

# AI 1103 - Challenging Problem 11

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[https://github.com/rohanthota/  
Challenging\\_problem/main.tex](https://github.com/rohanthota/Challenging_problem/main.tex)

## 1 PROBLEM

(UGC/MATH 2018 (June set-a)-Q.106) Let  $X_{i \geq 1}$  be a sequence of i.i.d. random variables with  $E(X_i) = 0$  and  $V(X_i) = 1$ . Which of the following are true?

- 1)  $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow 0$  in probability
- 2)  $\frac{1}{n^{3/4}} \sum_{i=1}^n X_i \rightarrow 0$  in probability
- 3)  $\frac{1}{n^{1/2}} \sum_{i=1}^n X_i \rightarrow 0$  in probability
- 4)  $\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow 1$  in probability

## 2 Solution

**Definition 1.** (Convergence in distribution)

A sequence of random variables  $Y, Y_1, Y_2 \dots$  converges in distribution to a random variable  $Y$ , if

$$\lim_{n \rightarrow \infty} F_{X_n}(a) = F_X(a) \quad \forall a \in \mathbb{R}. \quad (2.0.1)$$

**Definition 2.** (Convergence in probability)

A sequence of random variables  $Y, Y_1, Y_2 \dots$  is said to converge in probability to  $Y$ , if

$$\lim_{n \rightarrow \infty} \Pr(|Y_n - Y| > \epsilon) = 0 \quad \forall \epsilon > 0. \quad (2.0.2)$$

**Lemma 2.1.** If  $Y_n \rightarrow Y$  in probability,  $Y_n \rightarrow Y$  in distribution.

**Lemma 2.2.** (Strong Law of Large Numbers)

Let  $X_1, X_2, \dots, X_n$  be i.i.d. random variables with expected value  $E(X_i) = \mu < \infty$ , then,

$$\lim_{n \rightarrow \infty} \Pr\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| \geq \epsilon\right) = 0 \quad (2.0.3)$$

Or,  $\frac{1}{n} \sum_{i=1}^n X_i$  converges in probability to  $\mu$ .

**Lemma 2.3.** If  $X_i$  is a sequence of i.i.d. random variables, satisfying condition

$$F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x) = F_X(x) \quad (2.0.4)$$

then,

$$F_{X_1^2}(x) = F_{X_2^2}(x) = \dots = F_{X_n^2}(x) = F_{X^2}(x) \quad (2.0.5)$$

$\forall x \in \mathbb{R}$  where  $F_X(x)$  is the c.d.f. of  $X_i$ .

*Proof.*  $X_i$  is a sequence of i.i.d. random variables, which means it satisfies the following condition.

$$F_{X_1}(x) = F_{X_2}(x) = \dots = F_{X_n}(x) = F_X(x) \quad (2.0.6)$$

where  $F_X(x)$  is the c.d.f. of  $X_i$ .

Let  $Y_i = X_i^2$ . For  $y \geq 0$ ,

$$F_{Y_i}(y) = \Pr(Y_i \leq y) \quad (2.0.7)$$

$$\implies F_{Y_i}(y) = \Pr(X_i^2 \leq y) \quad (2.0.8)$$

$$\implies F_{Y_i}(y) = \Pr(-\sqrt{y} \leq X_i \leq \sqrt{y}) \quad (2.0.9)$$

$$\implies F_{Y_i}(y) = \Pr(X_i \leq \sqrt{y}) - \Pr(X_i \leq -\sqrt{y}) \quad (2.0.10)$$

$$\implies F_{Y_i}(y) = F_{X_i}(\sqrt{y}) - F_{X_i}(-\sqrt{y}) \quad (2.0.11)$$

Using (2.0.6),

$$F_{Y_i}(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \quad (2.0.12)$$

From (2.0.12),

$$F_{Y_1}(y) = F_{Y_2}(y) = \dots = F_{Y_n}(y) = F_Y(y) \quad (2.0.13)$$

where  $F_Y(y)$  is the c.d.f. of  $Y_i = X_i^2$ .  $\square$

**Lemma 2.4.** If  $X_i$  is a sequence of i.i.d. random variables, satisfying condition

$$F_{X_1, \dots, X_n}(x_1 \dots x_n) = F_X(x_1)F_X(x_2) \dots F_X(x_n) \quad (2.0.14)$$

where  $F_X(x)$  is the c.d.f. of  $X_i$ , then for  $Y_i = X_i^2$

$$F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) = F_Y(y_1)F_Y(y_2) \dots F_Y(y_n) \quad (2.0.15)$$

where  $F_Y(y)$  is the c.d.f. of  $Y_i = X_i^2$ .

*Proof.* Let  $Y_i = X_i^2$ . Now, for  $y_i \geq 0$ , consider  $F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n)$

$$= \Pr(Y_1 \leq y_1, Y_2 \leq y_2, \dots, Y_n \leq y_n) \quad (2.0.16)$$

$$= \Pr(X_1^2 \leq y_1, X_2^2 \leq y_2, \dots, X_n^2 \leq y_n) \quad (2.0.17)$$

$$= \Pr(-\sqrt{y_1} \leq X_1 \leq \sqrt{y_1}, \dots, -\sqrt{y_n} \leq X_n \leq \sqrt{y_n}) \quad (2.0.18)$$

