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Runge–Kutta methods for age-structured population models

L.M. Abia *, J.C. López-Marcos

Departamento de Matemática Aplicada y Computación, Facultad de Ciencias, Universidad de Valladolid, Valladolid, Spain

Abstract

Difference schemes based on Runge–Kutta methods are introduced for the numerical solution of age-structured population models. The schemes are completely analysed: consistency, stability, existence and convergence are established. Also reported are some numerical experiments in order to show numerically the results proved in our analysis.

Keywords: Age-structured population models; Runge–Kutta methods

1. Introduction

In this paper, we consider difference schemes for the numerical solution of age-structured population models. More precisely, we are interested in models in which the governing equations are nonclassical first-order hyperbolic initial boundary value problems. In these equations, the integral appears over the spatial domain of a function of the solution. If we have such an integral in the boundary condition we say that the boundary condition is non-local, and when the integral appears in the partial differential equation then we have a partial integrodifferential equation. As a prototype of such problems, we consider in this paper the Gurtin–MacCamy equation [7]

$$u_x + u_t = f\left(x, \int_0^A u(a, t)\, \mathrm{d}a\right)u, \qquad 0 \leqslant x \leqslant A, \quad 0 \leqslant t, \tag{1.1}$$

with initial condition

$$u(x, 0) = u^0(x), \quad 0 \leqslant x \leqslant A, \tag{1.2}$$

and non-local boundary condition

$$u(0, t) = g\left(\int_0^A b\left(x, \int_0^A u(a, t)\, \mathrm{d}a\right)u(x, t)\, \mathrm{d}x, t\right), \quad 0 \leqslant t. \tag{1.3}$$

* Corresponding author. E-mail: abia@cpd.uva.es.

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Eqs. (1.1)–(1.3) are used in biology to describe the evolution in time of the age structure of a population. The independent variables x and t denote, respectively, age and time, and $u(x, t)$ is the age-specific density of individuals of age x at time t. By representing by $b(x, I(t))$ and $f(x, I(t))$ the age-specific fertility modulus (a nonnegative function) and mortality modulus (a nonpositive function), respectively, we are assuming that both also depend on the total size of the population $I(t) = \int_0^A u(a, t)\,da$. The boundary condition (1.3) is the birth law of the population, and we are assuming that the births are given by means of a nonnegative function $g(\int_0^A b(x, I(t))u(x, t)\,dx, t)$, which depends on time and on the weighted average population with the fertility modulus. An extensive study of nonlinear age-dependent population dynamics can be found in the work of Webb [14].

Several numerical methods have been proposed for treating problem (1.1)–(1.3). The schemes analysed by Douglas and Milner [4], Chichia Chiu [3], Kostova [9] and López-Marcos [11] are explicit schemes of first order. Second-order schemes based on the Keller box method have been considered by Fairweather and López-Marcos in [5,6], and the role played by the quadrature rule used to approximate the integral terms was also studied. Kwon and Cho [10] have proposed a second-order multistep scheme based on finite differences along the characteristics. In the present work, we study schemes based on Runge–Kutta methods of order greater or equal than two. Along each characteristic, Eq. (1.1) can be viewed as an ordinary differential equation (ODE), with the property that these ordinary differential equations are coupled due to the dependence on the total size of the population of the age-specific mortality modulus and also due to the boundary condition (1.3). Hence the use of Runge–Kutta methods is not trivial, and it is necessary to modify the Runge–Kutta methods in order to use them in the numerical integration of (1.1)–(1.3). The resulting schemes are implicit. We analyse the consistency, stability, convergence and existence of the numerical approximations of the proposed numerical schemes. This analysis is carried out by means of a general discretization framework introduced by López-Marcos and Sanz-Serna [12]. It should be noted that our ideas can be extended to the system of equations modelling epidemic populations [1,2,8]; population genetics [14], etc. Milner and Rabbiolo [13] considered numerical algorithms, based on Runge–Kutta methods of second and fourth order, for models of population dynamics. Second-order methods were analysed for the Gurtin–MacCamy model and for a two-sex model, however the explicit fourth-order method was studied only for linear equations: the McKendrick–von Foerster linear model. Our study can be considered as a generalization of their work because we study numerical schemes based on general Runge–Kutta methods for the nonlinear model (1.1)–(1.3).

2. The numerical scheme

If we consider the initial value problem

$$y'(t) = F(t, y(t)), \tag{2.1a}$$

$$y(t_0) = y_0, \tag{2.1b}$$

a Runge–Kutta (R–K) method takes the form

$$y_{n+1} = y_n + \sum_{i=1}^{m} b_i Y_i(y_n), \tag{2.2a}$$

$$Y_i(y_n) = hF\left(t_n + c_i h,\ y_n + \sum_{l=1}^{m} a_{il} Y_l(y_n)\right), \quad 1 \leqslant i \leqslant m, \tag{2.2b}$$

where h is the time step and y_n is the numerical approximation to $y(t_n)$, $t_n = t_0 + nh$. We shall assume that the Runge–Kutta method is of order s, $s \geqslant 2$.

If u is a solution of (1.1) and we consider the characteristics $x = t + \gamma$, we may introduce the functions

$$v_\gamma(t, \gamma) = u(t + \gamma, t), \quad t \geqslant t_\gamma,$$

where $t_\gamma := \max\{0, -\gamma\}$. The function v_γ is called the cohort function corresponding to age γ and, as time evolves, it keeps track of those members of the population who are initially of age γ. From (1.1), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} v_\gamma(t, \gamma) = f(t + \gamma, I(t)) v_\gamma(t, \gamma), \quad t \geqslant t_\gamma, \tag{2.3}$$

where, for each fixed γ, (2.3) is an ODE.

