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Article in Ecological Modelling · October 2005

DOI: 10.1016/j.ecolmodel.2005.05.007

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1 Introduction

Individuals in a structured population are distinguished by age, size, maturity or some other individual physical characteristic. The main assumption when modelling the evolution of such a population is that the structure of the population with respect to these individual physical characteristics at a given time, and possibly some environmental input as time evolves, completely determines the dynamical behavior of the population. Mathematical models describing this evolution have attracted a considerable amount of interest among researchers as a tool for modelling the interaction of different population communities in such diverse fields as demography, epidemiology, ecology, cell kinetics, tumor growth, etc.

Continuous time formulations of structured population models were first introduced in the pioneering works of Sharpe and Lotka (1911) and McKendrick (1926), who considered age as the only structuring variable in the population and assumed only age-dependence in the fertility and mortality rates of the population. For many years afterwards, age-dependent population models were studied exclusively, along with many generalizations, the most important being the introduction by Gurtin and MacCamy (1974), of a nonlinear age-structured model which allowed the mortality and fertility rates of the population to be affected also by the total population. Size-structured population models considering other physiological characteristics in the individuals of the population, such as length, weight, maturity level, biomass, etc. appeared in the sixties as a more meaningful way to model the dynamics of species where these characteristics play an essential role in the capacity to survive, grow and reproduce.

Mathematical models of structured populations typically involve a system of first order hyperbolic partial integro-differential equation for the time dependent density function with respect to the chosen structuring variables of each of the populations in the biological system. Each of the partial integro-differential equations stems from the appropriate balance of individuals of the corresponding population inside an infinitesimally small box element before and after it has moved during infinitesimally small time intervals $(t, t + dt)$, the dependence of the growth and the mortality rates of the population on appropriate functionals of the density functions being the cause of the nonlocal nonlinearities. Moreover, a set of nonlocal boundary conditions describing the reproduction process of each population, and initial distributions of each of these populations close the system. Although these ideas allow highly realistic models to be constructed, unfortunately for the most realistic cases these kinds of models cannot be solved analytically. Therefore, numerical methods are unavoidable in order to obtain any quantitative information from the models. However, standard techniques for the numerical solution of hyperbolic partial differential equations should be adapted to cope with the nonlocal nonlinearities included in the governing equations, and also the nonlocal boundary condition which describes the corresponding birth law.

This paper is the first of a series of two and its goal is to review from a historical point of view the numerical methods considered for the numerical approximation of age-structured population models. The second in the series will review also the numerical methods for the numerical approximation of size-dependent population models [1]. In the second Section we carry out a brief historical revision of the continuous time age-structured population models, ending with the full nonlinear one. The third Section is devoted to giving a complete historical perspective of the more significant numerical methods that have been considered in the literature for the numerical approximation of age-structured population models. In Sections 4 and 5, we will consider separately each technique employed, introducing the equations of each numerical method, pointing out their features and their mathematical contributions. Finally, in Section 6, we will consider some numerical examples in which we show the behaviour of some methods in several situations.

2 Population Dynamics Models

The first mathematical model regarding the study of population dynamics was proposed and solved by Leonardo Pisano, Fibonacci, in 1202 in the Liber Abaci [61]. In this study, the well-known *Fibonacci sequence* was introduced. During the XVIIIth century, L. Euler [18] studied the modelization of populations and suggested that the human population grew geometrically. Using this idea, T.R. Malthus proposed the most simple of the models on population dynamics in his famous treatise about the growth of the human population [51], published at the end of XVIIIth century. He predicted an exponential growth of the human population through time. The Malthusian model considers homogeneous populations, i.e. it supposes that all the individuals of such a population are physiologically identical. Thus, the state of the population can be described by means of a few statistical data based only on the total quantity of individuals. Therefore, the only variable that this model takes into account is $P(t)$ (the size of the population in each instant of time). Also, in this model, it was supposed that the population lives isolated in an invariable habitat and with limitless resources, so the population depended, respectively, on constant fertility and mortality rates that we will denote with α and μ ($\delta = \alpha - \mu$ is usually named *the Malthusian parameter* of the population). Thus, the population dynamics was determined by the following linear ordinary differential equation called *Malthus' Law* :

$$\frac{d}{dt}P(t) = \alpha\, P(t) - \mu\, P(t) = \delta\, P(t). \tag{2.1}$$

Its only possible usefulness is the study of population growth during a time interval in which these hard hypotheses can be accepted. It is undoubtedly true that, although we assume that variations do not take place in the external habitat, the population itself causes changes in life conditions due to competition for the survival resources. Consequently, we could admit that the fertility and mortality rates depend on the total size of the population, replacing the linear model of Malthus by a nonlinear model.

The first model of this type was proposed by Verhulst (*logistic model*) and it corrected the most significant objections against the Malthusian model [65, 66, 67] (electronic version at http://gdz.sub.unigoettingen.de). Verhulst imposed a maximum size K for the total population size (*carrying capacity*) and considered this nonlinear ordinary differential equation,

$$\frac{d}{dt}P(t) = \delta_0 \left(1 - \frac{P(t)}{K}\right) P(t). \tag{2.2}$$

The model could be interpreted as a Malthusian model with constant fertility rate, $\alpha = \delta_0$, and a mortality rate proportional to the relative size of the total population with respect to the carrying capacity, $\mu = \delta_0 \dfrac{P(t)}{K}$.

Originally, the logistic model was thoroughly debated and considered as a universal law that governed the growth of a population that was left to develop under constant environmental conditions, with an absence of predators and with a sufficient quantity of food [59]. The model was successfully checked by means of laboratory experiments with colonies of bacteria, yeast and other simple organisms under those environmental conditions. Moreover, it was applied by Lotka [49, 50] to fit the growth curve of the United States population in the thirties. These experiments provided growth curves that agreed with the predictions of the logistic equation. However, this similarity was not observed for other organisms with a complex life cycle (eggs–larva–pupa–adult), such as the fly of the fruit, beetle and others. For these organisms, an approximation of the total population toward the carrying capacity was not observed, but rather in its place these populations exhibit fluctuations after a period of logistic growth. An interesting discussion regarding the logistic model can be found in Krebbs [43].

This former kind of modelization led us towards mathematical models with a very limited scope. This unsuitability to simulate the biological reality in most of the cases encouraged the development of more elaborate mathematical models in which the individuals are distinguished within the population by some physiological property. So, although it would be impossible to pick up each individual's singularity and its contribution to the dynamics of the population to which it belongs, it seems clear that there existed a relationship between certain individual physiological states and their role in the population dynamics.

The first work in which the population was structured in terms of age was carried out by Sharpe and Lotka [60]. They developed an integral formulation where the fertility and mortality rates depended on the age variable. This idea was used by McKendrick [52] to present his model by means of the following first order and linear partial differential equation, which described the dynamics of an age-structured population

$$u_t(a,t) + u_a(a,t) = -\mu(a)\, u(a,t), \quad t > 0, \quad a > 0, \tag{2.3}$$

where the function $u(a,t)$ represents the age-density function for the individuals of the population at time t. Therefore, the number of individuals which, at time t, have ages in the interval $[a_1, a_2]$ is given by the integral

$$\int_{a_1}^{a_2} u(a,t)\, da,$$

and the total population in the same instant of time is

$$P(t) = \int_0^\infty u(a,t)\,da, \quad t > 0. \tag{2.4}$$

The value $\mu(a)\,u(a,t)$ represents the loss of individuals due to mortality, where $\mu(a)$ is a non-negative function determined by age, called *age-specific mortality function of the population.* A complete description of the population dynamics should take into account the number of births that take place in the population at every instant of time. This number is given by the following integral equation (*birth law*) which was implicitly described in the work of Lotka [49]

$$u(0,t) = \int_0^\infty \alpha(a)\,u(a,t)\,da, \quad t > 0, \tag{2.5}$$

where $\alpha(a)$ is a non-negative function called the *age-specific fertility function of the population*. This equation plays the role of a nonlocal boundary condition for the problem. Finally, the age-density function is determined after considering the initial state of the population

$$u(a,0) = u_0(a), \quad a \geq 0, \tag{2.6}$$

where the non-negative function $u_0(a)$ represents the initial age-distribution of the population. This model was also associated with von Förster [21] because he considered the equation (2.3) in his study on cellular populations. Detailed information on this model can be found in the books of Keyfitz [33] and Miller [54].

The McKendrick-von Förster model presents the same inconveniences as described for the Malthusian model. The vital functions are independent of the total population, therefore they are not able to describe the changes in life conditions of the individuals caused by the size of the total population. It would seem reasonable to suppose that there exist situations in which the life conditions for the individuals are more difficult and, consequently, the mortality increases and the fertility diminishes respectively when the population reaches a certain size. In 1974, bearing in mind these considerations, Gurtin and MacCamy [22] and Hoppensteadt [24] introduced the first nonlinear continuous models to describe the dynamics of an age-structured population. In the study of Gurtin and MacCamy, the size of the total population was incorporated in the model, for the fact that the vital processes, birth and mortality, were nonlinear functionals of the population density, with explicit dependence on the total population, $P(t)$.

The model consists of a nonlinear partial differential equation with nonlocal terms

$$u_t(a,t) + u_a(a,t) = -\mu(a, P(t))\,u(a,t), \quad a > 0, \quad t > 0, \tag{2.7}$$

where the mortality rate, $\mu(a, P(t))$ is a non-negative function. The birth law of the Gurtin-MacCamy model is given by

$$u(0,t) = \int_0^\infty \alpha(a, P(t))\,u(a,t)\,da, \quad t > 0, \tag{2.8}$$

representing a nonlinear and nonlocal boundary condition, where the fertility rate $\alpha(a, P(t))$ is also a non-negative function. The model is completed with the initial condition (2.6). Besides introducing it, Gurtin and MacCamy carried out a wide theoretical analysis of the model showing that the nonlinear models present a richer dynamic than the linear ones.

Research concerning the Gurtin-MacCamy's model had been extensive considering both specific situations and possible generalizations. In the monographs of Webb [68], Metz and Dieckman [53], Iannelli [26] and Cushing [16] a broad relationship with the most significant results can be found.

We conclude the revision of the age-structured models with a problem where the vital functions depend on the size of a weighted average population to reflect the different influence that individuals of different ages have on the change of life conditions of the population. On the other hand, they also depend on time to include the environmental variations such as, for example, permanent or periodic modifications in the conditions of life. Lastly, the model considers a maximum age A for the individuals in the population. Thus, if we introduce the following weighted average quantities

$$I_\mu(t) = \int_0^A \gamma_\mu(a)\, u(a,t)\, da, \quad I_\alpha(t) = \int_0^A \gamma_\alpha(a)\, u(a,t)\, da, \quad t > 0, \tag{2.9}$$

the model is determined by the following equation

$$u_t(a,t) + u_a(a,t) = -\mu(a, I_\mu(t), t)\, u(a,t), \quad 0 < a < A, \quad t > 0, \tag{2.10}$$

and the boundary condition,

$$u(0,t) = \int_0^A \alpha(a, I_\alpha(t), t)\, u(a,t)\, da, \quad t > 0, \tag{2.11}$$

where the mortality and fertility rates are non-negative functions of three independent variables. The model is completed with the initial condition (2.6). It is clear that if $\gamma_\mu(a) = \gamma_\alpha(a) = 1$, $a \in [0, A]$, we get the nonautonomous Gurtin-MacCamy model with finite age. The monographs of Iannelli [26] and Cushing [16] provide a detailed analysis of the autonomous case with the vital functions depending on an arbitrary number of functionals like (2.9).

Next, we focus our attention on the maximum age A. In a first approach, A is the maximum age a_+ which an individual in the population could reach. Thus, the assumption

$$\int_0^{a_+} \mu(a,p,t)\, da = \infty, \tag{2.12}$$

is made because the probability of an individual surviving to age a (*the survival probability*) is given by the function

$$\prod(a,p,t) = \exp\left(- \int_0^a \mu(\sigma,p,t)\, d\sigma \right),$$

and, in order to be congruent, $\prod(a_+, p, t) = 0$ must be held (for more details we refer to the monograph of Iannelly [26]). There is no contradiction between (2.12) and the boundedness of μ when $a_+ = \infty$. However, when the maximum age is finite, we understand that A is a suitable age chosen in such a way that the individuals with ages between A and the maximum age a_+ do not exercise a meaningful influence at ages less than A, either for the vital rates or for newborns. More precisely, if we assume that

$$\gamma_\mu(a) = 0, \qquad \gamma_\alpha(a) = 0, \quad a \in [A, a_+], \tag{2.13}$$

$$\alpha(a,p,t) = 0, \quad a \in [A, a_+], \tag{2.14}$$

then the solution to the problem (2.9)-(2.11) is the restriction to $[0, A]$ of the solution for the same model problem on the whole $[0, a_+)$. Note that the conditions (2.13)-(2.14) are realistic from a biological point of view in the situation previously described. On the other hand, if $a_+ = \infty$ the above hypotheses allow us to work in the same finite age-interval $[0, A]$ for all the times $t > 0$ and without considering the age-interval containing the initial age function support, which is a clear advantage from a computational point of view.

