D8: Random Signals with Unknown Parameters

We consider the detection of a Gaussian random signal with unknown parameters in WGN. The extension to colored Gaussian noise is simple.

Incompletely Known Signal Covariance

$\mathcal{H}_0: x[n] = w[n]$ $\quad n = 0, \ldots, N-1$

$\mathcal{H}_1: x[n] = s[n] + w[n]$ $\quad$ "

$s[n]$ is a Gaussian random process with 0-mean, cov C_s.

$w[n] \sim$ WGN with var σ^2. $s[n] \perp\!\!\!\perp w[n]$.

The signal covariance C_s depends on some unknown parameters. Consider an example with $N=2$ and $C_s = r_{ss}[0]\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, where the $\rho = r_{ss}[1]/r_{ss}[0]$ is the corr. coef. and is known. The signal power $r_{ss}[0]$ is unknown. The pdf under $\mathcal{H}_1$ is

$$p(x; r_{ss}[0], \mathcal{H}_1) = \frac{1}{2\pi \det^{1/2}(C_s + \sigma^2 I)} e^{-\frac{1}{2} x^T (C_s + \sigma^2 I)^{-1} x}$$

Let $r_{ss}[0] = P_0$ (for notation simplicity). Then, with $C = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$

$\det(P_0 C + \sigma^2 I) = \prod_{i=1}^{N} (P_0 \lambda_i + \sigma^2)$ (easily seen using the eigendecomposition $C = V \Lambda V^{-1}$ $(V^{-1} = V^T)$. Also, we have

$(P_0 C + \sigma^2 I)^{-1} = V(P_0 \Lambda + \sigma^2 I)^{-1} V^T$.

$$\therefore \ln p(x; P_0, \mathcal{H}_1) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{N} \ln(P_0 \lambda_i + \sigma^2) - \frac{1}{2} x^T V (P_0 \Lambda + \sigma^2 I)^{-1} V^T x$$

$$= -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{N}\ln(P_0\lambda_i+\sigma^2) - \frac{1}{2}\sum_{i=1}^{N}\frac{(v_i^T x)^2}{P_0\lambda_i+\sigma^2}$$

To find the MLE of P_0, we can minimize

$$J(P_0) = \sum_{i=1}^{N}\left[\ln(P_0\lambda_i+\sigma^2) + \frac{(v_i^T x)^2}{P_0\lambda_i+\sigma^2}\right].$$

Ex] Unknown Signal Power (white signal)

$$\lambda_i = \lambda \ \forall i : \ J(P_0) = N\ln(P_0\lambda+\sigma^2) + \frac{1}{(P_0\lambda+\sigma^2)}\sum_{i=1}^{N}(v_i^T x)^2$$

But $\sum_{i=1}^{N}(v_i^T x)^2 = x^T V V^T x = x^T x$, so $J(P_0) = N\ln(P_0\lambda+\sigma^2) + \dfrac{x^T x}{P_0\lambda+\sigma^2}$

Differentiating and equating to zero...

$$\hat{P}_0 = \begin{cases} \dfrac{1}{N\lambda}\sum_{n=0}^{N-1} x^2[n] - \dfrac{\sigma^2}{\lambda} \ \leftarrow \hat{P}_0^+ & \text{if } \hat{P}_0^+ > 0 \\ 0 & \text{if } \hat{P}_0^+ \le 0 \end{cases}$$

$$= \max(0, \hat{P}_0^+)$$

The GLRT decides $\mathcal{H}_1$ if $\ln L_G(x) = \ln\dfrac{p(x;\hat{P}_0,\mathcal{H}_1)}{p(x;\mathcal{H}_0)} > \ln\gamma$.

$$\ln L_G(x) = -\frac{1}{2}J(\hat{P}_0) + \frac{N}{2}\ln\sigma^2 + \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]$$

$$\Rightarrow \ln L_G(x) = -\frac{N}{2}\ln(\hat{P}_0\lambda+\sigma^2) - \frac{1}{2}\left(\sum_{n=0}^{N-1}x^2[n]\right)\Big/(\hat{P}_0\lambda+\sigma^2) + \frac{N}{2}\ln\sigma^2 + \frac{1}{2\sigma^2}\sum_{n=0}^{N-1}x^2[n]$$

$$= \cdots = \frac{N}{2}\left[\left(\frac{\hat{P}_0^+\lambda+\sigma^2}{\hat{P}_0\lambda+\sigma^2}\right)\frac{\hat{P}_0\lambda}{\sigma^2} - \ln\left(\frac{\hat{P}_0\lambda}{\sigma^2}+1\right) - 1\right]$$

If $\hat{P}_0 = 0$, then $\ln L_G(x) = 0$ ($\ln \gamma > 0$), so we choose $\mathcal{H}_0$.