Given a positive integer J, if $h = A/J$ and $N = [T/h]$, we introduce a grid

$$\{(x_j, t_n):\ x_j = jh;\ j = 0, \ldots, J;\ t_n = nh;\ n = 0, \ldots, N\}$$

on $[0, A] \times [0, T]$. We refer to the grid point x_j by a subscript j and to the time level t_n by a superscript n. If we apply the R–K method (2.2) to problem (2.3) at the grid points, we obtain the following equations, noting that the grid points (x_j, t_n) and (x_{j+1}, t_{n+1}) belong to the same characteristic,

$$U_{j+1}^{n+1} = U_j^n + \sum_{i=1}^{m} b_i \overline{Y}_i(U_j^n), \tag{2.4a}$$

$$\overline{Y}_i(U_j^n) = hf(x_j + c_i h,\ I(t_n + c_i h)) \left[U_j^n + \sum_{l=1}^{m} a_{il} \overline{Y}_l(U_j^n) \right], \tag{2.4b}$$

$1 \leqslant i \leqslant m$, $0 \leqslant j \leqslant J - 1$, $0 \leqslant n \leqslant N - 1$, where U_j^n is the numerical approximation to $u(x_j, t_n)$. However these relations are not sufficient to define a numerical method because the values $I(t_n)$ and $I(t_n + c_i h)$ are unknown and the time level $t_n + c_i h$ is not, in general, a node of the grid defined on the time interval $[0, T]$. To circumvent this problem, we use a numerical quadrature Q_h to approximate the integral terms, and we take into account that if the solution $u(x, t)$ is sufficiently smooth then, for each $i = 1, \ldots, m$, there exist constants $c_i^0, c_i^1, \ldots, c_i^{s-1}$, such that

$$I(t_n + c_i h) = c_i^0 I(t_n) + c_i^1 I(t_{n-1}) + \cdots + c_i^{s-1} I(t_{n-s+1}) + \mathrm{O}(h^s), \tag{2.5}$$

$1 \leqslant i \leqslant m$, $s - 1 \leqslant n \leqslant N - 1$, where s is the order of the R–K method (2.2).

Now if we denote

$$U^n = (U_0^n, U_1^n, \dots, U_J^n), \quad 0 \leqslant n \leqslant N,$$
$$b(U^n)_j = b\big(x_j, Q_h(U^n)\big), \quad 0 \leqslant j \leqslant J,$$
$$\boldsymbol{b}(U^n) = (b(U^n)_0, b(U^n)_1, \dots, b(U^n)_J),$$

and

$$Q_h(U^n) = \sum_{j=0}^{J} hq_jU_j^n, \quad 0 \leqslant n \leqslant N, \tag{2.6}$$

with $\sup_{J\in\mathbb{N}}(\max_{0\leqslant j\leqslant J}|q_j|) \leqslant B < \infty$, then the scheme takes the form

$$U_{j+1}^{n+1} = U_j^n + \sum_{i=1}^{m} b_iY_i(U_j^n), \tag{2.7a}$$

$$Y_i(U_j^n) = hf\left(x_j + c_ih, \sum_{l=0}^{s-1} c_i^lQ_h(U^{n-l})\right)\left[U_j^n + \sum_{l=1}^{m} a_{il}Y_l(U_j^n)\right], \tag{2.7b}$$

$1 \leqslant i \leqslant m$, $0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$, with given initial conditions

$$U^p = (U_0^p, \dots, U_J^p), \quad 0 \leqslant p \leqslant s-1, \tag{2.8}$$

and boundary condition

$$U_0^{n+1} = g\big(Q_h(\boldsymbol{b}(U^{n+1})U^{n+1}), t_{n+1}\big), \quad s-1 \leqslant n \leqslant N-1, \tag{2.9}$$

where in (2.9) and henceforth we denote by $\boldsymbol{b}(V)V$ the componentwise product of the vectors $\boldsymbol{b}(V)$ and V.

To analyse the numerical schemes defined by (2.7)–(2.9), we shall employ the discretization framework introduced by López-Marcos and Sanz-Serna [12]. To describe this framework, we consider the set $H = \{h > 0\colon h = A/J,\ J \in \mathbb{N}\}$, and for each h in H, we define the vector spaces

$$X_h = (\mathbb{R}^{J+1})^{N+1}, \qquad Y_h = (\mathbb{R}^{J+1})^s \times (\mathbb{R}^{N-s+1}) \times (\mathbb{R}^J)^{N-s+1}.$$

If $V = (V_0, \dots, V_J) \in \mathbb{R}^{J+1}$ and

$$\| V \|_1 = \sum_{j=0}^{J} h|V_j|,$$

then, for $(V^0, V^1, \dots, V^N) \in X_h$, we define

$$\| (V^0, V^1, \dots, V^N) \|_{X_h} = \max_{0\leqslant n\leqslant N} \| V^n \|_1.$$

In addition, if $\boldsymbol{P} \in \mathbb{R}^{N-s+1}$, $\boldsymbol{W} \in \mathbb{R}^J$, and

$$\| \boldsymbol{P} \| = \sum_{n=s}^{N} h|P^n|, \qquad \| \boldsymbol{W} \|_1^* = \sum_{j=1}^{J} h|W_j|,$$

then, for $(\boldsymbol{P}^0, \dots, \boldsymbol{P}^{s-1}, \boldsymbol{P}_0, \boldsymbol{P}_*^s, \dots, \boldsymbol{P}_*^N) \in Y_h$, we define

$$\left\| (\boldsymbol{P}^0, \dots, \boldsymbol{P}^{s-1}, \boldsymbol{P}_0, \boldsymbol{P}_*^s, \dots, \boldsymbol{P}_*^N) \right\|_{Y_h} = \sum_{p=0}^{s-1} \| \boldsymbol{P}^p \|_1 + \| \boldsymbol{P}_0 \| + \sum_{n=s}^{N} h \| \boldsymbol{P}_*^n \|_1^*.$$

For each h in H, we consider the element $\boldsymbol{u}_h \in X_h$,

$$\boldsymbol{u}_h \doteq (\boldsymbol{u}^0, \boldsymbol{u}^1, \dots, \boldsymbol{u}^N) \in X_h,$$

with

$$\boldsymbol{u}^n = (u_0^n, u_1^n, \dots, u_J^n) \in \mathbb{R}^{J+1}, \quad u_j^n = u(x_j, t^n), \tag{2.10}$$

$0 \leqslant j \leqslant J$, $0 \leqslant n \leqslant N$, where u is the theoretical solution of (1.1)–(1.3).

