Divide and Conquer (Part 1)

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Divide and Conquer

- Reduce problem to one or more sub-problems of the same type
- Typically, each sub-problem is at most a constant fraction of the size of the original problem
- Subproblems typically disjoint
- Often gives significant, usually polynomial, speedup
- Examples: Quick Sort, Mergesort, Binary Search, Strassen’s Algorithm, Quicksort. FFT, etc.

Steps involved

- Divide-and-conquer algorithm has three parts:

- Divide: the problem into a number of subproblems that are smaller instances of the same problem.

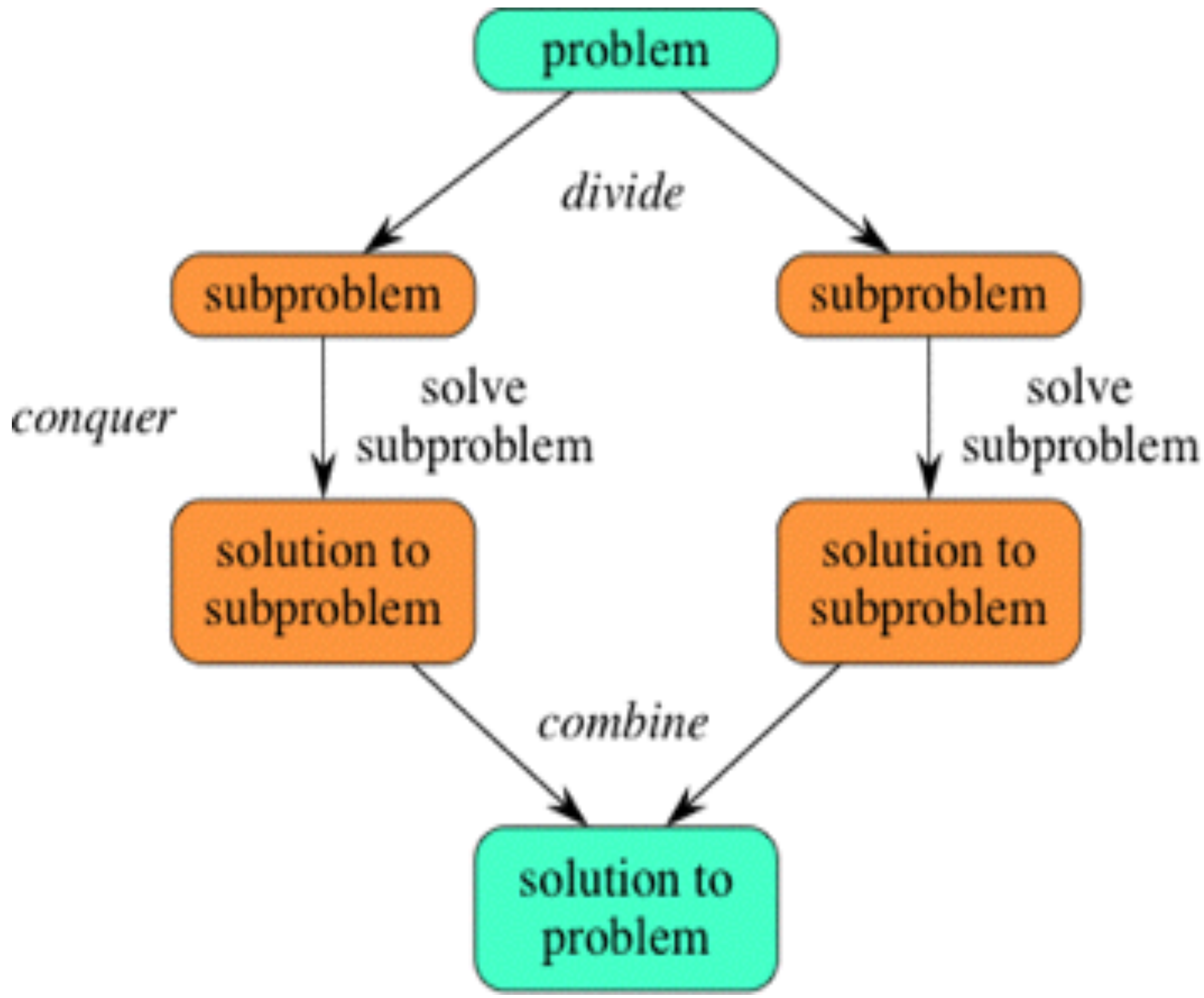
- Conquer: the subproblems by solving them recursively. If they are small enough, solve the subproblems as base cases.

- Combine: the solutions to the subproblems into the solution for the original problem.Consequence:

Benefit

- Brute force / naïve solution: N^2 (typically)
- Divide-and-conquer: $N \log N$

Algorithm sketch - two subproblems



Algorithm sketch - more than two subproblems

Finding word in dictionary

Suppose you want to find **"janissary"** in a dictionary:

- open the book near the middle
- the heading on the top left page is **"kiwi"**,
- so move back a small number of pages
- here you find **"hypotenuse"**, so move forward
- find **"ichthyology"**, move forward again

The number of pages you move gets smaller (or at least adjusts in response to the words you find)

Common D&C algorithms

- **Mathematics**
    - Polynomial & Matrix Multiplication
    - Exponentiation
    - Large Integer Manipulation
    - FFT
- **Geometry**
    - Convex Hull
    - Closest Pair
- **Searching**
    - Binary Search
- **Sorting**
    - Merge Sort
    - Quick Sort

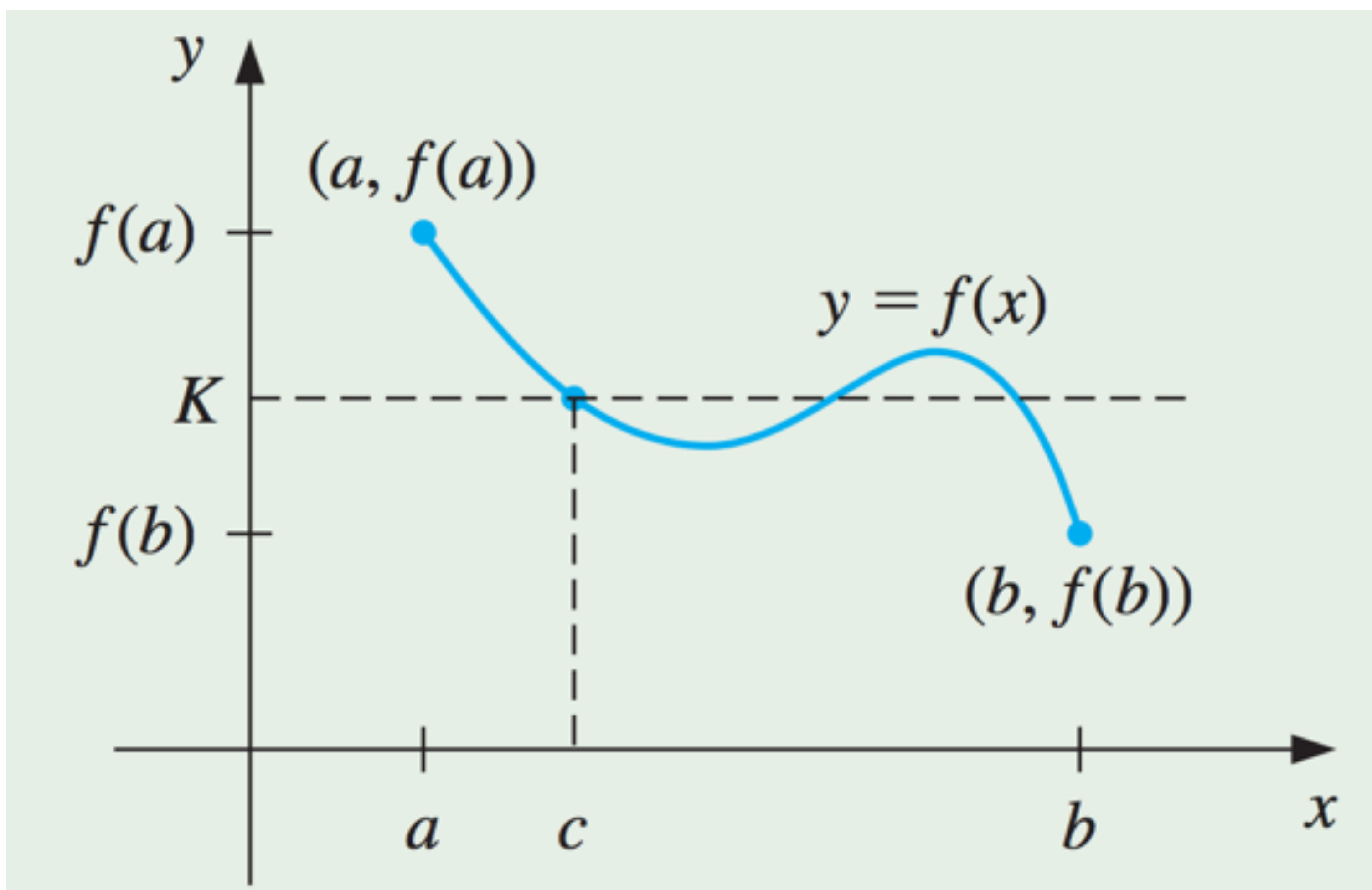
Root finding - bisection

Root finding problem

- Finding Zero of function f(x)
- This process involves finding a root, or solution, of an equation of the form $f(x) = 0$ for a given function f

Intermediate value theorem (IVT)

It simply states that for any value L between f(a) & f(b), there's a value c in [a,b] for which f(c)=L.



Example

- Show that $f(x) = x^3 + 4x^2 - 10 = 0$ has a root in [1, 2] and use the Bisection method to determine an approximation to the root that is accurate to at least within 10^{-4}

Solution sketch

- Because $f(1) = -5$ and $f(2) = 14$ the IVT ensures that this continuous function has a root in $[1, 2]$.

Solution sketch continued …

- For the first iteration of the Bisection method we use the fact that at the midpoint of [1, 2] we have $f(1.5) = 2.375 > 0$.
- This indicates that we should select the interval [1, 1.5] for our second iteration.
- Then we find that $f(1.25) = -1.796875$ so our new interval becomes [1.25, 1.5], whose midpoint is 1.375.
- Continuing in this manner gives the values shown in the following table

Iterations

Iter	a_n	b_n	p_n	$f(a_n)$	$f(p_n)$	RelErr
1	1.000000	2.000000	1.500000	-5.000	2.375	0.33333
2	1.000000	1.500000	1.250000	-5.000	-1.797	0.20000
3	1.250000	1.500000	1.375000	-1.797	0.162	0.09091
4	1.250000	1.375000	1.312500	-1.797	-0.848	0.04762
5	1.312500	1.375000	1.343750	-0.848	-0.351	0.02326
6	1.343750	1.375000	1.359375	-0.351	-0.096	0.01149
7	1.359375	1.375000	1.367188	-0.096	0.032	0.00571
8	1.359375	1.367188	1.363281	-0.096	-0.032	0.00287
9	1.363281	1.367188	1.365234	-0.032	0.000	0.00143
10	1.363281	1.365234	1.364258	-0.032	-0.016	0.00072
11	1.364258	1.365234	1.364746	-0.016	-0.008	0.00036
12	1.364746	1.365234	1.364990	-0.008	-0.004	0.00018
13	1.364990	1.365234	1.365112	-0.004	-0.002	0.00009

Precision

- The number of iterations depends on the precision being looked for

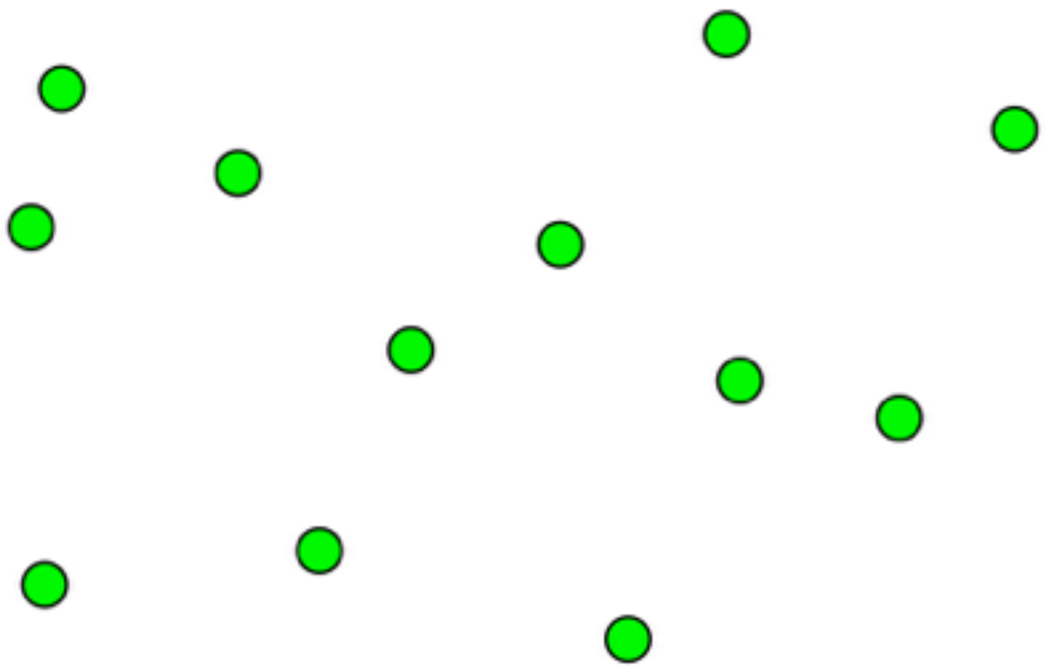
- if $c_1 = (a+b)/2$ is the midpoint of the initial interval, and c_n is the midpoint of the interval in the n^{th} step, then the difference between c_n and a solution c is bounded by $|c_n - c| <= |b - a|/2^n$

- One should be able to perform the back calculation

Closest Pair

Closest pair - problem statement

- **Given a set of points $\{p_1, \ldots, p_n\}$ find the pair of points $\{p_i, p_j\}$ that are closest**

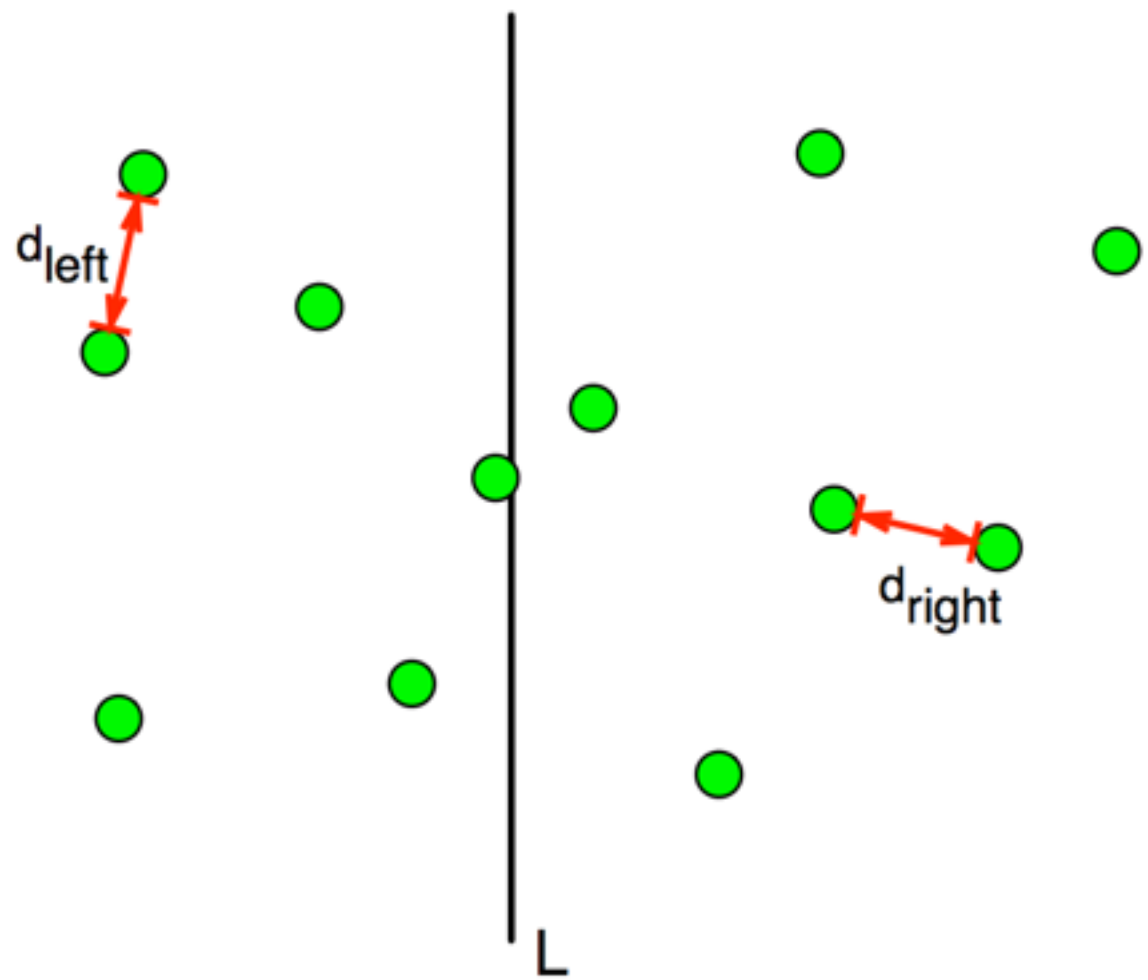


Brute force vs. divide and conquer

- Brute force gives an $O(n^2)$ algorithm to just check ever pair of points.
- Can we do it faster? Seems like difficult without checking every pair.
- In fact, we can find the closest pair in $O(n \log n)$ time.

