4. $\dot{[B]} = [B](\beta - \gamma) - \frac{\beta [B]^2}{N}$

Factorising by $\frac{1}{[B]^2}$

$\frac{1}{[B]^2} \dot{[B]} = (\beta - \gamma) \frac{1}{[B]} - \frac{\beta}{N}$ — (i)

let $y = \frac{1}{[B]}$

$\dot{y} = \frac{-1}{[B]^2} \dot{[B]}$ — (ii)

Subt (ii) in (i)

$-\dot{y} = (\beta - \gamma) y - \frac{\beta}{N}$

$\dot{y} = \frac{\beta}{N} - (\beta - \gamma) y$

$y = \frac{\beta}{N} - \frac{1}{N} (N\beta - N\gamma) y$

$\dot{y} = \frac{\beta}{N} - \frac{\beta}{N} \left(N - \frac{N\gamma}{\beta} \right) y$

$$\dot{y} = \frac{\beta}{N} - \frac{\beta}{N} B^* y \qquad \left[B^* = N - \frac{Nr}{\beta} \right]$$

$$\dot{y} = -\frac{\beta B^*}{N} y + \frac{\beta}{N}$$

let $\frac{\beta B^*}{N} = \lambda$ & $\frac{\beta}{N} = I$.

$$\dot{y} = -\lambda y + I.$$

Assuming the following

$$y(t) = u(t) \cdot v(t) \quad \text{—} \quad (ii)$$

Solution to the homogeneous equation

differentiating (ii)

$$\frac{dy}{dt} = u'(t) v(t) + u(t) v'(t)$$

From (iii), $u'(t) = -\lambda u$

$$\frac{dy}{dt} = -\lambda u(t) v(t) + u(t) v'(t)$$

$$\frac{dy}{dt} = -\lambda y(t) + u(t) v'(t) \quad \left[y = u(t) v(t) \right]$$

According to (iii)

$$I = u(t) v'(t)$$

$$v'(t) = \frac{I}{u(t)} \quad \text{—— (iv)}$$

$$u'(t) = -\lambda u$$

Integrating

$$\int \frac{du}{-\lambda u} = \int dt$$

$$\frac{-1}{\lambda} \log |u| + c = t \quad \text{—— (v)}$$

at $t = 0$, $u = u_0$

$$\frac{1}{-\lambda} \log |u| = t - \frac{1}{\lambda} \log |u_0|$$

$$\log |u| = -\lambda t + \log |u_0|$$

$$u = e^{-\lambda t} u_0 \quad \text{—— (vi)}$$

subt (vi) in (iv)

$$v'(t) = \frac{I e^{\lambda t}}{u_0}$$

Integrating wrt t

$$v(t) = \frac{I}{u_0 \lambda} e^{\lambda t} + C$$

From (ii)

$$y(t) = u_0 e^{-\lambda t}\left(\frac{I e^{\lambda t}}{u_0 \lambda} + C\right)$$

$$y(t) = \frac{I}{\lambda} + C e^{-\lambda t}$$

Resubstituting I & λ

$$y(t) = \frac{\cancel{\beta}}{\cancel{N}} \times \frac{\cancel{N}}{\cancel{\beta} \times B^*} + C e^{-\frac{\beta B^* t}{N}}$$

$$y(t) = \frac{1}{B^*} + C e^{-\frac{\beta B^* t}{N}}$$

Resubt $y = \frac{1}{[B]}$

$$\frac{1}{[B]} = 1 + B^* C e^{-\frac{\beta B^*}{N} t}$$

$$[B] = \frac{B^*}{1 + B^* C e^{-\frac{\beta B^*}{N} t}}$$

$$[B] = \frac{B^*}{1 + B^* C e^{-\frac{\cancel{\beta} \times \cancel{N} (\beta - \gamma) t}{\cancel{N} \times \cancel{\beta}}}}$$

$$[B] = \frac{B^*}{1 + B^* C e^{-(\beta - \gamma) t}} \quad \text{— (vii)}$$

at $t=0$ $[B] = B_0$ [given]

$$B_0 = \frac{B^*}{1 + B^* C}$$

$$(1 + B^* C) = \frac{B^*}{B_0}$$

$$C = \left(\frac{B^*}{B_0} - 1\right) \frac{1}{B^*}$$

Subt C in (VII)

$$[B] = \frac{B^*}{1 + B^* \frac{1}{B^*}\left(\frac{B^*}{B_0} - 1\right) e^{-(\beta - \gamma)t}}$$

$$\boxed{[B](t) = \frac{B^*}{1 + \left(\frac{B^*}{B_0} - 1\right) e^{-(\beta - \gamma)t}}}$$