Population growth

Continuous system- Ordinary differential equations

Population growth: Differential equations

- $dN/dt = bN - mN$ *(density dependent)*

  b be the birth rate per capita;
  m (for mortality) the death rate per capita.

- This equation is usually written as

  $dN/dt = rN$ $(r = b-m)$

  $dN/N = r\,dt$

- Taking the integrals both sides with limits $t = 0$ to $t=T$ gives

  $\ln(N(T)) - \ln(N(0)) = rT$

  So $N(T) = N(0)e^{rt}$.

if *$r = 0$, the population size is stationary;*
if *$r > 0$, the population grows exponentially without bound.*
if *$r < 0$, the population approaches 0.*

Comparison with discrete time

- The discrete equation $N_{t+1} = R\, N_t$ has a solution
$$N_t = R^t No.$$

- For *R > 1, the solution increases without bound*
- for *R = 1, the solution is a constant*
- for *R < 1, the solution approaches 0.*

$$N(T) = N(0)e^{rt} \textit{ (for continous)}$$
$$N_t = R^t No \quad \textit{(For discrete)}$$

- we see that R^t *and* $e^{rt} = (e^r)^t$ play analogous roles.
- Thus we conclude that
- $$R = e^r$$
- The Taylor series for e^r *is* $1 + r + r^2 + \cdots.$
- And therefore $R \sim 1 + r$

Continous Logistic model

- $dN/dt = N\, f(N)$
- *f(N) is the per capita growth rate.*
- $f(N) = r(l - N/K)$
- $dN/dt = r\,N\,(l - N/K)$
- $dN/N(1-N/k) = rdt$
- We then integrate both sides of the equation from $t = 0$ to $t = T$:
- $1/(N(1-N/K))$ *is written as*

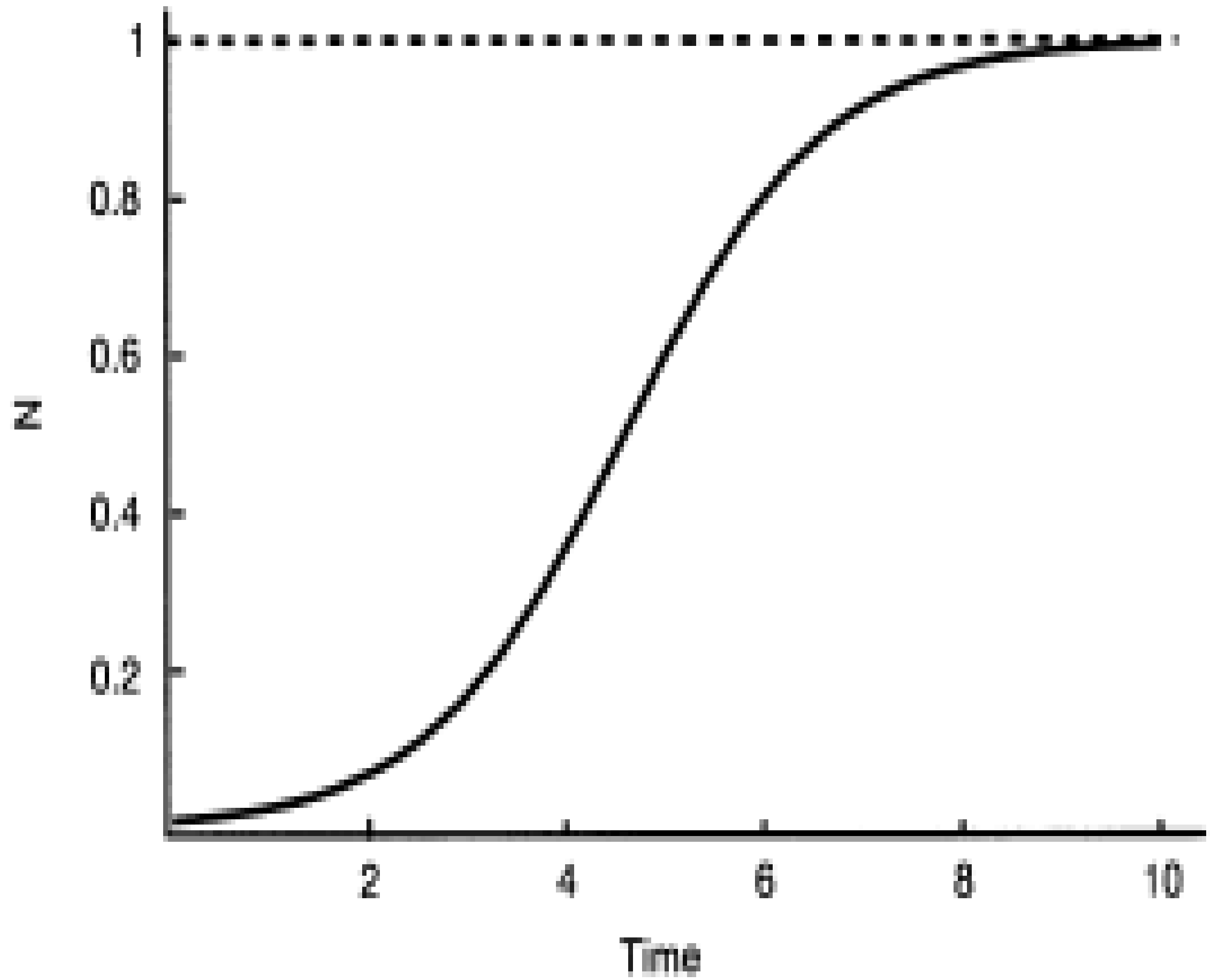
$1/N + (1/K)/(1-N/K)$ *(Partial fraction)*