Let M be a fixed positive constant and denote by $B(\boldsymbol{u}_h, Mh)$ the open ball with center $\boldsymbol{u}_h$ and radius Mh of the space X_h endowed with the norm $\|\cdot\|_{X_h}$ given above. We introduce the mapping

$$\phi_h : B(\boldsymbol{u}_h, Mh) \to Y_h,$$

defined by the equations

$$\phi_h(\boldsymbol{V}^0, \boldsymbol{V}^1, \dots, \boldsymbol{V}^N) = (\boldsymbol{P}^0, \dots, \boldsymbol{P}^{s-1}, \boldsymbol{P}_0, \boldsymbol{P}^s, \dots, \boldsymbol{P}^N), \tag{2.11}$$

$$\boldsymbol{P}^p = \boldsymbol{V}^p - \boldsymbol{U}^p, \quad 0 \leqslant p \leqslant s-1, \tag{2.12}$$

$$P_{j+1}^{n+1} = \frac{V_{j+1}^{n+1} - V_j^n}{h} - h^{-1} \sum_{i=1}^{m} b_i Y_i(V_j^n), \tag{2.13a}$$

$$Y_i(V_j^n) = hf\left(x_j + c_i h, \sum_{l=0}^{s-1} c_i^l Q_h(\boldsymbol{V}^{n-l})\right)\left[V_j^n + \sum_{l=1}^{m} a_{il} Y_l(V_j^n)\right], \tag{2.13b}$$

$1 \leqslant i \leqslant m$, $0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$;

$$P_0^n = V_0^n - g(Q_h(\boldsymbol{b}(\boldsymbol{V}^n)\boldsymbol{V}^n), t_n), \tag{2.14}$$

$s \leqslant n \leqslant N$. It is clear that $\boldsymbol{U}_h = (\boldsymbol{U}^0, \boldsymbol{U}^1, \dots, \boldsymbol{U}^N) \in X_h$ is a solution of the scheme (2.7)–(2.9) if and only if

$$\phi_h(\boldsymbol{U}^0, \boldsymbol{U}^1, \dots, \boldsymbol{U}^N) = \boldsymbol{0}. \tag{2.15}$$

Note that we have to prove that the operator ϕ_h is well defined. This is immediate if the R–K method is explicit, but in the general case we have to show that the elements $Y_i(V_j^n)$ are well defined by the linear system (2.13b). From now on, C will denote a positive constant which is independent of h, n $(0 \leqslant n \leqslant N)$, and j $(0 \leqslant j \leqslant J)$; C has possibly different values in different places.

Proposition 2.1. *Assume that the functions f and u are sufficiently smooth, and that the quadrature formula Q_h is convergent. Let $(\boldsymbol{V}^0, \dots, \boldsymbol{V}^N)$, $(\boldsymbol{W}^0, \dots, \boldsymbol{W}^N)$ be in the ball B $(\boldsymbol{u}_h, Mh)$ of the space X_h. For h sufficiently small, the linear system* (2.13b) *has a unique solution such that*

$$|Y_i(V_j^n)| \leqslant Ch, \tag{2.16}$$

$$|Y_i(V_j^n) - Y_i(W_j^n)| \leqslant Ch\left(\sum_{l=0}^{s-1} \|\boldsymbol{V}^{n-l} - \boldsymbol{W}^{n-l}\|_1 + |V_j^n - W_j^n|\right), \tag{2.17}$$

$1 \leqslant i \leqslant m$, $0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$.

Proof. Because $(V^0,\dots,V^N)\in B(\boldsymbol{u}_h, Mh)$, it is easy to show that there exists a positive constant C such that, for h sufficiently small,

$$\left|\sum_{l=0}^{s-1} c_i^l Q_h(V^{n-l}) - I(t_n + c_i h)\right| \leqslant C, \quad 0 \leqslant i \leqslant m, \tag{2.18}$$

$$\| V^n \|_\infty \leqslant C, \tag{2.19}$$

$$| Q_h(V^n - W^n) | \leqslant C \| V^n - W^n \|_1, \tag{2.20}$$

$0 \leqslant n \leqslant N-1$. Now we can assume that the arguments of the mortality moduli f always belong to $[0, A]\times D_1$, where D_1 is a compact neighbourhood of $\{I(t)\colon 0\leqslant t\leqslant T\}$.

We define

$$\boldsymbol{Y}(V_j^n) = \left(Y_1(V_j^n),\dots,Y_m(V_j^n)\right)^{\mathrm{T}}, \tag{2.21}$$

$$\boldsymbol{F}(V_j^n) = V_j^n\left(f\left(x_j + c_1 h, \sum_{l=0}^{s-1} c_1^l Q_h(V^{n-l})\right),\dots, f\left(x_j + c_m h, \sum_{l=0}^{s-1} c_m^l Q_h(V^{n-l})\right)\right)^{\mathrm{T}}, \tag{2.22}$$

$0\leqslant j\leqslant J-1$, $s-1\leqslant n\leqslant N-1$. We also consider the matrix

$$\mathscr{A}(V_j^n) = \left(f\left(x_j + c_i h, \sum_{l=0}^{s-1} c_i^l Q_h(V^{n-l})\right) a_{il}\right)_{i,l=1}^{m} \tag{2.23}$$

$0\leqslant j\leqslant J-1$, $s-1\leqslant n\leqslant N-1$. Next we have from (2.13b)

$$\left(\mathscr{I} - h\mathscr{A}(V_j^n)\right)\boldsymbol{Y}(V_j^n) = h\boldsymbol{F}(V_j^n), \tag{2.24}$$

$$\begin{aligned}&\left(\mathscr{I} - h\mathscr{A}(V_j^n)\right)\left(\boldsymbol{Y}(V_j^n) - \boldsymbol{Y}(W_j^n)\right)\\ &\quad = h\left(\mathscr{A}(V_j^n) - \mathscr{A}(W_j^n)\right)\boldsymbol{Y}(W_j^n) + h\left(\boldsymbol{F}(V_j^n) - \boldsymbol{F}(W_j^n)\right),\end{aligned} \tag{2.25}$$

$0\leqslant j\leqslant J-1$, $s-1\leqslant n\leqslant N-1$. Taking into account that f is bounded on $[0, A]\times D_1$, we deduce that, for h sufficiently small, the matrix $(\mathscr{I} - h\mathscr{A}(V_j^n))$ has an inverse (which implies that the mapping ϕ_h is well defined) and

$$\left\|\left(\mathscr{I} - h\mathscr{A}(V_j^n)\right)^{-1}\right\| \leqslant \frac{1}{1 - h\|\mathscr{A}(V_j^n)\|} \leqslant C, \tag{2.26}$$

$0\leqslant j\leqslant J-1$, $s-1\leqslant n\leqslant N-1$. Hence, from (2.24) and using (2.19) and the boundedness of f, we obtain (2.16).

