5.3 Legendre Polynomials

Applications: Laplace Eq^n / BVP / Elliptic

(I) $$\underbrace{(1-x^2)}_{h} y'' \underbrace{- 2x}_{p} y' + \underbrace{n(n+1)}_{q} y = 0$$

Assume $y = \sum a_m x^m$

$$\Rightarrow (1-x^2) \sum_{m=2}^{\infty} (a_m)\, m(m-1)\, x^{m-2} - 2x \sum_{m=1}^{\infty} (a_m)\, m\, x^{m-1} + n(n+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow (1-x^2) \sum_{s=0}^{\infty} a_{s+2} (s+2)(s+1) x^s + (-2x) \sum_{s=0}^{\infty} a_{s+1} (s+1) x^s + n(n+1) \sum_{m=0}^{\infty} a_m x^s = 0$$

For x^0: $\quad 2a_2 + n(n+1)a_0 = 0$

For $x^{(1)}$: $\quad 3 \cdot 2\, a_3 - 2a_1 + n(n+1)a_1 = 0$

For $s = 2, 3, \ldots$: $\quad (s+2)(s+1)\, a_{s+2} + \left[-s(s-1) - 2s + n(n+1)\right] a_s = 0$

$$a_2 = -\frac{n(n+1)}{2} a_0$$

$$a_4 = -\frac{(n-2)(n+3)}{4 \cdot 3} a_2$$

$$= +\frac{(n-2)\, n\, (n+1)(n+3)}{4!} a_0$$

$$a_3 = -\frac{(n-1)(n+2)}{3!} a_1$$

$$a_5 = -\frac{(n-3)(n+4)}{5!} a_3$$

$$= \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$$

$$y(x) = a_0 y_1 + a_1 y_2$$

$$\left[y_1 = 1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-1)\,n\,(n+1)(n+3)}{4!} x^4 - \cdots \right]$$

$$y_2 = x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots$$

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y_1, y_2 : L.I.

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1 Special case : n : non-neg. integer.

$$S = n \Rightarrow a_{n+2} = 0 = a_{n+4} = a_{n+6}$$

1a $n =$ even y_1 : finite order poly.

1b $n =$ odd y_2 : " " "

(1) : $y(x) = P_n(x)$: Legendre poly.

$$(s+2)(s+1)\, a_{s+2} = -\left[\underbrace{s(s-1) + 2s - n(n+1)}_{s(s+1)}\right] a_s .$$

$$\cancel{a_{s+2} = -\left[\right]}$$

$$\cancel{a_s = -\frac{\left[(s-2)(s-1) - n(n+1)\right]}{s(s-1)}\, a_{s-2}}$$

$$\Rightarrow \quad a_s = -\frac{\left[(s+2)(s+1)\right]}{\underbrace{s(s+1) - n(n+1)}_{\tau}}\, a_{s+2}$$

$$\boxed{a_s = +\frac{(s+2)(s+1)\, a_{s+2}}{(n-s)(n+s+1)}}$$

$a_i \neq 0$

$i = 0, 1, \ldots n$

$s \longrightarrow (n-2)$!

$$a_{n-2} = \frac{n(n-1)}{2(2n-1)} a_n .$$

$s \longrightarrow (n-4)$

$$a_{n-4} = \quad . \quad . \quad . \quad a_{n-2}$$

$$= \frac{(2n-4)!}{2^n \, 2! \, (n-2)! \, (n-4)!} a_n$$

$$\boxed{a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n (m!)(n-m)!(n-2m)!}}$$

$$\boxed{P_n(x) = y(x) = \sum_{m=1}^{M} a_{n-2m} x^{n-2m}}$$

$$M = \frac{n}{2}$$

or $\frac{(n-1)}{2}$ whichever is an integer.