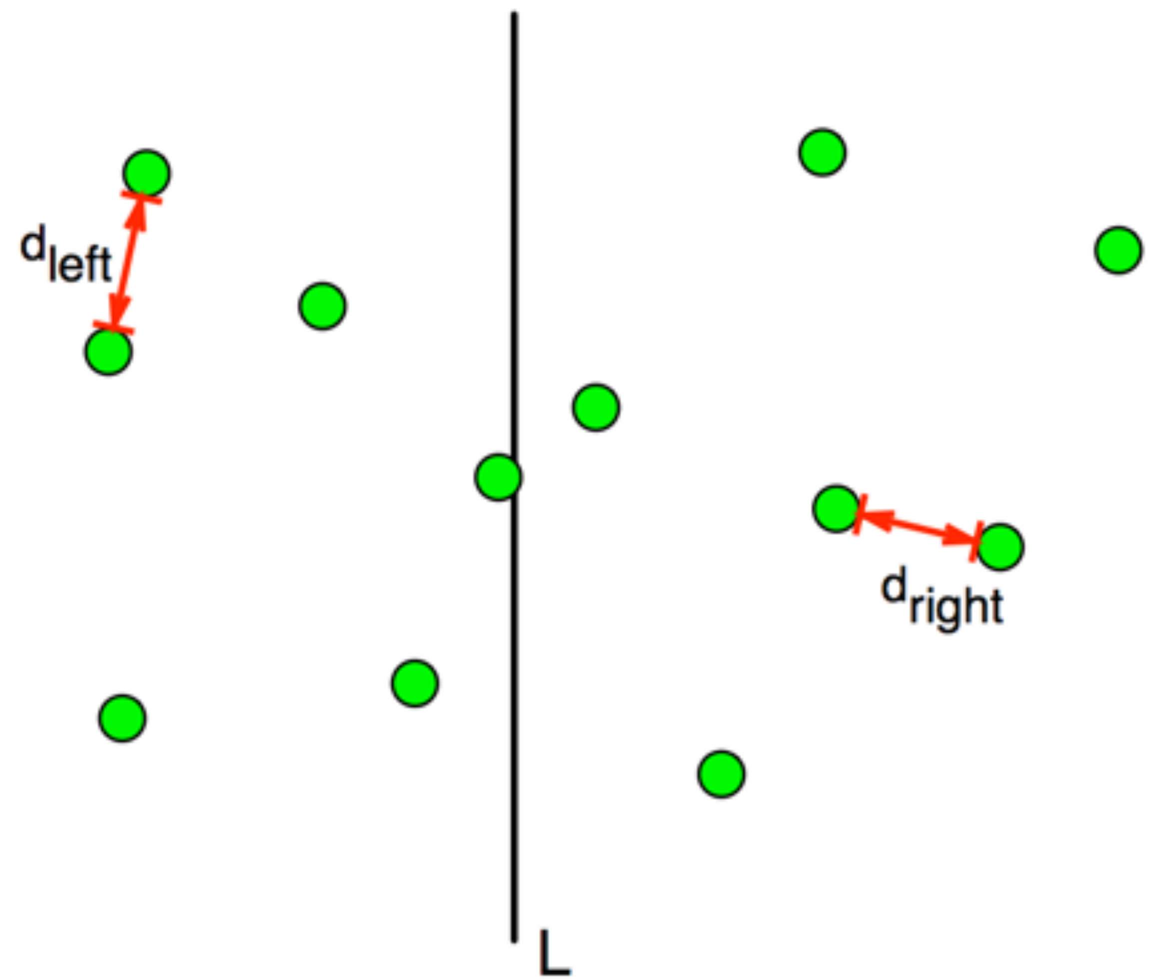
Divide

- Split the points with line L so that half the points are on each side.
- Recursively find the pair of points closest in each half.



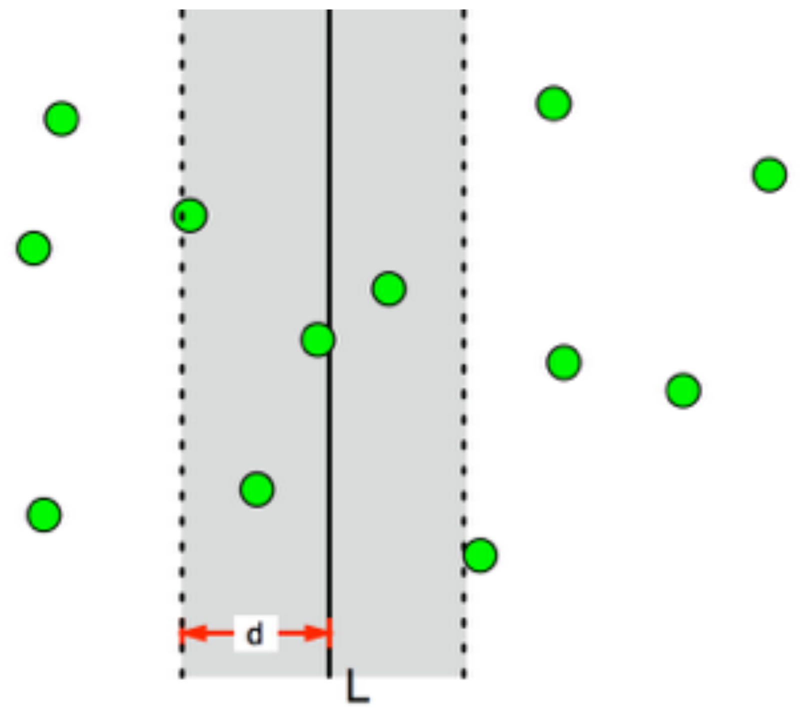
Merge

- Let $d = \min\{d_{left}, d_{right}\}$
- d would be the answer, except maybe L split a close pair!



Region near L

- **If the closest pair exists across the L, it should contained within d margins on both sides**



A life saver observation

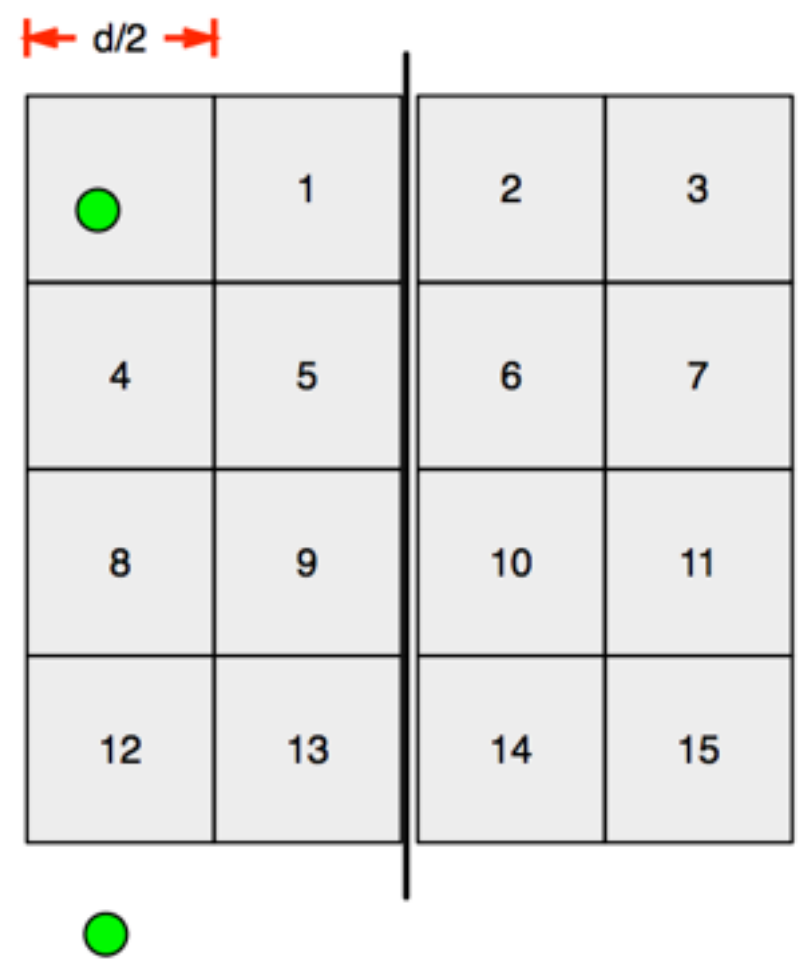
- Let S_y be an array of the points in that region, sorted by decreasing y-coordinate value.
- S_y might contain all the points, so we can't just check every pair inside it.

Theorem

Suppose $S_y = p_1, \ldots, p_m$. If $dist(p_i, p_j) < d$ then $j - i \leq 15$.

Constant number of checks for each point is enough

- We can split the entire margin area into d/2 sided square boxes
- Each box can have max 1 point
- Suppose 2 points are separated by > 15 indices.
  - Then, at least 3 full rows separate them (the packing shown is the smallest possible).
  - But the height of 3 rows is > 3d/2, which is > d.
  - So the two points are further than d apart.



Conclusion

- Starting from the bottom most point, for each point, it is enough to check next 15 data points up in the sequence of increasing value in S_y
- Constant (15) time for each of the n points

The algorithm

```
ClosestPair(Px, Py):
   if |Px| == 2: return dist(Px[1],Px[2])    // base

   d1 = ClosestPair(FirstHalf(Px,Py))      // divide
   d2 = ClosestPair(SecondHalf(Px,Py))
   d = min(d1,d2)

   Sy = points in Py within d of L     // merge
   For i = 1,...,|Sy|:
       For j = 1,...,15:
           d = min( dist(Sy[i], Sy[j]), d )
   Return d
```

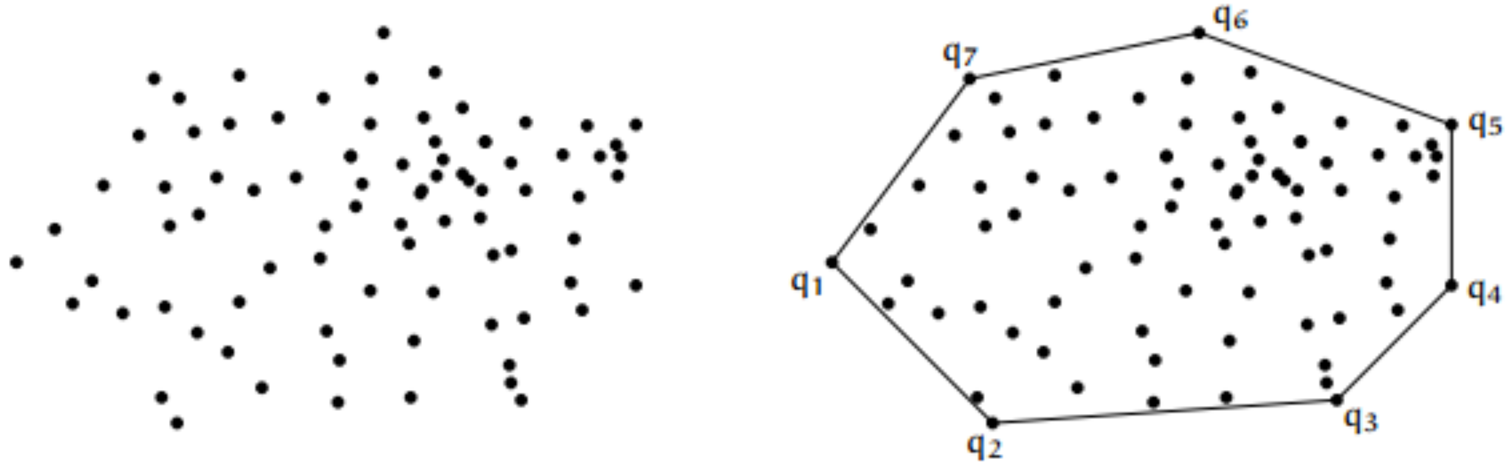
Running time

- Divide set of points in half each time: O(log n) depth recursion
- Merge takes O(n) time.
- Recurrence: $T(n) = 2T(n/2) + cn$
- Analysis is same as MergeSort ⇒ O(n log n) time

Convex Hull

Convex Hull

- Studied in the field of Computational Geometry.
- A convex hull, of a set of N points is the smallest perimeter fence enclosing all these points.
- Convex Hull Output - Sequence of vertices in counterclockwise/ clockwise order.



(a) Input. (b) Output.

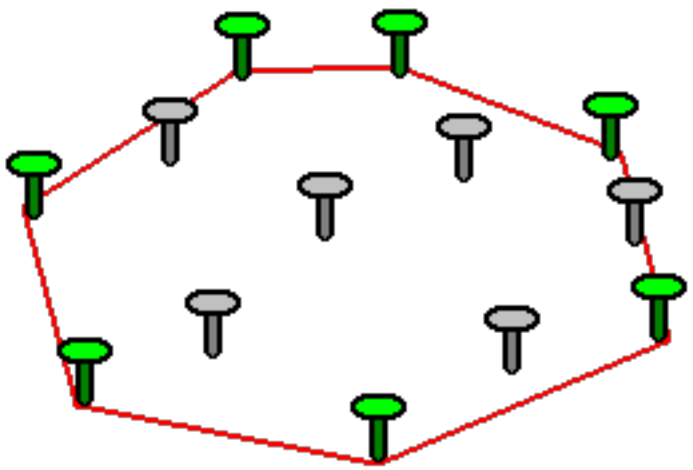
Motion planning

- Robot Motion Planning - Find the shortest path in the plane from a starting point s to an ending point t, that avoids a polygonal obstacle.

