Dynamical Systems Model of Images

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Dynamical System Model of Medical Imaging

Controller: Linear Shift Invariant System

$p_t(x)$

input

Control
LSI: Kernel K

$v_t(x) = \int k(x - \phi_t(y)) p_t(y) dy$

output

Dynamical System Constraint

$v_t(x), t \in [0, T]$

input

Non-Linear
Dynamical
System

$$\phi_t(x) = x + \int_0^t v_s \circ \phi_s(x) ds$$

output

Image Transformation (large):

$\phi_t(x)$

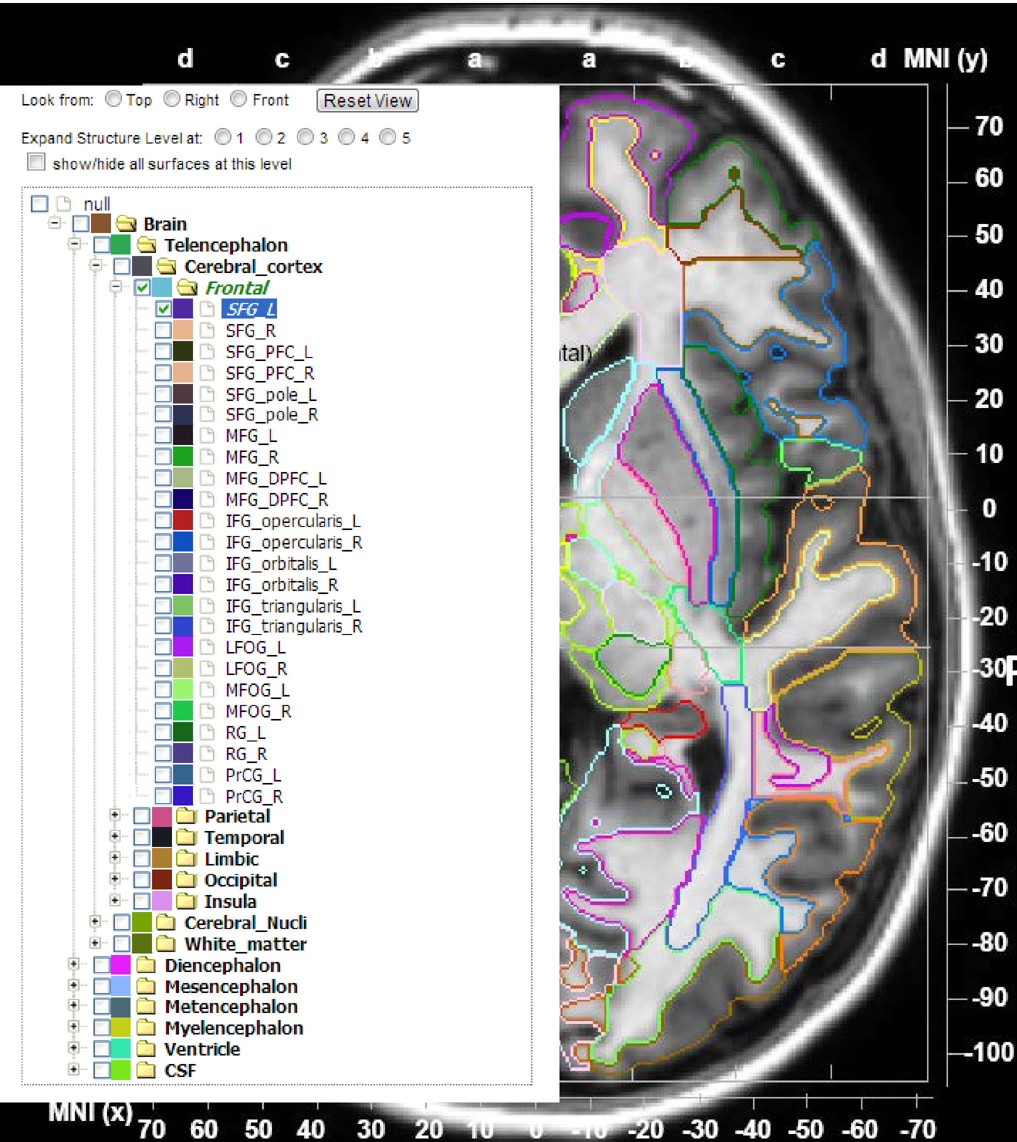
input

Non-Linear
Observer Equation

$I_{temp} \circ \phi_t^{-1}(x), x \in X$

output

High Throughput Neuroinformatics



Miller, Faria, Oishi, Mori, High Throughput Neuroinformatics”. Frontiers Neuroinfromatics,,

