Group Coursework Submission Form (PA)

Specialist Masters Programme

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SMM272 Risk Analysis

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March 20, 2025

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1 VaR Modelling

1.1 Statistical Analysis of Portfolio Returns

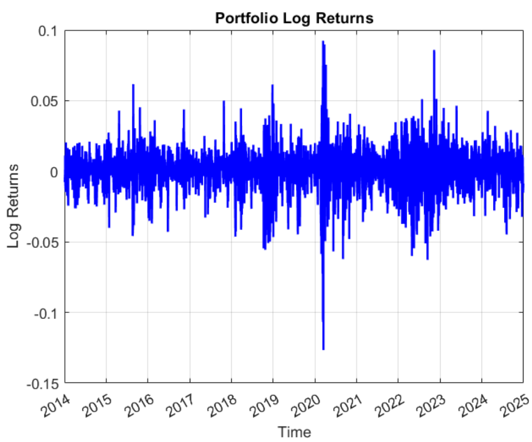
The adjusted closing prices of the six stocks - AAPL, AMZN, GOOG, IBM, MSFT and NVDA for the period spanning January 1, 2014 to December 31, 2024 were retrieved from Yahoo Finance ('MyStock_merge.csv') and an equally weighted portfolio was set up by allocating equal weights of 16.67% (=1/N, where N is the number of assets) to the stocks.

The daily log returns of the individual stocks were computed, and subsequently, the portfolio daily returns were obtained as the weighted sum of the individual log returns. Important to mention here is that, the portfolio return as the weighted sum of the individual returns is 'exact' when the returns calculated are simple, but in our case this works as a very good approximation of the exact result, due to the approximation of Taylor series expansion, where

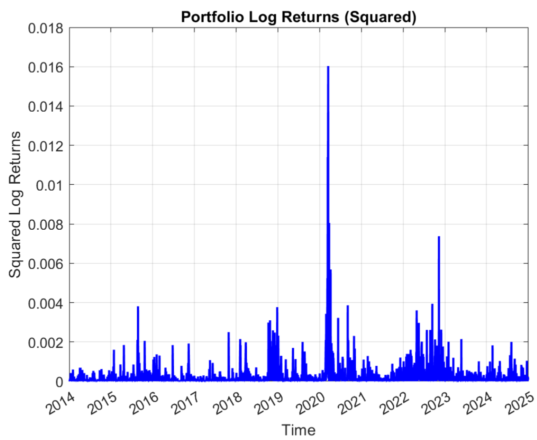
$$e^r = \sum_{n=0}^{\infty} \frac{r^n}{n!} \approx 1 + r, \quad \text{for small } r.$$

Hence, the portfolio log return formula was simplified as: $R_p = log(\sum_{i=1}^{N} w_i e^{r_i}) \approx \sum_{i=1}^{N} w_i log(e^{r_i})$.

The time series of the daily portfolio log-returns was plotted for visual inspection. As can be observed from the plot of the portfolio daily log-returns (Figure 1a), the return series appears to be stationary, with a small positive mean. Meanwhile, as for its squares (Figure 1b), there were periods where returns showed high volatility, followed by periods of low volatility (volatility clusters), with the presence of occasional large spikes, especially in early 2020 (the covid-19 crash) representing days of unusually positive or negative returns.



(a) Log-Returns



(b) Squared Log-Returns

Figure 1: Portfolio Daily Log-Returns

Descriptive statistical analyses were conducted on the portfolio daily log returns, and the results are presented below, in Table 1.
With 2766 return observations, the mean of the daily portfolio returns is 0.0010 (0.1%), indicating a slight positive drift, with a standard deviation of 0.0151. The negative skewness of 0.4330 indicates a longer left tail, meaning more frequent negative shocks. The excess kurtosis of 5.7589 shows the presence of heavy tails compared to a normal distribution, implying a higher probability of extreme returns.

To test for normality, the Jarque-Bera test was conducted. The JB-test statistic (3908.6559) was significantly higher than the critical value (5.9915), leading to the rejection of normality. This suggests that

Descriptive Statistic	Value
Number of Observations	2766
Mean	0.0010
Standard Deviation	0.0151
Skewness	-0.4330
Excess Kurtosis	5.7589
Jarque-Bera Statistic	3908.6559
Critical Value	5.9915
Jarque-Bera Test	Reject Normality (Non-Normal)

Table 1: Descriptive Analysis of Log Returns

the portfolio log returns exhibited non-normal characteristics, which is also conformed by the QQ plot in Figure 2 :

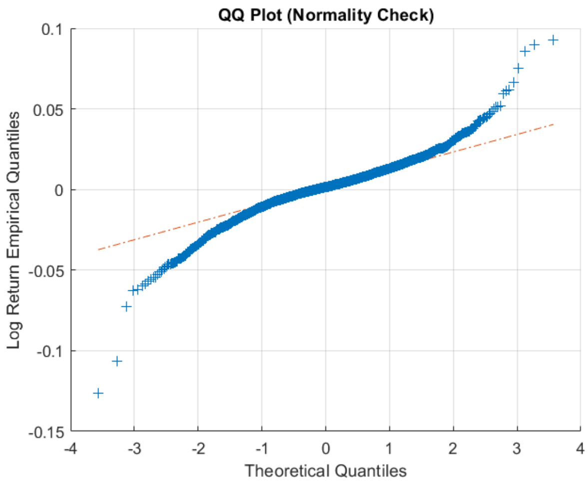
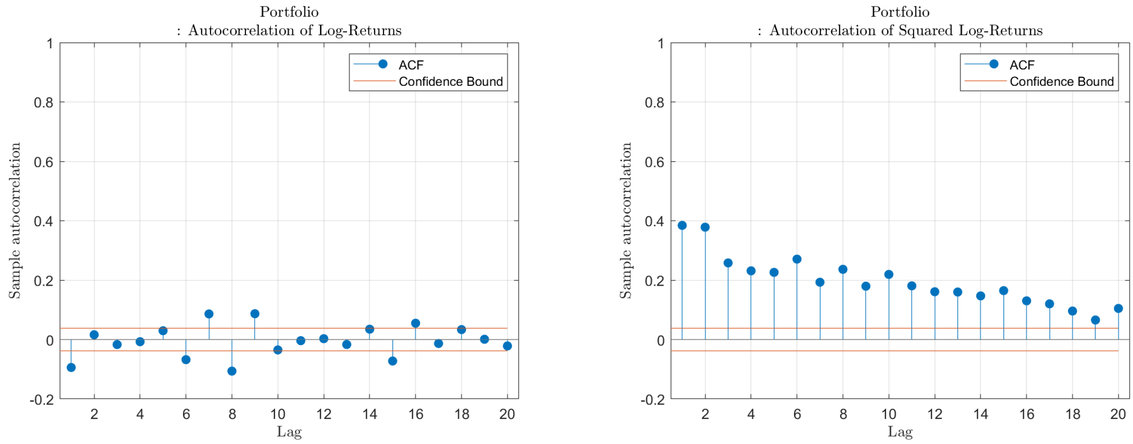


Figure 2: QQ Plot of Portfolio Daily Log-Returns

Finally, Figure 3 shows autocorrelations of log returns and squared log returns up to 20 lags. Even though they are marginally small values near zero, correlations exceeding confidence level (2 standard deviations) among log returns up to around ninth lag are apparently observed. On the other hand, as also confirmed from the volatility clustering, more evident and prolonged autocorrelations are confirmed in the squared log returns.



