Lecture 9

Bessel's fn. of 2nd kind

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

Assume $\nu = 0$

$$x y'' + y' + x y = 0 \longrightarrow (I)$$

$$y_1 = J_0(x)$$

$$y_2 = \underbrace{J_0(x)}_{y_1} \ln(x) + \sum_{m=1}^{\infty} A_m x^m$$

$$y_2' = J_0' \ln x + \frac{J_0}{x} + \sum_{m=1}^{\infty} m A_m x^{m-1}$$

$$y_2'' = J_0'' \ln x + \frac{2J_0'}{x} - \frac{J_0}{x^2} + \sum m(m-1) A_m x^{m-2}$$

$$\Rightarrow \underbrace{\cancel{\left(x J_0'' + J_0' + x J_0\right)}}_{=0} \ln x + 2J_0' + \sum m(m-1) A_m x^{m-1} + \sum m A_m x^{m-1} + \sum A_m x^{m+1} = 0$$

$$\left\} \sum m^2 A_m x^{m-1} \right.$$

<u>NOTE</u>: $J_0' = \sum_{m=1}^{\infty} \frac{(-1)^m \, 2m \, x^{2m-1}}{2^{2m} (m!)^2}$

$$\underbrace{2 \sum \frac{(-1)^m \, 2m \, x^{2m-1}}{2^{2m} (m!)^2}}_{S_1} + \underbrace{\sum m^2 A_m x^{m-1}}_{S_2} + \underbrace{\sum A_m x^{m+1}}_{S_3} = 0$$

$$\sum \frac{(-1)^m x^{2m-1}}{2^{2m-2} (m!)(m-1)!}$$

x^0: $A_1 \equiv 0$

Consider even powers:

In S_2: $\quad m-1 = 2s \Rightarrow m = 2s+1$

In S_3: $\quad m+1 = 2s \Rightarrow m = 2s-1$

$$\Rightarrow (2s+1)^2 A_{2s+1} + A_{2s-1} = 0$$

$$\Rightarrow A_{2s+1} = -\frac{A_{2s-1}}{(2s+1)^2} \qquad s = 1, 2, \ldots$$

$$0 = A_1 = A_3 = A_5 = \ldots$$

Consider Coeff. of $x^1 = -1 + 4A_2 = 0$

Similarly find A_4 A_6 ... $\quad A_2 = \frac{1}{4}$

In S_1 $\quad 2m-1 = 2s+1$

$$\Rightarrow \underline{\underline{m = s+1}} \qquad s = 0, 1, 2, \ldots$$

In S_2 : $\quad m-1 = 2s+1$

$$m = 2s+2 \qquad s = 0, 1, 2, \ldots$$

In S_3 : $\quad m+1 = 2s+1$

$$m = 2s \qquad s = 1, \ldots$$

$$\frac{(-1)^{s+1}}{2^{2s}(s+1)!\,s!} + (2s+2)^2 A_{2s+2} + A_{2s} = 0$$

$$A_{2s} = \frac{(-1)^{s-1}}{2^{2s}(s!)^2} \underbrace{\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{s}\right]}_{h_s} \qquad s = 2, 3, \ldots$$

$$A_2 = \frac{1}{4}$$

$$y_2 = J_0 \ln(x) + \sum \frac{(-1)^m h_m}{2^{2m} (m!)^2} x^{2m}$$

$$= J_0 \ln(x) + \frac{x^2}{4} - \frac{3}{128} x^4 + \frac{11}{13824} x^6$$

$= Y_0(x)$: Neumann fn.

: Bessel fn of 2nd kind of order 0.

Instead of y_2 : $\tilde{y}_2 = a\, y_2 + b\, J_0$

$= \tilde{Y}_0$

$a = \frac{2}{\pi}$, $b = \gamma - \ln 2$

γ = Euler const.