Registering Coordinate Systems via Large Deformation Image Matching (LDDMM)

https://en.wikipedia.org/wiki/Large_deformation_diffeomorphic_metric_mapping

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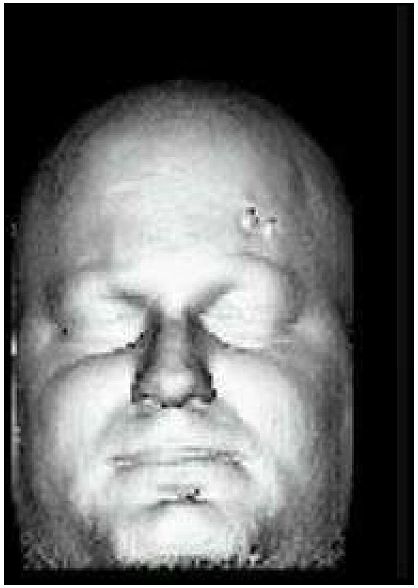
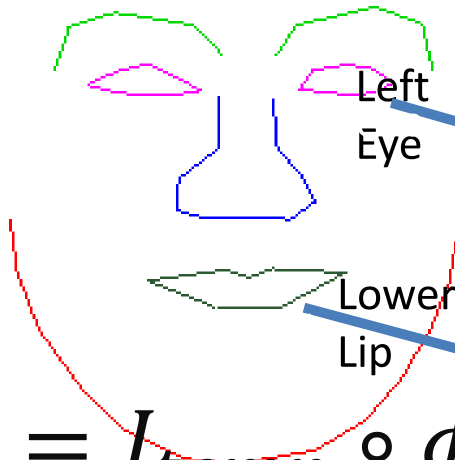
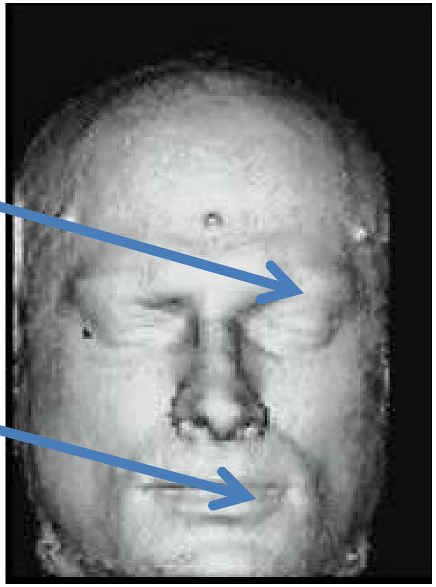
Anatomy is an element in the orbit; the flow positions the information.

