Computers Math. Applic. Vol. 27, No. 2, pp. 41–53, 1994
Printed in Great Britain.

0898-1221/94 $6.00 + 0.00

An Explicit Extrapolated Box Scheme for the Gurtin-Maccamy Equation

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(Received February 1993; revised and accepted May 1993)

Abstract—An explicit extrapolated box method is considered for a nonlinear partial integrodifferential equation, the Gurtin-MacCamy equation, subject to a nonlocal boundary condition. This problem describes the evolution in time of the age structure of a population. The consistency, stability and convergence properties of the method are studied, and its second order accuracy is demonstrated both analytically and numerically.

1. INTRODUCTION

Recently, much attention has been devoted to the formulation and analysis of methods for the numerical solution of the Gurtin-MacCamy equation [1]

$$u_x + u_t = f(x, I(t))u, \qquad 0 \le x \le A, \quad 0 \le t \le T, \tag{1}$$

subject to the initial condition

$$u(x, 0) = u^0(x), \qquad 0 \le x \le A, \tag{2}$$

and the non-local boundary condition

$$u(0, t) = g\left(\int_0^A b(a, I(t))\, u(a, t)\, da, t\right), \qquad 0 \le t \le T, \tag{3}$$

where

$$I(t) = \int_0^A u(a, t)\, da, \qquad 0 \le t \le T;$$

see [2–8]. This problem is used in biology to describe the evolution in time of the age structure of a population. In this case, $u(x, t)$ is the age-specific density of individuals of age x at time t. The functions b and f, the age-specific fertility modulus (a nonnegative function) and mortality modulus (a nonpositive function), respectively, depend on the total size $I(t)$ of the population. In the nonlocal condition (3), which is the birth law of the population, the function g is nonnegative. An extensive study of nonlinear age-dependent population dynamics is given in [9].

*Supported in part by funds from the National Science Foundation Grant CCR-9103451.
†Supported by project CICYT PB91-0200 and by Universidad de Valladolid, Programa 541 A-644.

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All of the numerical techniques proposed in [2–7] for the solution of (1)–(3), or special cases of this problem, are first order accurate in time. In [2] and [3], finite difference methods along the characteristic direction for (1) were considered, whereas in [4] a method of lines approach was employed. Kannan and Ortega [5] used a finite difference method to examine the existence of a unique solution of the differential equation and proved the convergence of the method. Kostova [6] examined the use of Rothe's method for the time discretization combined with a spatial discretization based on the trapezoidal rule. López-Marcos [7] formulated and rigorously analyzed an upwind finite difference method.

In [8], a box scheme was formulated for (1)–(3). Its stability and convergence properties were examined and the method was shown to be second order accurate in time, assuming that $k = rh$, where k and h denote the time and space step-lengths, respectively, and r is a fixed but arbitrary positive constant. Being implicit, this method requires the solution of a system of nonlinear equations at each time step. The purpose of this paper is to analyse an explicit box method which is obtained from the method of [8] by employing extrapolation in the nonlinear terms. Using the general discretization framework introduced in [10–12], we show that the extrapolated scheme is second order accurate under the same assumptions as required in the analysis of the box method of [8].

In the treatment of the integral terms, the accuracy of the quadrature rule in (1)–(3) must be compatible with that of the discretization of the derivative terms. In this paper, we concentrate on the use of the trapezoidal rule, but also consider other quadrature rules with order of accuracy greater than two. This feature could be important in practical situations where we cannot consider arbitrary grids on the age interval because the data of the problem (1)–(3) are only known on a fixed discrete set of points in the age interval.

It should be noted that Milner [13] formulated a second order in time finite element scheme for solving a system of equations of the form (1) which arises in a model of human populations in which partitioning into sexes is considered. The numerical technique employs a Crank-Nicolson time discretization for which second order accuracy is proved. A first order finite difference method for the solution of the same problem is discussed in [14].

An overview of this paper is as follows. In Section 2, we present the extrapolated box scheme and introduce the basic ingredients of the general framework used in the error analysis. Consistency, stability and convergence are proved in Section 3. In Section 4, we discuss some quadrature rules which can be used in the numerical integration without significantly altering the theoretical analysis. Some numerical results are presented in Section 5.

2. THE EXTRAPOLATED BOX SCHEME

The notation adopted in this paper is that of [8]. Let J and N be positive integers and with $h = A/J$ and $k = T/N$ set $x_j = j\,h$, $0 \le j \le J$, and $t_n = n\,k$, $0 \le n \le N$. Let U_j^n denote an approximation to $u(x_j, t_n)$ and define

$$D\,U_j^n = U_j^{n+1} - U_j^n, \qquad \nabla U_j^n = U_j^n - U_{j-1}^n, \qquad U_j^{n+\frac{1}{2}} = \frac{U_j^{n+1} + U_j^n}{2}, \qquad U_{j-\frac{1}{2}}^n = \frac{U_{j-1}^n + U_j^n}{2}.$$

Also, set

$$b(\mathbf{U}^n)_j = b(x_j, Q_h(\mathbf{U}^n)), \qquad \mathbf{b}(\mathbf{U}^n) = (b(\mathbf{U}^n)_0, b(\mathbf{U}^n)_1, \dots, b(\mathbf{U}^n)_J),$$

where Q_h denotes the trapezoidal rule, viz.,

$$Q_h(\mathbf{U}^n) = \sum_{j=0}^{J}{}'' h\, U_j^N, \tag{4}$$

and the double prime indicates that the first and last terms are halved. With this notation, the extrapolated box scheme takes the form

$$\frac{\nabla U_j^{n+1} + \nabla U_j^n}{2h} + \frac{D\,U_j^n + D\,U_{j-1}^n}{2k} - f\left(x_{j-\frac{1}{2}}, Q_h\left(\frac{3\mathbf{U}^n - \mathbf{U}^{n-1}}{2}\right)\right) \frac{U_j^{n+\frac{1}{2}} + U_{j-1}^{n+\frac{1}{2}}}{2} = 0, \tag{5}$$

