

# An Algebraic Study on Star Patterns.

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## 1 Introduction

Consider the problem of drawing star patterns. If you are asked to draw a 5 pointed star pattern, it is relatively straight forward. How about an 8 pointed star pattern?. It could be challenging at first. To do this, start by drawing a circle and then mark equally spaced 8 points and label them 0, 1, 2, 3,  $\dots$ , 7. Next, connect each point to another point which is at a fixed units away. For instance, let us decide to connect every point to points which are 3 units away. This means we can connect 0 to 3, 3 to 6,  $\dots$ , 5 to 0 to obtain Fig 1.1.

On the other hand, if the choice is to connect every point to points that are 2 units away, this choice forms a star pattern having two distinct components. To draw this pattern, we can connect 0 to 2, 2 to 4,  $\dots$ , 6 to 0 for the first component, and then connect 1 to 3, 3 to 5,  $\dots$ , 7 to 1 to obtain . Notice that the choice of turning by 3 results in construction without lifting pen; such a pattern is called as a *star polygon*. In contrast, when connecting 2 units away requires lifting the pen, and this pattern is referred to as a *star figure*. This means, star patterns are generally classified into star polygons and star figures.

These patterns are commonly denoted using Schläfli symbol  $\{n/k\}$ , where  $n$  is the number of vertices and  $k$  is turning number. The star polygon obtained from connecting every third is denoted as  $\{8/3\}$ , while the star figure drawn by connecting every two units apart is denoted as  $\{8/2\}$ .

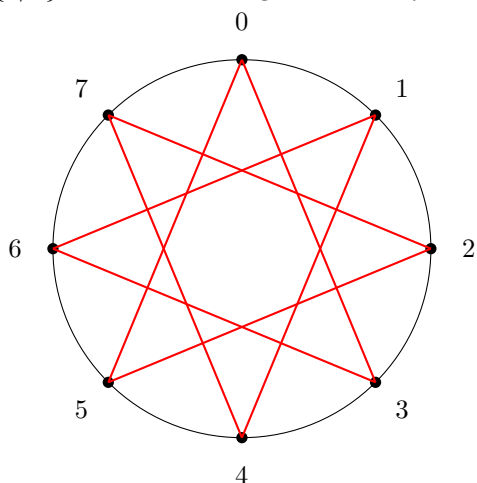


Fig 1.1 Star Polygon  $\{8/3\}$

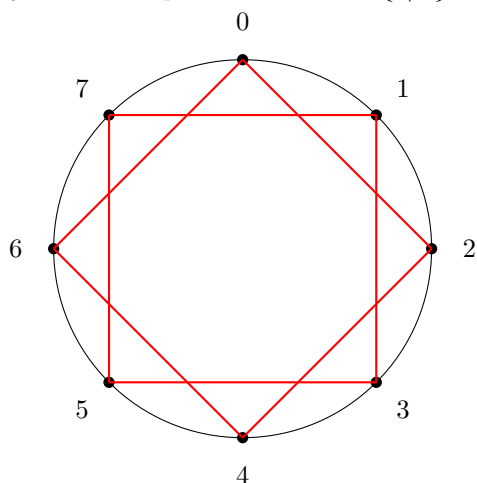


Fig 1.2 Star Figure  $\{8/2\}$

It's a fun activity to observe and make conjectures about these patterns. For instance, How many  $n$  pointed star polygons can be drawn , or How many distinct components are there in a star figure. We will discuss the algebraic connections in star polygons and figures to answer these questions. In fact, these patterns will help us to visualize abstract concepts from Group theory such as cyclic