Next, by the regularity hypotheses and (2.18)–(2.20), we obtain the following inequalities

$$\left| f\left(x_j + c_i h, \sum_{l=0}^{s-1} c_i^l Q_h(V^{n-l})\right) - f\left(x_j + c_i h, \sum_{l=0}^{s-1} c_i^l Q_h(W^{n-l})\right)\right| \leqslant C\sum_{l=0}^{s-1} \| V^{n-l} - W^{n-l} \|_1, \tag{2.27}$$

$$\begin{aligned}&\left| V_j^n f\left(x_j + c_i h, \sum_{l=0}^{s-1} c_i^l Q_h(V^{n-l})\right) - W_j^n f\left(x_j + c_i h, \sum_{l=0}^{s-1} c_i^l Q_h(W^{n-l})\right)\right|\\ &\quad \leqslant C\left(\sum_{l=0}^{s-1} \| V^{n-l} - W^{n-l} \|_1 + | V_j^n - W_j^n |\right),\end{aligned} \tag{2.28}$$

$0 \leqslant i \leqslant m$, $0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$. Therefore (2.17) is derived with the aid of (2.25)–(2.28) and (2.16). □

3. Consistency

We define the *local discretization error* as

$$l_h = \phi_h(u_h) \in Y_h, \tag{3.1}$$

and we say that the discretization (2.11) is *consistent* if, as $h \to 0$,

$$\lim_{h\to 0} \| \phi_h(u_h) \|_{Y_h} = \lim_{h \to 0} \| l_h \|_{Y_h} = 0. \tag{3.2}$$

The next theorem establishes the consistency of the numerical scheme (2.7)–(2.9).

Theorem 3.1. *Assuming that the functions f, g, b and u are sufficiently smooth, and that the quadrature formula Q_h used to approximate the integral terms is at least s-order accurate, then the local discretization error satisfies, as $h \to 0$,*

$$\| \phi_h(u_h) \|_{Y_h} = \sum_{p=0}^{s-1} \| u^p - U^p \|_1 + O(h^s). \tag{3.3}$$

Proof. Since the R–K scheme considered in (2.2) is s-order accurate, as $h \to 0$,

$$\tau_{j+1}^{n+1} = \frac{u_{j+1}^{n+1} - u_j^n}{h} - h^{-1} \sum_{i=1}^{m} b_i \bar{Y}_i(u_j^n) = O(h^s), \tag{3.4a}$$

$$\bar{Y}_i(u_j^n) = hf(x_j + c_i h, I(t_n + c_i h)) \left[u_j^n + \sum_{l=1}^{m} a_{il} \bar{Y}_l(u_j^n) \right], \tag{3.4b}$$

$1 \leqslant i \leqslant m$, $0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$, the estimate being uniform in n and in j because of the regularity assumptions.

On the other hand, the fact that Q_h is s-order accurate, the smoothness of u and f, and (2.5) imply

$$\left| f(x_j + c_i h, I(t_n + c_i h)) - f\left(x_j + c_i h, \sum_{l=0}^{s-1} c_i^l Q_h(u^{n-l}) \right) \right| = O(h^s). \tag{3.5}$$

We set

$$\bar{Y}(u_j^n) = \left(\bar{Y}_1(u_j^n), \ldots, \bar{Y}_m(u_j^n) \right)^{\mathrm{T}}, \tag{3.6}$$

$$\bar{F}(u_j^n) = u_j^n \left(f(x_j + c_1 h, I(t_n + c_1 h)), \ldots, f(x_j + c_m h, I(t_n + c_m h)) \right)^{\mathrm{T}}, \tag{3.7}$$

$0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$. We also consider the matrix

$$\bar{\mathscr{A}}(u_j^n) = \left(f(x_j + c_i h, I(t_n + c_i h)) a_{il} \right)_{i,l=1}^{m} \tag{3.8}$$

$0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$. Next we have from (3.4b)

$$\left(\mathscr{I} - h\bar{\mathscr{A}}(u_j^n)\right)\bar{Y}(u_j^n) = h\bar{F}(u_j^n), \tag{3.9}$$

$$\begin{aligned}&\left(\mathscr{I} - h\bar{\mathscr{A}}(u_j^n)\right)\left(\bar{Y}(u_j^n) - Y(u_j^n)\right)\\ &\quad = h\left(\bar{\mathscr{A}}(u_j^n) - \mathscr{A}(u_j^n)\right)Y(u_j^n) + h\left(\bar{F}(u_j^n) - F(u_j^n)\right),\end{aligned} \tag{3.10}$$

$0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$. By means of an argument similar to that used in the proof of Proposition 2.1, we have

$$|\bar{Y}_i(u_j^n)| \leqslant Ch, \tag{3.11}$$

$$|\bar{Y}_i(u_j^n) - Y_i(u_j^n)| = \mathrm{O}(h^{s+1}), \tag{3.12}$$

$1 \leqslant i \leqslant m$, $0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$.

