

AI 1103 - Challenging Problem 11

T. Rohan
CS20BTECH11064

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[https://github.com/rohanthota/
Challenging_problem/main.tex](https://github.com/rohanthota/Challenging_problem/main.tex)

1 PROBLEM

(UGC/MATH 2018 (June set-a)-Q.106) Let $X_{i \geq 1}$ be a sequence of i.i.d. random variables with $E(X_i) = 0$ and $V(X_i) = 1$. Which of the following are true?

- 1) $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow 0$ in probability
- 2) $\frac{1}{n^{3/4}} \sum_{i=1}^n X_i \rightarrow 0$ in probability
- 3) $\frac{1}{n^{1/2}} \sum_{i=1}^n X_i \rightarrow 0$ in probability
- 4) $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow 1$ in probability

2 Solution

Definition 1. (Convergence in distribution)

A sequence of random variables $Y, Y_1, Y_2 \dots$ converges in distribution to a random variable Y , if

$$\lim_{n \rightarrow \infty} F_{Y_n}(a) = F_Y(a) \quad \forall a \in \mathbb{R}. \quad (2.0.1)$$

Definition 2. (Convergence in probability)

A sequence of random variables $Y, Y_1, Y_2 \dots$ is said to converge in probability to Y , if

$$\lim_{n \rightarrow \infty} \Pr(|Y_n - Y| > \epsilon) = 0 \quad \forall \epsilon > 0. \quad (2.0.2)$$

Lemma 2.1. If $Y_n \rightarrow Y$ in probability, $Y_n \rightarrow Y$ in distribution.

Lemma 2.2. (Strong Law of Large Numbers)

Let X_1, X_2, \dots, X_n be i.i.d. random variables with expected value $E(X_i) = \mu < \infty$, then,

$$\lim_{n \rightarrow \infty} \Pr\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) = 0 \quad (2.0.3)$$

Or, $\frac{1}{n} \sum_{i=1}^n X_i$ converges in probability to μ .

Lemma 2.3. (Chebyshev's Inequality)

Let the random variable X have a finite mean μ and a finite variance σ^2 . For every $k > 0$,

$$\Pr(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2} \quad (2.0.4)$$

Lemma 2.4. (Central Limit Theorem)

Let X_1, X_2, \dots, X_n be i.i.d. random variables with expected value $E(X_i) = \mu < \infty$ and $0 < V(X_i) = \sigma^2 < \infty$. Then the random variable

$$Z_n = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{X_1 + X_2 + \dots + X_n - n\mu}{\sqrt{n}\sigma} \quad (2.0.5)$$

converges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n \rightarrow \infty} \Pr(Z_n \leq a) = \Phi(a) \quad \forall a \in \mathbb{R}. \quad (2.0.6)$$

where $\Phi(a)$ is the standard normal CDF.

1)

Lemma 2.5. Let $F_{X_i}(x)$ be the c.d.f. for the random variable X_i . If X_i is a sequence of i.i.d. random variables, it follows the following conditions $\forall x, x_i \in \mathbb{R}$:

- a) $F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x) = F_X(x)$
 - b) $F_{X_1, \dots, X_n}(x_1 \dots x_n) = F_X(x_1)F_X(x_2) \dots F_X(x_n)$
- where $F_X(x)$ is the c.d.f. of X_i .

Now, we know that $\{X_i\}$ is a sequence of i.i.d. random variables. We now try to prove $\{X_i^2\}$ is a sequence of i.i.d. random variables.

Proof. Let $Y_i = X_i^2$.

a) For $y \geq 0$,

$$F_{Y_i}(y) = \Pr(Y_i \leq y) \quad (2.0.7)$$

$$\Rightarrow F_{Y_i}(y) = \Pr(X_i^2 \leq y) \quad (2.0.8)$$

$$\Rightarrow F_{Y_i}(y) = \Pr(-\sqrt{y} \leq X_i \leq \sqrt{y}) \quad (2.0.9)$$

$$\Rightarrow F_{Y_i}(y) = \Pr(X_i \leq \sqrt{y}) - \Pr(X_i \leq -\sqrt{y}) \quad (2.0.10)$$

$$\Rightarrow F_{Y_i}(y) = F_{X_i}(\sqrt{y}) - F_{X_i}(-\sqrt{y}) \quad (2.0.11)$$

Using condition (a) for $\{X_i\}$,

$$F_{Y_i}(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad (2.0.12)$$

From (2.0.12),

$$F_{Y_1}(y) = F_{Y_2}(y) = \dots = F_{Y_n}(y) = F_Y(y) \quad (2.0.13)$$

where $F_Y(y)$ is the c.d.f. of $Y_i = X_i^2$.

b) Now, for $y_i \geq 0$, consider

$$\begin{aligned} & F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) \\ &= \Pr(Y_1 \leq y_1, Y_2 \leq y_2, \dots, Y_n \leq y_n) \end{aligned} \quad (2.0.14)$$

$$= \Pr(X_1^2 \leq y_1, X_2^2 \leq y_2, \dots, X_n^2 \leq y_n) \quad (2.0.15)$$

$$= \Pr(-\sqrt{y_1} \leq X_1 \leq \sqrt{y_1}, -\sqrt{y_2} \leq X_2 \leq \sqrt{y_2}, \dots, -\sqrt{y_n} \leq X_n \leq \sqrt{y_n}) \quad (2.0.16)$$