$1 \le j \le J$, $1 \le n \le N-1$, with the given initial conditions

$$\mathbf{U}^0 = (U_0^0, U_1^0, \dots, U_J^0) \in \mathcal{R}^{J+1}, \qquad \mathbf{U}^1 = (U_0^1, U_1^1, \dots, U_J^1) \in \mathcal{R}^{J+1}, \tag{6}$$

and the boundary values $\mathbf{U}_0 = (U_0^2, U_0^3, \dots, U_0^N) \in \mathcal{R}^{N-1}$ such that

$$U_0^{n+1} = g(Q_h(\mathbf{b}(2\mathbf{U}^n - \mathbf{U}^{n-1})\,(2\mathbf{U}^n - \mathbf{U}^{n-1})), t_{n+1}), \tag{7}$$

$1 \le n \le N-1$.

To analyse this scheme, we employ the discretization framework developed in [12–14]. We assume that $k = r\,h$, where r is an arbitrary but fixed positive constant, and h takes values in the set $H = \{h > 0 : h = A/J,\ J \in \mathcal{N}\}$. For each $h \in H$, we define the spaces X_h and Y_h by

$$X_h = (\mathcal{R}^{J+1})^{N+1},$$

where, for $0 \le n \le N$, each factor $\mathcal{R}^{J+1}$ refers to the points (x_j, t_n), $0 \le j \le J$, and

$$Y_h = \mathcal{R}^{J+1} \times \mathcal{R}^{J+1} \times \mathcal{R}^{N-1} \times (\mathcal{R}^J)^{N-1},$$

where $\mathcal{R}^{J+1}$ refers to the points $(x_j, 0)$, (x_j, k), $0 \le j \le J$, the factor $\mathcal{R}^{N-1}$, to the boundary points $(0, t_n)$, $2 \le n \le N$, and the factor $(\mathcal{R}^J)^{N-1}$ to the points $(x_j,\ t_n)$, $1 \le j \le J$, $2 \le n \le N$. We also introduce the mapping $\phi_h : X_h \to Y_h$ defined by

$$\phi_h(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N) = (\mathbf{P}^0, \mathbf{P}^1, \mathbf{P}_0, \mathbf{P}^2, \dots, \mathbf{P}^N), \tag{8}$$

where

$$\mathbf{P}^0 = \mathbf{V}^0 - \mathbf{U}^0 \in \mathcal{R}^{J+1}, \qquad \mathbf{P}^1 = \mathbf{V}^1 - \mathbf{U}^1 \in \mathcal{R}^{J+1},$$

and

$$P_0^{n+1} = V_0^{n+1} - g(Q_h(\mathbf{b}(2\mathbf{V}^n - \mathbf{V}^{n-1})\,(2\mathbf{V}^n - \mathbf{V}^{n-1})), t_{n+1}), \tag{9}$$

$1 \le n \le N-1$, and

$$P_j^{n+1} = \frac{\nabla V_j^{n+1} + \nabla V_j^n}{2h} + \frac{D\,V_j^n + D\,V_{j-1}^n}{2k} - f\left(x_{j-\frac{1}{2}}, Q_h\left(\frac{3\mathbf{V}^n - \mathbf{V}^{n-1}}{2}\right)\right)\frac{V_j^{n+\frac{1}{2}} + V_{j-1}^{n+\frac{1}{2}}}{2}, \tag{10}$$

$1 \le j \le J$, $1 \le n \le N-1$. Clearly $\mathbf{U}_h = (\mathbf{U}^0, \mathbf{U}^1, \dots \mathbf{U}^N) \in X_h$ is a solution of the extrapolated box scheme (5)–(7) if and only if

$$\phi_h(\mathbf{U}^0, \mathbf{U}^1, \dots, \mathbf{U}^N) = \mathbf{0}. \tag{11}$$

We endow the spaces X_h and Y_h with the norms

$$\|(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N)\|_{X_h} = \max\left\{\left(\sum_{n=0}^{N}{}''k(V_0^n)^2\right)^{\frac{1}{2}}, \max_{0\le n\le N}\left(\sum_{j=1}^{J} h(V_{j-\frac{1}{2}}^n)^2\right)^{\frac{1}{2}}\right\},$$

and

$$\|(\mathbf{P}^0, \mathbf{P}^1, \mathbf{P}_0, \mathbf{P}^2, \dots, \mathbf{P}^N)\|_{Y_h} = \left\{\sum_{j=0}^{J}{}''h(P_j^0)^2 + \sum_{j=0}^{J}{}''h(P_j^1)^2 + \sum_{n=2}^{N} k(P_0^n)^2 + \sum_{n=2}^{N} k\,\|\mathbf{P}^n\|^2\right\}^{\frac{1}{2}},$$

and, if $\mathbf{P} \in \mathcal{R}^J$, then

$$\|\mathbf{P}\| = \left(\sum_{j=1}^{J} h(P_j)^2 \right)^{\frac{1}{2}}.$$

For each $h \in H$, let

$$\mathbf{u}_h = (\mathbf{u}^0, \mathbf{u}^1, \dots, \mathbf{u}^N) \in X_h,$$

where

$$\mathbf{u}^n = (u_0^n, u_1^n, \dots, u_J^n) \in \mathcal{R}^{J+1}, \qquad u_j^n = u(x_j, t^n),$$

