AI 1103 - Challenging Problem 14

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Download all latex codes from

https://github.com/rohanthota/Challenging_problem/main.tex

Question

Which of the following conditions imply independence of the random variables X and Y ?

1) $\Pr(X > a|Y > a) = \Pr(X > a) \, \forall a \in \mathbb{R}$
2) $\Pr(X > a|Y < b) = \Pr(X > a) \, \forall a, b \in \mathbb{R}$
3) X and Y are uncorrelated.
4) $E[(X-a)(Y-b)] = E(X-a) \times E(Y-b) \forall a, b \in \mathbb{R}$

Solution

First, we want to show that, to show independence, it is enough to show that

$$F_{X,Y}(x, y) = F_X(x) F_Y(y) \tag{0.0.1}$$

Here, $F_X(x)$ and $F_Y(y)$ refer to the individual cdf's. $F_{X,Y}(x, y)$ refers to joint cdf. $f_{X,Y}(x, y)$ refers to the joint pdf. $f_X(x)$ and $f_Y(y)$ refer to individual pdf's. Considering (0.0.1) and applying $\frac{\partial^2}{\partial x \partial y}$ on both sides.

$$\frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} = \frac{\partial^2 F_X(x) F_Y(y)}{\partial x \partial y} \tag{0.0.2}$$

Now, we know

$$\frac{\partial^2 F_{X,Y}(x, y)}{\partial x \partial y} = f_{X,Y}(x, y) \tag{0.0.3}$$

We also know $F_X(x)$ is independent of 'y' and $F_Y(y)$ is independent of 'x' Therefore,

$$\frac{\partial^2 F_X(x) F_Y(y)}{\partial x \partial y} = \frac{\partial}{\partial x} \frac{\partial F_X(x) F_Y(y)}{\partial y} \tag{0.0.4}$$

$$= \frac{\partial F_X(x)}{\partial x} \times \frac{\partial F_Y(y)}{\partial y} \tag{0.0.5}$$

By definition, we know

$$\frac{\partial F_X(x)}{\partial x} = f_X(x) \tag{0.0.6}$$

$$\frac{\partial F_Y(y)}{\partial y} = f_Y(y) \tag{0.0.7}$$

Considering (0.0.1), from (0.0.3) (0.0.5) (0.0.6) (0.0.7)

$$f_{X,Y}(x, y) = f_X(x) \times f_Y(y) \tag{0.0.8}$$

Which is the condition for independence. Therefore it is enough to show $F_{X,Y}(x, y) = F_X(x) F_Y(y)$ for independence. Now, we solve the options.

1) Let C.D.F's of X, Y and joint C.D.F (X,Y) be represented as

$$F_X(a) = \Pr(X < a) \tag{0.0.9}$$

$$F_Y(a) = \Pr(Y < a) \tag{0.0.10}$$

$$F_{X,Y}(a, a) = \Pr(X < a, Y < a). \tag{0.0.11}$$

To show independence, we want to prove that,

$$F_X(a) F_Y(a) = F_{X,Y}(a, a) \forall a \in \mathbb{R} \tag{0.0.12}$$

Now, we know

$$\Pr(X > a|Y > a) = \frac{\Pr(X > a, Y > a)}{\Pr(Y > a)} \tag{0.0.13}$$

$$= \Pr(X > a) \tag{0.0.14}$$

$$\implies \Pr(X > a, Y > a) = \Pr(X > a) \Pr(Y > a) \tag{0.0.15}$$

We can write,

$$\Pr(X > a) = 1 - F_X \tag{0.0.16}$$

$$\Pr(Y > a) = 1 - F_Y \tag{0.0.17}$$

Now, we can write

$$F_Y = \Pr(X < a, Y < a) + \Pr(X > a, Y < a) \tag{0.0.18}$$

$$= F_{X,Y} + \Pr(X > a, Y < a) \tag{0.0.19}$$

$$\implies \Pr(X > a, Y < a) = F_Y - F_{X,Y} \quad (0.0.20)$$

Now, $\Pr(X > a) = \Pr(X > a, Y < a)$

$$+\Pr(X > a, Y > a) \quad (0.0.21)$$

From (0.0.16) and (0.0.20)