(a) Log-Returns (b) Squared Log-Returns

Figure 3: Portfolio Daily Log-Returns

1.2 Modelling VaR using Paramteric and Non-Parametric Methods

Value at Risk (VaR) is defined as the maximum expected loss of a portfolio (in value or returns), over a specified time horizon at a predetermined confidence level α such that the probability of the actual loss (in or returns) exceeding this threshold is 1 - α.

Among the several parametric and non-parametric approaches available to model the portfolio returns VaR, the methods used for this exercise were:

- VaR Estimation via Gaussian (Top-Down) - Parametric approach
- VaR Estimation via Historical Simulation - Non-Paramteric approach
- VaR Estimation via Monte Carlo Simulation (Bottom-Up) - Parametric approach
- VaR Estimation via Bootstrapping - Non-Paramteric approach
- VaR Estimation via Block Bootstrapping - Non-Paramteric approach

A rolling window of 6 months (120 past return observations) and two confidence levels 90% and 99% are used to estimate the daily rolling VaR for each of the methods. The first VaR estimate begins on July 1, 2014 and subsequent daily portfolio return VaRs are estimated using the previous 120 days' returns, with the last VaR estimate on December 31, 2024. Therefore, for each of the methods we have 2,643 VaR estimates.

The specific model features are described as follows:

1.2.1 VaR Estimation via Gaussian (Top-Down) - Parametric approach

In this approach, it is assumed that the portfolio returns follow a specific distribution, in this case, the Gaussian distribution. The portfolio's mean return and standard deviation are estimated from the historical returns data (here, the previous 120 days' portfolio returns) and VaR is computed based on the chosen confidence levels (90% and 99%). The parametric formula for VaR used is:

$$VaR_p = -(\mu_p + z_{1-\alpha}\sigma_p)$$

where:

- μ_p is the portfolio mean return.
- σ_p is the portfolio standard deviation.
- $z_{1-\alpha}$ is the critical value from the Gaussian distribution at confidence level $1-\alpha$.

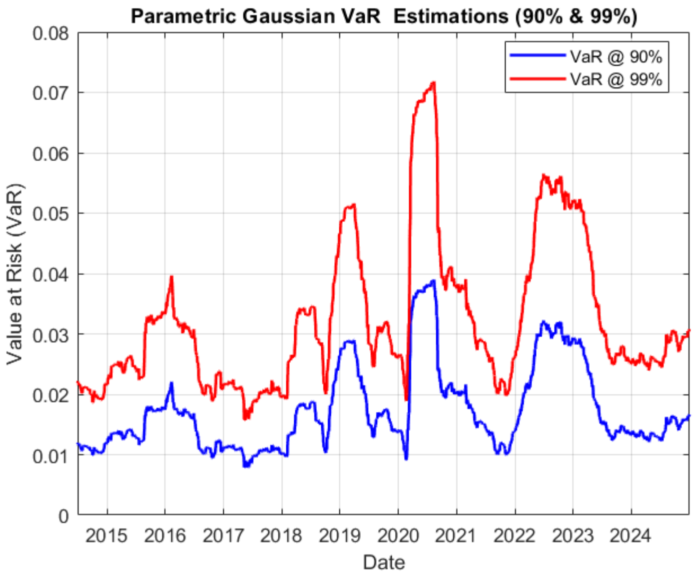


Figure 4: Gaussian VaR Estimates

Below is the algorithm of the code:

Algorithm: Gaussian VaR Estimation

Step 1: Identify the Start Row for the Rolling Window
Convert the date column of `stockData` into a MATLAB datetime format.
Locate the row number `rowNum` corresponding to `strt_date` = 01 July 2014. Subtract 1 from `rowNum` to align with MATLAB indexing.

Step 2: Compute Rolling Portfolio Mean and Standard Deviation
Initialize RpG, a matrix to store rolling mean and standard deviation of portfolio returns.
Loop through the data from $rowNum$ to the end:

- Compute the rolling (120 days) portfolio mean of the portfolio log returns
- Compute the rolling portfolio standard deviation.

Step 3: Compute Parametric Gaussian VaR
Convert confidence levels = [90 99] into probability thresholds:

$$p_1 = 1 - \frac{p}{100}$$

Compute parametric Gaussian VaR using the formula:

$$vAr\_G = -(\mu_p + z_\alpha \sigma_p)$$

where z_α is obtained from the normal inverse cumulative distribution function (`norminv(p1)` in MATLAB).

1.2.2 VaR Estimation via Historical Simulation - Non-Paramteric approach.

Portfolio return VaR estimation using the Historical Simulation approach is a non-parametric method and involves constructing a return distribution from historical portfolio returns over a lookback period (in our case, 120 days rolling window). The VaRs at a given confidence level (90% or 99%) are determined by selecting the corresponding percentiles from the sorted historical returns.

For a confidence level α, Historical Simulated VaR is computed as the (1-α) quantile of the past returns. While this method is model-free and is relatively simple to implement, it assumes that past market conditions represent future risks. Also, as seen in Figure 5 , the HS VaR levels remain same for certain periods and do not show the fluctuations like other VaR estimates, hinting it's heavy dependency on historical returns. Figure 5 plots the HS VaRs at 90% and 99%.

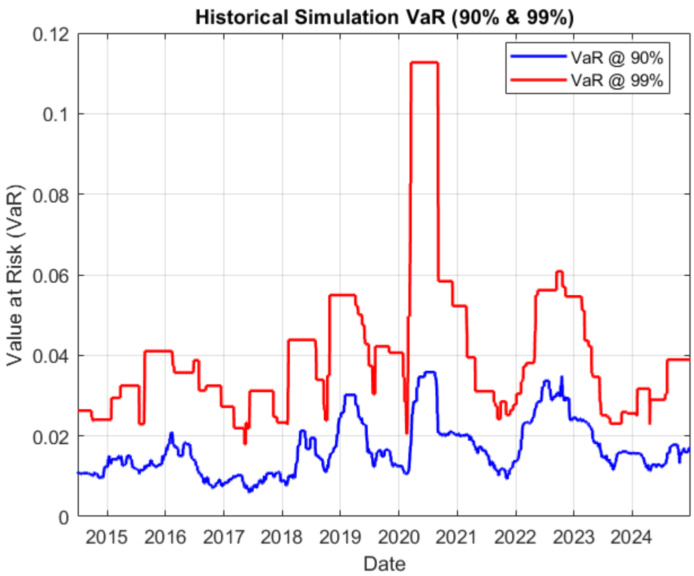


Figure 5: Historical Simulated VaR Estimates

Algorithm: Historical Simulation VaR Estimation

Step 1: Initialize VaR Storage
Create an empty matrix vAr_HS of size $(NObs - rowNum + 1) \times 2$ to store the 90% and 99% VaR values.

Step 2: Loop Through Rolling Window For each rolling window from $i = 0$ to $NObs - rowNum$:

- Extract the portfolio log returns over the rolling window.
- Compute Historical VaR at the 90% and 99% confidence levels by taking the negative percentiles of the past portfolio returns:

$$vAr\_HS(i+1,:) = -\text{prctile}(\text{logRet}(rowNum - rollWin + i : (rowNum-1)+i, 1 : end) \times w, (100-p))$$

Step 3: Store the Computed VaR Values
Save the computed VaR values for each rolling window in the vAr_HS matrix.

