$\Rightarrow x^3y^2 + \frac{x^2}{y} = c$ (Ans), where c is a constant.

19/04/22

Linear Equation

$\boxed{M\,dx + N\,dy = 0}$

(linear eq^n of y w.r.t x) $\frac{dy}{dx} + P(x)y = Q(x)$ — y on x

(linear eq^n ~~w.r.t~~ of x w.r.t y) $\frac{dx}{dy} + L(y)x = R(y)$ — x on y

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{—— ①}$$

linear eq^n of y w.r.t x

$$I.F = e^{\int P(x)dx}$$

multiply both side of ① by IF

$$I.F.\left[\frac{dy}{dx} + P(x).y\right] = I.F.Q(x)$$

$$\Rightarrow \frac{d}{dx}(y.IF) = I.F.Q(x)$$

$$\Rightarrow d(y.IF) = \int IF.Q(x)\,dx$$

$$\Rightarrow y.IF = \int [IF.Q(x)]dx$$

Ex:

① $\frac{dy}{dx} + \frac{4x}{x^2+1}y = \frac{1}{(x+1)^3}$

② $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(1+x)$

③ $\frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{\tan^{-1}y}{1+y^2}$

Sol^n ②

given, $\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(1+x)$ —— ①

$$\frac{dy}{dx} + P(x)y = Q(x) \quad \text{—— ②}$$

equation ① is a linear equation of y w.r.t x

comparing ① and ② we get

$P(x) = -\frac{1}{1+x}$ $\qquad Q(x) = e^{3x}(1+x)$

$$IF = e^{\int P(x)dx}$$

$$= e^{\int -\frac{1}{1+x}dx}$$

$$= e^{-\int \frac{1}{1+x}dx}$$

$$= e^{-\log(1+x)}$$

$$= e^{\log(1+x)^{-1}}$$

$$= (1+x)^{-1}$$

$$= \frac{1}{1+x}$$

Rough

$\frac{d}{dx}\left\{\underset{u}{y}\underset{v}{\left(\frac{1}{1+x}\right)}\right\}$

$= \frac{dy}{dx}\left(\frac{1}{1+x}\right) + y(-1)(1+x)^{-2}$

$= \frac{dy}{dx}\left(\frac{1}{1+x}\right) - \frac{y}{(1+x)^2}$

$= \frac{1}{1+x}\left[\frac{dy}{dx} - \frac{y}{1+x}\right]$

multiply both side of ① by IF, we get,

$$\frac{1}{1+x}\left(\frac{dy}{dx} - \frac{y}{1+x}\right) = \frac{1}{\cancel{1+x}}\left[e^{3x}\cancel{(1+x)}\right]$$

$$\Rightarrow \frac{d}{dx}\left\{y.\left(\frac{1}{1+x}\right)\right\} = e^{3x}$$

$$\Rightarrow \int d\left(y\left(\frac{1}{1+x}\right)\right) = \int e^{3x}dx$$

$\Rightarrow y\left(\frac{1}{1+x}\right) = \frac{e^{3x}}{3} + c$ (Ans) where c is a constant

Sol^n-①

given, $\frac{dy}{dx} + \frac{4x}{x^2+1} y = \frac{1}{(x^2+1)^3}$ —— ①

$\frac{dy}{dx} + P(x) y = Q(x)$ —— ⑪

equation ① is a linear equation of y w.r.t x

comparing ① and ②,

$P(x) = \frac{4x}{x^2+1}$ $\quad Q(x) = \frac{1}{(x^2+1)^3}$

$IF = e^{\int P(x)dx} = e^{\int \frac{4x}{x^2+1} dx}$ $\quad x^2+1 = z$, $\Rightarrow 2x\,dx = dz$

$= e^{2\int \frac{dz}{z}}$

$= e^{2\log z}$

$= e^{\log z^2}$

$= z^2$

$= (x^2+1)^2$

multiply both side of ① by IF we get

$(x^2+1)^2 \left[\frac{dy}{dx} + \frac{4x}{x^2+1} y\right] = \cancel{(x^2+1)^2} \left(\frac{1}{(x^2+1)^{3}}\right)$

