Recap: Picard's Thm.

$$y' = f(x,y) \quad ; \quad y(x_0) = y_0 \quad : \underline{IVP}$$

Chk: RHS: $f(x,y)$ in the neighborhood of (x_0, y_0)

① RHS is cont. : existence.

② " " " diff. w.r.t 'dependent variable = y' : uniqueness.

Eg 1. $\dot{x} = x^{1/3}$ $\quad x(0) = 0$

$\dot{x} = \frac{dx}{dt}$

$$x_1 = \begin{cases} C x^{\alpha} \\ 0 \end{cases}$$

Chap 2

2nd order ODE

$$(1)\quad y'' + \underline{p(x)}\, y' + \underline{q(x)}\, y = \underline{r(x)}$$: Standard form. → linear

a) Non-Homog. $(r(x) \not\equiv 0)$

b) Hom. $(r(x) \equiv 0)$

Eg1: $y'' + 25y = e^{-x}\cos x$: 2nd ord. / const. coeff. / non H.

Eg2: $xy'' + y' + xy = 0$

$$\Rightarrow \quad y'' + \frac{1}{x}y' + y = 0 \qquad (x \neq 0).$$

Eg3: $y''y + y'^2 = 0$

Thm (Superposition Principle): For homo. / 2nd order linear ODEs:

$$y'' + p(x)y' + q(x)y = 0 \longrightarrow \text{I}$$

↳ y_1, y_2 : solutions.

then $y(x) = c_1 y_1 + c_2 y_2$ (c_1, c_2: constants)

is also a solutions.

Eg 4: $y'' + y = 0$

$$\left.\begin{aligned} y_1 &= \cos x \\ y_2 &= \sin x \end{aligned}\right\} \text{ Chk: solutions}$$

$\Rightarrow y = A\cos x + B\sin x$: Sol^n

Eg5:

$$y'' + y = 1$$

$$\left.\begin{aligned} y_1 &= \cos x + 1 \\ y_2 &= \sin x + 1 \end{aligned}\right\} \quad \underline{\text{Chk}}:$$

$y = c_1 y_1 + c_2 y_2$: solution ? $\underline{\underline{X}}$

Eg6:

$$y''y - xy' = 0$$

$$\left.\begin{aligned} y_1 &= x^2 \\ y_2 &= 1 \end{aligned}\right\} \quad \underline{\underline{\text{Chk}}}$$

$y = c_1 y_1 + c_2 y_2$ not a $\underline{\underline{\text{sol}^n}}$ $\forall c_1, c_2$

Eg 7.

$$y'' - y = 0$$

$y_1 = e^x$ → Basis

$y_2 = e^{-x}$ } Sol^n

Chk : y_1, y_2 are L.I. since $\frac{y_1}{y_2} \neq$ const.

n^{th} order ODE : Exactly n L.I. sol^{n}s : Basis for the sol^n-space

Eg 8. Find the basis $(x^2 - x)y'' - xy' + y = 0$ if $y_1 = x$

y_2 (L.I.) $= u(x)\, y_1(x)$ ⟵ Put in ODE

$$y_2' = u' y_1 + u y_1' \longleftarrow$$

$$y_2'' = u'' y_1 + 2u' y_1' + u y_1'' \longleftarrow$$

$$\boxed{u = \ln|x| + \frac{1}{x}}$$

$$\Rightarrow \quad y_2 = u y_1 = x \ln|x| + 1$$

PP: (2.1) : 6, 14, 22, 24

2-2 Hom. 2nd order const. coeff.

ODE:

$$y'' + ay' + by = 0$$

Sol^n: Assume $y = e^{\lambda x}$

$$\underbrace{(\lambda^2 + a\lambda + b)}_{=0} \underbrace{e^{\lambda x}}_{\neq 0} = 0$$

$$\lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$a^2 - 4b > 0$: 2 real.

$a^2 - 4b = 0$: 2 real.

$a^2 - 4b < 0$: roots. Comp.