(HW: 2nd method Solve this through Bernoulli's method)

So that finally we get after integrating the above equation on both sides;

- $N(T) = N(O)e^{rT}/\,(1 + N(O)(e^{rT} - 1)/K)$

Sigmoidal curve



Stability for continuous systems

Box 4.1. Qualitative analysis of a model with a single differential equation.

We consider a model of the form

$$dN/dt = F(N).$$

The first step in the analysis is to determine the equilibria. Do this by setting $dN/dt = 0$ to obtain an equation for $\hat{N}$:

$$0 = F(\hat{N}).$$

Then solve this equation for $\hat{N}$. Note that this may be impossible to do for some functions F.

The next step is to determine the stability of these equilibria by approximating F. We define the deviation from equilibrium, n, by letting $N = \hat{N} + n$ and compute

$$dn/dt \approx \lambda n,$$

where

$$\lambda = \left.\frac{dF}{dN}\right|_{N=\hat{N}}.$$

- The equilibrium is stable, and is approached if the system starts nearby, if λ for that equilibrium is negative.
- The equilibrium is unstable, and the system moves away from the equilibrium if the system starts nearby, if λ for that equilibrium is positive.

The rate of return to the equilibrium, or the rate at which the system moves away from the equilibrium, is determined by λ.

Equilibrium analysis

- Determine the values of the population density, *N^*, which* are equilibria. Set
- $dN/dt = 0 = rN^*(1-N^*/K)$
- *Two equilibrium points $N^* = 0$ and $N^* = K$.*
- *How to determine the stability of equilibria?*

Stability Analysis

- We let *n represent the deviation from the equilibrium, so $N = N^* + n$ (n is perturbation).*

- So that $dN/dt = dN^*/dt = F(N) = rN(1-N/k)$

- We need to approximate *F(N) near the equilibrium, N. We use a* Taylor series to see that

- $F(N^* + n) \sim F(N^*) + n\, dF/dN$ *(at* $N=N^*$*)* $+ O(n^2)$... *(Taylors's Expansion)*

