5.1

# Power series Method.
(P.S.)

P.S of a fn. around '$x_0$':

$$\sum_{m=0}^{\infty} a_m (x-x_0)^m = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

$x_0 = 0$: P.S: $\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \cdots$

## P.S of known functions

① $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ $\forall x$ $a_m = \frac{1}{m!}$

② $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ $a_m = \frac{(-1)^m}{(2m)!}$

③ $\frac{1}{1-x} = [1-x]^{-1} = 1 + x + x^2 + \cdots$ $|x| < 1$

ODEs: $y'' + p(x)y' + q(x)y = 0$ } $p, q$: cont.

Assume: $y = \sum_{m=0}^{\infty} a_m (x - x_0)^m$

$$y' = \sum_{m=1}^{\infty} m a_m (x - x_0)^{m-1}$$

$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m (x - x_0)^{m-2}$$

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Eg 1: $y' = 2xy$ Use P.S. sol$^n$

$$\boxed{y = Ae^{x^2}}$$

Sol$^n$: Assume $y = \sum_{m=0}^{\infty} a_m x^m$

$y' = \sum m a_m x^{m-1}$

$$\Rightarrow \underbrace{\sum_{m=1}^{\infty} m a_m x^{m-1}}_{y'} = \underbrace{2x \sum_{m=0}^{\infty} a_m x^m}_{y}$$

$x^0$: $a_1 = 0$

$x^1$: $2a_2 = 2a_0$

$3a_3 = 2a_1$

$4a_4 = 2a_2$

$5a_5 = 2a_3$

$a_i = 0 \qquad i = 2k+1 \quad \underline{\underline{/\text{odd}}}$

$$\begin{cases} a_2 = a_0 \\ a_4 = \dfrac{a_2}{2} = \dfrac{a_0}{2!} \\ a_6 = \dfrac{a_0}{3!} \end{cases} \qquad a_{2m} = \frac{a_0}{m!}$$

<u>Sol$^n$</u> $y = \sum a_m x^m = a_0 + a_0 x^2 + \dfrac{a_0}{2!} x^4 + \dfrac{a_0}{3!} x^6$

$$= a_0 \underbrace{\left[1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots\right]}_{e^{x^2}}$$

$$= a_0 e^{x^2}$$

Eg 19 :

$$y' = 2xy$$

$$y = \sum a_m x^m$$

$$\sum_{m=1}^{\infty} m a_m x^{m-1} = 2x \sum a_m x^m$$

$$= 2 \sum_{m=0}^{\infty} a_m x^{m+1}$$

$m = s+2$

$$\Rightarrow \sum_{s+2=1}^{\infty} (s+2) a_{s+2} x^{s+1} = 2 \sum_{m=0}^{\infty} a_m x^{m+1} \longrightarrow \text{I}$$

$s = -1$

$\boxed{\text{At } s = -1}$ $\quad 1 \, a_1 x^0 = 0$

$$\text{I} : \quad a_1 + \sum_{s=0}^{\infty} (s+2) a_{s+2} x^{s+1} = \sum_{s=0}^{\infty} 2 a_s x^{s+1}$$

$a_1 = 0$ $\quad s = 0, 1, 2, \ldots$ $\quad \boxed{(s+2) a_{s+2} = 2 a_s}$

$$\boxed{(s+2)\,a_{s+2} = 2\,a_s}$$

$$a_1 = a_3 = a_5 = \cdots 0$$

$\underline{s=0}$ $\quad 2\,a_2 = 2\,a_0$

$s=2$ $\quad 4\,a_4 = 2\,a_2$

$s=4$ $\quad 6\,a_6 = 2\,a_4 \longrightarrow a_6 = \dfrac{a_4}{3} = \dfrac{a_2}{3\cdot 2} = \dfrac{a_0}{3!}$

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Eg 2. $\underline{\text{Solve}}$ : $y'' + y = 0$

