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Question 1

Optimization Calculation

Nomenclature

$$P_i = Price\ of\ bond\ i$$

$$x_i = Number\ of\ bonds\ i$$

$$F_i = Face\ value\ of\ bond\ i$$

$$c_i = Coupon\ payment\ of\ bond\ i$$

$$z_i = Excess\ cash\ in\ year\ i$$

$$L_i = Liability\ in\ year\ i$$

Minimize Equation:

$$P_1x_1 + P_2x_2 + P_3x_3 + P_4x_4 + P_5x_5 + P_6x_6 + P_7x_7 + P_8x_8 + P_9x_9 + P_{10}x_{10} + P_{11}x_{11} + P_{12}x_{12} + P_{13}x_{13} + P_{14}x_{14} + P_{15}x_{15} + P_{16}x_{16}$$

Subject To:

$$(F_1 + c_1)x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_8x_8 + c_9x_9 + c_{10}x_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} - z_1 = L_1$$

$$(F_2 + c_2)x_2 + (F_2 + c_2)x_3 + c_4x_4 + c_8x_8 + c_9x_9 + c_{10}x_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} + (1.02)z_1 - z_2 = L_2$$

$$(F_4 + c_4)x_4 + (F_5)x_5 + c_8x_8 + c_9x_9 + c_{10}x_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} + (1.02)z_2 - z_3 = L_3$$

$$(F_6)x_6 + c_8x_8 + c_9x_9 + c_{10}x_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} + (1.02)z_3 - z_4 = L_4$$

$$(F_7)x_7 + (F_8 + c_8)x_8 + c_9x_9 + c_{10}x_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} + (1.02)z_4 - z_5 = L_5$$

$$(F_9 + c_9)x_9 + (F_{10} + c_{10})x_{10} + c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} + (1.02)z_5 - z_6 = L_6$$

$$(F_{11} + c_{11})x_{11} + (F_{12} + c_{12})x_{12} + c_{13}x_{13} + c_{14}x_{14} + c_{15}x_{15} + (1.02)z_6 - z_7 = L_7$$

$$(F_{12} + c_{12})x_{12} + (F_{13} + c_{13})x_{13} + c_{15}x_{15} + (1.02)z_7 - z_8 = L_8$$

$$(F_{15} + c_{15})x_{15} + (F_{16})x_{16} + (1.02)z_9 = L_9$$

$$x_n \geq 0\ n = 1, \ldots, 16$$

$$z_n \geq 0\ z = 1, \ldots, 9$$

Results

Table 1 - Summary of Results Representing Number of Bonds to Buy to Meet Liabilities.

Bond Number	Number of units to buy (raw number)	Number of units to buy
1 (x_1)	147521.250613925	147522
2 (x_2)	175819.320962204	175820
3 (x_3)	2.24109781532533e-06	0
4 (x_4)	7.93877611560134e-06	0
5 (x_5)	204170.738707665	204171
6 (x_6)	204170.738691806	204171
7 (x_7)	4.88213955651684e-07	0
8 (x_8)	184170.738707680	184171
9 (x_9)	224760.556176796	224761
10 (x_{10})	4.86881365890923e-06	0
11 (x_{11})	270212.844413869	270213
12 (x_{12})	6.82249421579684e-06	0
13 (x_{13})	1.09071407927708e-06	0
14 (x_{14})	298114.445359939	298115
15 (x_{15})	322274.881508539	322275
16 (x_{16})	6.52174533133376e-06	0

Table 2 - Approximate Total Cost of Portfolio

Total Cost ($)	204673367.57

Question 2

Part A:

