Lecture-12 Section-6.2

☆ Laplace Transform, Inverse Transform

$$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$$

Thm: S-shifting thm.

$$\mathcal{L}(f) = F(s)$$

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

Existence & Uniqueness:

Existence:

① $|f(t)| \leq M e^{kt}$

② $f(t)$ is piecewise cts.

Laplace Transf of derivative & Integral :

Thm ① Laplace Transform of derivative

(1) $\mathcal{L}(f') = s\mathcal{L}(f) - f(0)$ ✓

(2) $\mathcal{L}(f'') = s^2\mathcal{L}(f) - s f(0) - f'(0)$

Pf ① $\mathcal{L}(f') = \int_0^\infty e^{-st} f' \, dt$

$$= e^{-st} f(t) \Big|_0^\infty - (-s) \underbrace{\int_0^\infty e^{-st} f(t)\, dt}_{\mathcal{L}(f)}$$

$$= -f(0) + s\mathcal{L}(f)$$

② $\mathcal{L}(f'') =$ Replace f by f' in part ①

Laplace Transf of derivative of nth order.

Let $f, f', \ldots f^{(n-1)}$ cts. $\forall t \geq 0$, satisfy growth restriction (ie $\leq Me^{kt}$)

Furthermore $f^{(n)}$: piecewise cts.

$$\mathcal{L}(f^{(n)}) = s^n \mathcal{L}(f) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

Eg ①

$f(t) = t\sin(\omega t)$, $f(0) = 0$ ✓

$\mathcal{L}(f'') = ?$

$f' = \sin(\omega t) + \omega t \cos(\omega t)$, $f'(0) = 0$ ✓

$f'' = 2\omega \cos \omega t - \omega^2 t \sin(\omega t)$

$$\mathcal{L}(f'') = 2\omega \mathcal{L}(\cos \omega t) - \omega^2 \mathcal{L}(t \sin \omega t)$$

$$= 2\omega \frac{s}{s^2+\omega^2} - \omega^2 \mathcal{L}(f) = s^2 \mathcal{L}(f) - s\cancel{f(0)}^{\,0} - \cancel{f'(0)}^{\,0}$$

$$(s^2+\omega^2)\mathcal{L}(f) = \frac{2\omega s}{s^2+\omega^2}$$

$$\mathcal{L}(f) = \frac{2\omega s}{(s^2+\omega^2)^2}$$

Eg ② $\mathcal{L}((\cos\omega t)'')$

$$(\cos\omega t)'' = \underline{-\omega^2\cos\omega t}$$

If $f = \cos\omega t$, then $\mathcal{L}(f'') = -\omega^2\mathcal{L}(f)$

$$\Downarrow$$

$$s^2\mathcal{L}(f) - s\underbrace{\cancel{f(0)}}_{1} - \underbrace{\cancel{f'(0)}}_{0}$$

Thm ② Laplace Transform of Integrals:

Let $F(s) = \mathcal{L}(f(t))$, piecewise cts for $\underline{t \geq 0}$ & satisfy growth restriction, then $s \geq 0$, $s > k$, $t \geq 0$

$$\boxed{\mathcal{L}\left[\int_0^t f(z)\,dz\right] = \frac{1}{s}F(s)}$$

$\mathcal{L}(f) = \int_0^\infty$

or $\underline{\int_0^t f(z)\,dz = \mathcal{L}^{-1}\left[\frac{1}{s}F(s)\right]}$

Pf: Let $g(t) = \int_0^t f(\tau)\,d\tau$

$$\Rightarrow |g(t)| = \left|\int_0^t f(\tau)\,d\tau\right| \le \int_0^t |f(\tau)|\,d\tau \le \int_0^t M e^{\tau k}\,d\tau$$

$$= \frac{M}{k}\left(e^{tk}-1\right) \le \frac{M}{k} e^{tk} \quad (k>0)$$

$\Rightarrow$ $g(t)$ also satisfy growth restriction

Also $g'(t) = f(t)$

$$\Rightarrow \mathcal{L}(f) = \mathcal{L}(g') = s\mathcal{L}(g) - \underbrace{g(0)}_{0}$$

$$= s\mathcal{L}(g)$$

$$\Rightarrow \mathcal{L}(g) = \frac{1}{s}\mathcal{L}(f) = \frac{F(s)}{s}$$

$\square$

Ex. 3 Find the inverse of $\frac{1}{s(s^2+\omega^2)}$ & $\frac{1}{s^2(s^2+\omega^2)}$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{\sin\omega t}{\omega}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+\omega^2)}\right] = \int_0^t \frac{\sin\omega\tau}{\omega}\,d\tau = \checkmark$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+\omega^2)}\right] = \int_0^t \underbrace{\frac{1}{\omega^2}(1-\cos\omega\tau)}\,d\tau = \checkmark$$