$0 \le j \le J$, $0 \le n \le N$. We say that the discretization (8) is *consistent* if

$$\lim_{h \to 0} \|\phi_h(\mathbf{u}_h)\| = 0,$$

and *convergent* if there exists $h_0 > 0$ such that, for each $h \in H$ with $h \le h_0$, (11) has a solution $\mathbf{U}_h$ for which

$$\lim_{h \to 0} \|\mathbf{u}_h - \mathbf{U}_h\| = 0.$$

As one would expect, $\phi_h(\mathbf{u}_h) \in Y_h$ is called the *local discretization error.* For each $h \in H$, let M_h be a real number *(the stability threshold)* with $0 < M_h \le \infty$. We say that the discretization (8) is *stable* for $\mathbf{u}_h$ restricted to the thresholds M_h, if there exist two positive constants h_0 and S *(the stability constant)* such that, for any $h \in H$ with $h \le h_0$, the open ball $B(\mathbf{u}_h, M_h)$ is contained in the domain of ϕ_h and for all $\mathbf{V}_h, \mathbf{W}_h$ in that ball

$$\|\mathbf{V}_h - \mathbf{W}_h\| \le S \|\phi_h(\mathbf{V}_h) - \phi_h(\mathbf{W}_h)\|.$$

This notion of stability was introduced by Sanz-Serna and López-Marcos [11], who proved the following theorem which states that, with this definition, consistency and stability imply convergence

THEOREM 2.1. *Assume that (8) is consistent and stable with thresholds M_h. If ϕ_h is continuous in $B(\mathbf{u}_h, M_h)$ and $\|\phi_h(\mathbf{u}_h)\| = o(M_h)$ as $h \to 0$, then:*

(i) *for h sufficiently small, the difference equations (11) possess a unique solution in $B(\mathbf{u}_h, M_h)$.*

(ii) *as $h \to 0$, the solutions converge. Furthermore, the order of convergence is not smaller than the order of consistency.*

3. ERROR ANALYSIS

We first examine the consistency of the extrapolated box scheme.

THEOREM 3.1. *If the functions f and g have continuous derivatives, the function b is twice continuously differentiable, and the solution u of (1)–(3) is three times continuously differentiable, then as $h \to 0$ the local discretization error satisfies*

$$\|\phi_h(\mathbf{u}_h)\|_{Y_h} = \left\{ \sum_{j=0}^{J}{}'' h \left(U_j^0 - u_j^0\right)^2 + \sum_{j=0}^{J}{}'' h \left(U_j^1 - u_j^1\right)^2 + O\left(h^2 + k^2\right)^2 \right\}^{\frac{1}{2}}.$$

PROOF. Following the corresponding analysis in [8], it is easy to bound the values in (10) with $(\mathbf{u}^0, \mathbf{u}^1, \dots, \mathbf{u}^N)$ in place of $(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N)$ as desired.

With regard to the boundary discretization errors (9), the smoothness requirements on g, b and u, the accuracy of the trapezoidal rule, the condition (3), and the estimate

$$|b(x_j, Q_h(2\mathbf{u}^n - \mathbf{u}^{n-1})) - b(x_j, I(t_{n+1}))| = O(h^2 + k^2),$$

enable us to write, for $1 \le n \le N-1$,

$$
\begin{aligned}
&|u_0^{n+1} - g(Q_h(\mathbf{b}(2\mathbf{u}^n - \mathbf{u}^{n-1})\,(2\mathbf{u}^n - \mathbf{u}^{n-1})), t_{n+1})| \\
&= \left| g\left(\int_0^A b(a, I(t_{n+1}))\, u(a, t_{n+1})\, da, t_{n+1}\right) - g(Q_h(\mathbf{b}(2\mathbf{u}^n - \mathbf{u}^{n-1})\,(2\mathbf{u}^n - \mathbf{u}^{n-1})), t_{n+1}\right) \Bigg| \\
&\le C \left| \int_0^A b(a, I(t_{n+1}))\, u(a, t_{n+1})\, da - Q_h(\mathbf{b}(2\mathbf{u}^n - \mathbf{u}^{n-1})\,(2\mathbf{u}^n - \mathbf{u}^{n-1})) \right| \\
&\le C \left\{ \left| \int_0^A b(a, I(t_{n+1}))\, u(a, t_{n+1})\, da - \sum_{j=0}^{J}{}'' h b(x_j, I(t_{n+1}))\, u_j^{n+1} \right| \right. \\
&\quad + \left| \sum_{j=0}^{J}{}'' h\, b(x_j, I(t_{n+1}))\, (u_j^{n+1} - 2u_j^n + u_j^{n-1}) \right| \\
&\quad \left. + \left| \sum_{j=0}^{J}{}'' h (b(x_j, I(t_{n+1})) - b(x_j, Q_h(2\mathbf{u}^n - \mathbf{u}^{n-1})))\, (2u_j^n - u_j^{n-1}) \right| \right\} \\
&= O(h^2 + k^2),
\end{aligned}
$$

which completes the proof. ∎

In order to establish the stability of the extrapolated box scheme, we need the inequality [8]

$$
\sum_{j=0}^{J} (V_j)^2 \le \frac{4(J+1)\,(2+J\varepsilon)}{\varepsilon} \left\{ \sum_{j=1}^{J} \left(V_{j-\frac{1}{2}}\right)^2 + \frac{\varepsilon}{4}(V_0)^2 \right\}, \tag{12}
$$

which is valid for any positive number ε . Also we need the following properties of the trapezoidal rule, which are easily derived. Let $\mathbf{V}, \mathbf{W} \in \mathcal{R}^{J+1}$, then

$$
|Q_h(\mathbf{V})| \le \sqrt{A} \left(\sum_{j=1}^{J} h \left(V_{j-\frac{1}{2}}\right)^2 \right)^{\frac{1}{2}}, \tag{13}
$$

$$
|Q_h(\mathbf{V}\ \mathbf{W})| \le \sqrt{A}\, \|\mathbf{V}\|_\infty \left(\sum_{j=1}^{J} h \left(W_{j-\frac{1}{2}}\right)^2 \right)^{\frac{1}{2}} + \frac{\sqrt{A}}{2} \left(\max_{1 \le j \le J} |V_j - V_{j-1}| \right) \left(\sum_{j=1}^{J} h (W_j)^2 \right)^{\frac{1}{2}}, \tag{14}
$$

$$
|Q_h(\mathbf{V}\mathbf{W})| \le \sqrt{A}\, \|\mathbf{V}\|_\infty \left(\sum_{j=0}^{J} h (W_j)^2 \right)^{\frac{1}{2}}, \tag{15}
$$

where

$$
\|\mathbf{V}\|_\infty = \max_{0 \le j \le J} |V_j|.
$$

THEOREM 3.2. *Under the hypotheses of Theorem 3.1, the discretization (8)–(10) is stable for* $\mathbf{u}_h$ *with thresholds* $M_h = Mh$, *where* M *is a fixed positive constant.*