1.2.3 VaR Estimation via Monte Carlo Simulation (Bottom-Up) - Parametric approach.

Portfolio return VaR estimation using Monte Carlo simulation involves generating a large number (here, M= 10^4) of simulated portfolio return scenarios based on statistical properties of historical returns, assuming that the stock returns follow a joint Gaussian distribution and uses the mean vector of individual asset returns and the covariance matrix to model dependencies between the stocks.
For estimating the MC VaR, the rolling means of the individual stocks along with the covariance matrices are computed and these values were used to generate a sample of returns which jointly follow the (mul-

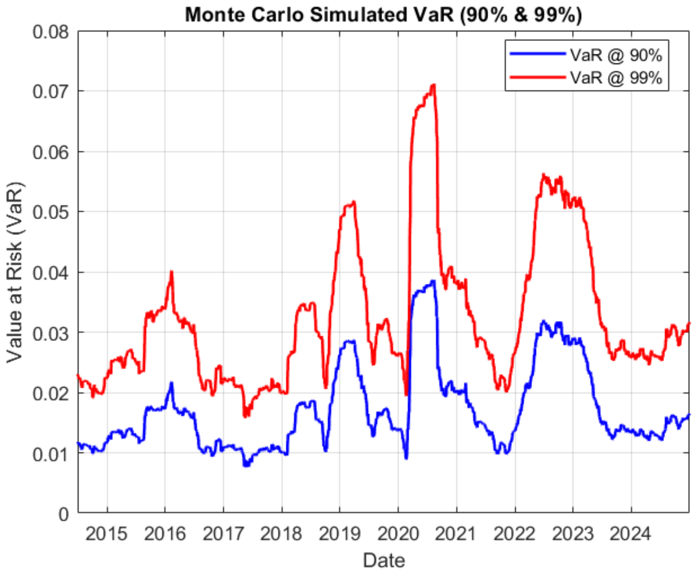


Figure 6: Monte Carlo VaR Estimates

tivariate) normal distribution. From the generated returns, the 90% and 99% VaRs are then calculated for each of the periods(1 day horizons). Figure 6 plots the VaRs for 90% and 99%.

Algorithm: Monte Carlo VaR Estimation

Step 1: Initialize Variables

- Create a matrix $meanRet$ of size $(NObs - rowNum + 1) \times NAsset$ to store rolling mean returns for each asset.
- Create a matrix vAr_MC of size $(NObs - rowNum + 1) \times \text{length}(p)$ to store the Monte Carlo VaR estimates.

Step 2: Loop Through Rolling Window
For each rolling window from $i = 0$ to $NObs - rowNum$:

- Set the random seed using `rng(1234)` for reproducibility.
- Compute the **rolling mean return vector** for all assets:

$$meanRet(i + 1, :) = \text{mean}(\text{logRet}[rowNum - rollWin + i : (rowNum - 1) + i, 1 : end])$$

- Compute the **rolling covariance matrix** for asset returns:

$$covarRoll = \text{cov}(\text{logRet}[rowNum - rollWin + i : (rowNum - 1) + i, 1 : end])$$

Step 3: Generate Monte Carlo Simulated Returns

- Generate M simulated **multivariate normal** asset returns using the estimated mean vector and covariance matrix:
$$simRi = \text{mvnrnd}(meanRet(i + 1, :), covarRoll, M)$$
- Convert simulated log returns into **simulated portfolio returns**:

$$simRP = \log(\exp(simRi) \times w)$$

Step 4: Compute Monte Carlo VaR

- Compute **Monte Carlo VaR** at different confidence levels by taking the **negative percentile** of the simulated portfolio return distribution:

$$vAr\_MC(i + 1, :) = -\text{prctile}(simRP, 100 - p)$$

1.2.4 VaR Estimation via Bootstrapping (Efron) - Non-Paramteric approach.

Portfolio return VaR estimation using the bootstrapping method, introduced by Efron (1979), is a non-parametric approach that resamples historical portfolio returns to generate a simulated return distribution. Unlike Monte Carlo simulation, which assumes a specific distribution for returns, bootstrapping randomly draws returns samples of the same size as the original sample (here, 120) with replacement from historical data ans estimates the VaRs at the chosen confidence levels (90% and 99%). This procedure is repeated multiple times (Nb = 1000) and the best VaR estimate from the full dataset is obtained by averaging all the sampled VaRs. This method allows the generation of a hypothetical distribution using past returns, without imposing any assumptions on them. Figure 7 plots the Bootstrapped VaR estimates at 90% and 99%.

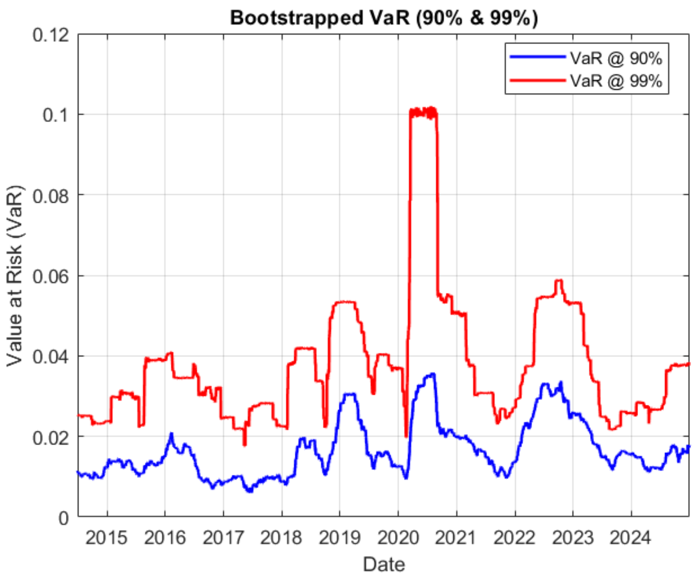


Figure 7: Bootstrapped VaR Estimates

Algorithm: Bootstrapped VaR Estimation

Step 1: Initialize Variables

- Create a matrix vAr_BS of size $(NObs - rowNum + 1) \times \text{length}(p)$ to store the **bootstrapped VaR estimates**.
- Create a vector sRp of size $rollWin \times 1$ to store rolling portfolio returns.

Step 2: Loop Through Rolling Window For each rolling window from $i = 0$ to $NObs - rowNum$:

- Extract the **rolling portfolio returns** over the last $rollWin$ days:

$$sRp = \text{logRet}[rowNum - rollWin + i : (rowNum - 1) + i, 1 : end] \times w$$

- Set the **number of bootstrap resamples** $Nb = 1000$.
- Create a matrix $bsVar$ of size $Nb \times 2$ to store bootstrapped VaR estimates.

