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Discrete Structures

3rd Semester

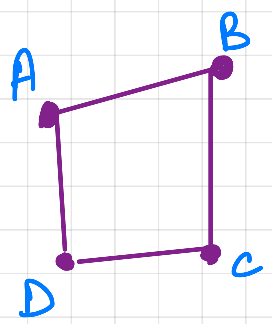
2/Aug/2022

DISCRETE STRUCTURES

① Graph Theory

Defination : By a graph we mean,

a) We need some points
b) Criteria
c) edges



Vertices → A, B, C, D

By a graph $\boxed{G = (V, E)}$

V = Vertex set , E = edge set

A graph (G) is finite if its vertex set is finite and edges set is finite.

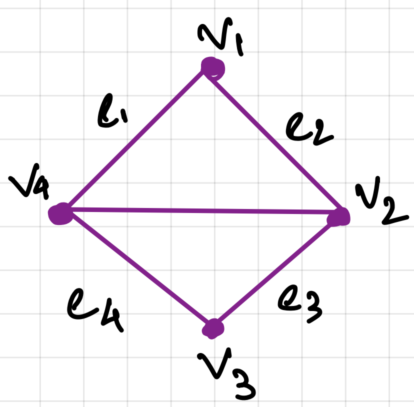
$V = \{ v_1, v_2, v_3, v_4, \ldots \}$

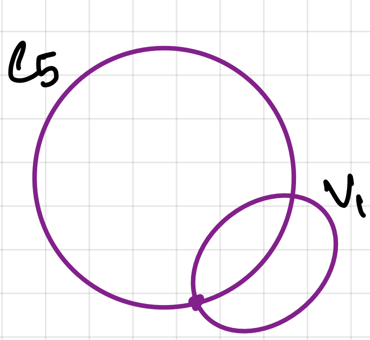
$E = \{ e_1, e_2, e_3, e_4 \ldots \}$

Defination 2 : Two vertices V_i and V_j are said to be adjacent if there exists an edge between V_i and V_j. If no such edge exist then V_i is not adjacent to V_j.

$\boxed{V_i \text{ and } V_j \text{ are neighbours}}$

Result 1 : The degree of vertex is the total numbers of neighbours of a vertex. $\boxed{d(V_1) = 2}$ $\boxed{d(V_2) = 2}$





V_1 and V_2 are the end vertices of e_1 edge.

Loop : An edge is said to be a loop if its end vertices are the same.

Simple Graph : A graph is simple if it has no loops.

Isolated vertex : A vertex which has no neighbours.

Pendant Vertex : A vertex which has exactly one neighbour.

Problem 1 Find all the simple graphs on 2 and 3 vertices.

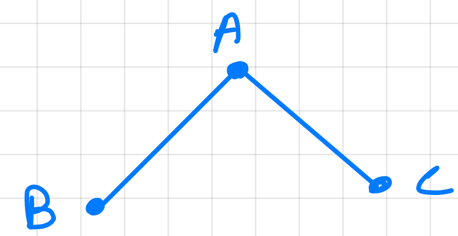
Case 1 : 2 vertices

Case 2 : 3 vertices

Ans :

Case 1

Case 2



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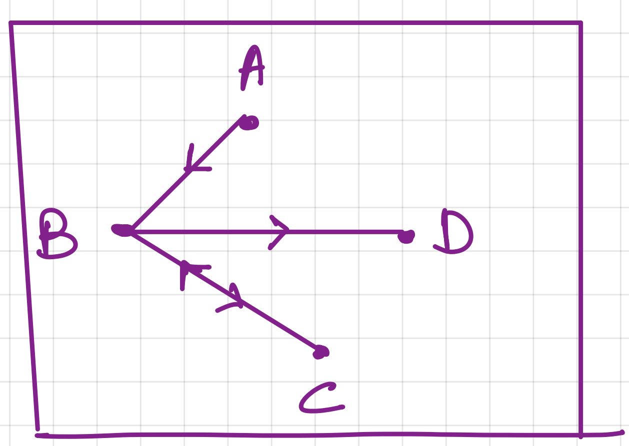
Degree of Vertex : The number of vertices that are adjacent to a given vertex.

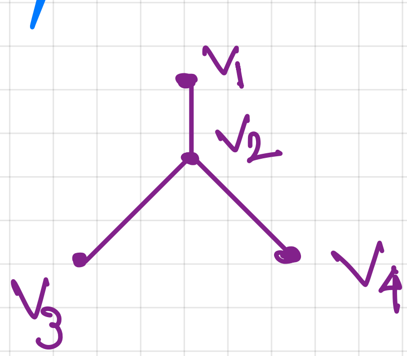


Directed graph

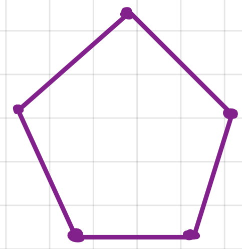
A directed graph consists of a set of vertices & a set of directed edges.

$G = (V, E)$

Degree Sequence : The degree sequence of a graph G is the set of all degrees of vertices of graph which are arranged in a decreasing order.



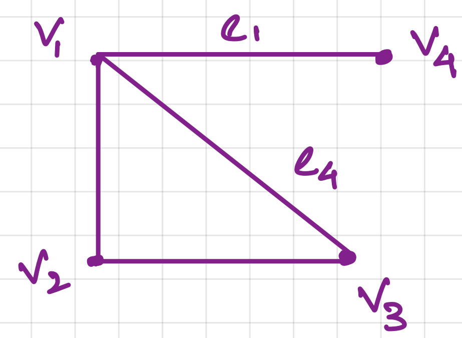
Weighted Graph : When we add a certain weight to the edges of a graph, we obtain a weighted graph.



Theorem 1

Let G be a graph then the sum of the degrees of the graph is equals to twice the number of edges in G.

V_1 —— e —— V_4



$deg(v_1) = 1+$

$deg(v_2) =$

Proof : $V_4 - e - V_1$

$deg(v_1) = 1+1+1$ $\quad$ $deg(v_2) = 1+1$

$deg(v_4) = 1+$

$deg(v_3) = 1+1$

Let e be an edge of the graph G, every edge 'e' adds 1 to the degree of vertices it is joining. Hence the sum of the degree of vertices equals twice the number of edges. [Proved]

Theorem 2 : No of odd degree vertices in a graph is always even

Proof : From theorem 1,

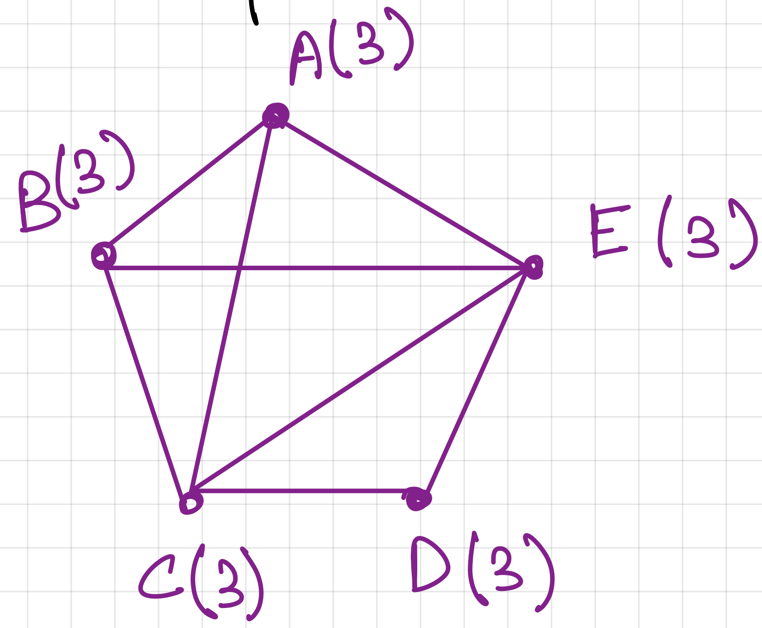
Sum of degree of vertices,

$$\sum degree(v) = 2 \times (\text{no. of edges})$$
$$= 2e$$

Problem: Draw 5 line segments which intersect exactly 3 other line segments.

(graph theory)

Solve: line 1 → A
line 2 → B
line 3 → C
line 4 → D
line 5 → E



Sum of degrees = 2 × no of edges = even

$\Rightarrow 3 \times 5 = 15 \rightarrow$ odd

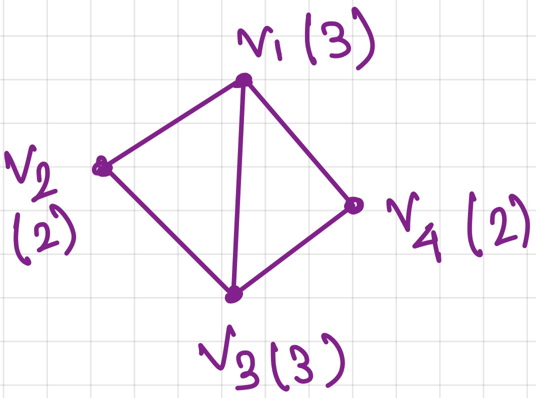
Soln: Sum of degrees = $3+3+3+3+3 = 15$ → odd number which it is an odd number so we can not draw the graph.

Problem ①: Tell me names of 7 students who have exactly 5 friends.

Model ①

The graph G has friends as vertices and if 2 vertices are adjacent then they are friends. Then each vertex has the degree 5. Then sum of degrees $5 \times 7 = 35$

$$\sum_{\deg = odd} \deg(v) = 2e - \sum_{\deg = even} \deg(v)$$



degree sequence = (3, 3, 2, 2)

IV **Complete Graph :** If all the pair of vertices of a graph are adjacent are called complete graph.

V **Regular Graph :** If degree of each vertex of a graph is same.

Problem : Does there exist a simple graph with no. degree sequence (4, 4, 3, 1)

Soln : Bcz in a simple graph on 4 vertices maximum degree can be 3.

Problem : Simple graph with degree sequence. (2, 2, 2, 1) → 7 odd.

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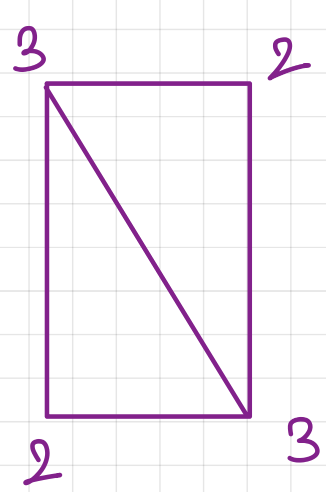
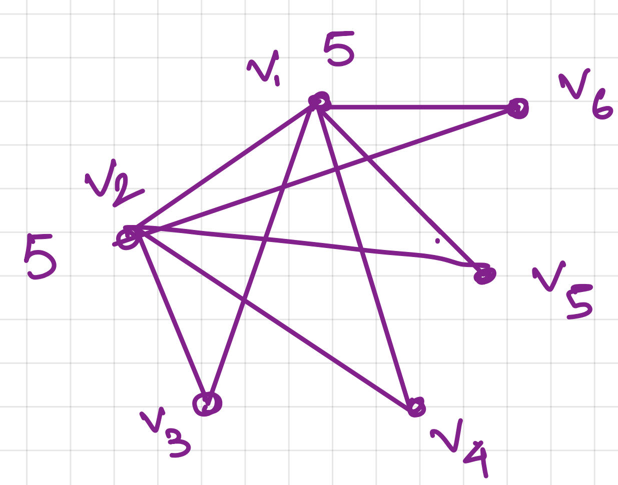
(3, 3, 2, 2)



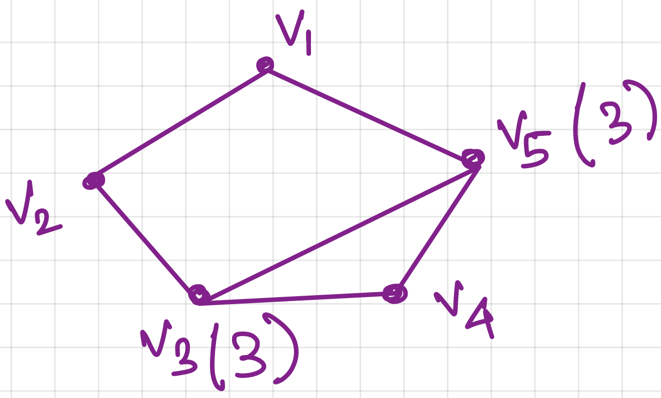


(5, 5, 4, 2, 2, 2) → not possible

Reasoning :-

Both V_1 and V_2 have degree 5 so all other vertices have degree atleast 2. But V_3 has degree 4. Also V_4 can't be adjacent to other vertices.

$(3,3,2,2,2) \longrightarrow$ Possible



Pendent vertex $\rightarrow$ degree 1

Isolated vertex $\rightarrow$ degree 0

Connected Graph :-

Path : A path is a sequence of vertices and edges.

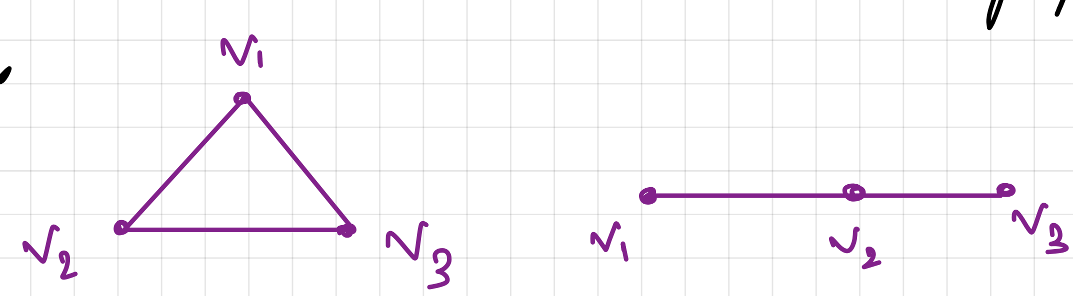
A graph is called connected graph if there exists a path between any two vertices.

Problem : Can we have an isolated graph in an connected graph?

$\Rightarrow$ No, not possible

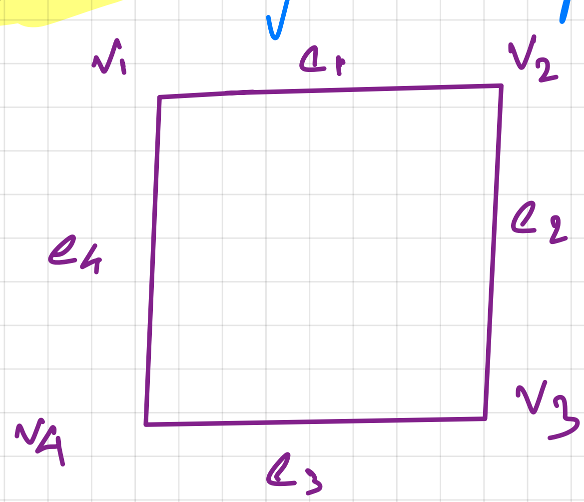
Problem : Draw all simple connected graphs on 3 vertices.

$\Rightarrow$

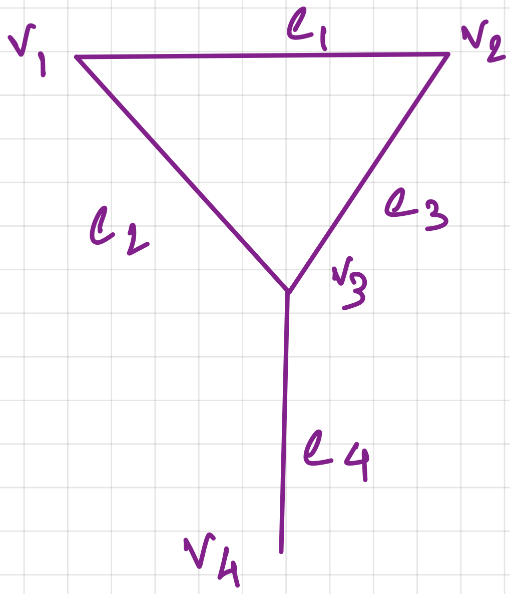


Trail : If all the edges in a path are distinct we call it a trail.

Circuit : If the end points of a path are same, we call it a circuit.



Path = $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4$

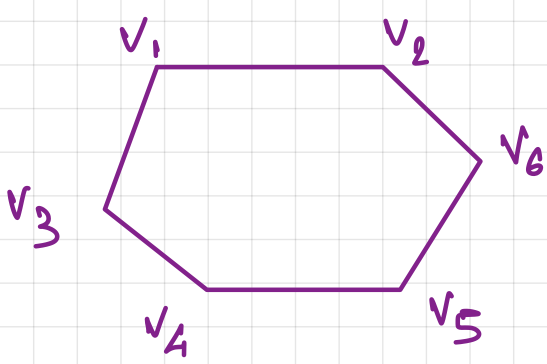


v_1 to v_2

$v_1 e_1 v_2$, $v_1 e_2 v_3 e_3 v_2$

Bipartite Graph :-

A graph G is bipartite if the vertex set $V(G)$ of G can be partitioned as $V(G) = E_1 \cup E_2$. Such that e is an edge of G then e has one end point in E_1 and other in E_2.