PROOF. Let $(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N)$ and $(\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N)$ be in the ball $B(\mathbf{u}_h, M_h)$ of the space X_h, and set

$$
\begin{aligned}
\mathbf{E}^n &= \mathbf{V}^n - \mathbf{W}^n \in \mathcal{R}^{J+1}, \qquad 0 \le n \le N, \\
\phi_h(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N) &= (\mathbf{P}^0, \mathbf{P}^1, \mathbf{P}_0, \mathbf{P}^2, \dots, \mathbf{P}^N), \\
\phi_h(\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N) &= (\mathbf{R}^0, \mathbf{R}^1, \mathbf{R}_0, \mathbf{R}^2, \dots, \mathbf{R}^N).
\end{aligned}
$$

Then it is easy to show that there exists a positive constant C such that, for h sufficiently small,

$$\left|Q_h\left(\frac{3\mathbf{V}^n-\mathbf{V}^{n-1}}{2}\right)-I(t_{n+\frac{1}{2}})\right|\le C, \tag{16}$$

$$|Q_h(2\mathbf{V}^n-\mathbf{V}^{n-1})-I(t_{n+1})|\le C, \tag{17}$$

$$\left(\sum_{j=1}^{J}h(W_j^{n+\frac{1}{2}}+W_{j-1}^{n+\frac{1}{2}})^2\right)^{\frac{1}{2}}\le C. \tag{18}$$

From (10), we may write

$$\begin{aligned}\frac{DE_j^n+DE_{j-1}^n}{2}&=k(P_j^{n+1}-R_j^{n+1})-\frac{r}{2}(\nabla E_j^{n+1}+\nabla E_j^n)\\&+\frac{k}{2}\left\{f\left(x_{j-\frac{1}{2}},Q_h\left(\frac{3\mathbf{V}^n-\mathbf{V}^{n-1}}{2}\right)\right)\left(E_j^{n+\frac{1}{2}}+E_{j-1}^{n+\frac{1}{2}}\right)\right.\\&+\left[f\left(x_{j-\frac{1}{2}},Q_h\left(\frac{3\mathbf{V}^n-\mathbf{V}^{n-1}}{2}\right)\right)\right.\\&\left.\left.-f\left(x_{j-\frac{1}{2}},Q_h\left(\frac{3\mathbf{W}^n-\mathbf{W}^{n-1}}{2}\right)\right)\right]\left(W_j^{n+\frac{1}{2}}+W_{j-1}^{n+\frac{1}{2}}\right)\right\},\end{aligned} \tag{19}$$

$1\le n\le N-1$, $1\le j\le J$. We now multiply (19) by $h(E_j^{n+\frac{1}{2}}+E_{j-1}^{n+\frac{1}{2}})$ and sum on j. On the left hand side of the resulting expression, we obtain

$$\sum_{j=1}^{J}\frac{h}{2}(DE_j^n+DE_{j-1}^n)\left(E_j^{n+\frac{1}{2}}+E_{j-1}^{n+\frac{1}{2}}\right)=\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n+1}\right)^2-\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n}\right)^2. \tag{20}$$

By following the corresponding analysis in [8], it is easy to show that the sum of the first three terms on the right hand side is bounded by

$$\frac{k}{2}\{(E_0^{n+1})^2+(E_0^n)^2+\|\mathbf{P}^{n+1}-\mathbf{R}^{n+1}\|^2\}+Ck\left\{\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n+1}\right)^2+\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n}\right)^2\right\}. \tag{21}$$

The last term on the right hand side can be bounded using (13), (16) and (18) and the fact that f is Lipschitz continuous on compact sets to obtain

$$\begin{aligned}&\frac{k}{2}\sum_{j=1}^{J}h\left[f\left(x_{j-\frac{1}{2}},Q_h\left(\frac{3\mathbf{V}^n-\mathbf{V}^{n-1}}{2}\right)\right)-f\left(x_{j-\frac{1}{2}},Q_h\left(\frac{3\mathbf{W}^n-\mathbf{W}^{n-1}}{2}\right)\right)\right]\\&\quad\cdot\left(W_j^{n+\frac{1}{2}}+W_{j-1}^{n+\frac{1}{2}}\right)\left(E_j^{n+\frac{1}{2}}+E_{j-1}^{n+\frac{1}{2}}\right)\\&\le Ck\left|Q_h\left(\frac{3\mathbf{E}^n-\mathbf{E}^{n-1}}{2}\right)\right|\sum_{j=1}^{J}h\left(W_j^{n+\frac{1}{2}}+W_{j-1}^{n+\frac{1}{2}}\right)\left(E_j^{n+\frac{1}{2}}+E_{j-1}^{n+\frac{1}{2}}\right)\\&\le Ck\left|Q_h\left(\frac{3\mathbf{E}^n-\mathbf{E}^{n-1}}{2}\right)\right|\left(\sum_{j=1}^{J}h\left(W_j^{n+\frac{1}{2}}+W_{j-1}^{n+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{J}h\left(E_j^{n+\frac{1}{2}}+E_{j-1}^{n+\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}\\&\le Ck\left\{\left(\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n-1}\right)^2\right)^{\frac{1}{2}}\right\}\\&\quad\times\left\{\left(\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n+1}\right)^2\right)^{\frac{1}{2}}+\left(\sum_{j=1}^{J}h\left(E_{j-\frac{1}{2}}^{n}\right)^2\right)^{\frac{1}{2}}\right\}\\&\le Ck\sum_{j=1}^{J}h\left[\left(E_{j-\frac{1}{2}}^{n+1}\right)^2+\left(E_{j-\frac{1}{2}}^{n}\right)^2+\left(E_{j-\frac{1}{2}}^{n-1}\right)^2\right].\end{aligned} \tag{22}$$