Step 3: Perform Bootstrapping For each bootstrap iteration $j = 1$ to Nb:

- Randomly sample $rollWin$ indices **with replacement** from historical returns:

$$\text{indices} = \text{randi}([1, rollWin], 1, rollWin)$$

- Compute **bootstrapped VaR** by taking the **negative percentile** of the sampled portfolio returns:

$$bsVar(j, :) = -\text{prctile}(sRp(\text{indices}), 100 - p)$$

Step 4: Compute Final Bootstrapped VaR

- Compute the **mean of all bootstrap samples** to obtain the final VaR estimate:

$$vAr\_BS(i + 1, :) = \text{mean}(bsVar)$$

1.2.5 VaR Estimation via Block Bootstrapping - Non-Paramteric approach.

As an extension to the normal bootstrap approach, the block bootstrap approach is also applied, since the normal bootstrap method has limitations in capturing autocorrelation structure, and the estimate tends to give the same value as the historical estimate.

Thus, given the weak autocorrelation in returns observed in the statistical analysis, here we adopt the block bootstrapping method, which draws an arbitrary number of consecutive observations at a time instead of a single observation. In our case, while weak but notable autocorrelations were partly observed up to ninth lag, we conservatively set the block size as two. This procedure is repeated multiple times (Nb = 1000) and the best VaR estimate from the full dataset is obtained by averaging all the sampled VaRs.

Figure 8 plots the Block Bootstrapped VaR estimates at 90% and 99%.

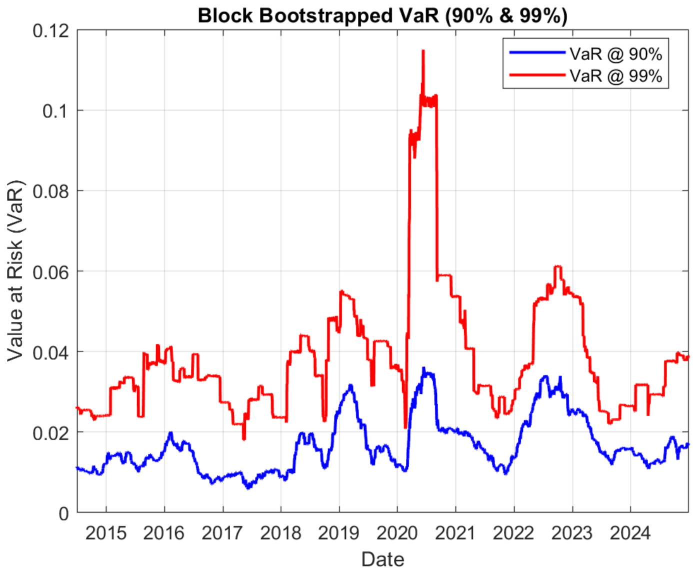


Figure 8: Block Bootstrapped VaR Estimates

Algorithm: Block Bootstrapped VaR Estimation

Step 1: Initialize Variables

- Create a matrix vAr_BS of size $(NObs - rowNum + 1) \times \text{length}(p)$ to store the **bootstrapped VaR estimates**.
- Create a vector sRp of size $rollWin \times 1$ to store rolling portfolio returns.

Step 2: Loop Through Rolling Window
For each rolling window from $i = 0$ to $NObs - rowNum$:

- Extract the **rolling portfolio returns** over the last $rollWin$ days:

$$sRp = \text{logRet}[rowNum - rollWin + i : (rowNum - 1) + i, 1 : end] \times w$$

- Set the **number of bootstrap resamples** $Nb = 1000$, **block size** $Bsz = 2$, and **number of blocks in each sample** $Blockn = length(sRp)/Bsz$

Create a matrix $bbsVar$ of size $Nb \times 2$ to store bootstrapped VaR estimates.

Step 3: Perform Bootstrapping
For each bootstrap iteration $j = 1$ to Nb:

- Construct a set of observations of blocks for the given rolling window: $blocks = reshape(sRp, [Blockn, Bsz])'$
- tmpOpt = bootstrp(Nb, @(x)x', blocks)
- Randomly sample dataset **with replacement** from $blocks$:

$$tmpOpt = bootstrp(Nb, @(x)x', blocks)$$

- Compute **bootstrapped VaR** by taking the **negative percentile** of the sampled portfolio returns:

$$bbsVar(j, :) = -\text{prctile}(sRp(\text{indices}), 100 - p)$$

Step 4: Compute Final Bootstrapped VaR

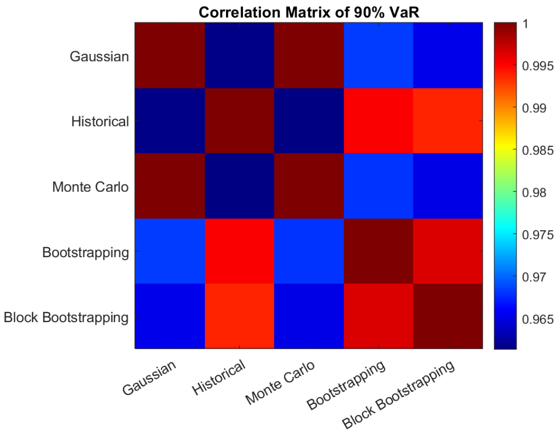
- Compute the **mean of all bootstrap samples** to obtain the final VaR estimate:

$$vAr\_BBS(i + 1, :) = \text{mean}(bsVar)$$

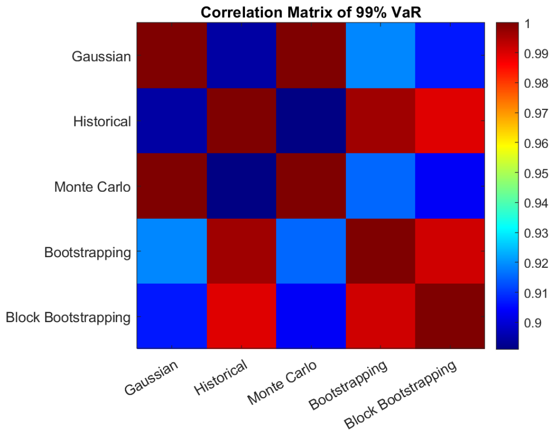
1.3 VaR Forecasts and VaR Violations

VaR forecasts among the five different approaches show some variability. The parametric approaches show greater convergence in their estimates than the non-parametric ones. Figure 9a and Figure 9b show the correlation matrix of the five VaR estimates over time for 90% and 99% confidence levels, respectively.

VaR violations occur when the actual return of the portfolio is less than the negative VaR estimated from the historical data. Based on this, the VaR violations for each of the 5 methods are tabulated in Table 2:



(a) Correlation matrix for 90% VaR estimates



(b) Correlation matrix for 99% VaR estimates

Figure 9: Correlation matrix - VaR estimates

VaR Estimation Method	**Total Obs**	**Violations @ 90%**	**Violations @ 99%**
Gaussian (Top-Down)	2643	254	75
Historical (Non-Parametric)	2643	271	36
Monte Carlo (Bottom-Up)	2643	259	74
Bootstrapping (Non-Parametric)	2643	275	43
Block Bootstrapping (Non-Parametric)	2643	273	41

Table 2: VaR Violations Across Different Methods

1.4 Backtesting Performance of the VaR Models

In order to check the accuracy of the five VaR models, three statistical tests such as the Kupiec test (Likelihood Ratio test for unconditional convergence), Conditional Convergence test (Christoffersen's test) and Kolmogorov-Smirnov test, were applied. The results of these test are tabulated below:

Model	**Kupiec Test p-value**	**Cond. Coverage Test p-value**	**KS Test p-value**
Gaussian	0.5017	0.0167	0.1900
Historical	0.6652	0.0112	0.2012
Monte Carlo	0.7303	0.0216	0.1933
Bootstrap	0.4904	0.0252	0.2039
Block Bootstrap	0.5745	0.0221	0.2025

Table 3: 90% VaR Violations - Backtest Results

1.4.1 Kupiec test (Likelihood Ratio test for Unconditional Convergence)

The Kupiec test is a likelihood ratio test that is used to evaluate whether the number of VaR violations statistically matches the expected frequency based on the given confidence level, by comparing the observed proportion of violations to the theoretical probability. It rejects the model if the number of deviations are statistically significant. A low p-value (here, < 0.05) indicates that the VaR model

Model	Kupiec Test p-value	Cond. Coverage Test p-value	KS Test p-value
Gaussian	0.0000	0.0000	0.0977
Historical	0.0762	0.0002	0.0843
Monte Carlo	0.0030	0.0000	0.0865
Bootstrap	0.0084	0.0006	0.0858
Block Bootstrap	0.0084	0.0006	0.0858

Table 4: 99% VaR Violations - Backtest Results

underestimates or overestimates risk, making it unreliable, while a high p-value suggests that the model correctly predicts expected violations.

1.4.1.1 Comments on the Kupiec test results

Table 3: At 90% VaR,

- All the models have high p-values (0.4904 to 0.7303), meaning they do not significantly deviate from expected violations, indicating better model performance.

Table 4: At 99% VaR,

- The Gaussian, Monte Carlo and the Bootstrap approaches (normal and Block) have p-values close to zero (though the bootstaps are marginnaly better), thus indicating poor performance at capturing tail risk.
- However, Historical Simulation approach ($p = 0.0762$) performs the best, as its p-value is above the 5% threshold, meaning it adequately estimates extreme risks.

Conclusion: All the models perform well at 90% VaR, while only the Historical simulation approach is the most reliable at 99% VaR.

1.4.2 Conditional Convergence test (Christoffersen's test)

The Conditional Coverage Test checks whether VaR violations occur independently or if they cluster over time. A low p-value (< 0.05) suggests that violations are not independent and past violations increase the likelihood of future violations.

1.4.2.1 Comments on the Christoffersen's test results

Table 3: At 90% VaR,

- All the models perform inaccurately (p-values < 0.05) suggesting some clustering of VaR violations.

Table 4: At 99% VaR,

- Similar to 90% VaR case, all models fail to properly estimate the risk (volatility clusters), though the bootstrap methods do marginally better.

1.4.3 Kolmogorov-Smirnov test

The Kolmogorov-Smirnov test measures the distributional fit of the VaR violations, and checks whether the empirical distribution of the violations significantly deviates from a uniform distribution.

1.4.3.1 Comments on the KS test results

Table 3: At 90% VaR, and Table 4: At 99% VaR, show that all models perform reasonably well, although the models perform relatively worse in the extreme cases (weaker fit of 99% VaRs).

2 The Risk Parity Portfolio

The risk parity portfolio is a portfolio in which each asset contributes equally to the Component Value at Risk (VaR). This portfolio is constructed by selecting weights that minimize the dispersion (or standard deviation) of the individual Component VaRs (CVaRs).

Using the dataset from Question 1, we compute the Component VaR using both the parametric approach, which relies on the sample covariance matrix, and the nonparametric approach. We also analyze two additional portfolio construction methodologies: the maximum diversification portfolio and the equally weighted portfolio.

2.1 Portfolio Construction

2.1.1 Equally-Weighted Portfolio

The equally weighted portfolio assigns identical weights to all assets:

$$w_i = \frac{1}{N} \quad \forall i \in \{1, 2, ..., N\} \tag{1}$$

where:

- w_i is the weight of asset i
- N is the total number of assets

This is the simplest allocation strategy that does not require an estimation of returns, volatilities, or correlations.

2.1.2 Maximum Diversification Portfolio

The maximum diversification portfolio aims to maximize the diversification ratio.

$$\max_w \frac{w^T \sigma}{\sqrt{w^T \Sigma w}} \tag{2}$$

subject to:

- $w^T 1 = 1$ (weights sum to 1)
- $w \geq 0$ (non-negative weights)

where:

- w is the vector of portfolio weights
- σ is the vector of asset volatilities (standard deviations)
- Σ is the covariance matrix of returns

The numerator represents the weighted average of individual asset volatilities, while the denominator is the portfolio's overall volatility. A higher ratio indicates greater diversification benefits from imperfect correlations.

2.1.3 Risk Parity Portfolio

The risk parity portfolio equalizes risk contribution across assets. The optimization problem is the following.

$$\min_w \operatorname{std}(\mathrm{CVaR}_i(w)) \tag{3}$$

subject to: $w^T 1 = 1$ and $w \geq 0$, where $\mathrm{CVaR}_i(w)$ is the component value at risk of the asset i.
The Component VaR is calculated as:

$$\mathrm{CVaR}_i(w) = w_i \times \mathrm{MVaR}_i(w) \tag{4}$$

with Marginal VaR (MVaR) defined as:

$$\mathrm{MVaR}_i(w) = \frac{\Sigma_i w}{\sqrt{w^T \Sigma w}} \tag{5}$$

where Σ_i is the i-th row of the covariance matrix and $w^T \Sigma w$ represents the portfolio variance. The Marginal VaR quantifies the sensitivity of the portfolio's risk to small changes in the weight of a specific asset.
In a true risk-parity portfolio, all assets have equal risk contributions:

$$w_i \times \mathrm{MVaR}_i(w) = w_j \times \mathrm{MVaR}_j(w) \quad \forall i, j \in \{1, 2, ..., N\} \tag{6}$$

This creates a portfolio where each asset contributes equally to the total portfolio risk. In practice, achieving perfect risk parity may not always be feasible due to constraints such as the non-negativity of weights. Therefore, we formulate the optimization problem as minimizing the dispersion of Component VaRs across assets, which approximates the risk parity objective.

We split the dataset into two parts, using the first half to determine the portfolio compositions and the second half to compute the daily log-returns for each portfolio. This out-of-sample testing approach provides a more realistic assessment of portfolio performance, as it evaluates strategies based on their ability to perform on unseen data rather than data used for optimization.

For each portfolio, we evaluate three key performance metrics: Sharpe Ratio (assuming zero risk-free rate), Maximum Drawdown, and VaR Violations at 95% confidence level. When constructing these portfolios, we apply constraints to ensure the weights sum to one (fully invested constraint) and are non-negative (long-only constraint). Optimization problems are solved using constrained optimization with appropriate initial values.
Figure 10 shows the optimal weights for each portfolio strategy. The equally-weighted portfolio assigns exactly 16.67% to each asset. The risk parity portfolio shows a more balanced risk allocation, though not equal in capital terms, as it equalizes risk contribution rather than capital allocation. The maximum diversification portfolio exhibits more concentrated positions, particularly favoring assets that have lower correlation with the rest of the portfolio, resulting in a less uniform weight distribution.

Table 5 summarizes the key performance metrics for each portfolio strategy:

Metric	**Equally-Weighted**	**Risk Parity**	**Maximum Diversification**
Sharpe Ratio	0.0642	0.0622	0.0647
Maximum Drawdown	0.3997	0.3531	0.3314
VaR Violations	61	60	60

Table 5: Performance metrics comparing the three portfolio strategies, calculated using the out-of-sample testing period.