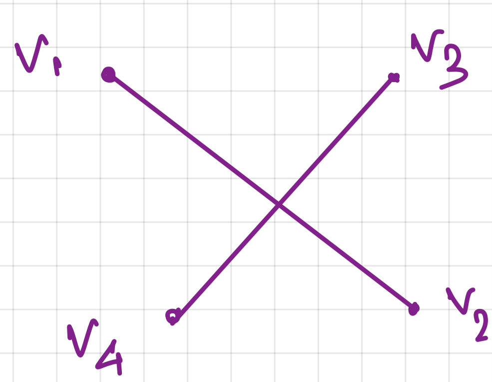
bipartite

$V(G) = E_1 \cup E_2$

$E_1 = \{v_1, v_3, v_5\}$

$E_2 = \{v_2, v_4, v_6\}$

Problem ② : Check if the graph is bipartite.

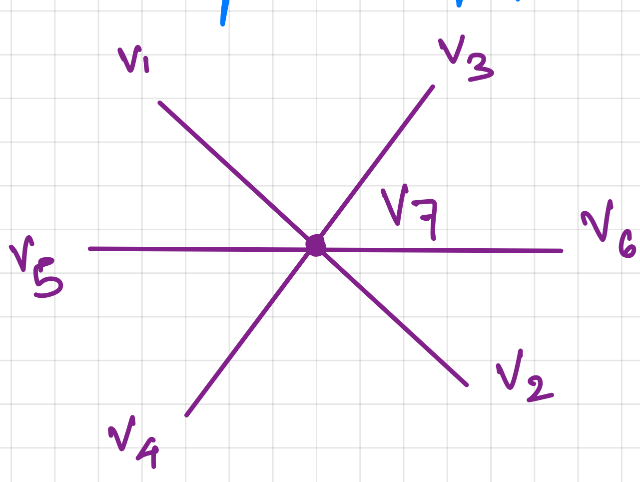


$V(G) = \{v_1, v_2, v_3, v_4\}$

$E_1 = \{v_1, v_3\}$

$E_2 = \{v_2, v_4\}$

$\therefore$ Bipartite graph



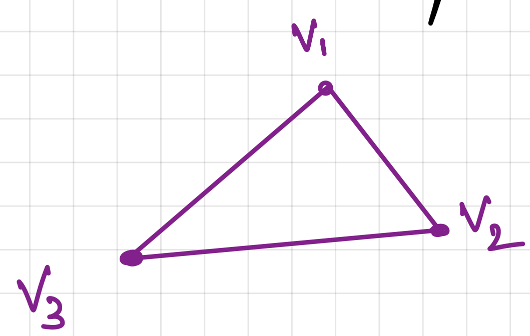
$V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$

$E_1 = \{v_7\} \cup E_2 = \{v_2, v_3, v_4, v_5, v_6, v_1\}$

$\therefore$ Bipartite Graph

Problem: Can a bipartite graph have a triangle?

$\Rightarrow$



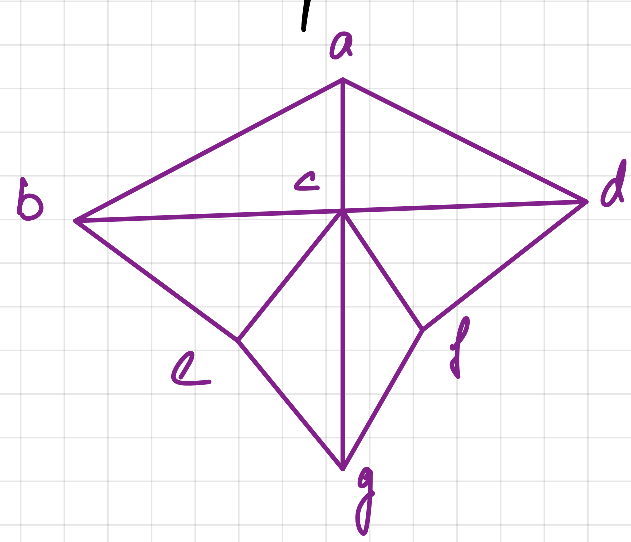
$V(G) = \{v_1, v_2, v_3\}$

$E_1 = \{v_1\} \cup E_2 = \{v_2, v_3\}$

$\therefore$ Not bipartite graph.

All the vertices are adjacent to each other so we can't partition.
Therefore triangle can not be bipartite graph.

Q. Is this bipartite?

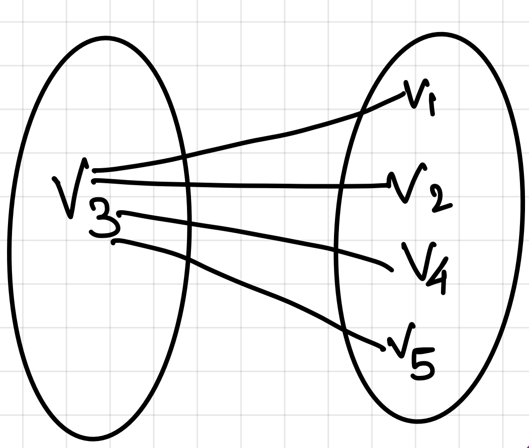


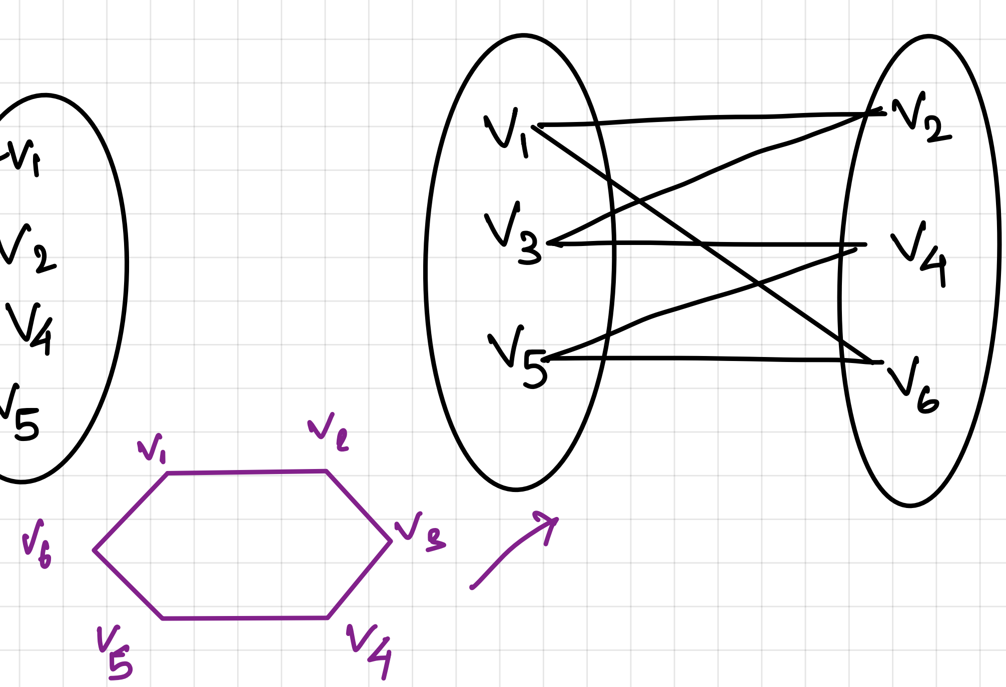
This graph contains a triangle so it is not a bipartite graph.

- **Complete bipartite graph :-**

(I) There exists a partition of $V(G)$ as $E_1 \cup E_2$ i.e,
$V(G) = E_1 \cup E_2$, $E_1 \cap E_2 = \phi$

(II) There exists a edge between every pair of vertices V_1 and V_2 such that $V_2 \in E_2$.

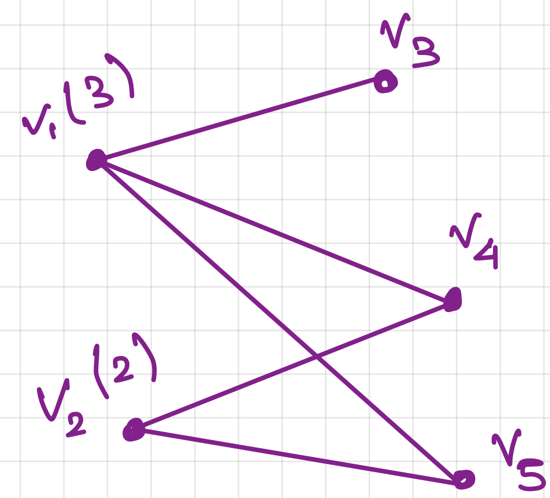




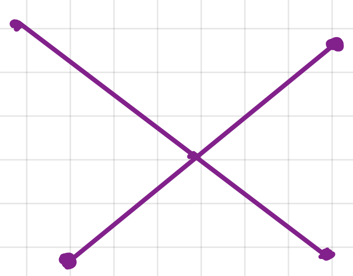
1) We denote a complete bipartite graph on m & n vertices by K_{mn}.

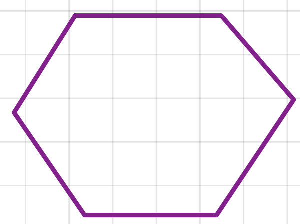
$m = 2$ $\quad K_{2,3}$ $\quad E_1 = \{v_1, v_2\}$

$n = 3$ $\quad E_2 = \{v_3, v_4, v_5\}$



Q. Give an example of bipartite graph which is not complete bipartite?





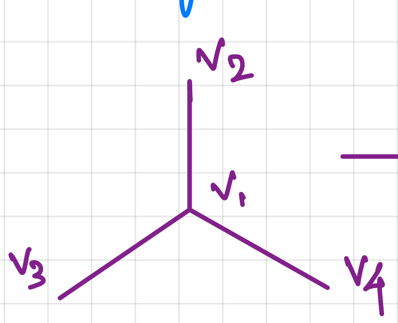
Graph Representation :-

matrix $\longrightarrow$

$$\begin{array}{c|cccc|}
 & v_1 & v_2 & v_3 \cdots & v_m \\
\hline
v_1 & a_{11} & a_{12} & a_{13} & a_{1m} \\
v_2 & a_{21} & a_{22} & a_{23} & a_{2m} \\
v_3 & \vdots & \text{---} & \cdots & \text{- -} \\
\vdots & \vdots & & & \\
v_n & a_{n1} & a_{n2} & a_{n3} & a_{nm}
\end{array} \longrightarrow \boxed{a_{ij} = 0} \text{ for all } i$$

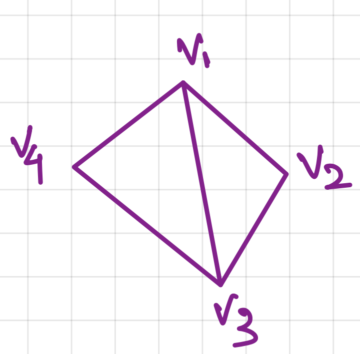
Let G be a graph with n vertices adjacent matrix.

$$A(G) = (a_{ij}) = \begin{cases} 1 \text{ if } v_i \text{ is adjacent to } v_j \\ 0 \text{ else} \end{cases}$$



$\longrightarrow$

$$\begin{array}{c|cccc|}
 & v_1 & v_2 & v_3 & v_4 \\
\hline
v_1 & 0 & 1 & 1 & 1 \\
v_2 & 1 & 0 & 0 & 0 \\
v_3 & 1 & 0 & 0 & 0 \\
v_4 & 1 & 0 & 0 & 0
\end{array}$$

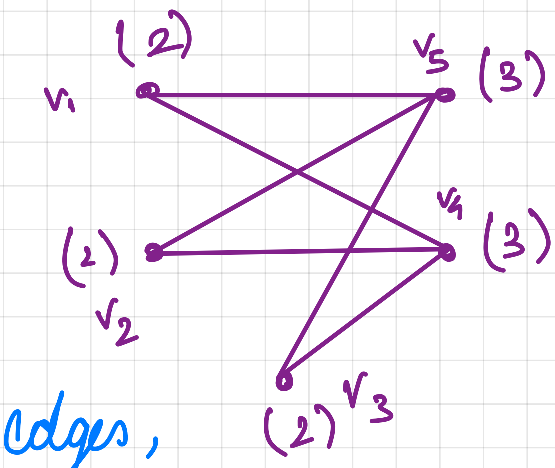


$\longrightarrow$

$$\begin{array}{c|cccc|}
 & v_1 & v_2 & v_3 & v_4 \\
\hline
v_1 & 0 & 1 & 1 & 1 \\
v_2 & 1 & 0 & 1 & 0 \\
v_3 & 1 & 1 & 0 & 1 \\
v_4 & 1 & 0 & 1 & 0
\end{array}$$

Q. Find the degree sequence of $K_{m,n}$ where $m>n$.

$K_{3,2}$ $(3,3,2,2,2)$



number of vertices and number of edges,

$$(\underbrace{m,m,\dots m}_{n\text{ times}},\ \underbrace{n,n,\dots n}_{m\text{ times}})$$

- **Adjacent matrix :-**

① If $A(G)=\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$ find the degree of the 2nd vertex.

take the sum

Ans: 2 (take the sum)

② If $A(G)=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is the graph connected?

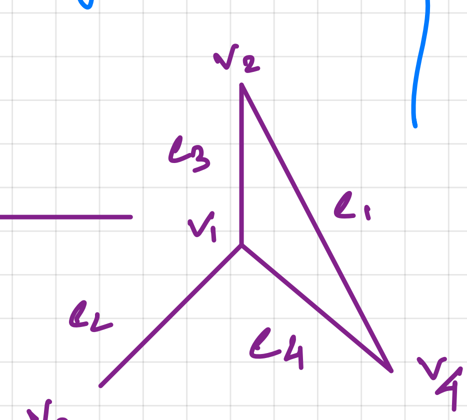
$\Rightarrow$ here the vertex v_4 is isolated point. Therefore the graph is disconnected.

- **Incidence Matrix :-**

Let G be a graph with 'n' vertices and 'e' edge. Then the incidence matrix of G is given by : $M(G)=(m_{ij})_{n\times e}$ $= \begin{cases} 1 & \text{if } e_i \text{ is incidence on } v_j \\ 0 & \text{elsewhere,} \end{cases}$

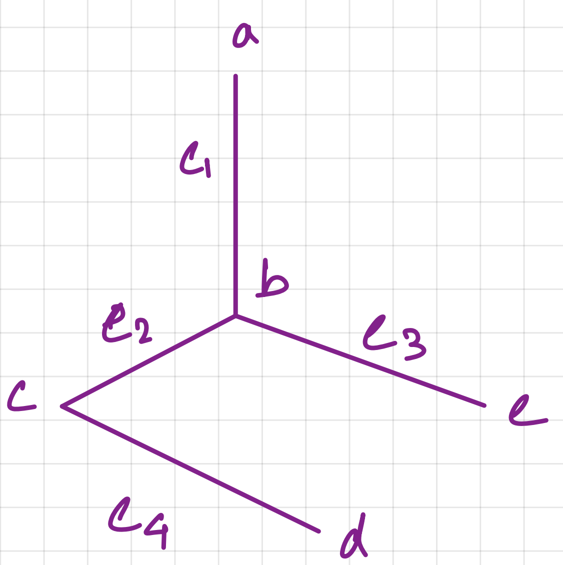
	e_1	e_2	e_3	e_4
v_1	0	1	1	1
v_2	1	0	1	0
v_3	0	1	0	0
v_4	1	0	0	1

$\leftarrow$



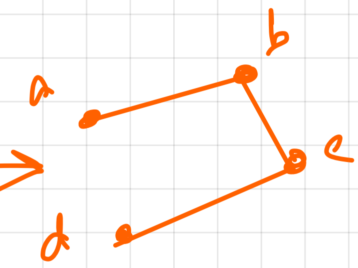
• Sub Graph :-

Let G be a graph with vertex set V(G) & edge set E(G). Let G be a graph A graph. H is said to be a sub graph of G if $V(H) \leq V(G)$ or $E(H) \leq E(G)$



$V(G) = \{a, b, c, d, e\}$

$E(G) = \{e_1, e_2, e_3, e_4\}$

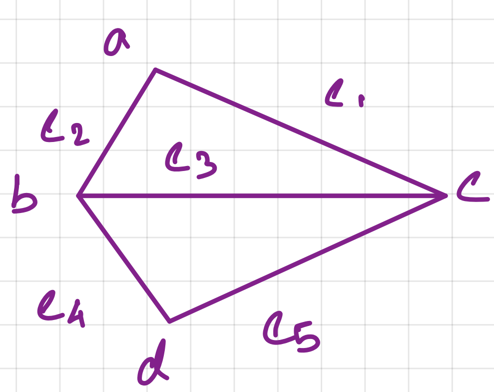


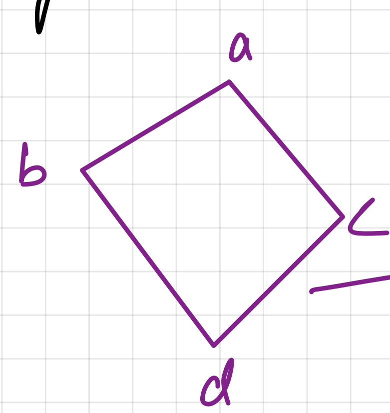
no of vertices

no of edges.