Now, if we denote

$$\phi_h(u_h) = \left(L^0, \dots, L^{s-1}, L_0, L^s, \dots, L^N\right), \tag{3.13}$$

we have by (3.4a), (3.11) and (3.12) that

$$|L_{j+1}^{n+1}| \leqslant |L_{j+1}^{n+1} - \tau_{j+1}^{n+1}| + |\tau_{j+1}^{n+1}| = \mathrm{O}(h^s), \tag{3.14}$$

$0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$. Therefore to derive (3.3) we have to bound the boundary terms. By taking into account the regularity of g, b and u, the accuracy of Q_h, and (1.3), we obtain

$$\begin{aligned}|L_0^{n+1}| &= \left|u_0^{n+1} - g\left(Q_h(b(u^{n+1})u^{n+1}), t_{n+1}\right)\right|\\ &= \left|g\left(\int_0^A b(x, I(t_{n+1}))u(x, t_{n+1})\,\mathrm{d}x, t_{n+1}\right) - g\left(Q_h(b(u^{n+1})u^{n+1}), t_{n+1}\right)\right|\\ &\leqslant C\left(\left|\int_0^A \left(b(x, I(t_{n+1})) - b(x, Q_h(u^{n+1}))\right)u(x, t_{n+1})\,\mathrm{d}x\right|\right.\\ &\qquad \left. + \left|\int_0^A b(x, Q_h(u^{n+1}))u(x, t_{n+1})\,\mathrm{d}x - Q_h(b(u^{n+1})u^{n+1})\right|\right)\\ &= \mathrm{O}(h^s),\end{aligned} \tag{3.15}$$

$s-1 \leqslant n \leqslant N-1$. Hence (3.3) holds and the proof is complete. □

4. Stability

Another notion that plays an important role in the analysis of our numerical methods is *stability*. For each h, let M_h be a real number (*the stability thresholds*) with $0 < M_h \leqslant \infty$; we say that the discretization (2.11) is *stable* for u_h restricted to the thresholds M_h, if there exist two positive constants h_0 and S (*the stability constant*) such that, for any h in H with $h \leqslant h_0$, the open ball $B(\boldsymbol{u}_h, M_h)$ is contained in the domain of ϕ_h and, for all $\boldsymbol{V}_h$, $\boldsymbol{W}_h$ in that ball,

$$\| \boldsymbol{V}_h - \boldsymbol{W}_h \|_{X_h} \leqslant S \| \phi_h(\boldsymbol{V}_h) - \phi_h(\boldsymbol{W}_h) \|_{Y_h}. \tag{4.1}$$

Theorem 4.1. *Under the hypotheses of Theorem* 3.1, *if* M *is a fixed positive constant then the discretization* (2.11)–(2.14) *is stable for* $\boldsymbol{u}_h$ *with thresholds* Mh.

Proof. Let $(\boldsymbol{V}^0, \boldsymbol{V}^1, \dots, \boldsymbol{V}^N)$ and $(\boldsymbol{W}^0, \boldsymbol{W}^1, \dots, \boldsymbol{W}^N)$ be in the ball $B(\boldsymbol{u}_h, Mh)$ of the space X_h. We set

$$\boldsymbol{E}^n = \boldsymbol{V}^n - \boldsymbol{W}^n \in \mathbb{R}^{J+1}, \quad 0 \leqslant n \leqslant N,$$
$$\phi_h(\boldsymbol{V}^0, \boldsymbol{V}^1, \dots, \boldsymbol{V}^N) = \left(\boldsymbol{P}^0, \dots, \boldsymbol{P}^{s-1}, \boldsymbol{P}_0, \boldsymbol{P}^s, \dots, \boldsymbol{P}^N\right),$$
$$\phi_h(\boldsymbol{W}^0, \boldsymbol{W}^1, \dots, \boldsymbol{W}^N) = \left(\boldsymbol{R}^0, \dots, \boldsymbol{R}^{s-1}, \boldsymbol{R}_0, \boldsymbol{R}^s, \dots, \boldsymbol{R}^N\right).$$

By (2.13)

$$E_{j+1}^{n+1} = E_j^n + \sum_{i=1}^{m} b_i\left(Y_i(V_j^n) - Y_i(W_j^n)\right) + h\left(P_{j+1}^{n+1} - R_{j+1}^{n+1}\right), \tag{4.2}$$

$0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$. Taking into account (2.17), we obtain

$$\begin{aligned} \| \boldsymbol{E}^{n+1} \|_1^* &\leqslant \| \boldsymbol{E}^n \|_1 + C \sum_{i=1}^{m} \sum_{j=0}^{J-1} h \,| Y_i(V_j^n) - Y_i(W_j^n) | + h \| \boldsymbol{P}^{n+1} - \boldsymbol{R}^{n+1} \|_1^* \\ &\leqslant \| \boldsymbol{E}^n \|_1 + Ch \sum_{i=1}^{m} \left(\| \boldsymbol{E}^n \|_1 + \left(\sum_{l=0}^{s-1} \| \boldsymbol{E}^{n-l} \|_1 \right) \right) + h \| \boldsymbol{P}^{n+1} - \boldsymbol{R}^{n+1} \|_1^* \\ &\leqslant \| \boldsymbol{E}^n \|_1 + Ch \left(\sum_{l=0}^{s-1} \| \boldsymbol{E}^{n-l} \|_1 \right) + h \| \boldsymbol{P}^{n+1} - \boldsymbol{R}^{n+1} \|_1^*, \end{aligned} \tag{4.3}$$

$s-1 \leqslant n \leqslant N-1$.

Next we have to bound the boundary terms. Because of the use of stability thresholds, we can assume that the arguments of the age-specific fertility b are varying on $[0, A] \times D_1$, where D_1 is a compact neighbourhood of $\{I(t)\colon 0 \leqslant t \leqslant T\}$, and that the arguments of function g in (1.3) belong to $D_2 \times [0, T]$, with D_2 a compact neighbourhood of $\{\int_0^A b(a, I(t)) u(a, t)\, \mathrm{d}a\colon 0 \leqslant t \leqslant T\}$. We shall use the fact that

$$| Q_h(\boldsymbol{V}^n \boldsymbol{W}^n) | \leqslant C \| \boldsymbol{V}^n \|_\infty \| \boldsymbol{W}^n \|_1, \tag{4.4}$$

$0 \leqslant n \leqslant N$.