Combining (20)–(22), we obtain

$$\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{n+1}\right)^2 - \sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{n}\right)^2 \le \frac{k}{2}\Bigg\{\|\mathbf{P}^{n+1}-\mathbf{R}^{n+1}\|^2 + (E_0^{n+1})^2 + (E_0^n)^2 + C\left[\sum_{j=1}^{J} h\left(\left(E_{j-\frac{1}{2}}^{n+1}\right)^2 + \left(E_{j-\frac{1}{2}}^{n}\right)^2 + h\left(E_{j-\frac{1}{2}}^{n-1}\right)^2\right)\right]\Bigg\},$$

$1 \le n \le N-1$, and summation on n yields

$$\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{n}\right)^2 \le (1+Ck)\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{1}\right)^2 + Ck\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{0}\right)^2 + \sum_{m=1}^{n}{}'' k(E_0^m)^2 + \sum_{m=2}^{n} k\|\mathbf{P}^m-\mathbf{R}^m\|^2 + Ck\sum_{m=2}^{n}\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^2, \tag{23}$$

$2 \le n \le N$.

For the boundary terms, we can write by (9) and the smoothness properties of g and b,

$$\begin{aligned}
|E_0^{m+1}| &\le |P_0^{m+1} - R_0^{m+1}| + |g(Q_h(\mathbf{b}(2\mathbf{V}^m - \mathbf{V}^{m-1})\,(2\mathbf{V}^m - \mathbf{V}^{m-1})), t_{m+1}) \\
&\quad - g(Q_h(\mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1})\,(2\mathbf{W}^m - \mathbf{W}^{m-1})), t_{m+1})| \\
&\le |P_0^{m+1} - R_0^{m+1}| + C\,|Q_h(\mathbf{b}(2\mathbf{V}^m - \mathbf{V}^{m-1})\,(2\mathbf{V}^m - \mathbf{V}^{m-1})) \\
&\quad - Q_h(\mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1})\,(2\mathbf{W}^m - \mathbf{W}^{m-1}))| \\
&\le |P_0^{m+1} - R_0^{m+1}| \\
&\quad + \{|Q_h((\mathbf{b}(2\mathbf{V}^m - \mathbf{V}^{m-1}) - \mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1}))\,(2\mathbf{V}^m - \mathbf{V}^{m-1}))| \\
&\quad + |Q_h(\mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1})\,(2\mathbf{E}^m - \mathbf{E}^{m-1}))|\},
\end{aligned} \tag{24}$$

$1 \le m \le N-1$. Now using (15), the fact that b is Lipschitz continuous on compact sets, (13), (12) with $\varepsilon = 1$, and the stability thresholds, we obtain

$$\begin{aligned}
&|Q_h((\mathbf{b}(2\mathbf{V}^m - \mathbf{V}^{m-1}) - \mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1}))\,(2\mathbf{V}^m - \mathbf{V}^{m-1}))| \\
&\le \sqrt{A}\,\|\mathbf{b}(2\mathbf{V}^m - \mathbf{V}^{m-1}) - \mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1})\|_\infty \left(\sum_{j=0}^{J} h\left(2V_j^m - V_j^{m-1}\right)^2\right)^{\frac{1}{2}} \\
&\le C\,|Q_h(2\mathbf{E}^m - \mathbf{E}^{m-1})| \left\{ 2\left(\sum_{j=0}^{J} h\left(V_j^m - u_j^m\right)^2\right)^{\frac{1}{2}} \right. \\
&\left. + \left(\sum_{j=0}^{J} h\left(V_j^{m-1} - u_j^{m-1}\right)^2\right)^{\frac{1}{2}} + \left(\sum_{j=0}^{J} h\left(2u_j^m - u_j^{m-1}\right)^2\right)^{\frac{1}{2}} \right\} \\
&\le C\left\{ \left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^2\right)^{\frac{1}{2}} + \left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m-1}\right)^2\right)^{\frac{1}{2}} \right\} \\
&\times \left\{ \sqrt{4(J+1)(2+J)} \left(\sum_{j=1}^{J} h\left(V_{j-\frac{1}{2}}^m - u_{j-\frac{1}{2}}^m\right)^2\right)^{\frac{1}{2}} \right. \\
&\left. + \left(\sum_{j=1}^{J} h\left(V_{j-\frac{1}{2}}^{m-1} - u_{j-\frac{1}{2}}^{m-1}\right)^2\right)^{\frac{1}{2}} + h^{\frac{1}{2}}|V_0^m - u_0^m| + h^{\frac{1}{2}}|V_0^{m-1} - u_0^{m-1}| \right) + C \Bigg\}
\end{aligned}$$

$$\leq C\left\{\left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^{2}\right)^{\frac{1}{2}}+\left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m-1}\right)^{2}\right)^{\frac{1}{2}}\right\}, \tag{25}$$

