2 x^2 - n^2) y = 0$$

$$y_n = J_n(kx)$$

$$\begin{cases} p(x) = x \\ q(x) = -\dfrac{n^2}{x} \\ r(x) = x \end{cases}$$

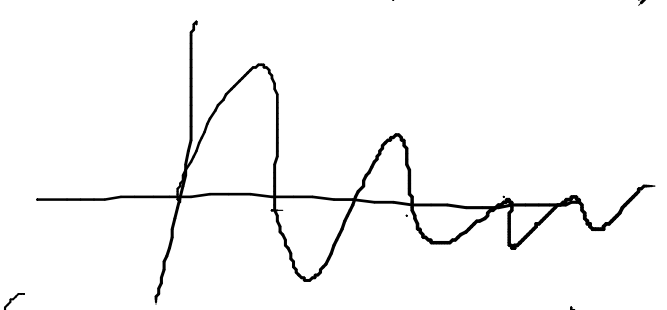
$$\underline{p(0) = 0}, \quad p(R) \neq 0$$

$$\boxed{J_n(kR) = 0}$$

$$\Rightarrow J_n(kR)\, J_n'(kR) = \frac{J_n(kR)}{J_n'(kR)}$$

$$\Rightarrow (J_n' \propto J_n) \; ??$$

Suppose m^{th} root of $J_n \equiv \alpha_{n,m}$



$$\Rightarrow kR = \alpha_{n,m}$$

$$\Rightarrow \boxed{k_{n,m} = \frac{\alpha_{n,m}}{R}}$$

$$\Rightarrow \{J_n(k_{n,m} x)\}_{n,m}$$

orthogonal.

Thm: Orthogonality of Bessel fns.

$\forall n \in I^+$ the Bessel fns of 1st kind $\{J_n(k_{n,m} x)\}_{n,m=0}^{\infty}$ forms an orthogonal set on $0 \le x \le R$.

Eg 3: Solve the S-L problem:

$$y'' + \lambda y = 0 \rightarrow (1)$$

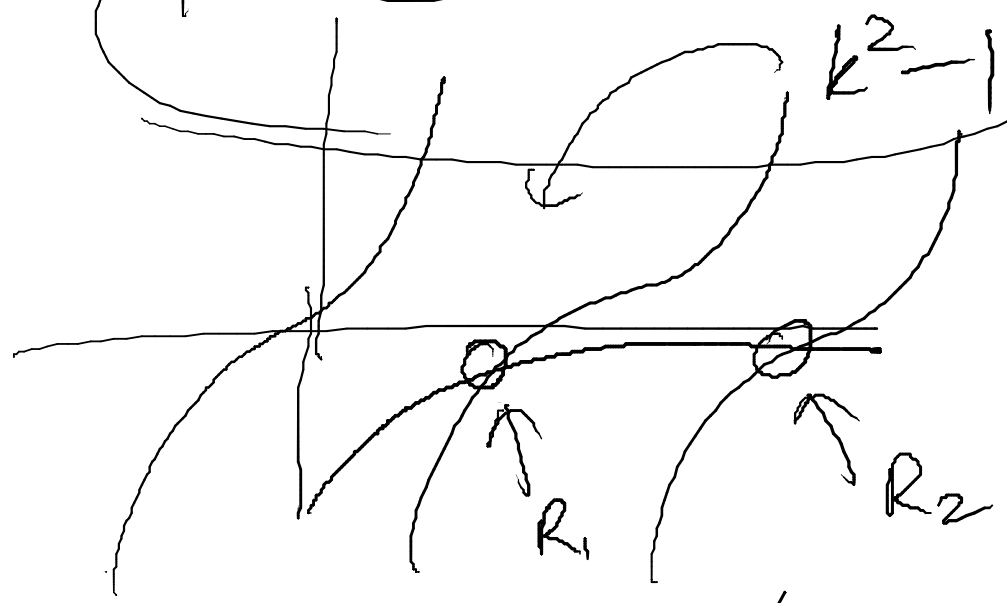
$$(2) \begin{cases} y(0) = -y'(0) \\ y(\pi) = y'(\pi) \end{cases}$$