Nomenclature

$$r_i = Return\ of\ assest\ i$$

$$\mu_i = Expected\ return\ of\ asset\ i$$

$$\sigma_i = Standard\ deviation\ of\ asset\ i$$

$$\sigma_{ij} = Corvariance\ between\ asset\ i\ and\ j$$

Equations Used

$$r_{iT} = \frac{(end\ of\ month\ adj\ price - beginning\ of\ month\ adj\ price)}{beginning\ of\ month\ adj\ price}\quad i = asset, T = month\ (1, \dots, 24)$$

$$\bar{r_i} = \frac{1}{24}\sum_{T=1}^{24} r_{iT}$$

$$\mu_i = \left(\prod_{T=1}^{24}(1 + r_{iT})\right)^{\frac{1}{24}} - 1$$

$$\sigma_{ij} = \frac{1}{24}\sum_{T=1}^{24}(r_{iT} - \bar{r}_i)(r_{jT} - \bar{r}_j)$$

$$\sigma_i = \sqrt{\sigma_{ii}}$$

Results

Table 3 - Computed Expected Returns

ETF	Value
SPY (μ_1)	0.00361373099239737
GOVT (μ_2)	0.001219991841418
EEMV (μ_3)	-0.00401165508692958

Table 4 - Computed Standard Deviations

ETF	Value
SPY (σ_1)	0.030122977518631
GOVT (σ_2)	0.011788466120886
EEMV (σ_3)	0.033383166756711

Table 5 - Computed Covariances

σ_{12}	-0.000122444785495764
σ_{13}	0.000500223863055122
σ_{23}	0.000107725243220972

Part B

Nomenclature

$$x_i = Weight\ of\ asset\ i\ within\ portfolio$$

$$R = expected\ return\ goal$$

Optimization Calculations

Minimize

$$\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \sigma_{ij}$$

Subject To

$$\sum_{i=1}^{3} \boldsymbol{x_i \mu_i} = R$$

$$\sum_{i=1}^{3} \boldsymbol{x_i} = 1$$

Results

Table 6 - MVO with Short Selling Results

Expected Return Goal $(\boldsymbol{R})$	**SPY Weight (x_1)**	**GOVT Weight $(\boldsymbol{x_2})$**	**EEMV Weight $(\boldsymbol{x_3})$**	**Variance $(\boldsymbol{\sigma})$**
0.0005	0.2749	0.8681	-0.1430	0.005982
0.001	0.2749	0.8681	-0.1431	0.005982
0.0015	0.2749	0.8681	-0.1430	0.005982
0.002	0.2749	0.8681	-0.1431	0.005982
0.0025	0.2749	0.8682	-0.1432	0.005982
0.003	0.3132	0.8838	-0.1971	0.006074
0.0035	0.3643	0.9047	-0.2691	0.006466
0.004	0.4155	0.9257	-0.3412	0.007121
0.0045	0.4667	0.9466	-0.4133	0.007972
0.005	0.5179	0.9675	-0.4855	0.008964
0.0055	0.5691	0.9885	-0.5576	0.010055
0.006	0.6203	1.0094	-0.6298	0.011217
0.0065	0.6715	1.0303	-0.7019	0.012429
0.007	0.7227	1.0513	-0.7741	0.013679
0.0075	0.7739	1.0722	-0.8462	0.014957
0.008	0.8251	1.0932	-0.9183	0.016255
0.0085	0.8763	1.1141	-0.9905	0.017571
0.009	0.9275	1.1350	-1.0626	0.018899
0.0095	0.9788	1.1560	-1.1348	0.020239
0.01	1.0300	1.1769	-1.2069	0.021586