The models that we have described serve as a base for problems which appear in other different areas. Now we carry out a brief description of some of these works that does not seek to be exhaustive for the diverse fields. We find models of population which distinguish between sexes, formulated by Hoppensteadt [25]. Also, there are models that describe the interaction among several species, such as the one introduced by Venturino [64], that models different situations as predator-prey, symbiosis, parasitism, etc. The work made in the space diffusion models, proposed by Langlais is important [45, 46]. And finally, the use of models in the field of epidemic control, such as those of Hoppensteadt [24], Busenberg and Iannelli [13], Busenberg *et al.* [12], [14], and Iannelli *et al.* [30], etc, are also important.

3 Numerical solution of age-structured models

The explicit analytical solution of the models outlined in the previous section is unfeasible and only in very special cases could it be done. In the last three decades, different numerical schemes have been proposed for the numerical integration of these models and, in this section we will carry out a brief outline of the most significant numerical methods considered in the literature, describing some of their main features. We will consider them in chronological order because we will group them in terms of the type of discretization in Section 4 in which we formulate explicitly the equations for each method.

To the best of the authors' knowledge, the first work in which the numerical solution of age-structured population models was dealt with was made by Douglas and Milner [17]. The authors studied a nonlinear model with the mortality rate depending on the total population, $\mu(P(t))$. First, they solved a model with a Dirichlet boundary condition, i.e. the birth law was given explicitly. Later, they studied a model with a nonlocal and linear boundary condition. The proposed numerical methods consisted of a linearly implicit finite difference method of first order which integrated the equation along the characteristic curves. This method employed extrapolation in the nonlinear term, a technique that would be used in later numerical methods. The same problem, with a Dirichlet boundary condition, was considered by Kostova [40] who introduced a first order numerical method based on a kind of the method of lines. She discretized the time variable in order to get an ordinary differential equation for the restrictions to the lines $t = cte$ of the solution, which she solved analytically to obtain an integral expression which was finally discretized using a quadrature rule. The convergence analysis showed that the scheme converged whenever $\frac{\Delta a}{(\Delta t)^2}$ remained bounded, which had been so restrictive for this kind of problem. Next, Kannan and Ortega [31] considered a numerical method whose interest was not to approach the theoretical solution but to provide an analytical tool to prove the existence and uniqueness of solutions for the Gurtin-MacCamy model. They proved the convergence of a finite difference scheme and employed the discrete equations to establish qualitative properties for the solution to the continuous problem.

In order to solve the Gurtin-MacCamy model (2.7)-(2.8) in an infinite interval, Chiu [15] introduced the first method which approximated the solution using the integration along the characteristic curves. The method was derived by discretizing a representation of the theoretical solution that gave the number of individuals of the population at time $t + \Delta t$ with age $a + \Delta a$ in terms of the number of individuals in the population at time t with age a and the conditional probability for individuals of age a to survive to age $a + \Delta a$,

$$u(a + \Delta a, t + \Delta t) = u(a, t) \exp\left(- \int_0^{\Delta a} \mu(a + \tau, I_\mu(t + \tau), t + \tau) d\tau \right). \tag{3.1}$$

She proposed and analyzed a first order explicit method by approximating the exponential function by its Taylor polynomial $1 + x$ and using the composite mid-point quadrature rule to approximate the integrals. Next, Kostova [41] proposed and provided a convergence analysis of a first order characteristics method for a generalization of the Gurtin-MacCamy model, in the sense of considering a system of equations which described the nonlinear interaction of age-structured population dynamics. This method was employed by Kostova and Chipev [42] to solve a system of two coupled equations for an intramoluscan trematode population dynamics. Barr [11] proposed a method based on a projection technique over spline spaces to discretize the linear case (2.3)-(2.5) over an infinite interval.

For the Gurtin-MacCamy model with a more general birth-law, López-Marcos [48] proposed and analyzed a first order scheme based on the upwind discretization. Also, Fairweather and López-Marcos [19] employed an implicit numerical method based on the box scheme to solve the same problem. This scheme was the first second order method analyzed to solve such a general model. Three years later, the same authors considered an explicit extrapolated box method for the same problem [20]. They studied the convergence properties of the method and demonstrated its second order accuracy. They also made a comparative study of the use of different quadrature rules for the approximation of the integral terms. These rules were of high order and with a special form.

Milner and Rabbiolo made an analysis of different methods for distinct situations in [57]. They proposed and analyzed algorithms both in one- and special two- sex versions. For the solution of (2.3)-(2.5) they proposed a fourth order method based on an adaptation of the classic explicit Runge-Kutta fourth order formula combined with the composite Simpson quadrature rule to approximate the integral in the birth law. They also considered a linearly implicit second order scheme based on a centered finite difference second order formula for the characteristics derivative. They also analyzed a second order scheme for a simplified version of the Gurtin-MacCamy model with a fertility function independent of the total population size. This scheme was based on the mid-point formula for the derivative along the characteristics and resulted in an explicit three level second order finite difference method. The algorithm was started with the Euler explicit method in the first step. Finally, they discretized a special nonlinear model that distinguished between sexes based on the Gurtin-MacCamy model using this second order scheme.

Explicit second order methods with two steps, based on the central difference operator along the characteristics, were also studied by Kwon and Cho [44]. The authors made a numerical comparison between this method and those proposed by Kostova [40] and Douglas and Milner [17]. Another comparison of different methods was made by Sulsky [62]. She presented two different schemes, the two-step Lax-Wendroff scheme and the total-variation diminishing (TVD) scheme developed by Harten and Osher [23]. She did not carry out the convergence analysis of any of them, but she made a comparative study with academic test problems and proposed two problems with real data. Another approximation technique to solve the problem was proposed by Kappel and Zhang [32]. They studied the linear problem (2.3)-(2.5) and proposed an approximation using modified Legendre polynomials.

Numerical methods of arbitrary order based on Runge-Kutta techniques were analyzed by Abia and López-Marcos [2] to solve the Gurtin-MacCamy's nonlinear model with a more general boundary condition. These authors generalized the work of Milner and Rabbiolo [57] in two senses. First, they used a completely nonlinear problem, with both mortality and fertility rates depending on the population size. Secondly, they analyzed Runge-Kutta methods of arbitrary high order. The authors considered both implicit and explicit schemes, where the method of obtaining the latter were based on extrapolation techniques (already used by Fairweather and López-Marcos in [20] for the box method). It was also possible to make the boundary condition explicit applying the same techniques employed in [20]. Lastly, they presented several numerical test problems which confirmed the accuracy of the method. Another method, in this case a first order one, was proposed for the Gurtin-MacCamy's model. Kim and Park [38] considered the upwind technique [48] with the use of extrapolation on the source term [17]. Also, taking into account the idea proposed by Chiu [15] about the use of a representation of the theoretical solution, Abia and López-Marcos in [3], studied second order implicit finite difference schemes using Padé rational approximation to the exponential function. The authors also introduced several numerical test problems that allowed them to show the accuracy of such methods. Next, Ianelli *et al.* [27] employed splitting methods that took advantage of the peculiar form of the mortality rate $\mu(a) + M(a, p)$ to derive a first order finite difference scheme for the Gurtin-MacCamy model. This numerical scheme was based on an upwind finite difference scheme with extrapolation. The work included an application for S-I-S and S-I-R models of epidemiology.

We should point out that the design of explicit numerical methods of high order is difficult due to the existence of nonlocal terms in (2.7) and (2.8). In [4] Abia and López-Marcos proposed the first numerical methods for the more general model (2.9)-(2.11). In this work, two-step schemes together with open quadrature rules were employed to obtain explicit second order numerical methods. They used the representation of the solution (3.1) to deal numerically with the problem, following the ideas presented in Chiu [15] and Abia and López-Marcos [3]. On the other hand, from the numerical experiments carried out by Fairweather and López-Marcos [20] and Abia and López-Marcos [2], there was enough numerical evidence to believe that the dominant error in the numerical integration of problems like (2.9)-(2.11) were due to the approximation of non-local terms. Thus, they also presented numerical results obtained with different quadrature formulae with different convergence rates. The methods proposed were analyzed obtaining an optimal order of convergence. The authors introduced two types of two-step schemes, one of which made the integration with two steps of length $\frac{\Delta t}{2}$ which required changes in both the grid and the quadrature rule. The other avoided the change in the quadrature rule by means of the use of a different formulation. They also proposed new different schemes employing several high order quadrature rules.

On the other hand, the interest of such techniques is based on the successfully application, in the past, of numerical methods to real life populations such as in demography [57], with intramolluscan trematode populations [42], Nicholson's blowflies (L. curpina) and gray squirrel population (Sciurus carolinensis) [62], with the sexual phase of the reproductive cycle of monogonont rotifera [8, 7] and mosquitofish populations (Gambusia affinis) [63, 5]. Favourable comparisons between the results of the application of such schemes to these models and the real data obtained from the populations indicate that they are valid tools to investigate the biological systems.

At this point in the discussion, most of the authors considered focus their attention on an age window of maximum finite age A. The usual approach for the convergence analysis of numerical methods for the problem (2.9)-(2.11), through the concepts of consistency and stability requires the assumption that the mortality rate was bounded, at least on $[0, A] \times D_1 \times [0, T]$, where D_1 is a compact neighborhood of $\{I_\mu(t) : 0 \leq t \leq T\}$. Next, we shall comment on two recently published papers that take into account populations with a finite life span. Iannelli and Milner [29] considered the linear Lotka-McKendrick equation and discussed in detail how to solve the problem of the breakdown of the standard finite-difference methods when the mortality rate satisfied the finite life span condition $\int_0^{a^+} \mu(a)\, da = \infty$. Usual error bounds required some derivatives of the mortality rate to be bounded across all ages. Their approach worked for a model class of mortality rates and they showed that not all methods were compatible with whatever mortality function. The model employed autonomous vital functions ($\alpha(a)$, $\mu(a)$) and they studied the explicit Euler and the Crank-Nicholson methods. Also, Kim and Kwon [36] considered for the nonlinear Gurtin-MacCamy model a special choice of the mortality rate $\mu(a, P(t)) = m(a) + M(a, P(t))$, where all the singularities were included in the $m(a)$ term. They employed a characteristic curves scheme based on the implicit Gauss-Legendre Runge-Kutta method (as in [48, 2, 3, 4]).

The most popular technique to integrate numerically age-structured problems is the characteristics method, although there are other possibilities. Among them, the most common is the use of standard finite difference schemes. However, we find alternatives to these two approaches which have not been completely developed like the method of lines, the use of projection methods and collocation schemes. All of these techniques are common in the numerical study of partial differential equations and some of them are usually employed in the hyperbolic first order class of these equations.

As we have already pointed out in the previous section, the models that we have described serve as a basis for other problems which appear in other different areas. For these, different numerical approaches have also been considered. Now we carry out a brief description of these works which does not seek to be exhaustive for the diverse fields. In population models which distinguish between sexes, formulated by Hoppensteadt [25], the linear model was numerically integrated by Milner [55] by means of the finite element method, while the nonlinear case was studied by Arbogast and Milner [10] using a first order difference method, and by Milner and Rabbiolo [57] by means of a second order scheme already described. In models for interaction among several species, such as the one introduced by Venturino [64], which modelized different situations such as predator-prey, symbiosis, parasitism, etc., Kostova [41] has carried out numerical approximations by means of a first order characteristics method. The space diffusion models, proposed by Langlais [45, 46], have been numerically treated combining a difference scheme with a finite element method in the space variables for Milner [56], Kim [34] and Kim and Park [37, 39]. Diverse epidemic models, such as those of Hoppensteadt [24], Busenberg and Iannelli [13], Busenberg *et al.* [12], [14] and Iannelli *et al.* [30], have been numerically analyzed by Kim [35] and Iannelli *et al.* [28] using schemes closely related to some of the methods already mentioned at the beginning of this section.

4 Characteristic Curves Numerical Methods for Age-Structured Population Models

In the present section we shall concern ourselves with the numerical solution to the initial-boundary value problem (2.6), (2.9)-(2.11) in a finite time interval $[0,T]$ with methods based on the characteristic curves of the partial differential equations. The section is divided into different subsections to distinguish between different types of schemes.