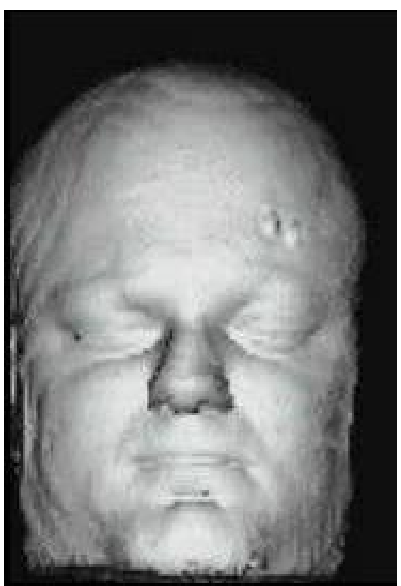


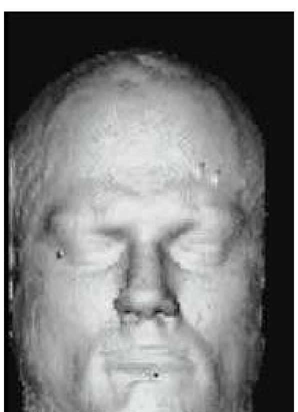


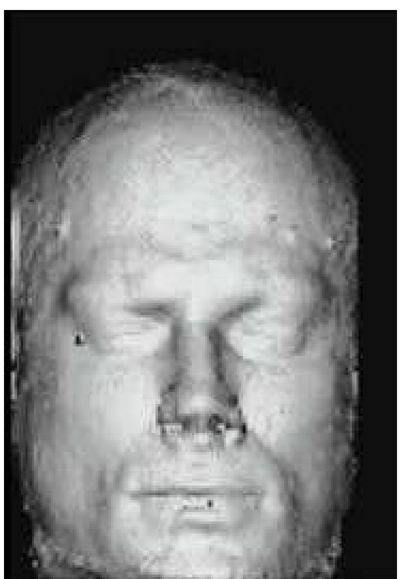


$$I = I_{temp} \circ \phi_1^{-1}$$

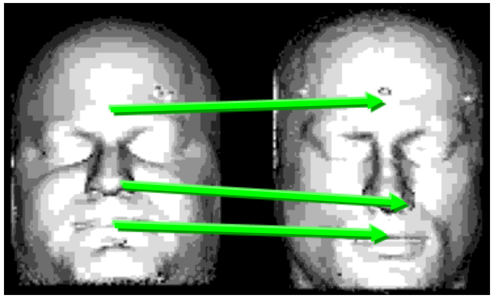
$$\phi_1(x) = x + \int_0^1 v_s \circ \phi_s(x) ds$$







Large Deformation Image Matching



- **Problem:**

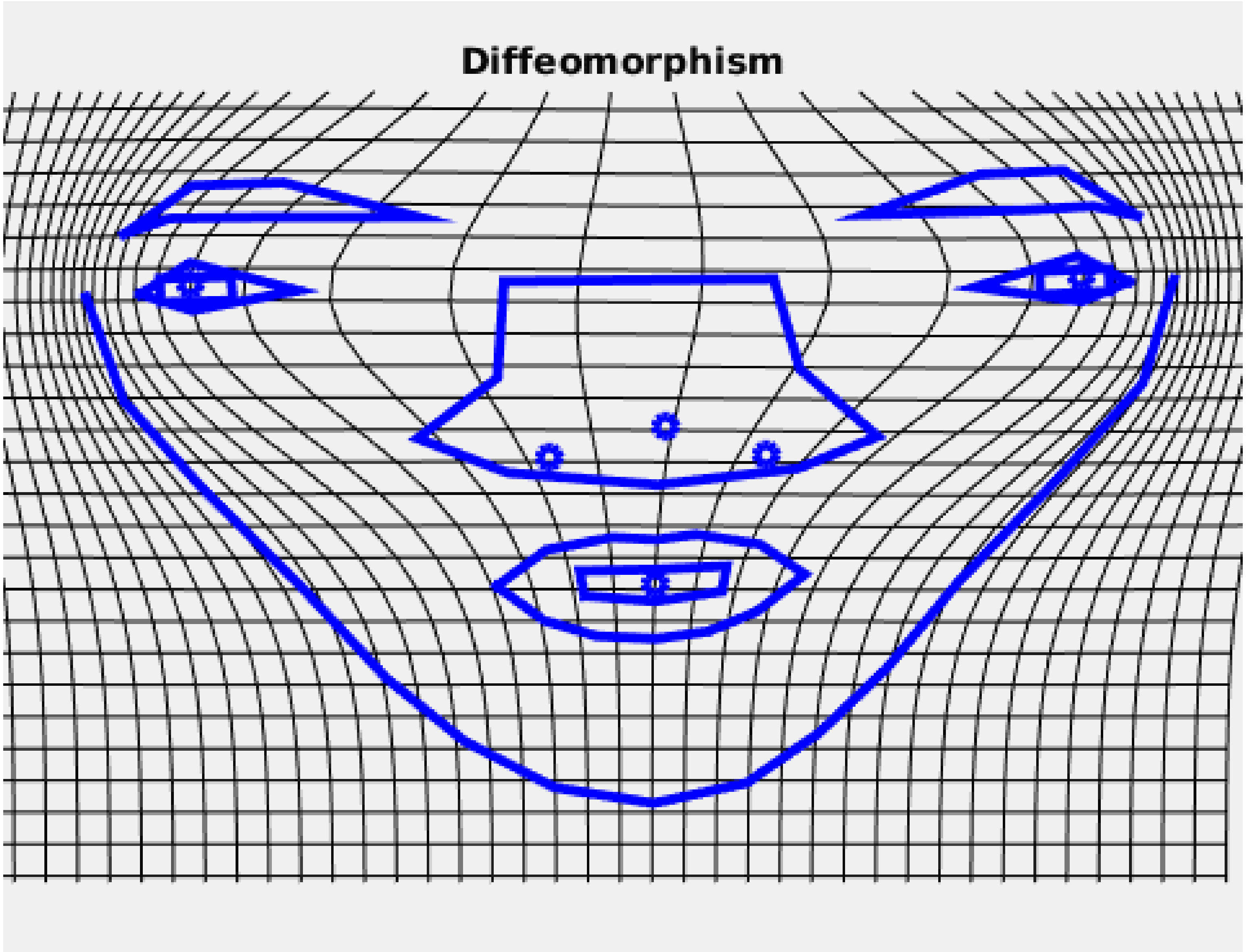
Estimate $\phi_1(x) = x + \int_0^1 v_s \circ \phi_s(x) ds$, $x \in \mathbb{R}^2$