Since  $X_1, X_2, \dots, X_n$  are independent,

$$\begin{aligned} F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) &= \\ \Pr(-\sqrt{y_1} \leq X_1 \leq \sqrt{y_1}) \Pr(-\sqrt{y_2} \leq X_2 \leq \sqrt{y_2}) \\ \dots \Pr(-\sqrt{y_n} \leq X_n \leq \sqrt{y_n}) \end{aligned} \quad (2.0.19)$$

From (2.0.9) and (2.0.13),

$$\begin{aligned} F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) &= \\ &= F_{Y_1}(y_1) F_{Y_2}(y_2) \dots F_{Y_n}(y_n) \end{aligned} \quad (2.0.20)$$

$$= F_Y(y_1) F_Y(y_2) \dots F_Y(y_n) \quad (2.0.21)$$

So,

$$\begin{aligned} F_{Y_1, Y_2, \dots, Y_n}(y_1, y_2, \dots, y_n) &= \\ &= F_Y(y_1) F_Y(y_1) \dots F_Y(y_n) \end{aligned} \quad (2.0.22)$$

□

**Lemma 2.5.** If  $X_i$  is a sequence of i.i.d. random variables, it follows that  $X_i^2$  is also a sequence of i.i.d. random variables.

*Proof.* From Lemma 2.3 and Lemma 2.4,  $X_i^2$  is also a sequence of i.i.d. random variables. □

**Lemma 2.6.** If  $X_1, X_2, \dots, X_n$  are independent random variables,

$$V\left(\sum_{i=1}^n X_i\right) = V(X_1) + V(X_2) + \dots + V(X_n) \quad (2.0.23)$$

**Lemma 2.7.** (Chebyshev's Inequality)

Let the random variable  $X$  have a finite mean  $\mu$  and a finite variance  $\sigma^2$ . For every  $\epsilon > 0$ ,

$$\Pr(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \quad (2.0.24)$$

**Lemma 2.8.** (Central Limit Theorem)

Let  $X_1, X_2, \dots, X_n$  be i.i.d. random variables with expected value  $E(X_i) = \mu < \infty$  and  $0 < V(X_i) = \sigma^2 < \infty$ . Then the random variable

$$Z_n = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma} \quad (2.0.25)$$

converges in distribution to the standard normal random variable as  $n$  goes to infinity, that is

$$\lim_{n \rightarrow \infty} \Pr(Z_n \leq a) = \Phi(a) \quad \forall a \in \mathbb{R}. \quad (2.0.26)$$

where  $\Phi(a)$  is the standard normal CDF.

1) From Lemma 2.5,  $\{X_i^2\}$  is a sequence of i.i.d. random variables. Now, we know,

$$E(X_i^2) = V(X_i) + (E(X_i))^2 \quad (2.0.27)$$

Putting given values, we get,

$$E(X_i^2) = 1 \quad (2.0.28)$$

From Lemma 2.2,

$$\frac{1}{n} \sum_{i=1}^n X_i^2 \rightarrow 1 \quad (2.0.29)$$

Therefore, option 1 is incorrect.

2)

**Lemma 2.9.** If  $\{X_i\}$  is a sequence of i.i.d. random variables with  $E(X_i) = 0$  and  $V(X_i) = 1$ , then the random variable  $Y_n = \frac{1}{n^{3/4}} \sum_{i=1}^n X_i$  has  $E(Y_n) = 0$  and  $V(Y_n) = \frac{1}{n^{1/2}}$

*Proof.*

$$E(Y_n) = \frac{1}{n^{3/4}} E\left(\sum_{i=1}^n X_i\right) = 0 \quad (2.0.30)$$

Since  $E(X_i) = 0$ . Now,

$$V(Y_n) = V\left(\frac{1}{n^{3/4}} \sum_{i=1}^n X_i\right) \quad (2.0.31)$$

$$V(Y_n) = \frac{1}{n^{3/2}} V\left(\sum_{i=1}^n X_i\right) \quad (2.0.32)$$

From Lemma 2.6, and since  $V(X_i) = 1$ ,

$$\Rightarrow V(Y_n) = \frac{1}{n^{3/2}} \times n = \frac{1}{n^{1/2}} \quad (2.0.33)$$

□

Now, from Lemma 2.7 and from Lemma 2.9

$$\lim_{n \rightarrow \infty} \Pr(|Y_n - 0| \geq \epsilon) \leq \lim_{n \rightarrow \infty} \frac{1}{n^{1/2} \epsilon^2} (= 0) \quad (2.0.34)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \Pr\left(\left|\frac{1}{n^{3/4}} \sum_{i=1}^n X_i - 0\right| \geq \epsilon\right) = 0 \quad (2.0.35)$$

From Definition 2,

$$\frac{1}{n^{3/4}} \sum_{i=1}^n X_i \rightarrow 0 \quad (2.0.36)$$

Thus, option 2 is correct.

- 3) Writing the random variable  $Z_n$  from Lemma 2.8 for  $\{X_i\}$  where  $\mu = 0$  and  $\sigma = 1$ ,

$$Z_n = \frac{1}{n^{1/2}} \sum_{i=1}^n X_i \quad (2.0.37)$$

From Lemma 2.8

$$\frac{1}{n^{1/2}} \sum_{i=1}^n X_i \rightarrow N(0, 1) \quad (2.0.38)$$

whereas the option states that

$$\frac{1}{n^{1/2}} \sum_{i=1}^n X_i \rightarrow 0 \quad (2.0.39)$$

from Lemma 2.1.

Therefore, option 3 is incorrect.

- 4) From (2.0.29), option 4 is correct.

Therefore, options 2 and 4 are correct.