$\Rightarrow \frac{d}{dx} \{y \cdot (x^2+1)^2\} = \frac{1}{(x^2+1)}$

$\Rightarrow \int d\left(y(x^2+1)^2\right) = \int \frac{1}{x^2+1} dx$

$\Rightarrow y(x^2+1)^2 = \tan^{-1}x + C$ (Ans) where c is a constant

Ex:

1) $\frac{dy}{dx} + y = 1$

2) $\frac{dy}{dx} + \frac{1}{1+x^2} y = \frac{\tan^{-1}x}{1+x^2}$

—— x ——

Sol^n-1

given, $\frac{dy}{dx} + y = 1$ —— ①

$\frac{dy}{dx} + P(x) y = Q(x)$ —— ⑪

equation ① is a linear equation of y w.r.t x

comparing ① and ⑪ we get,

$P(x) = 1$ $\quad Q(x) = 1$

$I.F = e^{\int P(x)dx} = e^{\int 1 dx} = e^x$

multiply both side of ① by IF we get

$e^x \left[\frac{dy}{dx} + y\right] = e^x (1)$

$\Rightarrow \frac{d}{dx} \{y \cdot e^x\} = e^x$

$\Rightarrow \int d(y \cdot e^x) = \int e^x dx$

$\Rightarrow ye^x = e^x + c$, where c is a constant. (Ans)

—— x ——

Sol^n-2

given, $\frac{dy}{dx} + \frac{1}{1+x^2} y = \frac{\tan^{-1}x}{1+x^2}$ —— ①

$\frac{dy}{dx} + P(x) y = Q(x)$ —— ②

equation ① is a linear equation of y w.r.t x

comparing ① and ② we get.

$\therefore P(x) = \frac{1}{1+x^2}$ $\quad Q(x) = \frac{\tan^{-1}x}{1+x^2}$

$I.F = e^{\int P(x)dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1}x}$

multiply both side of ① by I.F we get,

$e^{\tan^{-1}x} \left[\frac{dy}{dx} + \frac{1}{1+x^2} y\right] = e^{\tan^{-1}x} \left(\frac{\tan^{-1}x}{1+x^2}\right)$

(P.T.O)

$\tan^{-1}x = z$

$\frac{1}{1+x^2}dx = dz$

$\Rightarrow \frac{d}{dx}\{y \cdot e^{\tan^{-1}x}\} = e^{\tan^{-1}x}\left(\frac{\tan^{-1}x}{1+x^2}\right)$

$\Rightarrow \int d\left(y e^{\tan^{-1}x}\right) = \int e^{\tan^{-1}x}\left(\frac{\tan^{-1}x}{1+x^2}\right)dx$

$\Rightarrow y e^{\tan^{-1}x} = \int e^z \cdot z\,dz$

$\Rightarrow y e^{\tan^{-1}x} = z e^z - e^z + c$

$\Rightarrow y e^{\tan^{-1}x} = \tan^{-1}x \cdot e^{\tan^{-1}x} - e^{\tan^{-1}x} + C$ (Ans)

where c is a constant

(A) $(x+y+1)dy = dx$

$\Rightarrow \frac{dx}{dy} = x+y+1$

$\Rightarrow \frac{dx}{dy} - x = y+1$

which is linear equation of x w.r.t. y

(F) $(1+y^2)dx + (x - \tan^{-1}y)dy = 0$

$\Rightarrow (1+y^2)\frac{dx}{dy} + x - \tan^{-1}y = 0$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$