Problem :- Find a Subgraph of G with order 4 and size 3.

$\Rightarrow$





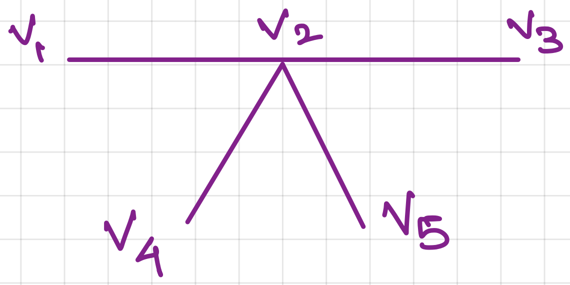
Not Spanning Subgraph.

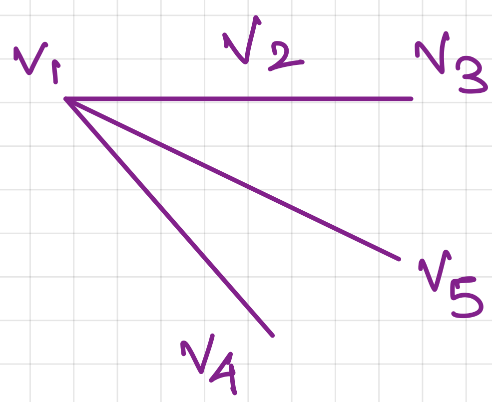
• Spanning Subgraph :-

If H is a sub graph of G such that $V(H) = V(G)$ then H is a spanning graph.

H.W If G is a graph such that G has 2 components G_1 and G_2. Find adjacent matrix of G in terms of adjacent matrix of G_1 and G_2.

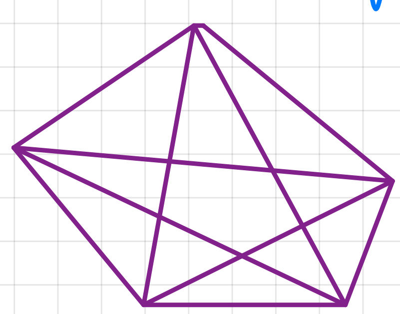
- **Complement :-**





Problem : Find the complement of K_5 (complete graph)

Complete graph : Any two vertices are adjacent.

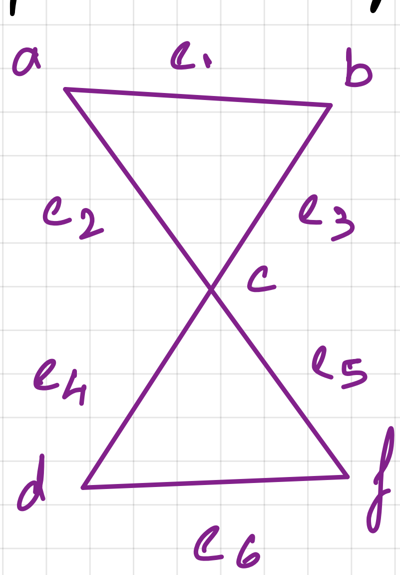


Null graph : A graph whose all vertices have degree zero.

Euler Graph : A graph which has an euler circuit.

Euler Circuit : A circuit which goes through all the edges in the graph.

Q. Explain Euler graph with example.



• Isomorphism :-

Two graphs G & H are isomorphic if the following –

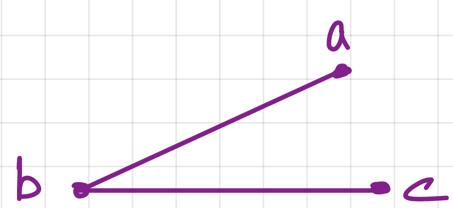
(i) There exist a bijection between G & H.

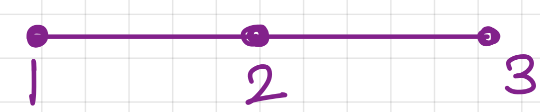
(ii) Two vertices a & b are adjacent in G if and only if f(a) & f(b) are adjacent in H.

V(G) = vertex set of G

V(H) = vertex set of H

$f : V(G) \longrightarrow V(H)$





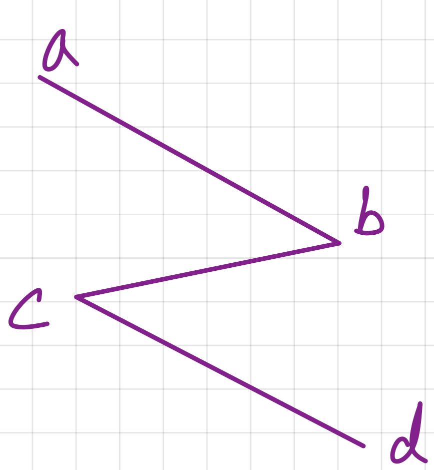
$a \to 1$
$b \to 2$
$c \to 3$

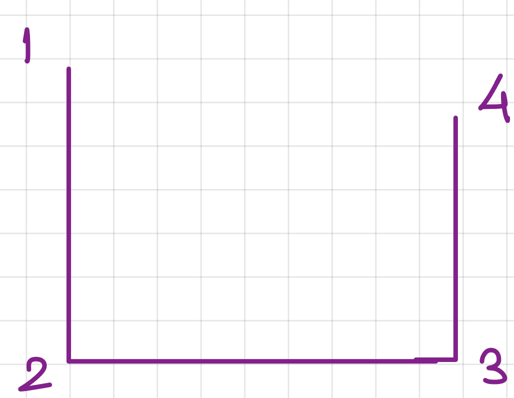
$f(a) = 1$
$f(b) = 2$
$f(c) = 3$

$f(a) \to 2$
$f(b) \to 1$
$f(c) \to 3$

$\times$

1)



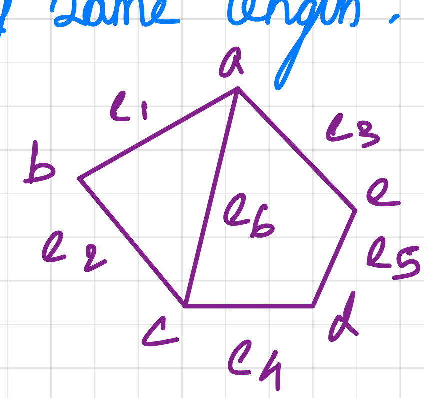


$a = 1$
$b = 2$
$c = 3$
$d = 4$

If G and H are isomorphic then the following points hold ——

- No of vertices & edges of G and H are same.
- Degree sequence of G and H are same.
- Both G and H have equal no. of circuits of same length.

[Length is no of edges in a circuit.]



$a e_1 b e_3 d e_6 a$
length = 3

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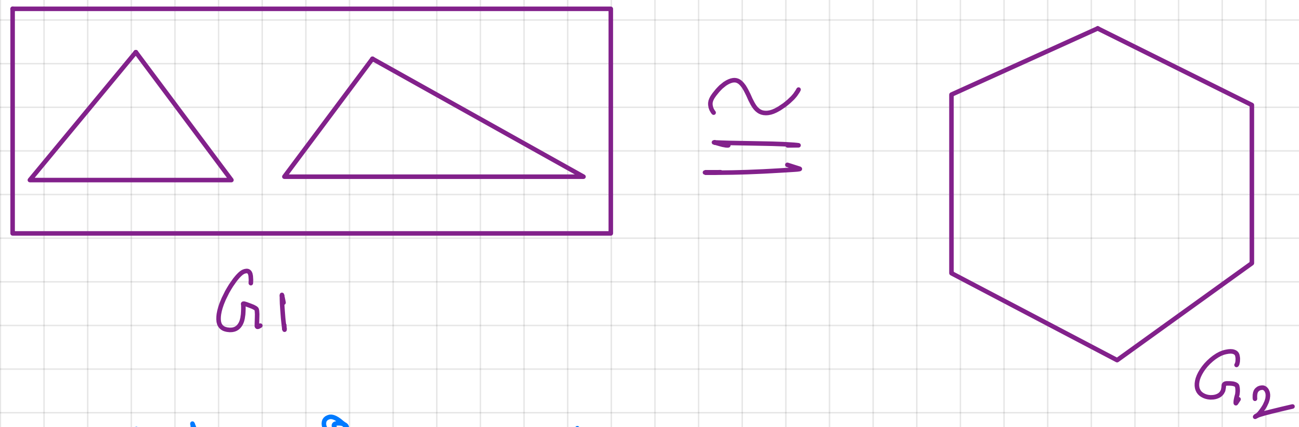
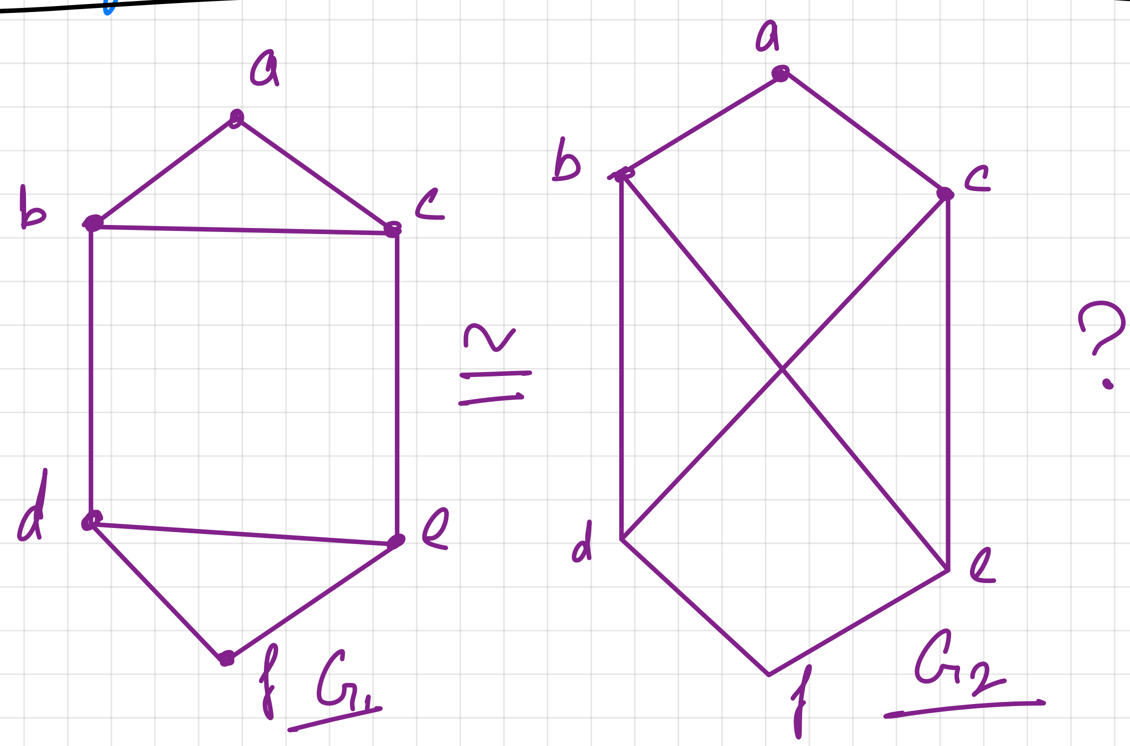


⇒ Not isomorphic, Because of not equal circuits of same length. And one is disconnected and other one is connected.



⇒ Not isomorphic, Not same no of equal length circuits.

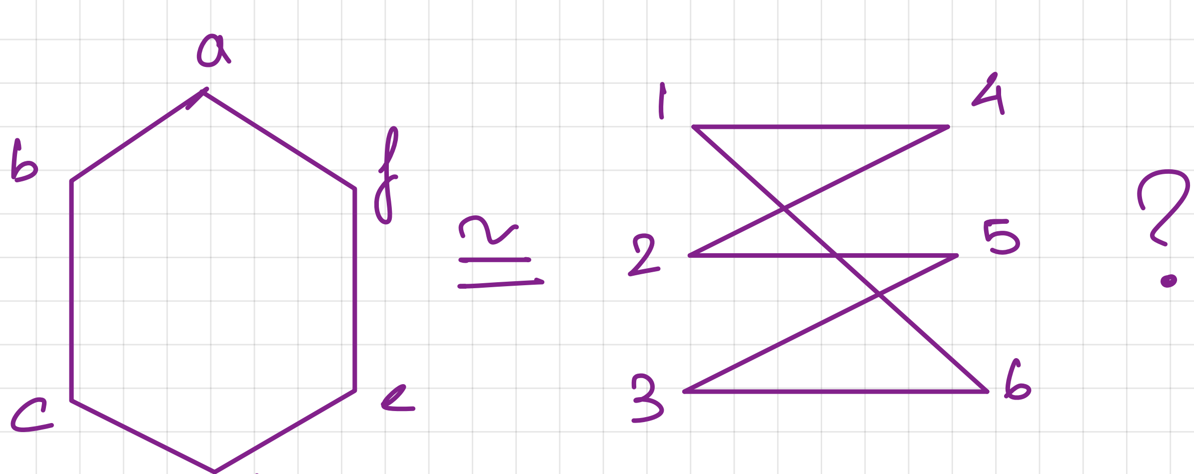
abca

$a e_1 b e_3 c e_2 a$

3 edges

a b e c a

4 edges



⇒

$a \rightarrow 1$
$b \rightarrow 4$
$c \rightarrow 2$
$d \rightarrow 5$
$e \rightarrow 3$
$f \rightarrow 6$

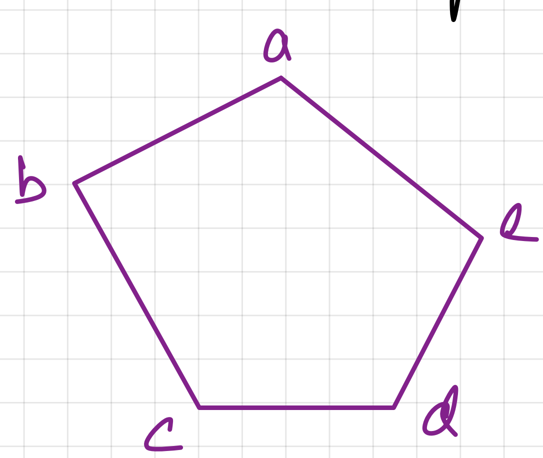
∴ The two graphs are isomorphic.

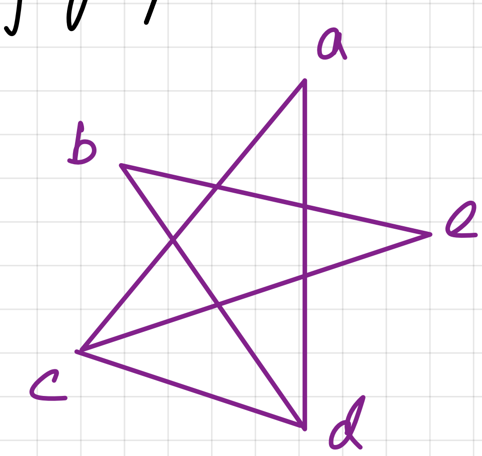
① To show if two vertices are adjacent in G, then they are adjacent in H.

② To show if two vertices are not adjacent in G, then they are not adjacent in H.

1. Assignment :-

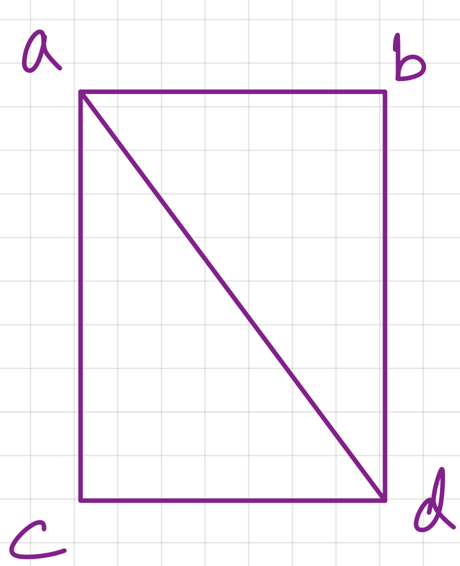
① Show that the following graphs are isomorphic.