- Here, we can see that the shortest path is either the straight line from s to t, or one of the two polygon chains of the convex hull.

Mechanical algorithm

Hammer nails perpendicular to the plane and stretch elastic rubber band around these points.



Brute force

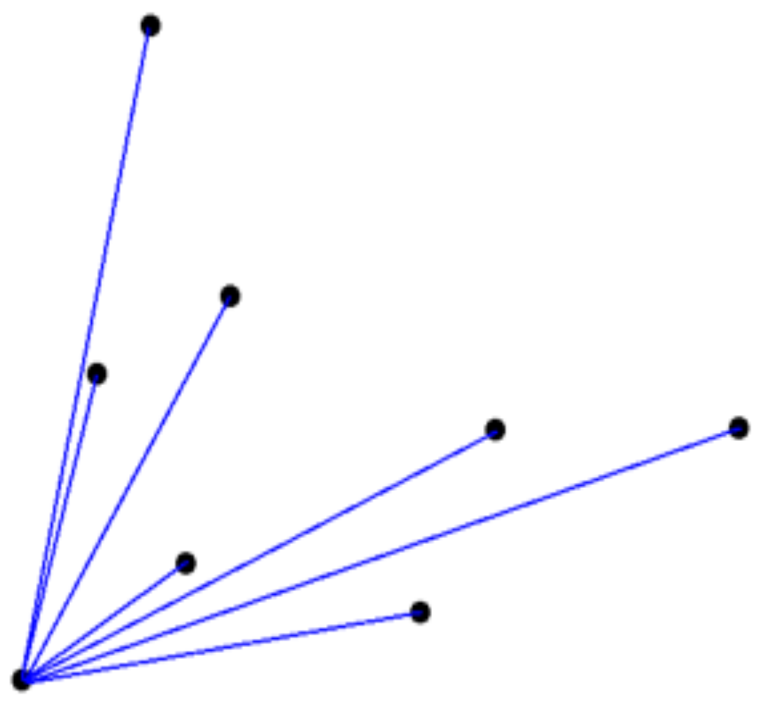
- Create a list of all possible line segments
  - N choose 2 line segments for N points
- For each line segment check if both sides have points
  - If that's not the case, include the segment in the solution set
- Quadratic complexity algorithm

Graham's scan

- Start at point guaranteed to be on the hull. (the point with the minimum y value)
- Sort remaining points by polar angles of vertices relative to the first point.
- Go through sorted points, keeping vertices of points that have left turns and dropping points that have right turns.

Graham's scan example

Graham's scan example



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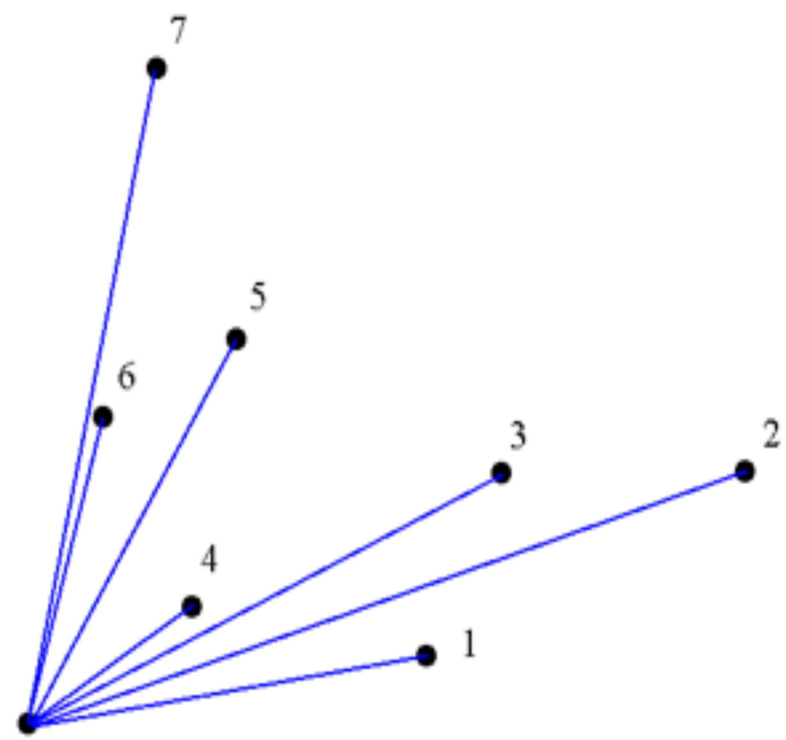

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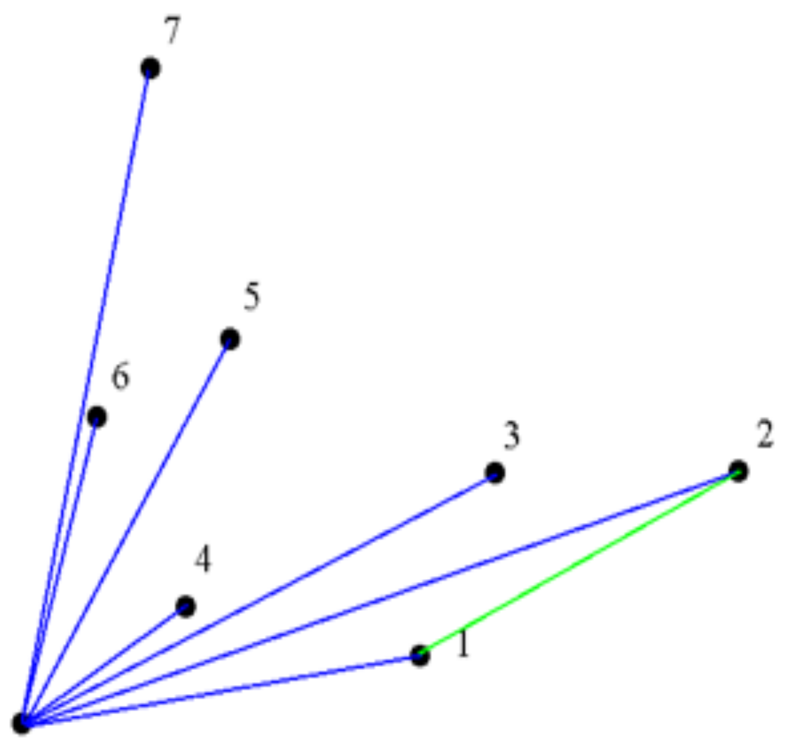

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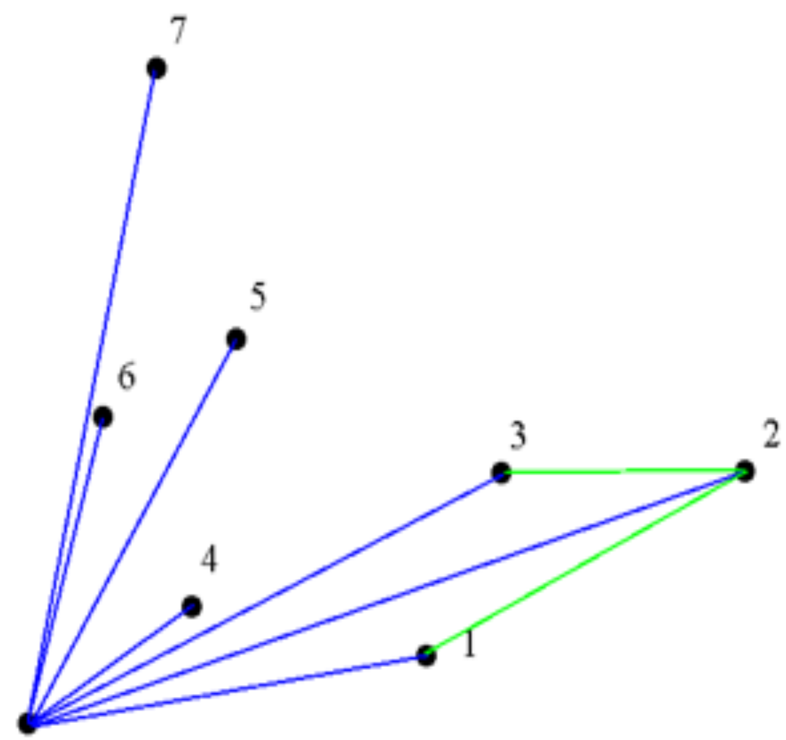

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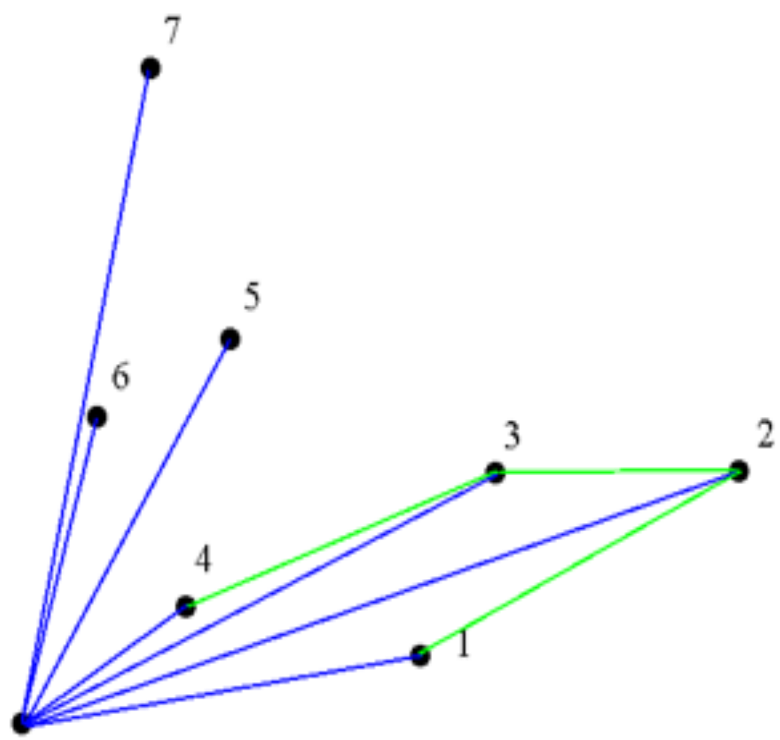

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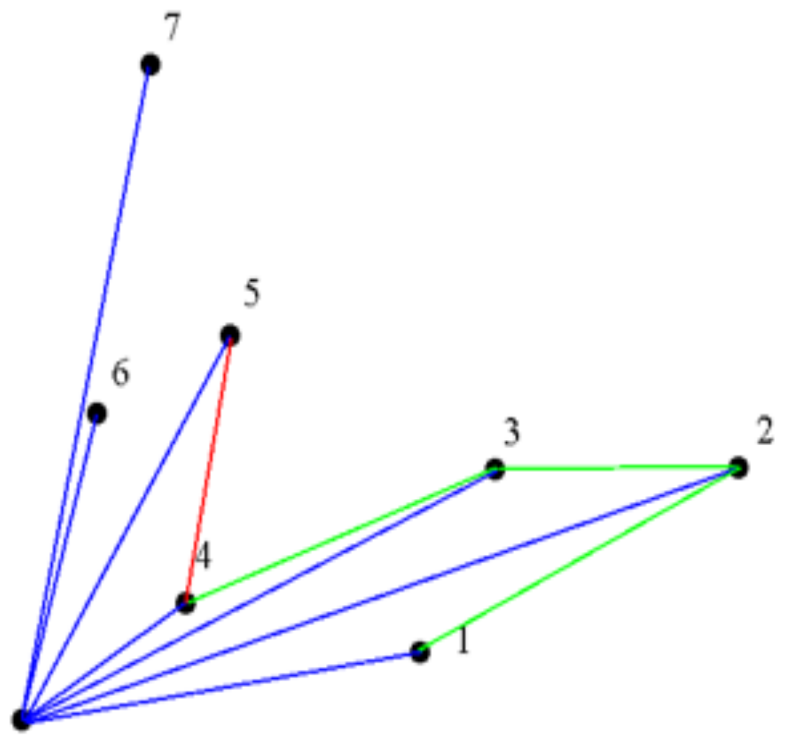

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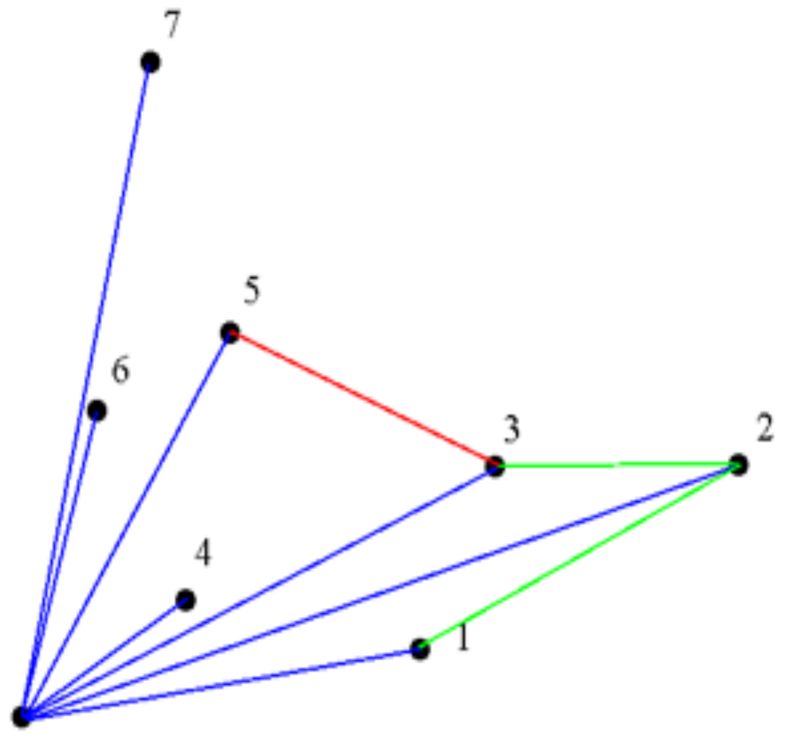

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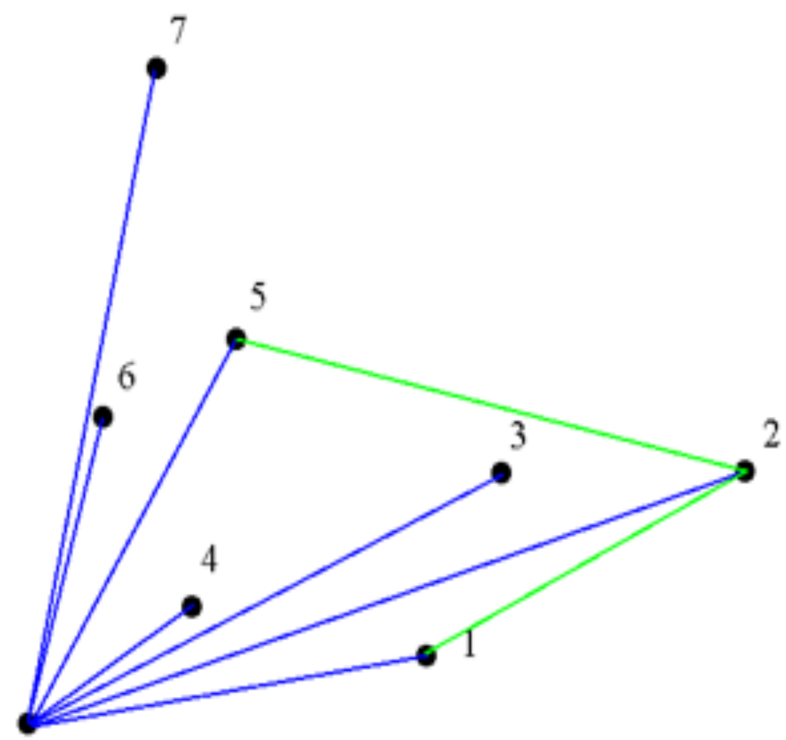

