AI 1103 - Assignment 8

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Download latex codes from

https://github.com/rohanthota/Assignment_8/Assignment 8.tex

Question

Suppose n units are drawn from a population of N units sequentially as follows. A random sample

$$U_1, U_2, ...U_N \text{ of size N, drawn from } U(0,1) \tag{0.0.1}$$

The k-th population unit is selected if

$$U_k < \frac{n - n_k}{N - k + 1}, k = 1, 2, ..N.\text{where, } n_1 = 0, n_k = \tag{0.0.2}$$

number of units selected out of first k-1 units for each k = 2, 3, ..N. Then,

1) The probability of inclusion of the second unit in the sample
$$\text{is } \frac{n}{N} \tag{0.0.3}$$
2) The probability of inclusion of the first and the second unit in the sample
$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{0.0.4}$$
3) The probability of not including the first and including the second unit in the sample
$$\text{is } \frac{n(N-n)}{N(N-1)} \tag{0.0.5}$$
4) The probability of including the first and not including the second unit in the sample
$$\text{is } \frac{n(n-1)}{N(N-1)} \tag{0.0.6}$$

Solution

$$\text{Defining random variable } X \in \{0, 1, 2, ..N\} \tag{0.0.7}$$
$$\text{Where, } X = i \text{ when ith unit is included.} \tag{0.0.8}$$

The first unit in the sample is included if

$$U_1 < \frac{n - n_1}{N - 1 + 1} \tag{0.0.9}$$
$$\text{Here, } n_1 = 0 \text{ is given in the qn.} \tag{0.0.10}$$
$$\therefore \Pr(X = 1) = \frac{n}{N} \tag{0.0.11}$$

1) For k=2,

$$n_2 = 1 \text{ when, first unit is included.} \tag{0.0.12}$$
$$U_2 < \frac{n - n_2}{N - 2 + 1} \left(= \frac{n-1}{N-1}\right) \tag{0.0.13}$$
$$\therefore \Pr(X = 2 \mid X = 1) = \frac{n-1}{N-1} \tag{0.0.14}$$
$$\Pr(X = 1, X = 2) = \Pr(X = 2 \mid X = 1) \times \Pr(X = 1) \tag{0.0.15}$$
$$\therefore \Pr(X = 1, X = 2) = \frac{n(n-1)}{N(N-1)} \tag{0.0.16}$$
$$n_2 = 0 \text{ when, first unit is not included.} \tag{0.0.17}$$
$$U_2 < \frac{n - n_2}{N - 2 + 1} \left(= \frac{n}{N-1}\right) \tag{0.0.18}$$
$$\therefore \Pr(X = 2 \mid X \neq 1) = \frac{n}{N-1} \tag{0.0.19}$$
$$\Pr(X \neq 1, X = 2) = \Pr(X = 2 \mid X \neq 1) \times \Pr(X \neq 1) \tag{0.0.20}$$
$$\therefore \Pr(X \neq 1, X = 2) = \left(1 - \frac{n}{N}\right) \times \frac{n}{N-1} \tag{0.0.21}$$
$$\therefore \Pr(X \neq 1, X = 2) = \frac{n(N-n)}{N(N-1)} \tag{0.0.22}$$

From (0.0.16) and (0.0.22)

$$\Pr(X = 2) = \frac{n(n-1)}{N(N-1)} + \frac{n(N-n)}{N(N-1)} = \frac{n}{N} \tag{0.0.23}$$

Hence, option 1 is correct.

2) From (0.0.16)

$$\Pr(X = 1, X = 2) = \frac{n(n-1)}{N(N-1)} \tag{0.0.24}$$

Hence, option 2 is correct.

3) From (0.0.22)

$$\Pr(X \neq 1, X = 2) = \frac{n(N-n)}{N(N-1)} \tag{0.0.25}$$

Hence, option 3 is correct.

4)

$$\Pr(X = 1, X \neq 2) = \frac{n}{N} \times \left(1 - \frac{n}{N}\right) = \frac{n(N-n)}{N^2} \tag{0.0.26}$$

Hence, option 4 is incorrect.

Therefore, Options 1, 2, 3 are correct