P.P (5.3): 4, 5, 8, 13

Q. What about the sol^n existence in case $p(x)$, $q(x)$ are not analytic?

Ans. A series sol^n still exists. (P.S. multiplied with frac. power or log)

(I) $$y'' + p(x)y' + q(x)y = 0$$

5.4 Frobenius Method.

Thm [Frobenius Meth.]: Let $b(x)$, $c(x)$ be any fns. that are analytic at $x=0$. Then the ODE:

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

has at least sol^n of the form:

$$y(x) = x^r \sum a_m x^m = x^r \left[a_0 + a_1 x + a_2 x^2 \ldots \right]$$

Eg 1. $x^2 y'' + x\, b(x)\, y' + c(x)\, y = 0.$

Expand
$$\begin{cases} b(x) = b_0 + b_1 x + b_2 x^2 + \ldots \\ c(x) = c_0 + c_1 x + c_2 x^2 + \ldots \end{cases}$$

Assume: $y = \sum_{m=0}^{\infty} a_m x^{m+r}$

$$y' = \sum_{m=0}^{\infty} a_m (m+r) x^{m+r-1}$$

$$y'' = \sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r-2}$$

$$\left[x^2 y'' + x\, b(x)\, y' + c(x)\, y = 0 \right]$$

$$x^r \underbrace{\left[r(r-1) + b_0 r + c_0 \right]}_{=0} a_0 = 0$$

Indicial Equation

Eg 1. Euler – Cauchy Eq^n.

$$x^2 y'' + b_0 x y' + c_0 y = 0$$

$$y = \sum a_m x^{r+m}$$

$$x^r \underbrace{\left[r(r-1) + b_0 r + c_0 \right]}_{=0} a_0 = 0$$

Case 1: $r_1 \neq r_2$

Case 2: $r_1 = r_2$

$$y_2 = (y_1) \ln(x)$$

$$x^r / x^r \ln(x)$$

Eg 2. $x(x-1)y'' + (3x-1)y' + y = 0$

Sol^n: $y = \sum a_m x^{m+r}$

$$\Rightarrow x(x-1)\sum a_m (m+r)(m+r-1)x^{m+r-2}$$
$$+(3x-1)\sum a_m (m+r)x^{m+r-1}$$
$$+\sum a_m x^{m+r} = 0$$

$$\Rightarrow \text{(A)} \quad \sum_0^\infty (m+r)(m+r-1)x^{m+r} - \sum_0^\infty (m+r)(m+r-1)a_m x^{m+r-1}$$
$$+3\sum a_m (m+r)x^{m+r} - \sum a_m (m+r)x^{m+r-1}$$
$$+\sum a_m x^{m+r} = 0$$

$x^{(r-1)}$: $[-r(r-1) - r]a_0 = 0 \Rightarrow r = 0$ double root

Use $r=0$ in (A):

Coeff of x^s (Sub $s=m-1$ where necessary)

$$s(s-1)a_s - s(s+1)a_{s+1} + 3s\,a_s - (s+1)a_{s+1} + a_s = 0$$

$$\Rightarrow (s+1)^2\underbrace{\left[a_s - a_{s+1}\right]}_{\neq 0} = 0$$

$$\Rightarrow \boxed{a_{s+1} = a_s} \qquad \Rightarrow \quad a_0 = a_1 = a_2 = \cdots$$

$$y_1 = x^r \sum_{m=0}^{\infty} a_m x^m = \boxed{a_0 \sum_{m=0}^{\infty} x^m}$$

$$= a_0 \frac{1}{1-x} \qquad |x| < 1$$

Chk: If $|x|<1$ $\quad y_1 = \dfrac{a_0}{1-x}$

VoP $\quad \boxed{y_2 = u(x)\,y_1(x) = a_1 \dfrac{\ln(x)}{1-x}}$

Eg3: $(x^2-x)y'' - xy' + y = 0$

Sol^n: Coeff x^{r-1} : $r(r-1)=0$

$r = 0, 1$

$r=1$: $\quad a_{s+1} = \dfrac{s^2}{(s+2)(s+1)} a_s \longrightarrow a_1 = a_2 = \cdots a_i = 0$

$$\begin{cases} y_1 = x \\ y_2 = \cancel{(x\ln x + 1)}\left(\ln x + 1/x\right) \end{cases}$$

PP: 4, 8, 12, 16, 19 (5.4)