2-8 Forced Simple Harmonic Oscillator.

$$my'' + cy' + ky = r(t)$$

$r=0$: Done

$r \neq 0$: $F_0 \cos(\omega t)$ → forced frequency

Sol^n: $y = y_h + y_p$ (Done)

$$y_p = A\cos(\omega t) + B\sin(\omega t)$$

$$A = \frac{F_0(k - m\omega^2)}{(k-\omega^2 m)^2 + (\omega c)^2} \quad ; \quad B = \frac{F_0\,\omega c}{(k-\omega^2 m)^2 + (\omega c)^2}$$

$\omega_0 = \sqrt{\frac{k}{m}}$: natural freq.

$$A = \frac{F_0\, m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2-\omega^2)^2 + (\omega c)^2} \quad ; \quad B = \frac{F_0\,\omega c}{m^2(\omega_0^2-\omega^2)^2 + (\omega c)^2}$$

Case 1: Undamped, forced oscillation.

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

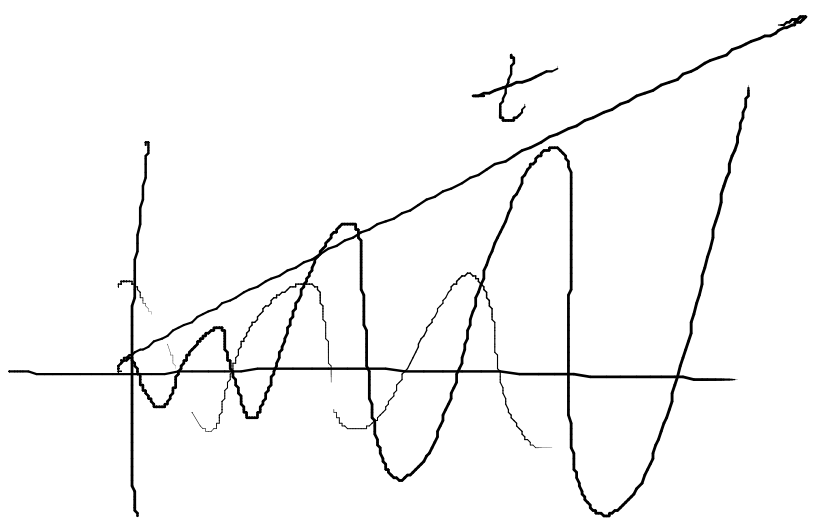
$$c \cos(\omega_0 t - \delta)$$

$$c = \sqrt{a^2 + b^2}$$

$$\delta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$y = y_h + y_p = \boxed{a \cos(\omega_0 t) + b \sin(\omega_0 t)} + \boxed{\frac{F}{m(\omega_0^2 - \omega^2)} \cos(\omega t)}$$

$\boxed{\text{If } \omega \longrightarrow \omega_0}$: $y_p = \left[\frac{F}{m(\omega_0^2 - \omega^2)} \cos(\omega t)\right] t$: Resonance



