Introduction to Binomial Trees

A Simple Binomial Model

- A stock price is currently $20
- In three months it will be either $22 or $18

Stock Price = $22

Stock price = $20

Stock Price = $18

A Call Option

A 3-month call option on the stock has a strike price of 21.

Stock Price = \$22
Option Price = \$1

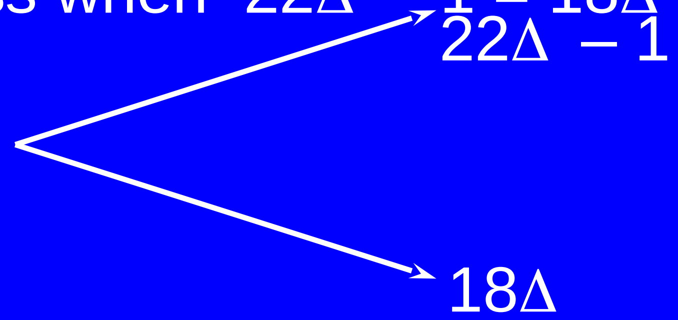
Stock price = \$20
Option Price=?

Stock Price = \$18
Option Price = \$0

Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ shares short 1 call option

- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$



Valuing the Portfolio
(Risk-Free Rate is 12%)

- The riskless portfolio is:

  long 0.25 shares
  short 1 call option

- The value of the portfolio in 3 months is

$$22\times0.25 - 1 = 4.50$$

- The value of the portfolio today is

$$4.5e^{-0.12\times0.25} = 4.3670$$

Valuing the Option

- The portfolio that is

  long 0.25 shares

  short 1 option

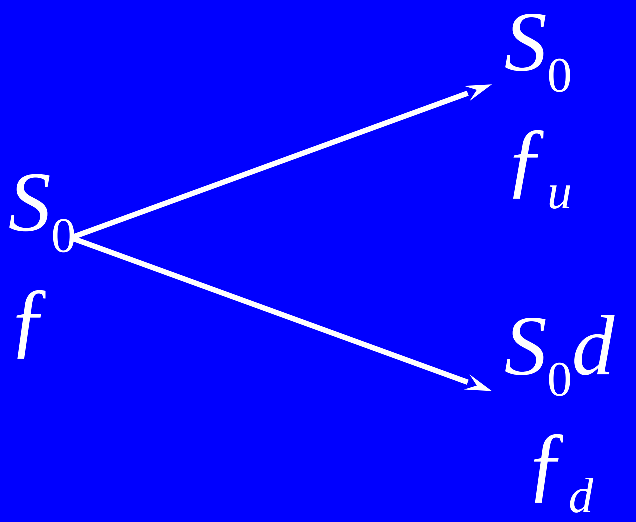
  is worth 4.367
- The value of the shares is

  5.000 (= 0.25×20)
- The value of the option is therefore

  0.633 (= 5.000 – 4.367)

Generalization

- A derivative lasts for time T and is dependent on a stock



Generalization
(continued)

- Consider the portfolio that is long Δ shares and short 1 derivative

$$S_0 u\Delta - f_u$$

$$S_0 - f$$

$$S_0 d\Delta - f_d$$

- The portfolio is riskless when $S_0 u\Delta - f_u = S_0 d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Generalization
(continued)

- Value of the portfolio at time T is

  $S_0 u \Delta - f_u$

- Value of the portfolio today is

  $(S_0 u \Delta - f_u)e^{-rT}$

- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence $f = S_0\Delta - (S_0 u \Delta - f_u)e^{-rT}$

Generalization
(continued)

- Substituting for Δ we obtain

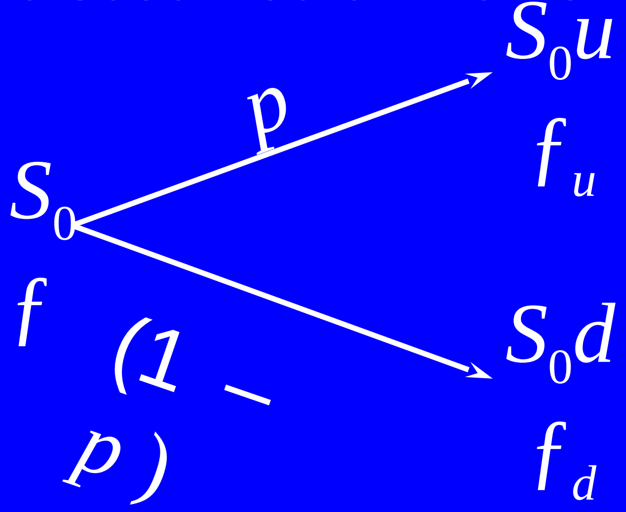
$$f = [\,p f_u + (1 - p) f_d\,] e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