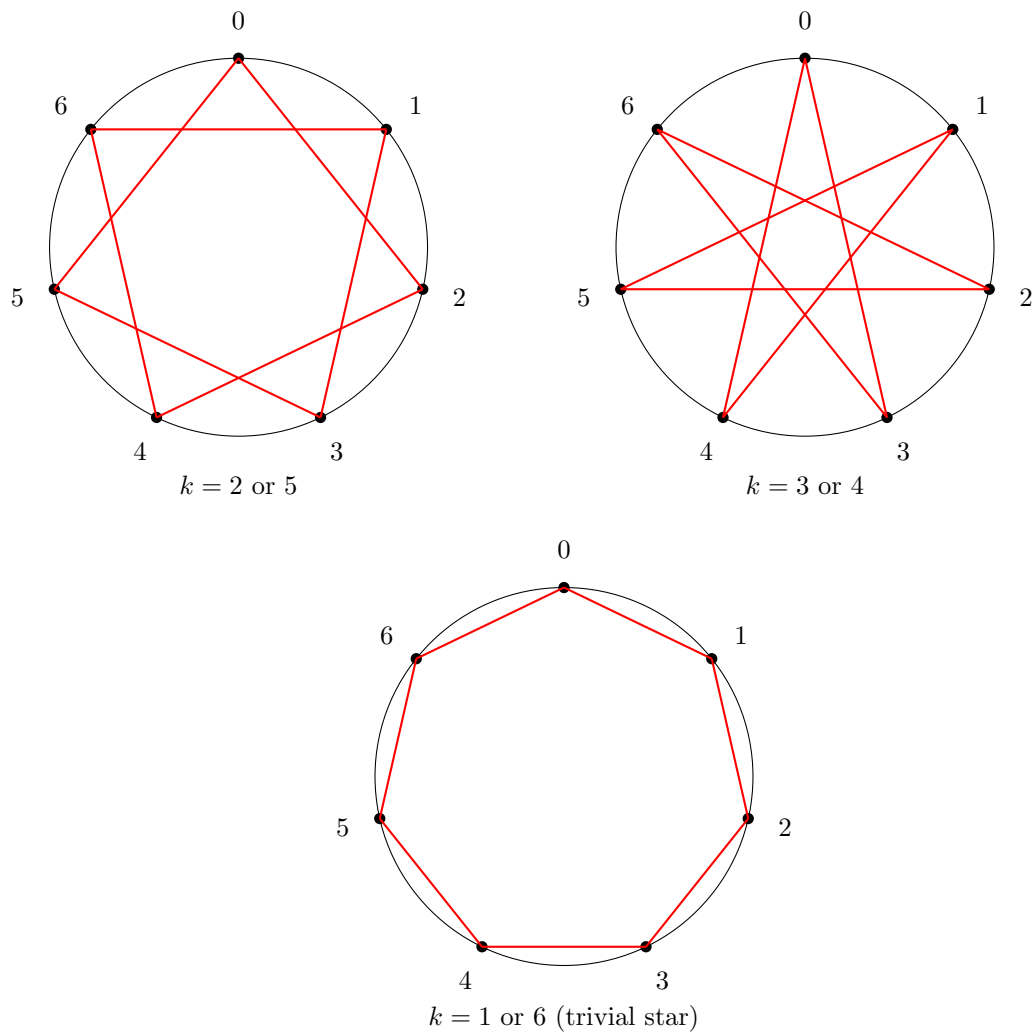
groups, isomorphism and cosets. Let us begin by considering the construction and discussion of star patterns without lifting the pen or better known as star polygons.

## 2 Star Polygons: Construction and Their Cyclic Structure

As suggested in the introduction, when constructing a star polygon, we need to choose the turning number such that it doesn't loop in a smaller subset of the labels but reach all them. This is exactly the situation when we choose  $k$  such that  $\gcd(k, n) = 1$

Draw a circle and mark  $n$  points and label them with numbers from 0 to  $n - 1$ . Choose any number  $k$  such that  $\gcd(k, n) = 1$ . Join each point  $x$  to  $(x + k) \pmod{n}$ .

For instance, The following are possible 7 pointed star polygons based on this procedure:



In this section, we will mainly address following questions:

- Why the condition  $\gcd(k, n) = 1$ ?
- How many  $n$  pointed star polygons can be drawn. ?
- Why do star polygon symmetries pair up ?