First, we will derive a theoretical representation of the solution along the characteristic curves such as Abia and López-Marcos proposed in [3, 4]. The characteristic curves of equation (2.10) are the lines $x - t = c$, c constant, and along those characteristics the solution $u(x,t)$ satisfies

$$\frac{du}{dt} = -\mu(x, I_\mu(t), t)\, u. \tag{4.1}$$

If we integrate along the characteristics, we find that the solutions to the hyperbolic integro-differential equation (2.10) have the following property: for each x_0, with $0 < x_0 < A$, so that $a + h < A$, then

$$u(x_0 + h, t_0 + h) = u(x_0, t_0)\, \exp\left(- \int_0^h \mu(x_0 + \tau, I_\mu(t_0 + \tau), t_0 + \tau) d\tau \right), \tag{4.2}$$

where $t_0 > 0$.

Thus, we divide the Section in two parts. In the first, methods that employ the finite difference discretizations of the differential equations along the characteristics (4.1) are considered and in the second one, we will review methods which employ the integral representation of the theoretical solution along them (4.2). We will describe each numerical method completely in the most general way possible included by the original authors and their main features paying spetial attention to their mathematical properties. The numerical schemes are identified with the name of the authors.

Next, we will introduce the notation used to describe each numerical method. We will try to obtain a numerical approximation to the values of the theoretical solution u of (2.6), (2.9)-(2.11) in a time interval $[0,T]$. Given a positive integer J, if $h = A/J$, and $N = [T/h]$, we introduce the grid points $x_j = j\,h, j = 0, \ldots, J$ and time levels $t_n = n\,h, 0 \le n \le N$. Thus $(x_{j+1}, t_{n+1}) = (x_j + h, t_n + h), 0 \le j \le J-1, 0 \le n \le N-1$. We refer to the grid point x_j by a subscript j and to the time level t_n by a superscript n. Let U_j^n be a numerical approximation to $u(x_j, t_n)$, $0 \le j \le J$, $0 \le n \le N$, and $\mathbf{U}^n = (U_0^n, U_1^n, \ldots, U_J^n)$, $0 \le n \le N$. We also consider that an approximation to the initial condition (2.6), $\mathbf{U}^0$, is given.

The use of numerical quadrature rules is the natural way to approximate the integrals which appear in the boundary condition (2.11) and in the weighted average population (2.9). The authors usually look for quadrature rules Q_h with the form

$$Q_h(\mathbf{f}) = \sum_{j=0}^{J} q_j^h\, f_j\ , \tag{4.3}$$

where $\mathbf{f} = (f(x_0), f(x_1), \ldots, f(x_J))$, the nodes x_j, $0 \leq j \leq J$, are defined, so that, if f is sufficiently smooth, then as $h \to 0$,

$$\int_0^A f(x)\,dx = Q_h(\mathbf{f}) + O(h^s)\,, \tag{4.4}$$

(remember that we will describe numerical methods of order s). Furthermore, we take into account that, from now on, $\mathbf{a}\,\mathbf{b}$ denotes the componentwise product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

4.1 Methods which discretize the set of ODEs

The methods which we include in this section discretize the identity (4.1). These methods use the differential representation of the right-hand size of the partial differential equation which can be represented as

$$Du(a,t) = \lim_{h\to 0} \frac{u(a+h, t+h) - u(a,t)}{h}. \tag{4.5}$$

4.1.1 Douglas and Milner [17]

The authors considered two different models but, here, we describe the most complex one. The numerical scheme is defined by the following system of equations,

$$\begin{aligned} U_{j+1}^{n+1} &= U_j^n - h\,\mu(x_j, Q_h(\mathbf{U}^n), t_{n+1})\,U_j^{n+1}, & (4.6)\\ U_0^{n+1} &= Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}), & (4.7)\end{aligned}$$

$0 \leq j \leq J-1$, $0 \leq n \leq N-1$, where $\boldsymbol{\alpha}^n = (\alpha(x_0, t_n), \alpha(x_1, t_n), \ldots, \alpha(x_J, t_n))$, $0 \leq n \leq N$, and Q_h is the composite rectangular quadrature rule

$$Q_h(\mathbf{U}^n) = \sum_{j=0}^{J-1} h\,U_j^n, \quad 0 \leq n \leq N.$$

Note that the method is linearly implicit and an explicit form of the equation (4.6) can be found because the nonlinear term is interpolated.

They obtained a first order method in the maximum norm, i.e.

$$\sup_{0\leq n\leq N}\ \sup_{0\leq j\leq J} \{|u(x_j, t_n) - U_j^n|\} \leq C\,h.$$

The proof of convergence was made by means of the properties of the Taylor expansion of the theoretical solution. Finally, we should note that this method was originally designed to work in the age interval $[0,\infty)$.

4.1.2 Kannan and Ortega [31]

This method was proposed for the Gurtin-MacCamy model (2.7)-(2.8). They considered $u_{0,h}$ a non-negative step function, with the property $\lim_{h\to 0} u_{0,h} = u_0$ in $L^1(0,\infty)$. Thus, this explicit method is defined by the following equations

$$\begin{aligned} U_{j+1}^{n+1} &= U_j^n - h\,\mu(x_j, Q_h(\mathbf{U}^n))\,U_j^n, & (4.8)\\ U_0^{n+1} &= Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}), & (4.9)\end{aligned}$$

$0 \leq j \leq J-1$, $0 \leq n \leq N-1$, where $\boldsymbol{\alpha}^n = (\alpha(x_0, t_n), \alpha(x_1, t_n), \ldots, \alpha(x_J, t_n))$, $0 \leq n \leq N$, and Q_h is, again, the composite rectangular quadrature rule

$$Q_h(\mathbf{U}^n) = \sum_{j=0}^{J-1} h\, U_j^n, \quad 0 \leq n \leq N.$$

Next, they defined $u_h(a,t)$ and $P_h(t)$ step functions

$$u_h(a,t) = U_j^n, \qquad P_h(t) = \int_0^\infty u_h(a,t)\, da, \quad \text{for} \quad x_{j-1} \leq a \leq x_j, \quad t_{j-1} \leq t \leq t_j. \tag{4.10}$$

And they established that the quantities defined by (4.10) are well defined and that u_h was an ε-approximate solution for the suitable choice of h. Finally, they stated that u_h converged to u in $L_\infty([0,T], L^1(0,\infty))$, when $h \to 0^+$.

They found that the method is first order, i.e.

$$|u(x_j, t_n) - U_j^n| \leq C\, h, \quad 0 \leq j \leq J, \quad 0 \leq n \leq N.$$

Finally, we should point out that this method originally considered the age interval $[0, \infty)$.

4.1.3 Kostova [41]

This method was proposed originally for a system of partial integro-differential equations describing a nonlinear interaction of age-dependent populations dynamics. We will introduce it for a more general Gurtin-MacCamy model (2.7)-(2.8), in which the mortality and fertility rates depend on a weighted average of the density function of the population. Also, she considered the contribution of newborns due to immigration or invasion, a quantity that was given by $C(t,p)$. Thus, the method is defined by the following equations

$$\begin{aligned} U_{j+1}^{n+1} &= U_j^n - h\, \mu(x_j, Q_h(\boldsymbol{\gamma}\, \mathbf{U}^n))\, U_j^n, & (4.11)\\ U_0^{n+1} &= C(t_{n+1}, Q_h(\boldsymbol{\gamma}\, \mathbf{U}^n)) + Q_h(\boldsymbol{\alpha}^{n+1}\, \mathbf{U}^{n+1}), & (4.12) \end{aligned}$$

$0 \leq j \leq J-1$, $0 \leq n \leq N-1$, where

$$\boldsymbol{\alpha}^n = (\alpha(x_0, Q_h(\boldsymbol{\gamma}\, \mathbf{U}^n)), \alpha(x_1, Q_h(\boldsymbol{\gamma}\, \mathbf{U}^n)), \ldots, \alpha(x_J, Q_h(\boldsymbol{\gamma}\, \mathbf{U}^n))),$$

$0 \leq n \leq N$, and $\boldsymbol{\gamma} = (\gamma(x_0), \gamma(x_1), \ldots, \gamma(x_J))$, where γ is the average function. Q_h is the composite trapezoidal quadrature rule

$$Q_h(\mathbf{U}^n) = \frac{h}{2}\, (U_0^n + U_J^n) + \sum_{j=1}^{J-1} h\, U_j^n, \quad 0 \leq n \leq N.$$

Note that the method is explicit and she obtained first order convergence, i.e.

$$|u(x_j, t_n) - U_j^n| \leq C\, h, \quad 0 \leq j \leq J, \quad 0 \leq n \leq N.$$

4.1.4 Milner and Rabbiolo [57]

The authors proposed two numerical methods in order to solve the linear model given by the equations (2.3)-(2.5), one of second order and the other of fourth order, and a numerical scheme of second order to solve the nonlinear model (2.7)-(2.8), in which the fertility rate did not depend on the total population size. Also, they considered a special system modelling a two-sex age-structured population and their numerical approximation with the method introduced. Here, we only comment on the method applied to the Gurtin-MacCamy model. This scheme consists of a two-step method, which requires a starting procedure in the first time step. Thus,

$$U_{j+1}^1 = U_j^0 - h\,\mu(x_j, Q_h(\mathbf{U}^0))\,U_j^0, \quad 0 \le j \le J-1, \tag{4.13}$$

$$U_0^1 = Q_h(\boldsymbol{\alpha}^1\,\mathbf{U}^1). \tag{4.14}$$

The general recursion is defined by the following equations:

$$U_{j+2}^{n+2} = U_j^n - 2\,h\,\mu(x_{j+1}, Q_h(\mathbf{U}^{n+1}))\,\frac{U_j^n + U_{j+2}^{n+2}}{2}, \tag{4.15}$$

$$U_1^{n+2} = U_0^{n+1} - h\,\mu(x_0, Q_h(\mathbf{U}^{n+1}))\,U_0^{n+1}, \tag{4.16}$$

$$U_0^{n+2} = Q_h(\boldsymbol{\alpha}^{n+2}\,\mathbf{U}^{n+2}), \tag{4.17}$$

$0 \le j \le J-2$, $0 \le n \le N-2$, where $\boldsymbol{\alpha}^n = (\alpha(x_0), \alpha(x_1), \ldots, \alpha(x_J))$, $0 \le n \le N$ and Q_h is, again, the composite trapezoidal quadrature rule

$$Q_h(\mathbf{U}^n) = \frac{h}{2}\,(U_0^n + U_J^n) + \sum_{j=1}^{J-1} h\,U_j^n, \quad 0 \le n \le N.$$

Note that the method is linearly implicit and its explicit formulation can be easily found. They obtained second order accuracy, i.e.

$$\sup_{0\le n\le N}\ \sup_{0\le j\le J} \{|u(x_j,t_n) - U_j^n|\} \le C\,h^2.$$

They tested the simulators with a steady state solution for the age distribution but these numerical examples were only performed for the linear models. They also verified the linear model (and linear numerical method) with a real example based on the female population in the USA.

4.1.5 Kwon and Cho [44]

These authors considered two different models but we will only take into account the most complex one: they solved the Gurtin-MacCamy model with a simplification in the fertility function. The explicit numerical method which they developed is a two-step scheme, thus they initialized the first step in time with,

$$U_{j+1}^1 = U_j^0 - h\,\mu(x_j, Q_h(\mathbf{U}^0), t_0)\,U_j^0, \quad 0 \le j \le J-1, \tag{4.18}$$

$$U_0^1 = Q_h(\boldsymbol{\alpha}^1\,\mathbf{U}^1). \tag{4.19}$$

The method was defined by the following equations

$$\begin{aligned} U_{j+2}^{n+2} &= U_j^n - 2\,h\,\mu(x_{j+1}, Q_h(\mathbf{U}^{n+1}), t_{n+1})\,U_{j+1}^{n+1}, && (4.20)\\ U_1^{n+2} &= U_0^{n+1} - h\,\mu(x_0, Q_h(\mathbf{U}^{n+1}), t_{n+1})\,U_0^{n+1}, && (4.21)\\ U_0^{n+2} &= Q_h(\boldsymbol{\alpha}^{n+2}\,\mathbf{U}^{n+2}), && (4.22) \end{aligned}$$

$0 \le j \le J-2$, $0 \le n \le N-2$, where $\boldsymbol{\alpha}^n = (\alpha(x_0, t_n), \alpha(x_1, t_n), \ldots, \alpha(x_J, t_n))$, $0 \le n \le N$ and Q_h is the composite trapezoidal quadrature rule

$$Q_h(\mathbf{U}^n) = \frac{h}{2}\,(U_0^n + U_J^n) + \sum_{j=1}^{J-1} h\,U_j^n, \quad 0 \le n \le N.$$

They proved second order accuracy, i.e.

$$\sup_{0\le n\le N}\ \sup_{0\le j\le J} \{|u(x_j, t_n) - U_j^n|\} \le C\,h^2,$$

where Taylor expansions properties were employed.

