Midsem

Course: MTH204 **Duration: 1 hour** **M. Marks: 30**

Ques 1(a) If $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at $t = a(> 0)$, satisfies the growth condition and $f(t)$ is piecewise continuous on every finite interval on $t \geq 0$. Prove that:

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a+0) - f(a-0)]e^{-as}.$$

[4 marks]

(b) Find Laplace transform of e^{t^2}, if exists. Give quantitative reasons for its existence or non existence. **[3 marks]**

(c) If $\mathcal{L}(f(t)) = F(s)$ and c is any positive constant, show that

$$\mathcal{L}(f(ct)) = \frac{1}{c}F(\frac{s}{c}).$$

[3 marks]

Ques 2(a) Hermite Polynomial (a special case of Strum-Liouville eigenfunctions) are given by $He_n(x) = (-1)^n e^{(\frac{x^2}{2})} \frac{d^n}{dx^n} e^{\frac{-x^2}{2}}$, $n = 1, 2, \ldots$ and $He_0(x) = 1$. A generating function of these polynomials is $e^{tx - \frac{t^2}{2}} = \sum_{n=0}^{\infty} a_n(x)t^n$. Prove that $a_n(x) = \frac{1}{n!}He_n(x)$. **[6 marks]**

(b) Show that $He_n'(x) = nHe_{n-1}(x)$. **[3 marks]**

(c) Show that the Hermite polynomials are orthogonal on $-\infty < x < \infty$ with respect to the weight function $r(x) = e^{-\frac{x^2}{2}}$. **[3 marks]**

Ques 3 Find an electric analog of mass spring system with mass 0.5kg, spring constant 40kg/sec^2, damping constant 9kg/sec and driving force 102 cos$(6t)$N. Solve the analog using 0 initial current & charge. **[4 marks]**

Ques 4 Derive a second linearly independent solution of $x^2y'' + axy' + by = 0$ by variation of parameter in the case of repeated roots of characteristic equation. Assume one solution is x^m. **[4 marks]**

Ans 1(a).

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^a e^{-st} f'(t)\,dt + \int_a^\infty e^{-st} f'(t)\,dt.$$

$\because$ $f(t)$ has discontinuity at $t=a$.

$$= \lim_{h\to 0}\int_0^{a-h} e^{-st} f'(t)\,dt + \lim_{h\to 0}\int_{a+h}^\infty e^{-st} f'(t)\,dt \quad \Big] \text{ 1 mark}$$

Integrating by parts:

1 mark

$$= \lim_{h\to 0}\left[e^{-st} f(t)\right]_0^{a-h} + s\lim_{h\to 0}\int_0^{a-h} e^{-st} f(t)\,dt + \lim_{h\to 0}\left[e^{-st} f(t)\right]_{a+h}^\infty$$

$$+ s\lim_{h\to 0}\int_{a+h}^\infty e^{-st} f(t)\,dt$$

0.5 mark

($\because$ f satisfies growth restriction).

$$= e^{-as} f(a-0) - f(0) + 0 - e^{-as} f(a+0) + s\int_0^\infty e^{-st} f(t)\,dt$$

1 mark

(0.5, 0.5)

$$= s\mathcal{L}(f) - f(0) - [f(a+0) - f(a-0)]e^{-as}$$

0.5

Ans 1(b) Laplace transform of e^{t^2} does not exist. — 1 mark

Suppose $\exists$ the constants M & k such that

$$|e^{t^2}| < Me^{kt} \quad , \forall\, t > t_0 \text{ for some } t_0$$ — 1 mark

$M > 0,\ k > 0$

M 1 mark

$$e^{t^2} < Me^{kt}$$

$$t^2 < \log_e M + (kt) \Rightarrow t^2 - kt < \log_e M.$$

$\Leftarrow$ contradiction.

$\Rightarrow$ which puts restriction on t.

$\because$ M & k are fixed but t can be any no. $\in [0, \infty)$

Thus e^{t^2} does not satisfy growth restriction for existence of laplace transform. — 1 mark

Ans 1(c) $$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

$$L(f(ct)) = \int_0^\infty e^{-st} f(ct)\, dt.$$ — 1

0.5 Algebra

let $ct = x \Rightarrow dt = \frac{1}{c}dx$. $\rightarrow$ 1

$$L(f(ct)) = \frac{1}{c}\int_0^\infty e^{(-\frac{s}{c})x} f(x)\, dx = \frac{1}{c}F\left(\frac{s}{c}\right)$$ — 0.5

1(b) Alternative soln.