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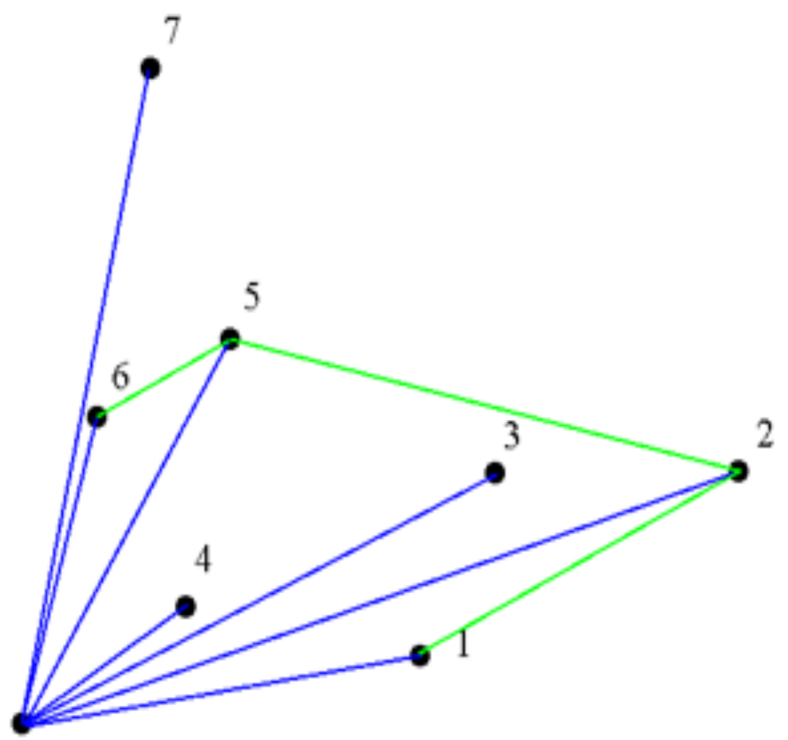

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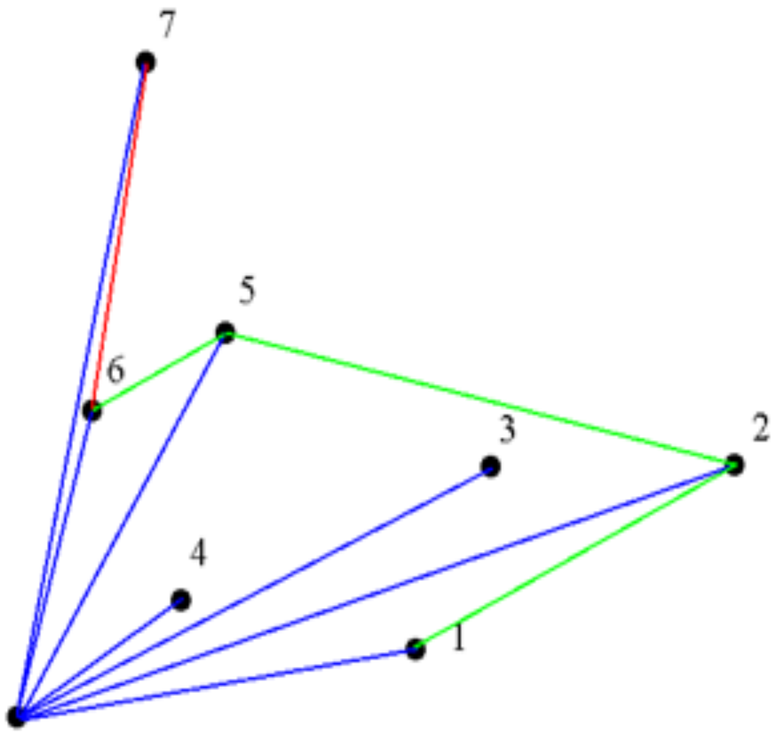

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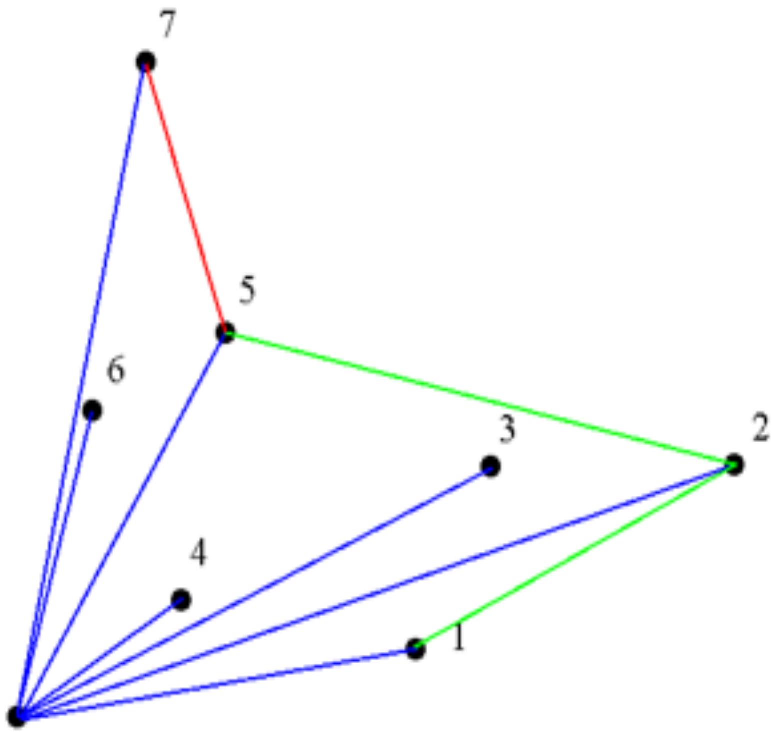

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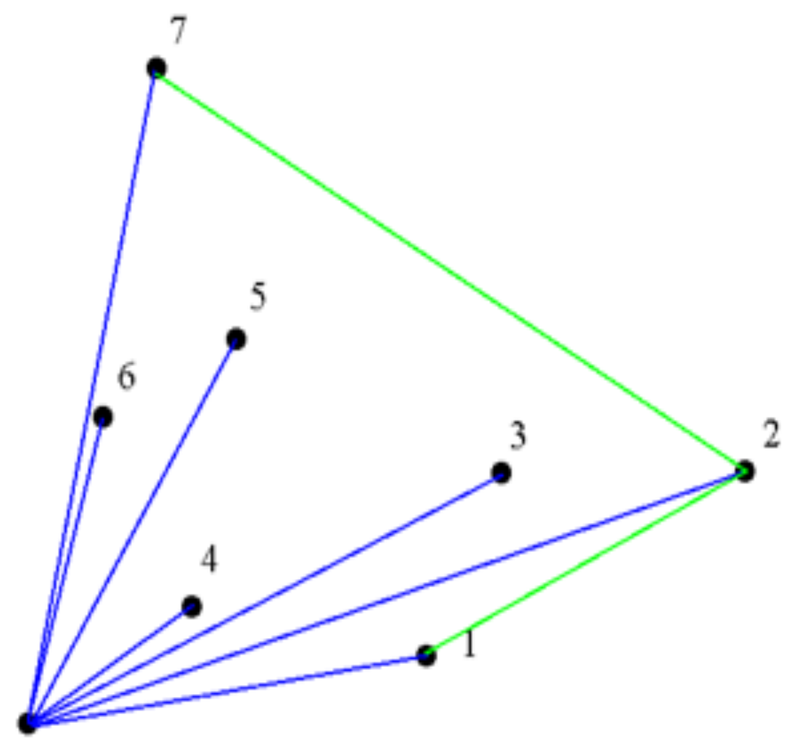

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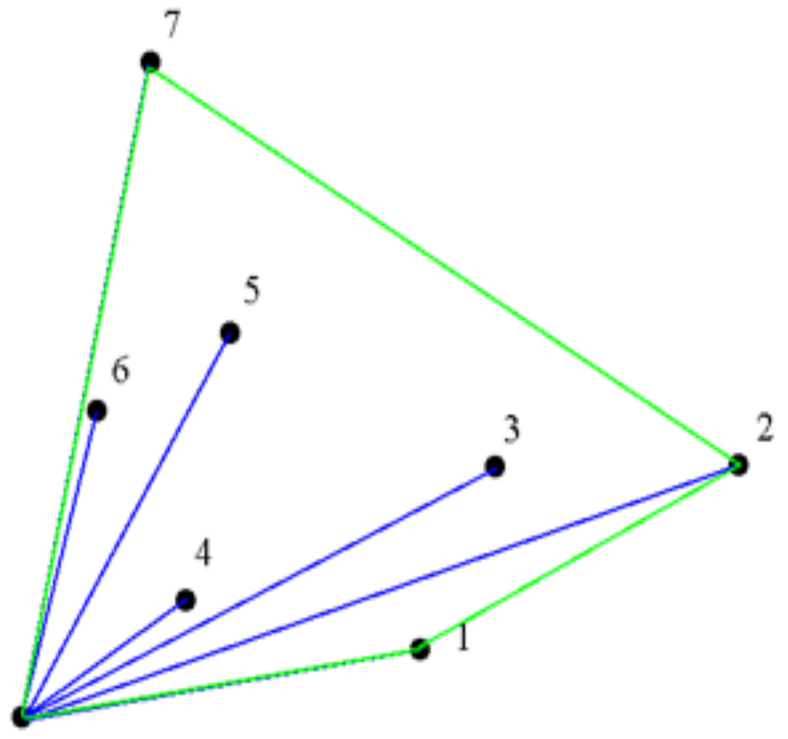
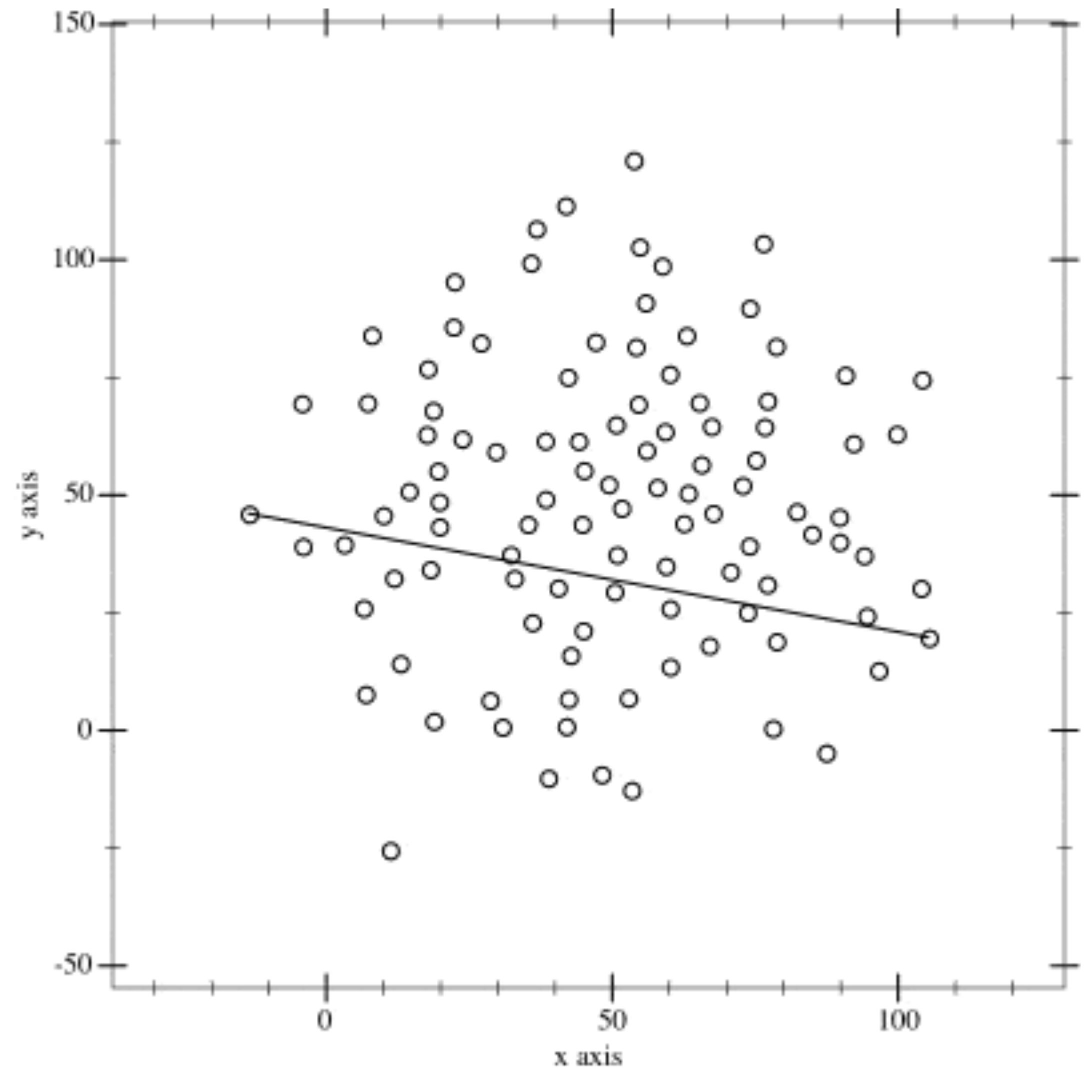


Runtime of Graham's can

- Graham's scan is O(n log n) due to initial sort of angles.