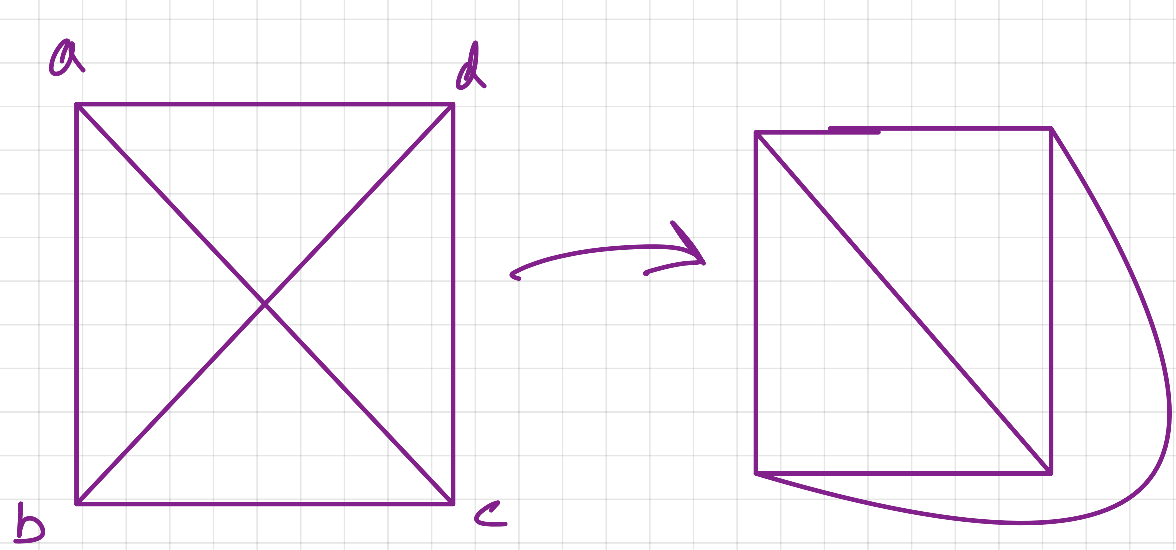




② Draw all simple connected graphs on 4 vertices.

PLANAR GRAPH

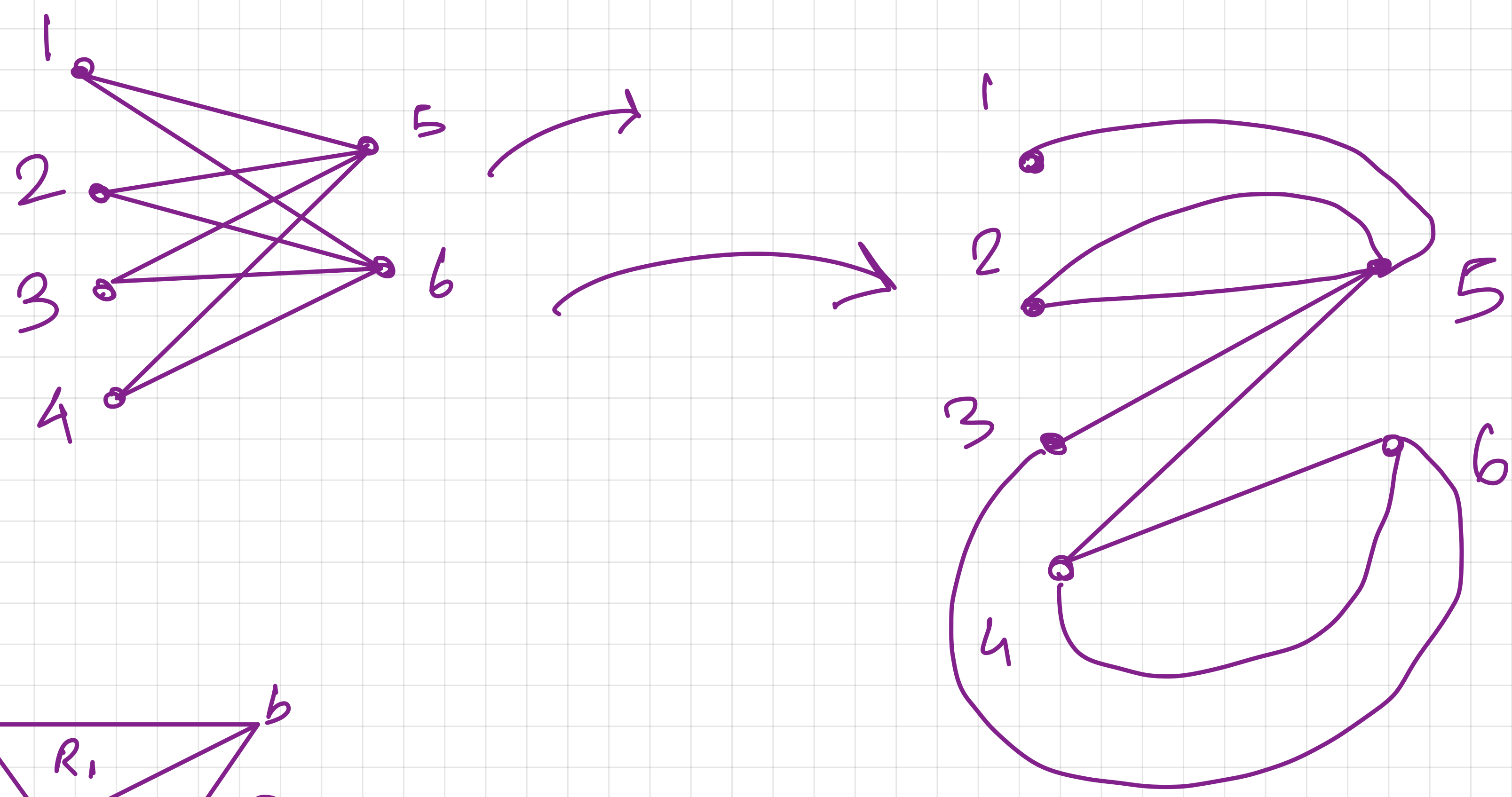


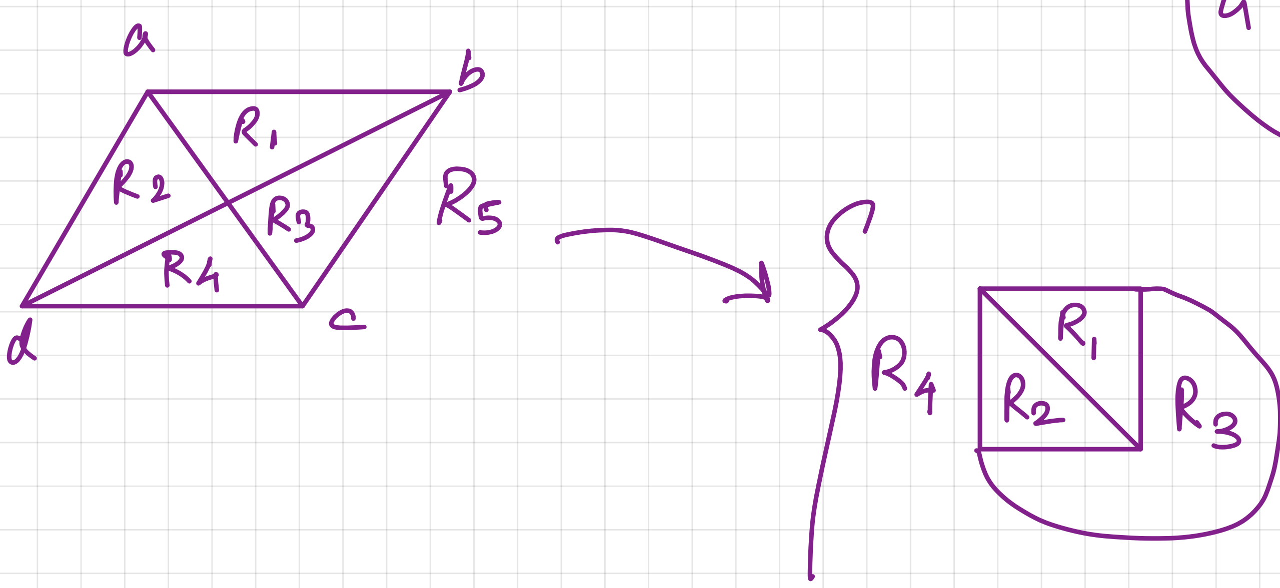


① Draw the graph in the plane in such a way such that no edges cross each other.

A graph which can be drawn on the plane in such a way such that no two edges cross each other.

Q ① Is $K_{4,2}$ planar?





If a graph is planar then the following condition holds:-

i) $\boxed{n - e + r = 2}$ ⇒ Euler Formula

ii) $e \leq 3n - 6$

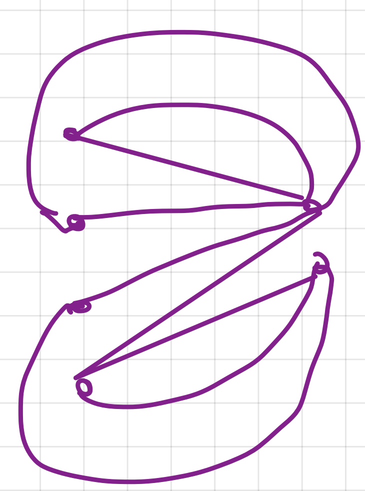
iii) $e \leq 2n - 4$

if G has no 3 length circuit

no of vertices = n
edges = e
region = r

Q. Check Euler formula. $K_{4,2}$

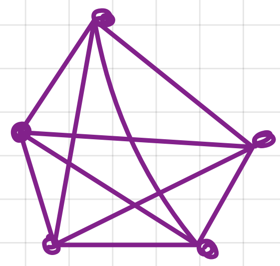
⇒ $n = 6$, $e = 8$, $r = 4$



Q. Show K_5 is not planar.

$\Rightarrow$ Here $n = 5$,

$e = 10$,



$e \leq 3n - 6$

$e \leq 15 - 6$

$\therefore 10 \leq 9$ [Not possible] Proved

Q. Is $K_{3,3}$ graph planar?

$\Rightarrow$ $K_{3,3}$ graph has no circuit of length 3

Now, $e \leq 2n - 4$

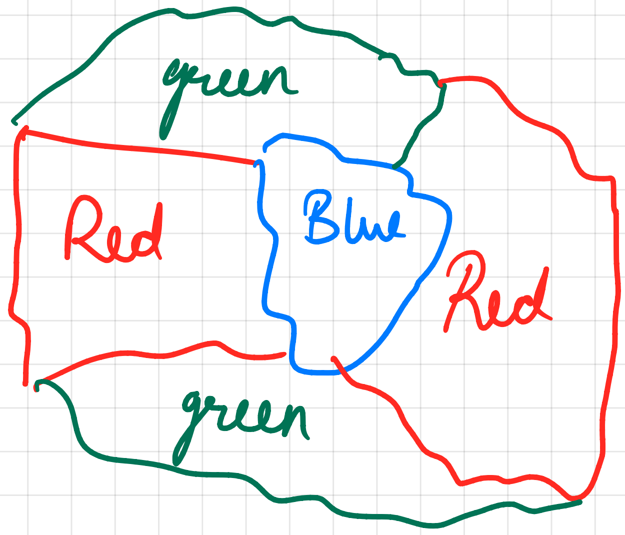
$\Rightarrow 9 \leq 2 \times 6 - 4$

$\Rightarrow 9 \leq 12 - 4$

$\Rightarrow 9 \leq 8$ $\therefore$ Condition not fulfilled.

$\therefore$ The graph is not planar.

GRAPH COLOURING :-

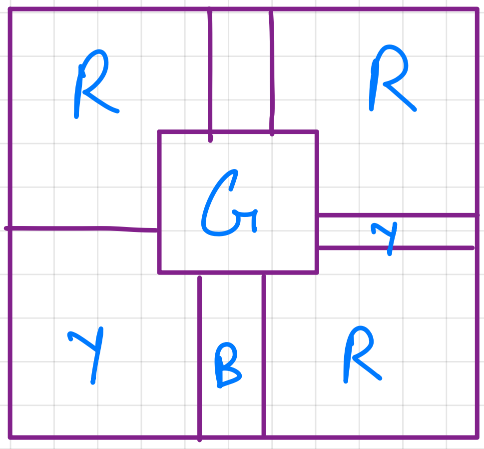


Vertices $\longrightarrow$ regions

edges $\rightarrow$ common boundary

Planar graph, then at most 4 distinct colours

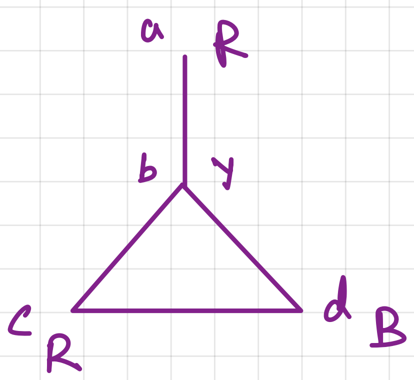
Colour the graph such that no adjacent side have same colour.

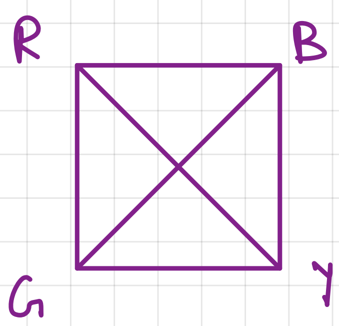


Chromatic Number :-

The minimum number of colours required to colour a graph such that no adjacent vertices have same colours.

Problem: Q. Find chromatic no. of K_n

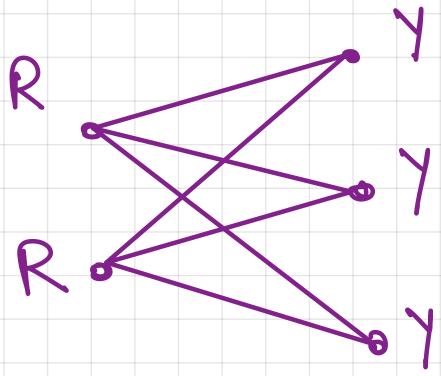




4

Q. Find chromatic number of $K_{m,n}$ → Complete bipartite graph.

⇒ eg: $K_{2,3}$

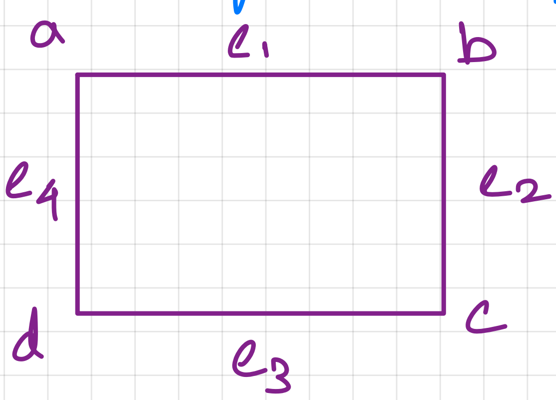


∴ $\boxed{CN = 2}$

{ For any complete bipartite graph CN = 2

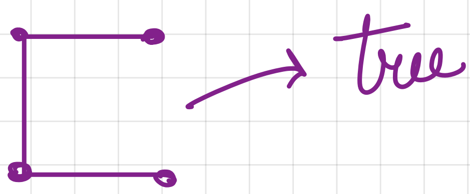
Trees :-

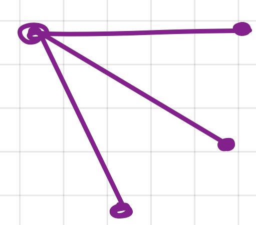
A cycle is a path whose end vertices are same.



$a e_1 b e_2 c e_3 d e_4 a$

A connected graph which has no cycle is called tree.





Properties of trees :-

① A tree of n vertices has $(n-1)$ edges.

② In a tree there exists exactly one path between any two vertices.

Q. Let G be a tree which has no vertices of degree more than 4. If G has 2 vertices of degree 2, 2 vertices of degree 3 and 1 vertex of degree 4, find the number of degree 1 vertices.

Assume degree 1 vertex $\rightarrow x$

No of vertices $= X + 2 + 2 + 1$

$= X + 5$

$\because$ G is a tree, $\therefore$ No of edges $= (n-1) = (X-5-1)$

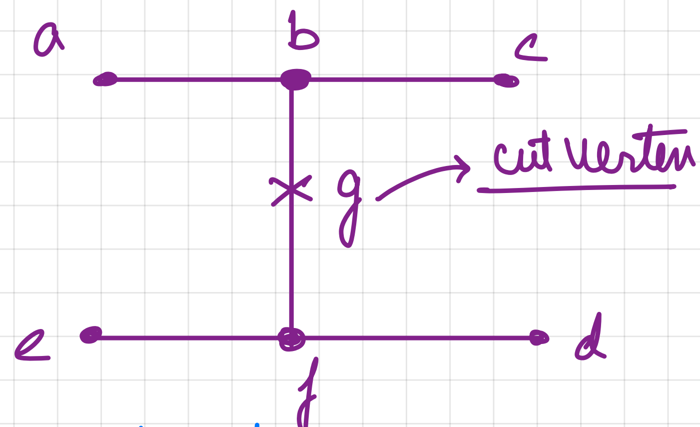
P.T.O $\longrightarrow$

$$
\begin{aligned}
\text{Total degree} &= x.1 + 2.2 + 2.3 + 1.4 \\
&= x + 4 + 6 + 4 \\
&= x + 14 = 2 \times (\text{no. of edges}) \\
\Rightarrow x + 14 &= 2 \times (x + 4) \\
\Rightarrow x + 14 &= 2x + 8 \\
\Rightarrow x &= 14 - 8 = 6
\end{aligned}
$$

$\therefore \boxed{x = 6}$ Ans

Cut Vertices :-

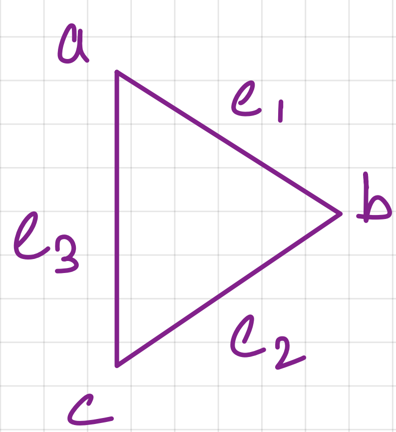
A vertex is said to be a cut vertex if removal of that vertex disconnects the graph.