Also, they tested their numerical method against the method of Kostova [40] and the method of Douglas and Milner [17] making a comparison of their computational complexities.

4.1.6 Abia and López-Marcos [2]

The authors considered difference schemes based on Runge-Kutta methods for the solution of a model more general in the boundary condition than the Gurtin-MacCamy model. In order to be consistent with this Section, we will consider a simplified version of the method, taking into account the fact that we will not consider the boundary condition as did the authors. The numerical schemes take the form

$$\begin{aligned} U_{j+1}^{n+1} &= U_j^n + \sum_{i=1}^{m} b_i\,Y_i(\mathbf{U}^n), && (4.23)\\ &\quad Y_i(\mathbf{U}^n) = -h\,\mu\left(x_j + c_i\,h, \sum_{l=0}^{s-1} c_i^l\,Q_h(\mathbf{U}^{n-l})\right)\left(U_j^n + \sum_{l=1}^{m} a_{il}\,Y_l(\mathbf{U}^n)\right), \quad 1 \le i \le m, && (4.24)\\ U_0^{n+1} &= Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}), && (4.25) \end{aligned}$$

$0 \le j \le J-1$, $s-1 \le n \le N-1$, where the coefficients b_i, a_{il} and c_i, $1 \le i, l \le m$, correspond to an m stages Runge-Kutta method of order s, $s \ge 2$. The values c_i^l, $0 \le l \le s-1$, are the weights of the extrapolation formulae to approach the values $P(t_n + c_i\,h)$ with order s, $1 \le i \le m$, $0 \le n \le N$. In this kind of schemes, Q_h is a composite quadrature rule of order s and $\boldsymbol{\alpha}^n = (\alpha(x_0, Q_h(\mathbf{U}^n)), \alpha(x_1, Q_h(\mathbf{U}^n)), \ldots, \alpha(x_J, Q_h(\mathbf{U}^n)))$, $0 \le n \le N$. The derived numerical methods are one-step explicit and implicit schemes, but the methods need an initial condition in the first s steps in order to be completely defined. Thus,

$$\mathbf{U}^p = (U_0^p, U_1^p, \ldots, U_J^p), \quad 0 \le p \le s-1, \tag{4.26}$$

are defined with different techniques in which the Richardson extrapolation is included.

They provided a complete analysis of such schemes in which consistency, stability, existence of discrete solutions and convergence were established. In order to make the analysis, they employed the discretization framework developed by López-Marcos [47]. Thus, they obtained arbitrary high order convergence

$$\max_{0\le n\le N}\max_{0\le j\le J}\{|u(x_j,t_n)-U_j^n|\}\le C\left(h^s+\sum_{l=0}^{s-1}\max_{0\le j\le J}\{|u(x_j,t_l)-U_j^l|\}\right).$$

In this paper, they reported extensive numerical experiments on test problems that featured all the nonlinearities of the mathematical model to analyze numerically both implicit and explicit methods of the class, and second, third and fourth order schemes. They used Simpson's quadrature rule to approximate the integral terms, and obtain the initial conditions with a second-order scheme. In order to obtain the highest order method in the class they also employed Richardson's extrapolation. They concluded that schemes based on high-order algorithms were in general preferable when the data were smooth. They also considered an extrapolation for the quadratures in the boundary condition as a way to build an explicit and cheaper method in terms of cpu-time.

4.1.7 Iannelly, Kim and Park [27]

This explicit scheme was considered for a Gurtin-MacCamy model with the mortality function of the form $\mu(a,p)=m(a)+M(a,p)$. They developed a splitting method given by the equations

$$\begin{aligned}
\frac{U_j^{n-\frac{1}{2}}-U_{j-1}^{n-1}}{h} &= m(a_j)\,U_j^{n-\frac{1}{2}}, &(4.27)\\
\frac{U_j^{n}-U_{j}^{n-\frac{1}{2}}}{h} &= -M(x_j,Q_h(\mathbf{U}^{n-1}))\,U_j^n, &(4.28)\\
U_0^{n+1} &= Q_h(\boldsymbol{\alpha}^{n-1}\,\mathbf{U}^n), &(4.29)
\end{aligned}$$

for $1\le j\le J$, $1\le n\le N$, where

$$\boldsymbol{\alpha}^n=(\alpha(x_0,Q_h(\mathbf{U}^n)),\alpha(x_1,Q_h(\mathbf{U}^n)),\dots,\alpha(x_J,Q_h(\mathbf{U}^n))),$$

$0\le n\le N$. The composite trapezoidal quadrature rule is again considered for the approximations of the integrals

$$Q_h(\mathbf{U}^n)=\frac{h}{2}\,(U_0^n+U_J^n)+\sum_{j=1}^{J-1}h\,U_j^n,\quad 0\le n\le N.$$

The authors proved first order convergence in the discretization parameter,

$$\max_{0\le n\le N}\max_{0\le j\le J}|u(x_j,t_n)-U_j^n|\le C\,h.$$

They also modified the method to fit it to epidemiological models and provided numerical experiments which confirmed the expected order of convergence.

4.1.8 Kim and Kwon [36]

The authors considered a scheme for the solution of the Gurtin-MacCamy model with a finite life span. Let J be an even number, then the numerical scheme is defined by the following system of equations,

$$\begin{aligned} U_{j+1}^{n+1} &= U_j^n - \frac{h}{2}\left(\mu(a_{j+c_1}, Q_h(\mathbf{U}^{n+c_1}))\,\varepsilon_1 + \mu(a_{j+c_2}, Q_h(\mathbf{U}^{n+c_2}))\,\varepsilon_2\right), \qquad (4.30)\\ &\quad \varepsilon_1 = U_j^n - h\left(\frac{1}{4}\,\mu(a_{j+c_1}, Q_h(\mathbf{U}^{n+c_1}))\,\varepsilon_1 + \left(\frac{1}{4} - \frac{\sqrt{3}}{6}\right)\mu(a_{j+c_2}, Q_h(\mathbf{U}^{n+c_2}))\,\varepsilon_2\right),\\ &\quad \varepsilon_2 = U_j^n - h\left(\left(\frac{1}{4} - \frac{\sqrt{3}}{6}\right)\mu(a_{j+c_1}, Q_h(\mathbf{U}^{n+c_1}))\,\varepsilon_1 + \frac{1}{4}\,\mu(a_{j+c_2}, Q_h(\mathbf{U}^{n+c_2}))\,\varepsilon_2\right),\\ U_0^{n+1} &= Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}), \qquad (4.31)\end{aligned}$$

$0 \le j \le J-1$, $0 \le n \le N-1$, where $a_{j+c_i} = a_j + c_i\,h$, and $Q_h(\mathbf{U}^{n+c_1}))$ is extrapolated with the same technique employed in [2], $i = 1, 2$. Q_h is the composite Simpson quadrature rule

$$Q_h(\mathbf{U}^n) = \sum_{j=0}^{\frac{J}{2}-1} \frac{h}{6}\left(U_{2j}^n + 4\,U_{2j+1}^n + U_{2j+2}^n\right), \quad 0 \le n \le N,$$

and $\boldsymbol{\alpha}^n = (\alpha(x_0, Q_h(\mathbf{U}^n)), \alpha(x_1, Q_h(\mathbf{U}^n)), \ldots, \alpha(x_J, Q_h(\mathbf{U}^n)))$, $0 \le n \le N$. The method is completely defined with initial conditions that are obtained with the method introduced in [2].

Note that the method is implicit and they have to solve a nonlinear system of equations. They obtain fourth order convergence by transforming the implicit Gauss-Lobato Runge-Kutta method in a collocation scheme, i.e.

$$\sup_{0\le n\le N}\ \sup_{0\le j\le J} \{|u(x_j, t_n) - U_j^n|\} \le C\,h^4.$$

They also reported some numerical experiments that showed the expected order of convergence.

4.2 Methods which employ the theoretical representation of the solution

The methods which we include in this section discretize the identity (4.2). So, we have

$$U_{j+1}^{n+1} = U_j^n\,\exp\left(-\int_0^h \mu(x_j+\tau, I_\mu(t_n+\tau), t_n+\tau)d\tau\right). \qquad (4.32)$$

$0 \le j \le J-1$, $0 \le n \le N-1$.

4.2.1 Chiu [15]

This author considered a method to solve the Gurtin-MacCamy model. This explicit method is defined by the following equations

$$\begin{aligned} U_{j+1}^{n+1} &= \frac{U_j^n}{1 + h\,\mu(x_j, Q_h(\mathbf{U}^n))}, \qquad (4.33)\\ U_0^{n+1} &= Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}), \qquad (4.34)\end{aligned}$$

$0 \leq j \leq J-1$, $0 \leq n \leq N-1$, where $\boldsymbol{\alpha}^n = (\alpha(x_0, Q_h(\mathbf{U}^n)), \alpha(x_1, Q_h(\mathbf{U}^n)), \ldots, \alpha(x_J, Q_h(\mathbf{U}^n)))$, $0 \leq n \leq N$. The quadrature rule Q_h employed is the composite mid-point quadrature rule, which is used in order to increase the accuracy of the scheme,

$$Q_h(\mathbf{U}^n) = \sum_{j=0}^{m} 2\,h\,U^n_{2\,j+1}, \quad 0 \leq n \leq N, \quad J = 2\,m+2.$$

She obtained first order convergence, i.e.

$$2\,h \sum_{j=0}^{J} |u(x_j, t_n) - U_j^n|\} \leq C\,\left(h + \frac{h}{\varepsilon}\right), 0 \leq n \leq N,$$

where $\varepsilon > 0$, is a given value with the next property: if we fix $T > 0$, there exist $a^* > 0$ and $\sigma > 0$, such that

$$\int_{a^*}^{\infty} u(x,t)\,da \leq \varepsilon, \quad \text{and} \quad K \geq P(t) \geq \sigma > 0, \quad 0 \leq t \leq T.$$

She also reported a numerical example to test the algorithm, but in the example the fertility rate was not depending on the total population.

Finally, we should point out that the method was designed to solve the problem in an infinite interval and that the author employed a Padé $(0,1)$-rational approximation to the exponential, an approach which would be developed by Abia and López-Marcos in [3].

4.2.2 Abia and López-Marcos [3]

The authors considered different methods defined by the following system of equations,

$$U_{j+1}^{n+1} = U_j^n\,R\left(-\frac{h}{2}\,\left(\mu\left(x_j, Q_h\left(\boldsymbol{\gamma}_\mu\,\mathbf{U}^n\right)\right) + \mu\left(x_{j+1}, Q_h\left(\boldsymbol{\gamma}_\mu\,\mathbf{U}^{n+1}\right)\right)\right)\right), \tag{4.35}$$

$$U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}), \tag{4.36}$$

$0 \leq j \leq J-1$, $0 \leq n \leq N-1$, where

$$\boldsymbol{\alpha}^n = (\alpha(x_0, Q_h\left(\boldsymbol{\gamma}_\alpha\,\mathbf{U}^n\right), t_n), \alpha(x_1, Q_h\left(\boldsymbol{\gamma}_\alpha\,\mathbf{U}^n\right), t_n), \ldots, \alpha(x_J, Q_h\left(\boldsymbol{\gamma}_\alpha\,\mathbf{U}^n\right), t_n)),$$

$\boldsymbol{\gamma}_\alpha = (\gamma_\alpha(x_0), \gamma_\alpha(x_1), \ldots, \gamma_\alpha(x_J))$, $\boldsymbol{\gamma}_\mu = (\gamma_\mu(x_0), \gamma_\mu(x_1), \ldots, \gamma_\mu(x_J))$, $0 \leq n \leq N$. And Q_h is the composite rectangular quadrature rule

$$Q_h(\mathbf{U}^n) = \frac{h}{2}\,(U_0^n + U_J^n) + \sum_{j=1}^{J-1} h\,U_j^n, \quad 0 \leq n \leq N.$$

The different methods developed were described in terms of the function $R(z)$ which denoted a Padé (m,n)-rational approximation to the exponential $\exp(z)$ of at least second order. This meant that $R(z) = \dfrac{P_m(z)}{Q_n(z)}$, $P_m(z)$, $Q_n(z)$ polynomials on the variable z of degrees m and n respectively, so that

$$|\exp(z) - R(z)| = O(z^3), \quad z \to 0.$$

For instance, the Padé (1,1), Padé (0,2), Padé (2,0), and Padé (2,2) rational approximations to the function $\exp(z)$, are given respectively by

$$\frac{2+z}{2-z}, \quad \frac{1}{1-z+\dfrac{1}{2}z^2}, \quad 1+z+\frac{1}{2}z^2, \quad \frac{12+6z+z^2}{12-6z+z^2}.$$

Note that in (4.35), the authors used the trapezoidal rule to approximate the integral inside the exponential in (4.2). This was mainly the cause of the implicitness of the formula (4.35). Also the composite trapezoidal rule made (4.36) implicit. Thus, the equations (4.35)-(4.36) were solved at each time step by means of a fixed-point iteration which was discussed by the authors.