20/04/22

~~Bernoulli's Equation~~

Solve $(x+y+1)dy = dx$

$\Rightarrow \frac{dx}{dy} = x+y+1$

$\Rightarrow \frac{dx}{dy} - x = y+1$ — ①

which is a linear equation of x w.r.t. y

$\frac{dx}{dy} + P(y)x = Q(y)$ — ②

comparing ① and ②

$P(y) = -1$ $\quad Q(y) = y+1$

$I.F = e^{\int P(y)dy}$

$= e^{\int -1 dy} = e^{-y}$

multiply both side of ① by IF we get,

$e^{-y}\left[\frac{dx}{dy} - x\right] = e^{-y}(y+1)$

$\Rightarrow \frac{d}{dx}(x \cdot e^{-y}) = e^{-y}(y+1)$

$\Rightarrow \int d(x \cdot e^{-y}) = \int e^{-y}(y+1)dy$

$\Rightarrow x e^{-y} = (y+1)\left(\frac{e^{-y}}{-1}\right) - (1)\left(\frac{e^{-y}}{1}\right) + c$

$\Rightarrow x e^{-y} = -(y+1)e^{-y} - e^{-y} + c$ (Ans)

where c is a constant

Bernouli's Equation $\rightarrow$ linear $\downarrow$ solve it

$\frac{dy}{dx} + P(x)y = Q(x)[f(y)]$

Ex)

① $\frac{dy}{dx} - 2xy = x^3y^2$

V.V.I ⑪ $\frac{dy}{dx} - \frac{y\log y}{x} = \frac{y(\log y)^2}{x^2}$

V.V.I ⑪ $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

⑫ $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

V.V.I ⑤ $\frac{dy}{dx} + y = (\cos x - \sin x)y^3$

Sol-1

Given; $\frac{dy}{dx} - 2xy = x^3y^2$ —— ①

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} - \frac{2x}{y} = x^3$ —— ②

Let, $\frac{1}{y} = z$

$\Rightarrow (-1)y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = -\frac{dz}{dx}$

substituting in ② we get

$-\frac{dz}{dx} - 2xz = x^3$

$\Rightarrow \frac{dz}{dx} + 2xz = -x^3$ —— ③

which is a linear equation of z w.r.t x

$\frac{dz}{dx} + P(x)z = Q(x)$ —— ④

Comparing ③ and ④ we get,

$P(x) = 2x$ $\quad Q(x) = -x^3$

$I.F = e^{\int P(x)dx}$

$= e^{\int 2x dx}$

$= e^{x^2}$

multiply on both side of ③ by IF we get

$e^{x^2}\left[\frac{dz}{dx} + 2xz\right] = -x^3e^{x^2}$

$x^2 = t$

$2xdx = dt$

$xdx = \frac{dt}{2}$

$\Rightarrow \frac{d}{dx}(z.e^{x^2}) = -x^3e^{x^2}$

$\Rightarrow \int \frac{d}{dx}(z.e^{x^2}) = \int x^3e^{x^2}dx$

$\Rightarrow ze^{x^2} = -\int te^t\frac{dt}{2}$

$\Rightarrow ze^{x^2} = -\frac{1}{2}\left[te^t - e^t\right] + c$

$\Rightarrow \frac{1}{y}e^{x^2} = -\frac{1}{2}(x^2e^{x^2} - e^{x^2}) + c$ (Ans)

where c is a constant

Soln-2

$\frac{dy}{dx} + \frac{y\log y}{x} = \frac{y(\log y)^2}{x^2}$ —— ①

$\Rightarrow \frac{1}{y(\log y)^2}\frac{dy}{dx} + \frac{1}{x\log y} = \frac{1}{x^2}$ —— ②

Let, $\frac{1}{\log y} = z$

$\Rightarrow (-1)(\log y)^{-2}\left(\frac{1}{y}\right)\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{1}{-y(\log y)^2}\frac{dy}{dx} = \frac{dz}{dx}$

