UGC/MATH 2018 (June set-a)-Q.106

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Question

Let X_i, $i \geq 1$, be a sequence of i.i.d. random variables with $E(X_i) = 0$ and $V(X_i) = 1$. Which of the following are true?

1. $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \rightarrow 0$ in probability
2. $\frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i \rightarrow 0$ in probability
3. $\frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \rightarrow 0$ in probability
4. $\frac{1}{n} \sum_{i=1}^{n} X_i^2 \rightarrow 1$ in probability

Definitions

Definition

A sequence of random variables Y, Y_1, $Y_2 \ldots$ converges in distribution to a random variable Y, if

$$\lim_{n \to \infty} F_{Y_n}(a) = F_Y(a) \; \forall a \in \mathbb{R}. \tag{1}$$

Definition

A sequence of random variables Y, Y_1, $Y_2 \ldots$ is said to converge in probability to Y, if

$$\lim_{n \to \infty} \Pr\left(Y_n - Y > \epsilon\right) = 0 \; \forall \epsilon > 0. \tag{2}$$

Lemmas

Lemma

If $Y_n \to Y$ in probability, $Y_n \to Y$ in distribution.

Strong Law of Large Numbers

Let $X_1, X_2, \ldots X_n$ be i.i.d. random variables with expected value $E(X_i) = \mu < \infty$, then,

$$\lim_{n\to\infty} \Pr\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i - \mu\right| \geq \epsilon\right) = 0 \tag{3}$$

Or, $\frac{1}{n}\sum_{i=1}^{n} X_i$ converges in probability to μ.

Lemma

If X_i is a sequence of i.i.d. random variables, satisfying condition

$$F_{X_1}(x) = F_{X_2}(x) = \ldots = F_{X_n}(x) = F_X(x) \tag{4}$$

then,

$$F_{X_1^2}(x) = F_{X_2^2}(x) = \ldots = F_{X_n^2}(x) = F_{X^2}(x) \tag{5}$$

$\forall x \in \mathbb{R}$ *where $F_X(x)$ is the c.d.f. of X_i.*

Proof

X_i is a sequence of i.i.d. random variables, which means it satisfies the following condition.

$$F_{X_1}(x) = F_{X_2}(x) = \ldots = F_{X_n}(x) = F_X(x) \tag{6}$$

Proof contd.

Let $Y_i = X_i^2$. For $y \geq 0$,

$$F_{Y_i}(y) = \Pr\left(Y_i \leq y\right) = \Pr\left(X_i^2 \leq y\right) \tag{7}$$

$$\implies F_{Y_i}(y) = \Pr\left(-\sqrt{y} \leq X_i \leq \sqrt{y}\right) \tag{8}$$

$$\implies F_{Y_i}(y) = \Pr\left(X_i \leq \sqrt{y}\right) - \Pr\left(X_i \leq -\sqrt{y}\right) = F_{X_i}(\sqrt{y}) - F_{X_i}(-\sqrt{y}) \tag{9}$$

Using (6),

$$F_{Y_i}(y) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \tag{10}$$

From (10),

$$F_{Y_1}(y) = F_{Y_2}(y) = \ldots = F_{Y_n}(y) = F_Y(y) \tag{11}$$

where $F_Y(y)$ is the c.d.f. of $Y_i = X_i^2$.

Lemma

If X_i is a sequence of i.i.d. random variables, satisfying condition

$$F_{X_1,\dots X_n}(x_1 \dots x_n) = F_X(x_1)F_X(x_2)\dots F_X(x_n) \tag{12}$$

where $F_X(x)$ is the c.d.f. of X_i, then for $Y_i = X_i^2$

$$F_{Y_1,Y_2,\dots,Y_n}(y_1, y_2, \dots, y_n) = F_Y(y_1)F_Y(y_1)\dots F_Y(y_n) \tag{13}$$

where $F_Y(y)$ is the c.d.f. of $Y_i = X_i^2$.

Proof

Let $Y_i = X_i^2$. Now, for $y_i \geq 0$, consider
$F_{Y_1,Y_2,\dots,Y_n}(y_1, y_2, \dots, y_n) = \Pr(Y_1 \leq y_1, Y_2 \leq y_2, \dots, Y_n \leq y_n)$

$$= \Pr\left(X_1^2 \leq y_1, \dots, X_n^2 \leq y_n\right) \tag{14}$$
$$= \Pr\left(-\sqrt{y_1} \leq X_1 \leq \sqrt{y_1}, \dots, -\sqrt{y_n} \leq X_n \leq \sqrt{y_n}\right) \tag{15}$$

Proof contd.