Quick Hull Algorithm

Quick hull example



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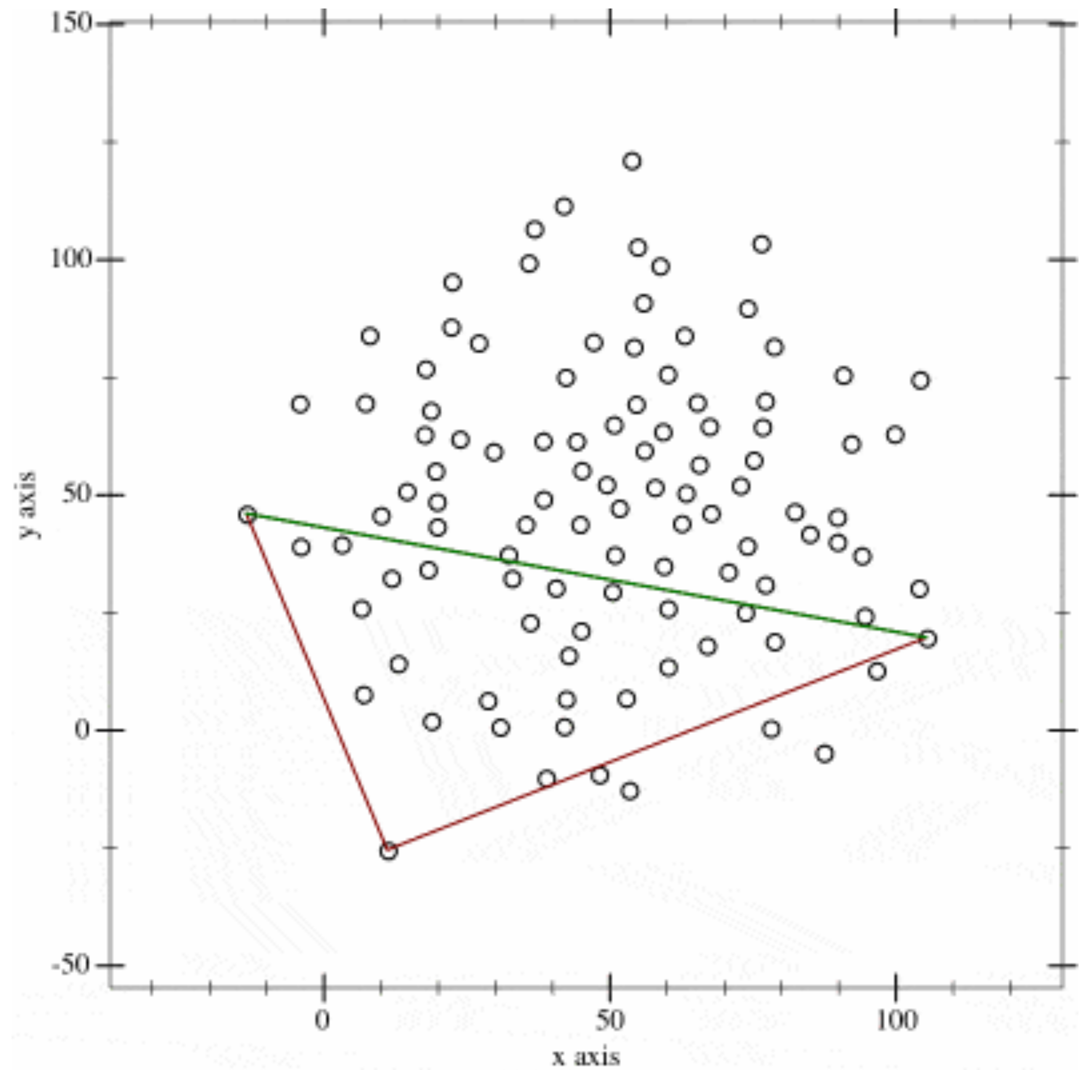

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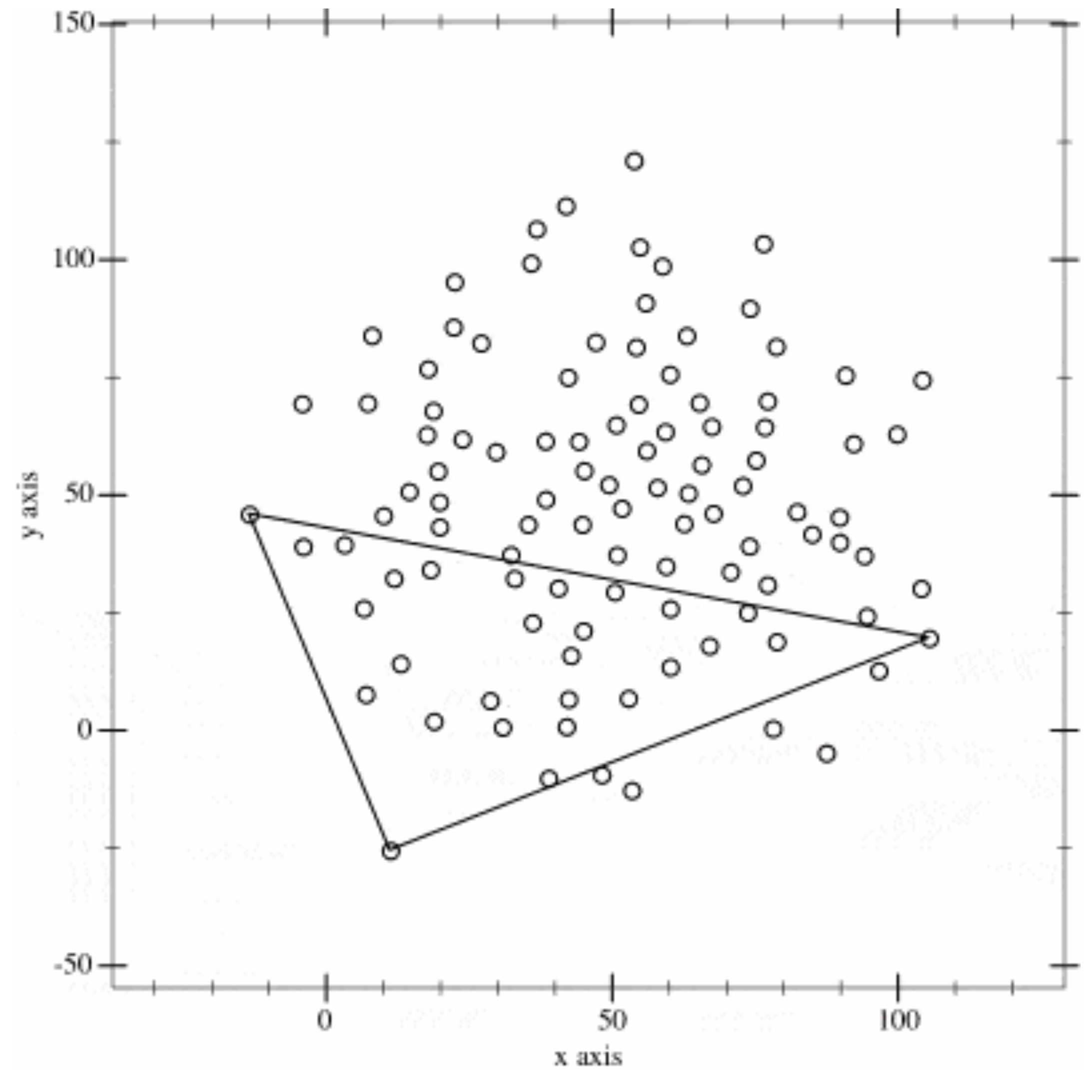

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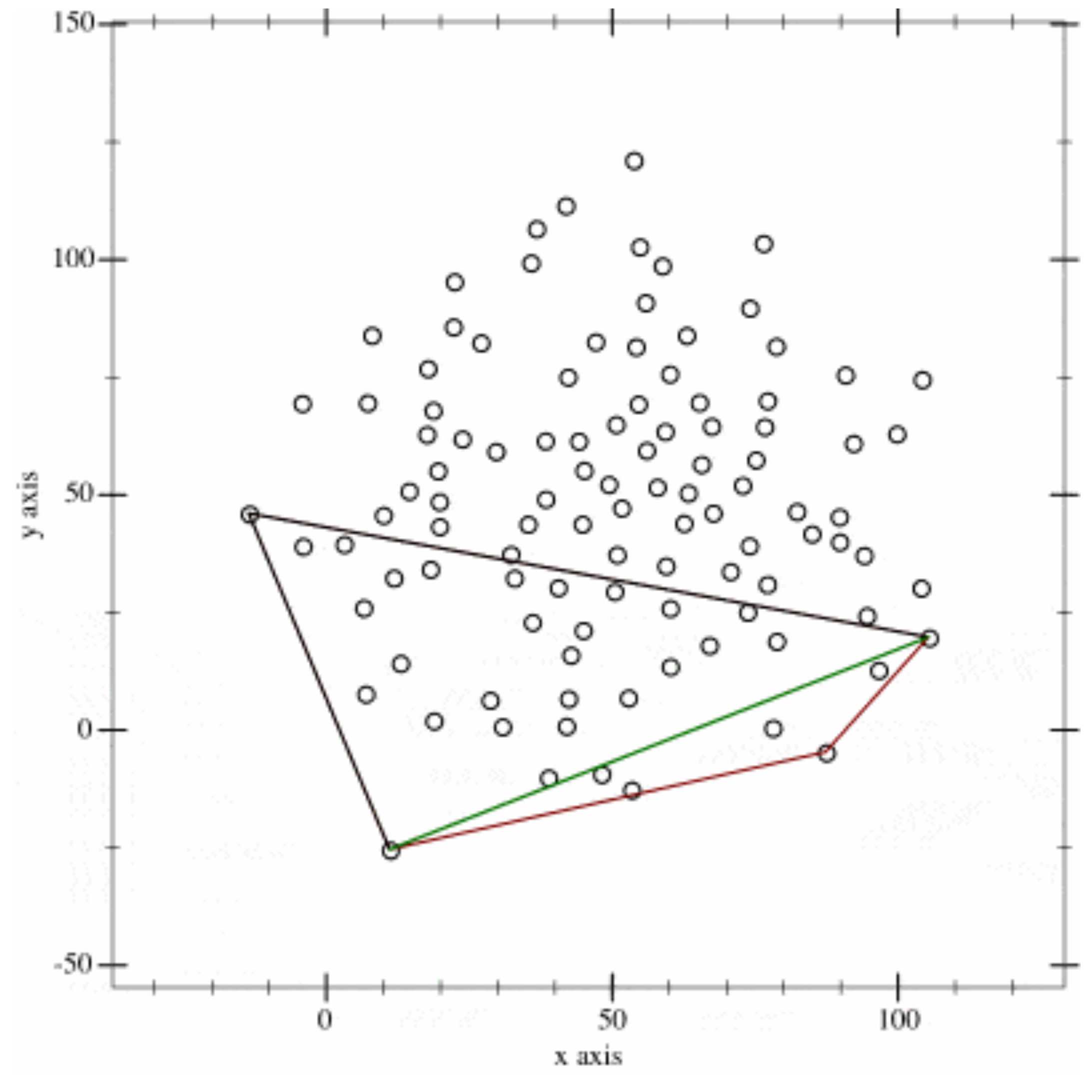

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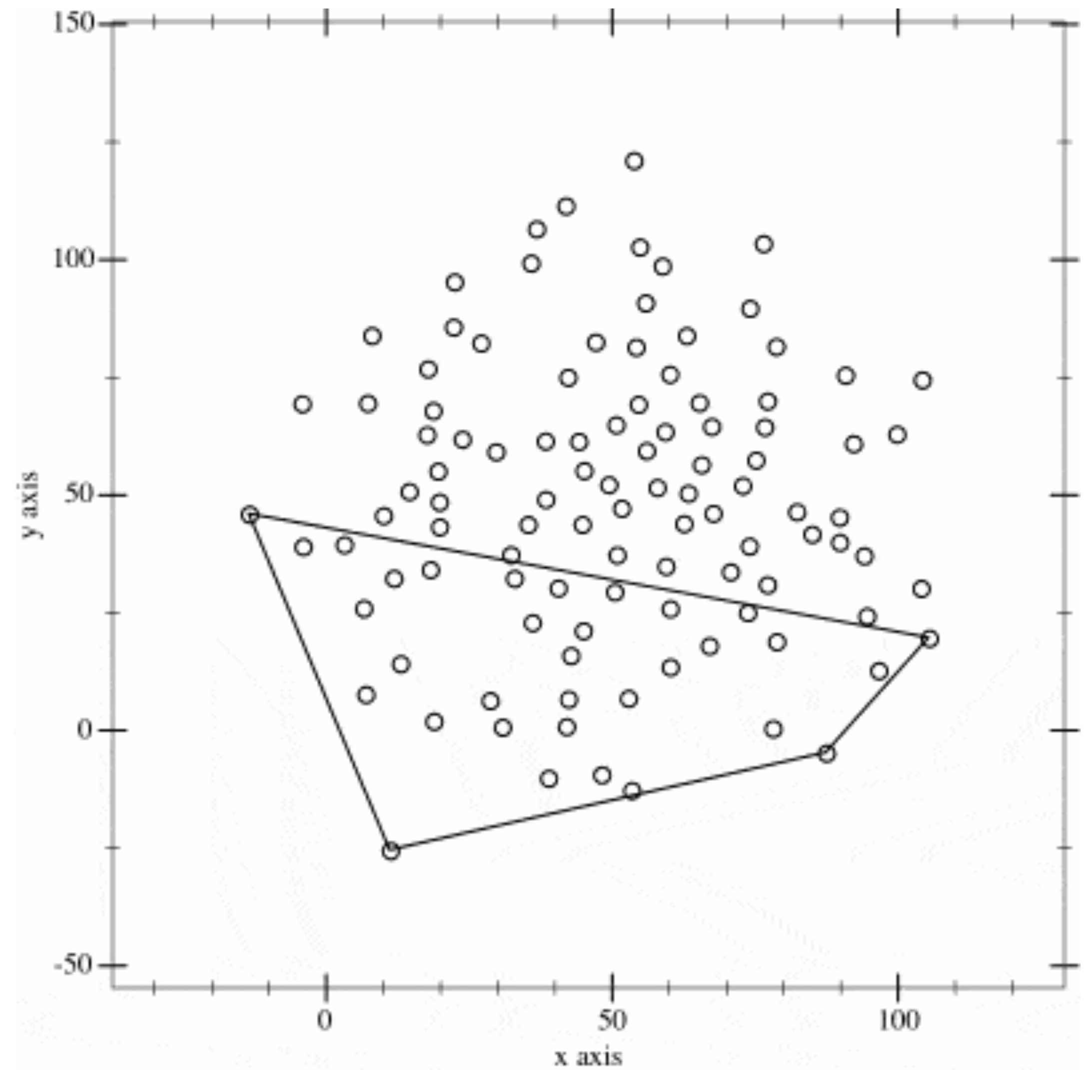

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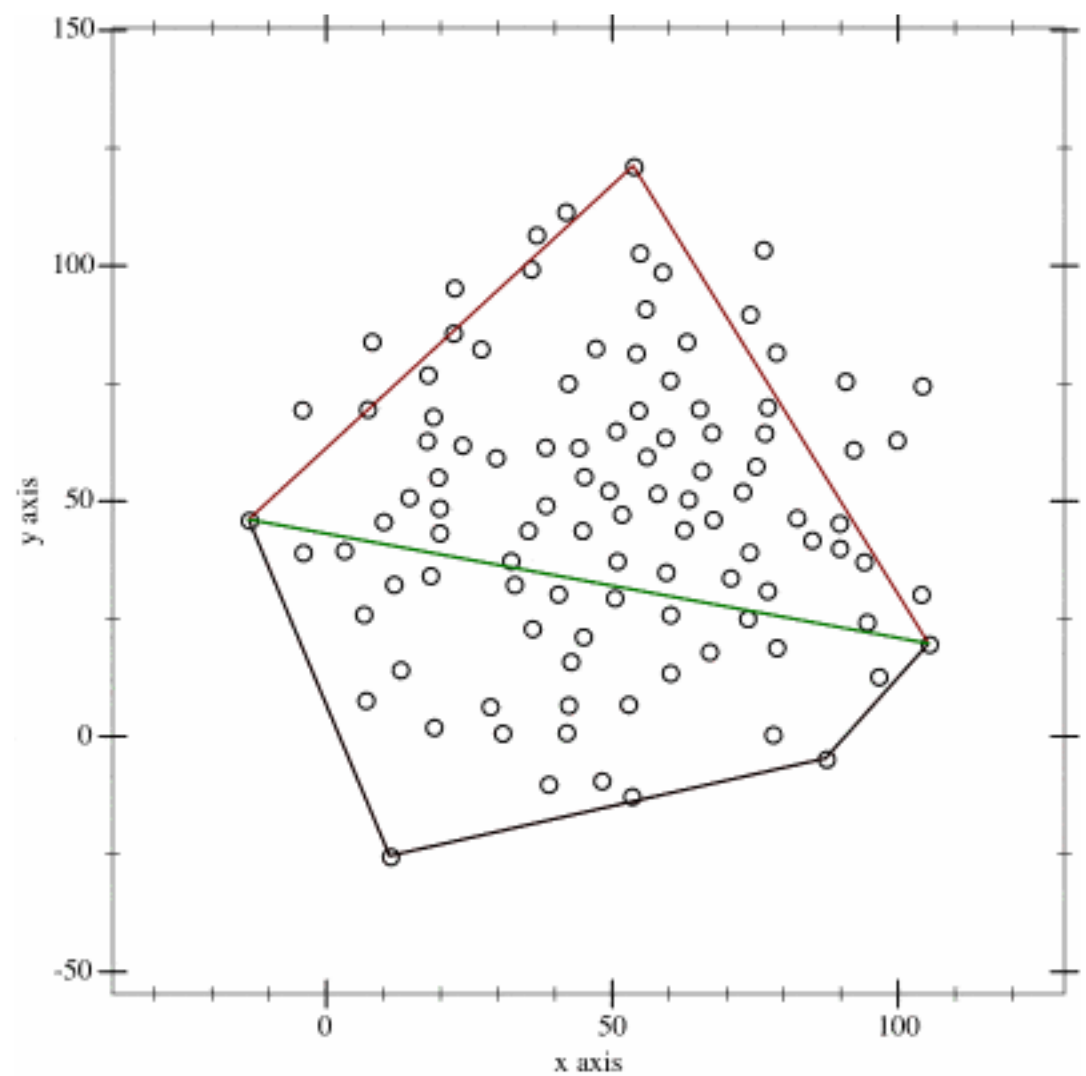