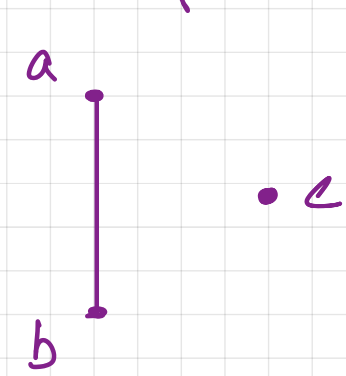
Cut-set : A set of edges E is said to be a cut-set of a graph G if —

(I) Removal of all elements of E disconnects the graph.

(II) Removal of some elements of E does not disconnect the graph.

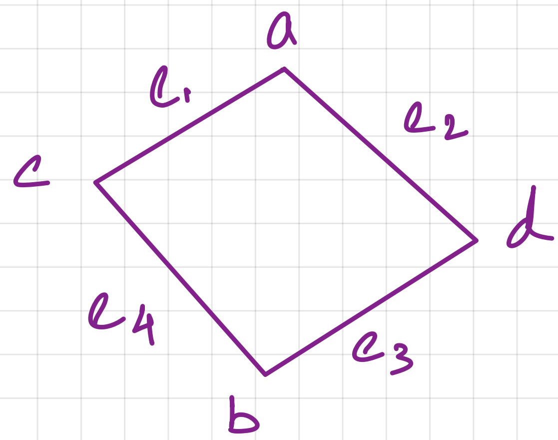


$$
\begin{aligned}
E &= \{e_1, e_2\} \\
&= \{e_1, e_3\} \\
&= \{e_2, e_3\}
\end{aligned}
\qquad \{e_1, e_2, e_3\}
$$

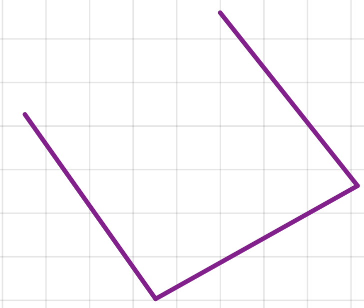


Deletion of Vertex :-

Removal of vertex means deleting that vertex & all other edges incident on that vertex.



$G\{e_1\} \Rightarrow$



06/Sep/2022

Problem: Q. Let G be a planar connected graph of 7 vertices and 14 edges.

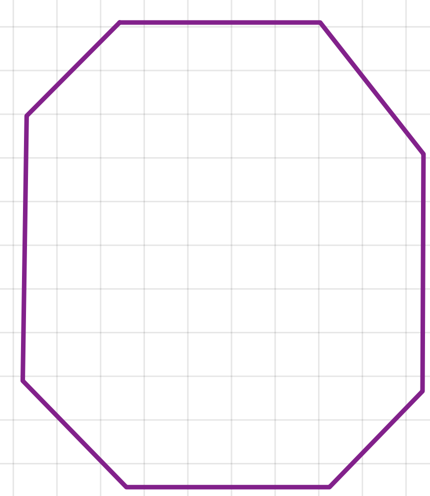
i) Find the no. of regions in which G divides the plane.

ii) Find the no. of bounded regions in which G divides the plane.

Soln: (i) $n - e + r = 2$

$\Rightarrow 7 - 14 + r = 2$

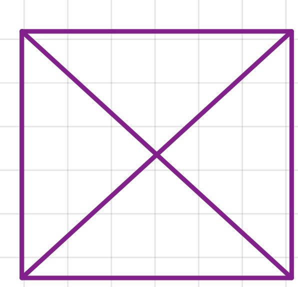
$\Rightarrow r = 2 + 7 = 9$

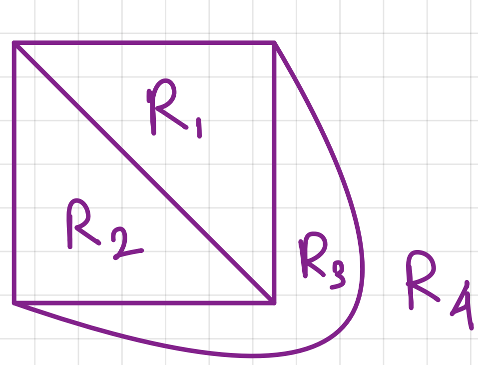


$\therefore$ The no. of regions are 9. **Ans**

(ii) The total no of bounded regions

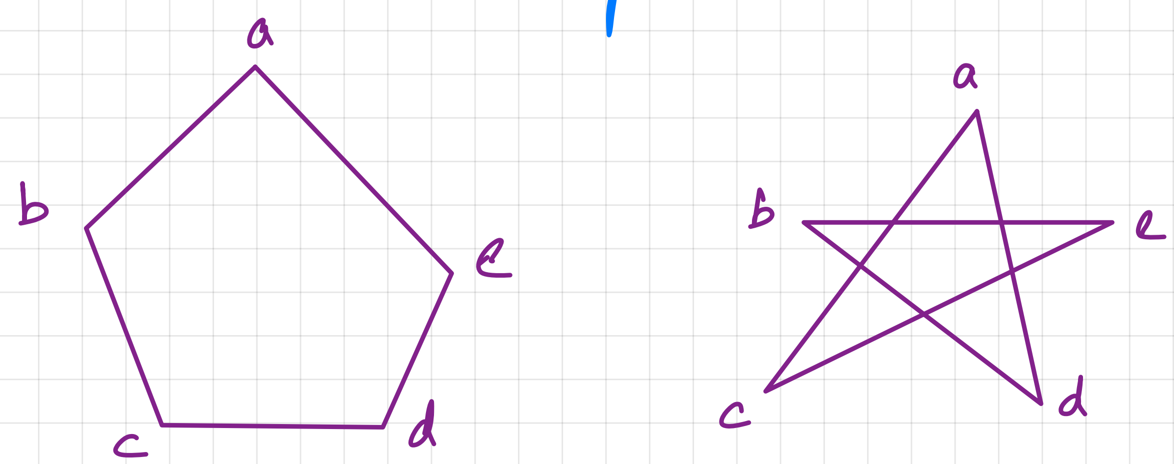
$= r - 1 = 9 - 1 = 8$



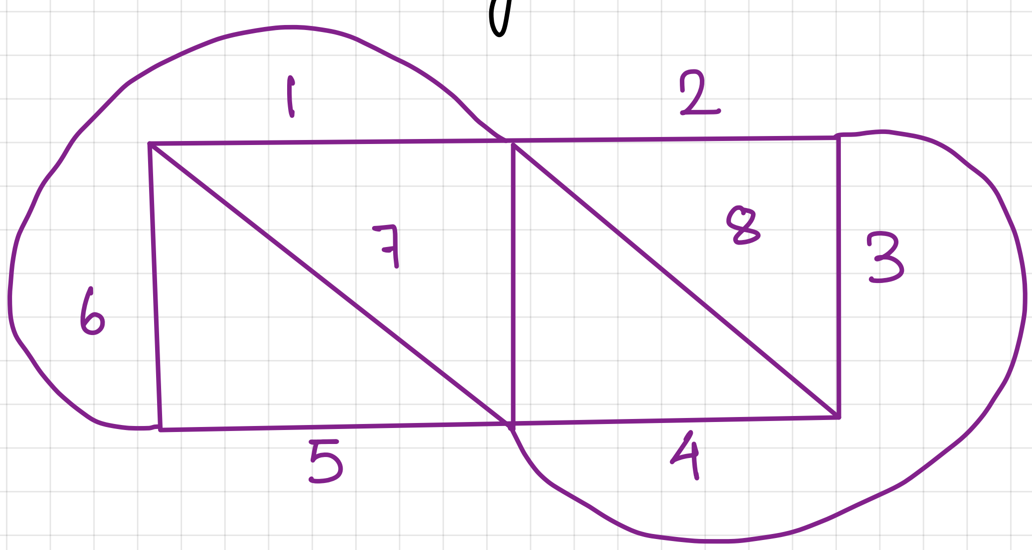


Chocolate problem: Q. Let G be a graph. If G is disconnected, then G^c is disconnected. (True / False)? {where G^c is complement of G).

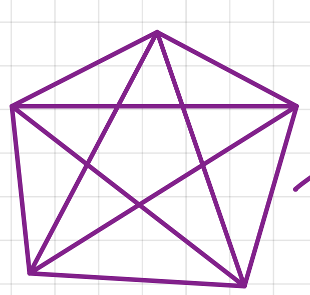
Soln: The statement is false.



Problem: Q. Draw a simple connected planar graph with 6 vertices and 10 edges. Can we have a simple disconnected non planar graph on 6 vertices and 10 edges?



$\Rightarrow$ $\boxed{\text{6 vertices and 10 edges}}$

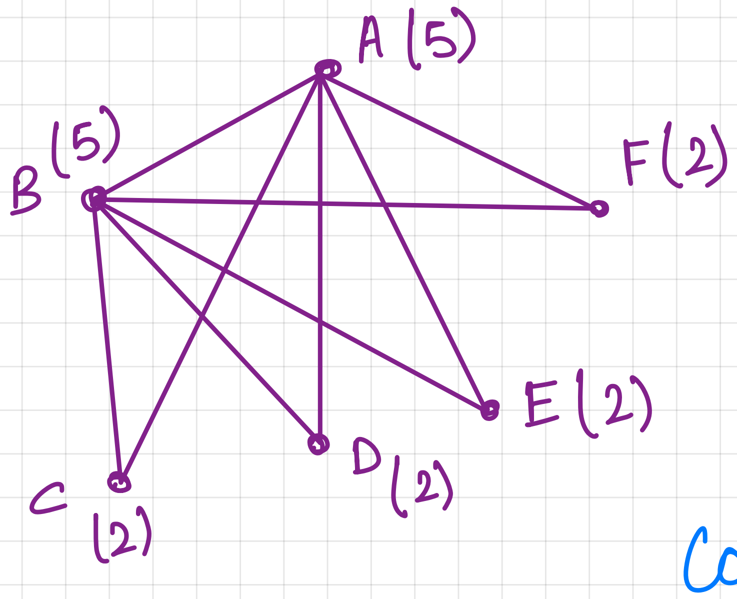


$\rightarrow$ Simple disconnected non-planar graph.

Q. Does there exist a simple connected graph with degree sequence, $(5, 5, 4, 2, 2, 2)$?

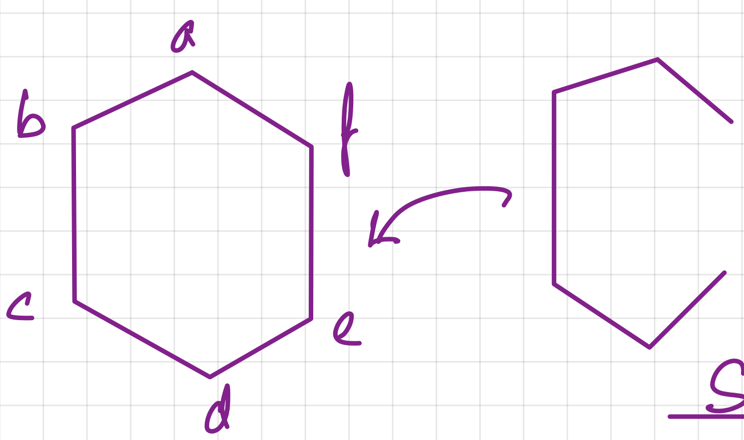
P.T.O

Soln:



The degree sequence says the degree of the third vertex is 4. But the 4th, 5th and 6th vertex has already degree 2. So we can't make the 3rd vertex adjacent to the 4th, 5th and 6th vertex.

Spanning Subgraph :-



Spanning Sub graph

MODULE 3

Mathematical Induction :-

i) Sum of 1st n natural numbers : $\frac{n(n+1)}{2}$

$\Rightarrow 1+2+3+\ldots+n = \frac{n(n+1)}{2}$

ii) $1^3+2^3+3^3+\ldots+n^3 = \frac{n^2(n+1)^2}{4}$

iii) $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Math Induction

(i) 1st check the identity for $n=1$

(ii) Assume that the identity is true for m.

(iii) Prove it for $n+1$.

Step 1 : $1+2+3+\dots+n = \frac{n(n+1)}{2}$

$n=1$;

$LHS = 1$;

$RHS = \frac{1(1+1)}{2} = \frac{2}{2} = 1$

Step 2 : $1+2+3+\dots\dots+n = \frac{n(n+1)}{2}$

Assumption

Step 3 : $(1+2+3+\dots+n)+n+1$

$= \frac{n(n+1)}{2} + n+1$

$= \frac{n(n+1)+2n+2}{2}$

$= \frac{n^2+n+2n+2}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)(n+2)}{2}$ [Proved]

$\because$ LHS = RHS in step 3, the statement is true for all $n \in N$.

Q. Prove that 3^n-1 is divisible by 2

Soln : $3^n - 1$

$RHS = \frac{2}{2} = 1$

Step-1 : Let $n=1$,

$LHS = 3^1 - 1 = 2$, which is divisible by 2.

Step-2 : $3^n - 1$ is divisible by 2

$\longrightarrow$ Assumption, $3^n - 1 = 2m$

Step-3 : We need to check if $3^{n+1} - 1$ is divisible by 2.

$\therefore 3^{n+1} - 1$

$= 3^n \cdot 3 - 1$

$= 3^n \cdot 3 - (3 - 2)$

$= 3^n \cdot 3 - 3 + 2$

$= 3(3^n - 1) + 2$

$= 3(2m) + 2 \quad \{\because 3^n - 1 = 2m\}$

$= 2(3m + 1) \longrightarrow$ which is divisible by 2 [Proved]

09/Sep/2022

Q. Show that $n^3 - 7n + 3$ is divisible by 3.

Soln : Let $n = 1$,

LHS $= 1 - 7 + 3$

$= -3$, which is divisible by 3

RHS $= -3/3 = -1$

Step-2 : Assume that $n^3 - 7n + 3 = 3m$ is divisible by 3

Step-3 : Proving $(n+1)^3 - 7(n+1) + 3$ is " " 3

$= n^3 + 3n^2 + 3n + 1 - 7n - 7 + 3$

$= n^3 + 3n^2 + 3n - 7n - 3$

$= 3m + 3n^2 - 3 = 3(m + n^2 - 1) \quad \{$ which is divisible by 3 $\}$

Chocolate problem Q. $3^n > n^2$

Soln: Step-1

$n = 1$

$3 > 1$

Step-2: Assume $3^n > n^2$

Step-3: Prove $3^{n+1} > (n+1)^2$

$\Rightarrow 3^{n+1} - (n+1)^2 > 0$

$\Rightarrow 3^n . 3 - n^2 - 2n - 1 > 0$

$\left[\because 3^n > n^2 \quad \therefore 3.3^n > 3.n^2 \right]$

$\Rightarrow 2n^2 - 2n - 1 > 0$

$\Rightarrow 2n(n-1) - 1 > 0 \quad \forall n$

So, $3^n > n^2 \quad \forall n$

13/Sep/2022

Q. Using Mathematical induction show that $7^n - 2^n$ is divisible by 5 $\forall n$.

Soln: Step-1, $n = 1$.

$7^1 - 2^1 = 5$

Step-2: "Assume" $7^n - 2^n = 5K$

Step-3: $7^{n+1} - 2^{n+1}$ (Prove)

$= 7^n . 7 - 2^n . 2$

$= 7^n . 7 - 2^n (7 - 5)$

$= 7^n . 7 - 2^n . 7 + 2^n . 5$

$= 7(7^n - 2^n) + 5 . 2^n$

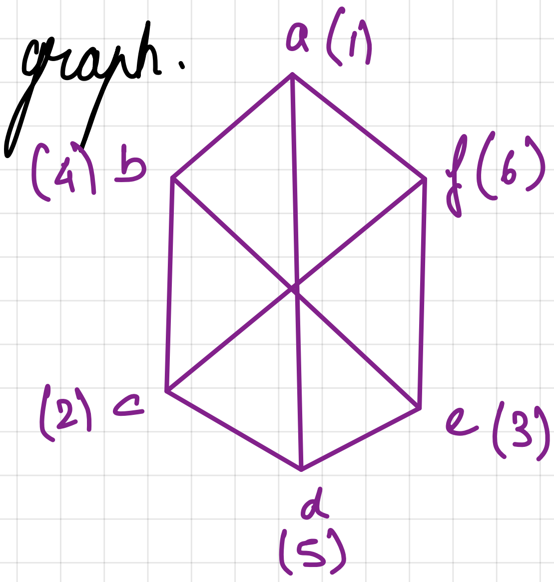
$= 7(5K) + 5 . 2^n \quad [\because 7^n - 2^n = 5k]$

$= 5(7K + 2^n)$

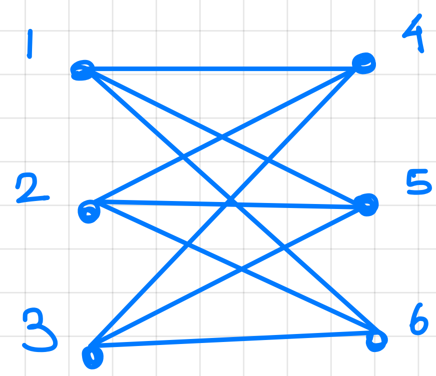
$\Rightarrow$ which is divisible by 5

[Proved]

Q. Is $K_{3,3}$ isomorphic to the following graph.



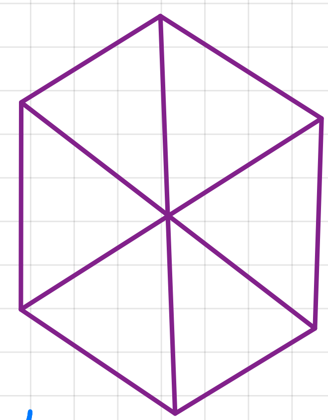
Soln:



$a \to 1$
$b \to 4$
$c \to 2$
$d \to 5$
$e \to 3$
$f \to 6$

Q. Is the following graph planar?

Soln: The given graph is isomorphic



$\cong K_{3,3}$, $e \leq 2n-4$

to $K_{3,3}$. But $K_{3,3}$ is a non planar. So the given graph is non planar.