Sol^n: $y(x) = A\cos kx + B\sin kx \qquad k = \sqrt{\lambda}$

$A = B = 0 \Rightarrow y \equiv 0$ Trivial sol^n

$\Rightarrow B \neq 0 \Rightarrow \underline{-2k + \tan(\pi k) - k^2 \tan(k\pi) = 0}$

$$\Rightarrow \tan(k\pi) = \frac{-2k}{k^2-1} \qquad k = ? \quad \underline{\underline{E.V.}}$$



<u>E. fns</u>: $\sin(kx)$, $\cos(kx)$

<u>E.V</u>: 'k'

<u>5.7</u> (9th) : 4, 6, 12, 18, 20b.

5.8 **Orthogonal Eig-fn Expansion.**

Suppose $y_0, y_1, y_2, \ldots$ O.g. w.r.t weight $r(x)$

$$\Rightarrow \langle y_m, y_n \rangle = 0 \quad n \neq m.$$

$$\|y_m\| = \sqrt{\langle y_m, y_m \rangle} = \sqrt{\int_a^b r(x) y_m^2 \, dx}$$

Consider Any cont. fn. $f(x)$ can be represented as:

$$f(x) = \sum a_m y_m(x)$$

To find $a_m = \langle f, y_m \rangle = \int_a^b r(x) f(x) y_m(x) \, dx$

$$= \int_a^b r(x) \left(\sum a_n y_n\right) y_m(x) \, dx$$

$$\Rightarrow \langle f, y_m \rangle = \sum_{n=0}^{\infty} a_n \underbrace{\int_a^b r(x)\, y_m(x)\, y_n(x)\, dx}_{n = m}$$

$$= a_m \|y_m\|^2$$

$$\Rightarrow \boxed{a_m = \frac{\langle f, y_m \rangle}{\|y_m\|^2}}$$

Eg 1. Revisit SL: Eq^n: $y'' + \lambda y = 0$

$-\pi \le x \le \pi$ $\qquad y(\pi) = y(-\pi)$

$y'(\pi) = y'(-\pi)$

$$y_m = \left\{ \frac{\sin mx}{\sqrt{\pi}} \right\} \text{ or } \left\{ \frac{\cos mx}{\sqrt{\pi}} \right\} \qquad \underline{\underline{\lambda = m^2}}$$

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(mx)\, dx$$

Eg 2.

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) \; : \; \text{Legendre } \underline{\text{Series}}$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f(x)\, P_m(x)\, dx$$

$$\checkmark \; \|P_m\|^2 = \frac{2}{2m+1} = \sqrt{\int_{-1}^{1} P_m^2\, dx}$$

<u>Chk</u>: $P_n(x) = \frac{1}{2^n} \frac{1}{n!} \left[(x^2-1)^n \right]^{(n)}$ $\quad n \in \mathbb{I}^+ \,\&\, 0$

(Rodrigues formula)

$$\langle P_n, P_n \rangle = \frac{2}{2n+1}$$

Suppose $f(x) = \sin \pi x$

I want to write Legendre series

$$f(x) = \sum a_m P_m(x)$$

$$a_m = \frac{1}{\|P_m\|^2} \int_{-1}^{1} f \, P_m(x) \, dx$$

$$a_0 = \int_{-1}^{1} \sin(\pi x) \, 1 \, dx = 0$$

$$a_1 = \frac{1}{\|P_1\|^2} \int_{-1}^{1} \sin(\pi x) \, \underline{P_1(x)} \, dx = \frac{3}{\pi}$$

$a_2 = 0$, $a_3 \neq 0$ . . . .

$$f(x) = \frac{3}{\pi} P_1(x) - 1.1582 P_3(x) + 0.2143 P_5(x) - \ldots$$