Eg).

$$y'' + 0.4y' + 9.04y = 0 \leftarrow$$

$$\begin{cases} y(0) = 0 \\ y'(0) = 3 \end{cases}$$

$$y = e^{\lambda x}$$

$$\left.\begin{matrix} A = 0 \\ B = 1 \end{matrix}\right\}$$

I.C.S

PP (2.2):

8, 13, 19, 26, 34A

Sol^n:

Characteristic Eq^n $\left\{ \lambda^2 + 0.4\lambda + 9.04 = 0 \right.$

$$\boxed{\lambda = -0.2 \pm 3i}$$

$$y = c_1 e^{(-0.2+3i)x} + c_2 e^{(-0.2-3i)x}$$

$$= e^{-0.2x}\left[c_1 e^{3ix} + c_2 e^{-3ix}\right]$$

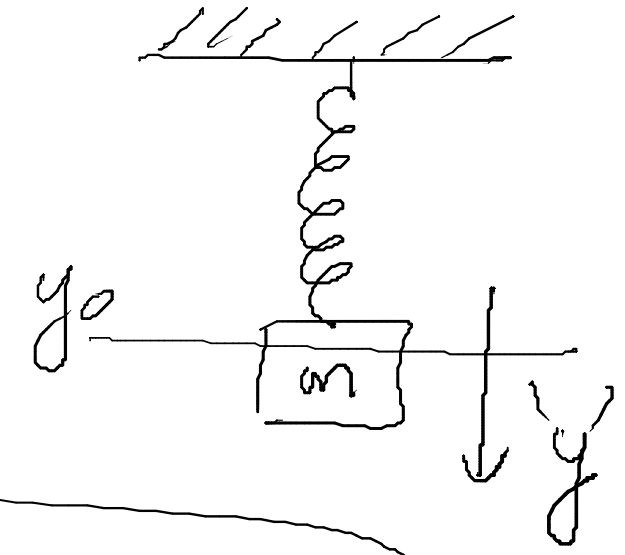
$$= e^{-0.2x}\left[\overbrace{(c_1 + c_2)}^{A}\cos 3x + \overbrace{i(c_1 - c_2)}^{B}\sin 3x\right]$$

$$y = e^{-0.2x}\left[\sin 3x\right]$$

2.4 Modeling "mass-spring" Oscillations

Object of mass : 'm'

Vertical displacement : 'y'



Newton's 2nd law:

$$\Rightarrow \quad m\frac{dv}{dt} = \boxed{m\frac{d^2y}{dt^2} = -ky}$$

$$\Rightarrow \quad y'' + \frac{k}{m}y = 0 \longrightarrow \text{I}$$

$$\boxed{y(x) = A\cos\omega t + B\sin\omega t \qquad \omega = \sqrt{\frac{k}{m}}}$$

Eg1. An iron ball of weight $W = 98\,N$ stretches a spring by $1.09\,m$. How many cycles/min will this spring-mass system execute? What will be its motion if we pull down the weight by an additional 16 cm with zero initial velocity.

Sol^n: ~~1.09~~ $W = kx$

At equi $98 = k(1.09)$

$$k = \frac{98\,N}{1.09\,m} = 90\ N/m$$

$$\boxed{y = A\cos\omega t + B\sin\omega t}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{90}{10}} = 3\ /\text{sec}$$

A, B to be found using $y(0) = 0.16$, $y'(0) = 0$

Damped oscillations: $F_d \propto$ velocity

$$\Rightarrow \boxed{F_d = -cy'} \qquad \underline{\underline{c > 0}}$$

$$\underline{my'' = -ky - cy'}$$

$$\Rightarrow my'' + cy' + ky = 0$$

Sol^n: $y = e^{\lambda t}$

$$\underbrace{\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0}_{\text{Char. Eq}^n}$$

$$\lambda = -\frac{c}{2m} \pm \frac{1}{2m}\sqrt{c^2 - 4mk}$$

Case I: $c^2 - 4mk > 0$: Overdamped.

Case II: $c^2 - 4mk = 0$ Critically damp.

III : $c^2 - 4mk < 0$ Underdamped.

Eg2. find sol^n to damped case with $c = 100$ kg/s, $c = 60$ kg/s, $c = 10$ kg/s.

with $m = 90$, $k = 10$

Sol^n: $c^2 - 4mk = (100)^2 - 4(90)(10)$

$= 6400 > 0$

① $y = e^{\lambda t} = e^{\left(-\frac{c}{2m} \pm \sqrt{c^2 - 4mk}\right)t}$

$= e^{-\frac{100}{20}t}\left[Ae^{4t} + Be^{-4t}\right]$

② $c = 60$ kg/s $\quad y = e^{-\frac{60}{20}t}\left[A + Bt\right]$

③ $c = 10 \quad y = e^{-\frac{10}{20}t}\left[A\cos\sqrt{\frac{35}{4}}t + B\sin\sqrt{\frac{35}{4}}t\right]$

PP: (2.4): 1, 4, 6, 8, 12