AI 1103 - Challenging Problem 14

T. Rohan
CS20BTECH11064

Download all latex codes from

https://github.com/rohanthota/Challenging_problem/main.tex

1 Problem

(UGC/MATH 2018 (June set-a)-Q.106) Let $X_{i i \geq 1}$ be a sequence of i.i.d. random variables with $E(X_i) = 0$ and $V(X_i) = 1$. Which of the following are true?

1) $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \to 0$ in probability
2) $\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i \to 0$ in probability
3) $\frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \to 0$ in probability
4) $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \to 1$ in probability

Solution

CONVERGENCE IN PROBABILITY :
Let Y, Y_1, Y_2, $\ldots$ be random variables. We say that the sequence $Y_n \to Y$, if

$$\lim_{n\to\infty} \Pr(Y_n \leq a) = \Pr(Y \leq a)\, \forall a \in \mathbb{R}. \tag{1.0.1}$$

CENTRAL LIMIT THEOREM(CLT):
Let $X_1, X_2, \ldots X_n$ be i.i.d. random variables with expected value $E(X_i) = \mu < \infty$ and $0 < V(X_i) = \sigma^2 < \infty$. Then the random variable

$$Z_n = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma} \tag{1.0.2}$$

converges in distribution to the standard normal random variable as n goes to infinity, that is

$$\lim_{n\to\infty} \Pr(Z_n \leq a) = \Phi(a)\ \forall a \in \mathbb{R}. \tag{1.0.3}$$

where $\Phi(a)$ is the standard normal CDF.

3) We need to check if $\frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \to 0$ in probability. Writing the random variable Z_n from CLT, for the i.i.d random variables,

$$Z_n = \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sqrt{n}\sigma} \tag{1.0.4}$$

$$= \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}} \tag{1.0.5}$$

$$= \frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \tag{1.0.6}$$

Since $\mu = 0$ and $\sigma = 1$.
According to Central limit theorem,

$$Z_n \to Z, \text{where, } Z \sim N(0,1) \tag{1.0.7}$$

which is not what the option states. Therefore, option 3 is incorrect.