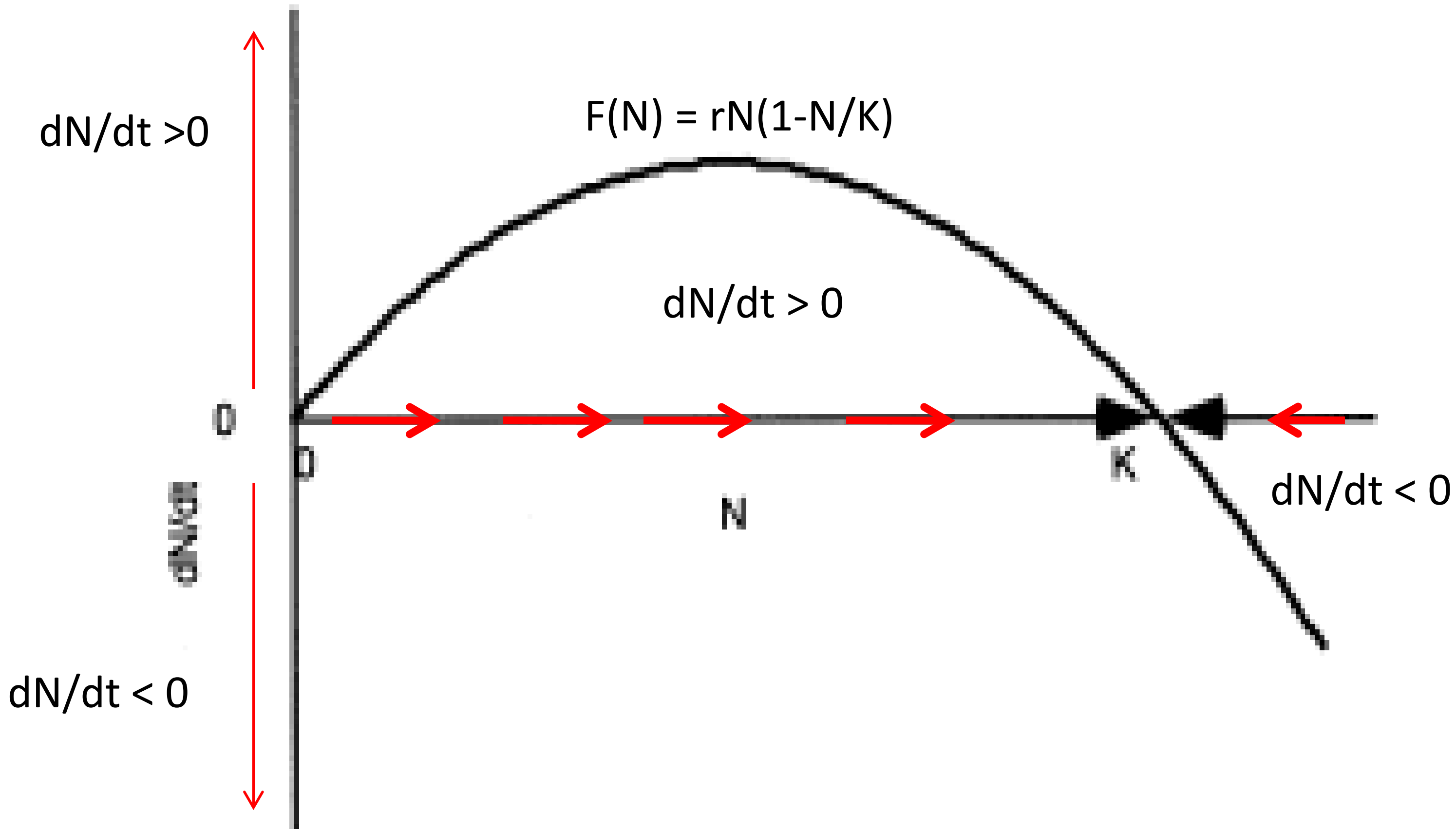
- We note that since *N^* is an equilibrium, $F(N^*)$,*

- We conclude that $dn/dt \sim n\, dF/dN$ *(at* $N=N^*$*)* ·

At equilibrium of N* = 0 and K

- In the logistic model with
- *$F(N) = rN(l- N/K) = rN- rN^2/K$*
- dF/dN = F'(N*) = *r- 2rN*/K (at equilibrium N=N*).*
- Near the equilibrium *N* = 0,*
- $dn/dt \sim n\,(r- 2rN/K)|_{N=O} = r$ *(unstable r >0)*
- Near the equilibrium *N* = K,.*
- $dn/dt \sim n\,(r- 2rN/K)|_{N=K} =$ -r n (stable)
- The solution $n(t) = n_0\, e^{-rt}$
- (stable since it approaches n(0) = K, as t-> infinity)

Graphical method



dn/dt > 0, arrow moves towards right

dn/dt <0 arrow moves towards left