Table 7 - MVO without Short Selling Results

Expected Return Goal (R)	SPY Weight (x_1)	GOVT Weight (x_2)	EEMV Weight (x_3)	Variance (σ)
0.00025	0.2024	0.7975	1.0077e-06	0.006559
0.0005	0.2024	0.7975	1.6177e-05	0.006559
0.00075	0.2023	0.7974	0.0001056	0.006559
0.001	0.2024	0.7975	1.8513e-07	0.006559
0.00125	0.2024	0.7975	1.4158e-06	0.006559
0.0015	0.2024	0.7975	1.4475e-05	0.006559
0.00175	0.2215	0.7784	5.8054e-07	0.006577
0.002	0.3258	0.6741	1.3183e-08	0.007270
0.00225	0.4303	0.5696	2.8023e-07	0.008748
0.0025	0.5347	0.4652	1.7066e-08	0.010691
0.00275	0.6391	0.3608	9.0388e-13	0.012890
0.003	0.7436	0.2563	1.4129e-07	0.015234
0.00325	0.8480	0.1519	1.0125e-09	0.017666
0.0035	0.9525	0.0474	5.1673e-06	0.020155

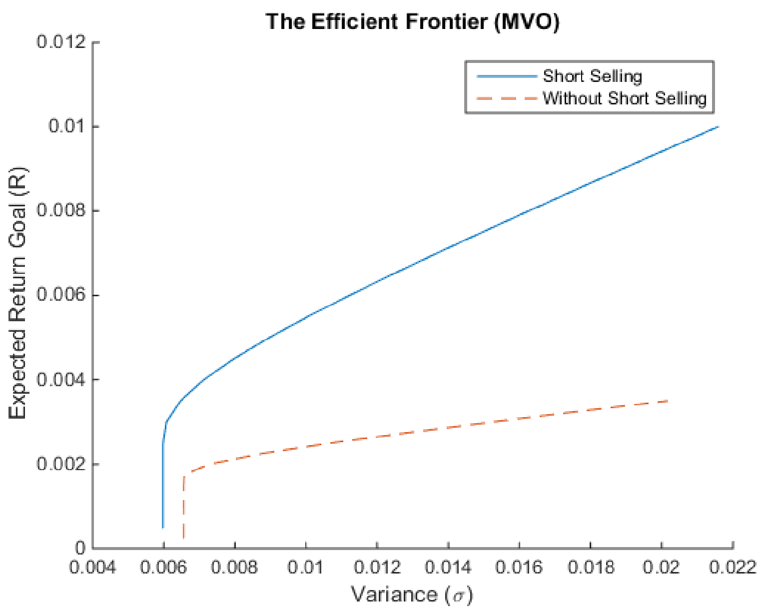

Figure 1 - Efficient Frontier of Portfolio

Figure 1 represents the 'risk' associated with each expected return goal. It is evident that the efficient frontier of the scenario without short selling does not return as much as short selling. In fact, it is caped off at 0.0035% because this is the maximum return that these three ETFs can give together. After this point, the added weights of the portfolio surpass 1 which is not possible. As well to get a higher return the calculation recommends that only the SPY asset be invested in. On the other hand, a max value of 0.01% is shown for the efficient frontier with short selling due to aesthetic reasons and it can in fact get a higher expected return goal if desired.

Part C

Equations Used

$$Realized\ Return = \sum_{i=1}^{3} \boldsymbol{x_i \mu_i}$$

Results

Table 8 - Calculated Realized Return of Each Asset for month of February

ETF	Realized Return
SPY $(\boldsymbol{\mu_1})$	0.03888
GOVT $(\boldsymbol{\mu_2})$	0.00521
EEMV $(\boldsymbol{\mu_3})$	0.01951

Table 9 - Weight of Portfolio with Short Selling for Expected Return Goal of 0.003%

Asset	Weight
SPY $(\boldsymbol{x_1})$	0.3132
GOVT $(\boldsymbol{x_2})$	0.8838
EEMV $(\boldsymbol{x_3})$	-0.1971

Table 10 - Weight of Portfolio without Short Selling for Expected Return Goal of 0.003%

Asset	Weight
SPY $(\boldsymbol{x_1})$	0.7436
GOVT $(\boldsymbol{x_2})$	0.2563
EEMV $(\boldsymbol{x_3})$	1.4129e-07

Table 11 - Realized Return of Portfolio

Realized Return (With Short Selling)	0.01293
Realized Return (Without Short Selling)	0.03024