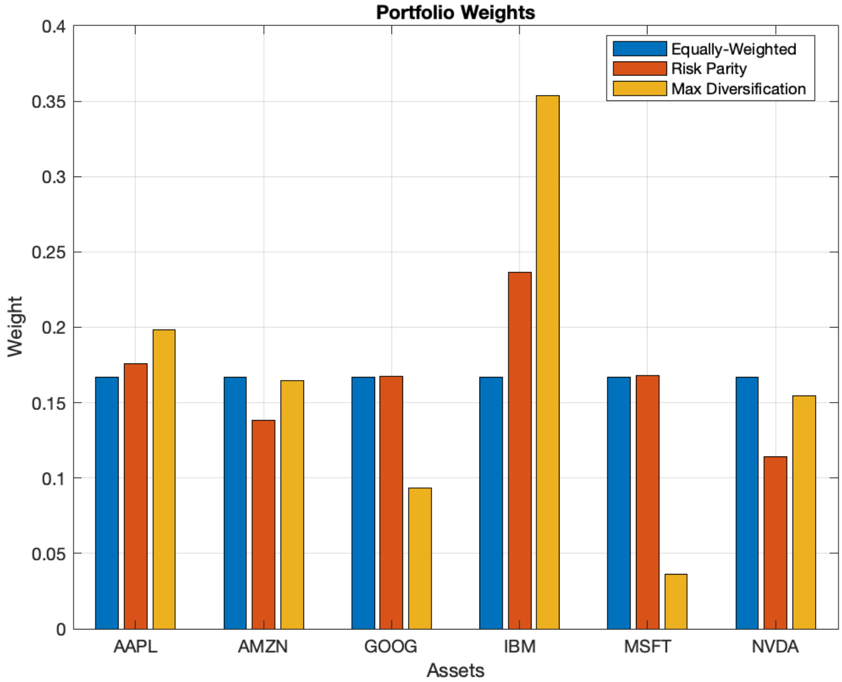


Figure 10: Portfolio Weights

2.2 Discussion

Performance metrics reveal several important insights about the three portfolio construction methodologies. Sharpe ratios show that the Maximum Diversification portfolio achieved the highest risk-adjusted returns (0.0647), followed closely by the equally weighted portfolio (0.0642), with the Risk Parity portfolio showing the lowest risk-adjusted performance (0.0622). However, the differences in Sharpe ratios are relatively small, suggesting that all three strategies provide similar risk-adjusted returns in the tested period.

More significant differences emerge when examining downside protection capabilities. The Maximum Diversification portfolio demonstrated the strongest downside protection with the lowest maximum drawdown (0.3314), followed by the Risk Parity portfolio (0.3531), while the Equally Weighted portfolio experienced the largest drawdown (0.3997). This represents a substantial improvement of approximately 17% in the reduction of drawdown for the maximum diversification portfolio compared to the equally weighted approach, highlighting the superior risk management capabilities of risk-optimized strategies during market downturns.

The VaR violations follow a similar pattern, with the Maximum Diversification portfolio having the fewest instances where realized losses exceeded the estimated VaR (1328 violations), followed by the Risk Parity portfolio (1333 violations), with the Equally-Weighted portfolio having the most violations (1335). Though the differences are small in absolute terms, they consistently favor the risk-optimized approaches, suggesting slightly better risk estimation accuracy.

The overall pattern across all metrics indicates a consistent ordering of the three strategies: the Maximum Diversification portfolio generally outperforms the Risk Parity portfolio, which in turn outperforms the Equally-Weighted portfolio, particularly in terms of downside risk protection. This consistency across different performance dimensions, especially in drawdown reduction, suggests that the benefits of risk-based optimization approaches are most evident in their ability to mitigate extreme losses rather than in generating superior average returns.

The superior downside protection of the Maximum Diversification portfolio can be attributed to its effective exploitation of the correlation structure among assets. By maximizing the ratio of weighted average volatilities to portfolio volatility, this approach creates a portfolio that optimally captures diversification benefits while minimizing overall risk. Theoretically, this looks particularly valuable during market stress periods when correlations between assets tend to change. However, there are several caveats to be noted for implementation in reality, such as: (1) the future correlations have to be estimated before it changes, and (2) in crisis the correlations tend to increase positively and the benefit will be marginal, along with the increased risk of estimation errors.

The Risk Parity approach, while not matching the performance of Maximum Diversification in terms of drawdown protection and showing a slightly lower Sharpe ratio than even the Equally-Weighted portfolio, still offers advantages through its intuitive risk allocation framework. By equalizing risk contributions, it prevents any single asset from dominating the portfolio's risk profile, leading to more balanced risk exposure. The relatively small difference in performance metrics suggests that the Risk Parity approach might still be valuable in environments where risk estimation is more important than return estimation. As such, it is still considered to be an alternative portfolio choice for Maximum Diversification portfolio to manage the tail risk, especially during market stress periods.

The Equally-Weighted portfolio, despite its simplicity, performed surprisingly well in terms of risk-adjusted returns, nearly matching the Sharpe ratio of the Maximum Diversification portfolio. This aligns with previous research suggesting that naive diversification strategies can be effective due to their robustness to estimation errors. However, its substantially higher maximum drawdown highlights a key weakness: naive capital diversification does not necessarily translate to effective risk diversification, particularly during market downturns.

It is worth noting that the VaR violations are relatively high across all strategies (over 1300 violations), which may indicate that the 95% confidence level VaR model used in this analysis underestimates the true risk. This could be due to non-normal return distributions, fat tails, or volatility clustering not captured by the model. The slightly better performance of the risk-optimized portfolios in this metric suggests they might be somewhat more robust to model misspecification, but all three approaches would benefit from more sophisticated risk models.

The practical implications of these findings are nuanced. While the Maximum Diversification portfolio offers the best overall performance, particularly in drawdown protection, the differences in Sharpe ratios are modest. This suggests that investors primarily concerned with average risk-adjusted returns might find the simplicity of the Equally-Weighted approach appealing, especially when considering implementation costs. However, investors with low risk tolerance or those particularly concerned with tail risk would likely benefit from the enhanced downside protection offered by the Maximum Diversification or Risk Parity approaches.

In practice, implementation considerations such as transaction costs, liquidity constraints, and parameter estimation errors would also influence the choice and effectiveness of these strategies. Higher turnover strategies may incur greater transaction costs, reducing their net performance. Similarly, strategies that allocate significant weights to less liquid assets may face implementation challenges in practice.

For the specific dataset used in this analysis, the Maximum Diversification approach provided the best overall performance, particularly in minimizing downside risk. This highlights the importance of considering not just the distribution of capital, but also the distribution of risk contributions and diversification benefits when constructing investment portfolios.

3 VaR of a Bond

3.1 Probability of a 10% decline in the bond price within a 30-day period

The price at the time of issuance of a bond with annual coupon payments is expressed as follows:

$$P(y) = \sum_{t=1}^{T} \frac{C}{(1+y)^t} + \frac{FV}{(1+y)^T} \tag{7}$$

Where:

- C : Coupon payment
- FV : Face value
- T : Maturity of the bond (in years)
- y : Yield to maturity
- P : Bond price.