In [15] an explicit first-order accurate method was derived (and analyzed) with the same approach because, in such a case, the integral inside the exponential in (4.2) was approximated with the rectangle quadrature rule and $R(z) = \dfrac{1}{1-z}$, was the Padé $(0,1)$-rational approximation to the exponential.

The authors provided a complete analysis of the schemes in which the consistency, stability, existence of discrete solutions and convergence of solutions were established. In order to make the analysis, they employed the discretization framework developed by López-Marcos [47]. Thus, they obtained second order accuracy

$$\max_{0\le n\le N}\max_{0\le j\le J}\{|u(x_j,t_n)-U_j^n|\} \le C\,h^2.$$

In this paper, they introduced a numerical experiment fitted to the model that was employed to analyze numerically four methods with the different Padé rational approximations given above. They tested the numerical schemes with both composite trapezoidal and Simpson quadrature rules to approximate the integral terms. They concluded that the global precision of the method improved with the use of higher order quadrature rules and the Padé rational approximations which seemed to indicate that schemes based on high-order algorithms were in general preferable when the data were smooth.

4.2.3 Abia and López-Marcos [4]

The authors considered methods to solve the model defined by the equations (2.6), (2.9)-(2.11). In order to describe these schemes, we introduce additional notation. Let $x_{j+1/2} = (j+1/2)\,h, 0 \le j \le J-1$, be intermediate grid points and $t_{n+1/2} = (n+1/2)\,h$ denote the intermediate time levels. Let $U^0_{j+1/2}, 0 \le j \le J-1$, denote approximations to the initial condition at the intermediate grid points. The numerical method which they developed is a two-step scheme, thus they needed to initialize the first step, in this case the time level $t_{\frac{1}{2}}$, by means of the following equations:

$$\begin{aligned}
U^{1/2}_{j+1} &= U^0_{j+1/2}\,\exp\left(-\frac{h}{2}\,\mu(x_{j+1/2}, Q_h(\boldsymbol{\gamma}^0_\mu\,\mathbf{U}^0)), t_0\right), \quad 0 \le j \le J-1,\\
U^{1/2}_{1/2} &= U^0_0\,\exp\left(-\frac{h}{2}\,\mu(x_0, Q_h(\boldsymbol{\gamma}^0_\mu\,\mathbf{U}^0), t_0)\right),\\
U^{1/2}_0 &= Q^*_h(\boldsymbol{\alpha}^{\frac{1}{2}}\,\mathbf{U}^{\frac{1}{2}}).
\end{aligned}$$

Then, the general recursion of the method is defined by the following equations

$$\begin{aligned} U^n_{j+\frac{1}{2}} &= U^{n-1}_{j-\frac{1}{2}} \exp(-h\,\mu(x_j, Q^*_h(\boldsymbol{\gamma}^{n-\frac{1}{2}}_\mu\, \mathbf{U}^{n-\frac{1}{2}}), t_{n-\frac{1}{2}}))\;, \quad 1 \le j \le J-1, \\ U^n_{\frac{1}{2}} &= U^{n-\frac{1}{2}}_0 \exp(-\frac{h}{2}\mu(x_{\frac{1}{4}}, \frac{3}{2}\, Q^*_h(\boldsymbol{\gamma}^{n-\frac{1}{2}}_\mu\, \mathbf{U}^{n-\frac{1}{2}}) - \frac{1}{2}\, Q_h(\boldsymbol{\gamma}^{n-1}_\mu\, \mathbf{U}^{n-1}), t_{n-\frac{1}{4}}))\;, \\ U^n_0 &= Q_h(\boldsymbol{\alpha}^n\, \mathbf{U}^n), \quad 1 \le n \le N, \end{aligned}$$

where

$$\boldsymbol{\alpha}^n = (\alpha(x_{\frac{1}{2}}, Q_h\left(\boldsymbol{\gamma}^n_\alpha \mathbf{U}^n\right), t_n), \alpha(x_{\frac{3}{2}}, Q_h\left(\boldsymbol{\gamma}^n_\alpha \mathbf{U}^n\right), t_n), \dots, \alpha(x_{J-\frac{1}{2}}, Q_h\left(\boldsymbol{\gamma}^n_\alpha \mathbf{U}^n\right), t_n)),$$

$1 \le n \le N$, and also

$$\begin{aligned} U^{n+\frac{1}{2}}_{j+1} &= U^{n-\frac{1}{2}}_j \exp(-h\;\mu(x_{j+\frac{1}{2}}, Q_h(\boldsymbol{\gamma}^n_\mu\, \mathbf{U}^n), t_n))\;, \quad 0 \le j \le J-1, \\ U^{n+\frac{1}{2}}_{\frac{1}{2}} &= U^n_0 \exp(-\frac{h}{2}\,\mu(x_{\frac{1}{4}}, \frac{3}{2}\, Q_h(\boldsymbol{\gamma}^n_\mu\, \mathbf{U}^n) - \frac{1}{2} Q^*_h(\boldsymbol{\gamma}^{n-\frac{1}{2}}_\mu\, \mathbf{U}^{n-\frac{1}{2}}), t_{n-\frac{1}{4}}))\;, \\ U^{n+\frac{1}{2}}_0 &= Q^*_h(\boldsymbol{\alpha}^{n+\frac{1}{2}}\, \mathbf{U}^{n+\frac{1}{2}}), \quad 1 \le n \le N-1, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\alpha}^{n+\frac{1}{2}} &= \left(\alpha\left(x_{\frac{1}{2}}, Q^*_h\left(\boldsymbol{\gamma}^{n+\frac{1}{2}}_\alpha\, \mathbf{U}^{n+\frac{1}{2}}\right), t_{n+\frac{1}{2}}\right),\right. \\ & \left. \alpha\left(x_1, Q^*_h\left(\boldsymbol{\gamma}^{n+\frac{1}{2}}_\alpha\, \mathbf{U}^{n+\frac{1}{2}}\right), t_{n+\frac{1}{2}}\right), \dots, \alpha\left(x_J, Q^*_h\left(\boldsymbol{\gamma}^{n+\frac{1}{2}}_\alpha\, \mathbf{U}^{n+\frac{1}{2}}\right), t_{n+\frac{1}{2}}\right)\right), \end{aligned}$$

$1 \le n \le N-1$. We also denote (with $\nu = \mu, \alpha$) $\boldsymbol{\gamma}^n_\nu = (\gamma_\nu(x_{\frac{1}{2}}), \gamma_\nu(x_{\frac{3}{2}}), \dots, \gamma_\nu(x_{J-\frac{1}{2}}))$, $0 \le n \le N$, and $\boldsymbol{\gamma}^{n+\frac{1}{2}}_\nu = (\gamma_\nu(x_{\frac{1}{2}}), \gamma_\nu(x_1), \dots, \gamma_\nu(x_J))$, $0 \le n \le N-1$. Finally, Q_h was the composite midpoint quadrature rule, given by

$$Q_h(\mathbf{U}^n) = \sum_{j=0}^{J-1} h\, U^n_{j+1/2},$$

and Q^*_h was the composite rule which uses the midpoint rule in the first subinterval and, in the rest of the intervals, the trapezoidal quadrature rule

$$Q^*_h(\mathbf{U}^{n+\frac{1}{2}}) = h\, U^{n+1/2}_{1/2} + \sum_{j=2}^{J} h\, \frac{U^{n+1/2}_{j-1} + U^{n+1/2}_j}{2}.$$

Thus, the careful design of the quadratures makes it possible to obtain an explicit second-order accurate method. Other quadrature schemes of higher order are possible.

The fact of choosing $h/2$ as the time step makes the integration in time more accurate but the method considers two different grids in the age interval and, hence, two different quadrature formulae. It is possible to consider h as the time step and the same grid for each discrete level of time and only a quadrature rule. This method was also proposed by the authors and we can find it in [4] with several quadrature rules of high order.

The authors made a complete convergence analysis of such schemes. They established the consistency, stability, existence of discrete solutions and the convergence by means of the discretization framework developed by López-Marcos [47]. And, thus, they obtained second order convergence

$$
\begin{aligned}
&\max_{0\le n\le N}\max_{1\le j\le J-1}\{|u(x_{j+\frac{1}{2}},t_n)-U^n_{j+\frac{1}{2}}|\}\le C\,h^2,\\
&\max_{0\le n\le N-1}\left\{|u(x_{\frac{1}{2}},t_{n+\frac{1}{2}})-U^{n+\frac{1}{2}}_{\frac{1}{2}}|,\ \max_{1\le j\le J}\{|u(x_j,t_{n+\frac{1}{2}})-U^{n+\frac{1}{2}}_j|\}\right\}\le C\,h^2,\\
&\max_{0\le n\le N}\{|u(0,t_n)-U^n_0|\}\le C\,h^2.
\end{aligned}
$$

Also in this paper, we can find numerical experiments in which the authors compared each method and concluded that the global precision was improved with methods based on the use high order quadrature rules.

5 Finite Difference Methods for Age-Structured Population Models

In this Section we describe numerical methods which try to obtain a numerical approximation to the values of the theoretical solution u of (2.6), (2.9)-(2.11) in a time interval $[0,T]$ by means of classical finite difference methods which usually appear in the literature of the first order hyperbolic partial differential equations.

We will describe each numerical method completely and their main features paying special attention to their mathematical properties. Each method is identified with the name of the type of discretization.

Next, we will introduce the notation employed in the descriptions. Given a positive integer J, if $h=A/J$, we introduce the grid points $x_j=j\,h, j=0,\dots,J$. We denote the step-length in time by k, $N=[T/k]$, and time levels $t_n=n\,k, 0\le n\le N$. We refer to the grid point x_j by a subscript j and to the time level t_n by a superscript n. Let U^n_j be a numerical approximation to $u(x_j,t_n)$, $0\le j\le J$, $0\le n\le N$, and $\mathbf{U}^n=(U^n_0,U^n_1,\dots,U^n_J)$, $0\le n\le N$. We also consider that an approximation to the initial condition (2.6) is given by $\mathbf{U}^0$.

Furthermore, we consider the quadrature rules Q_h with the form

$$
Q_h(\mathbf{f})=\sum_{j=0}^{J} q^h_j\,f_j\ , \tag{5.37}
$$

where $\mathbf{f}=(f(x_0),f(x_1),\dots,f(x_J))$, the nodes x_j, $0\le j\le J$, are defined above, so that, if f is sufficiently smooth then, as $h\to 0$,

$$
\int_0^A f(x)\,dx=Q_h(\mathbf{f})+O(h^s)\,, \tag{5.38}
$$

(note that we will describe a numerical method of order s).

We should observe that this kind of numerical schemes have two discretization parameters, h and k, (instead of one as with the characteristics methods) which can be used to find a balance between the time discretization error and the age-discretization error. Also, we comment that some of these schemes (the upwind and Lax-Wendroff methods) are conditioned to satisfy the Courant-Friedrichs-Levy (CFL) condition [58].

5.1 The upwind scheme [48]

López-Marcos considered the upwind method for the Gurtin-MacCamy model. We will modify the equations for the model (2.6), (2.9)-(2.11), thus the upwind scheme takes the form

$$\frac{U_j^{n+1}-U_j^n}{k}+\frac{U_j^n-U_{j-1}^n}{h} = -\mu(x_j, Q_h(\boldsymbol{\gamma}_\mu\, \mathbf{U}^n), t_n)\, U_j^n, \tag{5.39}$$

$$U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\, \mathbf{U}^{n+1}), \tag{5.40}$$

for $1 \le j \le J$, $0 \le n \le N-1$, where

$$\boldsymbol{\alpha}^n = (\alpha(x_1, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n), \alpha(x_2, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n), \ldots, \alpha(x_J, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n)),$$

$0 \le n \le N$, and $\boldsymbol{\gamma}_\alpha = (\gamma_\alpha(x_1), \gamma_\alpha(x_2), \ldots, \gamma_\alpha(x_J))$, $\boldsymbol{\gamma}_\mu = (\gamma_\mu(x_1), \gamma_\mu(x_2), \ldots, \gamma_\mu(x_J))$. Q_h is the composite rectangular quadrature rule

$$Q_h(\mathbf{U}^n) = \sum_{j=1}^{J} h\, U_j^n, \quad 0 \le n \le N.$$

The method derived is explicit and very easy to implement. The author carried out the analysis with the use of the general analytic framework introduced in [47]. He employed an energy argument to obtain proof of the main theorems and he concluded that, if $k = r\,h$ and $0 < r \le 1$, (the CFL condition), the convergence was of first order in time and age. Thus,

$$\max_{0\le n\le N}\left(\sum_{j=1}^{J} |u(x_j, t_n) - U_j^n|^2\right)^{\frac{1}{2}} \le C\,h,$$

$$\left(\sum_{n=0}^{N}{}'' k\, |u(0, t_n) - U_0^n|^2\right)^{\frac{1}{2}} \le C\,h,$$

where the double prime means that the first and last terms are halved.