To determine the performance of the portfolio created in Part B it had to be used for a data set outside of Jan 2015 – Dec 2016. Therefore, it was used for Feb 2017 and Table 8 summarizes the realized return for each of the three assets used in the portfolio for this month. These returns were than multiplied by the weights in Table 9 and Table 10. Note that with these weights the expected return goal was 0.003%. However, evidently from Table 11 this goal was surpassed and thus the portfolio performed well. The realized return with short selling for the portfolio was 0.01293% which is about four times more than the expected. As well, the return for the portfolio is between the smallest and largest return values of the three assets. Compared to the individual asset returns the overall return of the portfolio is low but this is mainly because the majority of the portfolio consists of the GOVT ETF which had the lowest return for the month of February. On the other hand, the return of the portfolio without short selling had better success and was closer to the higher return asset. This was because three quarters of the portfolio is made up of SPY asset which had the highest return. However, the portfolio without short selling had a higher variance and therefore is riskier. This is evident from these results because it depends so much on how one asset performs whereas the portfolio with short selling depends a little more on the other two ETFs. Nevertheless, the portfolio performed well as it gives a profitable return but it is important to consider that the month could be an anomaly because of how well the SPY and EEMV performed relative to the previous months.

Appendix (MATLAB Code + Output)

Part 1

```
% Initial Conditions
Liability = [24e6; 26e6; 28e6; 28e6; 26e6; 29e6; 32e6; 33e6; 34e6];
Price = [102.44 99.95 100.02 102.66 87.90 85.43 83.42 103.82 110.29 ...
    108.85 109.95 107.36 104.62 99.07 103.78 64.66 0 0 0 0 0 0 0 0 0]';
% Note Price includes reinvestment for later use
Coupon_payments = [5.625 4.75 4.25 5.25 0 0 0 5.75 6.875 6.5 6.625 ...
    6.125 5.265 4.75 5.5 0]';
Maturity_year = [1 2 2 3 3 4 5 5 6 6 7 7 8 8 9 9]';
reinvestment_rate = 0.02;
% Calculating Face Value
n = 0;
Face_Value = zeros(16,1);
while n < 16
    n = n + 1;
    Face_Value(n,1) = 100 + Coupon_payments(n,1);
end
% Seting up LP calculation
cp = Coupon_payments;
F = Face_Value;
% Bond Matrix
A  = zeros(9,16);
A(1,:)= [F(1,1) cp(2,1) cp(3,1) cp(4,1) cp(5,1) cp(6,1) cp(7,1) ...
    cp(8,1) cp(9,1) cp(10,1) cp(11,1) cp(12,1) cp(13,1) cp(14,1) ...
    cp(15,1) cp(16,1)];
A(2,2:16)= [F(2,1) F(3,1) cp(4,1) cp(5,1) cp(6,1) cp(7,1) cp(8,1) ...
    cp(9,1) cp(10,1) cp(11,1) cp(12,1) cp(13,1) cp(14,1) cp(15,1) ...
    cp(16,1)];
A(3,4:16)= [F(4,1) F(5,1) cp(6,1) cp(7,1) cp(8,1) cp(9,1) cp(10,1) ...
    cp(11,1) cp(12,1) cp(13,1) cp(14,1) cp(15,1) cp(16,1)];
A(4,6:16)= [F(6,1) cp(7,1) cp(8,1) cp(9,1) cp(10,1) cp(11,1) cp(12,1) ...
    cp(13,1) cp(14,1) cp(15,1) cp(16,1)];
A(5,7:16)= [F(7,1) F(8,1) cp(9,1) cp(10,1) cp(11,1) cp(12,1) cp(13,1) ...
    cp(14,1) cp(15,1) cp(16,1)];
A(6,9:16)= [F(9,1) F(10,1) cp(11,1) cp(12,1) cp(13,1) cp(14,1) cp(15,1) ...
    cp(16,1)];
A(7,11:16)= [F(11,1) F(12,1) cp(13,1) cp(14,1) cp(15,1) cp(16,1)];
A(8,13:16)= [F(13,1) F(14,1) cp(15,1) cp(16,1)];
A(9,15:16)= [F(15,1) F(16,1)];
% Reinvestment Matrix
R = zeros(9,9);
R(1,1) = -1;
R(2,1:2) = [1.02 -1];
R(3,2:3) = [1.02 -1];
R(4,3:4) = [1.02 -1];
R(5,4:5) = [1.02 -1];
R(6,5:6) = [1.02 -1];
R(7,6:7) = [1.02 -1];
R(8,7:8) = [1.02 -1];
R(9,8:9) = [1.02 0];
```

```
B = zeros(9,25);
B(:,1:16) = A;
B(:,17:25) = R;
% LP Function
c = Price;
b = [];
A = [];
Aeq = B;
beq = Liability;
lb = zeros(25,1);
ub = (lb+1)*inf;
[x_part1,cost] = linprog(c, A, b, Aeq, beq, lb, ub)
```

```
Optimization terminated.

x_part1 =

   1.0e+05 *

    1.4752
    1.7582
    0.0000
    0.0000
    2.0417
    2.0417
    0.0000
    1.8417
    2.2476
    0.0000
    2.7021
    0.0000
    0.0000
    2.9811
    3.2227
    0.0000
    0.0000
    0.0000
    0.0000
    0.0000
    0.0000
    0.0000
    0.0000
    0.0000
         0


cost =

   2.0467e+08
```