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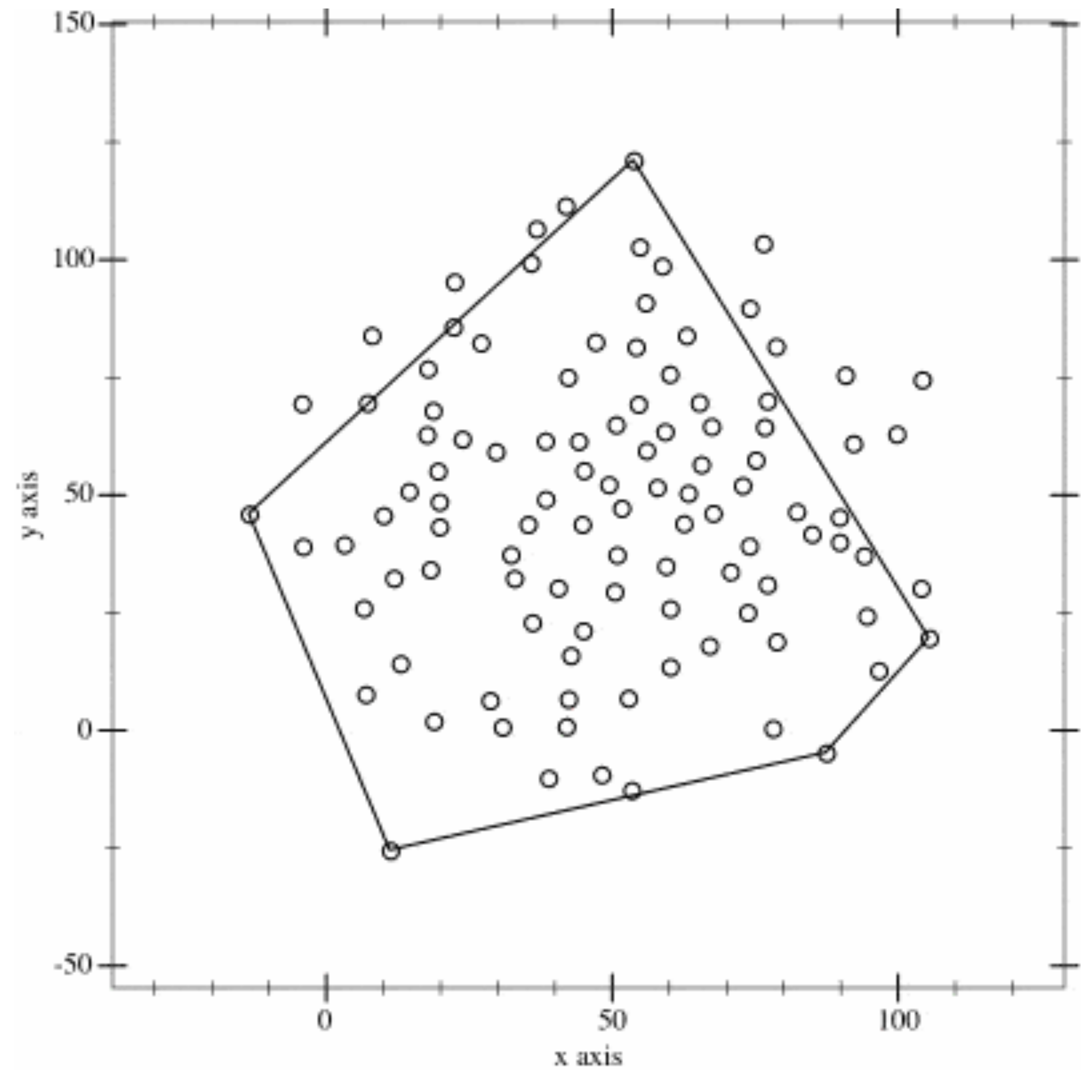

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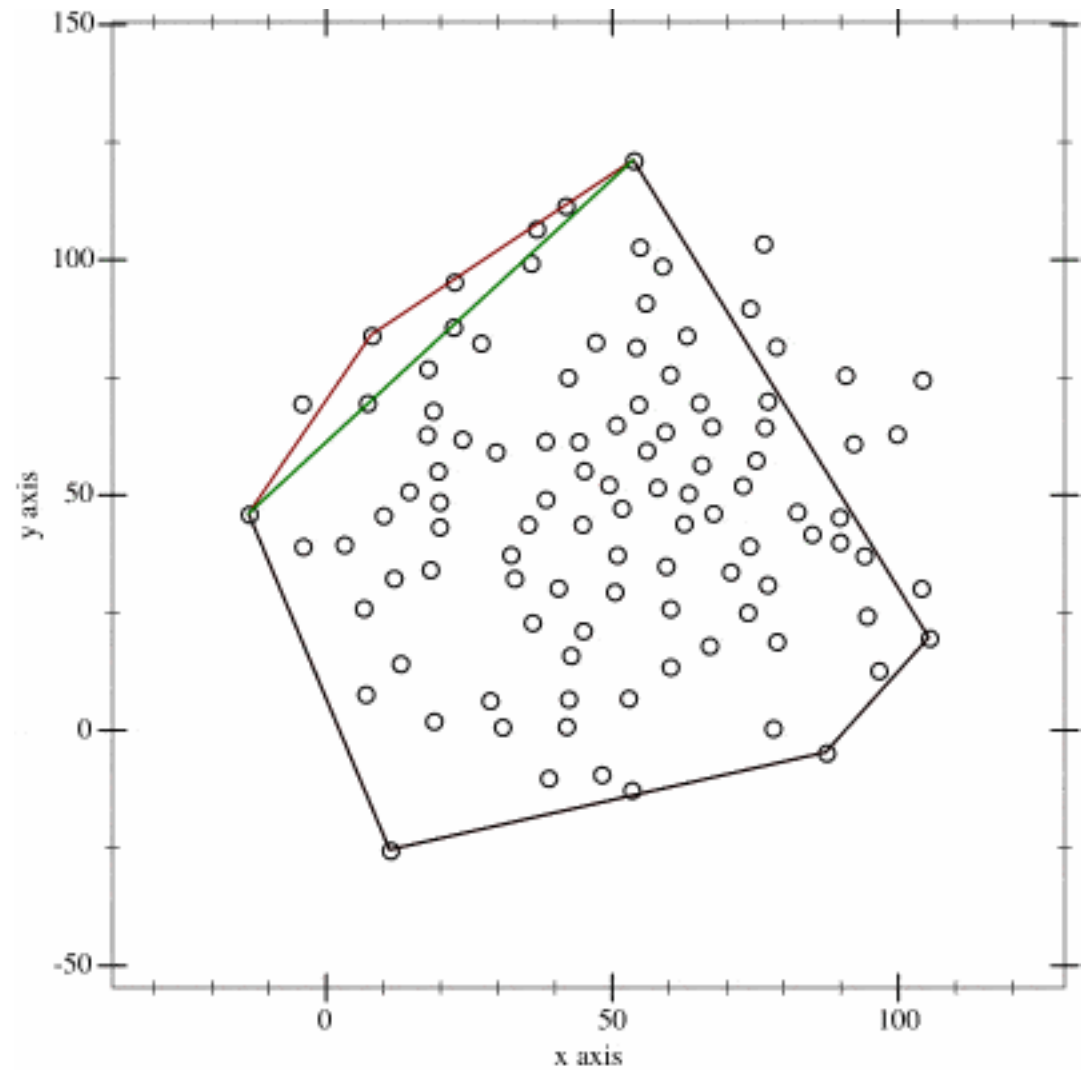

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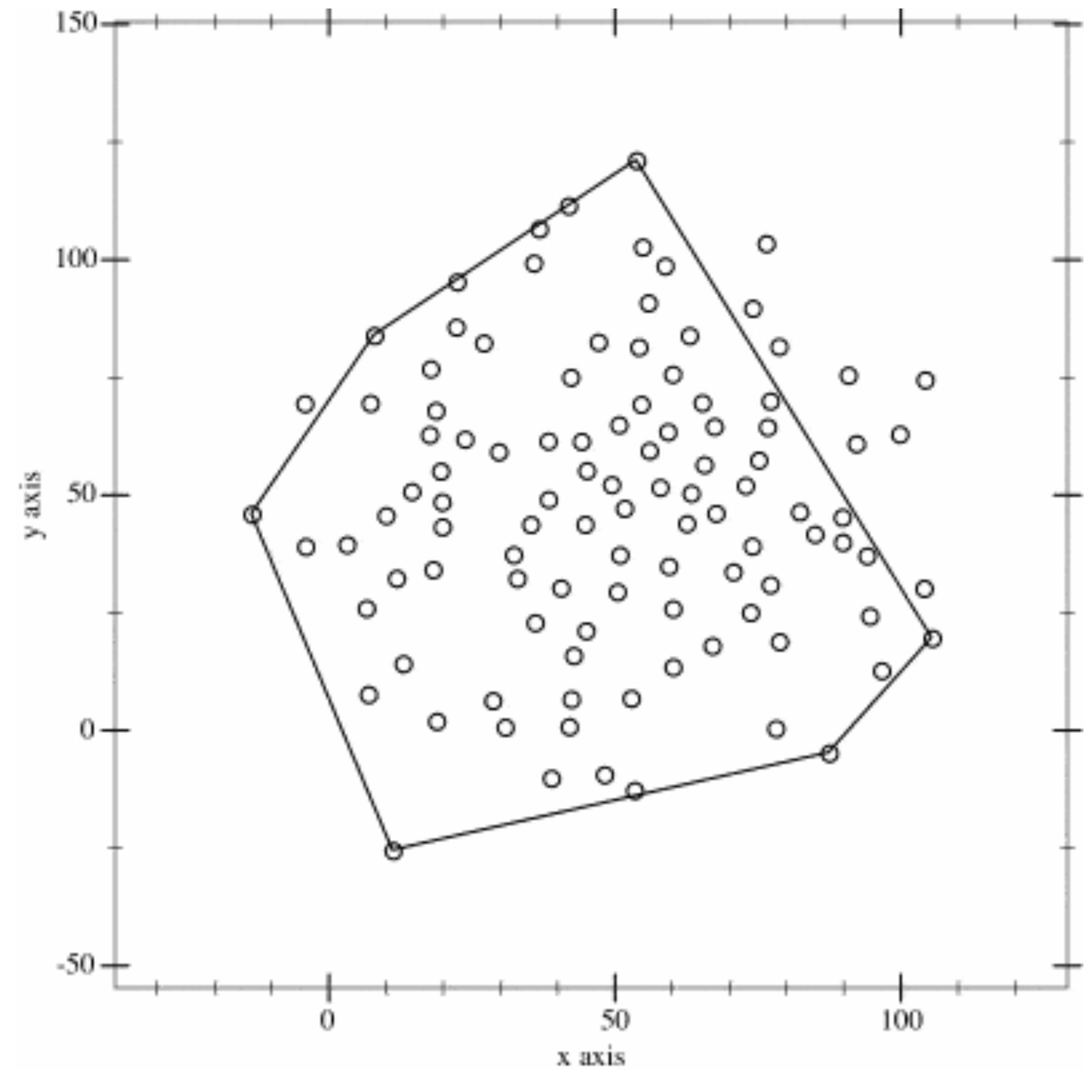

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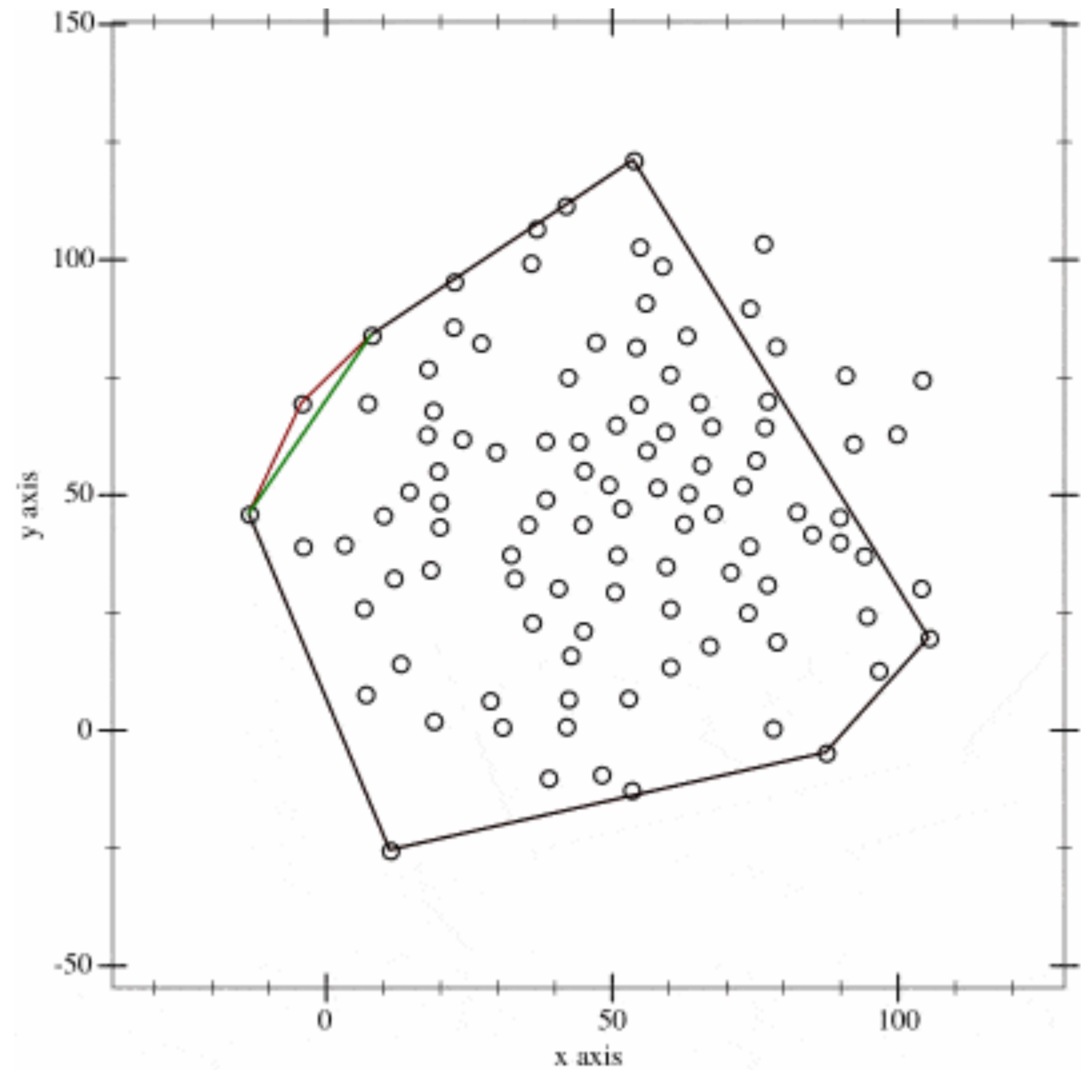

