2.5

Euler – Cauchy ODEs

$$x^2 y'' + a x y' + b y = 0 \qquad a, b \in \mathbb{R}$$

Choose $\boxed{y = x^m}$

$$\Rightarrow \quad m(m-1)x^m + a m x^m + b x^m = 0$$

$$\Rightarrow \quad \underbrace{x^m}_{\neq 0} \underbrace{\left[m(m-1) + a m + b \right]}_{=0} = 0$$

$$m_{1,2} = \frac{1-a}{2} \pm \sqrt{\underbrace{\left(\frac{1-a}{2}\right)^2 - b}}$$

$$\boxed{y = A x^{m_1} + B x^{m_2}}$$

$$b = \left(\frac{1-a}{2}\right)^2 \Rightarrow m_1 = m_2 = \frac{1-a}{2}$$

Eg1. $x^2 y'' + 0.6xy' + 16.04y = 0$

Sol^n : $y = x^m$

$$m = \frac{1-a}{2} \pm \sqrt{\left(\frac{1-a}{2}\right)^2 - b}$$

$$= \frac{1-0.6}{2} \pm \sqrt{\left(\frac{1-0.6}{2}\right)^2 - 16.04}$$

$$= 0.2 \pm 4i$$

$$y = A x^{0.2+4i} + B x^{0.2-4i}$$

$$= x^{0.2}\left[A \underbrace{x^{4i}}_{e^{(4\ln x)i}} + B \underbrace{x^{-4i}}_{e^{(-4\ln x)i}}\right]$$

$$e^{\ln x^a} = e^{a \ln x}$$

$$y_1 = x^{\frac{1-a}{2}}$$

$$y_2 = u(x)\, x^{\frac{1-a}{2}}$$

$$y'' + \frac{a}{x} y' + \frac{(1-a)^2}{4x^2} y = 0$$

$$-\int \frac{a}{x}\, dx$$

$$u(x) = \int U\, dx$$

$$= \int \frac{1}{x}\, dx$$

$$= \ln|x|$$

where

$$U = \frac{1}{y_1^2} e^{-a \ln x}$$

$$= \frac{1}{x^{1-a}} e$$

$$= \frac{1}{x^{1-a}} e^{\ln x^{-a}}$$

$$= \frac{1}{x^{1-a}} x^{-a}$$

$$= \frac{1}{x}$$

$$y_2 = \ln|x|\, y_1$$

Prac Prob (2.5): 6, 10, 16c

2.6. Existence, uniqueness of sol^n: Wronskian.

$$\begin{cases} \text{I} \quad y'' + p(x)y' + q(x)y = 0 \\ \text{II} \quad y(x_0) = K_0, \quad y'(x_0) = K_1 \end{cases}$$

IVP

Thm If $p(x)$, $q(x)$ are continuous on some open interval I and $x_0 \in I$, then I, II has a unique sol^n on I given by $y = c_1 y_1 + c_2 y_2$; y_1, y_2 : basis.

Proof: ODE (I) with, $p(x)$, $q(x)$ on I

then solutions y_1, y_2 are linearly dep. iff

Wronskian $= W(y_1, y_2) = \boxed{y_1 y_2' - y_2 y_1' = 0}$

If Lin. Dep

$\Rightarrow \boxed{y_1(x) = k\, y_2(x)}$ / $y_2 = k\, y_1(x)$

$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = k y_2 y_2' - k y_2 y_2'$

$= 0$

($\Leftarrow$) Suppose $W(y_1(x_0), y_2(x_0)) = 0$ at $x_0 \in I$

Consider $k_2(k_1 y_1 + k_2 y_2) = 0 \leftarrow$ Diff.