Constraint $I' \simeq I \circ \phi_1^{-1}$

$$\min_{\phi_1} \int_X (I'(x) - I \circ \phi_1^{-1}\,(x))^2 dx$$

The computational algorithm for the GPS is called
Large Deformation Diffeomorphic Metric Mapping.

https://en.wikipedia.org/wiki/Large_deformation_diffeomorphic_metric_mapping



There are many possible flows.

We use Hamilton’s Principle of Least Action.

https://en.wikipedia.org/wiki/Computational_anatomy#The_action_integral_for_Hamilton.27s_principle_on_diffeomorphic_flows

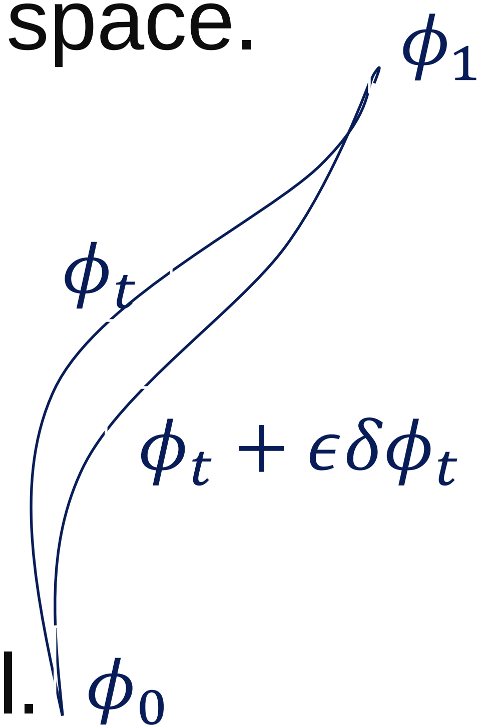
Hamilton's Principle of Least Action & the Lagrangian

- Coordinates ϕ_t of system at time t. Temporal evolution is a curve in configuration space.

- Lagrangian $L\left(\phi_t, \dot{\phi}_t\right)$ = K.E.-P.E., action integral $J = \int_0^1 L(\phi_t, \dot{\phi}_t)dt$

- PRINCIPLE: true evolution of system is stationary for action integral.

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} J(\epsilon) = 0$$



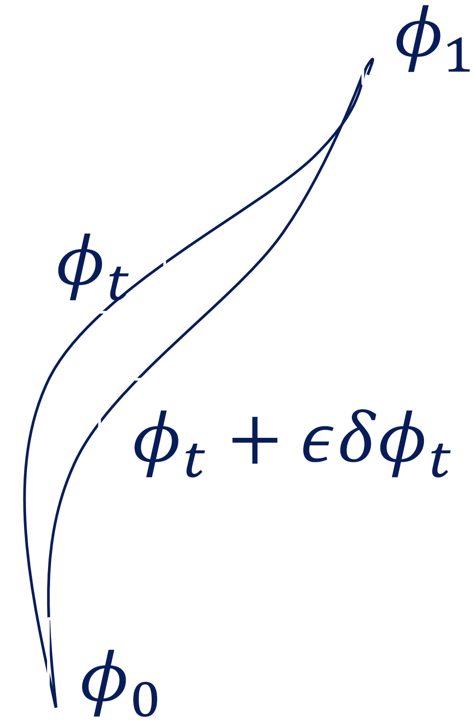
Euler-Lagrange Equation for Hamilton's Principle