$\Rightarrow \frac{1}{y(\log y)^2}\frac{dy}{dx} = -\frac{dz}{dx}$

Substitute all in ② we get,

$-\frac{dz}{dx} + \frac{z}{x} = \frac{1}{x^2}$

$\Rightarrow \frac{dz}{dx} - \frac{z}{x} = -\frac{1}{x^2}$ —— ③

which is a linear equation of z w.r.t. x

$\frac{dz}{dx} + P(x)z = Q(x)$ —— ④

comparing ③ and ④ we get,

$P(x) = -\frac{1}{x}$ $\quad Q(x) = -\frac{1}{x^2}$

$IF = e^{\int P(x)dx} = e^{-\int\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$

multiply both side of ③ by I.F we get,

$$\frac{1}{x}\left[\frac{dz}{dx}-\frac{z}{x}\right]=-\frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx}\left(z\cdot\frac{1}{x}\right)=-\frac{1}{x^3}$$

$$\Rightarrow \int d\left(z\cdot\frac{1}{x}\right)=-\int\frac{1}{x^3}\,dx$$

$$\Rightarrow \frac{z}{x}=-\frac{x^{-3+1}}{-3+1}+c$$

$$\Rightarrow \frac{1}{x\log y}=\frac{1}{2x^2}+c$$ (Ans) where c is a constant

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25/4/22

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]$$

$$\int e^{ax}\cos bx\,dx=\frac{e^{ax}}{a^2+b^2}\left[a\cos bx+b\sin bx\right]$$

$$\int e^{-2x}\sin x\,dx-\int e^{-2x}\cos x\,dx$$

Laplace

Scale change property

If $L[f(t)]=F(s)$

then $L[f(at)]=\frac{1}{a}F(s/a)$

Ex: 1) using scale change property, evaluate ⓐ $L[\cos 3t]$ ⓑ $L[\sin 5t]$

ⓐ $L[\cos 3t]=\frac{s}{s^2+9}$

ⓑ $L[\sin 5t]=\frac{5}{s^2+25}$

Solⁿ ⓐ

Here, $L[\cos 3t]=?$

$\therefore a=3$

Let, $f(t)=\cos t$

$\therefore L[f(t)]=L[\cos t]=\frac{s}{s^2+1}=F(s)$

using scale change property,

$$L[\cos 3t]=L[f(3t)],\ a=3$$

$$=\frac{1}{a}F(s/a)$$

$$=\frac{1}{3}F(s/3)$$

$$=\frac{1}{3}\cdot\frac{s/3}{(s/3)^2+1}$$

$$=\frac{1}{3}\left[\frac{\frac{s}{3}}{\frac{s^2+9}{9}}\right]$$

$$=\frac{1}{3}\left[\frac{s\times 9}{3(s^2+9)}\right]$$

$$=\frac{s}{s^2+9}$$ Ans

Solⁿ ⓑ

Here, $L[\sin 5t]=?$

$a=5$

Let, $f(t)=\sin t$

$\therefore L[f(t)]=L[\sin t]=\frac{1}{s^2+1}=F(s)$

Using scale change property,

$$L[\sin 5t]=L[f(5t)],\ a=5$$

$$=\frac{1}{a}F(s/a)$$

$$=\frac{1}{5}F(s/5)$$

$$=\frac{1}{5}\cdot\left(\frac{1}{(s/5)^2+1}\right)$$

$$=\frac{1}{5}\left(\frac{1}{\frac{s^2+25}{25}}\right)$$

$$=\frac{1}{5}\left(\frac{25}{s^2+25}\right)=\frac{5}{s^2+25}$$ Ans

Q) If $L[\sin 3t] = \frac{3}{s^2+9}$, using scale change property prove that $L[\sin 2t] = \frac{2}{s^2+4}$

Ans Here, $L[\sin 3t] = \frac{3}{s^2+9} = F(s)$ $\quad f(t) = \sin 3t$

Now,

$$L[\sin 2t] = L\left[\sin \tfrac{2}{3}(3t)\right], \; a = \tfrac{2}{3} \quad (*)$$