Direct Proof:-

If n is odd, show that n^2 is odd.

- **Proof:**

$\Rightarrow$ A series of mathematical arguments to establish a mathematical statement.

If n is odd, then $n = 2k+1$

$n^2 = (2k+1)^2$

$n^2 = (2k+1)^2$

$= \underbrace{4k^2 + 4k}_{} + 1$

even number + 1 = odd number

$\Rightarrow$ $p \Rightarrow q$

$\underbrace{n \text{ is odd}}_{p} \Rightarrow \underbrace{n^2 \text{ is odd}}_{q}$

Proof by Contradiction :-

- **Proof** : If n^2 is odd, prove n is odd.

$p \Rightarrow q$

$p =$ If n^2 is odd.

$q =$ n is odd.

$p \Rightarrow q$

Assume, q is not true.

$\Rightarrow$ n is not odd.

$\Rightarrow$ n is even

$\Rightarrow$ $n = 2k$

$\Rightarrow$ $n^2 = 4k^2$ $\Rightarrow$ Contradicting P

Since P is given,

So, P can't be false.

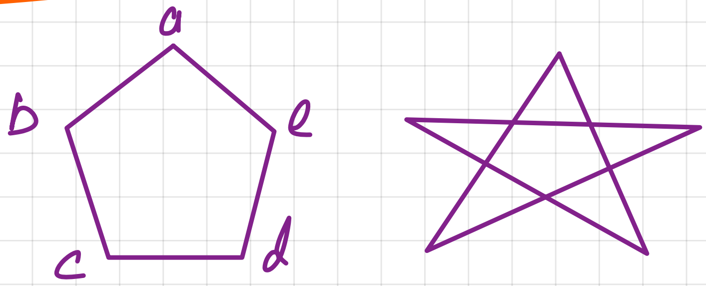
$\therefore$ Our assumption that q is false is not true. So q is true.

$\therefore$ n is odd

Assume q is false and arrive at a contradiction.

TO DO

Assignment ② Q1. Show that the following graphs are isomorphic



Q2. Draw all simple connected graphs on 4 vertices.

Neccenary and Sufficient :-

If Virat Kohli scores a ton, India wins.

p : Virat Kohli scores a ton

q : India wins.

$$p \Rightarrow q$$

"Neccenary condition but not a sufficient condition."

$p \Rightarrow q$ and $q \Rightarrow p$ is denoted by $p \Leftrightarrow q$

Example : n is even if and only if n^2 is even.

n is odd if and only if 3n+2 is odd

$\Rightarrow$ p : n is odd & q : 3n+2 is odd.

$p \Rightarrow q$ $\quad n = 2k+1$

$$3n+2 = 3(2k+1)+2 = 6k+3+2$$

$$= \underbrace{6k}_{\text{even}} + \underbrace{5}_{\text{odd}} = \text{odd}.$$

even + odd = odd.

$\underline{p \Leftrightarrow q}$

Relation & Elements :- 16/Sep/2022

A set is a well defined collection of elements.

- Null set
- Universal set
- Power Set

let, $A = \{1, 2, 3\}$

$\Rightarrow P(A) = \{\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \phi\}$

- Power Set is a set consisting of all subsets of A.

Q.1. If $A = \{\phi\}$ $|A| = 1$
find $P(A) = \{\{\phi\}, \phi\}$ $|P(A)| = 2^1 = 2$

Soln: If $A = \phi$ $|A| = 0$

$\therefore P(A) = \{\phi\}$ $|P(A)| = 2^0 = 1$

Q. If $A \cap B = A \cup B$ is $A = B$? {when $A \subseteq B$ and $B \subseteq A$ then $A = B$}

Soln: 1) $\underline{A \subseteq B}$ ie A subset B

$X \in A$

$\Rightarrow X \in A \cup B$

$\Rightarrow X \in A \cap B$

$\Rightarrow X \in B$

$\Rightarrow A \subseteq B$

2) $B \subseteq A$

$x \in B$

$\Rightarrow x \in A \cup B$

$\Rightarrow x \in A \cap B$

$\Rightarrow x \in A$

$\Rightarrow B \subseteq A$

Q. If $A = B$, show that $P(A) = P(B)$?

Soln: 1) $P(A) \subseteq P(B)$

Let, $x \in P(A)$, so x contain an element of A

But $A = B$ so x contains all elements of B.

So,

$A = B$

$\{A\} \in P(A)$

$\{B\} \in P(B)$

If $A = B$

$\{A\} = P(B)$

$\therefore A \subseteq B$ $\therefore P(A) \subseteq P(B)$

$\Rightarrow P(A) = P(B)$ Ans

Q. If $P(A) = P(B)$, then Show that $A = B$.

Soln: Let $x \in A$

then $\{x\} \in P(A) = P(B)$

then $\{x\} \in P(B)$

$\Rightarrow x \in B$

$\Rightarrow A \subseteq B$ —— ①

To show $B \subseteq A$

Let, $b \in B$

$\Rightarrow \{b\} \in P(B) = P(A)$

$\Rightarrow \{b\} \in P(A)$

$\Rightarrow b \in A$

$\Rightarrow B \subseteq A$ $\therefore$ This proves $A = B$.

TO DO

H.W Q.1. If $(A \cup B) = (A \cup C)$ & $(A \cap B) = A \cap C$

Show that $B = C$?

Q.2. If $A = B$, then Show that $P(A) = P(B)$

- **Relation** :-

Let A be a set then a relation R is a subset of $A \times A$.

$A \times A$ = Cartesian product of A with A.

Let, $A = \{1, 2, 3\}$, $\therefore R = \{(1,1), (2,1), (3,1), (1,3)\} \subseteq A \times A$

R is a relation on A.

- **Reflexive**: If $(a,a) \in R$ for all $a \in A$.
- **Symmetric**: If $(a,b) \in R$, implies $(b,a) \in R$ for all $(a,b) \in R$, then R is symmetric
- **Transitive**: If $(a,b) \in R$ & $(b,c) \in R$ implies that $(a,c) \in R$ for all $a,b,c \in A$

$A = \{1, 2, 3\}$

$R = \{(1,1), (2,2), (1,3), (1,2)\}$ Not transitive

$R = \{(1,2)\ (2,3)\}$

$(1,3) \notin R$ $\quad$ $(1,2)$

$(2,3)$

$(1,3)$

So, R is not transitive.

- **Equivalence**: A relation R is said to be equivalence if R is reflexive, symmetric and transitive.

Q. Show that $aRb \Leftrightarrow |a-b| =$ even is an equivalence relation on N.

Soln:

Q. ① $R = \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,a), (c,b), (c,c)\}$ check if R is reflexive, symmetric & transitive.

Soln:- Reflexive: $(a,a), (b,b), (c,c) \in R$ so, R is reflexive.

Symmetric: $(a,b) \in R$ but $(b,a) \notin R$
So, R is not symmetric.

Transitive: $(a,c) \notin R$ ∴ Not transitive

Q. ② $aRb \iff 1+ab>0$

Soln:- Reflexive (a,a)

$1+a \cdot a = 1+a^2 > 0$

So R is reflexive.

- Symmetric: aRb

$1+ab>0 \Rightarrow 1+ba>0$

$\Rightarrow (b,a) \in R$

So, R is symmetric

- Transitive: aRb & bRc

i) $1+ab>0$ To check $1+ac>0$ or not for every $(a,b,c) \in R$

ii) $1+bc>0$ Let, $a=1$, $b=-1/2$, $c=-2$

∴ $1+ab>0$
$= 1-1/2>0$
$= 1/2>0$

$1+bc>0$
$1+bc>0$
$3>0$

So, $1+ac$
$1+(1)(-2)$
$= 1-2$
$= -1<0$

So, R is not transitive.

So, R is not equivalence.

Q.③ S = Set of all lines in plane let $L_1 R L_2$ if and only if L_1 is perpendicular to L_2. Is R reflexive symmetric & trans.

Q.④ $aRb \iff a \leq b^2$ $aRb \iff a \leq b^2$

Reflexive: (a, a) $a = \frac{1}{2} = 0.5$ $a^2 = \frac{1}{4} = 0.23$

$a \leq a^2$, So R is not reflexive.

Symmetric: aRb

When $a \leq b^2$ to check $b \leq a^2$,

let, $a = 1$, $b = 5$ So, $1 \leq 5^2$ but $5 \leq 1^2$

So, R is not symmetric.

Transitive: aRb bRc

When, $a \leq b^2$, $b \leq c^2$ then to check $a \leq c^2$.

let, $a = 1$, $b = -5$, $c = \frac{1}{4}$

$a \leq b^2 \rightarrow 1 \leq (-5)^2 = 25$

$b \leq c^2 \longrightarrow -5 \leq (\frac{1}{4})^2$

$a \leq c^2 \longrightarrow 1 \leq (\frac{1}{4})^2$ (not true)

So R is not transitive

Soln ③ $L_1 R L_2 \iff L_1 \perp L_2$

Reflexive: Since L_1 can't be perpendicular to itself

Symmetric: if $L_1 \perp L_2$ then $L_2 \perp L_1$ So R is symmetric.

Transitive: $L_1 \perp L_2$ $L_2 \perp L_3$ but $L_1 \not\perp L_3$ because it will be perpendicular to L

TO DO

H.W Q. Check for reflexive, symmetric and transitive

- $R = \{(a,b)\ (b,a), (a,c), (c,a)\}$

- $R_3 = \{(a,b)\ (b,c), (c,a)\}$

- $R = \{(1,1)\ (2,2)\ (3,3)\ (1,2)\ (2,3)\}$

N.Deo → graph theory.

Problem: On the set of real number check

- $a R b \Leftrightarrow a \leq b$
- $a R b \Leftrightarrow a = 1/b$

Q. $aRb \Leftrightarrow a$ is divisible by b.

Soln: Reflexive: (a, a)

Given a is divisible by b then check a is divisible by a.

Let $a = 2$, So $2/2 = 1$

$\therefore$ R is reflexive.

Symmetric :- Let, $a = 4$ and $b = 2$,

$\therefore a/b = 4/2 = 2$ & $b/a = 2/4 = 0.5$

$\therefore$ Not natural number. $\therefore$ Not symmetric.

Transitive :-

When a is divisible by b, b is divisible by a then to check a is divisible by c.,

Let $a = 8$, $b = 4$, $c = 6$

(aRb)

Let, $a = mb$.

(bRc)

Let, $b = nc$

now, $a/c = \frac{mb}{c} = \frac{mnc}{c} = mn$

$\therefore$ a is divisible by c $\forall (a,c) \in R$ $\therefore$ Transitive

Antisymmetric

If for all elements $a, b, c \in A$ $(a,b) \in R$ & $(b,a) \in R$ implies that $a = b$, then R is antisymmetric.

Example: Consider the set of all natural numbers & $aRb \Leftrightarrow a$ is divisible by b. Check if antisymmetric.

Soln: aRb, then a is divisible by b.

$\Rightarrow a = Kb \rightarrow$ (i)

bRA, then b is divisible by a

$\Rightarrow b = ma \longrightarrow$ (ii)

from (i) & (ii)

$a = Kma \Rightarrow Km = 1 \Rightarrow K = m = 1$

Anti symmetric → If there exists an arrow between vertices it should be one sided arrow. If not its not antisymmetric

Important!

Example: Consider the set of all integers $aRb \Leftrightarrow a$ is divisible by b. check it antisymmetric.

Soln: let, $a = -4$ $\quad b = 4$

a is divisible by b.

b is divisible by a. but $a \neq b$.

Partial Ordering Relation

A relation R which is reflexive, antisymmetric & transitive

Example: Take the set of natural numbers where $aRb \Leftrightarrow a$ is divisible by b.

- ## Relation

Q. Let A be a given set and P(A) be the set of all subset of A. Define a relation R on P(A) by $aRb \Leftrightarrow a \subseteq b$. Check if R is Reflexive, Asymmetric or transitive is R a P.O?

Soln: ① (R) $\Rightarrow$ To show aRa

$a \subseteq a$ (+), since any set is subset of itself, so R is reflexive.

② Symmetric : $a \subseteq b$ $\quad a = \{1\}$

$b \not\subseteq a$ $\quad b = \{1, 2, 3\}$

So, not symmetric.

③ Anti symmetric : Let ARb & bRa

$\Downarrow$

$a \subseteq b$ & $b \subseteq a$

$\Rightarrow a = b$ $\quad$ So, R is Anti symmetric.

Transitive : Given aRb, bRc

$\Rightarrow a \subseteq b$ & $b \subseteq c$

To show : $a \subseteq c$

Let, $x \in a$

$\Rightarrow x \in b$

$\Rightarrow x \in c$

$\Rightarrow a \subseteq c$ $\quad \therefore$ R is transitive.

Q. Let $A = \mathbb{R}$ (real no.). Define $aRb \Longleftrightarrow a \leq b$ in R a Reflexive, Symmetric and transitive. Is R equivalent & P.O ?

Soln: Reflexive: aRa

$a \leq a \Rightarrow aRa\ (R)$

Symmetric: Assume aRb

$a \leq b$

To show bRa, $b \leq a$

$a = 1$, $b = 2$ ∴ Not symmetric.

Transitive: aRb $a \leq b$

bRc $b \leq c$

So, $a \leq b \leq c$

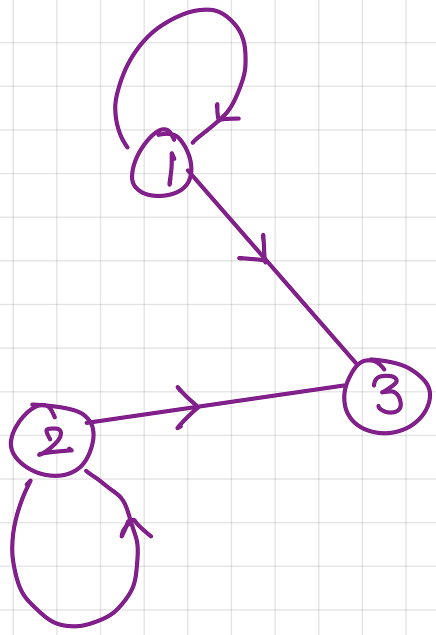
$\Rightarrow aRc$ ∴ Transitive

Antisymmetric: aRb & bRa

- **Relation Using Graph** :-

$R = \{(1,1), (2,2), (2,3), (1,3), (3,1)\}$

$A = \{1, 2, 3\}$



Reflexive: If there exist a loop in every vertex.

Symmetric : If there exist a directed edge from a to b, and there exist a directed edge from b to a, for every pair of a and b.

Transitive : If there exist a directed edge from a to b & b to c, then there exists a directed edge from a to c.

Function : Let A & B be two sets. A relation $R \subseteq A \times B$ is said to be a function if the following hold

- for each $a \in A$, there exists $b \in B$ such that $(a, b) \in R$.
- if $(A, b) = (a, c)$ then $b = c$,

$A = \{1, 2\}$, $B = \{a, b, c\}$

- $R = \{(o, a), (1, b)\}$ R but not F.
- $R = \{(1, a), (1, b), (2, a)\}$
- $R = \{(1, a)\ (2, b)\}$

11/10/2022

Finite Set : A set which has finite number of elements.

Infinite Set : A set which is not finite. → Countable, → Uncountable

- **Countable Set** :-

A set is said to be countable if there exists and bijection from from S to $\mathbb{N}$.

- **Uncountable Set :** A set which is not countable.

Example : 1) Set of all integers (Countable set)

2) Set of all real numbers in b/w $[0,1]$ (Uncountable Set)

- **Binary Operation :-**

A function $f: S \times S \to S$ is said to be a binary operation on S.