- $J(\phi) = \int_0^1 \underbrace{\int_X L\left(\phi_t(x), \dot{\phi}_t(x)\right) dx}_{Lagrangian\ L\left(\phi_t, \dot{\phi}_t\right)} dt$

- Hamiltonian Minimizer Satisfies

$$\phi^\epsilon = \phi + \epsilon\delta\phi, \delta\phi_0 = \delta\phi_1 = 0$$

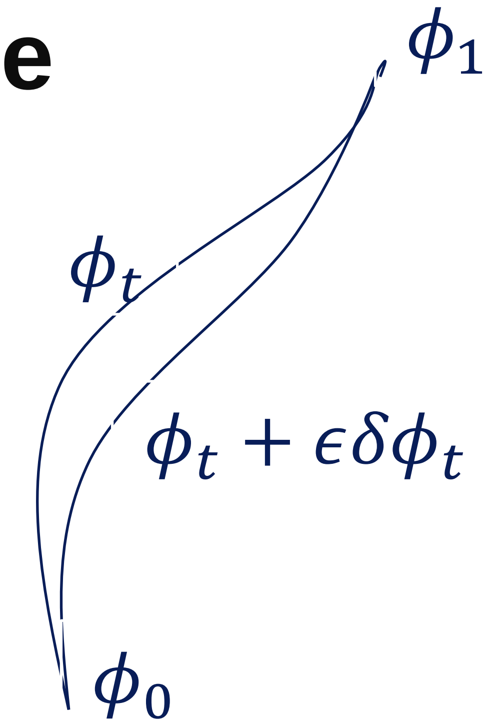
$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} J(\phi^\epsilon) = 0$$



- Euler-Lagrange Equation

$$-\frac{d}{dt}\frac{\partial L\left(\phi_t(x), \dot{\phi}_t(x)\right)}{\partial \dot{\phi}(x)} + \frac{\partial L(\phi_t(x), \dot{\phi}_t(x))}{\partial \phi(x)} = 0$$

Hamiltonian Momentum from Hamilton's Principle



- Lagrangian $\quad L(\phi_t, \dot{\phi}_t)$

- Hamiltonian Momentum

$$p_t(x) \doteq \frac{\partial L(\phi_t(x), \dot{\phi}_t(x))}{\partial \dot{\phi}(x)}$$

- Euler-Lagrange Equation

$$-\frac{d}{dt} p_t(x) + \frac{\partial L(\phi_t(x), \dot{\phi}_t(x))}{\partial \phi(x)} = 0$$

Euler-Lagrange proof

- $\phi^{\epsilon} = \phi + \epsilon\delta\phi, \dot{\phi}^{\epsilon} = \dot{\phi} + \epsilon\frac{d}{dt}\delta\phi,\ \delta\phi(0) = \delta\phi(1) = 0$

$$\frac{d}{d\epsilon}J(\epsilon)\Big|_{\epsilon=0} = \int_0^1\int_X \frac{d}{d\epsilon}L\left(\phi_t^{\epsilon}(x), \dot{\phi}_t^{\epsilon}(x)\right)\Big|_{\epsilon=0} dxdt$$

$$= \int_0^1\int_X \left(\frac{\partial L\left(\phi_t^{\epsilon}, \dot{\phi}_t^{\epsilon}\right)}{\partial\phi}\cdot\frac{d\phi_t^{\epsilon}}{d\epsilon} + \frac{\partial L\left(\phi_t^{\epsilon}, \dot{\phi}_t^{\epsilon}\right)}{\partial\dot{\phi}}\cdot\frac{d\dot{\phi}_t^{\epsilon}}{d\epsilon}\right)\Big|_{\epsilon=0} dx\, dt$$

$$= \int_0^1\int_X \left(\frac{\partial L\left(\phi_t, \dot{\phi}_t\right)}{\partial\phi}\cdot\delta\phi_t + \frac{\partial L\left(\phi_t, \dot{\phi}_t\right)}{\partial\dot{\phi}}\cdot\frac{d}{dt}\delta\phi_t\right) dx\, dt$$

$$= \int_0^1\int_X \underbrace{\left(\frac{\partial L(\phi_t, \dot{\phi}_t)}{\partial\phi} - \frac{d}{dt}\frac{\partial L(\phi_t, \dot{\phi}_t)}{\partial\dot{\phi}}\right)}_{=0\ \ Euler-Lagrange}\cdot\,\delta\phi_t\, dxdt$$