Risk-Neutral Valuation

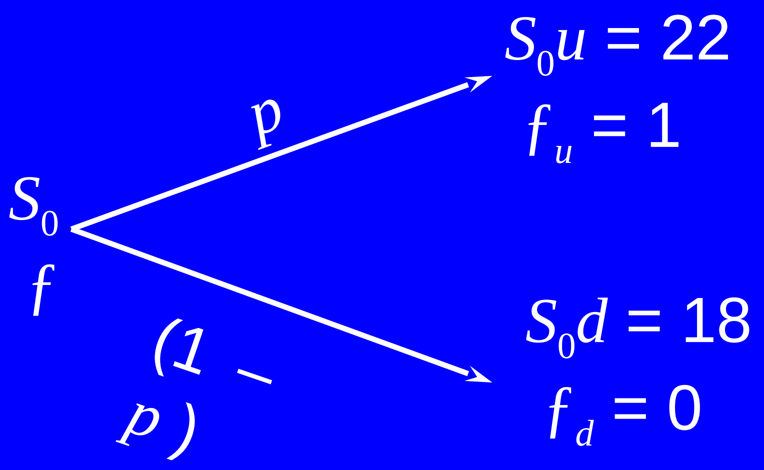
- $f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$
- The variables p and $(1 - p)$ can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



Irrelevance of Stock's Expected Return

When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant

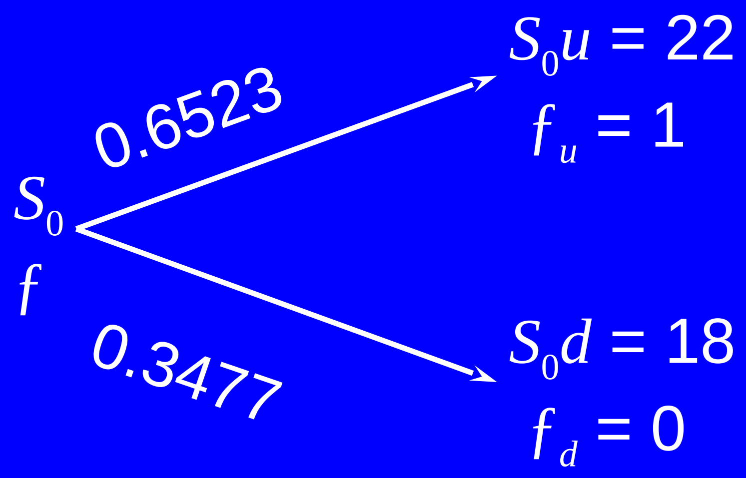
Original Example Revisited



- Since *p* is a risk-neutral probability $20e^{0.12\times0.25} = 22p + 18(1 - p)$; $p = 0.6523$
- Alternatively, we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12\times0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option