$$= L[f(\tfrac{2}{3}t)]$$

$$= \frac{1}{a} F(s/a)$$

$$= \frac{1}{\frac{2}{3}} F\left(\frac{s}{2/3}\right)$$

$$= \frac{3}{2} F\left(\frac{3s}{2}\right) \qquad F(s) = \frac{3}{s^2+9}$$

$$= \frac{3}{2}\left(\frac{3}{\left(\frac{3s}{2}\right)^2+9}\right)$$

$$= \frac{9}{2}\left(\frac{1}{\frac{9s^2+4}{9}}\right)$$

$$= \frac{\cancel{9}}{2\times\cancel{9}}\left[\frac{1}{\frac{s^2}{4}+1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{\frac{s^2+4}{4}}\right)$$

$$= \frac{1}{2}\left(\frac{4}{s^2+4}\right)$$

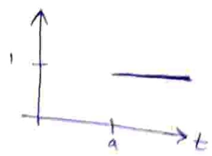
$$= \frac{2}{s^2+4} \quad \text{Ans}$$

$$\frac{3}{2}\left(\frac{3\times \cancel{4}2}{9s^2+36}\right) \qquad \frac{\cancel{9}}{\cancel{9}}\left(\frac{2}{s^2+4}\right)$$

$$\left(\frac{3s}{2}\right)^2+9 \qquad \frac{9}{9}\left(\frac{4\cdot 2}{9s^2+36}\right) = \frac{2}{s^2+4}$$

Unit Step Function

$$u(t-a) = 1, \quad t > a$$
$$= 0, \quad t < a$$



(*) $L[u(t-a)] = \frac{e^{-as}}{s}$

(**) $L[f(t)\,u(t-a)] = e^{-as} L[f(t+a)]$

Express the general function in terms of Unit Step function:-

type-1 $\quad f(t) = f_1 \qquad 0 < t < a$
$\qquad\quad = f_2 \qquad a < t < \infty$

Soln The required function in term of Unit Step function in

$f(t) = f_1 + (f_2 - f_1)\,u(t-a)$ $\quad$ ($\because$ a is common in both limits)

Ex: Express the following function in term of Unit Step Function

$f(t) = \sin t \qquad 0 < t < \pi/4$
$\quad\;\; = t \qquad\;\; \pi/4 < t < \infty$

Ans $f(t) = \sin t + (t - \sin t)\,u(t - \pi/4)$ Ans
$\quad = \sin t + t u(t-\pi/4) - \sin t\, u(t - \pi/4)$ Ans

---

type-2

$$f(t) = f_1, \quad 0 < t < a$$
$$= f_2, \quad a < t < b$$
$$= f_3, \quad b < t < c$$
$$= f_4, \quad c < t < \infty$$

The required function in terms unit step function is

$$f(t) = f_1 + (f_2 - f_1) u(t-a) + (f_3 - f_2) u(t-b) + (f_4 - f_3) u(t-c)$$

Ex: Express the following f^n in terms of unit step f^n.

$$f(t) = t^2, \quad 0 < t < 2$$
$$= \sin 3t, \quad 2 < t < 4$$
$$= e^{2t}, \quad 4 < t < \infty$$

Sol^n

$$f(t) = f_1 + (f_2 - f_1) u(t-a) + (f_3 - f_2) u(t-b)$$
$$= t^2 + (\sin 3t - t^2) u(t-2) + (e^{2t} - \sin 3t) u(t-4)$$
$$= t^2 + \sin 3t\, u(t-2) - t^2 u(t-2) + e^{2t} u(t-4) - \sin 3t\, u(t-4)$$

Ex: (i) $L[u(t-3)] = \frac{e^{-3s}}{s}$

(ii) $L[e^{2t} u(t-3)] = L[f(t) u(t-3)], \; f(t) = e^{2t}$

$$= e^{-3s} L[f(t+3)]$$
$$= e^{-3s} L[e^{2(t+3)}]$$
$$= e^{-3s} L[e^{2t} \cdot e^{6}]$$
$$= e^{-3s} \cdot e^{6} L[e^{2t}]$$
$$= e^{-3s} \cdot e^{6} \times \frac{1}{s-2} \quad \text{Ans}$$

