1.5 Linear ODE $\quad y' + p(x)y = r(x)$ } — (1)

linear — (1) $y, D^i y$ } degree 1 $\quad y' + y^2 + 1$

(2) $y, D^i y$ ~~product~~ of $\quad yy' + x = 0$

Soln. $\quad y(x) = \dfrac{1}{e^{\int p\,dx}}\left[\int e^{\int p\,dx} r(x) + c\right]$ }

$F(x)$

$$Fy' + pFy = rF \longrightarrow$$

$$(Fy)' = F'y + Fy' \quad \text{if} \quad pFy = F'y$$

$$pF = F'$$

$$\text{Sol}^n \quad \int \frac{dF}{F} = \int p\,dx \Rightarrow F = e^{\int p\,dx}$$

$h = \int p\,dx \quad (\text{sy})$ $\qquad F = e^{\int p\,dx}$

$\Rightarrow h' = p.$

LHS: $e^h y' + h' e^h y$ $\qquad (F y' + p F y = \gamma F)$

$e^h y' + (e^h)' y$

$\Rightarrow (e^h y)' = \gamma e^h$

By Int $\quad e^h y = \int \gamma e^h + c$

$$\boxed{y = \frac{1}{e^h}\left[\int \gamma e^h dx + c\right]}$$

eg ①

$$y' - y = e^{2x}$$

Find $y(x)$

$$p(x) = -1$$

$$r(x) = e^{2x}$$

$$y(x) = \frac{1}{IF}\left[\int (r(x) \times IF)\,dx + c\right]$$

$$IF = e^{\int p\,dx} = e^{\int -1\,dx} = e^{-x}$$

$$y(x) = \frac{1}{e^{-x}}\left[\int e^{2x} \cdot e^{-x}\,dx + c\right]$$

$$= \frac{1}{e^{-x}}\left[\int e^{x}\,dx + c\right]$$

Eg 2 Hormone varies with time

time rate of change = difference b/w {sinusoidal input of 24 hr-period from the thyroid gland & the removal rate proportional to present level}

Sol^n Let $y(t)$ be the hormone level at time 't'

Removal rate $\propto y(t)$

$= k\,y(t)$

Input rate $A + B\cos\omega t$, $\omega = \frac{2\pi}{24}$

$$\Rightarrow \boxed{\frac{dy}{dt} = IR - RR = A + B\cos\omega t - k\,y(t)}$$

$p(t) = k$

$r(t) = A + B\cos\omega t$

find $y(t)$

General sol^n

$$y(t) = \underbrace{\frac{A}{k}}_{\text{steady-state sol}^n} + \underbrace{\frac{B}{144k^2+\pi^2}\left[144k\cos\left(\frac{\pi}{12}t\right) + 12\pi\sin\left(\frac{\pi}{12}t\right)\right]}_{\text{steady-state sol}^n} + \underbrace{ce^{-kt}}_{\text{Transient sol}^n}$$

$t \to \infty$

Bernoulli's eq.:-

$$y' + p(x)y = r(x)y^a$$

reduced to linear ode.

IF $Mdx + Ndy = 0$

let $u(x) = [y(x)]^{1-a}$

$$u' = (1-a)\, y(x)^{-a} y'$$

$$= (1-a)\, y^{-a}\left[r y^a - p y \right]$$

$$= (1-a)\left[r - p y^{1-a} \right]$$

$$\Rightarrow \boxed{u' + (1-a)\, p(x)\, u = (1-a)\, r(x)}$$

linear

---

eg ② $y' = Ay - By^2$ (logistic eq.)

$y = 0$ is also a sol.

linear

$a = 2$

choose $u = y^{1-2} = y^{-1}$

$$\boxed{u' + Au = B} \Rightarrow \boxed{u = ce^{-At} + \frac{B}{A}}$$

$u = y^{-1} = \frac{1}{y}$

$$y = \frac{1}{\left(ce^{-At} + \frac{B}{A}\right)}$$

$y \to \frac{A}{B}$

$t \to \infty$

1.7 Existence & Uniqueness of Sol^n.

Eg ①

$|y'| + |y| = 0$, $y(0) = 1$

$y = x$ $\quad |1| + |x| = 0$ ✗

$y = 3$ $\quad |0| + |3| = 0$ ✗

$y = 0$ $\quad 0 + 0 = 0$ ✗

No Sol^n

② $y' = 2x$, $y(0) = 1$

$y = x^2 + c$

$1 = 0 + c \Rightarrow c = 1$

Unique Sol^n.