Part 2A

```
% Import SPY, GOVT, EEMV data into matrix
% SPY
r1 = zeros(48,1);
t = 0;
while t < 47
    t = t + 1;
    r1(t,1) = (SPY((t+1),1) - SPY(t,1))/SPY(t,1); %Calculating
    ... return of asset
end
r1 = r1(1:2:end,:);
```

```
r1bar = mean(r1); %Average of asset
r1a = 1+r1;
mew1 = (prod(r1a))^(1/24) - 1; %Geometric expected return of asset
% GOVT
r2 = zeros(48,1);
t = 0;
while t < 47
    t = t + 1;
    r2(t,1) = (GOVT((t+1),1) - GOVT(t,1))/GOVT(t,1);
end
r2 = r2(1:2:end,:);
r2bar = mean(r2);
r2a = 1+r2;
mew2 = (prod(r2a))^(1/24) - 1;
% EEMV
r3 = zeros(48,1);
t = 0;
while t < 47
    t = t + 1;
    r3(t,1) = (EEMV((t+1),1) - EEMV(t,1))/EEMV(t,1);
end
r3 = r3(1:2:end,:);
r3bar = mean(r3);
r3a = 1+r3;
mew3 = (prod(r3a))^(1/24) - 1;

% Corivance
sig = zeros(3);
t = 0;
s11 = zeros(24,1);
while t < 24
    t = t + 1;
    s11(t,1) = (r1(t,1) - r1bar) * (r1(t,1) - r1bar);
end
sig(1,1) = (1/24)*sum(s11);
t = 0;
s22 = zeros(24,1);
while t < 24
    t = t + 1;
    s22(t,1) = (r2(t,1) - r2bar) * (r2(t,1) - r2bar);
end
sig(2,2) = (1/24)*sum(s22);
t = 0;
s33 = zeros(24,1);
while t < 24
    t = t + 1;
    s33(t,1) = (r3(t,1) - r3bar) * (r3(t,1) - r3bar);
end
sig(3,3) = (1/24)*sum(s33);
t = 0;
s12 = zeros(24,1);
while t < 24
    t = t + 1;
    s12(t,1) = (r1(t,1) - r1bar) * (r2(t,1) - r2bar);
```

```
end
sig(1,2) = (1/24)*sum(s12);
t = 0;
s13 = zeros(24,1);
while t < 24
    t = t + 1;
    s13(t,1) = (r1(t,1) - r1bar) * (r3(t,1) - r3bar);
end
sig(1,3) = (1/24)*sum(s13);
t = 0;
s23 = zeros(24,1);
while t < 24
    t = t + 1;
    s23(t,1) = (r2(t,1) - r2bar) * (r3(t,1) - r3bar);
end
sig(2,3) = (1/24)*sum(s23);
```