Ex: $L[e^{-3t} u(t-2)]$

$$= L[f(t) u(t-2)], \; f(t) = e^{-3t}$$
$$= e^{-2s} L[f(t+2)]$$
$$= e^{-2s} L[e^{-3(t+2)}]$$
$$= e^{-2s} L[e^{-3t} \cdot e^{-6}]$$
$$= e^{-2s} \cdot e^{-6} \cdot L[e^{-3t}]$$
$$= e^{-2s} \cdot e^{-6} \left(\frac{1}{s+3}\right) \quad \text{(Ans)}$$

Ex: $L[t^2 u(t-2)]$

$$= L[f(t) u(t-2)], \; f(t) = t^2$$
$$\cancel{= t^2 L[f(t+2)]}$$
$$= e^{-2s} [f(t+2)]$$
$$= e^{-2s} L[(t+2)^2]$$
$$= e^{-2s} L[t^2 + 4t + 4]$$
$$= e^{-2s} \{L(t^2) + 4L(t) + 4L(1)]$$
$$= e^{-2s} \left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$$

Ex: $L[\sin t\, u(t-2)]$

$$= L[f(t) u(t-2)], \; f(t) = \sin t$$
$$= e^{-2s} [f(t+2)]$$
$$= e^{-2s} L[\sin(t+2)]$$
$$= e^{-2s} L[\sin t \cos 2 + \sin 2 \cos t]$$
$$= e^{-2s} \{\cos 2 L[\sin t] + \sin 2 L[\cos t]\}$$
$$= e^{-2s} \left\{(\cos 2)\left(\frac{1}{s^2+1}\right) + (\sin 2)\left(\frac{s}{s^2+1}\right)\right\} \quad \text{Ans}$$

Q> Express the following function

$$f(t) = e^t, \quad 0 < t < 2$$
$$= t, \quad 2 \le t < \infty$$

in terms of unit step function and hence find its Laplace Transform.

Sol^n

The required unit step function is

$$f(t) = e^t + (t - e^t)\, u(t-2)$$

$$\Rightarrow f(t) = e^t + t u(t-2) - e^t u(t-2) \text{ --- } ① \text{ (Ans)}$$

~~tak~~ taking Laplace Transform on both side of ① we get,

$$L[f(t)] = L[e^t] + L[t u(t-2)] - L[e^t u(t-2)]$$

$$= \frac{1}{s-1} + e^{-2s} L[t+2] - e^{-2s} L[e^{t+2}]$$

$$= \frac{1}{s-1} + e^{-2s}\{L(t) + 2L(1)\} - e^{-2s}[L(e^t \cdot e^2)]$$

$$= \frac{1}{s-1} + e^{-2s}\left\{\frac{1}{s^2} + \frac{2}{s}\right\} - e^{-2s} \cdot e^2 \cdot \left(\frac{1}{s-1}\right) \text{ (Ans)}$$

Ex:

Express the following f^n

$$f(t) = e^{-t}, \quad 0 < t < 3$$
$$= 0, \quad 3 < t < \infty$$

in terms of unit step function and find its Laplace Transform.

Sol^n

The required unit step function is

$$f(t) = e^{-t} + (0 - e^{-t})\, u(t-3)$$

$$\Rightarrow f(t) = e^{-t} - e^{-t} u(t-3) \text{ --- } ①$$

taking Laplace Transform on both side of ① we get,

$$L[f(t)] = L[e^{-t}] - L[e^{-t} u(t-3)]$$

$$= \frac{1}{s+1} - e^{-3s} L[e^{-(t+3)}] \Rightarrow \frac{1}{s+1} - e^{-3s} e^{-3} L[e^{-t}]$$

$$= \frac{1}{s+1} - e^{-3s} \cdot e^{-3} \cdot \frac{1}{s+1} \text{ (Ans)}$$

Periodic function

$\sin x, \cos x \rightarrow$ periodic f^n with period 2π

$\tan x \rightarrow$ " " " " π

Any f^n $f(t)$ is said to be a periodic function with period T if it can be expressed as —