$1 \leq m \leq N-1$. Next, (14), the smoothness of b, and (12) with $\varepsilon = h$ enable us to write

$$\begin{aligned}
&|Q_h(\mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1})(2\mathbf{E}^m - \mathbf{E}^{m-1}))| \\
&\leq \sqrt{A}\|\mathbf{b}(2\mathbf{W}^m - \mathbf{W}^{m-1})\|_\infty \left(\sum_{j=1}^{J} h\left(2E_{j-\frac{1}{2}}^{m} - E_{j-\frac{1}{2}}^{m-1}\right)^{2}\right)^{\frac{1}{2}} \\
&+ \frac{\sqrt{A}}{2}\left(\max_{1\leq j\leq J}\left|\mathbf{b}\left(2\mathbf{W}^m - \mathbf{W}^{m-1}\right)_j - \mathbf{b}\left(2\mathbf{W}^m - \mathbf{W}^{m-1}\right)_{j-1}\right|\right)\left(\sum_{j=1}^{J} h\left(2E_j^m - E_j^{m-1}\right)^2\right)^{\frac{1}{2}} \\
&\leq C\left\{\left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^{2}\right)^{\frac{1}{2}}+\left(\sum_{j=1}^{J} h(E_{j-\frac{1}{2}}^{m-1})^{2}\right)^{\frac{1}{2}}\right\} \qquad (26) \\
&+ Ch\sqrt{\frac{4(J+1)(2+Jh)}{h}}\left\{\sum_{j=1}^{J} h\left(2E_{j-\frac{1}{2}}^{m} - E_{j-\frac{1}{2}}^{m-1}\right)^{2} + \frac{h^2}{4}\left(2E_0^m - E_0^{m-1}\right)^2\right\}^{\frac{1}{2}} \\
&\leq C\left\{\left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^{2}\right)^{\frac{1}{2}}+\left(\sum_{j=1}^{J} h(E_{j-\frac{1}{2}}^{m-1})^{2}\right)^{\frac{1}{2}} + h\left(|E_0^m| + |E_0^{m-1}|\right)\right\},
\end{aligned}$$

$1 \leq m \leq N-1$. Substituting (25) and (26) in (24), and as $k = r\,h$, we obtain

$$\begin{aligned}
|E_0^{m+1}| &\leq |P_0^{m+1} - R_0^{m+1}| \\
&+ C\left\{\left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^{2}\right)^{\frac{1}{2}}+\left(\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m-1}\right)^{2}\right)^{\frac{1}{2}} + k(|E_0^m| + |E_0^{m-1}|)\right\},
\end{aligned}$$

$1 \leq m \leq N-1$. Now, by squaring and summing on m, we have

$$\sum_{m=2}^{n} k|E_0^m|^2 \leq C\left\{\sum_{m=2}^{n} k|P_0^m - R_0^m|^2 + \sum_{m=0}^{n-1} k\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^{2} + k^2\sum_{m=0}^{n-1}|E_0^m|^2\right\},$$

and hence, for k sufficiently small,

$$\sum_{m=2}^{n} k|E_0^m|^2 \leq C\left\{\sum_{m=2}^{n} k|P_0^m - R_0^m|^2 + \sum_{m=0}^{n-1} k\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^{2} + k^2(|E_0^0|^2 + |E_0^1|^2)\right\}, \tag{27}$$

$2 \leq n \leq N$. Thus, with (27) in (23), we obtain

$$\begin{aligned}
\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{n}\right)^{2} &\leq (1+C\,k)\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{1}\right)^{2} + C\,k\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{0}\right)^{2} \\
&+ C\,k(E_0^1)^2 + C\,k(E_0^0)^2 + \sum_{m=2}^{n} k\|\mathbf{P}^m - \mathbf{R}^m\|^2 \\
&+ C\,k\sum_{m=2}^{n}\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^{m}\right)^{2} + C\sum_{m=2}^{n} k|P_0^m - R_0^m|^2,
\end{aligned}$$

$2 \le n \le N$. Using the discrete Gronwall lemma, it follows that

$$\sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^n\right)^2 \le C\left\{\sum_{j=0}^{J}{}'' h(E_j^0)^2 + \sum_{j=0}^{J}{}'' h(E_j^1)^2 + \sum_{m=2}^{n} k\|\mathbf{P}^m - \mathbf{R}^m\|^2 \sum_{m=2}^{n} k|P_0^m - R_0^m|^2\right\}, \tag{28}$$

$0 \le n \le N$. This completes the derivation of the stability estimate for the interior grid points.

Finally, for the boundary terms we have, by taking into account (27) and (28),

$$\begin{aligned}
\sum_{n=0}^{N}{}'' k|E_0^n|^2 &\le \frac{k}{2}|E_0^0|^2 + k|E_0^1|^2 + C\Bigg\{\sum_{n=2}^{N} k|P_0^n - R_0^n|^2 \\
&\quad + \sum_{n=0}^{N-1} k \sum_{j=1}^{J} h\left(E_{j-\frac{1}{2}}^n\right)^2 + k^2(|E_0^0|^2 + |E_0^1|^2)\Bigg\} \\
&\le C\Bigg\{k|E_0^0|^2 + k|E_0^1|^2 + \sum_{n=2}^{N} k|P_0^n - R_0^n|^2 \\
&\quad + \Bigg[\sum_{n=0}^{N-1} k\Bigg(\sum_{j=0}^{J}{}'' h(E_j^0)^2 + \sum_{j=0}^{J}{}'' h(E_j^1)^2 \\
&\quad + \sum_{m=2}^{N} k\|\mathbf{P}^m - \mathbf{R}^m\|^2 + \sum_{m=2}^{N} k|P_0^m - R_0^m|^2\Bigg)\Bigg]\Bigg\} \\
&\le C\left\{\sum_{j=0}^{J}{}'' h(E_j^0)^2 + \sum_{j=0}^{J}{}'' h(E_j^1)^2 + \sum_{m=2}^{N} k\|\mathbf{P}^m - \mathbf{R}^m\|^2 + \sum_{m=2}^{N} k|P_0^m - R_0^m|^2\right\},
\end{aligned} \tag{29}$$

and hence, we have shown that the discretization (8) is stable. That is, by (28) and (29), we have proved that, for fixed T and r, there exists a positive constant S, the stability constant, such that, for h sufficiently small, and for $(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N)$, $(\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N)$ in the ball $B(\mathbf{u}_h, M_h)$,

$$\begin{aligned}
&\|(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N) - (\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N)\|_{X_h} \\
&\qquad \le S\|\phi_h(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N) - \phi_h(\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N)\|_{Y_h}. \quad \blacksquare
\end{aligned}$$

REMARK 3.1. Note that since the stability thresholds $M_h = M\,h$ are only used in (25), as in [8], the stability thresholds can be chosen independent of the discretization parameter if the fertility function b only depends on the age and not on the size of the total population.