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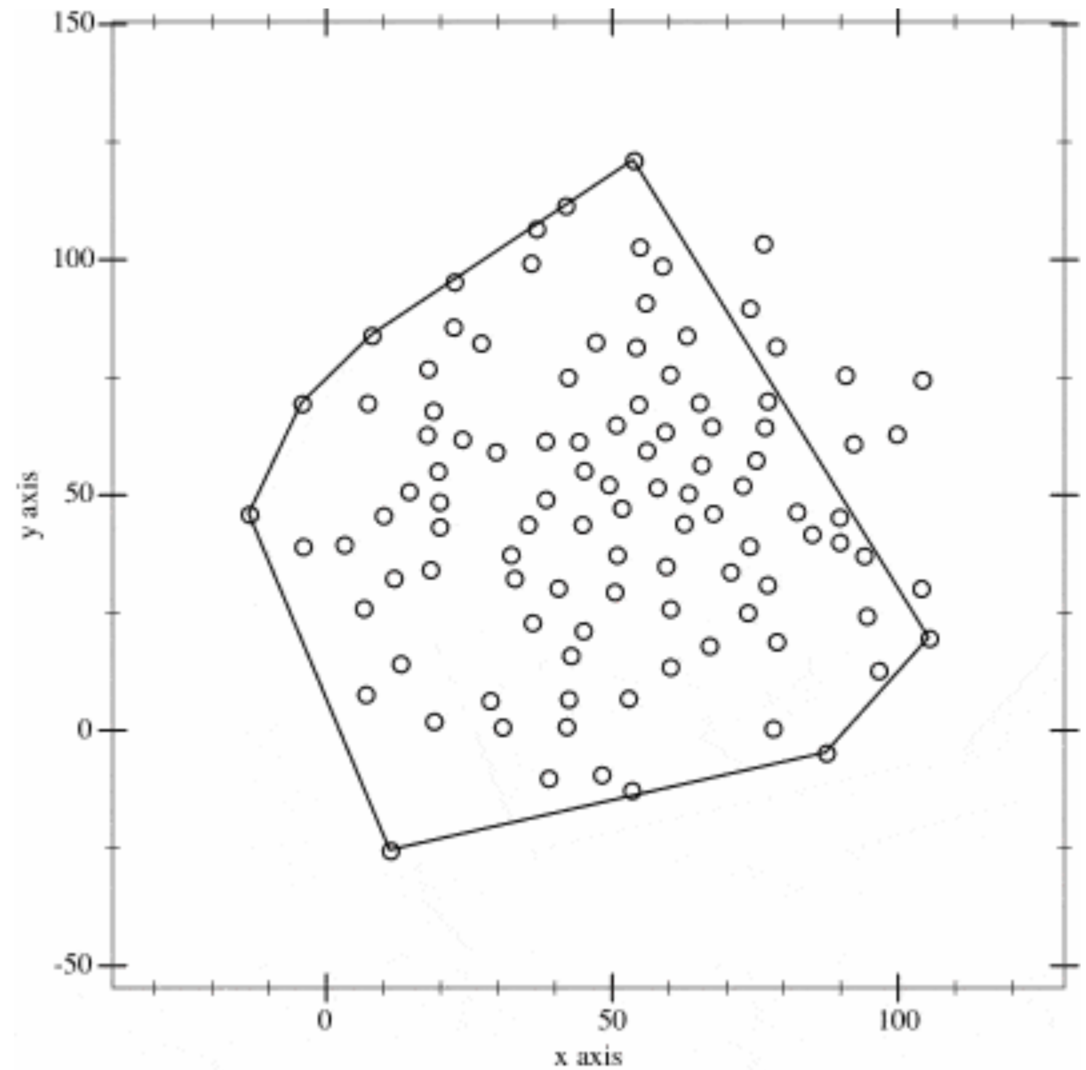

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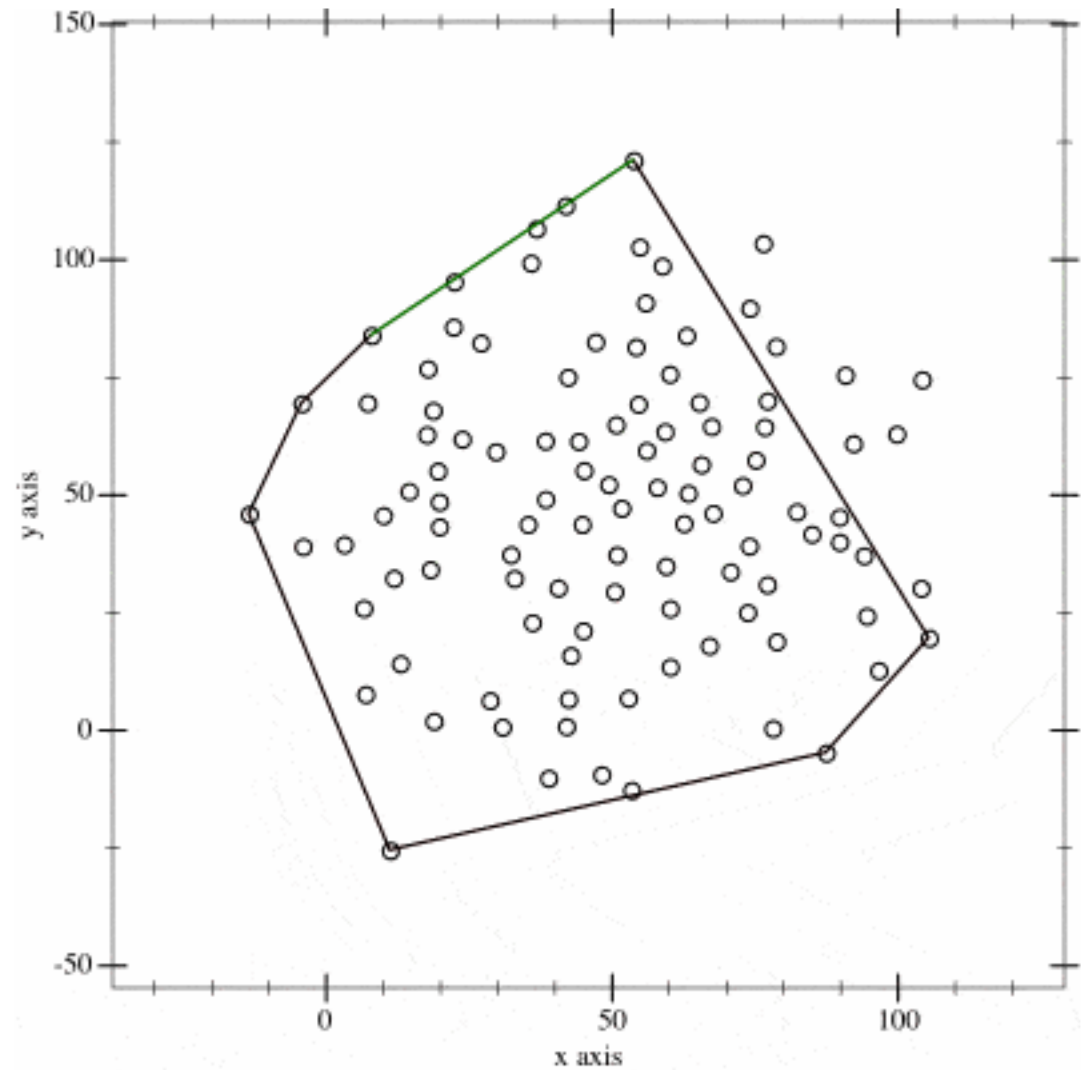

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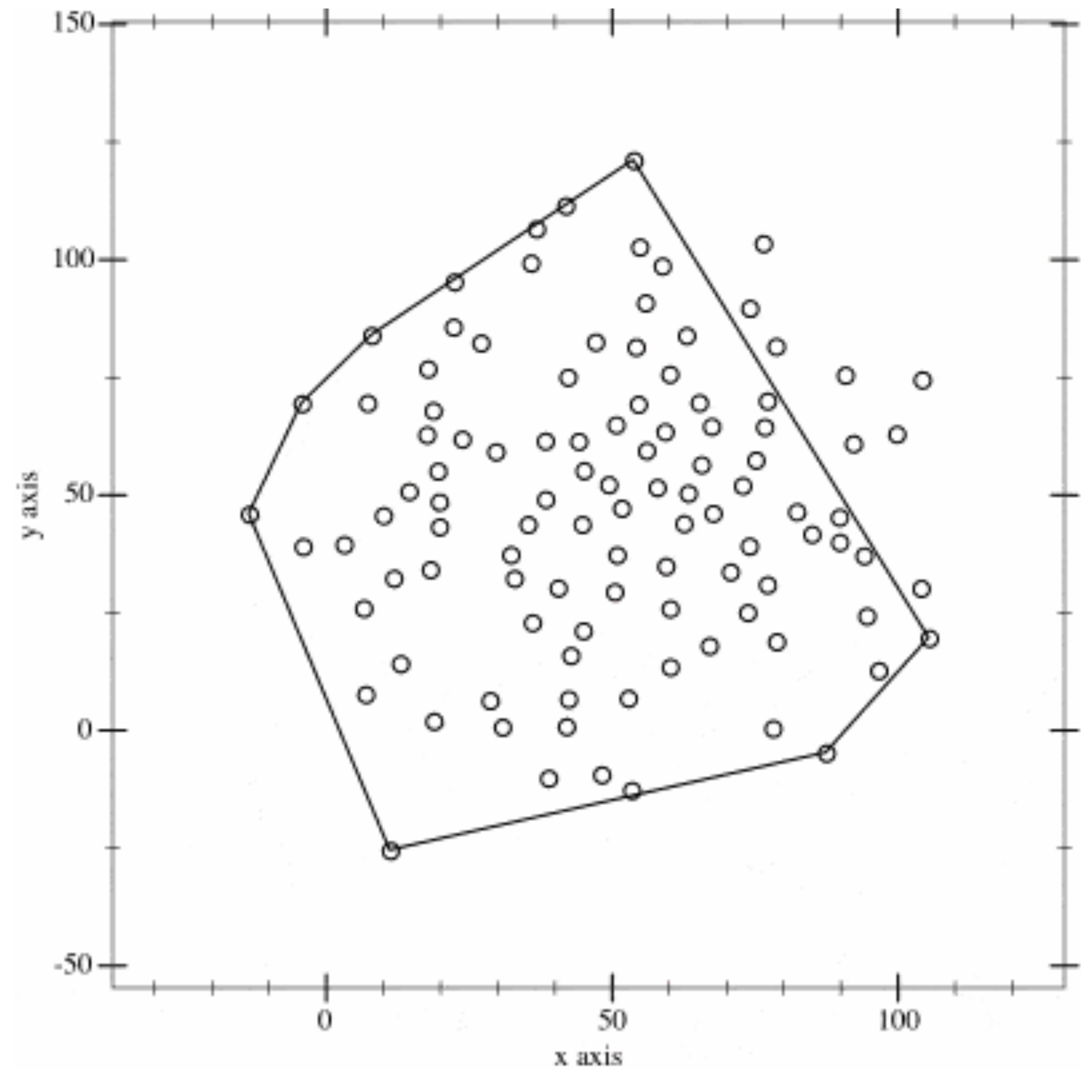

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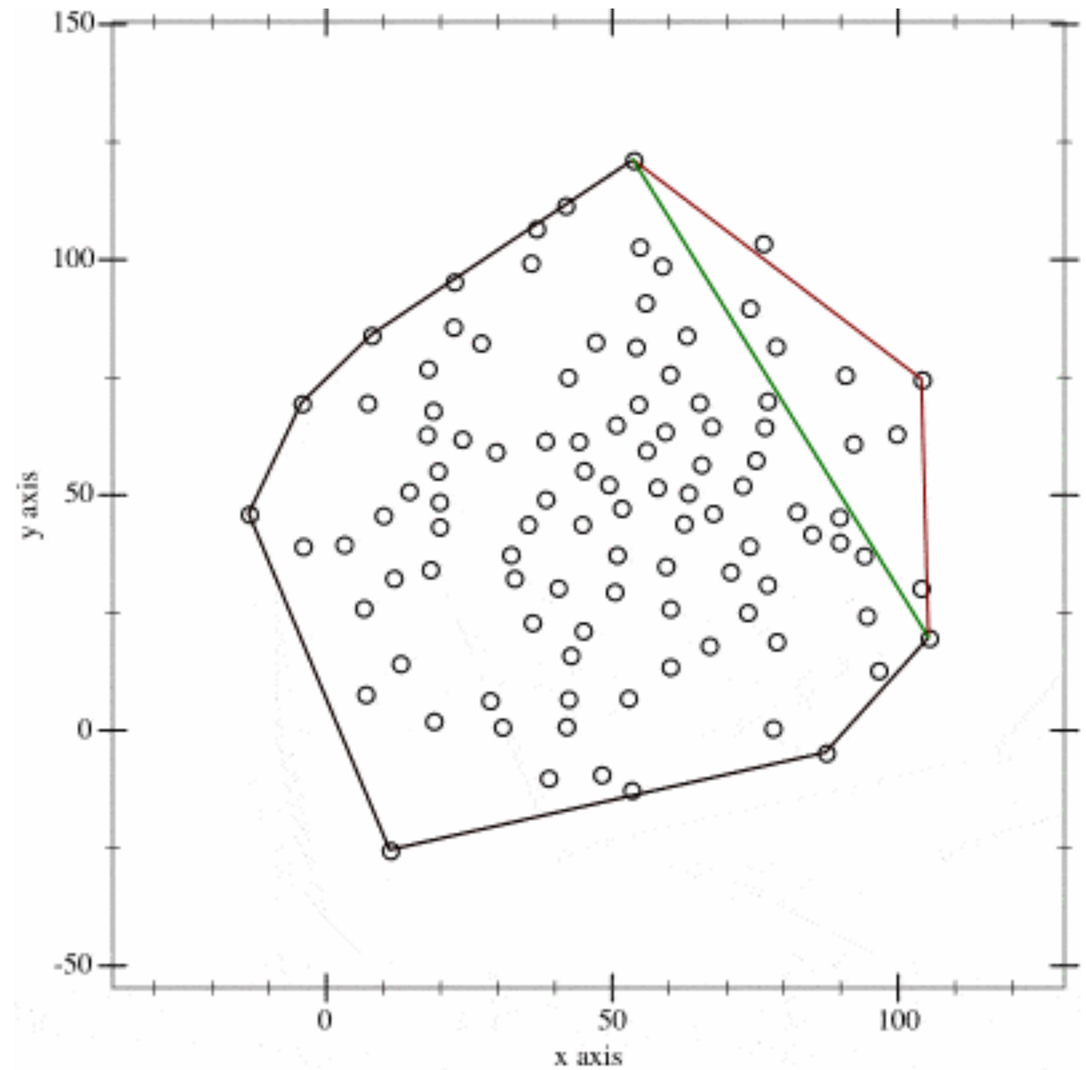

