EECE 5644 : Machine Learning **September 28, 2019**

Homework #1

Name: Rohin Arora *NUID:* 001302812

Problem 1 (10 %)

(1) Variance(x) - $Var(x)$; Expectation(x) - $E(x) - \mu$

$$\begin{aligned}
Var(x) &= E[(x-\mu)^2] \\
&= E[x^2 + \mu^2 - 2*x*\mu] \\
&= E[x^2] + E[\mu^2] - 2*E[x*\mu] && \text{linearity of expectation} \\
&= E[x^2] + \mu^2 - 2*\mu*E[x] && \text{linearity of expectation} \\
&= E[x^2] + \mu^2 - 2*\mu^2 \\
&= E[x^2] - \mu^2
\end{aligned}$$

■

(2) $E[\vec{x}] = \vec{\mu}$; Covariance($\vec{x}$)=$cov(\vec{x})$

$$\begin{aligned}
cov(\vec{x}) &= E[(\vec{x}-\vec{\mu})((\vec{x}-\vec{\mu})^T] \\
&= E[(\vec{x}*\vec{x}^T - \vec{x}*\vec{\mu}^T - \vec{\mu}*\vec{x}^T + \vec{\mu}*\vec{\mu}^T] \\
&= E[(\vec{x}*\vec{x}^T] - E[\vec{x}*\vec{\mu}^T] - E[\vec{\mu}*\vec{x}^T] + E[\vec{\mu}*\vec{\mu}^T] && \text{linearity of expectation} \\
&= E[(\vec{x}*\vec{x}^T] - E[\vec{x}]*\vec{\mu}^T - \vec{\mu}*E[\vec{x}^T] + E[\vec{\mu}*\vec{\mu}^T] && \text{linearity of expectation} \\
&= E[(\vec{x}*\vec{x}^T] - \vec{\mu}*\vec{\mu}^T - \vec{\mu}*\vec{\mu}^T + \vec{\mu}*\vec{\mu}^T && \text{linearity of expectation} \\
&= E[(\vec{x}*\vec{x}^T] - \vec{\mu}*\vec{\mu}^T
\end{aligned}$$

■

Problem 2 (20 %)

(1)

- Class conditional probabilities shall sum to 1
- Let the constant of integration with class l be k_l

$$\int_{-\infty}^{\infty} P(X|L=l)dx = k_l * \int_{-\infty}^{\infty} e^{-\frac{|x-a_l|}{b_l}} dx$$

- This integral should sum to 1.
- Consider a modified integral.

$$f(x) = k_l * \int_{a_l}^{\infty} e^{-\frac{x-a_l}{b_l}} dx$$

- The above integral should sum to 1/2.
- Continuing with this $f(x)$

$$\begin{aligned}
f(x) &= k_l * \int_{a_l}^{\infty} e^{-\frac{x-a_l}{b_l}} dx \\
&= k_l * e^{\frac{a_l}{b_l}} * \int_{a_l}^{\infty} e^{-\frac{x}{b_l}} dx \\
&= k_l * e^{\frac{a_l}{b_l}} * (-b_l) * e^{-\frac{x}{b_l}} \Big|_{a_l}^{\infty} \\
&= k_l * e^{\frac{a_l}{b_l}} * (-b_l) * (0 - e^{-\frac{a_l}{b_l}}) \\
&= k_l * e^{\frac{a_l}{b_l}} * (b_l) * (e^{-\frac{a_l}{b_l}}) \\
&= k_l * b_l \\
&= \frac{1}{2}
\end{aligned}$$

$$k_l = \frac{1}{2 * b_l}$$

$$P(X|L=1) = \frac{1}{2 * b_1} * e^{-\frac{|x-a_1|}{b_1}}$$

$$P(X|L=2) = \frac{1}{2 * b_2} * e^{-\frac{|x-a_2|}{b_2}}$$

(2)

$$\begin{aligned}
l(x) &= \ln P(X|L=1) - \ln P(X|L=2) \\
&= \ln \frac{1}{2 * b_1} - \ln \frac{1}{2 * b_2} - \frac{|x-a_1|}{b_1} + \frac{|x-a_2|}{b_2} \\
&= \frac{|x-a_2|}{b_2} - \frac{|x-a_1|}{b_1} + \ln \frac{b2}{b_1}
\end{aligned}$$

(3) Setting the values $a_1 = 0, b_1 = 1, a_2 = 1, b_2 = 2$

$$l(x) = \frac{|x-1|}{2} - |x| + \ln 2$$

This function is plotted in the following notebook- Q2_Git_Repo

Problem 3 (20 %)

Minimum probability of error classification rule implies 0 - 1 loss. Also, the 2 classes have equal priors. Therefore, the decision rule simplifies to maximum likelihood estimator. Suppose we have 2 labels, 1 and 2. In such a scenario, the decision rule is-

Decide label 1 if $P(x|L = 1) > P(x|L = 2)$ **Else decide label 2** (contentions resolved arbitrarily)

Rephrasing-
When $a < x < r$ - **class 1**
When $r < x < b$ - **class 1** if $1/(b - a) > 1/(t - r)$ else **class 2**
When $b < x < t$ - **class 2**

$$P(x|L = 1) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \quad (0.1) \\ 0 & elsewhere \quad (0.2) \end{cases}$$

$$P(x|L = 2) = \begin{cases} \dfrac{1}{t-r} & r \leq x \leq t \quad (0.3) \\ 0 & elsewhere \quad (0.4) \end{cases}$$

$$\frac{P(x|L = 1)}{P(x|L = 2)} \gtrless^{class1}_{class2} 1$$

$$\frac{1}{b-a} \gtrless^{class1}_{class2} \frac{1}{t-r}$$

The accompanying code for this question with visual example is present here- Q3_Git_Repo

Problem 4 (30 %)

Problem 5 (20 %)

References and Acknowledgements

1. Python
2. Scientific python stack
3. Variance
4. Latex
5. Lecture Notes
6. Iridescent
7. DON'T PANIC