The existence and convergence of the approximate solution now follow from the consistency, Theorem 3.1, the stability, Theorem 3.2, and Theorem 2.1. The desired results are stated in the following theorem.

THEOREM 3.3. *Under the hypotheses of Theorem 3.1, if the initial values* $\mathbf{U}^0, \mathbf{U}^1$ *given in (6) are such that*

$$\left(\sum_{j=0}^{J}{}'' h\left(U_j^0 - u_j^0\right)^2\right)^{\frac{1}{2}} = o(h), \quad \textit{and} \quad \left(\sum_{j=0}^{J}{}'' h\left(U_j^1 - u_j^1\right)^2\right)^{\frac{1}{2}} = o(h),$$

as $h \to 0$, *then, for* h *sufficiently small, the solution of (5)–(7),*

$$(\mathbf{U}^0, \mathbf{U}^1, \dots, \mathbf{U}^N),$$

is in the ball $B(\mathbf{u}_h, M_h)$ of X_h, and there exists a positive constant S such that

$$\left(\sum_{j=1}^{J} h\left(U_{j-\frac{1}{2}}^{n}-u_{j-\frac{1}{2}}^{n}\right)^{2}\right)^{\frac{1}{2}} \leq S\left\{\sum_{j=0}^{J}{}'' h\left(U_{j}^{0}-u_{j}^{0}\right)^{2}+\sum_{j=0}^{J}{}'' h\left(U_{j}^{1}-u_{j}^{1}\right)^{2}+O(h^{2}+k^{2})^{2}\right\}^{\frac{1}{2}},$$

for $0 \leq n \leq N$, and

$$\left(\sum_{n=0}^{N}{}'' k\left(U_{0}^{n}-u_{0}^{n}\right)^{2}\right)^{\frac{1}{2}} \leq S\left\{\sum_{j=0}^{J}{}'' h\left(U_{j}^{0}-u_{j}^{0}\right)^{2}+\sum_{j=0}^{J}{}'' h\left(U_{j}^{1}-u_{j}^{1}\right)^{2}+O\left(h^{2}+k^{2}\right)^{2}\right\}^{\frac{1}{2}}.$$

If $\mathbf{U}^0$ and $\mathbf{U}^1$ are taken as the grid restriction $\mathbf{u}^0$ of the initial condition (2) and an approximation of second order to $\mathbf{u}^1$, respectively, then the extrapolated scheme is second order accurate.

REMARK 3.2. (cf. [7, Remark 5.1]; [8, Remark 3.2].) The use of stability thresholds determines that the properties of f, g and b away from the theoretical solution are of no consequence for the stability and convergence of the scheme. Therefore, it is possible to carry out the analysis if we assume that the smoothness requirements on f and b hold on $[0, A] \times D_1$, where D_1 is a compact neighborhood of $\{I(t) : 0 \leq t \leq T\}$, and that the hypothesis on g holds on $D_2 \times [0, T]$, where D_2 is a compact neighborhood of

$$\left\{\int_0^A b(a, I(t))\, u(a,t)\, da : 0 \leq t \leq T\right\}.$$

Note that if the stability thresholds can be chosen independently of h, then the size of the constant M, the stability threshold, plays an important role to ensure that we are working in the sets D_1 and D_2. It is also possible to carry out the above analysis with weaker regularity assumptions than those listed in Theorem 3.1, since the essential features are the continuity and the uniform Lipschitz continuity with respect to one of the arguments in the above domains, and the convergence of the trapezoidal rule. Less regularity is necessary for stability than for consistency. Therefore, if stability holds and the local truncation error is such that $\|\phi_h(\mathbf{u}_h)\| = o(M_h)$, then by Theorem 2.1 we can ensure that the numerical approximations converge at a rate of at least $\|\phi_h(\mathbf{u}_h)\|$. On the other hand, under these weaker assumptions, the existence of solutions for the extrapolated scheme (5)–(7) is not immediate because it might happen that when $\mathbf{U}^{n-1}$ and $\mathbf{U}^n$ have been computed, the computation of $\mathbf{U}^{n+1}$ requires the evaluation of f, g and b out of their domains. Theorem 2.1 establishes that, for h small enough, the numerical solution is defined.

REMARK 3.3. When $A = \infty$, only initial conditions with compact support have biological meaning. The solution of (1)–(3) with u^0 having compact support has also compact support for each fixed t. Moreover, if the support of u^0 is contained in the interval $[0, A^*]$ then the support of $u(x,t)$ is contained in $[0, A^* + t]$. Therefore, if we seek numerical approximations to such solutions on a time interval $[0, T]$ then we have to consider $[0, A^* + T]$ as the age interval.

4. QUADRATURE RULES

An interesting question in the study of discretizations of the equations (1)–(3) is the role played by the approximations to the integral terms $I(t) = \int_0^A u(a,t)da$ and $\int_0^A b(a, I(t))u(a,t)da$. As was mentioned in Section 1, the choice of the quadrature rule is very important since the accuracy of the rule must be compatible with that of the discretization of the derivative terms. The use of the trapezoidal rule is natural in our situation and is also effective, as the numerical experiments in the next section demonstrate. However, it seems reasonable to ask if the use of more accurate quadrature rules could improve the approximations obtained by the numerical method. It is possible that, in some practical situations, the values of the age-specific fertility and mortality moduli, $b(x,z)$ and $f(x,z)$, respectively, might be determined by empirical methods only on a discrete set of the age interval. Therefore, when we solve such problems numerically, it would be more suitable to use more accurate quadrature rules than to use more refined grids on the age interval.