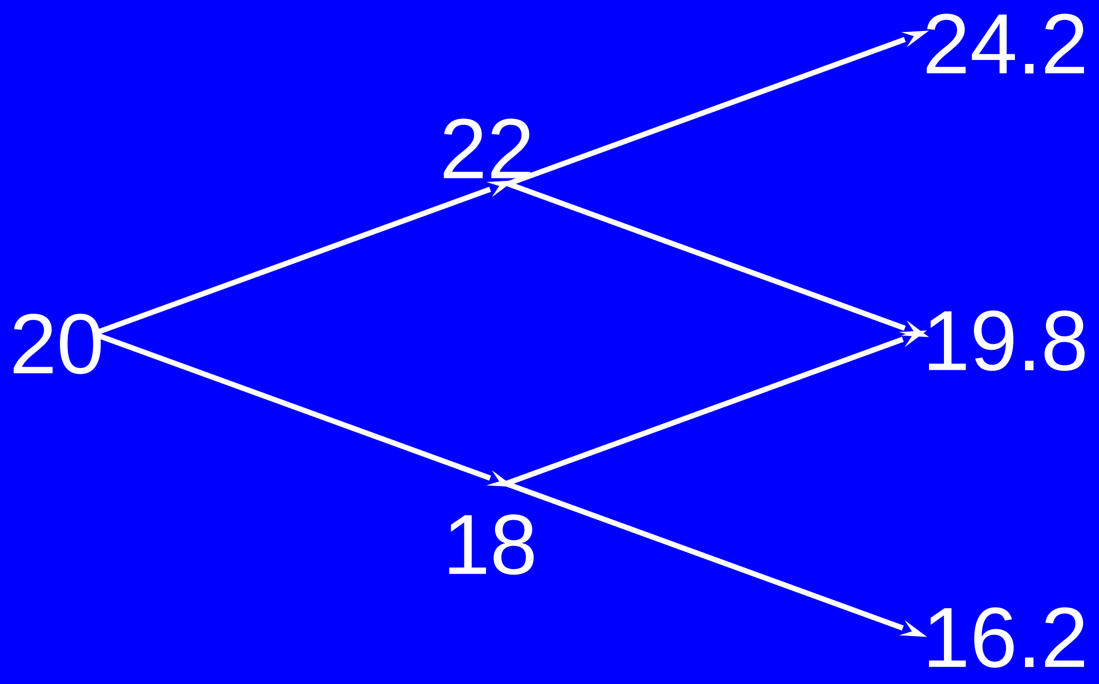


The value of the option is

$$e^{-0.12\times 0.25}[0.6523\times 1 + 0.3477\times 0]$$

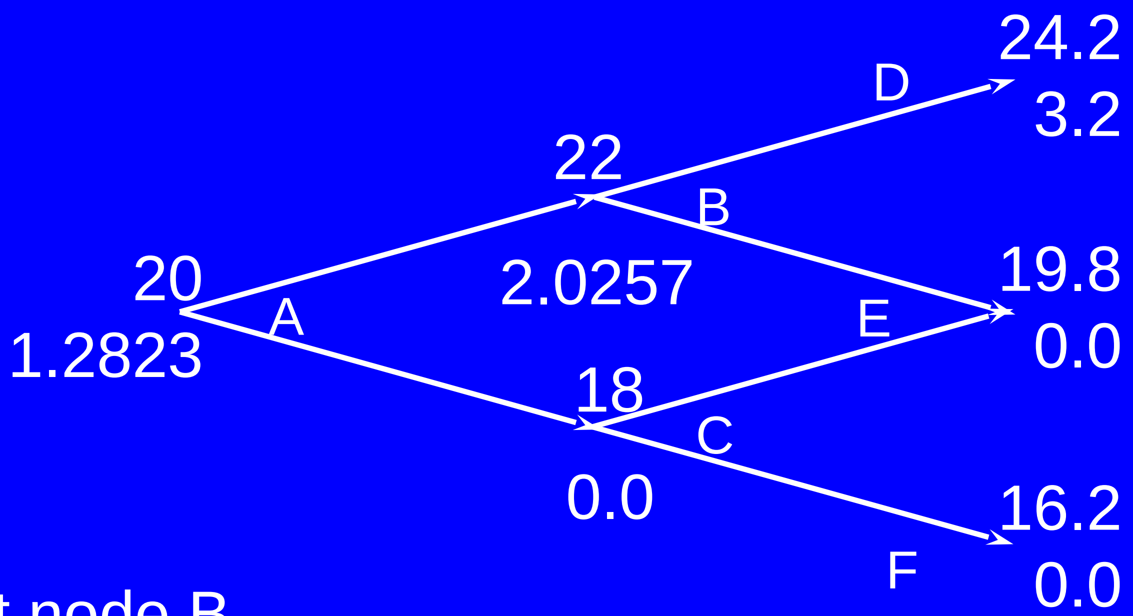
$$= 0.633$$

A Two-Step Example



- Each time step is 3 months

Valuing a Call Option



- Value at node B

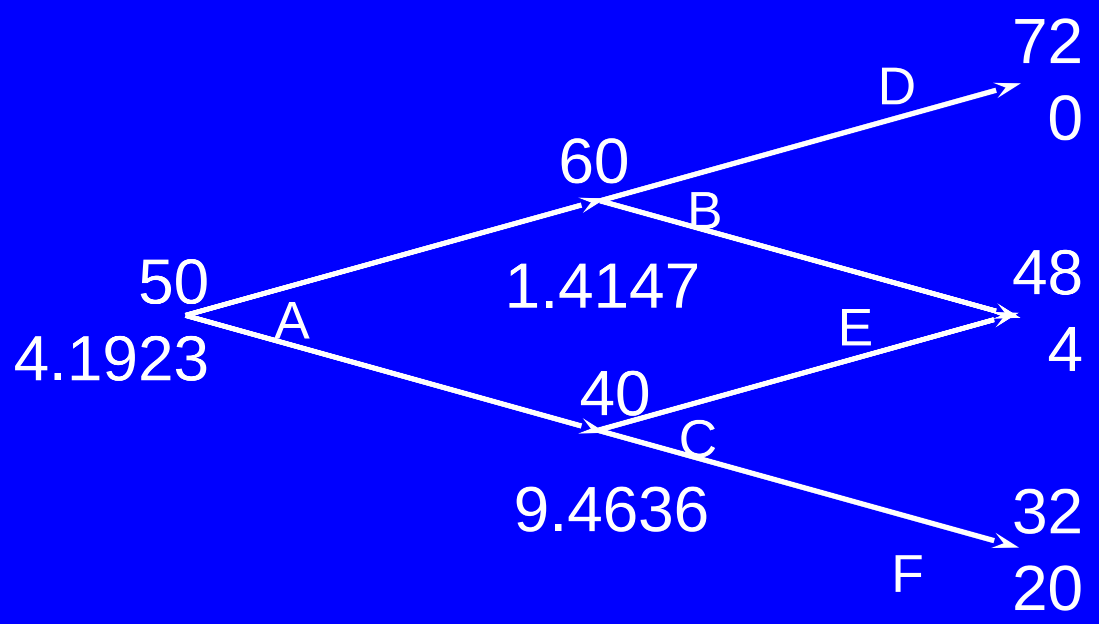