Large Deformation Geodesic Flows as Least Action

Euler-Lagrange Large Deformations

https://en.wikipedia.org/wiki/Computational_anatomy#Landmark_or_pointset_geodesics
https://en.wikipedia.org/wiki/Computational_anatomy#Surface_geodesics
https://en.wikipedia.org/wiki/Computational_anatomy#Volume_geodesics

Velocity: $\dot{\phi}_t = v_t \circ \phi_t$

Action Integral: $\int_0^1 L(\phi_t, \dot{\phi}_t) dt$

Lagrangian: $L(\phi_t, \dot{\phi}_t) = \frac{1}{2} \int_X A(\dot{\phi}_t \circ \phi_t^{-1}(x)) \cdot \dot{\phi}_t \circ \phi_t^{-1}(x) dx$

Canonical Momentum

$$p_t(x) = \frac{\partial L(\phi_t(x), \dot{\phi}_t(x))}{\partial \dot{\phi}} = A v_t \circ \phi_t(x) \, |\partial_X \phi_t(x)|$$

- Euler-Lagrange and Geodesic vector fields

$$\dot{p}_t(x) = - \left. (\partial_X v_t) \right|_{\phi_t(x)}^T p_t, \quad p_0 = A v_0$$

$$v_t(x) = \int_X k(x - \phi_t(y)) p_t(y) dy$$

Geodesic Dynamical Systems Model of Medical Imagery

Geodesic Momentum Dynamics
with Linear Controller $\dot{p}_t(x) = (\partial_X v_t)\Big|^T_{\phi_t(x)} p_t(x),\ p_0\ i.c.$

$p_t(x)$ input $\rightarrow$ LSI Kernel k $\rightarrow$ output $v_t(x) = \int k\big(x - \phi_t(y)\big)p_t(y)dy$

Eulerian Dynamics

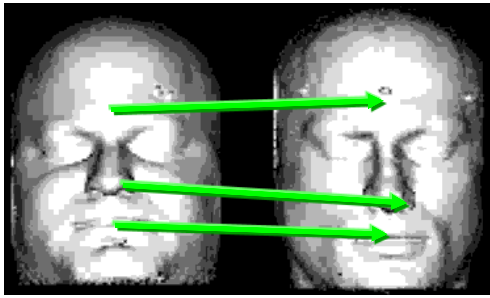
$v_t(x)$ input $\rightarrow$ Non-Linear Dynamics $\rightarrow$ output $\phi_t(x) = x + \int_0^t v_s \circ \phi_s(x)ds$

Image Transformation (large):

$\phi_t(x)$ input $\rightarrow$ Non-Linear Observer Equation $\rightarrow$ output $I_{temp} \circ \phi_t^{-1}(x), x \in X$