Relation :-

A relation $f \subseteq A \times B$ is a function from A to B.

i) for each $a \in A$, there, exists $b \in B$, such that $(a,b) \in R$

ii) $(a,b) \in R$ & $(a,c) \in R \Rightarrow b = c$

Q. On $\mathbb{N}$, define $a * b = a + b \to$ (Binary Operation) true for $\mathbb{Z}$ also.

$a * b = a - b \to$ (Not binary operation) for $\mathbb{N}$ true for $\mathbb{Z}$

Q. On $\mathbb{Z}$, define $a * b = \frac{a}{b} \longrightarrow$ (Not Binary Operation)

Q. On $\mathbb{Q}$, define $a * b = \frac{a}{b} \longrightarrow$ (Not Binary Operation)

↳ (rational numbers)

$\frac{3}{0}$ = undefined.

Q. Let $A = \{1, 2, 3, 4\}$ $a * b$ = lcm of a and b.

↳ (Not a Binary Operation)

Associative :-

A binary operation "$*$" is said to be an associative if
$(a*b)*c = a*(b*c)$

Q. Define "$*$" on $\mathbb{N}$ as $a*b = a+b$

Soln: $a*(b*c) = a*(b+c)$

$= (a+b+c)$ — (i)

$(a*b)*c = (a+b)*c$

$= (a+b)+c$ — (ii)

(i) = (ii) $\therefore$ associative Binary Operation. [Ans]

Q. $a*b = 2^{ab}$

$\Rightarrow$ $\therefore$ $a*(b*c) = a*(2^{bc})$

$= 2^{a.2^{bc}}$ — (i)

$(a*b)*c = 2^{ab}*c = 2^{2^{ab}.c}$ — (ii)

$\therefore$ (i) $\neq$ (ii), so $*$ is not associative. [Ans]

Q. Define '$*$' on $\mathbb{N}$ as $a*b = ab+1$

Soln:

- **Identity** :-

If there exists an element $e \in S$, such that,

$$a * e = e * a = a, \text{ for all } a \in S,$$

then, e is said to be identity.

Q. On $\mathbb{N}$, define $a * b = a + b$, check if $(\mathbb{N}, *)$ has an identity.

Q. On $\mathbb{N}$, define $a * b = ab$, find identity.

Soln: $a * e = e * a = a$

$\Rightarrow ae = a$

$\Rightarrow e = 1$

- **Sets Relations & Functions** :-

01/Nov/2022

i) Commutative $\rightarrow$ $a * b = b * a$

ii) Associative $\rightarrow$ $a * (b * c) = (a * b) * c$

iii) Identity $\rightarrow$ $a * e = a \ \forall a \in A$

iv) Inverse $\rightarrow$ $a * b = e$

$* : S \times S \rightarrow S$

$a * b = a + b$. $S = \mathbb{N}$

i) $a * b = a + b$

$b * a = b + a$ $\therefore$ So commutative

(11) $a * (b * c) = (a * b) * c$

LHS : $a * (b * c)$

$= a * (b + c)$

$= a + (b + c)$

RHS : $(a * b) * c$

$= (a + b) * c$

$= (a + b) + c$

So, Associative

Q. $a * b = a + b - ab$ over N.

Commutative : LHS $\Rightarrow$ $a * b = a + b - ab$

RHS $\Rightarrow$ $b * a = b + a - ba$

$= a + b - ab = \text{LHS}$

- Associative : LHS $\rightarrow$

$a * (b * c)$

$= a * (b + c - bc)$

$= a + b + c - bc - a(b + c - bc)$

$= a + b + c - bc - ab - ac + abc$

RHS : $(a * b) * c$

$= (a + b - ab) * c$

$= a + b - ab + c - (a + b - ab)c$

$= a + b - ab + c - ac - bc + abc$

$= a + b + c - ab - ac - bc + abc. = \text{LHS}$

Identity :- To find 'e' such that, $a * e = a$

$a * e = a + e - ae$

$\therefore \cancel{a} + e - ae = \cancel{a}$

$\Rightarrow e - ae = 0$

$\Rightarrow e(1-a) = 0$

$\Rightarrow e = 0 \quad (\because a \neq 1 \; \forall a)$

- Inverse : Given $a \in S$,

We need to find $b \in S$

such that $a * b = e$,

Now, $a * b = e = 0$

$\Rightarrow a + b - ab = 0$

$\Rightarrow b - ab = -a$

$\Rightarrow b(1-a) = -a$

$\Rightarrow b = -\dfrac{a}{1-a}$

Q. $a * b = a + b + ab$ over $\mathbb{N}$.

Soln:- Commutative : LHS

$a * b = a + b + ab$

RHS

$b * a = b + a + ba$

$= a + b + ab = LHS$

- Associative : $a * (b * c) \longrightarrow LHS$

$= a * (b + c + bc)$

$= a + b + c + bc + a(b + c + bc)$

$= a + b + c + bc + ab + ac + abc$

RHS: $(a * b) * c$

$= (a + b + ab) * c$

$= a + b + ab + c + (a + b + ab)c$

$= a + b + ab + c + ac + bc + abc = LHS$

- **Identity**: To find e such that $a * e = a$

$$a * e = a$$

$$a * e = a + e + ae$$

$\Rightarrow \cancel{a} + e + ae = \cancel{a}$

$\Rightarrow e + ae = 0$

$\Rightarrow e(1 + a) = 0$

$\Rightarrow e = 0$

- **Inverse**: Given $a \in S$, we need to find $b \in S$, such that

$$a * b = e$$

$$a * b = e = 0$$

$$a + b + ab = 0$$

$\Rightarrow b + ab = -a$

$\Rightarrow b(1 + a) = -a$

$\Rightarrow b = -\dfrac{a}{1+a} \Rightarrow$ So, it is inverse.

- **Group**:–

Let G be a non empty set. Let us consider binary operation "$*$" on G. If $*$ satisfies the following conditions :–

i) $*$ is associative.

ii) G has identity with respect to $*$

iii) For each element $a \in G$, there exists $b \in G$, such that $a * b = e$, then. $(G, *)$ is a group.

- **G has an identity with respect to $*$.** $\langle ii \rangle$

$G * e = G$

$\Rightarrow G + e = G$

$\Rightarrow e = 0 \notin N$

- **$*$ is associative :-** $\langle i \rangle$

$a * (b * c)$ | $(a * b) * c$
$= a * (b + c)$ | $= (a+b) * c$
$= a + (b + c)$ | $= (a+b) + c$
$= a + b + c$ | $= a + b + c$

$\therefore$ It is associative

Q. Is $(\mathbb{Z} ; +)$ a group?

$\mathbb{Z} = \{-3, -2, -1, 0, 1, 2, 3\}$

Identity $= 0$ $\quad a * e = a$

$\Rightarrow a + e = a$

$\Rightarrow e = 0$

$\therefore$ So, it is an identity.

Q. Is $(\mathbb{Z} ; \cdot)$ a group?

i) $a * e = a$

$\Rightarrow \cancel{a} \cdot e = \cancel{a}$

$\Rightarrow e = 1$ $\quad \therefore$ So, it is an identity.

ii) $a * (b * c)$ | $(a * b) * c$
$= a * (b \cdot c)$ | $= (a \cdot b) * c$
$= a \cdot b \cdot c$ | $= a \cdot b \cdot c$

$\therefore$ So, it is associative

iii) $a * b = e = 1$

$\Rightarrow a b = 1$

$\Rightarrow b = \frac{1}{a}$ $\Rightarrow$ So, it is not inverse.

Ans: Take $a = 2$; $b = \frac{1}{2} \notin \mathbb{Z}$

$\therefore (\mathbb{Z} ; \cdot)$ is not a group.

Q③ Is $(\mathbb{Q};\cdot)$ a group?

Q.④ $M=\left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x\in\mathbb{R}\right\}$, Does, $(M;\cdot)$ form a group?

Soln ③

$a*(b*c)$
$= a*(b.c)$
$= a.b.c$

$(a*b)*c$
$= (a.b)*c$
$= a.b.c$

$\therefore$ So, it is associative.

$a*e=a$
$\Rightarrow \cancel{a}.e=\cancel{a}$
$\Rightarrow e=1\notin\mathbb{Q}$ $\therefore$ So, it is not an identity.

Ans: $(\mathbb{Q};\cdot)$ cannot be a group.

Soln ④ $M=\left\{\begin{pmatrix} x & x \\ x & x \end{pmatrix} : x\in\mathbb{R}\right\}$

$A*B=AB$, $A*B=B*A$, $A=\begin{bmatrix} x & x \\ x & x \end{bmatrix}$, $B=\begin{bmatrix} y & y \\ y & y \end{bmatrix}$

$$A*B=\begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} y & y \\ y & y \end{bmatrix}$$
$$=\begin{bmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{bmatrix}$$
$$=\begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix}$$

$$B*A=\begin{bmatrix} y & y \\ y & y \end{bmatrix}\begin{bmatrix} x & x \\ x & x \end{bmatrix}$$
$$=\begin{bmatrix} xy+xy & xy+xy \\ xy+xy & xy+xy \end{bmatrix}=\begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix}$$

$\therefore$ $A*B=B*A$; So, it is commutative.

$$A * (B * C) = (A * B) * C \qquad \left\{ A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}; B = \begin{bmatrix} y & y \\ y & y \end{bmatrix}; C = \begin{bmatrix} z & z \\ z & z \end{bmatrix} \right\}$$

$$A * B = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} y & y \\ y & y \end{bmatrix}$$

$$= \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix}$$

$$[A * B] * C = \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \begin{bmatrix} z & z \\ z & z \end{bmatrix}$$

$$= \begin{bmatrix} 2xyz + 2xyz & 2xyz + 2xyz \\ 2xyz + 2xyz & 2xyz + 2xyz \end{bmatrix} = \begin{bmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{bmatrix}$$

Similarly, $A * (B * C)$

$$B * C = \begin{bmatrix} y & y \\ y & y \end{bmatrix} \begin{bmatrix} z & z \\ z & z \end{bmatrix}$$

$$= \begin{bmatrix} 2yz & 2yz \\ 2yz & 2yz \end{bmatrix}$$

$$A * (B * C) = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} 2yz & 2yz \\ 2yz & 2yz \end{bmatrix}$$

$$= \begin{bmatrix} 4xyz & 4xyz \\ 4xyz & 4xyz \end{bmatrix}$$

$\therefore$ So, it is associative.

Identity :-

$$A * e = A$$

$$e = \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}, \quad A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$\begin{bmatrix} x & x \\ x & x \end{bmatrix} \begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2x\alpha & 2x\alpha \\ 2x\alpha & 2x\alpha \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$$

$$\therefore 2x\alpha = x$$

$$\Rightarrow 2x\alpha - x = 0$$

$$\Rightarrow x(2\alpha - 1) = 0$$

$$\Rightarrow 2\alpha - 1 = 0$$

$$\Rightarrow \alpha = 1/2$$

$$\therefore e = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

So, it is an identity.

04/Nov/2022

LOGIC (Module 1)

Proposition :-

A statement which has either a true or a false value.

Eg. i) $2+7=11 \longrightarrow$ False proposition

ii) $2+7=9 \longrightarrow$ True proposition

iii) The sky is red. $\longrightarrow$ False proposition

iv) Let us go to sleep $\longrightarrow$ Neither false nor true, so not proposition.

Logical Operators :-

i) OR $\vee$ ii) AND $\wedge$ iii) Negation/Not $\sim$

Truth Table :-

OR Gate :

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

NOT :

P	$\sim P$
T	F
F	T

AND :

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Q. Find the truth table of $(P \wedge \sim Q) \vee (\sim R)$?

p	q	r	$\sim q$	$\sim r$	$p \wedge \sim q$	$(p \wedge \sim q) \wedge (\sim r)$
T	T	T	F	F	F	F
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	T	F	F	T	F	T
F	F	T	T	F	F	F
F	F	F	T	T	F	T

Q. Find the truth table for $(p \wedge \sim q) \wedge (\sim r)$. (Home Work)

- Tautology :-

The proposition all of whose output values are true.

- Contradiction :-

The proposition all of whose output values are false.

- Contingency :-

If some of the output values are true & some of them are false, then the proposition is known as contingency.

Q. Check if the following are tautography, contradiction or contingency.

i) $p \vee \sim p$

ii) $p \wedge \sim p$

iii) $p \vee (\sim (p \wedge q))$

Soln: i)

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

$\therefore$ $P \vee \sim P$ is a tautography [Proved]

ii)

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

$\therefore$ $P \wedge \sim P$ is a Contradiction [Proved]

iii)

P	q	$P \wedge q$	$\sim(P \wedge q)$	$P \vee (\sim P \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

$\therefore$ $P \vee (\sim(P \wedge q))$ is a tautology. [Proved]

- **Laws:-**

→ **Indempotent:** $a \vee a = a$
$a \wedge a = a$

→ **Commutative:** $a \wedge b = b \wedge a$
$a \vee b = b \vee a$

→ **Associative:** i) $a \vee (b \vee c) = (a \vee b) \vee c$
ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

→ **Distributive:** i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

→ **De Morgan:** i) $\sim (a \vee b) = \sim a \wedge \sim b$
ii) $\sim (a \wedge b) = \sim a \vee \sim b$

i) $a \vee T = T$; $a \vee F = a$

$a \wedge T = a$; $a \wedge F = F$

Implication :-

$p \to q$ (If p then q)

Truth Table :-

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	F

p : It is cloudy.

q : It will rain.

$p \to q$

Bi-implication :-

$p \to q \wedge q \to p$ denoted by $p \leftrightarrow q$ (p if and only if q)

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

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Q. Check if $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$ Using truth table.

Soln : $p \leftrightarrow q$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	
T	T	T	T	T	
T	F	F	T	F	
F	T	T	F	F	
F	F	T	T	T	

Q2. S.T $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

Soln : $p \to q \wedge q \to p$

$\Rightarrow (\sim p \vee q) \wedge (\sim q \vee p)$

let, $r = (\sim p \vee q)$ assume

$r \wedge (\sim q \vee p)$

$= (r \wedge \sim q) \vee (r \wedge p)$

$= (\sim p \vee q) \wedge \sim q$

$= (\sim p \wedge \sim q) \vee (q \wedge \sim q)$

$= (\sim p \wedge \sim q) \vee F = (\sim p \wedge \sim q)$ ——— (i)

(ii) $r \wedge p$

$= (\sim p \vee q) \wedge p$

$= (\sim p \wedge p) \vee (q \wedge p)$

$= F \vee (p \wedge q) = p \wedge q$

From, (i) & (2), LHS = RHS.

H.W (i) $p \leftrightarrow p \wedge q \equiv p \to q$

TO DO

(ii) $(p \to q) \wedge (r \to q) \equiv (p \vee r) \to q$

Soln (ii) LHS

$(p \to q) \wedge (r \to q)$

$\equiv (\sim p \vee q) \wedge (\sim r \vee q)$

$\equiv (q \vee \sim p) \wedge (q \vee \sim r)$ (using commutative law)

$\equiv q \vee (\sim p \wedge \sim r)$ (Using distributive law)

$\equiv q \vee \sim(p \vee r)$ (Using De Morgan law)

$\equiv \sim(p \vee r) \vee q$

$\equiv (p \vee r) \rightarrow q = \text{RHS}$

$\left[\because p \rightarrow q = \sim p \vee q \quad \therefore (\sim p \vee r) \rightarrow q = \sim(p \vee r) \vee q\right]$

[Proved]

Soln (i) $p \longleftrightarrow p \wedge q \equiv p \longrightarrow q$

LHS

$p \longleftrightarrow p \wedge q$

$\equiv (p \longrightarrow p \wedge q) \wedge (p \wedge q \longrightarrow p)$

$\equiv \{\sim p \vee (p \wedge q)\} \wedge \{\sim(p \wedge q) \vee p\}$

$\sim p \vee (p \wedge q)$

$\equiv (\sim p \vee p) \wedge (\sim p \vee q)$ (using distributive law)

$\equiv T \wedge (\sim p \vee q)$

$\equiv (\sim p \vee q)$ —— (i)

$\sim(p \wedge q) \vee p$

$\equiv (\sim p \vee \sim q) \vee p$ (Using de margan theory)

$\equiv p \vee (\sim p \vee \sim q)$

$\equiv (p \vee \sim p) \vee \sim q$ (Using associative law)

$\equiv T \vee \sim q$

$\equiv T$ —— (ii)

From (i) and (ii) we get —

LHS

$p \longleftrightarrow p \wedge q =$

• Recurrence Relation :-

Important!

(i) A relation in which a term is related to it's previous terms.

$a_1 = 1,\ a_2 = 2,\ a_3 = 3,\ a_4 = a_3 - 1 = 3 - 1 = 2,\ a_5 = a_4 - 1$
$= 2 - 1 = 1$

$\Rightarrow a_n = a_{n-1} - 1 \quad n \geq 4$

$a_n = \frac{n}{n+1}$

$a_0 = \frac{0}{1} = 0$

$a_1 = \frac{1}{1+1} = \frac{1}{2}$

$a_2 = \frac{2}{2+1}$

$= \frac{2}{3}$

→ **Order & degree :-**

Order :-

$a_r - 6a_{r-1} + 7a_{r-2} = 0$

$a_0 = 1$

$a_1 = 2$

$V = 3,$

$a_2 - 6a_1 + 7a_0 = 0$

$\Rightarrow a_2 = -7a_0 + 6a_1$

$= -7 + 12 = 5$

Order = Highest Subscript − lowest subscript

$= r - (r-2) = r - r + 2 = 2$

Q. Find the order, $a_r - 7a_{r-1} + 6a_{r-3} = 0$

Soln: $r - (r-3) = r - r + 3$

$= 3$

• **Degree** : Power of the term of highest subscript.