$= 0.577215$

Using Frobenius Method, Neumann fn. of any order $n \in \mathbb{Z}$

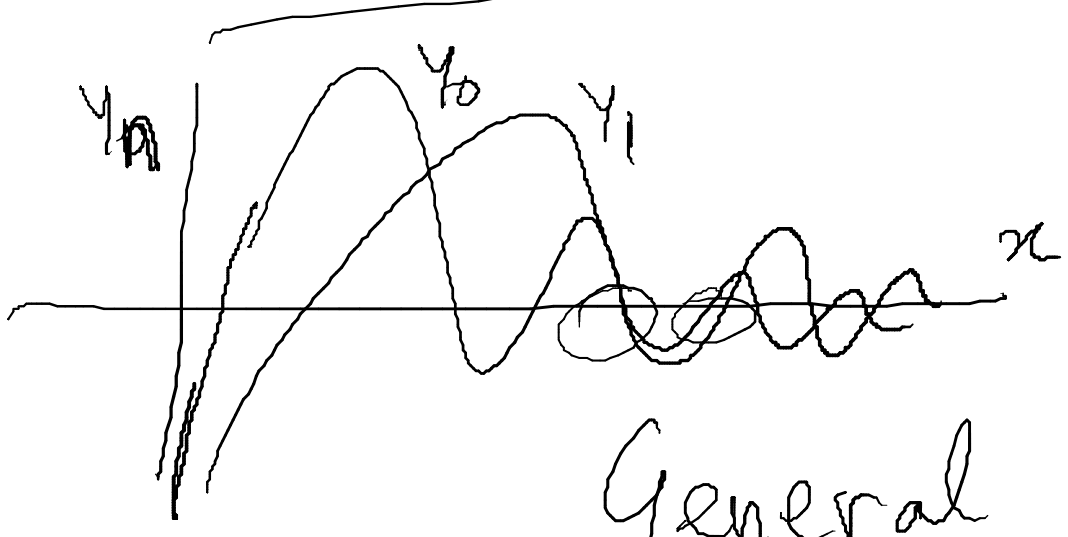
$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x)$$

$$Y_\nu(x) = \frac{1}{\sin(\nu\pi)}\left[J_\nu \cos\nu\pi - J_{-\nu}\right]$$

Chk: J_ν & Y_ν : L.I.

In series form: $Y_\nu(x) = \frac{2}{\pi} J_n(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right]$ nearest integer.

$$+ \frac{x^n}{\pi} \sum \frac{(-1)^{m-1}}{2^{2m+n}} \frac{(h_m + h_{m+n})}{(m!)(m+n)!} x^{2m}$$



General sol^n of Bessels Eq^n:

$$y = c_1 J_\nu(x) + c_2 Y_\nu(x)$$

Sometimes we need ~~complete~~ complex solution of Bessel's Eq^n.

$$H_\nu^{(1)}(x) = J_\nu + i\,Y_\nu$$

$$H_\nu^{(2)}(x) = J_\nu - i\,Y_\nu$$

Bessel fn. of 3rd kind: Hankel fn.

Prac. Prob: 3, 6, 9, 13. (5.6)

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

5.7. (SL) Strum-Liouville Problems:

Orthogonal fns.

$$\rightarrow (p(x) y')' + [q(x) + \lambda r(x)] y = 0$$

E.V.

B.C.

$$k_1 y(a) + k_2 y'(a) = 0 \quad a \le x \le b$$

$$l_1 y(b) + l_2 y'(b) = 0$$

Eg1. Legendre Eq^n:

$p = (1 - x^2)$

$q = 0$

$r = 1$

$$(1 - x^2) y'' - 2x y' + n(n+1) y = 0$$

Eg 2.

Bessel's Eqn:

$$x^2 y'' + x y' + (k^2 x^2 - n^2) y = 0$$

$$\Rightarrow x y'' + y' + \left[\frac{-n^2}{x} + k^2 x\right] y = 0 \qquad \tilde{x} = kx$$

$$\Rightarrow \left.\begin{aligned} p &= x \\ q &= -\frac{n^2}{x} \\ r &= x \end{aligned}\right\} \leftarrow$$

$\underline{y=0}$: <u>Trivial sol^n</u>

$\underline{y \neq 0}$: Eigen-functions (λ = Eigen-value)

$\underline{\text{Eg } 3}$

$$\underline{y'' + \lambda y} = 0$$

$$p = 1, \quad r = 0, \quad q = 0$$

$$y = A\cos(kx) + B\sin(kx) \qquad k = \sqrt{\lambda}$$

$\underline{B.C}$: $A, B = ?$

Existence of Eigen-values/fns: Guaranteed if p, q, r, p' : cont.

Also: Real-valued fns : if p, q, r, p' are real valued

Def^n: Orthogonal fns $\quad r \geq 0$

$y_1, y_2, y_3 \ldots y_n$ $(a \leq x \leq b)$ are orthogonal on this interval w.r.t weight fn. $r(x) > 0$ if for $\forall m, n$

$$(*) \quad \int_a^b r(x)\, y_m(x)\, y_n(x)\, dx = 0$$

Orthonormal fns: Orthogonal & $\| \ \| = 1$ $\quad n \neq m$.

$$\Rightarrow \sqrt{\int r\, y_n^2\, dx} = 1$$