  $= e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$

- Value at node A

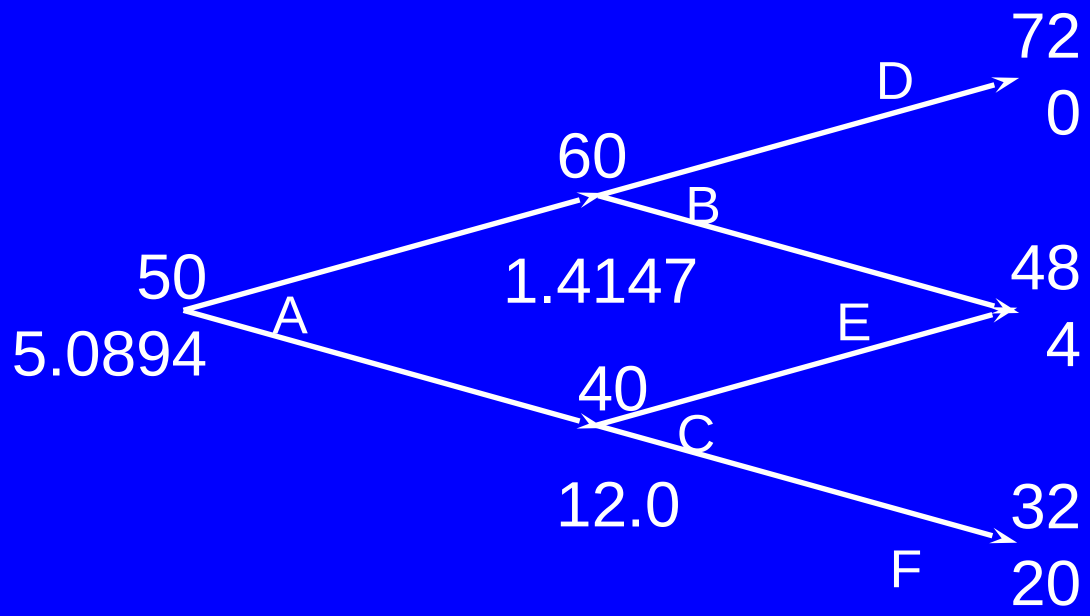
  $= e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0)$

  $= 1.2823$

A Put Option Example; *K*=52



What Happens When an Option is American



Delta

- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node