Eg → $a_r^2 - 6a_{r-1} - 7a_{r-2} + 8a_{r-3} = 0$

degree = 2

Q. Solve the recurrence relation.

1. $a_r - 6a_{r-1} + 9a_{r-2} = 0$ $\longrightarrow$ order $= r-(r-2) = \cancel{r} - \cancel{r} + 2$
$= 2$

The characteristic equation –

$$x^{\text{order}} - 6x^{\text{order}-1} + 9x^{\text{order}-2} = 0$$

$\Rightarrow x^2 - 6x^1 + 9x^0 = 0$

$\Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x-3)^2 = 0 \Rightarrow x = 3$

• **Working Rule** :-

i) If the roots are distinct, say (x_1, x_2) then the general solution is –

$$a_r = A_0 x_1^r + A_1 x_2^r \qquad (A_0, A_1 \text{ are const})$$

ii) If the roots are same, then the general solution is

$$x_1 = x_2,$$
$$a_r = (A_0 + A_1 r) x_1^r$$

$\therefore$ General soln : $a_r = (A_0 + A_1 r) 3^r$

Q. Solve. $a_r - 7a_{r-1} + 12a_{r-2} = 0$ Order $= r-(r-2)$
$= 2$

Soln: $x^{\text{order}} - 7x^{\text{order}-1} + 12x^{\text{order}-2} = 0$

$\Rightarrow x^2 - 7x + 12x^0 = 0$

$\Rightarrow x^2 - 7x + 12 = 0$

$\Rightarrow (x-3)(x-4) = 0$

$\therefore x = 3 \; ; \; x = 4,$

So, the general soln is $\rightarrow a_r = A_0 3^r + A_1 4^r$

TO DO

Q.① $a_r - 6a_{r-1} + 9a_{r-2} = 0$

Q.② $a_r - 5a_{r-1} - 14a_{r-2} = 0$

Q.③ $a_r - 9a_{r-1} + 20a_{r-2} = 0$

Soln ① Order = highest subscript − lowest subscript

$= r-(r-2) = \cancel{r}-\cancel{r}+2 = 2$

$a_r - 6a_{r-1} + 9a_{r-2} = 0$

$x^{order} - 6x^{order-1} + 9x^{order-2} = 0$

$\Rightarrow x^2 - 6x + 9x^0 = 0$

$\Rightarrow x^2 - 6x + 9 = 0$

$\Rightarrow (x-3)^2 = 0$

$\Rightarrow x = 3$ $\therefore$ The general soln is $\Rightarrow a_r = (A_0 + A_1 r)\,3^r$

Ans

Soln ② Order = highest subscript − lowest subscript

$= r-(r-2) = \cancel{r}-\cancel{r}+2 = 2$

$a_r - 5a_{r-1} - 14a_{r-2} = 0$

$\Rightarrow x^{order} - 5x^{order-1} - 14x^{order-2} = 0$

$\Rightarrow x^2 - 5x - 14x^0 = 0$

$\Rightarrow x^2 - 5x - 14 = 0$

$\Rightarrow x^2 - 7x + 2x - 14 = 0$

$\Rightarrow x(x-7) + 2(x-7) = 0$

$\Rightarrow (x-7)(x+2) = 0$

$\therefore x = 7;\ x = -2$ $\therefore$ The general soln is $\rightarrow a_r = A_0 7^r + A_1(-2)^r$

Ans

Soln ③ Order $= r-(r-2) = \cancel{r}-\cancel{r}+2 = 2$

$$a_r - 9a_{r-1} + 20a_{r-2} = 0$$

$$\Rightarrow x^{order} - 9x^{order-1} + 20x^{order-2} = 0$$

$$\Rightarrow x^2 - 9x + 20x^0 = 0$$

$$\Rightarrow x^2 - 9x + 20 = 0$$

$$\Rightarrow x^2 - 5x - 4x + 20 = 0$$

$$\Rightarrow x(x-5) - 4(x-5) = 0$$

$$\Rightarrow (x-5)(x-4) = 0$$

$\therefore$ $x = 5$; $x = 4$ $\qquad$ $\therefore$ The general soln, $a_r = A_0 5^r + A_1 4^r$ Ans

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15/Nov/2022

Q.1. Show that $S = \{1, \omega, \omega^2\}$ is an abelian group under multiplication.

Q.2) Show that $S = \{1, -1, i, -i\}$ is an abelian group under multiplication.

Soln 1) Here $S = \{1, \omega, \omega^2\}$

$\omega^3 = 1$

$1 + \omega + \omega^2 = 0$

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Here closer property is satisfied under multiplication because all resultant elements also belong to the set.

(ii) Associative Property :-

$1 \times (\omega \times \omega^2) = (1 \times \omega) \times \omega^2$, we want to verify this —

LHS $\quad 1 \times (\omega \times \omega^2)$

$= 1 \times 1$

$= 1$

RHS $\quad (1 \times \omega) \times \omega^2$

$= \omega \times \omega^2 = 1$

$\therefore$ LHS = RHS $\quad$ $\therefore$ Associative property is satisfied under multiplication

(iii) Identity element :-

From the table it is clear that 'one' is identity element.

(iv) Inverse element :-

The inverse element for $1, \omega, \omega^2$ are $1, \omega^2, \omega$.

(v) Commutative property :-

From the table it is clear, that table is symmetric about principle diagonal element hence commutative property is satisfied under multiplication.

Hence $S = \{1, \omega, \omega^2\}$ is an abelian/commutative group under multiplication.

Soln 2) Here, $S = \{1, -1, i, -i\}$

×	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

(i) Here closure property is satisfied under multiplication because all resultant element are also belong to the set.

(ii) Associative property

$1 \times (-i \times i) = (1 \times -1) \times i$, we want to verify this.

LHS

$1 \times (-1 \times i)$

$= -i$

RHS

$(1 \times -1) \times i$

$= -1 \times i = -i$

$\therefore$ LHS = RHS

$\therefore$ Hence, associative property is satisfied under multiplication.

(iii) Identity Element :-

From the table it is clear that '1' is the identity element.

(iv) Inverse element : The inverse element for $1, -1, i, -i$ are $1, -1, -i, i$

(v) Commutative element :-

From the table it is cleared that the table is symmetric about principle diagonal element hence commutative element is satisfied under multiplication.
Hence $S = \{1, -1, i, -i\}$ is an abelian / commutative group under multiplication.

Q. Prove that by mathematical induction $3^n - 1$ is divisible by 2.

Soln: Step-1: Let $P(n) = 3^n - 1$ is divisible by 2

$P(1) = 3 - 1 = 2$ is divisible by 2.

$\therefore$ $P(n)$ is true for $n = 1$

Step-2: We consider $P(n)$ is true for $n = m$

ie, $P(m) = 3^m - 1$ is divisible by 2.

$\Rightarrow 3^m - 1 = 2k$ —— (1)

Step-3: We want to prove $P(n)$ is true for $n = m+1$

ie $P(n+1) = 3^{m+1} - 1$

$= 3^m \cdot 3 - 1$

$= 3(3^m - 1) + 3 - 1 = 3(2k) + 2$

$= 2(3k+1)$, divisible by 2.

$\therefore$ $P(n)$ is true for $n = m+1$.

Hence by mathematical induction $P(n)$ is true for all n.

Q. Show that $n^3 - 7n + 3$ is divisible by 3.

Soln: Step 1: Let $P(n) = n^3 - 7n + 3$ is divisible by 3

Now, $P(1) = n^3 - 7n + 3$ is divisible by 3

$= 1^3 - 7.1 + 3 = 3$ is divisible by 3

$\therefore$ $P(n)$ is true for $n = 1$

Step-2: We consider $P(n)$ is true for $n = m$

$\therefore P(m) = m^3 - 7m + 3$ is divisible by 3

$\Rightarrow m^3 - 7m + 3 = 3K$ —— (1)

Step-3: We want to prove $P(n)$ is true for $n = m+1$

ie, $P(m+1) = (m+1)^3 - 7(m+1) + 3$

$= m^3 + 3m + 3m^2 + 1 - 7m - 7 + 3$

$= \underline{m^3 - 7m + 3} + 3m + 3m^2 - 6$

$= 3K + 3m + 3m^2 + 6$

$= 3(K + m + m^2 + 2)$, it is divisible by 3

$\therefore P(n)$ is true for $n = m+1$

Hence by mathematical induction $P(n)$ is true for all n.

Q.(2) $a_r - 5a_{r-1} + 6a_{r-2} = 0$ 17/Nov/2022

Soln: Order $= r - (r+2) = 2$

Characteristic eqn is —

$x^{order} - 5x^{order-1} + 6x^{order-2}$

$\Rightarrow x^2 - 5x + 6 = 0$

$\Rightarrow x^2 - 3x - 2x + 6 = 0$

$\Rightarrow (x-2)(x-3) = 0$

$\therefore$ General solution —

$a_r = A_0 2^r + A_1 3^r$ —— (1)

$a_0 = 2$, $a_1 = 5$, Putting $r = 0$ in (1) we get

$a_0 = A_0 2^0 + A_1 3^0$

$\Rightarrow 2 = A_0 + A_1$ —— (II)

Putting $r = 1$ in (1),

$a_1 = A_0 2 + A_1 3$

$\Rightarrow 5 = 2A_0 + 3A_1$ —— (III)

$\therefore$ The general soln is: $a_r = 2^r + 3^r$.

Q. 3 Find the general solution,

$$a_r = a_{r-1} + a_{r-2} \quad \text{where } a_0 = 1 \ \& \ a_1 = 1$$

Soln: Order $= r - (r-2) = 2$

Characteristic equation,

$$x^{\text{order}} - x^{\text{order}-1} - x^{\text{order}-2} = 0$$

$$\Rightarrow x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1-4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

So, the solution is —

$$a_r = A_0\left(\frac{1+\sqrt{5}}{2}\right)^r + A_1\left(\frac{1-\sqrt{5}}{2}\right)^r$$

Putting $r = 0$,

$$a_0 = A_0\left(\frac{1+\sqrt{5}}{2}\right)^0 + A_1\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$\Rightarrow a_0 = A_0 + A_1$$

$$\Rightarrow 1 = A_0 + A_1 \quad \text{———} \quad (ii)$$

Putting $r = 1$ in eqn (i)

$$a_1 = A_0\left(\frac{1+\sqrt{5}}{2}\right) + A_1\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow 1 = A_0\left(\frac{1+\sqrt{5}}{2}\right) + A_1\left(\frac{1-\sqrt{5}}{2}\right) \quad \text{———} \quad (iii)$$

from (iii) $\quad A_0 = 1 - A_1$

$$\therefore 1 = (1-A_1)\left(\frac{1+\sqrt{5}}{2}\right) + A_1\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow 1 = \frac{1+\sqrt{5}}{2} - A_1\left(\frac{1+\sqrt{5}}{2}\right) + A_1\left(\frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow 1 = \frac{1+\sqrt{5}}{2} - A_1\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right)$$

$$\Rightarrow A_1\left(\frac{2\sqrt{5}}{\cancel{2}}\right) = \frac{1+\sqrt{5}}{2} - 1$$

$$\Rightarrow A_1 = -\frac{1}{2\sqrt{5}}$$

$$\Rightarrow A_0 = \frac{1+2\sqrt{5}}{2\sqrt{5}}$$

$\therefore$ So the general solution is —

$$a_r = A_0\left(\frac{1+\sqrt{5}}{2}\right)^r + A_1\left(\frac{1-\sqrt{5}}{2}\right)^r$$

$$a_r = \left(\frac{1+2\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^r - \left(\frac{1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^r$$

Generating Function :-

If $a_0, a_1, a_2 \ldots \ldots, a_n$ is an infinite sequence of numbers, then the generating function of $(a_0, a_1, \ldots a_n)$ is the function —

$$G(x) = \sum_{r=0}^{\infty} a_r x^r = a_0 x^0 + a_1 x^1 + a_2 x^2 + \ldots + a_n x^n$$

TO DO Module-3 → Solve all problems in sheet

Q. Find the generating function (g.f) of (1, 1, 1, 1, 1, 1)

Soln: $a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 1,\ a_4 = 1,\ a_5 = 1,\ a_6 = 1$

$$G.F = 1.x^0 + 1.x^1 + 1.x^2 + 1.x^3 + 1.x^4 + 1.x^5$$

$$= 1 + 1x + x^2 + x^3 + x^4 + x^5$$

$$= \frac{x^{5+1} - 1}{x - 1} = \frac{x^6 - 1}{x - 1}$$

{G.P series formula $a + ar + ar^2 + ar^3 + ar^n = \frac{a(r^{n+1} - 1)}{r - 1}$ }

Important! Infinite G.P series summation →

$$a + ar + ar^2 + ar^3 + \ldots\ldots = \frac{a}{r-1}$$

Q. Find G.F of (1, 1, 1, 1, ...)

Soln: $1 + x^1 + x^3 + x^4 + x^5 + \ldots\ldots = \frac{1}{x-1}$

Q.1. Find the G.F of $a_r - 3a_{r-1} + 2a_{r-2} = 0$. $a_0 = 2$ & $a_1 = 3$

Soln: $a_r - 3a_{r-1} + 2a_{r-2} = 0$

Multiplying both sides by x^r & taking summation for $r \geq 2$, we get.

$$\sum_{r=2}^{\infty} a_r x^r - 3\sum_{r=2}^{\infty} a_{r-1} x^r + 2\sum_{r=2}^{\infty} a_{r-2} x^r = 0$$

Again $\sum_{r=2}^{\infty} a_r x^r$

$= a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$

$= G(x) - (a_0 x^0 + a_1 x^1)$

$= G(x) - a_0 x^0 - a_1 x^1 \longrightarrow (2)$

$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$

$\Rightarrow G(x) - a_0 x^0 - a_1 x^1$

Again, $\sum_{r=2}^{\infty} a_{r-1} x^r$

$= a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots$

$= x(a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)$

$= x(G(x) - a_0 x^0)$

$= x(G(x) - a_0) \longrightarrow (3)$

$\sum_{r=2}^{\infty} a_{r-2} x^r$

$= a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$

$\Rightarrow x^2(a_0 + a_1 x + a_2 x^2 + \cdots)$

$= x^2 G(x) \longrightarrow (4)$

Putting the values of (2), (3), (4) in (1) we get,

$(G(x) - a_0 x^0 - a_1 x) - 3x(G(x) - a_0) + 2x^2 G(x) = 0$

$a_0 = 2$ & $a_1 = 3$

$\Rightarrow (G(x) - 2 - 3x) - 3x(G(x) - 2) + 2x^2 G(x) = 0$

$\Rightarrow G(x) - 2 - 3x - 3x G(x) + 6x + 2x^2 G(x) = 0$

$\Rightarrow G(x) = \dfrac{2 - 3x}{2x^2 - 3x + 1}$

Ans

Q.(2) Find the general function for fibonacci sequence $a_r = a_{r-1} + a_{r-2}$

Where $a_0 = 1$, $a_1 = 1$

Soln: $a_{r-1} + a_{r-2}$

Multiplying both sides by x^r & taking summation $r \geq 2$,

$$\sum_{r=2}^{\infty} a_r x^r = \sum_{r=2}^{\infty} a_{r-1} x^r + \sum_{r=2}^{\infty} a_{r-2} x^r \quad \text{——— (1)}$$

$$\sum_{r=2}^{\infty} a_r x^r = a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots$$

$$\sum_{r=2}^{\infty} a_r x^r = G(x) - (a_0 x^0 + a_1 x^1)$$

$$= G(x) - a_0 x^0 - a_1 x^1 \quad \text{——— (2)}$$

$$\sum_{r=2}^{\infty} a_{r-1} x^r = a_1 x^2 + a_2 x^3 + a_3 x^4 + a_4 x^3 + \cdots$$

$$= x(a_1 x + a_2 x^2 + a_3 + x^3 + a_3 x^2)$$

$$= x(G(x) - a_0 x^0) \quad \text{——— (3)}$$

$$\sum_{r=2}^{\infty} a_{r-2} x^r = a_0 x^2 + a_1 x^3 + a_2 x^4 + \cdots$$

$$= x^2(a_0 + a_1 x^1 + a_2 x^2 +)$$

$$\Rightarrow x^2(G(x)) \quad \text{——— (4)}$$

Putting value of , (2), (3), (4) in 1,

$$G(x) - a_0 x^0 - a_1 x^1 - x(G(x) - a_0 x^0) - x^2(G(x)) = 0$$

$$\Rightarrow G(x) - 1 - x - x G(x) + x - x(G(x)) = 0$$

$$\Rightarrow G(x)(1 - x - x^2) - 1 = 0$$

$$\Rightarrow G(x) = \frac{1}{1 - x - x^2}$$

Ans