Basic Concepts : 1st order ODEs

$$m\frac{dv}{dt} = mg - bv^2 \;:$$

1st order / 2nd

forces

Resis.

$$\frac{\partial}{\partial t} v(x,t)$$

Eg1. $y' = \frac{dy}{dx} = \cos x$

Solution to an ODE

Suppose $F(x, y, y') = 0$ → (1) ODE

A fn. $h(x)$ is a solution to (1) on an open interval $a < x < b$ if $h(x)$ is differentiable in the domain & $F(x, h, h') = 0$ is an identity.

Eg1: $\boxed{y_0 = \frac{c}{x}}$ (c = const.) $x \neq 0$

ODE: $\boxed{y' = -\frac{c}{x^2}}$ → (2)

$y_0' = -\frac{c}{x^2}$

For 'y_0' (2) is true $\forall x$

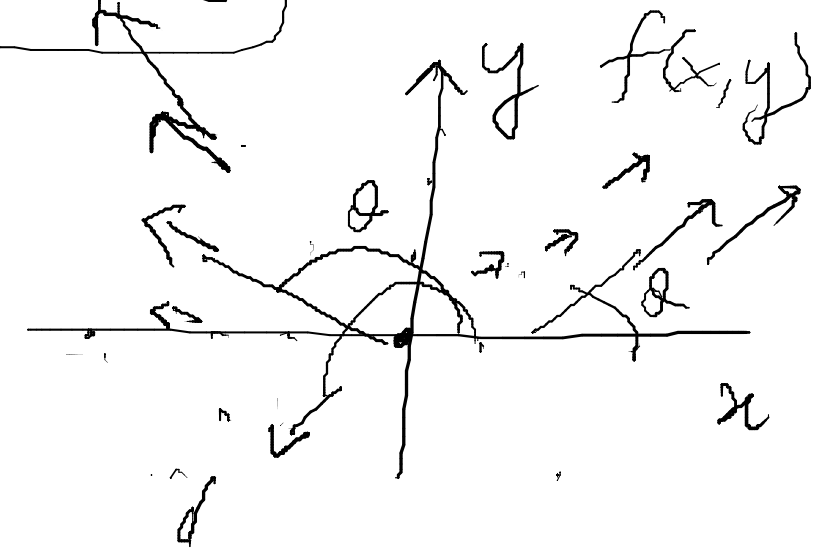
1.1 Initial Value Problems (IVP)

IVP $\boxed{y' = f(x,y) \qquad y(x_0) = y_0}$

Existence / Uniqueness ! Picard's Thm.

1.2 Geometric meaning of $\boxed{y' = f(x,y)}$

Eg. 1 $y' = xy$: Dir^n field



Prac 1.1: 4, 9, 15, 17

Prac 1.2: 10, 15, 16.

1.3. Seperable ODE.

$$\frac{dy}{dx} = y' = \frac{f(x)}{g(y)} \Rightarrow \int \underline{g(y)}\, dy = \int \underline{f(x)\, dx} + c$$

Eg1. $$\frac{dy}{dx} = 1+y^2 \Rightarrow \int \frac{dy}{1+y^2} = \int dx$$

$$\boxed{\tan^{-1} y = x + c}$$

Eg 2. A tank contains 1000 gal of water in which 100 lb of salt is dissolved. Brine runs in at a rate of 10 gal/min and each gallon contains 5 lbs of dissolved salts. Brine runs out at 10 gal/min. Mixture is kept uniform. Find the amount of salt at any time t.

Sol^n: Assume 'amount' of salt is $y(t)$

$$\Rightarrow \frac{dy}{dt} = 10(5) - y\frac{10}{1000}$$

$$\Rightarrow y = 5000 + c e^{-0.01t} \qquad (t=0,\ y=100)$$

Eg2. $t=0$

Suppose the daytime temp. inside an office is maintained at 70°F. Heating is shut off at 10 PM & restarted at 6AM next day. The temp. inside the building at 2AM was found 65°F. [The temp. outside was 50°F at 10 PM & 40°F at 6AM]. What was the temp. inside at 6 AM?

Soln:

$$\frac{dT}{dt}\left[\alpha\left(T-T_0\right)\right] = k\left(T-T_0\right)$$

$$\Rightarrow \frac{dT}{k\left(T-T_0\right)} = dt \quad \Rightarrow \boxed{\frac{1}{k}\ln\left|T-T_0\right| = t + c}$$

On avg. (10PM – 6AM) : $T_0 = \frac{50+40}{2} = 45$

$$\frac{1}{k}\ln|T-T_0| = t+c$$

$$\Rightarrow \ln|T-T_0| = k(t+c)$$

$$\Rightarrow \boxed{T = T_0 + Ce^{kt}}$$

25

* At $t=0$, $T=70$

$$70 = 45 + C$$

$$C = 25$$

* At $t=4$, $T=65$

$$k = \frac{1}{4}\ln 0.8 = -0.056$$

* What is T at $t=8$?

ODE of the type: $y' = f\left(\frac{y}{x}\right)$

Sol^n: Choose $u = \frac{y}{x}$

Eg 4. $\frac{2xy\,y' = y^2 - x^2}{xy}$

PP: 9, 18, 24, 30, 33 (1.3).