If exists, $$L(e^{t^2}) = \int_0^\infty e^{-st} e^{t^2} dt = \int_0^\infty e^{t^2 - st} dt.$$

$$= \int_0^\infty e^{t^2 - st + \frac{s^2}{4} - \frac{s^2}{4}} dt = \int_0^\infty e^{(t-\frac{s}{2})^2 - \frac{s^2}{4}} dt = e^{-\frac{s^2}{4}} \int_0^\infty e^{(t-\frac{s}{2})^2} dt.$$

let $t - \frac{s}{2} = x \quad dt = dx$

$$= e^{-\frac{s^2}{4}} \int_{-s/2}^{\infty} e^{x^2} dx. \quad \text{diverges.}$$

Sol^n 2 (a).

$$f(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow a_n = \frac{f^n(0)}{n!} = \frac{d^n f(t)}{dt^n}\Big|_{t=0}.$$ 1 mark

$$f(t) = e^{tx - \frac{t^2}{2}}.$$

$$\frac{d^n f(t)}{dt^n}\Big|_{t=0} = \frac{d^n}{dt^n} e^{\left(tx - \frac{t^2}{2}\right)}\Big|_{t=0} = \frac{d^n}{dt^n} e^{\left[tx - \frac{t^2}{2} + \frac{x^2}{2} - \frac{x^2}{2}\right]}\Big|_{t=0}.$$

$$= \frac{d^n}{dt^n} e^{\frac{x^2}{2}} \cdot e^{-(x^2+t^2-2xt)/2}\Big|_{t=0}$$

$$= \frac{d^n}{dt^n} e^{\frac{x^2}{2}} e^{-(x-t)^2/2}\Big|_{t=0}$$

$$= e^{\frac{x^2}{2}} \frac{d^n}{dt^n} e^{-(x-t)^2/2}\Big|_{t=0}.$$

3 marks

Put $x - t = x_1$

$$\frac{d^n}{dt^n} e^{-\frac{(x-t)^2}{2}}\Big|_{t=0} = (-1)^n \frac{d^n}{dx_1^n} e^{-(x_1)^2/2}\Big|_{x=x_1} = (-1)^n \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}$$

thus

$$f^n(0) = e^{x^2/2} (-1)^n \frac{d^n}{dx^n} e^{-x^2/2} = He_n(x).$$

$$a_n = \frac{f^n(0)}{n!} = \frac{He_n(x)}{n!}$$

2 marks

2(b): from Part (a). $He_n(x) = n!\, a_n(x)$.

$$e^{tx - t^2/2} = \sum_{n=0}^{\infty} a_n(x) t^n.$$

$$\frac{d}{dx} e^{\left(tx - \frac{t^2}{2}\right)} = \frac{d}{dx} \sum_{n=0}^{\infty} a_n(x) t^n$$

1 mark

$$t\, e^{tx - \frac{t^2}{2}} = \sum_{n=0}^{\infty} a_n'(x) t^n$$

$$t \sum_{n=0}^{\infty} a_n(x) t^n = \sum_{n=0}^{\infty} a_n'(x) t^n$$

$$\sum_{n=0}^{\infty} a_n(x) t^{n+1} = \sum_{n=0}^{\infty} a_n'(x) t^n$$

$$\sum_{n=1}^{\infty} a_{n-1}(x) t^n = \sum_{n=0}^{\infty} a_n'(x) t^n$$

1.5 Mark

$$\Rightarrow a_n'(x) = a_{n-1}(x), \quad n = 1, 2, \ldots$$

0.5 mark

$$\left(\frac{He_n(x)}{n!}\right)' = \frac{He_{n-1}(x)}{(n-1)!}$$

$$He_n'(x) = n\, He_{n-1}(x).$$

Q(c). $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} H_n(x) H_m(x)\, dx$.

$\left(H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \right)$

If $n > m$,

$= \int_{-\infty}^{\infty} (-1)^n \frac{d^n}{dx^n} e^{-x^2/2} H_m(x)\, dx$. —— ✪.

0.5 Mark

Integrating By Parts:

$= (-1)^n \underbrace{\left[H_m(x) \frac{d^{n-1}}{dx^{n-1}} e^{-\frac{x^2}{2}} \right]_{-\infty}^{\infty}}_{= 0} - (-1)^n \int_{-\infty}^{\infty} \frac{d}{dx} H_m(x) \frac{d^{n-1}}{dx^{n-1}} e^{-\frac{x^2}{2}}\, dx$.

1.5 marks

Integrating $(n-1)$ times:

$= (-1)^n (-1)^n \int_{-\infty}^{\infty} \frac{d^n}{dx^n} H_m(x)\, e^{-\frac{x^2}{2}}\, dx$.

$\because$ $H_m(x)$ is a Poly of degree m. $\frac{d^n}{dx^n} H_m(x) = 0$ $(\because n > m)$.

$= 0$

1 mark. Similarly for $m > n$ proceed from ✪ & interchange positions of $H_n(x)$ & $H_m(x)$.