<u>IVP using Laplace Transform:</u>

$$y'' + ay' + by = r(t), \quad y(0) = k_0, \; y'(0) = k_1$$

<u>Idea:</u> ① Let $Y = \mathcal{L}(y)$

$$\Rightarrow \mathcal{L}[y''] + a\mathcal{L}[y'] + b\mathcal{L}(y) = \left(s^2Y - s\,y(0) - y'(0)\right) + a\left(sY - y(0)\right) + bY = R(s)$$

$$(s^2+as+b)Y - (a+s)y(0) - y'(0) = R(s)$$

$$\boxed{Y = \frac{R(s) + (a+s)y(0) + y'(0)}{(s^2+as+b)}}$$

② $y = \mathcal{L}^{-1}[Y]$

Q ① solve $y'' - y = t$; $y(0)=1$; $y'(0)=1$

comparing with $y'' + ay' + by = r(t)$, $\left.\begin{array}{l} a=0 \\ b=-1 \\ r(t)=t \end{array}\right\}$

$$Y = \frac{sy(0) + y'(0) + 1/s^2}{s^2-1} = \underbrace{\frac{1}{s-1} + \frac{1}{s^2(s^2-1)}} = \underbrace{\frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}}$$

$$y = \mathcal{L}^{-1}[Y] = \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = \checkmark$$

Ex ⑤ Solve $y'' + y' + 9y = 0$; $y(0) = 0.16$, $y'(0) = 0$

$$\mathcal{L}[\text{Lhs}] = \left[s^2 Y - s(0.16) - 0\right] + \left[sY - 0.16\right] + 9Y = 0$$

$$\Rightarrow \quad (s^2 + s + 9)Y = 0.16(s+1)$$

$$\Rightarrow \quad Y = \frac{0.16(s+1)}{(s^2+s+9)} = \frac{(0.16)\left(s+\frac{1}{2}\right) + 0.08}{\left(s+\frac{1}{2}\right)^2 + \frac{35}{4}}$$

$$\Rightarrow y = \mathcal{L}^{-1}(Y) = e^{-t/2}\left[0.16\cos\sqrt{\tfrac{35}{4}}\,t + \frac{0.08}{\sqrt{35/4}}\sin\sqrt{35/4}\,t\right]$$

Ex. $y'' + y = 2t$, $y(\pi/4) = \pi/2$, $y'(\pi/4) = 2\sqrt{2}$

I.C. at $t_0 = \pi/4$

we $\boxed{\tilde{t} = t - t_0}$ $\Rightarrow$ I.C. at $\tilde{t} = 0$

$t = \tilde{t} + t_0 = \tilde{t} + \pi/4$

$$\tilde{y}'' + \tilde{y} = 2\left(\tilde{t} + \pi/4\right) \quad ; \quad \tilde{y}(0) = \pi/2 , \quad \tilde{y}'(0) = 2\sqrt{2}$$

$$\mathcal{L}\left[\ \downarrow\ \right] \Rightarrow \quad \tilde{Y}(s) = \frac{2}{(s^2+1)s^2} + \frac{\pi/2}{(s^2+1)s} + \frac{\pi/2\, s}{s^2+1} + \frac{2\sqrt{2}}{s^2+1}$$

$$\boxed{\tilde{y}(\tilde{t}) = 2\tilde{t} + \pi/2 - \sqrt{2}\sin\tilde{t}}$$

$$\boxed{\begin{aligned} 2\tilde{t} + \pi/2 &= 2(t - \pi/4) + \pi/2 = 2t \\ \sin\tilde{t} &= \sin(t - \pi/4) = \frac{\sin t - \cos t}{\sqrt{2}} \end{aligned}}$$

$$\rightarrow \underbrace{2t - (\sin t - \cos t)}_{= y(t)}$$

Prac. Prob: 7, 12, 17, 24, 33