

AI 1103 - Assignment 8

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Download latex codes from

https://github.com/rohanthota/Assignment_8/Assignment_8.tex

Question

Suppose n units are drawn from a population of N units sequentially as follows. A random sample

$$U_1, U_2, \dots, U_N \text{ of size } N, \text{ drawn from } U(0, 1) \quad (0.0.1)$$

The k -th population unit is selected if

$$U_k < \frac{n - n_k}{N - k + 1}, k = 1, 2, \dots, N. \text{ where, } n_1 = 0, n_k = \quad (0.0.2)$$

number of units selected out of first $k-1$ units for each $k = 2, 3, \dots, N$. Then,

- 1) The probability of inclusion of the second unit in the sample

$$\text{is } \frac{n}{N} \quad (0.0.3)$$

- 2) The probability of inclusion of the first and the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \quad (0.0.4)$$

- 3) The probability of not including the first and including the second unit in the sample

$$\text{is } \frac{n(N-n)}{N(N-1)} \quad (0.0.5)$$

- 4) The probability of including the first and not including the second unit in the sample

$$\text{is } \frac{n(n-1)}{N(N-1)} \quad (0.0.6)$$

Solution

Defining random variable $X \in \{0, 1, 2, \dots, N\}$ (0.0.7)

Where, $X = i$ when i th unit is included. (0.0.8)

The first unit in the sample is included if

$$U_1 < \frac{n - n_1}{N - 1 + 1} \quad (0.0.9)$$

Here, $n_1 = 0$ is given in the qn. (0.0.10)

$$\therefore \Pr(X = 1) = \frac{n}{N} \quad (0.0.11)$$

- 1) For $k=2$,

$$n_2 = 1 \text{ when, first unit is included.} \quad (0.0.12)$$

$$U_2 < \frac{n - n_2}{N - 2 + 1} \left(= \frac{n - 1}{N - 1} \right) \quad (0.0.13)$$

$$\therefore \Pr(X = 2 | X = 1) = \frac{n - 1}{N - 1} \quad (0.0.14)$$

$\Pr(X = 1, X = 2)$

$$= \Pr(X = 2 | X = 1) \times \Pr(X = 1) \quad (0.0.15)$$

$$\therefore \Pr(X = 1, X = 2) = \frac{n(n-1)}{N(N-1)} \quad (0.0.16)$$

$n_2 = 0$ when, first unit is not included.

$$(0.0.17)$$

$$U_2 < \frac{n - n_2}{N - 2 + 1} \left(= \frac{n}{N - 1} \right) \quad (0.0.18)$$

$$\therefore \Pr(X = 2 | X \neq 1) = \frac{n}{N - 1} \quad (0.0.19)$$

$\Pr(X \neq 1, X = 2)$

$$= \Pr(X = 2 | X \neq 1) \times \Pr(X \neq 1) \quad (0.0.20)$$

$$\therefore \Pr(X \neq 1, X = 2) = \left(1 - \frac{n}{N} \right) \times \frac{n}{N - 1} \quad (0.0.21)$$

$$\therefore \Pr(X \neq 1, X = 2) = \frac{n(N-n)}{N(N-1)} \quad (0.0.22)$$

From (0.0.16) and (0.0.22)

$$\Pr(X = 2) = \frac{n(n-1)}{N(N-1)} + \frac{n(N-n)}{N(N-1)} = \frac{n}{N} \quad (0.0.23)$$

Hence, option 1 is correct.

2) From (0.0.16)

$$\Pr(X = 1, X = 2) = \frac{n(n-1)}{N(N-1)} \quad (0.0.24)$$

Hence, option 2 is correct.

3) From (0.0.22)

$$\Pr(X \neq 1, X = 2) = \frac{n(N-n)}{N(N-1)} \quad (0.0.25)$$

Hence, option 3 is correct.

4)

$$\Pr(X = 1, X \neq 2) = \frac{n}{N} \times \left(1 - \frac{n}{N}\right) = \frac{n(N-n)}{N^2} \quad (0.0.26)$$

Hence, option 4 is incorrect.

Therefore, Options 1, 2, 3 are correct