$\Rightarrow 2y' = \frac{y}{x} - \frac{x}{y}$ Choose $u = \frac{y}{x}$

$\Rightarrow y = ux$

$\Rightarrow 2y' = u - \frac{1}{u}$

$\frac{dy}{dx} = \left(u + x\frac{du}{dx}\right)$

$\Rightarrow u'x = -\frac{u}{2} - \frac{1}{2u}$] Variable sep.

$\Rightarrow y^2 + x^2 = cx$! Circles.

1.4

Integrating factors

Consider: 1st order ODE: $M(x,y)\,dx + N(x,y)\,dy = 0$ ①

Suppose

$$\boxed{\begin{aligned} M &= \frac{\partial u}{\partial x} \\ N &= \frac{\partial u}{\partial y} \end{aligned}} \quad ②$$

Necessary cond. for ②

$$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}} \qquad \begin{matrix} u_{xy} \\ \| \\ u_{yx} \end{matrix}$$

① :

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0$$

$$\Rightarrow \quad du = 0$$

Int. $\Rightarrow$ $\boxed{u = c}$

$$\boxed{M\,dx + N\,dy = 0}$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$: Necessary & suff. cond. for existence of sol^n.

Eg1: Solve: $\underbrace{\cos(x+y)}_{M}dx + \underbrace{\left[3y^2 + 2y + \cos(x+y)\right]}_{N}dy = 0.$

$M = \frac{\partial u}{\partial x}$

Sol^n :

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= -\sin(x+y) \\ \frac{\partial N}{\partial x} &= -\sin(x+y) \end{aligned}\right\} \Rightarrow \begin{aligned} &\frac{\partial u}{\partial x} = M \longrightarrow u = \int M\,dx + k(y) \\ &\frac{\partial u}{\partial y} = N \end{aligned}$$

$$u(x,y) = \int M\,dx + k(y)$$

$$= \int \cos(x+y)\,dy + k(y)$$

$$= \boxed{\sin(x+y) + k(y)}$$

$$\frac{\partial u}{\partial y} = \cos(x+y) + \frac{dk}{dy} = N = \cos(x+y) + 3y^2 + 2y$$

$$\boxed{u(x,y) = \cos(x+y) + y^3 + y^2 + c.}$$

Eg 2. $-y\,dx + x\,dy = 0 \longrightarrow \text{I}$

Sol^n: $M = -y$, $N = x$ $\Big\}$ $\underline{M_y \neq N_x}$

$\Rightarrow$ Multiply I by $\frac{1}{x^2}$: $-\frac{y}{x^2}\,dx + \frac{x}{x^2}\,dy = 0$

Chk: $\boxed{M_y = N_x}$

$d\left(\underbrace{\frac{y}{x}}_{u}\right) = 0$

$\boxed{\frac{y}{x} = c}$

Consider

$$\underbrace{FP}_{M}dx + \underbrace{FQ}_{N}dy = 0$$

Goal: Find F $\qquad$ $\Vert$ exact differential.

$\Rightarrow$ For exactness $\Rightarrow$ $\dfrac{\partial}{\partial y}(FP) = \dfrac{\partial}{\partial x}(FQ)$

$$(\mathrm{II}) \Rightarrow \boxed{F_y P + P_y F = F_x Q + Q_x F}$$

Case 1: Suppose $F = F(x)$

$$\Rightarrow P_y F = F_x Q + Q_x F$$

$$\Rightarrow \frac{P_y - Q_x}{Q} = \frac{F_x}{F}$$

$$\Rightarrow \begin{cases} F = e^{\int R\,dx} \\ R = \dfrac{P_y - Q_x}{Q} \end{cases}$$

Case 2: Suppose $F = F(y)$ only.

$$F = e^{\int R\,dy}$$

$$R = \frac{Q_x - P_y}{P}$$

Eg 3: Solve $\underbrace{(e^{x+y} + ye^y)}_{P}\,dx + \underbrace{(xe^y - 1)}_{Q}\,dy = 0$ & $y(0) = -1$

Sol^n: If

$$\begin{cases} P_y = e^{x+y} + e^y + ye^y \\ Q_x = e^y \end{cases}$$

Method does not work directly

$$\Rightarrow P_y - Q_x = e^y[e^x + y] \longrightarrow (A)$$

$$\frac{A}{Q} = \text{Case 1 NOT work.}$$

$$\frac{A}{P} = -1 = R$$

Case: $F = e^{\int R\,dy} = e^{-y}$

$$M = FP = e^x + y = \frac{\partial u}{\partial x}$$

$$N = FQ = \boxed{x - e^{-y}} = \frac{\partial u}{\partial y}$$

$$\Rightarrow \quad u = \int M\,dx + k(y) = \boxed{e^x + yx + e^{-y} + c}$$

$$= \int (e^x + y)\,dx + k(y) = e^x + yx + k(y)$$

$$\frac{\partial u}{\partial y} = x + \frac{dk}{dy} = N$$

$$\frac{dk}{dy} = -e^{-y} \quad \Rightarrow \quad k(y) = e^{-y} + c$$

<u>PP</u> (1.4) : 9, 18, 20, 21