Since $X_1, X_2, \ldots, X_n$ are independent,

$$F_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = \Pr\left(-\sqrt{y_1} \le X_1 \le \sqrt{y_1}\right)\Pr\left(-\sqrt{y_2} \le X_2 \le \sqrt{y_2}\right) \ldots \Pr\left(-\sqrt{y_n} \le X_n \le \sqrt{y_n}\right) \tag{16}$$

From (8) and (11),

$$F_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = F_{Y_1}(y_1)F_{Y_2}(y_2)\ldots F_{Y_n}(y_n) \tag{17}$$

$$= F_Y(y_1)F_Y(y_2)\ldots F_Y(y_n) \tag{18}$$

So,

$$F_{Y_1,Y_2,\ldots,Y_n}(y_1, y_2, \ldots, y_n) = F_Y(y_1)F_Y(y_1)\ldots F_Y(y_n) \tag{19}$$

Some other lemmas

Lemma

If X_i is a sequence of i.i.d. random variables, it follows that X_i^2 is also a sequence of i.i.d. random variables.

Proof.

From Lemma 4 and Lemma 5, X_i^2 is also a sequence of i.i.d. random variables. $\square$

Lemma

If $X_1, X_2, \ldots X_n$ are independent random variables,

$$V(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} V(X_i) \tag{20}$$

Chebyshev's Inequality

Let the random variable X have a finite mean μ and a finite variance σ^2. For every $\epsilon > 0$,

$$\Pr\left(X - \mu \geq \epsilon\right) \leq \frac{\sigma^2}{\epsilon^2} \tag{21}$$

Central Limit Theorem

Let $X_1, X_2, \ldots X_n$ be i.i.d.r.v with $E(X_i) = \mu$ and $V(X_i) = \sigma^2$. Then

$$Z_n = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \tag{22}$$

converges in distribution to standard normal random variable as n $\to \infty$

$$\lim_{n \to \infty} \Pr\left(Z_n \leq a\right) = \Phi(a) \ \forall a \in \mathbb{R}. \tag{23}$$

where $\Phi(a)$ is the standard normal CDF.

Option 1

From Lemma 6, $\{X_i^2\}$ is a sequence of i.i.d. random variables. Now, we know,

$$E(X_i^2) = V(X_i) + (E(X_i))^2 \tag{24}$$

Putting given values, we get,

$$E(X_i^2) = 1 \tag{25}$$

From Lemma 4,

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2 \to 1 \tag{26}$$

Therefore, option 1 is incorrect.

Option 2

Lemma

If $\{X_i\}$ *is a sequence of i.i.d. random variables with* $E(X_i) = 0$ *and* $V(X_i) = 1$, *then the random variable* $Y_n = \frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i$. *has* $E(Y_n) = 0$ *and* $V(Y_n) = \frac{1}{n^{1/2}}$

Proof

$$E(Y_n) = \frac{1}{n^{3/4}} E \sum_{i=1}^{n} X_i = 0 \tag{27}$$

Since $E(X_i) = 0$. Now, $V(Y_n) = V \frac{1}{n^{3/4}} \sum_{i=1}^{n} X_i = \frac{1}{n^{3/2}} V \sum_{i=1}^{n} X_i$

Proof contd.

From Lemma 7, and since $V(X_i) = 1$,

$$\implies V(Y_n) = \frac{1}{n^{3/2}} \times n = \frac{1}{n^{1/2}} \tag{28}$$

Now, from Lemma 10 and from Lemma 8

$$\lim_{n\to\infty} \Pr\left(|Y_n - 0| \geq \epsilon\right) \leq \lim_{n\to\infty} \frac{1}{n^{1/2}\epsilon^2} = 0 \tag{29}$$

$$\implies \lim_{n\to\infty} \Pr\left(\left|\frac{1}{n^{3/4}}\sum_{i=1}^{n} X_i - 0\right| \geq \epsilon\right) = 0 \tag{30}$$

From Definition 2,

$$\frac{1}{n^{3/4}}\sum_{i=1}^{n} X_i \to 0 \tag{31}$$

Thus, option 2 is correct.

Option 3

Writing the random variable Z_n from Lemma 11 for $\{X_i\}$ where $\mu = 0$ and $\sigma = 1$,

$$Z_n = \frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \tag{32}$$

From Lemma 11

$$\frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \to N(0, 1) \tag{33}$$

whereas the option states that

$$\frac{1}{n^{1/2}} \sum_{i=1}^{n} X_i \to 0 \tag{34}$$

from Lemma 3.

Therefore, option 3 is incorrect.

Option 4

From (26), option 4 is correct.
Therefore, options 2 and 4 are correct.