LDDMM Image Matching

Estimate $\phi_1(x) = x + \int_0^1 v_s \circ \phi_s(x) ds$, $x \in \mathbb{R}^2$



Constraint $I' \simeq I \circ \phi_1^{-1}$

$$E(\phi_1) = \left\| I' - I \circ \phi_1^{-1} \right\|^2$$

$$= \int_X (I'(x) - I \circ \phi_1^{-1}(x))^2 dx$$

- **Variational Problem:**

$$\min \int_0^1 L(\phi_t, \dot{\phi}_t) dt + E(\phi_1)$$

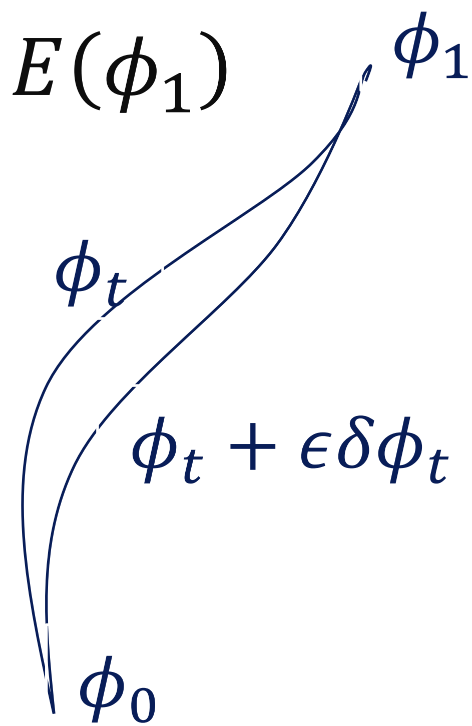
LDDMM Image Matching

https://en.wikipedia.org/wiki/Large_deformation_diffeomorphic_metric_mapping

- $J(\phi) = \int_0^1 \int_X L\big(\phi_t(x), \dot{\phi}_t(x)\big) dx\, dt + E(\phi_1)$

$$\phi^\epsilon = \phi + \epsilon\delta\phi, \delta\phi_0 = 0, \delta\phi_1 = free$$

$$\frac{d}{d\epsilon}\Big|_{\epsilon=0} J(\phi^\epsilon) = 0$$



- $p_t(x) = \frac{\partial L\big(\phi_t(x), \dot{\phi}_t(x)\big)}{\partial \dot{\phi}(x)}$

$$-\frac{d}{dt} p_t(x) + \frac{\partial L\big(\phi_t(x), \dot{\phi}_t(x)\big)}{\partial \phi(x)} = 0$$

$$p_1(x) + \frac{\partial E(\phi_1(x))}{\partial \phi_1} = 0$$

Euler-Lagrange proof

- $J = \int_0^1 \int_X L\big(\phi_t(x), \dot{\phi}_t(x)\big) dxdt + \frac{1}{2}\int_X (I \circ \phi_1^{-1} - I')^2 dx$
- $\phi_t^\epsilon = \phi_t + \epsilon\delta\phi_t, t \in [0,1),\ \delta\phi(0) = 0$
- $\phi_1^\epsilon = \phi_1 + \epsilon\delta\phi_1$
- $\frac{d}{d\epsilon}J(\epsilon)|_{\epsilon=0} = \int_0^1 \int_X \frac{d}{d\epsilon} L\big(\phi_t^\epsilon(x), \dot{\phi}_t^\epsilon(x)\big)|_{\epsilon=0} dxdt$

$$+ \frac{1}{2}\int_X \frac{d}{d\epsilon}(I \circ \phi_1^{\epsilon-1} - I')^2 dx\,|_{\epsilon=0}$$

$$= \int_0^1 \int_X \left(\frac{\partial L\big(\phi_t, \dot{\phi}_t\big)}{\partial \phi} - \frac{d}{dt}\frac{\partial L\big(\phi_t, \dot{\phi}_t\big)}{\partial \dot{\phi}} \right) \cdot \delta\phi_t\, dxdt$$

$$+ \int_X \frac{\partial L(\phi_1, \dot{\phi}_1)}{\partial \dot{\phi}} \cdot \delta\phi_1\, dx$$

$$+ \int_X (I \circ \phi_1^{-1} - I')\nabla I\,\Big|_{\phi_1^{-1}} \frac{d}{d\epsilon}\phi_1^{\epsilon-1}\,\Big|_{\epsilon=0}$$

Daniel Tward Examples

Euler-Lagrange of Lumped Mass-Spring Mechanical System

Position ϕ_t $\mathrm{P.E.} = \frac{1}{2}k\phi_t^2$ Velocity $v_t = \dot{\phi}_t$ $\mathrm{K.E.} = \frac{1}{2}m\dot{\phi}_t^2$

Action Integral: $J = \int \underbrace{(K.E. - P.E)}_{\text{Lagrangian } L(\phi, \dot{\phi}_t)} dt$

