

AI 1103 - Assignment 5

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Download all latex codes from

[https://github.com/rohanthota/
Challenging_problem/main.tex](https://github.com/rohanthota/Challenging_problem/main.tex)

Question

Which of the following conditions imply independence of the random variables X and Y ?

- 1) $\Pr(X > a|Y > a) = \Pr(X > a) \forall a \in \mathbb{R}$
- 2) $\Pr(X > a|Y < b) = \Pr(X > a) \forall a, b \in \mathbb{R}$
- 3) X and Y are uncorrelated.
- 4) $E[(X-a)(Y-b)] = E(X-a) \times E(Y-b) \forall a, b \in \mathbb{R}$

Solution

- 1) Let C.D.F's of X, Y and joint C.D.F (X,Y) be represented as

$$F_X(a) = \Pr(X < a) \quad (0.0.1)$$

$$F_Y(a) = \Pr(Y < a) \quad (0.0.2)$$

$$F_{X,Y}(a, a) = \Pr(X < a, Y < a). \quad (0.0.3)$$

To show independence, we want to prove that,

$$F_X(a)F_Y(a) = F_{X,Y}(a, a) \forall a \in \mathbb{R} \quad (0.0.4)$$

Now, we know

$$\Pr(X > a|Y > a) = \frac{\Pr(X > a, Y > a)}{\Pr(Y > a)} \quad (0.0.5)$$

$$= \Pr(X > a) \quad (0.0.6)$$

$$\implies \Pr(X > a, Y > a) = \Pr(X > a) \Pr(Y > a) \quad (0.0.7)$$

We can write,

$$\Pr(X > a) = 1 - F_X \quad (0.0.8)$$

$$\Pr(Y > a) = 1 - F_Y \quad (0.0.9)$$

Now, we can write

$$F_Y = \Pr(X < a, Y < a) + \Pr(X > a, Y < a) \quad (0.0.10)$$

$$= F_{X,Y} + \Pr(X > a, Y < a) \quad (0.0.11)$$

$$\implies \Pr(X > a, Y < a) = F_Y - F_{X,Y} \quad (0.0.12)$$

Now, $\Pr(X > a) = \Pr(X > a, Y < a)$

$$+ \Pr(X > a, Y > a) \quad (0.0.13)$$

From (0.0.8) and (0.0.12)

$$1 - F_X = F_Y - F_{X,Y} + \Pr(X > a, Y > a) \quad (0.0.14)$$

$$\implies \Pr(X > a, Y > a) = 1 - F_X - F_Y + F_{X,Y}. \quad (0.0.15)$$

From (0.0.8), (0.0.9) and (0.0.15)

$$1 - F_X - F_Y + F_{X,Y} = (1 - F_X) \times (1 - F_Y) \quad (0.0.16)$$

$$\implies F_{X,Y}(a, a) = F_X(a)F_Y(a). \quad (0.0.17)$$

Therefore, option 1 is correct.

- 2) Similar to option 1, C.D.F.s are represented as below,

$$F_X(a) = \Pr(X < a) \quad (0.0.18)$$

$$F_Y(b) = \Pr(Y < b) \quad (0.0.19)$$

$$F_{X,Y}(a, b) = \Pr(X < a, Y < b) \quad (0.0.20)$$

To show independence, we want to prove that,

$$F_X(a)F_Y(b) = F_{X,Y}(a, b) \forall a, b \in \mathbb{R} \quad (0.0.21)$$

From conditional probability we know:

$$\Pr(X > a|Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \quad (0.0.22)$$

and so using the given condition in option 2),

we can write (0.0.21) as

$$\Pr(X > a) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \quad (0.0.23)$$

$$\Pr(X > a) \Pr(Y < b) = \Pr(X > a, Y < b) \quad (0.0.24)$$

We can write the C.D.F as

$$\Pr(X > a) = 1 - F_X(a) \quad (0.0.25)$$

$$\Pr(Y < b) = F_Y(b) \quad (0.0.26)$$

$$(0.0.27)$$

$$F_Y(b) = \Pr(X > a, Y < b) + \Pr(X < a, Y < b) \quad (0.0.28)$$

From (0.0.20)

$$\Pr(X > a, Y < b) = F_Y(b) - F_{X,Y}(a, b) \quad (0.0.29)$$

From (0.0.25), (0.0.26) and (0.0.29)

$$(1 - F_X(a))(F_Y(b)) = F_Y(b) - F_{X,Y}(a, b) \quad (0.0.30)$$

$$\implies F_X(a)F_Y(b) = F_{X,Y}(a, b) \quad (0.0.31)$$

Therefore, option 2 is correct.

- 3) Given random variables X and Y are uncorrelated which means that their correlation is 0, or, equivalently, $\text{Cov}(X, Y) = 0$.

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) \quad (0.0.32)$$

$$[\because \text{Cov}(X, Y) = 0] \quad (0.0.33)$$

$$E(XY) = E(X)E(Y) \quad (0.0.34)$$

Trying to solve through a counter example, Let there be three random variables X, Y and Z such that,

$$Z \sim \mathcal{N}(0, 1) \quad (0.0.35)$$

$$X = Z \quad (0.0.36)$$

$$Y = Z^2 \quad (0.0.37)$$

They are clearly independent, now let's check if they are uncorrelated

$$E(XY) = E(Z^3) = 0 \quad (0.0.38)$$

$$E(X)E(Y) = E(Z)E(Z^2) = 0 \quad (0.0.39)$$

$$\implies E(XY) = E(X)E(Y) \quad (0.0.40)$$

Therefore, they are dependent and uncorrelated.

lated. Hence, option 3 is incorrect.

4) We extend L.H.S $E[(X - a)(Y - b)] = E[XY]$

$$-aE[Y] - bE[X] + ab \quad (0.0.41)$$

and R.H.S to compare, $E(X - a)E(Y - b) = E[X]E[Y]$

$$-aE[Y] - bE[X] + ab \quad (0.0.42)$$

From (0.0.41) and (0.0.42)

$$E[XY] = E[X]E[Y] \quad (0.0.43)$$

Which was already dealt in option 3, Hence option 4 is incorrect.