At x_0 $\quad k_1(k_1 y_1' + k_2 y_2') = 0$

$k_1 k_2\, W(y_1(x_0), y_2(x_0)) = 0$

At x_0

$$y_2'(k_1 y_1 + k_2 y_2) = 0$$
$$-y_2(k_1 y_1' + k_2 y_2') = 0$$

$$\cancel{k_1 k_2 W(x_0) + k_2^2 y_2 - k_1^2 y_1'}$$

$$k_1 W(x_0) = 0$$

$k_1 = 0$ May / not

$k_2 = 0$??

$$W(x_0) = y_1 y_2' - y_2 y_1' \Big|_{x_0} = 0$$

y_1, y_2 : Sol^n to ODE.

LD follows at $x = x_0$

$\downarrow$ Extend

$\forall x \in I$

Eg 1: $y'' + \omega^2 y = 0$

$y_1 = \cos \omega x$

$y_2 = \sin \omega x$

LD? NO

Sol^n: $W = y_1 y_2' - y_2 y_1' = \omega \neq 0$

$$= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1, y_2 \cdots y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & & & y_n' \\ & & & \\ y_1^{(n-1)} & & & y_n^{(n-1)} \end{vmatrix}$$

Eg 2. $y'' - 2y' + y = 0$

Sol^n: Chk y_1 y_2 : LI / LD.

$y_1 = e^x$

$y_2 = xe^x$

$$W(y_1, y_2) = \begin{vmatrix} \overset{y_1}{e^x} & \overset{y_2}{xe^x} \\ \underset{y_1'}{e^x} & \underset{y_2'}{(x+1)e^x} \end{vmatrix} = e^{2x} \neq 0$$

$\forall x \in \mathbb{R}$.

Prac Prob (2.6) : 8, 16, 18b, 18d

2.7. **Non-homogeneous ODEs**

(I) $y'' + p(x)y' + q(x)y = \underline{r(x)}$ $\quad r \neq 0.$

y_h

A general sol^n of (I) on an interval I':

$$y = \underbrace{y_h}_{\text{homo.}} + \underbrace{y_p}_{\text{part.}}$$

$$y_h = c_1 y_1 + c_2 y_2$$

Choice of 'y_p'

$r(x)$	y_p
$ke^{\alpha x}$	$Ce^{\alpha x}$
Kx^n	$k_n x^n + k_{n-1} x^{n-1} + \dots + k_0$
$K_1 \cos\omega x$	$A\cos\omega x + B\sin\omega x$
$K_2 \sin\omega x$	
$K_1 e^{ax}\cos\omega x$	$e^{ax}[A\cos\omega x + B\sin\omega x]$
$K_2 e^{ax}\sin\omega x$	

known

unknown

Method of undetermined coeff.

Eg1. Solve $y'' + y = 0.001x^2$ $\quad y(0) = 0$, $y'(0) = 1.5$

Sol^n

$$y_h = A\cos x + B\sin x$$

$$y_p = Cx^2 + Dx + E$$

$$\Rightarrow y_p'' + y_p = (2C) + (Cx^2 + Dx + E) = 0.001x^2$$

$$\left.\begin{aligned} 2C + E &= 0 \\ C &= 0.001 \\ D &= 0 \end{aligned}\right\} \quad E = -2C = -0.002$$

$$y = y_h + y_p = \underbrace{A\cos x + B\sin x + 0.001x^2 - 0.002}$$

Eg2. $y'' + 3y' + 2.25y = -10e^{-1.5x}$

$$\begin{cases} y(0) = 1 \\ y'(0) = 0 \end{cases}$$

Sol^n: $y_h = Ae^{-1.5x} + Bxe^{-1.5x}$

$y_p = x^2 C e^{-1.5x}$ $\qquad$ $y_p = u(x)e^{-1.5x}$

complete.

Eg3. $y'' + 2y' + 5y = e^{0.5x} + 40\cos 10x - 190\sin 10x$

$y(0) = 0.16$

$y'(0) = 40.08$

Sol^n: $y_h = e^{-x}[A\cos 2x + B\sin 2x]$

$y_p = C\cos 10x + D\sin 10x + Ee^{0.5x}$

Prac Prob (2.7): 6, 14, 17, 20