He provided numerical experiments which confirmed the expected order and the instability when $r > 1$. He also reported the existence of spurious oscillations which frequently happened when $r = 1$, that might disappear as T grows. The spurious oscillations were connected by the author to the loss of the non-negative character of the numerical solution.

5.2 The extrapolated upwind scheme [38]

The authors modify the work of López-Marcos [48] to solve the Gurtin-MacCamy model by means of the extrapolation introduced by Douglas and Milner [17]. We will modify the equations for the actual model (2.6), (2.9)-(2.11), thus the extrapolated upwind scheme takes the form

$$\begin{aligned}
\frac{U_j^n - U_j^{n-1}}{k} + \frac{U_j^{n-1} - U_{j-1}^{n-1}}{h} &= -\mu(x_j, Q_h(\boldsymbol{\gamma}_\mu\, \mathbf{U}^{n-1}), t_{n-1})\, U_j^n, && (5.41)\\
U_J^n &= \frac{\frac{k}{h}\, U_{J-1}^{n-1}}{1 + k\, \mu(x_J, Q_h(\boldsymbol{\gamma}_\mu\, \mathbf{U}^{n-1}), t_{n-1})}, && (5.42)\\
U_0^n &= Q_h(\boldsymbol{\alpha}^n\, \mathbf{U}^n), && (5.43)
\end{aligned}$$

for $1 \le j \le J$, $1 \le n \le N$, where

$$\boldsymbol{\alpha}^n = (\alpha(x_1, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n), \alpha(x_2, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n), \ldots, \alpha(x_J, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n)),$$

$0 \le n \le N$; and $\boldsymbol{\gamma}_\alpha = (\gamma_\alpha(x_1), \gamma_\alpha(x_2), \ldots, \gamma_\alpha(x_J))$, $\boldsymbol{\gamma}_\mu = (\gamma_\mu(x_1), \gamma_\mu(x_2), \ldots, \gamma_\mu(x_J))$. Also, the composite rectangular quadrature rule is considered

$$Q_h(\mathbf{U}^n) = \sum_{j=1}^{J} h\, U_j^n, \quad 0 \le n \le N.$$

The method derived is explicit. The authors obtain first order convergence of the method by means of Taylor expansions

$$\max_{0 \le n \le N} \max_{0 \le j \le J} |u(x_j, t_n) - U_j^n| \le C\, h.$$

They also provided numerical experiments which confirmed the expected order and a comparison between this scheme and another one which has never been published.

5.3 The Lax-Wendroff scheme [62]

Sulsky introduced this method to solve the nonlinear model of Gurtin-MacCamy but did not provide a convergence proof for it. In order to solve the problem (2.6), (2.9)-(2.11), the method is defined by means of the following equations: for each $n = 0, 1, \ldots, N-1$,

$$U_{j-\frac{1}{2}}^{n+\frac{1}{2}} = U_{j-\frac{1}{2}}^{n} - \frac{k}{2\,h}\left(U_j^n - U_{j-1}^n\right) - \frac{k}{2}\,\mu\left(x_{j-\frac{1}{2}}, Q_h\left(\boldsymbol{\gamma}_\mu\, \mathbf{U}^n\right), t_n\right) U_{j-\frac{1}{2}}^n, \tag{5.44}$$

$j = 1, 2, \ldots, J$, with $x_{j-\frac{1}{2}} = \frac{1}{2}\left(x_{j-1} + x_j\right)$, $U_{j-\frac{1}{2}}^n = \frac{1}{2}\left(U_{j-1}^n + U_j^n\right)$, $j = 1, 2, \ldots, J$. Then,

$$U_j^{n+1} = U_j^n - \frac{k}{h}\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right) - k\,\mu\left(x_j, Q_h\left(\boldsymbol{\gamma}_\mu^*\, \mathbf{U}^{n+\frac{1}{2}}\right), t_n + \frac{k}{2}\right) U_j^{n+\frac{1}{2}}, \tag{5.45}$$

$j = 1, 2, \ldots, J-1$, with $U_j^{n+\frac{1}{2}} = \frac{1}{2}\left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} + U_{j-\frac{1}{2}}^{n+\frac{1}{2}}\right)$, $j = 1, 2, \ldots, J-1$. Finally,

$$U_0^n = Q_h(\boldsymbol{\alpha}^n\, \mathbf{U}^n), \tag{5.46}$$

where

$$\boldsymbol{\alpha}^n = (\alpha(x_0, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n), \alpha(x_1, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n), \ldots, \alpha(x_J, Q_h\left(\boldsymbol{\gamma}_\alpha\, \mathbf{U}^n\right), t_n)),$$

$0 \le n \le N$, $\boldsymbol{\gamma}_\alpha = (\gamma_\alpha(x_0), \gamma_\alpha(x_1), \ldots, \gamma_\alpha(x_J))$, $\boldsymbol{\gamma}_\mu = (\gamma_\mu(x_0), \gamma_\mu(x_1), \ldots, \gamma_\mu(x_J))$, $\boldsymbol{\gamma}_\mu^* = \left(\gamma_\mu(x_{\frac{1}{2}}), \gamma_\mu(x_{\frac{3}{2}}), \ldots, \gamma_\mu(x_{J-\frac{1}{2}})\right)$, $\boldsymbol{\gamma}_\mu = (\gamma_\mu(x_1), \gamma_\mu(x_2), \ldots, \gamma_\mu(x_J))$. The quadrature rules used are the composite midpoint and trapezoidal rules given, respectively, by

$$Q_h^*(\mathbf{U}^{n+\frac{1}{2}}) = \sum_{j=0}^{J-1} h\, U_{j+\frac{1}{2}}^{n+\frac{1}{2}}, \quad Q_h(\mathbf{U}^n) = \frac{1}{2}\,(U_0^n + U_J^n) + \sum_{j=1}^{J-1} h\, U_j^n.$$

The solution of the implicit equation (5.46) is carried out by means of a fixed point iteration method.

5.4 The Box Scheme [19]

We introduce the notation

$$D\,U_j^n = U_j^{n+1} - U_j^n, \quad \nabla U_j^n = U_j^n - U_{j-1}^n, \quad U_j^{n+\frac{1}{2}} = \frac{U_j^{n+1} + U_j^n}{2}, \quad U_{j-\frac{1}{2}}^n = \frac{U_j^n + U_{j-1}^n}{2}.$$

Thus, the box scheme for the problem (2.6), (2.9)-(2.11), takes the form

$$\frac{\nabla U_j^{n+1} + \nabla U_j^n}{2\,h} + \frac{D\,U_j^n + D\,U_{j-1}^n}{2\,k} = -\mu(x_{j-1/2}, Q_h(\boldsymbol{\gamma}_\mu\,\mathbf{U}^{n+\frac{1}{2}}), t_{n+\frac{1}{2}})\frac{U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}}}{2}, \tag{5.47}$$

$$U_0^{n+1} = Q_h(\boldsymbol{\alpha}^{n+1}\,\mathbf{U}^{n+1}), \tag{5.48}$$

$1 \le j \le J$, $0 \le n \le N-1$, where

$$\boldsymbol{\alpha}^n = (\alpha(x_0, Q_h\left(\boldsymbol{\gamma}_\alpha\,\mathbf{U}^n\right), t_n), \alpha(x_1, Q_h\left(\boldsymbol{\gamma}_\alpha\,\mathbf{U}^n\right), t_n), \ldots, \alpha(x_J, Q_h\left(\boldsymbol{\gamma}_\alpha\,\mathbf{U}^n\right), t_n)),$$

$0 \le n \le N$; and $\boldsymbol{\gamma}_\alpha = (\gamma_\alpha(x_0), \gamma_\alpha(x_1), \ldots, \gamma_\alpha(x_J))$, $\boldsymbol{\gamma}_\mu = (\gamma_\mu(x_0), \gamma_\mu(x_1), \ldots, \gamma_\mu(x_J))$, with

$$Q_h(\mathbf{U}^n) = \frac{h}{2}\,(U_0^n + U_J^n) + \sum_{j=1}^{J-1} h\, U_j^n, \quad 0 \le n \le N,$$

denoting the composite trapezoidal quadrature rule.

The box scheme in the form given by (5.47)-(5.48) is truly implicit and at each time step the nonlinear equations must be solved by some fixed-point iteration procedure. The numerical method is second order accurate as h tends to zero assuming that the time step $k = r\,h$, with r a fixed but arbitrary positive constant. The stability and convergence analysis were provided by the authors for the Gurtin-MacCamy model who obtained

$$\left(\sum_{j=1}^{J} h\,(U_{j-\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n)^2\right)^{\frac{1}{2}} \le S\left[O(h^2)^2 + O(h^2+k^2)^2\right]^{\frac{1}{2}},$$

$$\left(\sum_{n=0}^{N}{}'' k\,|u(0,t_n) - U_0^n|^2\right)^{\frac{1}{2}} \le S\left[O(h^2)^2 + O(h^2+k^2)^2\right]^{\frac{1}{2}},$$

where the double prime means that the first and last terms are halved. They employed the energy method with an average norm and the discretization framework developed in [47]. They also provided a numerical experiment which confirmed that the box scheme was of second order.

5.5 The Extrapolated Box Scheme [20]

This scheme was proposed as a modification of the box method in which first order extrapolated values of the numerical solution at the advanced time level are used in the nonlinear terms. With the notation introduced in the previous scheme, the method is

$$\frac{\nabla U_j^{n+1} + \nabla U_j^n}{2\,h} + \frac{D\,U_j^n + D\,U_{j-1}^n}{2\,k} =$$

$$-\mu\left(x_{j-\frac{1}{2}}, Q_h\left(\boldsymbol{\gamma}_\mu\,\frac{3\,\mathbf{U}^n - \mathbf{U}^{n-1}}{2}\right), t_{n+\frac{1}{2}}\right)\frac{U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}}}{2}, \tag{5.49}$$

$$U_0^{n+1} = Q_h\left(\boldsymbol{\alpha}^{n+1}\left(2\,\mathbf{U}^{n+1} - \mathbf{U}^n\right)\right), \tag{5.50}$$

$1 \le j \le J$, $0 \le n \le N-1$, where

$$\boldsymbol{\alpha}^n = \left(\alpha(x_0, Q_h\left(\boldsymbol{\gamma}_\alpha\,\left(2\,\mathbf{U}^n - \mathbf{U}^{n-1}\right)\right), t_n), \ldots, \alpha(x_J, Q_h\left(\boldsymbol{\gamma}_\alpha\,\left(2\,\mathbf{U}^n - \mathbf{U}^{n-1}\right)\right), t_n)\right),$$

$0 \le n \le N$; and $\boldsymbol{\gamma}_\alpha = (\gamma_\alpha(x_0), \gamma_\alpha(x_1), \ldots, \gamma_\alpha(x_J))$, $\boldsymbol{\gamma}_\mu = (\gamma_\mu(x_0), \gamma_\mu(x_1), \ldots, \gamma_\mu(x_J))$. The composite trapezoidal quadrature rule is considered for the approximation of the integral terms

$$Q_h(\mathbf{U}^n) = \frac{h}{2}\,(U_0^n + U_J^n) + \sum_{j=1}^{J-1} h\,U_j^n, \quad 0 \le n \le N.$$

The cost that is paid for the explicit character of (5.49)-(5.50) is that the resulting numerical scheme has three levels of time involved and therefore some numerical approximations $\mathbf{U}^1 = (U_0^1, \ldots, U_J^1)$ must be previously computed to give the starting values of the algorithm. This is obtained with the box scheme previously described. The authors proved convergence of second order in time and age for the Gurtin-MacCamy model

$$\left(\sum_{j=1}^{J} h\,(U_{j-\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n)^2\right)^{\frac{1}{2}} \le S\left[O(h^2)^2 + O(h^2+k^2)^2\right]^{\frac{1}{2}},$$

$$\left(\sum_{n=0}^{N}{}'' k\,|u(0,t_n) - U_0^n|^2\right)^{\frac{1}{2}} \le S\left[O(h^2)^2 + O(h^2+k^2)^2\right]^{\frac{1}{2}},$$

where the double prime means that the first and last terms are halved. They employed the energy method with an average norm and the discretization framework developed by [47]. Finally, they made a comparative study of this scheme with different quadrature rules of high order but with a special form.