Meanwhile, if we consider any elapsed days since issuance, (7) is generalized as follows:

$$P(y_X, X) = \sum_{t=1}^{T} \frac{C}{(1+y_X)^{t-X}} + \frac{FV}{(1+y_X)^{T-X}} \tag{8}$$

Where X is $0 \leq X \leq 1$ and denotes days elapsed in years. For simplifying the calculation, we use the 30/360 day-count convention in the following discussion.

By computationally solving (8) for y with $P = 99$, $T = 10$, $C = 5$, $FV = 100$ and $X = 0$, we obtain the YTM at issuance as $y_0 = 0.05130325$. Likewise, solving (8) for y with $P = 99(1-0.1) = 89.1$ and $X = 30/360$ and other conditions constant, we obtain the YTM as of 30th day with the 10% price decline as $y_{30/360drop} = 0.06588528$.

Hence, there will be more than 10% decline in the bond price in 30 days, if the yield goes up more than $y_{30/360drop} - y_0 = 0.06588528 - 0.05130325 = 0.01458203$.

Assuming that the daily yield change conforms to the Gaussian distribution with a mean of 0 and a standard deviation of 0.006, the change in yields in 30 days conforms to $N(0, (0.006 \cdot \sqrt{30})^2)$ as long as the daily yield change is i.i.d.

Thus, the probability of 30$^{\text{th}}$ day yield change exceeds 0.01458203 is gained as follows:

$$Prob(y_{30/360drop} \geq 0.01458203) = 1 - \Phi\left(\frac{0.01458203}{0.006 \cdot \sqrt{30}}\right) = 0.32862359$$

Where $Prob$ denotes the probability and $\Phi(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.

3.2 Value at Risk (VaR) of the bond at a 99% confidence level across various horizons (1, 10, 20, 30,...,90 days)

The Value at Risk (VaR) for the bond at a 99% confidence level across horizons of 1, 10, 20, 30,...,90 days are computed by six different methods with the details given below. The results are summarized in Table 7.

3.2.1 Exact formula

In the region $P > 0$, under the conditions of $T = 10$, $C = 5$ and $FV = 100$, (7) (or (8) likewise) gives a monotonically decreasing function of y, and P is determined one-to-one against y. Hence, given a 99th percentile value of the change in y, the corresponding price change gives the VaR at 99% confidence level in price change.

Therefore, for each X = 1/360,10/360, 20/360, ... , 90/360, the VaR at 99% confidence level in price change is obtained by (8) as follows:

$$VaR_{0.99,price}(X) = P(y_0 + VaR_{0.99,yield}(X), X) - P(y_0, 0) \tag{9}$$

Where:

$$VaR_{0.99,yield}(X) = 0.006 \cdot \sqrt{360X} \cdot \Phi^{-1}(0.99) \tag{10}$$

3.2.2 Exact formula via delta approximation (Taylor's formula truncated to the first order)

Using (8) and Taylor's formula truncated to the first order, the price change since issuance corresponding to the change in time ($dX = X$) and yield ($dy_0 = y_X - y_0$), is approximated as follows:

$$P(y_X, X) = P(y_0, 0) + \frac{\partial P(y_0, 0)}{\partial X} X + \frac{\partial P(y_0, 0)}{\partial y} dy_0 \tag{11}$$

Where:

$$\frac{\partial P(y_0, 0)}{\partial y} = -\sum_{t=1}^{T} \frac{tC}{(1+y_0)^{t+1}} - \frac{T \cdot FV}{(1+y_0)^{T+1}} \tag{12}$$

$$\frac{\partial P(y_0, 0)}{\partial X} = \sum_{t=1}^{T} \frac{C \cdot \ln(1+y_0)}{(1+y_0)^{t}} + \frac{FV \cdot \ln(1+y_0)}{(1+y_0)^{T}} \tag{13}$$

Hence, for each X = 1/360, 10/360, 20/360, ... , 90/360, the VaR at 99% confidence level in price change is obtained from (9) and (11), by imposing $dy_0 = VaR_{0.99,yield}(X)$:

$$\begin{aligned} VaR_{0.99,price}(X) &= P(y_0, 0) + \frac{\partial P(y_0, 0)}{\partial X} X + \frac{\partial P(y_0, 0)}{\partial y} VaR_{0.99,yield}(X) - P(y_0, 0) \\ &= \frac{\partial P(y_0, 0)}{\partial X} X + \frac{\partial P(y_0, 0)}{\partial y} VaR_{0.99,yield}(X) \end{aligned} \tag{14}$$

Where $VaR_{0.99,yield}(X)$ is obtained from (10).

3.2.3 Exact formula via delta-gamma approximation (Taylor's formula truncated to the second order)

Using (8) and Taylor's formula truncated to the first order for X and second order for y, the price change since issuance corresponding to the change in time and yield is approximated as follows:

$$P(y_X, X) = P(y_0, 0) + \frac{\partial P(y_0, 0)}{\partial X} X + \frac{\partial P(y_0, 0)}{\partial y} dy_0 + \frac{1}{2} \frac{\partial^2 P(y_0, 0)}{\partial y^2} dy_0^2 \tag{15}$$

Where:

$$\frac{\partial^2 P(y_0, 0)}{\partial y^2} = \sum_{t=1}^{T} \frac{t(t+1)C}{(1+y_0)^{t+2}} + \frac{T(T+1)FV}{(1+y_0)^{T+2}} \tag{16}$$

Hence, for each X = 1/360, 10/360, 20/360, ... , 90/360, the VaR at 99% confidence level in price change is obtained from (9) and (15) as:

$$VaR_{0.99,price}(X) = \frac{\partial P(y_0, 0)}{\partial X} X + \frac{\partial P(y_0, 0)}{\partial y} VaR_{0.99,yield}(X) + \frac{1}{2} \frac{\partial^2 P(y_0, 0)}{\partial y^2} VaR^2_{0.99,yield}(X) \tag{17}$$

3.2.4 Monte Carlo simulation with 10,000 simulations with delta approximation, delta-gamma approximation, and full revaluation

As we noted in 3.2.1, in the region $P > 0$, (7) and (8) give a monotonically decreasing function of y, and P and y are determined one-to-one. Hence, using $VaR_{0.99,yield}(X)$ obtained from the simulated observations, we can also obtain $VaR_{0.99,yield}(X)$ from any of the equations (9), (14), or (17). We simulated 10,000 samples for each equation.

3.2.5 Discussion of the results

Firstly, Table 6 shows the prices at different horizons calculated with the constant YTM at issuance. The table suggests there are increases in price about 0.0138 per day without any external factor. While this value is almost negligible in a short-horizon, it should be noted that in 90 days horizon it accrues more than one percent of the original price. In addition, the effect of time-lapse also affects the accuracy of price estimation via Taylor approximation as implied in (12) and (16). These facts suggest that the equation (8) provides more accurate estimates than (7) when we calculate the bond price, and the necessity to derive sensitivity against a time elapse, as we did in (13).