Sol$^n$ :

$$\sum_{m=2}^{\infty} m(m-1)\,a_m\,x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$m = s+2$$

$$\Rightarrow \sum_{s+2=2}^{\infty} (s+2)(s+1)\,a_{s+2}\,x^s + \sum_{s=0}^{\infty} a_s x^s = 0$$

$$\sum_{s=0}^{\infty} \sum_{s=0}^{\infty} \underbrace{\left[(s+2)(s+1)\,a_{s+2} + a_s\right]}_{=0} x^s = 0$$

$$(s+2)(s+1)\,a_{s+2} = -a_s$$

$s=0$ $\quad a_2 = -\dfrac{a_0}{2\cdot 1}$ $\qquad s=1$ $\quad a_3 = -\dfrac{a_1}{3\cdot 2}$

$s=2$ $\quad a_4 = -\dfrac{a_2}{4\cdot 3} = \dfrac{+a_0}{4\cdot 3\cdot 2\cdot 1}$ $\qquad s=3$ : $a_5 = \dfrac{+a_1}{5\cdot 4\cdot 3\cdot 2\cdot 1}$

$$y = \sum a_m x^m = \left[a_0 + a_2 x^2 + a_4 x^4 + \cdots\right] + \left[a_1 x + a_3 x^3 + a_5 x^5 + \cdots\right]$$

$$= \left[a_0 - \frac{a_0}{2!}x^2 + \frac{a_0}{4!}x^4 - \cdots\right] + \left[a_1 x - \frac{a_1 x^3}{3!} + \frac{a_1 x^5}{5!} - \cdots\right]$$

$$= a_0 \cos x + a_1 \sin x$$

Prac Prob. (5.1):

5, 10, 12, 16.

§2

# Theory of P.S. method

**P.S.** $S(x) = \sum_{m=0}^{\infty} a_m (x-x_0)^m = \left[a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n\right]$

$+ a_{n+1}(x-x_0)^{n+1} + \cdots$

$= S_n(x) + R_n(x) \longrightarrow (1)$

for convergence : $\forall \varepsilon > 0 \ \exists N(\varepsilon)$, such that

$|R_n(x)| = |S(x) - S_n(x)| < \varepsilon$

$\forall n > N$

Q where to apply (1) ?

Radius of Convergence. (R)

If $x = x_0$ ! Does $S(x)$ converges?

yes / 0

$y = \sum a_m (\bar{x} - \bar{x}_0)^m$

$\forall \bar{x}$ s.t $|\bar{x} - \bar{x}_0| < R$

P.S converges.

Eg 1: $y = e^x = \sum 1 + x + \frac{x^2}{2!} + \ldots = \sum \frac{x^n}{n!}$

$R = \infty$

In general: $y = \sum_{n=0}^{\infty} a_n x^n$

$$\boxed{R = \lim_{n\to\infty} \frac{1}{\left|\frac{a_{n+1}}{a_n}\right|} = \left|\frac{a_n}{a_{n+1}}\right|}; \quad \text{RoC}$$

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Eg 1: $R = \lim_{n\to\infty} \frac{1}{\left|\frac{1}{(n+1)!} n!\right|} = \frac{1}{\frac{1}{(n+1)}} = n+1 \xrightarrow{n\to\infty} \infty$

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Eg 2: $y = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + x^3 + \ldots$

$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{1}{1}\right| = 1$

Conv. $\forall x$ s.t. $|x| < 1$

Eg 3 : Find RoC $\sum \frac{(-1)^m}{8^m} x^{3m} = \sum \frac{(-1)^m}{8^m} z^m$ $z = x^3$

Sol$^n$: $R = \lim_{m\to\infty} \left| \frac{a_m}{a_{m+1}} \right| = \lim_{m\to\infty} \left| \frac{(-1)^m \, 8^{m+1}}{8^m \, (-1)^{m+1}} \right| = 8$

& Conv. $\forall x$ s.t $|x^3| < 8$

$\Rightarrow |x|^3 < 8$

$\Rightarrow |x| < 2$ — RoC

RoC $= R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{1}{\sqrt[n]{a_n}}$

Existence of P.S sol$^{n}$s: $y'' + p(x)y' + q(x)y = r(x)$

or $h(x)y'' + \bar{p}(x)y' + \bar{q}(x)y = \bar{r}(x)$

(I) has p.s. sol$^{n}$ provided (at $x = x_0$) $\rightarrow$ (I)

$p, q, r, h$ have P.S expansion at $(x = x_0)$

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5.2 (P.P.) : 4, 8, 12, 16, 22