Part 2B

```
R = 0.0;
Q = [sig(1,1) sig(1,2) sig(1,3);...
    sig(1,2) sig(2,2) sig(2,3); ...
    sig(1,3) sig(2,3) sig(3,3)];
c = [0 0 0]';
A = -[mew1 mew2 mew3];
b = -R;
Aeq = [1 1 1];
beq = [1];
ub = [inf inf inf]';
lb = -ub; %Short selling
lb_without = c; %Without short selling
[x, fval] = quadprog(Q,c,A,b,Aeq,beq,lb_without,ub) %Solving MVO
var = sqrt(fval) %Getting Variance
% Constructing points for graph (R with short selling)
iterations = 20
R_graph = zeros(iterations,1);
sigf = zeros(iterations,1);
weight = zeros(iterations,3)
i = 0;
while i < 20
    R = R + 0.0005;
    i = i + 1;
    R_graph(i,1)= R;
    b = -R;
    [x, fval] = quadprog(Q,c,A,b,Aeq,beq,lb,ub);
    sigf(i,1) = fval;
    weight(i,1) = x(1,1);
    weight(i,2) = x(2,1);
    weight(i,3) = x(3,1);
end
risk = sqrt(sigf);
% Constructing points for graph (R without short selling)
R = 0;
```

```
i = 0;
iterations_wo = 14;
R_graph_wo = zeros(iterations_wo,1);
sigf_wo = zeros(iterations_wo,1);
weight_wo = zeros(iterations_wo,3);
while i < 14
    R = R + 0.00025;
    i = i + 1;
    R_graph_wo(i,1)= R;
    b = -R;
    [x, fval] = quadprog(Q,c,A,b,Aeq,beq,lb_without,ub);
    sigf_wo(i,1) = fval;
    weight_wo(i,1) = x(1,1);
    weight_wo(i,2) = x(2,1);
    weight_wo(i,3) = x(3,1);
end
risk_wo = sqrt(sigf_wo);
%Plotting Graph
figure; hold on
a1 = plot(risk,R_graph);
a2 = plot(risk_wo,R_graph_wo,'--');
title('The Efficient Frontier (MVO)');
xlabel('Variance (\sigma)');
ylabel('Expected Return Goal (R)');
M1 = ('Short Selling');
M2 = ('Without Short Selling');
legend([a1;a2],M1,M2);
hold off
```

Part 2C

```
realized_return = zeros(3,1);
% Calculating realized return for Feb 2017
SPY_rr = (235.4457 - 226.634)/226.634;
GOVT_rr = (25.022 - 24.8923)/24.8923;
EEMV_rr = (51.71 - 50.72)/50.72;
realized_return(1,1) = SPY_rr;
realized_return(2,1) = GOVT_rr;
realized_return(3,1) = EEMV_rr;
R = 0.003; %Picked weights for expected return goal of 0.003
ham = weight(6,:);
ham_wo = weight_wo(12,:);
rr = dot(realized_return,ham); %Realized return of portfolio with ss
rr_wo = dot(realized_return,ham_wo); %without ss
```

```
Minimum found that satisfies the constraints.

Optimization completed because the objective function is non-decreasing in
feasible directions, to within the default value of the function tolerance,
and constraints are satisfied to within the default value of the constraint tolerance.



x =
```

```
    0.2024
    0.7976
    0.0000


fval =

   4.3023e-05


var =

    0.0066
```