AI 1103 - Challenging Problem 11

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Download all latex codes from

https://github.com/rohanthota/Challenging_problem/main.tex

1 Problem

(UGC/MATH 2018 (June set-a)-Q.106) Let $X_{i i\geq 1}$ be a sequence of i.i.d. random variables with $E(X_i) = 0$ and $V(X_i) = 1$. Which of the following are true?

1) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 0$ in probability
2) $\frac{1}{n^{3/4}}\sum_{i=1}^{n} X_i \to 0$ in probability
3) $\frac{1}{n^{1/2}}\sum_{i=1}^{n} X_i \to 0$ in probability
4) $\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 1$ in probability

Solution

CONVERGENCE IN DISTRIBUTION :
A sequence of random variables Y, Y_1, $Y_2\dots$ converges in distribution to a random variable Y, shown by $Y_n \to Y$, if

$$\lim_{n\to\infty} F_{X_n}(a) = F_X(a)\ \forall a \in \mathbb{R}. \tag{1.0.1}$$

CONVERGENCE IN PROBABILITY:
A sequence of random variables Y, Y_1, $Y_2\dots$ is said to converge in probability to Y, if

$$\lim_{n\to\infty} \Pr\left(|Y_n - Y| > \epsilon\right) = 0\ \forall\epsilon > 0. \tag{1.0.2}$$

THEOREM : If $Y_n \to Y$ in probability, $Y_n \to Y$ in distribution.
CENTRAL LIMIT THEOREM(CLT):
Let X_1, X_2, $\dots X_n$ be i.i.d. random variables with expected value $E(X_i) = \mu < \infty$ and $0 < V(X_i) = \sigma^2 < \infty$. Then the random variable

$$Z_n = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} = \frac{X_1 + X_2 + \dots + X_n - n\mu}{\sqrt{n}\sigma} \tag{1.0.3}$$

converges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n\to\infty} \Pr\left(Z_n \leq a\right) = \Phi(a)\ \forall a \in \mathbb{R}. \tag{1.0.4}$$

where $\Phi(a)$ is the standard normal CDF.

3) The option states that $\frac{1}{n^{1/2}}\sum_{i=1}^{n} X_i \to 0$ in probability. This statement implies that $\frac{1}{n^{1/2}}\sum_{i=1}^{n} X_i \to 0$ in distribution, from the theorem mentioned above.
Writing the random variable Z_n from CLT, for the i.i.d random variables,

$$Z_n = \frac{X_1 + X_2 + \dots + X_n - n\mu}{\sqrt{n}\sigma} \tag{1.0.5}$$
$$= \frac{X_1 + X_2 + \dots + X_n}{\sqrt{n}} \tag{1.0.6}$$
$$= \frac{1}{n^{1/2}}\sum_{i=1}^{n} X_i \tag{1.0.7}$$

Since $\mu = 0$ and $\sigma = 1$.
According to Central limit theorem,

$$Z_n \to Z, \text{where, } Z \sim N(0,1) \tag{1.0.8}$$

which is not what the option states. Therefore, option 3 is incorrect.