Table 7 summarizes VaRs obtained from each approach described in 3.2.1 to 3.2.4. Major findings from the results are as follows:

- With 10,000 simulations, monte carlo estimates of the VaR of yield change at 99% confidence level provide stable values with less than 3% errors in their magnitude at all horizons up to 90 days. Due to this fact, exact formula combined with monte carlo simulation ($Exact_{MC}$) provides overall the best estimate of the VaR of price change when evaluated with $Exact$ as benchmark. However, it should be noted that this comparative advantage may not be observed in the shortest horizon when the error caused by the simulation is prevalent, because the estimation errors in the shortest horizon are marginal .

- In general, first-order Taylor approximation ($T1st$ and $T1st_{MC}$) provided less precise estimate than the second-order approximation ($T2nd$ and $T2nd_{MC}$), while the former overestimating and the latter underestimating the price change, as the horizon gets longer. These results illustrate the limitation of Taylor approximation approach, where the estimate will be inaccurate as the risk factor (yield) gets larger in longer horizons. One method to avoid these setbacks and obtain more precise estimate is to implement the estimation day-by-day basis up to the end of horizon and accumulate the changes, then take the VaR of an arbitrary confidence level from the accumulated figure. However, we should note that this approach consumes a lot of computational time as this requires monte carlo simulation of monte carlo simulation as a result. In our case, the full revaluation can be done with the exact formula, there is no comparative merit to take this approach.

- As for $T2nd$, reversal of the impact was observed in horizons exceeding 50 days, as the convexity effect gets stronger with the larger yield changes. However, in the case of $T2nd_{MC}$, this reversal tendency was more muted as the horizon reaches 90 days. This is assumed to be attributed to the facts that the second-order component has non-linear impact against the yield change, and outliers with very strong second-order component were truncated when we see the percentile value. As a result, monte carlo approach provided better estimate of VaR of price change for long horizons when we use the second-oreder Taylor approximation than the parametric one.

	$P_{constYTM}$	$(P_{constYTM} - 99)$
Days		
1	99.0138	0.0138
10	99.1377	0.1377
20	99.2756	0.2756
30	99.4136	0.4136
40	99.5519	0.5519
50	99.6903	0.6903
60	99.8290	0.8290
70	99.9678	0.9678
80	100.1068	1.1068
90	100.2460	1.2460

Table 6: Price changes due to elapse of time

	$VaR_{0.99,yield}$		$VaR_{0.99,price}$					
	Parametric	MC	Exact	T1st	T2nd	$Exact_{MC}$	$T1st_{MC}$	$T2nd_{MC}$
Days								
1	0.0140	0.0140	9.9421	10.6295	9.9093	9.9411	10.6421	9.9083
10	0.0441	0.0449	27.2963	33.5240	26.3221	27.6876	34.2576	26.6607
20	0.0624	0.0616	35.5720	47.3335	32.9296	35.2156	46.9731	32.6760
30	0.0765	0.0768	40.9947	57.8994	36.2935	41.1297	58.5871	36.3639
40	0.0883	0.0894	45.0254	66.7863	37.9785	45.4144	68.2166	37.8691
50	0.0987	0.0961	48.2119	74.6003	38.5905	47.3961	73.3094	38.2777
60	0.1081	0.1066	50.8271	81.6521	38.4404	50.3846	81.3131	38.3782
70	0.1168	0.1161	53.0284	88.1264	37.7127	52.8370	88.5496	38.3169
80	0.1248	0.1248	54.9159	94.1434	36.5279	54.9092	95.2299	38.1957
90	0.1324	0.1289	56.5569	99.7868	34.9693	55.7195	98.3616	38.0920

Table 7: VaR by different methods

Finally, considerations on main features and notable differences of each methodology are given below:

Expected Shortfall (ES) vs Value at Risk (VaR) The **Expected Shortfall (ES)** captures the average tail loss beyond the **Value at Risk (VaR)** threshold, not just at the cut-off point. ES generally exceeds VaR as it takes into account the *average loss in the worst-case scenarios* beyond the VaR limit.

VaR provides only the minimum loss at a given confidence level and does not fully account for tail risk or the severity of losses beyond the threshold. While VaR identifies extreme losses, it does not give a complete picture of the distribution's tail. In contrast, ES incorporates these extreme losses, making it a more robust and coherent measure of downside risk under both stress testing and regulatory requirements.

Monte Carlo Simulation vs Parametric Methods The **Monte Carlo simulation method** offers greater flexibility and accuracy when compared to traditional parametric approaches. Parametric methods typically assume a Gaussian (normal) distribution of returns, which can limit their effectiveness in capturing real-world market behaviors such as fat tails, skewness, or nonlinear dynamics.
Our findings imply that Monte Carlo simulations are particularly effective for portfolios containing derivative instruments or the price based on the Gaussian risk factor exhibiting non-Gaussian characteristics.

By simulating a wide range of scenarios, Monte Carlo approaches can provide a comprehensive view of potential portfolio outcomes even when an analitical solution is not available.

Delta Approximation The **Delta approximation** method utilizes linear sensitivity—such as bond duration—to estimate how bond prices respond to changes in yield (Yield to Maturity, YTM). This first-order approximation is computationally efficient and suitable for large portfolios requiring rapid risk estimation.

However, the Delta approximation has a key limitation: it neglects **convexity**, which represents the curvature in the price-yield relationship. As a result, this method can overestimate risk, especially in volatile markets or for instruments with longer maturities. Despite its limitations, the Delta approximation remains a useful tool for quick and approximate risk assessments.

3.3 Expected Shortfall (ES) of the bond at a 99% confidence level across various horizons (1, 10, 20, 30,...,90 days)

There is no analytical solution of the bond price as well as the conditional expectation of it in our case, while the Taylor approximation performs poorly in longer horizons as we saw in 3.2. Thus, we adopted here the full revaluation approach by monte carlo simulation, combined with the exact formula. The specific procedures are as follows:

1. Draw 10,000 samples of yield change from $N(0, (0.006 \cdot \sqrt{360X})^2)$ where X denotes the corresponding horizon in years. We call this yield Δy.
2. Calculate the bond price for each yield change obtained from the step 1 using the formula (8), precisely $P(y_0 + \Delta y, X)$. Then, calculate $\Delta P = P(y_0 + \Delta y, X) - P(y_0, 0)$.
3. Sort the values of each ΔP and take average of the smallest 100 values out of 10,000 samples (i.e. take the average of samples up to and including 1st percentile values). The result is our estimation of the expected shortfall of the bond price at a 99% confidence level for the horizon of X.
4. Repeat 1 to 3 for each X = 1/360, 10/360, 20/360, ... 90/360.

The results are summarized in Table 8 below. the column $ES_{0.99,price}$ shows the estimated expected shortfalls of the bond price, and the column $VaR_{0.99,price,Exact}$ shows VaRs of the bond price calculated by Exact method for comparison.

	$\text{ES}_{0.99,price}$	$\text{VaR}_{0.99,price,Exact}$	Difference
Days			
1	11.2875	9.9421	1.3454
10	30.3492	27.2963	3.0528
20	39.0404	35.5720	3.4685
30	45.8662	40.9947	4.8715
40	50.1058	45.0254	5.0805
50	53.4984	48.2119	5.2865
60	56.0092	50.8271	5.1821
70	57.1819	53.0284	4.1534
80	59.2447	54.9159	4.3289
90	60.7755	56.5569	4.2186

Table 8: Expected shortfall estimated by monte carlo simulations with exact method