6 Numerical Results

In order to show the quantitative and qualitative behaviour of some numerical techniques reviewed in this paper, we will present different numerical experiments. We will employ two second order accurate methods because these kinds of schemes present a good compromise between the efficiency and the requirements of regularity to the solution. The first is a characteristic curves method belonging to the type presented in subsection 4.1.6 based on the modified Euler method ($m = 2$, $b_1 = 0$, $b_2 = 1$, $c_2 = \frac{1}{2}$, $a_{12} = \frac{1}{2}$). The other is a finite difference method introduced in Section 5: the box method. The test problems will show a comparison of the efficiency of both methods and also their behaviour depending on which compatibility conditions between the boundary and initial data are satisfied.

<table>
<tr><th>$k\backslash h$</th><th>3.125E-02</th><th>1.563E-02</th><th>7.813E-03</th><th>3.906E-03</th><th>1.953E-03</th></tr>
<tr><td rowspan="2">3.125E-2</td><td>8.170E-04</td><td>2.029E-04</td><td>5.064E-05</td><td>1.266E-05</td><td>5.905E-06</td></tr>
<tr><td>| 3.540</td><td>| 8.640</td><td>| 17.84</td><td>| 35.07</td><td>| 70.06</td></tr>
<tr><td rowspan="2">1.563E-2</td><td>8.192E-04</td><td>2.044E-04</td><td>5.073E-05</td><td>1.266E-05</td><td>3.165E-06</td></tr>
<tr><td>| 7.780</td><td>1.999 | 12.57</td><td>2.000 | 31.55</td><td>2.000 | 65.96</td><td>2.000 | 134.7</td></tr>
<tr><td rowspan="2">7.813E-3</td><td>8.198E-04</td><td>2.049E-04</td><td>5.110E-05</td><td>1.268E-05</td><td>3.165E-06</td></tr>
<tr><td>| 14.93</td><td>1.999 | 27.93</td><td>2.000 | 46.07</td><td>2.000 | 123.1</td><td>2.000 | 251.3</td></tr>
<tr><td rowspan="2">3.906E-3</td><td>8.200E-04</td><td>2.051E-04</td><td>5.124E-05</td><td>1.278E-05</td><td>3.171E-06</td></tr>
<tr><td>| 27.98</td><td>1.999 | 53.22</td><td>2.000 | 104.3</td><td>2.000 | 165.1</td><td>2.000 | 475.5</td></tr>
<tr><td rowspan="2">1.953E-3</td><td>8.201E-04</td><td>2.051E-04</td><td>5.128E-05</td><td>1.281E-05</td><td>3.194E-06</td></tr>
<tr><td>| 51.87</td><td>1.999 | 102.0</td><td>2.000 | 203.0</td><td>2.000 | 395.4</td><td>2.000 | 630.4</td></tr>
</table>

Table 1: Global error (upper number), order (lower-left one) and cpu-time (lower-right one) for the box method at $T = 20$ in the test problem with initial condition (6.1).

k	Error	Order	cpu-time
3.125E-2	4.445E-4		3.62
1.563E-2	1.120E-4	1.988	14.08
7.813E-3	2.811E-5	1.994	56.17
3.906E-3	7.042E-6	1.997	223.88
1.953E-3	1.762E-6	1.999	883.44

Table 2: Global error, order and cpu-time for the characteristic scheme at $T = 20$ in the test problem with initial condition (6.1).

For the numerical experiments with both schemes, we consider the following non-autonomous Gurtin-MacCamy problem taken from the work of Abia and López-Marcos [4]. The age-specific fertility and mortality moduli in (2.10)-(2.11) are given by

$$\begin{aligned} \mu(a,z,t) &= z \,, \\ \alpha(a,z,t) &= \frac{4\,a\,z\,\exp(-a)\,(2-2\,\exp(-A)+\exp(-t))^2}{(1+z)^2\,(1-\exp(-A))\,(1-(1+2A)\,\exp(-2A))\,(1-\exp(-A)+\exp(-t))}\,, \end{aligned}$$

with $\gamma_\mu \equiv \gamma_\alpha \equiv 1$, as the weight functions in (2.9). We should recall that A defines a suitable age-window $[0, A]$ in which to follow the evolution of the population. In our numerical experiments, we have chosen $A = 5$, $T = 20$.

In the first experiment, we choose as the initial age-specific density the function

$$u^0(a) = \frac{\exp(-a)}{2 - \exp(-A)}\,. \tag{6.1}$$

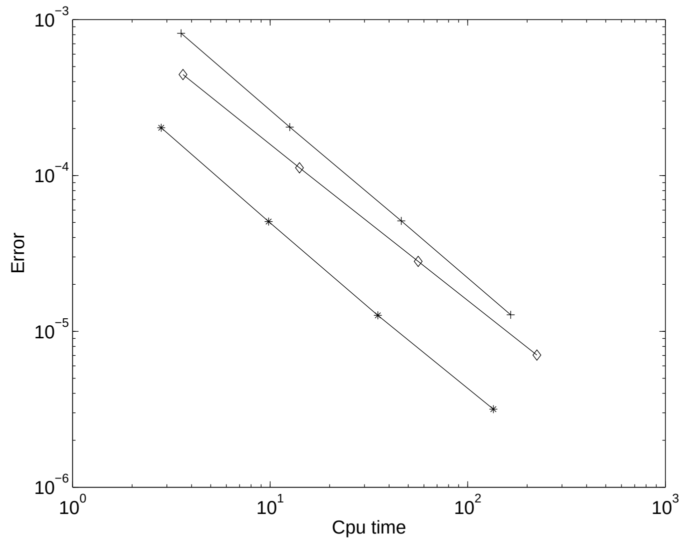

Figure 1: Efficiency plot. Characteristics method ($\diamondsuit$), box method, $r = 1$ (+) and $r = 8$ ($\star$).

The solution to the problem (2.10)-(2.11) is then given by

$$u(a,t) \; = \frac{\exp(-a)}{1 \, - \, \exp(-A) \, + \, \exp(-t)} \; .$$

We note that our test problem has meaningful nonlinearities and that the theoretical solution is known and smooth. Hence, from a numerical point of view, this test problem is suitable for showing quantitatively the behaviour of the numerical methods considered. On the other hand, from a biological point of view, we also note that the biological rates determine an interesting dynamics for the growth of the population because there is a nontrivial equilibrium.

In each entry in the columns of Table 1, the upper number represents the global error, $e_{k\,h}$, at $T = 20$,

$$e_{k\,h} = \max_{0 \le n \le N} ||\mathbf{u}^n - \mathbf{U}^n||_\infty \; ,$$

the lower left number represents the order s as computed from

$$s = \frac{\log\left(e_{2\,k,2\,h}/e_{k\,h}\right)}{\log(2)},$$

and the lower right number is the cpu-time in seconds. In Table 2, we show the same values for the characteristics scheme. The results in such tables clearly confirm the expected second order of convergence for both methods.

In Figure 1, we present an efficiency plot where we show the error (in the vertical axis) against the cpu-time spent (in the horizontal axis) in logarithmic scale. For the box method, we show the efficiency plots corresponding to numerical experiments with two values of the parameter r ($r = 8$ and $r = 1$). The value $r = 8$ represents, for this problem, the optimal relationship between the age-diameter h and the time-step k, and has been derived by means of the technique introduced in [9]. For the value $r = 1$, the box method computes the numerical

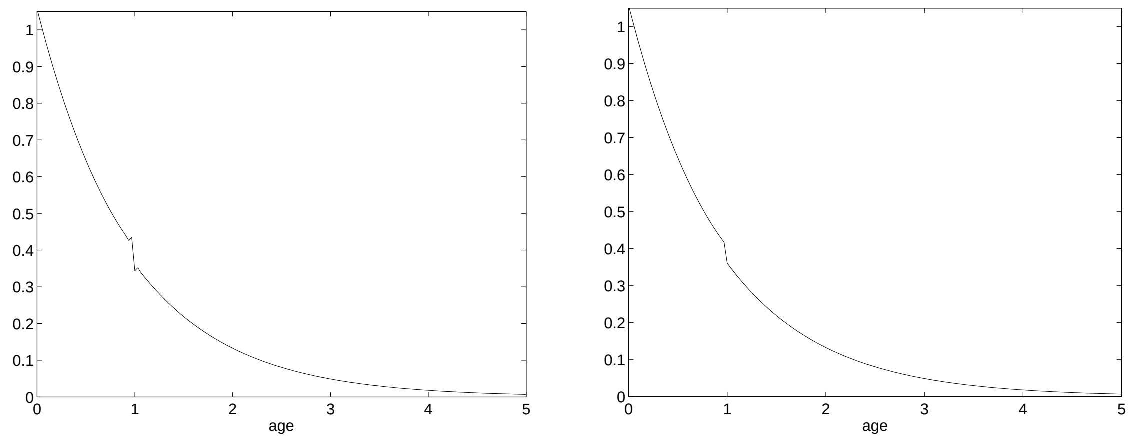

Figure 2: Numerical solution for problem (2.10)-(2.11) with initial condition (6.4) at $t = 1$. Left-hand side plot: box method, $r = 1$, $k = 0.03125$. Right-hand side plot: characteristic scheme, $k = 0.03125$.

solution on grid nodes along the characteristic curves of the problem, making the comparison with the numerical results of the characteristics method more meaningful.

In the figure, we show that the box method (with $r = 8$) is the most efficient for this problem. We should note that the determination of the optimal value of r is not a trivial task and that this value would probably change for different data of the problem. We should point out that, when the solution is not smooth enough, then among all the possible values of r, the choice of $r = 1$ provides numerical solutions with a better qualitative behaviour, in the sense that spurious oscillations in the numerical solution are minimized. However, the box method with $r = 1$ is less efficient than the characteristics method.

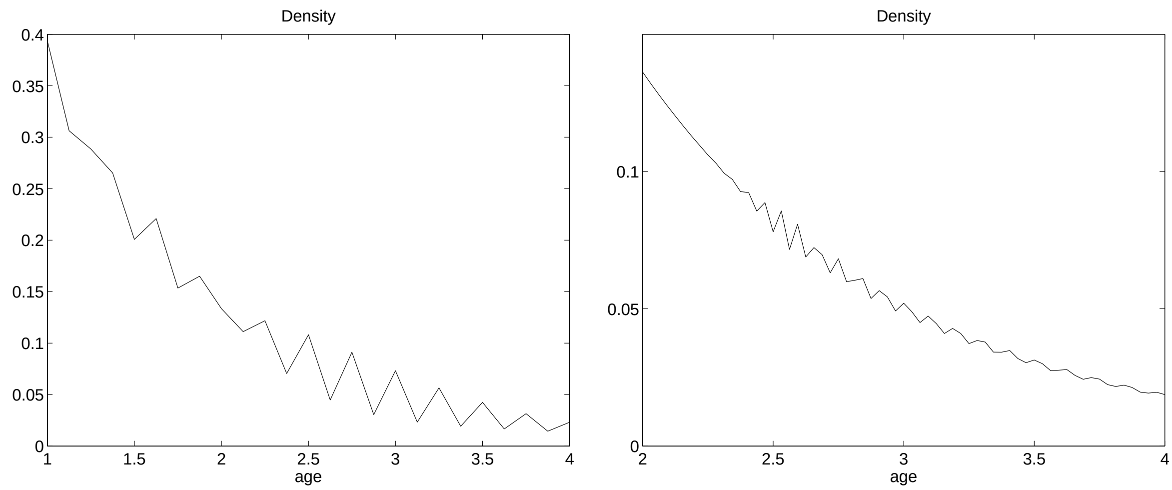

Figure 3: Spurious oscillations in the box method. Left hand plot: $r = 0.25$, $k = 0.03125$, at $t = 0.21875$. Right hand plot: $r = 2$, $k = 0.0625$, at $t = 10$.

