AI 1103 - Challenging Problem 14

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Download all latex codes from

https://github.com/rohanthota/Challenging_problem/main.tex

Question

Which of the following conditions imply independence of the random variables X and Y ?

1) $\Pr(X > a|Y > a) = \Pr(X > a)\,\forall a \in \mathbb{R}$
2) $\Pr(X > a|Y < b) = \Pr(X > a)\,\forall a, b \in \mathbb{R}$
3) X and Y are uncorrelated.
4) $E[(X-a)(Y-b)] = E(X-a)\times E(Y-b)\forall a, b \in \mathbb{R}$

Solution

1) Let C.D.F's of X, Y and joint C.D.F (X,Y) be represented as

$$F_X(a) = \Pr(X < a) \tag{0.0.1}$$
$$F_Y(a) = \Pr(Y < a) \tag{0.0.2}$$
$$F_{X,Y}(a, a) = \Pr(X < a, Y < a). \tag{0.0.3}$$

To show independence, we want to prove that,

$$F_X(a)F_Y(a) = F_{X,Y}(a, a)\forall a \in \mathbb{R} \tag{0.0.4}$$

Now, we know

$$\Pr(X > a|Y > a) = \frac{\Pr(X > a, Y > a)}{\Pr(Y > a)} \tag{0.0.5}$$
$$= \Pr(X > a) \tag{0.0.6}$$
$$\implies \Pr(X > a, Y > a) = \Pr(X > a)\Pr(Y > a) \tag{0.0.7}$$

We can write,

$$\Pr(X > a) = 1 - F_X \tag{0.0.8}$$
$$\Pr(Y > a) = 1 - F_Y \tag{0.0.9}$$

Now, we can write

$$F_Y = \Pr(X < a, Y < a) + \Pr(X > a, Y < a) \tag{0.0.10}$$
$$= F_{X,Y} + \Pr(X > a, Y < a) \tag{0.0.11}$$
$$\implies \Pr(X > a, Y < a) = F_Y - F_{X,Y} \tag{0.0.12}$$

Now, $\Pr(X > a) = \Pr(X > a, Y < a)$

$$+ \Pr(X > a, Y > a) \tag{0.0.13}$$

From (0.0.8) and (0.0.12)

$$1 - F_X = F_Y - F_{X,Y} + \Pr(X > a, Y > a) \tag{0.0.14}$$
$$\implies \Pr(X > a, Y > a) = 1 - F_X - F_Y + F_{X,Y}. \tag{0.0.15}$$

From (0.0.8), (0.0.9) and (0.0.15)

$$1 - F_X - F_Y + F_{X,Y} = (1 - F_X) \times (1 - F_Y) \tag{0.0.16}$$
$$\implies F_{X,Y}(a, a) = F_X(a)F_Y(a). \tag{0.0.17}$$

Therefore, option 1 is correct.

2) Similar to option 1, C.D.F.s are represented as below,

$$F_X(a) = \Pr(X < a) \tag{0.0.18}$$
$$F_Y(b) = \Pr(Y < b) \tag{0.0.19}$$
$$F_{X,Y}(a, b) = \Pr(X < a, Y < b) \tag{0.0.20}$$

To show independence, we want to prove that,

$$F_X(a)F_Y(b) = F_{X,Y}(a, b)\forall a, b \in \mathbb{R} \tag{0.0.21}$$

From conditional probability we know:

$$\Pr(X > a|Y < b) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \tag{0.0.22}$$

and so using the given condition in option 2), we can write (0.0.21) as

$$\Pr(X > a) = \frac{\Pr(X > a, Y < b)}{\Pr(Y < b)} \tag{0.0.23}$$

$$\Pr(X > a)\Pr(Y < b) = \Pr(X > a, Y < b) \tag{0.0.24}$$

We can write the C.D.F as

$$\Pr(X > a) = 1 - F_X(a) \tag{0.0.25}$$

$$\Pr(Y < b) = F_Y(b) \tag{0.0.26}$$

$$\tag{0.0.27}$$

$$F_Y(b) = \Pr(X > a, Y < b) + \Pr(X < a, Y < b) \tag{0.0.28}$$

From (0.0.20)

$$\Pr(X > a, Y < b) = F_Y(b) - F_{X,Y}(a, b) \tag{0.0.29}$$

From (0.0.25), (0.0.26) and (0.0.29)

$$(1 - F_X(a))(F_Y(b)) = F_Y(b) - F_{X,Y}(a, b) \tag{0.0.30}$$

$$\implies F_X(a)F_Y(b) = F_{X,Y}(a, b) \tag{0.0.31}$$

Therefore, option 2 is correct.

3) Given random variables X and Y are uncorrelated which means that their correlation is 0, or, equivalently, Cov(X, Y) = 0.

$$Cov(X, Y) = E(XY) - E(X)E(Y) \tag{0.0.32}$$

$$[\because Cov(X, Y) = 0] \tag{0.0.33}$$

$$E(XY) = E(X)E(Y) \tag{0.0.34}$$

Trying to solve through a counter example, Let there be three random variables X, Y and Z such that,

$$Z \sim \mathcal{N}(0, 1) \tag{0.0.35}$$

$$X = Z \tag{0.0.36}$$

$$Y = Z^2 \tag{0.0.37}$$

They are clearly independent, now let's check if they are uncorrelated

$$E(XY) = E\left(Z^3\right) = 0 \tag{0.0.38}$$

$$E(X)E(Y) = E(Z)E\left(Z^2\right) = 0 \tag{0.0.39}$$

$$\implies E(XY) = E(X)E(Y) \tag{0.0.40}$$

Therefore, they are dependent and uncorrelated. Hence, option 3 is incorrect.

4) We extend L.H.S $E[(X - a)(Y - b)] = E[XY]$

$$-aE[Y] - bE[X] + ab \tag{0.0.41}$$

and R.H.S to compare, $E(X - a)E(Y - b) = E[X]E[Y]$

$$-aE[Y] - bE[X] + ab \tag{0.0.42}$$

From (0.0.41) and (0.0.42)

$$E[XY] = E[X]E[Y] \tag{0.0.43}$$

Which was already dealt in option 3, Hence option 4 is incorrect.