In addition to answer these questions, we need to understand cyclic groups, the underlying algebra structure associated with these patterns. In fact they help to visualize cyclic groups. Let us first introduce the notion of a cyclic group.

## Cyclic Groups; An overview

Given a set  $G$  together with an binary operation  $*$ , we say  $G$  forms an algebraic structure named *group*, if it satisfies the following axioms,

1.  $*$  is a binary operation in  $G$
2.  $a * (b * c) = (a * b) * c, \forall a, b, c \in G$
3. there exist  $e \in G$  such that  $a * e = e * a = a$
4. for each  $a \in G$  there exist  $b$  such that  $a * b = e = b * a$

A group  $G$  is said to be *cyclic*, if there exist  $a \in G$  such that  $G = \{a^n : n \in \mathbb{Z}\}$ , where ' $a$ ' is called as the generator. In other words, a cyclic group posses an element that can generate all other elements in the group. An important example of cyclic group is an additive group. Let  $\mathbb{Z}_n = \{0, 1, 2, 3, \dots, (n-1)\}$  be the set of remainders upon division by  $n$ , we can easily verify  $\mathbb{Z}_n$  is a group under addition modulo  $n$ . Since  $1 \in \mathbb{Z}_n$ , clearly it is cyclic.

The construction procedure suggested earlier be thought as constructing over the cyclic group  $\mathbb{Z}_n$ . We can think of wrapping around elements of  $\mathbb{Z}_n$  around the circle, , choosing an element  $k$  from  $\mathbb{Z}_n$  and connecting the points by computing  $k, (k +_n k), (k +_n k +_n k), \dots, 0$ .

## Why the condition $\gcd(k, n) = 1$ ?

As our intuition suggests, an  $n$  pointed star polygons is only possible when the choice of  $k$  is a generator of the group  $\mathbb{Z}_n$ . If you observe Fig 2.1, we can note that when  $m = 8$  and  $k = 2$ , this choice reaches all the multiples of 2. Similarly,  $k = 6$  reaches the same labels. What's common between them ? They share the greatest common divisor of 2 with  $m = 8$ . In fact, when we choose some  $k$ , the labels it will reach are multiples  $\gcd(k, n)$ . So naturally to reach all the labels  $\gcd(k, n)$  simply has to be 1. The following theorem proves it formally using bezout's identity and congruence properties.

**Theorem 2.1**  $k \in \mathbb{Z}_n$  is a generator of  $\mathbb{Z}_n$ , if  $\gcd(k, n) = 1$

**Proof:** The idea of proof is similar to earlier discussion. We have to show that, if  $\gcd(k, n) = 1$ , then the successive elements  $k, (k +_n k), (k +_n k +_n k)$ , somehow reaches 1. If it can reach 1 then it must reach any arbitrary  $b \in \mathbb{Z}_n$ .

Applying bezout's identity in  $\gcd(a, n) = 1$ , we get  $x_0, y_0$  such that  $ax_0 + ny_0 = 1$ . Rearranging we get,  $ax_0 - 1 = n(-y_0)$ . From the definition of congruence, this will imply  $ax_0 \equiv 1 \pmod{n}$ . Now let  $b \in \mathbb{Z}_n$  be arbitrary , using the multiplication property of congruence we get

$$a(x_0 b) \equiv (b) \pmod{n} \implies \underbrace{a +_n a +_n \dots +_n a}_{x_0 b \text{ times}} = b$$

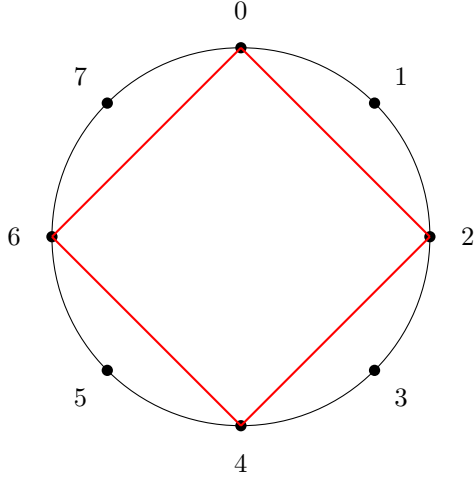


Fig 2.1 ( $k = 2$  or  $6$ )