Ans3. $m = 0.5\,kg$ $\quad k = 40\,kg/s^2$, $\quad c = 9\,kg/sec.$

$F(t) = 102\cos 6t.$

$my'' + cy' + ky = F(t)$ — mass-spring system

Corresponding Electrical system is:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)$$

$L = m$, $R = c$, $k = \frac{1}{C}$

1.5 Mark

i.e.

$$\left.\begin{array}{l} 0.5\dfrac{d^2Q}{dt^2} + 9\dfrac{dQ}{dt} + 40Q = 102\cos 6t. \\ Q(0) = 0 \text{ and } Q'(0) = 0 \end{array}\right\} \quad -(1)$$

$\because I = Q'$

Thus. System (1) can be re written as: -

$$0.5\, I'(t) + 9\, I(t) + 40\int I(t)\,dt = 102\cos(6t). \quad -*$$

where. $I(0) = Q'(0) = 0.$

using $t = 0$, $I(0) = 0$ in $*$.

$$I'(0) = \frac{102}{0.5} = 204.$$

So, we have

$$\left.\begin{array}{l} 0.5I'' + 9I' + 40I = -(102\times 6)\sin 6t. \\ I(0) = 0 \\ I'(0) = 204. \end{array}\right\} \quad (2)$$

Solve Either system (1) or system (2)

Lets us solve ① :-

$0.5y'' + 9y' + 40y = 102\cos 6t$, $y = Q(t)$.

$y(0) = 0$, $y'(0) = 0$

$y'' + 18y' + 80y = 204\cos 6t$ — $*_1$

y_h : ch. eqn: $\lambda^2 + 18\lambda + 80 = 0 \Rightarrow \lambda = -10, -8$.

$y_h = Q_h = c_1 e^{-8t} + c_2 e^{-10t}$.

1 mark

Let $y_p = a\cos 6t + b\sin 6t$.

$y_p' = -6a\sin 6t + 6b\cos 6t$.

$y_p'' = -36a\cos 6t - 36b\sin 6t$.

putting y_p, y_p', y_p'' in $*_1$.

$-36a\cos 6t - 36b\sin 6t + 18\times(-6)a\sin 6t + 18\times 6\, b\cos 6t$
$+ 80a\cos 6t + 80b\sin 6t = 20\cancel{4}\cos 6t$.

$\cos 6t(-36a + 108b + 80a) + \sin 6t(-36b - 108a + 80b) = 102\cos 6t$

$\Rightarrow$ $44a + 108b = \cancel{102}\ 204$
$-108a + 44b = 0.$

$a = 0.66.$
$b = 1.62.$

Approx Calculation are fine!

1 mark

Or

The coeff a & b can be directly be computed:

$S = wL - \frac{1}{wC}$, $w = 6$, $E_0 = \frac{102}{6} = 17$.

$a = \frac{-E_0 S}{R^2+S^2}$, $b = \frac{E_0 R}{R^2+S^2}$, $R = 9$, $\frac{1}{C} = 40$.

So, $y(t) = c_1 e^{-8t} + c_2 e^{-10t} + 0.66\cos 6t + 1.62\sin 6t$.

$y(0) = 0$, $y'(0) = 0$

$y(0) = c_1 + c_2 + 0.66 = 0 \Rightarrow c_1 + c_2 = -0.66$

$y'(0) = -8c_1 - 10c_2 + 6\times 1.62 \Rightarrow -8c_1 - 10c_2 = -9.72$.

$c_2 = 7.5$
$c_1 = -8.16$.

Approx Calculation are fine.

0.5 Marks

Ans. 4 $\quad x^2 y'' + axy' + by = 0 \quad —(1)$

$y = x^m, \; y' = mx^{m-1}, \; y'' = m(m-1)x^{m-2}$

$x^2(m(m-1))x^{m-2} + ax\, m x^{m-1} + bx^m = 0.$

$m(m-1) + am + b = 0$

$m^2 + (a-1)m + b = 0 \quad \leftarrow (*)$

0.5 mark

(*) has a double root iff.

$(1-a)^2 - 4b = 0, \quad b = \frac{1}{4}(1-a)^2$ → 1 mark

0.5 mark $\; m = \frac{1-a}{2}$ $\quad \Rightarrow$ (1) becomes $y'' + \frac{a}{x}y' + \frac{1}{4x^2}(1-a)^2 y = 0.$

A solution is $y_1 = x^{(1-a)/2}$.

To obtain a second L.I. solution, apply method of variation of parameter (~~[illegible]~~).

0.5 mark $\; y_2 = uy_1$

$u = \int v\,dx, \quad v = \frac{1}{y_1^2} e^{-\int P dx}, \quad P = \frac{a}{x}$

$e^{-\int \frac{a}{x}dx} = e^{-a \log_e x} = x^{-a} = \frac{1}{x^a}$

$v = \frac{1}{x^{(1-a)} \cdot x^a} = \frac{1}{x}.$

$u = \int \frac{1}{x}\,dx = \log_e x.$

1 mark

$\Rightarrow y_1 = x^m, \quad m = \frac{1}{2}(1-a).$

$y_2 = \log x \cdot x^m.$

0.5