$$f(t+T) = f(t) \quad \forall t$$

Ex: $\sin(t + 2\pi) = \sin t$

type-1 representation

$$f(t + 2\pi) = f(t) \quad \forall t$$

type-2 representation

$f(t)$ is a periodic function with period $T = 2\pi$

Laplace Transform of Periodic function:-

If $f(t)$ be a periodic function with period T then

$$L[f(t)] = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

Ex: Find the Laplace Transform of the following function

$$f(t) = t, \quad 0 < t < 4$$

and $f(t+4) = f(t), \forall$

periodic f^n of period $T=4$

Sol^n

Here the given f^n is a periodic function with period $T = 4$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-4s}} \int_0^4 e^{-st}\, t\, dt$$

$$= \frac{1}{1-e^{-4s}} \left[(t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right]_0^4$$

$$= \frac{1}{1-e^{-4s}} \left[-\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^4$$

$$= \frac{1}{1-e^{-4s}} \left[\left(-\frac{4}{s} e^{-4s} - \frac{1}{s^2} e^{-4s}\right) - \left(0 - \frac{1}{s^2} e^0\right) \right]$$

$$= \frac{1}{1-e^{-4s}} \left[-\frac{4}{s} e^{-4s} - \frac{1}{s^2} e^{-4s} + \frac{1}{s^2} \right] \text{(Ans)}$$

x

26/4/22

Q) Find the Laplace transform of the function –

$$f(t) = t, \quad 0 < t < 2$$
$$= 3, \quad 2 < t < 4$$

and $f(t+4) = f(t)\ \forall\, t$

Sol^n

Here $f(t)$ is a periodic function with period $T = 4$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt \qquad T = 4$$

$$= \frac{1}{1-e^{-4s}} \int_0^4 e^{-st} f(t)\, dt$$

$$= \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st} f(t)\, dt + \int_2^4 e^{-st} f(t)\, dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st}\, t\, dt + \int_2^4 e^{-st} \cdot 3\, dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\left\{ (t)\left(\frac{e^{-st}}{-s}\right) - (1)\left(\frac{e^{-st}}{s^2}\right) \right\}_0^2 + 3\left\{ \frac{e^{-st}}{-s} \right\}_2^4 \right]$$

$$= \frac{1}{1-e^{-4s}} \left[\left\{ \left(-\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s}\right) - \left(0 - \frac{1}{s^2} e^0\right) \right\} + 3\left\{ \frac{e^{-4s} - e^{-2s}}{-s} \right\} \right]$$

$$= \frac{1}{1-e^{-4s}} \left[-\frac{2}{s} e^{-2s} - \frac{1}{s^2} e^{-2s} + \frac{1}{s^2} - \frac{3}{s} e^{-4s} + \frac{3}{s} e^{-2s} \right] \text{(Ans)}$$

H/W

Find ~~a~~ Laplace Transform of the following periodic function with period $T=3$

$$f(t) = \sin 2t, \quad 0<t<3$$

x

Ordinary Differential Equation with first order But not first degree

Methods:-

① Solvable for p method (The given D.E can be factorized)

② Solvable for y method

$y = f(x,p)$ $p = \frac{dy}{dx}$

③ Solvable for x method

$x = f(y,p)$

Solvable for p method:-

Ex: ① Solve, $p^2 + 2xp - 3x^2 = 0$

② Solve, $xp^2 + (y-x)p - y = 0$

③ Solve; $p^2 - (e^x - e^{-x})p + 1 = 0$

Sol^n

① $p^2 + 2xp - 3x^2 = 0$

$\Rightarrow p^2 + (3-1)xp - 3x^2 = 0$

$\Rightarrow p^2 + 3xp - xp - 3x^2 = 0$

$\Rightarrow p(p+3x) - x(p+3x) = 0$

$\Rightarrow (p+3x)(p-x) = 0$

Now, $p + 3x = 0$

$\Rightarrow \frac{dy}{dx} = -3x$

$\Rightarrow \int dy = \int -3x\,dx$

$\Rightarrow y = -\frac{3}{2}x^2 + c$

$\Rightarrow (y + \frac{3}{2}x^2 - c) = 0$

Also,

$p - x = 0$

$\Rightarrow \frac{dy}{dx} = x$

$\Rightarrow \int dy = \int x\,dx$

$\Rightarrow y = \frac{x^2}{2} + c$

$\Rightarrow (y - \frac{x^2}{2} - c) = 0$

$\therefore$ The general solution is $(y + \frac{3}{2}x^2 - c)(y - \frac{x^2}{2} - c) = 0$ where C is a constant (Ans)