Again by the regularity hypotheses, (2.14), (2.18)–(2.20) and (4.4), we obtain

$$\begin{aligned} | E_0^{n+1} | &\leqslant | P_0^{n+1} - R_0^{n+1} | + | g\left(Q_h(\boldsymbol{b}(\boldsymbol{V}^{n+1})\boldsymbol{V}^{n+1}), t_{n+1}\right) \\ &\qquad - g\left(Q_h(\boldsymbol{b}(\boldsymbol{W}^{n+1})\boldsymbol{W}^{n+1}), t_{n+1}\right) | \\ &\leqslant | P_0^{n+1} - R_0^{n+1} | + C \,| Q_h(\boldsymbol{b}(\boldsymbol{V}^{n+1})\boldsymbol{V}^{n+1}) - Q_h(\boldsymbol{b}(\boldsymbol{W}^{n+1})\boldsymbol{W}^{n+1}) | \end{aligned}$$

$$\begin{aligned}
&\leqslant |P_0^{n+1} - R_0^{n+1}| + C\big(|Q_h(b(W^{n+1})E^{n+1})| + |Q_h((b(W^{n+1}) - b(V^{n+1}))V^{n+1})|\big)\\
&\leqslant |P_0^{n+1} - R_0^{n+1}| + C\big(\|b(W^{n+1})\|_\infty \|E^{n+1}\|_1\\
&\qquad\qquad + \|b(W^{n+1}) - b(V^{n+1})\|_\infty \|V^{n+1}\|_1\big)\\
&\leqslant |P_0^{n+1} - R_0^{n+1}| + C\|E^{n+1}\|_1, \qquad (4.5)
\end{aligned}$$

$s-1 \leqslant n \leqslant N-1$.

Now by (4.5) and (4.3), we obtain

$$\begin{aligned}
\|E^{n+1}\|_1 &\leqslant \|E^n\|_1 + Ch\left(\sum_{l=0}^{s} \|E^{n+1-l}\|_1\right)\\
&\qquad + h\|P^{n+1} - R^{n+1}\|_1^* + h|P_0^{n+1} - R_0^{n+1}|, \qquad (4.6)
\end{aligned}$$

$s-1 \leqslant n \leqslant N-1$, and summation on n yields

$$\begin{aligned}
\|E^n\|_1 &\leqslant \|E^{s-1}\|_1 + \sum_{m=s}^{n} h|P_0^m - R_0^m| + \sum_{m=s}^{n} h\|P^m - R^m\|_1^*\\
&\qquad + Ch\sum_{m=0}^{n} \|E^m\|_1, \qquad (4.7)
\end{aligned}$$

$s \leqslant n \leqslant N$. Using the discrete Gronwall lemma, it follows that

$$\|E^n\|_1 \le S\left(\sum_{l=0}^{s-1} \|E^l\|_1 + \sum_{m=s}^{n} h|P_0^m - R_0^m| + \sum_{m=s}^{n} h\|P^m - R^m\|_1\right), \qquad (4.8)$$

$s \leqslant n \leqslant N$. Therefore

$$\begin{aligned}
&\|(E^0, E^1, \dots, E^N)\|_{X_h}\\
&\quad \leqslant S\big\|\big((V-W)^0, \dots, (V-W)^{s-1}, (P-R)_0, (P-R)^s, \dots, (P-R)^N\big)\big\|_{Y_h},
\end{aligned}$$

as desired. □

5. Existence and convergence

We say that the discretization (2.11) is *convergent* if there exists $h_0 > 0$ such that, for each h in H with $h \leqslant h_0$, (2.15) has a solution U_h for which, as $h \to 0$,

$$\lim_{h\to 0} \|u_h - U_h\|_{X_h} = 0.$$

We define the *global discretization error* as

$$e_h = u_h - U_h \in X_h.$$

To derive the existence and convergence of solutions of our schemes (2.7)–(2.9), we shall use a result from the general discretization framework introduced by López-Marcos and Sanz-Serna [12].

Theorem 5.1. *Assume that* (2.11) *is consistent and stable with thresholds* M_h. *If* ϕ_h *is continuous in* $B(\boldsymbol{u}_h, M_h)$ *and* $\|\boldsymbol{l}_h\| = \mathrm{o}(M_h)$ *as* $h \to 0$, *then*:
(i) *for h sufficiently small, the discrete equations* (2.15) *possess a unique solution in* $B(\boldsymbol{u}_h, M_h)$;
(ii) *as* $h \to 0$, *the solutions converge.*
Furthermore, the order of convergence is not smaller than the order of consistency.

Now the existence and convergence is immediately obtained by means of the consistency (Theorem 3.1), the stability (Theorem 4.1) and Theorem 5.1.

Theorem 5.2. *Under the hypotheses of Theorem* 3.1, *if the numerical initial conditions* $\boldsymbol{U}^l$, $0 \leqslant l \leqslant s-1$, *given in* (2.8) *are such that*

$$\|\boldsymbol{U}^l - \boldsymbol{u}^l\|_\infty = \mathrm{o}(h),$$

as $h \to 0$, *then, for h sufficiently small, there exists a unique solution of equations* (2.7)–(2.9),

$$(\boldsymbol{U}^0, \boldsymbol{U}^1, \ldots, \boldsymbol{U}^N),$$

in the ball $B(\boldsymbol{u}_h, Mh)$ *of* X_h, *and*

$$\max_{0 \leqslant n \leqslant N} \|\boldsymbol{U}^n - \boldsymbol{u}^n\|_1 = \mathrm{O}\left(\sum_{l=0}^{s-1} \|\boldsymbol{U}^l - \boldsymbol{u}^l\|_1 + h^s\right). \tag{5.1}$$

Next, we establish the convergence of the numerical method in the maximum norm.

Theorem 5.3. *Under the hypotheses of Theorem* 5.2, *if*

$$(\boldsymbol{U}^0, \boldsymbol{U}^1, \ldots, \boldsymbol{U}^N)$$

is the solution of (2.7)–(2.9) *given by that theorem then, as* $h \to 0$,

$$\max_{0 \leqslant n \leqslant N} \|\boldsymbol{U}^n - \boldsymbol{u}^n\|_\infty = \mathrm{O}\left(\sum_{l=0}^{s-1} \|\boldsymbol{U}^l - \boldsymbol{u}^l\|_\infty + h^s\right). \tag{5.2}$$

Proof. We set

$$\boldsymbol{e}^n = \boldsymbol{U}^n - \boldsymbol{u}^n, \quad 0 \leqslant n \leqslant N.$$

By (4.5), (3.15) and (5.1) we obtain

$$\begin{aligned} |e_0^n| &\leqslant |L_0^n| + C\|\boldsymbol{e}^n\|_1 \\ &= \mathrm{O}\left(\sum_{l=0}^{s-1} \|\boldsymbol{U}^l - \boldsymbol{u}^l\|_1 + h^s\right), \end{aligned} \tag{5.3}$$