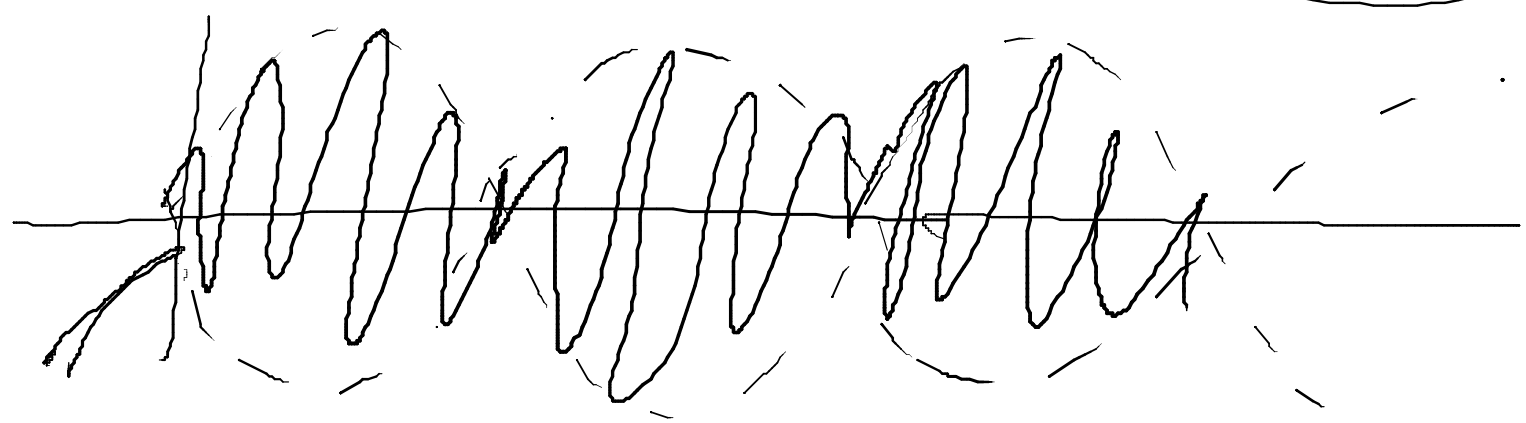
$$= \boxed{At \cos(\omega_0 t)}$$

Case 2 : Und- . . forced osc.

Suppose $y(t) = \underbrace{\frac{F}{m(\omega_0^2-\omega^2)}}_{A}\left[\cos\omega t - \cos(\omega_0 t)\right]$

$$= \cancel{2}\,A \underbrace{\sin\frac{(\omega+\omega_0)t}{2}}_{\text{fast}} \underbrace{\sin\frac{(\omega-\omega_0)t}{2}}_{\text{slow}}$$

$\omega_0 \neq \omega > 0$



Beats

Case 3: Damped, forced oscillations.

$\underline{\underline{c>0}}$: $y_h \xrightarrow{t\to\infty} e^{-ct} \longrightarrow 0$

$$y = y_h + y_p \xrightarrow{t\to\infty} y_p$$

$$y_p = \overbrace{\frac{F_0 m\left(\omega_0^2-\omega^2\right)}{\left(k-m\omega^2\right)^2+c^2\omega^2}}^{a}\cos(\omega t) + \overbrace{\frac{F_0\,\omega c}{\left(k-\omega^2 m\right)^2+(\omega c)^2}}^{b}\sin(\omega t)$$

$$= \underline{\underline{C}}\cos(t\omega-\delta)$$

C : amplitude

c : damping coeff

$$C = \sqrt{a^2+b^2} = \frac{F_0}{\sqrt{m^2\left(\omega_0^2-\omega^2\right)+(\omega c)^2}}$$

$$\left\{\begin{array}{l} C_{max} = \\ \omega_{max} = \end{array}\right\}$$

$$\frac{dC}{d\omega} = \frac{F_0}{-2(\text{Den})^{3/2}}\left[2m^2\left(\omega_0^2-\omega^2\right)(-2\omega) + 2\omega c^2\right] = 0$$

$$\Rightarrow c^2 = 2m^2(\omega_0^2 - \omega^2)$$

$$\Rightarrow \boxed{2m^2\omega^2 = 2mk - c^2} \longrightarrow f = 0$$

If $c^2 > 2mk$ $\qquad \omega =$ Purely imag.

Chk: $\dfrac{d^2C}{d\omega^2} < 0$

As $\omega \uparrow$ (s.t $c^2 > 2mk$)