The analysis of Section 3 requires little modification when certain quadrature rules other than trapezoidal rule are employed. While this analysis is based on properties (13)–(15) of the trapezoidal rule, it is not too difficult to prove that similar inequalities hold for any quadrature rule of the form

$$Q_h(\mathbf{V}) = \sum_{j=1}^{J} h\,\omega_j\,V_{j-\frac{1}{2}},$$

with

$$\sup_{J\in\mathcal{N}}\big(\max_{1\le j\le J}|\omega_j|\big) \le B < \infty.$$

In our numerical experiments, we shall use the following composite rules:

$$Q_h^*(\mathbf{V}) = \sum_{j=0}^{J-1} \frac{3h}{8}\left(V_{3j} + 3V_{3j+1} + 3V_{3j+2} + V_{3j+3}\right), \tag{30}$$

with $3J = A$, and

$$Q_h^{**}(\mathbf{V}) = \sum_{j=0}^{J-1} \frac{5h}{288}\left(19V_{5j} + 75V_{5j+1} + 50V_{5j+2} + 50V_{5j+3} + 75V_{5j+4} + 19V_{5j+5}\right), \tag{31}$$

with $5J = A$, which are fourth and sixth order accurate, respectively, [15]. Other standard quadrature rules could be considered, but (30), (31) and the trapezoidal rule are sufficient to illustrate the role played by the approximations to the integral terms.

5. NUMERICAL RESULTS

In this section, we present some numerical experiments carried out with the extrapolated box scheme with the quadrature rules (4), (30) and (31). The test problem is that considered in [8], in which $A = 5$, $T = 20$,

$$f(x,z) = -z, \qquad b(x,z) = \frac{x\,z\exp(-x)}{(1+z)^2},$$

$$g(z,t) = \frac{4z(2 - 2\exp(-A) + \exp(-t))^2}{(1-\exp(-A))\,(1-(1+2A)\exp(-2A))\,(1-\exp(-A)+\exp(-t))},$$

and

$$u^0(x) = \frac{\exp(-x)}{2-\exp(-A)}.$$

The solution of (1)–(3) is then

$$u(x,t) = \frac{\exp(-x)}{1-\exp(-A)+\exp(-t)}.$$

In each entry of Tables I-VI, the *first number* is

$$\max_{0\le n\le N}\left(\sum_{j=1}^{J} h\left(U_{j-\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n\right)^2\right)^{\frac{1}{2}},$$

the *second number* is

$$\left(\sum_{n=2}^{N} k\left(U_0^n - u_0^n\right)^2\right)^{\frac{1}{2}},$$

and the *third number* is the cpu time in seconds. We set $\mathbf{U}^0 = \mathbf{u}^0$, and determine $\mathbf{U}^1$ using the box scheme of [8]. In the results presented in Tables 1–3, we have used the trapezoidal rule, (30) and (31), respectively. The fact that the numerical method is asymptotically second order accurate is clear in each table. Note that, for fixed N, the error decreases as J increases until a limiting accuracy is achieved, when the time error dominates. On the other hand, for fixed J, the error decreases as N increases only when $J = 120$, 240 for the trapezoidal rule but for $J \geq 60$ when a more accurate quadrature rule is used. For most choices of N and J, the higher order rules give the more accurate results. Similar results were obtained using the box scheme and the three quadrature rules but with at least a 50% increase in execution time.

All of the calculations were carried out on a VAX-11/780.

Table 1. Extrapolated scheme with trapezoidal rule.

N	$J = 15$	$J = 30$	$J = 60$	$J = 120$	$J = 240$
60	0.05052 0.36007 0.20	0.01320 0.08963 0.41	0.00581 0.02582 0.74	0.00801 0.02237 1.38	0.00858 0.02383 2.58
120	0.05057 0.36672 0.43	0.01320 0.09508 0.76	0.00334 0.02272 1.32	0.00170 0.00676 2.54	0.00222 0.00600 4.81
240	0.05096 0.36854 0.79	0.01323 0.09696 1.46	0.00334 0.02410 2.82	0.00084 0.00570 4.90	0.00045 0.00173 9.71
480	0.05112 0.36900 1.56	0.01335 0.09747 2.99	0.00335 0.02459 4.99	0.00084 0.00605 9.79	0.00021 0.00143 18.94

Table 2. Extrapolated scheme with 4^{th} order quadrature rule.

N	$J = 15$	$J = 30$	$J = 60$	$J = 120$	$J = 240$
60	0.01335 0.08551 0.21	0.00674 0.02144 0.32	0.00833 0.02320 0.65	0.00866 0.02417 1.37	0.00874 0.02441 3.06
120	0.01331 0.09121 0.42	0.00212 0.01248 0.71	0.00199 0.00573 1.32	0.00230 0.00622 2.75	0.00237 0.00644 5.76
240	0.01356 0.09315 0.77	0.00212 0.01342 1.37	0.00044 0.00252 2.65	0.00053 0.00149 5.54	0.00060 0.00161 11.66
480	0.01381 0.09367 1.41	0.00217 0.01391 2.75	0.00044 0.00265 5.26	0.00011 0.00059 11.57	0.00013 0.00038 21.58

Table 3. Extrapolated scheme with 6^{th} order quadrature rule.

N	$J = 15$	$J = 30$	$J = 60$	$J = 120$	$J = 240$
60	0.32062 1.25199 0.20	0.00716 0.02272 0.37	0.00836 0.02329 0.70	0.00866 0.02417 1.37	0.00874 0.02441 2.80
120	0.44947 2.09819 0.40	0.01222 0.05106 0.75	0.00202 0.00574 1.32	0.00230 0.00623 2.50	0.00237 0.00644 5.42
240	0.00810 0.05099 0.77	0.00529 0.01876 1.40	0.00064 0.00295 2.60	0.00053 0.00149 5.52	0.00060 0.00161 10.29
480	0.00837 0.05152 1.51	0.00172 0.01031 2.70	0.00041 0.00241 5.11	0.00010 0.00058 10.56	0.00013 0.00038 20.55

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