$s \leqslant n \leqslant N$. On the other hand, taking into account (4.2), (2.17) and (5.1) we have

$$\begin{aligned} |e_{j+1}^{n+1}| &\leqslant |e_j^n| + h|L_{j+1}^{n+1}| + Ch\left(|e_j^n| + \sum_{l=0}^{s-1} \|\boldsymbol{e}^{n-l}\|_1\right) \\ &\leqslant (1+Ch)|e_j^n| + Ch\mathrm{O}\left(\sum_{l=0}^{s-1} \|\boldsymbol{U}^l - \boldsymbol{u}^l\|_1 + h^s\right), \end{aligned} \tag{5.4}$$

$0 \leqslant j \leqslant J-1$, $s-1 \leqslant n \leqslant N-1$. Hence, (5.2) is derived by means of a recursive argument. □

$$\begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

(*a*)

$$\begin{array}{c|c} \frac{1}{2} & \frac{1}{2} \\ \hline & 1 \end{array}$$

(*b*)

$$\begin{array}{c|ccc} 0 & & & \\ \frac{2}{3} & \frac{2}{3} & & \\ 0 & -1 & 1 & \\ \hline & 0 & \frac{3}{4} & \frac{1}{4} \end{array}$$

(*c*)

$$\begin{array}{c|cc} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & 1 & 0 \\ \hline & \frac{3}{4} & \frac{1}{4} \end{array}$$

(*d*)

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

(*e*)

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 1 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

(*f*)

Fig. 1. (a) Explicit Runge–Kutta method of second order. (b) Implicit Runge–Kutta method of second order. (c) Explicit Runge–Kutta method of third order. (d) Implicit Runge–Kutta method of third order. (e) Explicit Runge–Kutta method of fourth order. (f) Implict Runge–Kutta method of fourth order.

Note that, in particular, if $\| \boldsymbol{U}^l - \boldsymbol{u}^l \|_\infty = \mathrm{O}(h^s)$, $0 \leqslant l \leqslant s-1$, then our scheme is s-order accurate.

6. Numerical results

We have carried out numerical experiments with six Runge–Kutta schemes of various orders. The tableaux of these methods are given in Figs. 1(a)–1(f). In order to approximate the integral terms we have used the Simpson rule. When the values $(U_0^n, \ldots, U_J^n)$, $n \geqslant s-1$, are known we obtain the values $(U_1^{n+1}, \ldots, U_J^{n+1})$ by solving the linear systems (2.7b). Note that this is immediate when the R–K method considered is explicit. Next to obtain U_0^{n+1} we need to solve the nonlinear equation (2.9). To this end, we use the following iteration.

Step 1.

$$U_0^{n+1,0} = U_0^n.$$

Step 2.

$$U_0^{n+1,p+1} = g\big(Q_h\big(\boldsymbol{b}(\boldsymbol{U}^{n+1,p})\boldsymbol{U}^{n+1,p}\big),\ t_{n+1}\big),$$

where $\boldsymbol{U}^{n+1,p} = (U_0^{n+1,p}, U_1^{n+1}, \ldots, U_J^{n+1})$.

Step 3. If

$$| U_0^{n+1,p+1} - U_0^{n+1,p} | \geqslant \textit{tolerance}$$

then go to Step 2.

In our numerical experiments we have taken a tolerance of 10^{-8}.

We have obtained the initial conditions (2.8) with a second-order scheme based on the R–K method with the tableau given in Fig. 1(a). More precisely, this method is given by the equations

$$Y_1(U_j^n) = hf(x_j, Q_h(U^n))U_j^n, \tag{6.1}$$

$$Y_2(U_j^n) = hf\left(x_j + \frac{h}{2}, Q_h\left(\frac{U^{n+1} + U^n}{2}\right)\right)\left[U_j^n + \tfrac{1}{2}Y_1(U_j^n)\right], \tag{6.2}$$

$$U_{j+1}^{n+1} = U_j^n + Y_2(U_j^n), \quad 0 \leqslant j \leqslant J-1, \tag{6.3}$$

$$U_0^{n+1} = g\left(Q_h(b(U^{n+1})U^{n+1}), t_{n+1}\right), \quad 0 \leqslant n. \tag{6.4}$$

This method has only two time levels but (6.1)–(6.2) are nonlinear equations. The analysis of such method can be done in a very similar manner to the one above.

Now we can implement the methods of order two or three given by (2.7)–(2.9). Note that for such methods we only need two or three (respectively) initial conditions and that the approximations U^1 and U^2 given by (6.1)–(6.4) are $O(h^3)$. For fourth-order methods, we obtain the initial conditions U^i, $1 \leqslant i \leqslant 3$, in the following way: we use the scheme (6.1)–(6.4) to obtain two approximations to u^i, one with step h and another with step $h/2$, then Richardson extrapolation is used to obtain U^i. For schemes of order greater than four, we would work in an analogous way.

We have considered the following test problem. We choose the age-specific fertility and mortality moduli as

$$f(x, z) = -z, \qquad b(x, z) = \frac{xz \exp(-x)}{(1+z)^2},$$

the birth function as

$$g(z, t) = \frac{4z(2 - 2\exp(-A) + \exp(-t))^2}{(1 - \exp(-A))(1 - (1+2A)\exp(-2A))(1 - \exp(-A) + \exp(-t))},$$

and we consider as the initial age-specific density the function

$$u^0(x) = \frac{\exp(-x)}{2 - \exp(-A)}.$$

The solution of the problem (1.1)–(1.3) is given by

$$u(x, t) = \frac{\exp(-x)}{1 - \exp(-A) + \exp(-t)}.$$

In our numerical experiments, we have chosen $A = 5$ and $T = 20$.