$\Rightarrow C(\omega) \downarrow$

If $c^2 < 2mk$

$$\boxed{\omega_{max}^2 = \omega_0^2 - \frac{c^2}{2m}}$$

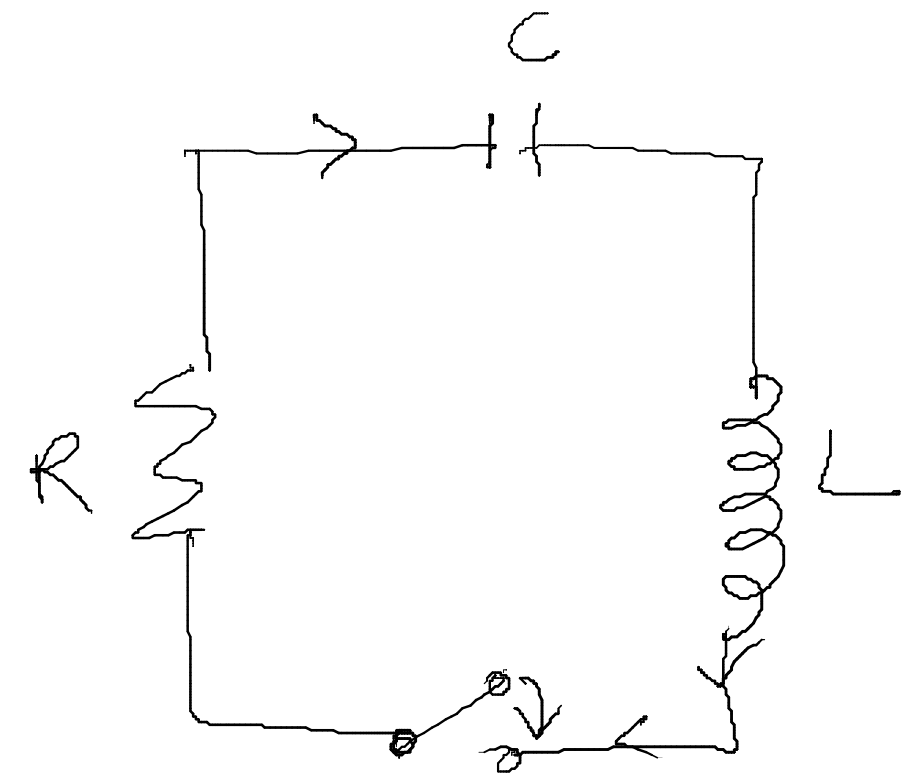
what happens when $c \longrightarrow 0$

$$C(\omega_{max}) = C_{max} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega_{max}^2) + \omega_{max}^2 c^2}} \longrightarrow \infty$$

$\omega_{max} = \omega_0 \longrightarrow C$ attains max $\longrightarrow$ Resonance

Prac. Prob (2-8) : 8, 14, 15, 19, 24

2.9 : Modeling Electric Circuits.



① Resistor: RI

② Inductor: $L\dfrac{dI}{dt}$

③ Capacitor: $\dfrac{Q}{C} = \dfrac{1}{C}\int I\,dt$

Net Voltage drop:

$$RI + L\frac{dI}{dt} + \frac{1}{C}\int I\,dt = E_0 \sin(\omega t)$$

$E(t) = E_0 \sin(\omega t)$

Diff once.

$$\Rightarrow L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = E_0\,\omega \cos(\omega t)$$

$$I_p = I_0 \sin(\omega t - \theta$$

$$I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$\lambda_{1,2} = \frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}}$$

$$S = \omega L - \frac{1}{\omega C}$$

$$\theta = \tan^{-1}\left(\frac{S}{R}\right)$$

Eg1. find $I(t)$ in an RLC-circuit, $R = 11\,\Omega$, $L = 0.1\,H$, $C = 10^{-2}\,F$

Assume current, charge $= 0$ when $t = 0$.

Sol^n:

$$0.1\,I'' + 11\,I' + \frac{I}{10^{-2}} = E$$

$$E = 100 \sin(400t)$$

$$(100)(400)\cos(400t)$$

$$I(0) = 0$$

$$I'(0) = 0$$

Comparison btwn Ele/Mech. case

$$LI'' + RI' + \frac{1}{C}I = F_0(\cos\omega t)\omega$$

$$my'' + cy' + ky = F_0 \cos\omega t$$

Inductor : L

Resistor : R

Capacitance : C

mass : m

damping : c

Spring const. $= k$

2.9 (PP): 1, 5, 9, 12, 15

2.10 Variation of parameters.

(I) $y'' + p(x)y' + q(x)y = r(x)$

$$y_p = -y_1 \int \frac{y_2 r}{W}\,dx + y_2 \int \frac{y_1 r}{W}\,dx$$

$$y_p = y_1\, u(x)$$

$$y_h = c_1 y_1(x) + c_2 y_2(x)$$

Eg1. $y'' + y = \sec x$: find y_p

Sol^n:

$y_1 = \sin x$

$y_2 = \cos x$

$W = y_1 y_2' - y_2 y_1' = -1$

$$y_p = -\sin x \int \frac{(\cos x)\sec x}{W}\,dx + \cos x \int \frac{(\sin x \sec x)}{W}\,dx$$

$$= +(\sin x)\,x^{-1} + \cos x \ln|\sec x|$$

2.10 (P.P.) : 5, 12, 16, 18a