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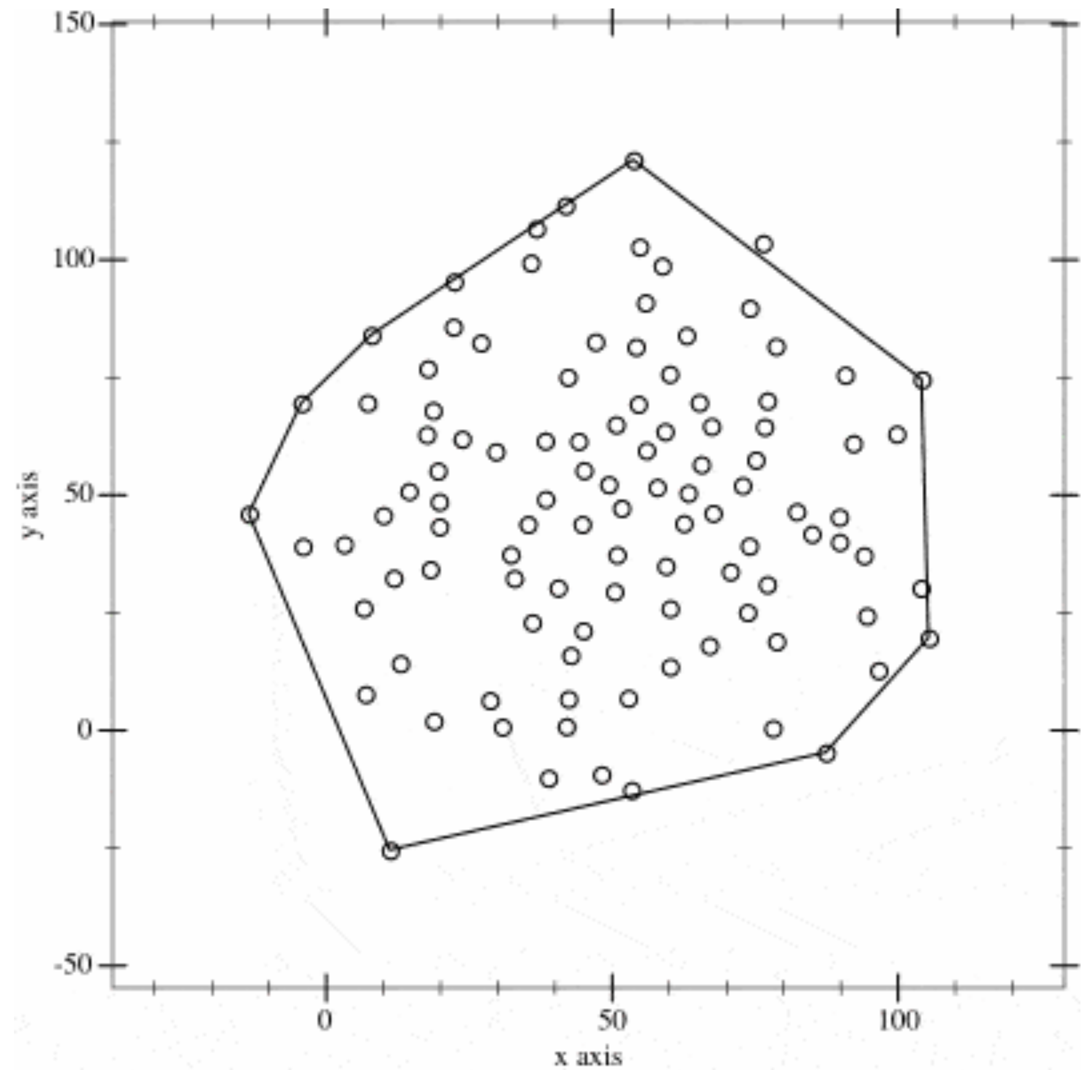

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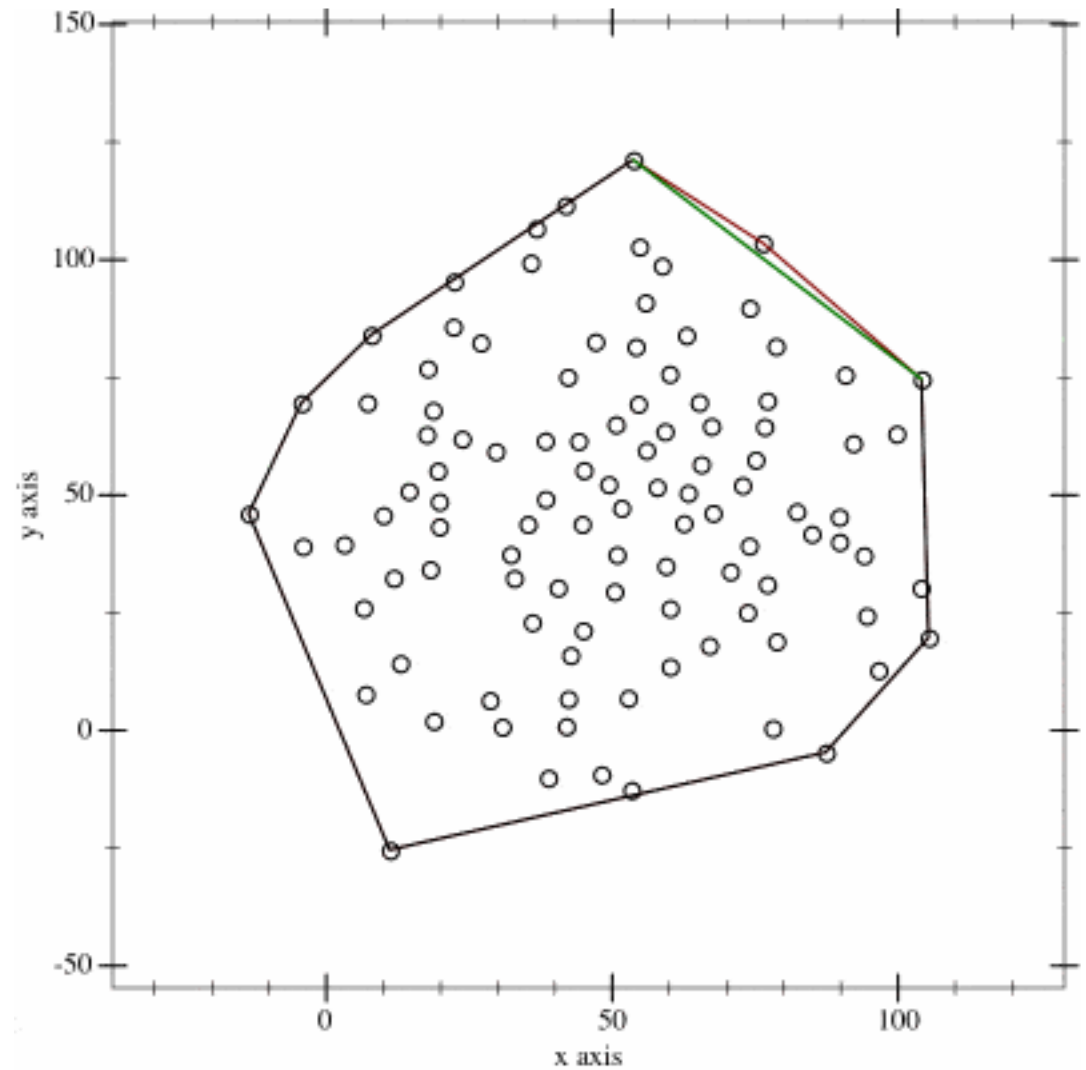

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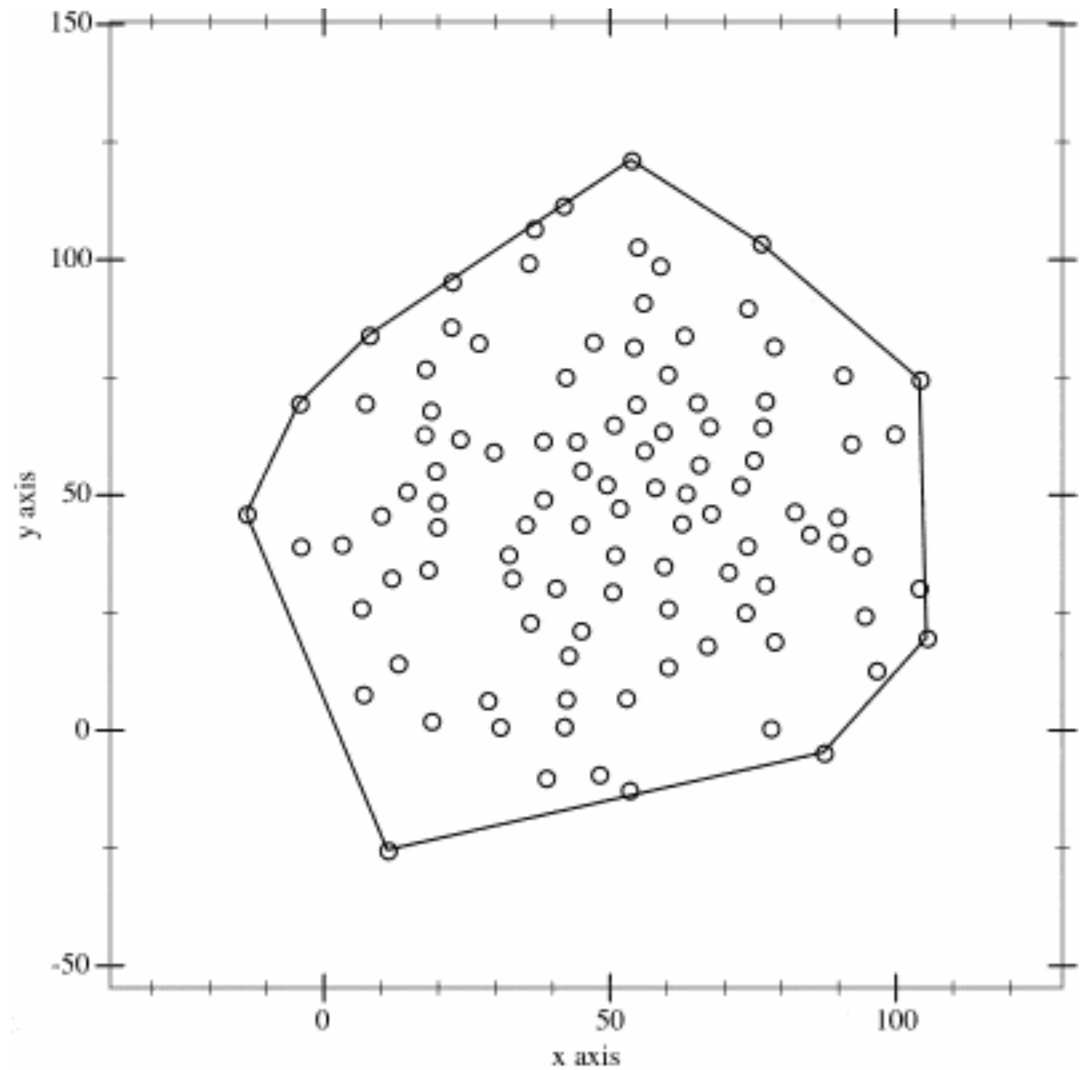

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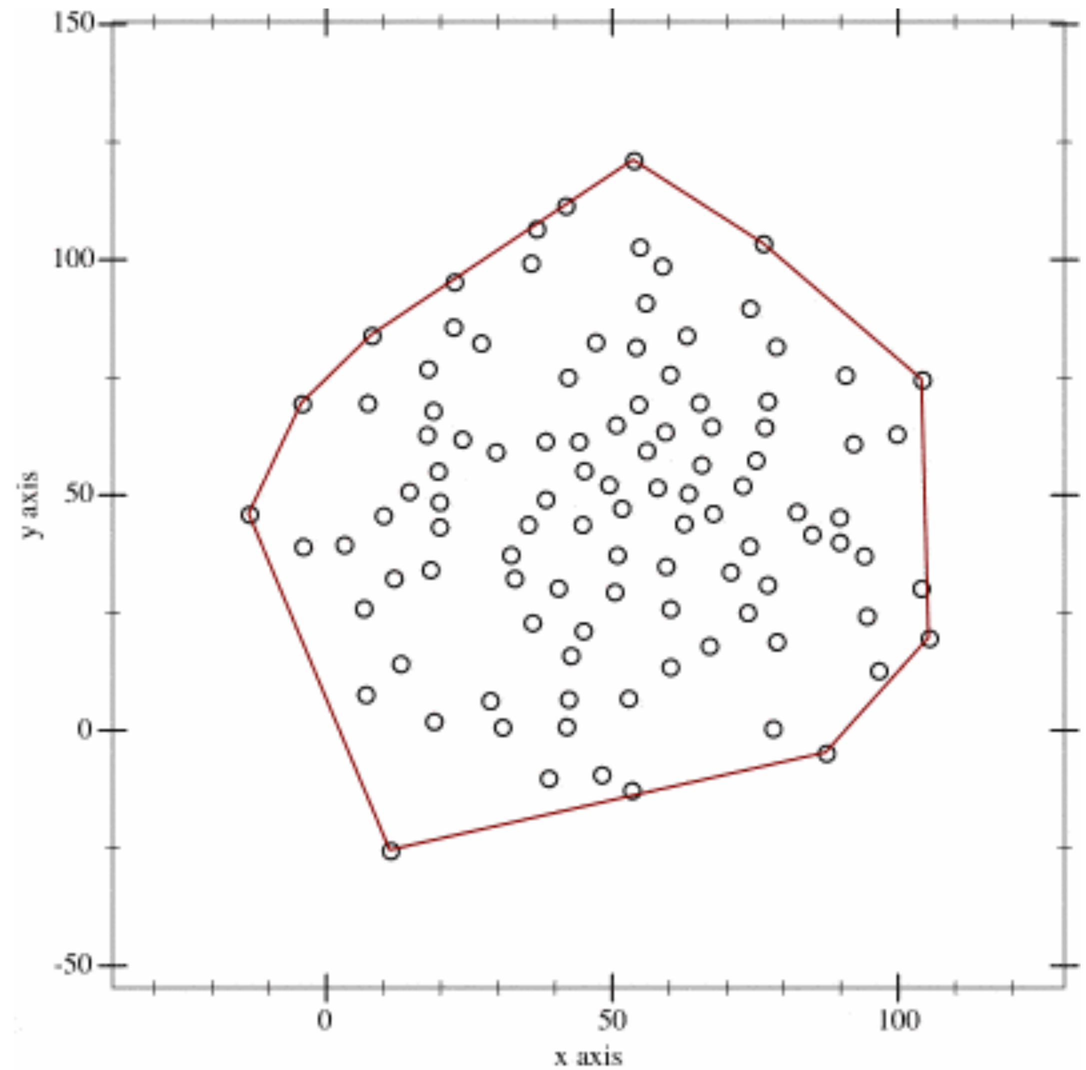

Practice

- Write down the also pseudocode

Solution

- Wiki: https://en.wikipedia.org/wiki/Quickhull

Runtime analysis

- $T(n) = T(n_1) + T(n_2) + O(n)$
  - Worst case $n_1 = n - k$ or $n_2 = n - k$, where k is a small constant (e.g., k=1)
    - $T(n) = O(n^2)$
  - Average case, $n_1 = \alpha n$ and $n_2 = \beta n$, where $\alpha < 1$ and $\beta < 1$
    - $T(n) = O(n \log n)$