In each entry in columns two to four of Tables 1 and 2, the upper number represents the global error

$$e_h = \max_{0 \leqslant n \leqslant N} \| u^n - U^n \|_\infty, \tag{6.5}$$

Table 1
Errors, CPU time (seconds) and order of schemes based on explicit Runge–Kutta method with implicit boundary condition

J	Second-order R–K	Third-order R–K	Fourth-order R–K
20	0.110654D−1 0.5 |	0.278228D−2 0.6 |	0.208397D−2 0.7 |
40	0.277192D−2 1.9 | 1.997	0.219513D−3 2.0 | 3.664	0.130470D−3 2.2 | 3.998
80	0.680003D−3 7.0 | 2.027	0.189044D−4 7.4 | 3.538	0.821526D−5 7.9 | 3.989
160	0.167314D−3 27.2 | 2.023	0.182239D−5 28.4 | 3.375	0.518105D−6 30.1 | 3.987
320	0.414216D−4 104.9 | 2.014	0.193917D−6 111.0 | 3.232	0.331754D−7 115.8 | 3.965

the lower number on the left the CPU time in seconds and the lower number on the right the order s of the method as computed from

$$s=\frac{\log(e_{2h}/e_h)}{\log(2)}. \tag{6.6}$$

Each column of the tables corresponds respectively to the use in the proposed method of a different R–K method for the stepping in time along the characteristics: explicit R–K methods of second, third and fourth order with tableaux given in Figs. 1(a), 1(c) and 1(e) were used in Table 1, while implicit R–K methods of the same order with tableaux given in Figs. 1(b), 1(d) and 1(f) were used in Table 2. In different rows we show numerical results corresponding to meshes of 20, 40, 80, 160 and 320 subintervals per time level. The results in the tables clearly

Table 2
Errors, CPU time (seconds) and order of schemes based on implicit Runge–Kutta methods with implicit boundary condition

J	Second-order R–K	Third-order R–K	Fourth-order R–K
20	0.778637D−2 0.6 |	0.206695D−2 0.5 |	0.211621D−2 0.7 |
40	0.157164D−2 1.9 | 2.309	0.150771D−3 1.9 | 3.777	0.132688D−3 2.2 | 3.995
80	0.367678D−3 6.7 | 2.096	0.130206D−4 7.4 | 3.533	0.834948D−5 7.9 | 3.990
160	0.902472D−4 26.3 | 2.026	0.131754D−5 28.1 | 3.305	0.525897D−6 29.6 | 3.989
320	0.224472D−4 102.9 | 2.007	0.147248D−6 108.3 | 3.162	0.335698D−7 114.6 | 3.969

Table 3
Errors, CPU time (seconds) and order of schemes based on explicit Runge–Kutta methods with explicit boundary condition

J	Second-order R–K	Third-order R–K	Fourth-order R–K
20	0.110654D−1	0.278228D−2	0.208523D−2
	0.3 |	0.4 |	0.5 |
40	0.277192D−2	0.219513D−3	0.130517D−3
	1.4 | 1.997	1.5 | 3.664	1.7 | 3.998
80	0.680003D−3	0.189044D−4	0.821651D−5
	5.2 | 2.027	5.7 | 3.538	6.2 | 3.990
160	0.167314D−3	0.182239D−5	0.518151D−6
	21.0 | 2.023	22.7 | 3.375	24.1 | 3.987
320	0.414216D−4	0.193917D−6	0.331760D−7
	85.0 | 2.014	89.9 | 3.232	95.3 | 3.965

confirm the expected order of convergence. We see that the processing times to obtain a fixed accuracy in the numerical solution for the schemes based on fourth-order R–K methods are always less than the processing times for the other schemes to obtain the same global precision. This is even more pronounced when a more demanding precision is sought. This seems to indicate that schemes based on high-order algorithms are in general preferable when the data are smooth.

We have also considered a modification of the numerical method (2.7)–(2.9) in order to make the boundary condition (2.9) explicit. We have noted above that when the values $(U_0^n, \ldots, U_J^n)$, $n \geqslant s-1$, are known we obtain the values $(U_1^{n+1}, \ldots, U_J^{n+1})$ by solving the linear

Table 4
Errors, CPU time (seconds) and order of schemes based on implicit Runge–Kutta methods with explicit boundary condition

J	Second-order R–K	Third-order R–K	Fourth-order R–K
20	0.775904D−2	0.205370D−2	0.211746D−2
	0.4 |	0.4 |	0.5 |
40	0.156788D−2	0.149598D−3	0.132725D−3
	1.3 | 2.307	1.5 | 3.779	1.7 | 3.996
80	0.367186D−3	0.129189D−4	0.835055D−5
	5.2 | 2.094	5.6 | 3.534	6.0 | 3.990
160	0.901833D−4	0.131008D−5	0.525938D−6
	20.3 | 2.026	22.2 | 3.302	23.2 | 3.989
320	0.224391D−4	0.146726D−6	0.335704D−7
	81.0 | 2.007	87.8 | 3.158	91.4 | 3.970

systems (2.7b). Hence we can compute an approximation $U_0^{*,n+1}$ by using the extrapolation formula

$$U_0^{*,n+1} = \sum_{p=1}^{s} (-1)^{p-1} \binom{s}{p} U_0^{n-p+1}, \tag{6.7}$$

of the same order s as that of the R–K method on which the scheme is based. Now we use as boundary condition

$$U_0^{n+1} = g\big(Q_h\big(b(U^{*,n+1})U^{*,n+1}\big), t_{n+1}\big), \quad s-1 \leqslant n \leqslant N-1, \tag{6.8}$$

where $U^{*,n+1} = (U_0^{*,n+1}, U_1^{n+1}, \ldots, U_J^{n+1})$. The analysis of such a method can be done in an analogous way to that for (2.7)–(2.9). for the implementation of (6.7), we need only use an additional array to store the values U_0^{n-p+1}, $1 \leqslant p \leqslant s$, and that must be updated after every step. The starting values are computed as explained for method (2.7)–(2.9).

Each entry of Tables 3 and 4 shows the global error (6.5) (upper number), the CPU time (lower number on the left) and the order (lower number on the right) computed from (6.6) in the numerical experiments with the modified method (2.7)–(2.8) and (6.7)–(6.8) on the same test problem as before. Each column of Table 3 (respectively 4) presents numerical results of the modified method based on the same explicit (respectively implicit) R–K method that was used for the corresponding column of Table 1 (respectively 2). Again, the expected order is apparent. The sizes of the errors are very close to those obtained with the method (2.7)–(2.9) and the CPU times are reduced. It should be noted that this improvement is becoming less significant as the mesh has more and more grid points per level of time because then the cost per step of the computation of the boundary value U_0^{n+1} is a lower fraction of the total cost of the step. For the most expensive run, the CPU time cost is improved by about 18%.

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