In the second kind of experiments, we study the effect of the use of initial conditions that are not compatible with the boundary condition data. In this case, the usual analysis of convergence should be carried out very carefully. This situation was taken into account first in the work of Angulo and López-Marcos [6] for the numerical solution of a fully nonlinear size-structured population model by means of a characteristics scheme. The necessary condition for the continuity at the points (t,t), $t \geq 0$, is

$$u_0(0) = \int_0^A \alpha\left(a, I_\alpha(0), 0\right)\, u_0(a)\, da. \tag{6.2}$$

From a biological point of view, this condition implies that the birth law is also valid at $t = 0$. Moreover, Milner and Rabbiolo [57] pointed out that the order of accuracy of the numerical scheme employed could be lost (and even the property of convergence) if such a condition is not satisfied. The second compatibility condition needed for the continuity of the first derivatives at the points (t,t), $t \geq 0$, is given by

$$-v(0) \;=\; \int_0^A \left[\left(\alpha_z\left(a, I_\alpha(0), 0\right)\, I'_\alpha(0) + \alpha_t\left(a, I_\alpha(0), 0\right)\right)\, u_0(a) - \alpha\left(a, I_\alpha(0), 0\right)\, v(a)\right]\, da, \tag{6.3}$$

where $I'_\alpha(0) = \int_0^A \gamma_\alpha(a)\, v(a)\, da$ and $v(a) = u'_0(a) + \mu\left(a, I_\mu(0), 0\right)\, u_0(a)$. If (6.2) holds but (6.3) does not, then the first derivatives have a finite jump at the points of the characteristic curve (t,t), $t \geq 0$. If both (6.2) and (6.3) hold then, in the worst case, the solution could have a jump in the second derivatives, if the compatibility condition for continuity of second derivatives is not satisfied.

	Box					Characteristics
$k \backslash h$	5.000E-01	2.500E-01	1.250E-01	6.250E-02	3.125E-02	
5.000E-1	1.273E-1	3.889E-2	5.049E-2	4.635E-2	4.669E-2	2.476E-1
		1.711E+0	-2.683E-1	-3.190E-1	-4.502E-1	
2.500E-1	1.273E-1	3.689E-2	4.887E-2	5.737E-2	5.426E-2	1.499E-2
		1.786E+0	-3.297E-1	-1.845E-1	-2.273E-1	4.046
1.250E-1	1.273E-1	5.441E-2	2.461E-2	5.625E-2	6.170E-2	5.253E-3
		1.226E+0	5.841E-1	-2.028E-1	-1.050E-1	1.513
6.250E-2	1.273E-1	7.990E-2	6.917E-2	2.648E-2	6.084E-2	1.915E-3
		6.715E-1	-3.462E-1	-1.058E-1	-1.133E-1	1.456
3.125E-2	1.273E-1	9.915E-2	9.275E-2	7.540E-2	2.717E-2	7.538E-4
		3.601E-1	-2.151E-1	-1.244E-1	-3.685E-2	1.345

Table 3: $\|\mathbf{e}(k) - \mathbf{e}(k/2)\|_{\infty,k}$ (upper number) and order (lower number) in problem (2.10)-(2.11) for the box method and the characteristic scheme with the initial condition (6.4) at $t = 20$.

We have considered two different initial conditions. The first one is

$$u^0(a) \;= \exp(-a)\;. \tag{6.4}$$

In this case, the problem (2.10)-(2.11) does not satisfy (6.2) and, therefore its solution is discontinuous along the characteristic curve (t,t), $t \geq 0$.

In Figure 2, we show the numerical solutions obtained by both the characteristics and the box method at $t = 1$. In the case of the characteristics method, we can affirm that the method is able to follow the discontinuity jump of the solution along the characteristic curve (t,t), $t \geq 0$, at the right speed. With the box method we only follow this jump when $r = 1$ but with some spurious oscillations located around the discontinuity point. In the case of $r < 1$, the speed of propagation of the discontinuity in the numerical solution is faster than the real one. Therefore, new spurious oscillations appear but they are located at points far away at the right of the discontinuity point. In the plot at the left in Figure 3, we show how these oscillations, for the numerical solution at $t = 0.21875$, are located for ages greater than 1.1. As time goes on, these oscillations will disappear through the right boundary of the age domain. On the other hand, when $r > 1$, we again have spurious oscillations in the numerical solution, but they travel at a speed lower than one. This phenomenon is clearly visible in the plot at the right in Figure 3. Eventually, all the numerical solutions provided by the box method tend to the asymptotic stationary state of the problem, as the characteristics scheme does. The existence of spurious oscillations in this kind of problem has been reported by López-Marcos [48] and Sulsky [63].

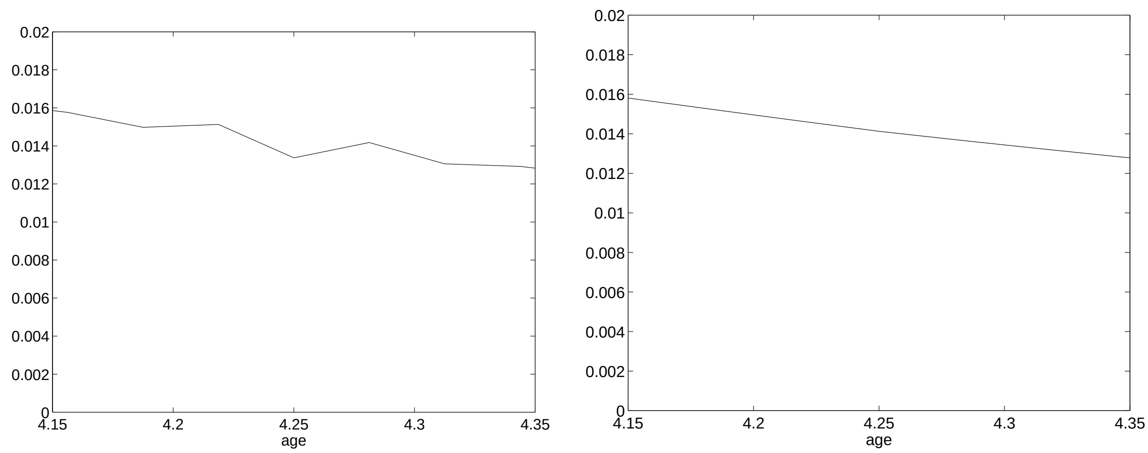

Figure 4: Numerical solution for problem (2.10)-(2.11) with initial condition (6.6) at $t = 4.25$. Left hand side plot: box method, $r = 1$, $k = 0.03125$. Right hand side plot: characteristic scheme, $k = 0.03125$.

We now compare the rate of convergence of the numerical solutions. The analytical solution to problem (2.10)-(2.11) with initial data (6.4) is not known and therefore we can only estimate experimentally the order of accuracy of the numerical solution. In each entry of Table 3, in the upper number we show the maximum magnitude of the differences over the coarsest grid between the two numerical solutions $\mathbf{U}(k/2)$, $\mathbf{U}(k)$, computed respectively with discretization parameters $k/2$ and k (for the box method h is given after we fix the constant r). This quantity agrees with the maximum magnitude over the coarsest grid of the differences of the global errors $\mathbf{e}(k/2)$ and $\mathbf{e}(k)$, for $\mathbf{U}(k/2)$ and $\mathbf{U}(k)$ respectively, denoted by

$$\|\mathbf{e}(k) - \mathbf{e}(k/2)\|_{\infty,k}.$$

The lower number in Table 3 represents the order of convergence s, which is then estimated from the formula

$$s = \frac{\log\left(\|\mathbf{e}(2\,k) - \mathbf{e}(k)\|_{\infty,2\,k} / \|\mathbf{e}(k) - \mathbf{e}(k/2)\|_{\infty,k}\right)}{\log(2)}. \tag{6.5}$$

The results in Table 3 confirm that the box method is not convergent and that the characteristics method is. In fact, the characteristics scheme would keep the convergence order if we used the rectangular quadrature rule for approximating the integrals on the age-intervals containing the discontinuity point of the solution.

A new test problem takes as an initial condition for (2.10)-(2.11)

$$u^0(a) \;=\frac{2-\exp(-A)}{(1-\exp(-A))^2}\;\exp(-a)\;. \tag{6.6}$$

In this case, the solution is continuous but with a jump in the first derivatives of the solution along the characteristic curve through the origin. We should note that, in real biological situations, it is not difficult to satisfy the first compatibility condition by choosing an appropriate approximation to the field data but, due to the complexity of the vital functions, it is usually a difficult task to satisfy the second compatibility condition.

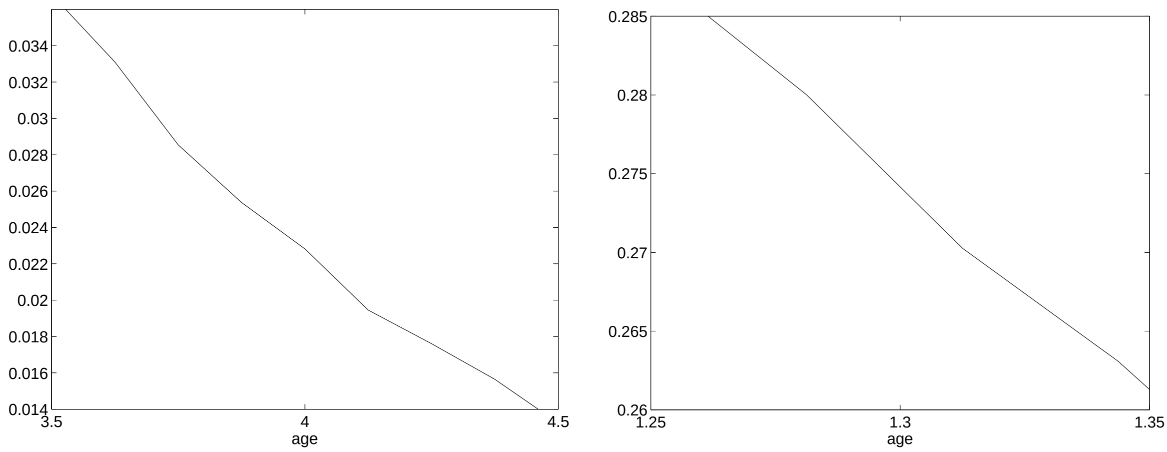

Figure 5: Spurious oscillations in the box method. Left hand plot: $r = 0.25$, $k = 0.03125$, at $t = 1$. Right hand plot: $r = 4$, $k = 0.0125$, at $t = 20$.

In this test problem, the box method with $r = 1$ has again the best behaviour among all other values of r. We show in Figure 4, the numerical approximations to the solution at $t = 4.25$ computed, respectively, with the box method and the characteristics scheme. We can observe that in the box method the numerical solutions again show spurious oscillations, in contrast to the good qualitative behaviour of the solution provided by the characteristics scheme. The scale in these plots has been enlarged in order to capture the magnitude of the oscillations which were only noticed after a numerical approximation of the derivative with respect to the age of the numerical solutions were computed by first order divided diferences.

The behaviour of the box method is still worse when $r \neq 1$ is considered. We show in Figure 5 that some oscillations appear, although they dissapear when the discontinuity which provokes them does after reaching the right boundary of the age-interval.

	Box					Characteristics
$k\backslash h$	5.000E-01	2.500E-01	1.250E-01	6.250E-02	3.125E-02	
5.000E-1	1.273E-1	5.055E-2	4.115E-2	3.855E-2	3.815E-2	1.758E+0
		1.745	1.392	1.293	1.257	
2.500E-1	1.273E-1	3.689E-2	1.718E-2	1.706E-2	1.567E-2	7.608E-2
		1.786	1.556	1.270	1.298	4.530
1.250E-1	1.273E-1	3.689E-2	9.594E-3	8.867E-3	7.318E-3	1.367E-2
		1.786	1.943	0.955	1.221	2.477
6.250E-2	1.273E-1	3.689E-2	9.594E-3	2.423E-3	4.320E-3	2.814E-3
		1.786	1.943	1.986	1.037	2.280
3.125E-2	1.273E-1	3.689E-2	9.594E-3	3.126E-3	6.120E-4	6.393E-4
		1.786	1.943	1.618	1.985	2.138

Table 4: $\|\mathbf{e}(k) - \mathbf{e}(k/2)\|_{\infty,k}$(upper number) and order (lower number) in problem (2.10)-(2.11) for the box method and the characteristic scheme with the initial condition (6.6) at $t = 20$.

Finally, we show in the entries of Table 4, the maximum magnitude of the differences over the coarsest grid between the two numerical solutions $\mathbf{U}(k/2)$, $\mathbf{U}(k)$, and the estimated order s of convergence using the formula (6.5). In this case, the convergence of both methods is observed but, while the characterisctics method keeps the second order of convergence, the box method only mantains this second order in case $r = 1$.

All of the calculations were carried out on a Workstation SUN Sparc2.

Acknowledgements: The authors were supported in part by the project of the Ministerio de Ciencia y Tecnología BFM2002-01250 and by the project of the Junta de Castilla y León and Unión Europea F.S.E. VA063/04.

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