Since X_1, X_2, \dots, X_n are independent,

$$\begin{aligned} & F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) = \\ & \Pr(-\sqrt{y_1} \leq X_1 \leq \sqrt{y_1}) \Pr(-\sqrt{y_2} \leq X_2 \leq \sqrt{y_2}) \\ & \dots \Pr(-\sqrt{y_n} \leq X_n \leq \sqrt{y_n}) \end{aligned} \quad (2.0.17)$$

From (2.0.9) and (2.0.13),

$$\begin{aligned} & F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) \\ &= F_{Y_1}(y_1) F_{Y_2}(y_2) \dots F_{Y_n}(y_n) \end{aligned} \quad (2.0.18)$$

$$= F_Y(y_1) F_Y(y_2) \dots F_Y(y_n) \quad (2.0.19)$$

So,

$$\begin{aligned} & F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) \\ &= F_Y(y_1) F_Y(y_1) \dots F_Y(y_n) \end{aligned} \quad (2.0.20)$$

By (2.0.13) and (2.0.20), $\{Y_i\} = \{X_i^2\}$ must also be a sequence of i.i.d. random variables. \square

We know,

$$E(X_i^2) = V(X_i) + (E(X_i))^2 \quad (2.0.21)$$

Putting given values, we get,

$$E(X_i^2) = 1 \quad (2.0.22)$$

From 2.2, $\frac{1}{n} \sum_{i=1}^n X_i^2$ converges in probability

to $E(X_i^2) = 1$.

Therefore, option 1 is incorrect.

2) Now, we define

$$Y_n = \frac{1}{n^{3/4}} \sum_{i=1}^n X_i \quad (2.0.23)$$

Then,

$$E(Y_n) = \frac{1}{n^{3/4}} E\left(\sum_{i=1}^n X_i\right) \quad (2.0.24)$$

$$\Rightarrow E(Y_n) = \frac{1}{n^{3/4}} (E(X_1) + E(X_2) \dots E(X_n)) \quad (2.0.25)$$

Since $E(X_i) = 0$

$$E(Y_n) = \frac{1}{n^{3/4}} (0) = 0 \quad (2.0.26)$$

Now,

$$V(Y_n) = V\left(\frac{1}{n^{3/4}} \sum_{i=1}^n X_i\right) \quad (2.0.27)$$

$$\Rightarrow V(Y_n) = \frac{1}{n^{3/2}} V(X_1 + X_2 + \dots + X_n) \quad (2.0.28)$$

As X_1, X_2, \dots, X_n are independent of each other,

$$V(Y_n) = \frac{1}{n^{3/2}} (V(X_1) + V(X_2) + \dots + V(X_n)) \quad (2.0.29)$$

Since $V(X_i) = 1$

$$V(Y_n) = \frac{1}{n^{3/2}} (1 + 1 + \dots + 1) = \frac{1}{n^{3/2}} \times n \quad (2.0.30)$$

$$V(Y_n) = \frac{1}{n^{1/2}} \quad (2.0.31)$$

Now for any $\epsilon > 0$, consider the probability

$$\Pr(|Y_n - 0| \geq \epsilon) = \Pr(|Y_n - E(Y_n)| \geq \epsilon) \quad (2.0.32)$$

Applying 2.3

$$\Pr(|Y_n - 0| \geq \epsilon) \leq \frac{V(Y_n)}{\epsilon^2} \left(= \frac{1}{n^{1/2} \epsilon^2}\right) \quad (2.0.33)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \Pr(|Y_n - 0| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \frac{1}{n^{1/2} \epsilon^2} = 0 \quad (2.0.34)$$

$$\implies \lim_{n \rightarrow \infty} \Pr \left(\left| \frac{1}{n^{3/4}} \sum_{i=1}^n X_i - 0 \right| \geq \epsilon \right) = 0 \quad (2.0.35)$$

So, $\frac{1}{n^{3/4}} \sum_{i=1}^n X_i \rightarrow 0$ in probability.

Thus, option 2 is correct.

- 3) The option states that $\frac{1}{n^{1/2}} \sum_{i=1}^n X_i \rightarrow 0$ in probability. This statement implies that $\frac{1}{n^{1/2}} \sum_{i=1}^n X_i \rightarrow 0$ in distribution, from 2.1.

Writing the random variable Z_n from 2.4, for $\{X_i\}$,

$$Z_n = \frac{X_1 + X_2 + \dots + X_n - n\mu}{\sqrt{n}\sigma} \quad (2.0.36)$$

$$= \frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}} \quad (2.0.37)$$

$$= \frac{1}{n^{1/2}} \sum_{i=1}^n X_i \quad (2.0.38)$$

Since $\mu = 0$ and $\sigma = 1$.

According to 2.4,

$$Z_n \rightarrow Z, \text{ where, } Z \sim N(0, 1) \quad (2.0.39)$$

which is not what the option states. Therefore, option 3 is incorrect.

- 4) As proved in option (1), $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow 1$ in probability. So option 4 is correct.

Therefore, options 2 and 4 are correct.