$$1 - F_X = F_Y - F_{X,Y} + \Pr(X > a, Y > a) \quad (0.0.22)$$

$$\implies \Pr(X > a, Y > a) = 1 - F_X - F_Y + F_{X,Y}. \quad (0.0.23)$$

From (0.0.16), (0.0.17) and (0.0.23)

$$1 - F_X - F_Y + F_{X,Y} = (1 - F_X) \times (1 - F_Y) \quad (0.0.24)$$

$$\implies F_{X,Y}(a, a) = F_X(a)F_Y(a). \quad (0.0.25)$$

Therefore, option 1 is correct.

2) Similar to option 1, C.D.F.s are represented as below,

$$F_X(a) = \Pr(X < a) \quad (0.0.26)$$

$$F_Y(b) = \Pr(Y < b) \quad (0.0.27)$$

$$F_{X,Y}(a, b) = \Pr(X < a, Y < b) \quad (0.0.28)$$

To show independence, we want to prove that,

$$F_X(a)F_Y(b) = F_{X,Y}(a, b) \forall a, b \in \mathbb{R} \quad (0.0.29)$$

From conditional probability we know:

$$\Pr(X > a | Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \quad (0.0.30)$$

and so using the given condition in option 2), we can write (0.0.29) as

$$\Pr(X > a) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \quad (0.0.31)$$

$$\Pr(X > a)\Pr(Y < b) = \Pr(X > a, Y < b) \quad (0.0.32)$$

We can write the C.D.F as

$$\Pr(X > a) = 1 - F_X(a) \quad (0.0.33)$$

$$\Pr(Y < b) = F_Y(b) \quad (0.0.34)$$

$$(0.0.35)$$

$$F_Y(b) = \Pr(X > a, Y < b) + \Pr(X < a, Y < b) \quad (0.0.36)$$

From (0.0.28)

$$\Pr(X > a, Y < b) = F_Y(b) - F_{X,Y}(a, b) \quad (0.0.37)$$

From (0.0.33), (0.0.34) and (0.0.37)

$$(1 - F_X(a))(F_Y(b)) = F_Y(b) - F_{X,Y}(a, b) \quad (0.0.38)$$

$$\implies F_X(a)F_Y(b) = F_{X,Y}(a, b) \quad (0.0.39)$$

Therefore, option 2 is correct.

3) Given random variables X and Y are uncorrelated which means that their correlation is 0, or, equivalently, Cov(X, Y) = 0.

$$Cov(X, Y) = E(XY) - E(X)E(Y) \quad (0.0.40)$$

$$[\because Cov(X, Y) = 0] \quad (0.0.41)$$

$$E(XY) = E(X)E(Y) \quad (0.0.42)$$

Trying to solve through a counter example, Let there be three random variables X, Y and Z such that,

$$Z \sim \mathcal{N}(0, 1) \quad (0.0.43)$$

$$X = Z \quad (0.0.44)$$

$$Y = Z^2 \quad (0.0.45)$$

They are clearly independent, now let's check if they are uncorrelated

$$E(XY) = E\left(Z^3\right) = 0 \quad (0.0.46)$$

$$E(X)E(Y) = E(Z)E\left(Z^2\right) = 0 \quad (0.0.47)$$

$$\implies E(XY) = E(X)E(Y) \quad (0.0.48)$$

Therefore, they are dependent and uncorrelated. Hence, option 3 is incorrect.

4) We extend L.H.S $E[(X - a)(Y - b)] = E[XY]$

$$-aE[Y] - bE[X] + ab \quad (0.0.49)$$

and R.H.S to compare, $E(X - a)E(Y - b) = E[X]E[Y]$

$$-aE[Y] - bE[X] + ab \quad (0.0.50)$$

From (0.0.49) and (0.0.50)

$$E[XY] = E[X]E[Y] \quad (0.0.51)$$

Which was already dealt in option 3, Hence option 4 is incorrect.