③ $xy' = y - 1$, $y(0) = 1$ ✓

$\boxed{y(x) = cx + 1}$

$1 = c \cdot 0 + 1 \Rightarrow c \in \mathbb{R}$

Infinite Sol^n.

Uniqueness thm - & Existence thm.

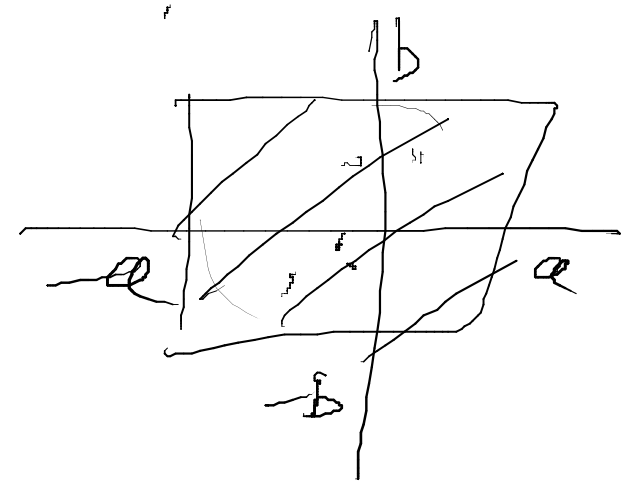
① **Existence thm.** Consider $y' = f(x,y)$, $y(x_0) = y_0$ } IVP

Picards Thm.

Let $f(x,y)$ be cts. inside some neighborhood $R: |x-x_0| < a$, $|y-y_0| < b$ }

& bdd in R, $\exists$ K s.t.

$$|f(x,y)| < K \quad \forall (x,y) \in R$$



Then ① IVP has atleast 1 sol^n.

② sol^n exist $\forall x: |x-x_0| < \alpha$

$$\alpha = \min\left(a, \frac{b}{K}\right) \}$$

② **Uniqueness thm** $y' = f(x,y)$; $y(x_0) = y_0$

Let f & its partial der. (f_y) be cts. inside R

then IVP has a unique sol^n in R $\forall x: |x-x_0| < \alpha$

(3) $xy' = y - 1$, $y(0) = 1$

$x_0 = 0$

$y_0 = 1$

$y' = \frac{y-1}{x}$ $\longrightarrow$ $y = x+1$

f

$f_y = \frac{1}{x}$ ✗

exist ✓

unique ✗

$|x - x_0| = |x| \le a$

$|y - 1| \le b$

---

(4) $y' = 1 + y^2$, $y(0) = 0$, $|x| \le 5$, $|y| \le 3$

$f = 1 + y^2$

$f_y = 2y$ $\Rightarrow$ unique sol. exist $\forall x$ st $|x| < \alpha$

$\alpha = \min\left(a, \frac{b}{K}\right)$

$\alpha = \min\left(5, \frac{3}{10}\right)$

$f = 1 + y^2$

$|f| = |1 + y^2| \le 10$ ✗

✓ $|f_y| = |2y| \le 6$

f_y → ok.

MVT: $f(x,y_2) - f(x,y_1) = (y_2 - y_1) f_y \Big|_{y=y_0}$

Uniform cont. → $|f| < R$, $|f_y| < M$

by $|f| \le R$, $|f(x,y_2) - f(x,y_1)| \le M(y_2 - y_1)$ } Lipschitz cond^n (weaker cont.)

---

eg. $y' = \sqrt{|y|}$, $y(0) = 0$

$y(x) = ?$

$y(x) = \begin{cases} x^2/4, & x \ge 0 \\ -x^2/4, & x < 0 \end{cases}$, $y = 0$

Check f is cont at $(0,0)$ ✓

$$\frac{|f(x,y_2) - f(x,y_1)|}{|y_2 - y_1|} = \frac{|\sqrt{y_2} - \sqrt{y_1}|}{|y_2 - y_1|}$$

choose $y_1 = 0$ ✓, $y_2 > 0$ ✓

$$= \frac{\sqrt{y_2}}{y_2} = \frac{1}{\sqrt{y_2}} > M \text{ for any } M.$$