x

Sol^n

3) $p^2 - (e^x + e^{-x})p + 1 = 0$

$\Rightarrow p^2 - e^x p - e^{-x}p + e^x \cdot e^{-x} = 0$

$\Rightarrow p(p - e^x) - e^{-x}(p - e^x) = 0$

$\Rightarrow (p - e^x)(p - e^{-x}) = 0$

$\therefore p - e^x = 0$

$\Rightarrow \frac{dy}{dx} = e^x$

$\Rightarrow \int dy = \int e^x dx$

$\Rightarrow y = e^x + C$

$\Rightarrow (y - e^x - c) = 0$

Also,

$p - e^{-x} = 0$

$\Rightarrow \frac{dy}{dx} = e^{-x}$

$\Rightarrow \int dy = \int e^{-x} dx$

$\Rightarrow y = \frac{+e^{-x}}{-1} + C$

$\Rightarrow (y + e^{-x} + c) = 0$

$\therefore$ The general solution is

$(y - e^x - c)(y + e^{-x} - c) = 0$

where c is a constant. (Ans)

Soln

$\Rightarrow xp^2 + (y-x)p - y = 0$

$\Rightarrow xp^2 + yp - xp - y = 0$

$\Rightarrow p(xp+y) - 1(xp+y) = 0$

$\Rightarrow (xp+y)(p-1) = 0$

$\therefore xp + y = 0$

$\Rightarrow \cancel{p} \frac{dy}{dx} = -\frac{y}{x}$

$\Rightarrow \int dy = -y \int \frac{dx}{x}$

$\Rightarrow y = -y \log(x) + c$

$\Rightarrow (y + y\log x - c) = 0$

Also

$p - 1 = 0$

$\Rightarrow \frac{dy}{dx} = 1$

$\Rightarrow \int dy = \int dx$

$\Rightarrow y = x + c$

$\Rightarrow (y - x - c) = 0$

$\therefore$ The general solution is $(y + y\log x - c)(y - x - c) = 0$ where c is a constant (Ans)

Periodic Function

26/4/22

Q) Express the following ~~in~~ function in terms of unit step fn and hence find its Laplace Transform.

$$f(t) = t \quad 0 < t < 2$$
$$= e^{2t} \quad 2 < t < \infty$$

Soln The required unit step function is

$f(t) = t + (e^{2t} - t)\,u(e^{2t} - 2)$

$\Rightarrow f(t) = t + e^{2t}u(e^{2t} - 2) - t\,u(e^{2t} - u)$ — ①

taking Laplace transform on both side of ① we get,

$L[f(t)] = L[t] + L[e^{2t}u(e^{2t} - 2)] - L[t\,u(e^{2t} - u)]$

$= \frac{1}{s^2} + e^{-}$

Soln The required unit step function is

$f(t) = t + (e^{2t} - t)\,u(t-2)$

$\Rightarrow f(t) = t + e^{2t}u(t-2) - t\,u(t-2)$ — ①

taking Laplace Transform on both side of ① we get,

$L[f(t)] = L[t] + L[e^{2t}u(t-2)] - L[t\,u(t-2)]$

$= \frac{1}{s^2} + e^{-2s}L[e^{2(t+2)}] - e^{-2s}L[t+2]$

$= \frac{1}{s^2} + e^{-2s}\{e^{2t}\cdot e^{4}\} - e^{-2s}[L(t) + 2L(1)]$

$= \frac{1}{s^2} + e^{-2s}\cdot e^{4}\left(\frac{1}{s-2}\right) - e^{2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$ (Ans)