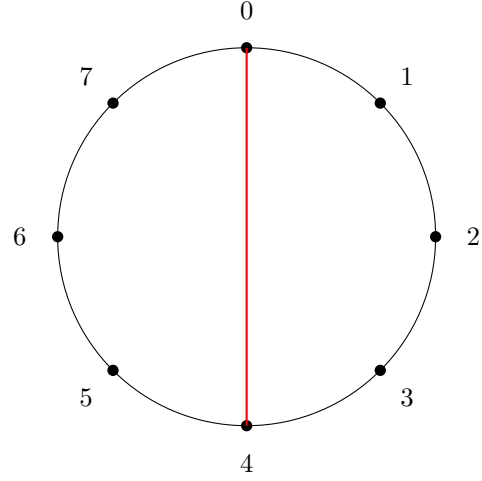


Fig 2.2 ( $k = 4$ )

Though the choice of  $k = 2$  and  $k = 4$  didn't draw the star but they help to visualize subgroups of  $\mathbb{Z}_8$ . Fig 2.1 represents the cyclic subgroup generated by 2 or  $\langle 2 \rangle = \{0, 2, 4, 6\}$  and Fig 2.2 represents  $\langle 4 \rangle = \{4, 0\}$ .

### How many $n$ pointed star polygons can be drawn ?

We expect to draw a star polygon for each  $k \in \mathbb{Z}_n$  such that  $\gcd(k, n) = 1$ . Therefore, counting those elements less than  $n$  and relatively prime to  $n$  will give us the possible choices of  $k$ . This is exactly same situation with Euler's Totient Function  $\phi(n)$ . Intuitively, we can see that each clockwise-construction is paired off with a counter-clockwise construction. Therefore, there are  $\frac{\phi(n)}{2}$  distinct star polygons (counting the trivial star polygon). For instance, when  $m = 8$ ,  $\phi(8) = 4$ , so the two distinct star patterns; The trivial star and Fig 1.1

### Why do symmetries pair up?

The fact that counter clock-wise constructions with  $k$  is always paired off with clockwise construction with  $(n - k)$  can be seen as a geometrical argument for why Euler's Totient Function is always even. Formally, if  $k \in \mathbb{Z}_n$  is a generator then  $(n - k)$  is also a generator of  $\mathbb{Z}_n$ .

**Theorem 2.2** if  $\gcd(k, n) = 1$  then  $\gcd(n - k, n) = 1$

**Proof** (By contra-position): Suppose  $\gcd(m - k, m) = d \neq 1$ . Thus  $d|(m - k)$  and  $d|m$ . from  $d|(m - k)$  we get  $(m - k) = dk$ , for some  $k \in \mathbb{Z}$ . also from  $d|m$  we get  $m = dl$ , for some  $l \in \mathbb{Z}$

We can easily replace  $m$  and see,  $dl - k = dk \implies k = d(l - k)$ . Thus  $d|k$ , but notice since  $d > 1$  this contradicts  $(k, m) = 1$  being greatest common divisor. Hence  $(m - k, m) = 1$

The proof effectively tells us that  $k$  and  $(n - k)$  as not just additive inverse in the additive group but they both generator of the group.

Finally, We are looking for the smallest possible non trivial star. Since the generators are always paired up. We need  $\phi(n) = 4$ , so that two choices belongs to trivial star and other two are our desired. This happens when we have  $n = 5$ , because  $\phi(5) = 4$  Therefore, the smallest possible star is our familiar 5 pointed star.