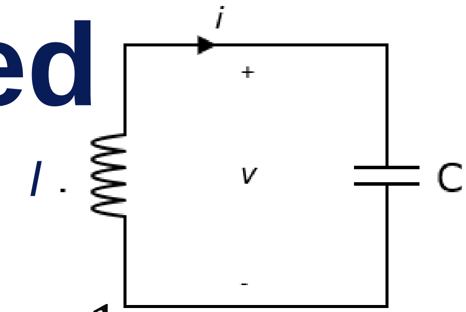
Lagrangian: $L(\phi_t, \dot{\phi}_t) = \frac{1}{2}m\dot{\phi}_t^2 - \frac{1}{2}k\phi_t^2$

$$-\frac{d}{dt}\frac{\partial L(\phi_t, \dot{\phi}_t)}{\partial \dot{\phi}} + \frac{\partial L(\phi_t, \dot{\phi}_t)}{\partial \phi} = -m\ddot{\phi}_t - k\phi_t = 0$$

Equation of Motion:

$$m\ddot{\phi}_t + k\phi_t = 0, \phi_t = \Phi_{init}\cos(\sqrt{k/m}\, t + \theta)$$

Euler-Lagrange of Lumped L-C Electrical System



Charge q_t P.E. $= \frac{1}{2c} q_t^2$ Current $i_t = \dot{q}_t$ K.E. $= \frac{1}{2} l\, \dot{q}_t^2$

Action Integral: $$J = \int \underbrace{(K.E. - P.E)}_{\text{Lagrangian } L(q, \dot{q}_t)} dt$$

Lagrangian: $$L(q_t, \dot{q}_t) = \frac{1}{2} l \dot{q}_t^2 - \frac{1}{2c} q_t^2$$

$$-\frac{d}{dt} \frac{\partial L(q_t, \dot{q}_t)}{\partial \dot{q}} + \frac{\partial L(q_t, \dot{q}_t)}{\partial q} = -l \ddot{q}_t - \frac{1}{c} q_t = 0$$

Kirchoff’s Voltage Law of Motion:

$$l \ddot{q}_t + \frac{1}{c} q_t = 0, q_t = Q_{init} \cos(\sqrt{1/lc}\, t + \theta)$$

Euler-Lagrange Cannonball

- Position-Velocity: $\phi_t = \begin{pmatrix} \phi_{tx} \\ \phi_{tz} \end{pmatrix}, \dot{\phi}_t = \begin{pmatrix} \dot{\phi}_{tx} \\ \dot{\phi}_{tz} \end{pmatrix} = v_t$

$$\phi_{0x} = \phi_{0z} = \dot{\phi}_{1z} = 0, \phi_{1x} = 1$$

- Lagrangian: Kinetic minus Potential

$$L(\phi_t, \dot{\phi}_t) = \frac{1}{2} m \, \dot{\phi}_t \cdot \dot{\phi}_t - mg \, \phi_{tz}$$

- Canonical Momentum:

$$p_t = \frac{\partial L(\phi_t, \dot{\phi}_t)}{\partial \dot{\phi}} = m \, \dot{\phi}_t$$

- Euler-Lagrange Equation

$$\dot{p}_{tx} = 0, \qquad \dot{p}_{tz} = -mg$$

- Geodesic Solution $\phi_{tx} = t, \quad \phi_{tz} = -\frac{1}{2} g t^2 + \frac{1}{2} g t$

Cannonball cont'd

- Given Least Action Euler-Lagrange Equation,

$$\dot{p}_{tx} = 0, \qquad \dot{p}_{tz} = -mg, \quad p_t = m\,\dot{\phi}_t$$

with B.C.'s $\phi_{0x} = \phi_{0z} = \phi_{1z} = 0, \phi_{1x} = 1$.

Prove geodesic: $\phi_{tx} = t, \quad \phi_{tz} = -\frac{1}{2}gt^2 + \frac{1}{2}gt$

$$\dot{p}_{tx} = m\ddot{\phi}_{tx} = 0 \Rightarrow \phi_{tx} = at + b \Rightarrow b = \phi_{00} = 0$$

$$\phi_{1x} = 1 \Rightarrow a = 1 \Rightarrow \phi_{tx} = t$$

$$\dot{p}_{tz} = -mg \Rightarrow m\ddot{\phi}_{tz} = -mg \Rightarrow \dot{\phi}_{tz} = -gt + c$$

$$\Rightarrow \phi_{tz} = -\frac{1}{2}gt^2 + ct + d$$

$$\phi_{0z} = 0 \Rightarrow d = 0, \phi_{1z} = 0 \Rightarrow c = \frac{1}{2}g$$