### 3 Multiplicative Construction and Isomorphism

In the previous section, we considered the successive elements in an additive fashion to construct the pattern. It's natural to ask, can we construct a star polygon in a multiplicative fashion ?. This means, starting with the labels  $\{1, 2, 3, \dots, n-1\}$  and choosing some label to repeatedly multiply and reach all elements of the set to draw a star. It turns out that we cannot do this all the time, but it's possible when  $n$  is some prime number. The interesting fact is that this construction doesn't even look anywhere close to a star until we apply some magic over it, the isomorphism.

Let us begin with  $U(p)$ , the of all positive integers less than  $p$  and relatively prime to  $p$ .

$$U(p) = \{x : \gcd(x, p) = 1\} = \{1, 2, \dots, p-1\}$$

We can easily see that  $U(p)$  is a group under multiplication modulo  $p$ . In general,  $U(n)$  is not cyclic that is why we don't get star patterns all the time but when  $n = p$ , it's cyclic group.

Consider  $U(11) = \{1, 2, 3, \dots, 10\}$ . Labels these points around a circle. We can verify that 7 is a generator of the group. Thus, compute the multiplicative powers of 7 in modulo 11 and join the labels. Connect 1 to 7, 7 to  $7^2 = 5$ , 5 to  $7^3 = 2$ , 2 to  $7^4 = 4$ ,  $\dots$ , and 8 to  $7^{10} = 1$  to obtain the Figure 2.1

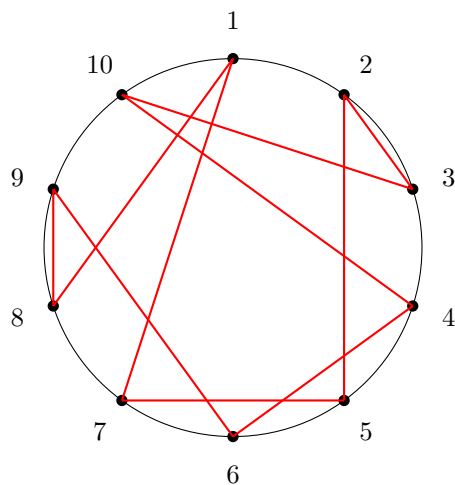


Fig 3.1

Clearly it doesn't look like a star, but let's relabel these points in some fashion preserving the multiplication operation

Let's observe, the following multiplication  $8 = 7^9$  and  $7 = 7^1$  in  $U(11)$

$$7^9 \cdot_{11} 7^1 = 7^{10} = 7^0 = 1$$

Notice that even though the multiplication is  $U(11)$ , in the exponents, an addition modulo 10 operation is happening under the hood. This means, the element  $27^9$  in  $U(11)$  can be corresponded to element 9 in  $\mathbb{Z}$ . Therefore we expect a structural similarity between the groups  $U(11)$  and  $\mathbb{Z}_n$ . This similarity can be established by an isomorphism.

Let's define the isomorphism,

$$\phi(7^k \pmod{11}) = k \pmod{10}$$

We have the following mappings from  $\mathbb{Z}_{10}$  to  $U(11)$ .  $0 \leftrightarrow 1, 1 \leftrightarrow 7, 2 \leftrightarrow 5, 3 \leftrightarrow 2, \dots, 9 \leftrightarrow 8$ .

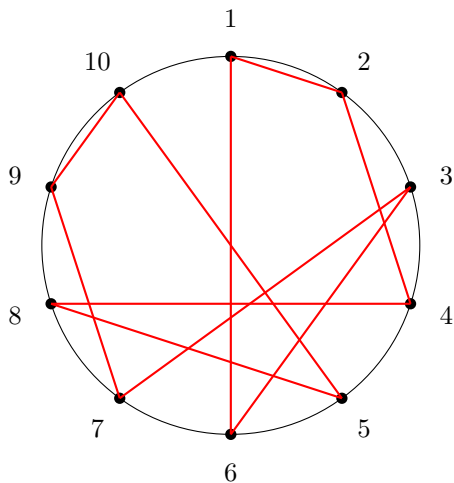


Fig 2.1

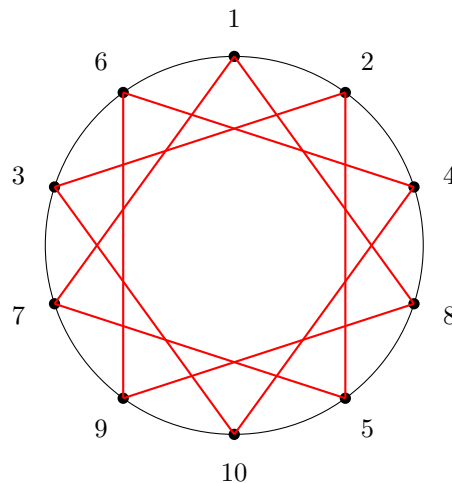


Fig 2.3

To obtain the star polygon from the distorted figure, we first found the isomorphism, then moved the points around by preserving its corresponding edges.

In the above example, we were able to easily find out an isomorphism to reveal our original pattern because, we were able to easily find out a generator or primitive root of the group  $U(p)$ . However, In general, It isn't an easy task to find a generator of the group  $U(p)$ . This is because, finding out generator of a group is better known as the **Discrete Logarithm Problem**, and it is known to be computationally hard, which is indeed the backbone of cryptographic algorithms that safe guard our daily transactions and conversations.

## 4 Star Figures: Cosets and Factor Group

In the introduction, we saw that the patterns requiring us to lift the pen is known as star figures. In the section we will count the number of components in a star figure, this will help to visualize a factor group.

What's a coset of  $H$  in  $G$  generated by ' $a$ '?. Think about first picking a subgroup  $H$  of  $G$ , then pick an another element  $a'$  from  $G$ , translate each element of  $H$  using ' $a$ ' to get the coset  $a + H$ . For instance,  $G = \mathbb{Z}_8$ ,  $H = \langle 2 \rangle = \{0, 2, 4, 6\}$ . The distinct cosets we can obtain are  $0 + \langle 2 \rangle$  and  $1 + \langle 2 \rangle$ . The key connection to the star figure is that each component corresponds to a coset.

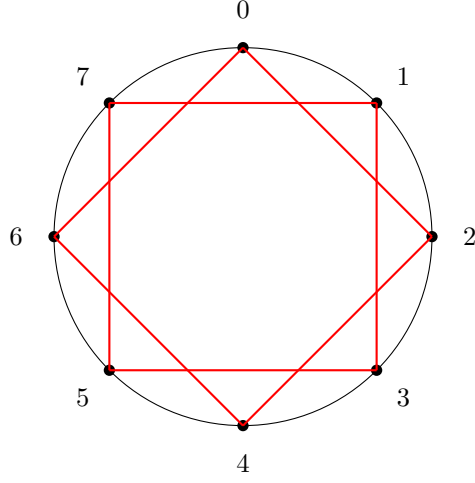


Fig 1.2 Star Figure  $\{8/2\}$

We can see when  $k = 2$ , we have  $H = \langle 2 \rangle$ . Therefore, the components are  $0 + H = \{0, 2, 4, 6\}$  and  $1 + H = \{1, 3, 5, 7\}$

If we were to collect all the cosets of group  $\mathbb{Z}_n$  and under the operation coset-addition it becomes the factor group  $\mathbb{Z}_n / \langle a \rangle = \{0 + \langle a \rangle, 1 + \langle a \rangle, \dots, (n-1) + \langle a \rangle\}$ .

## 1 How many components are there in an $\{n/k\}$ Star Figure ?

We saw that every each component correspond to each coset which is in turn an element of the factor group. Therefore, the total number of components is the cardinality of the factor group. Let  $n$  and  $k$  be given. There are

$$|\mathbb{Z}_n / \langle a \rangle|$$

components in a star figure.