If $\hat{P}_0 > 0$ so that $\hat{P}_0 = \hat{P}_0^+$, then

$$\ln L_G(x) = \frac{N}{2}\left[\left(\frac{\hat{P}_0 \lambda}{\sigma^2}+1\right) - \ln\left(\frac{\hat{P}_0\lambda}{\sigma^2}+1\right) - 1\right]$$

Note that $g(x) = x - \ln x - 1$ is monotonically increasing for $x > 1$, since $dg/dx = 1 - 1/x > 0$. Thus, for $x > 1$, g^{-1} exists. Letting $x = \frac{\hat{P}_0 \lambda}{\sigma^2} + 1 > 1$ (since $\hat{P}_0 > 0$), we decide $\mathcal{H}_1$, if $\frac{N}{2} g\left(\frac{\hat{P}_0\lambda}{\sigma^2}+1\right) > \ln\gamma$ or if $\hat{P}_0 > \frac{\sigma^2}{\lambda}\left[g^{-1}\left(\frac{2}{N}\ln\gamma\right) - 1\right] = \gamma'$.

Large Data Record Approximations

If the signal random process is WSS, then for large N,

$$l(x) = \ln \frac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = -\frac{N}{2}\int_{-1/2}^{1/2}\left[\ln\left(\frac{P_{ss}(f)}{\sigma^2}+1\right) - \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2}\,\frac{I(f)}{\sigma^2}\right]df$$

from Section D5. The PSD $P_{ss}(f;\theta)$ has unknown parameters. Let $\hat{\theta}$ be the MLE of θ under $\mathcal{H}_1$. Then GLRT decides $\mathcal{H}_1$ if $\ln L_G(x) = \ln \frac{p(x;\hat{\theta},\mathcal{H}_1)}{p(x;\mathcal{H}_0)} = -\frac{N}{2} J(\hat{\theta}) > \ln\gamma$, or if $-J(\hat{\theta}) > \gamma'$, or if $-\min_{\theta} J(\theta) > \gamma'$.

Ex) Unknown Signal Power

Assume $P_{ss}(f;P_0) = P_0 Q(f)$ where $\int_{-1/2}^{1/2} Q(f)\,df = 1$ so that P_0 is the total power of $s[n]$. The MLE is found by minimizing $J(P_0) = \int_{-1/2}^{1/2} \left[\ln\left(\frac{P_0 Q(f)}{\sigma^2} + 1 \right) - \frac{P_0 Q(f)}{P_0 Q(f) + \sigma^2} \frac{I(f)}{\sigma^2} \right] df$.

$\frac{\partial J(P_0)}{\partial P_0} = \int_{-1/2}^{1/2} \left[\frac{Q(f)}{P_0 Q(f) + \sigma^2} - \frac{Q(f)}{(P_0 Q(f) + \sigma^2)^2} I(f) \right] df = 0$ yields

an implicit equation for the MLE of P_0. This could be numerically solved in general.

For low SNR ($P_0 Q(f) << \sigma^2$), we have

$$\int_{-1/2}^{1/2} \frac{Q(f)(P_0 Q(f) + \sigma^2) - Q(f) I(f)}{\sigma^4} df \approx 0$$

which yields $\hat{P}_0 \approx \left[\int_{-1/2}^{1/2} Q(f)(I(f) - \sigma^2) df \right] / \left[\int_{-1/2}^{1/2} Q^2(f) df \right]$,

assuming $\hat{P}_0 > 0$. For $\hat{P}_0 \le 0$, we set $\hat{P}_0 = 0$.

The GLRT decides $\mathcal{H}_1$ if

$$T(x) = \int_{-1/2}^{1/2} \left[-\ln\left(\frac{\hat{P}_0 Q(f)}{\sigma^2} + 1 \right) + \frac{\hat{P}_0 Q(f)}{\hat{P}_0 Q(f) + \sigma^2} \frac{I(f)}{\sigma^2} \right] df > \gamma'.$$

Weak Signal Detection

$\mathcal{H}_0 : P_0 = 0$

$\mathcal{H}_1 : P_0 > 0$

This is a one-sided test. Assume P_0 is small in $\mathcal{H}_1$. Then the locally most powerful (LMP) test can be used. Omitting the normalizing factor $\sqrt{I(P_0)}$, this detector decides $\mathcal{H}_1$ if $T(x) = \left. \frac{\partial \ln p(x; P_0, \mathcal{H}_1)}{\partial P_0} \right|_{P_0 = 0} > \gamma$.

In this case, no MLE evaluation is required.

The LMP test is computationally very simple.

Ex| LMP Detecter for Unknown Power Signal

Under $\mathcal{H}_1$: $\ln p(x; P_0, \mathcal{H}_1) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln\det(P_0 C + \sigma^2 I)$
$$- \frac{1}{2} x^T (P_0 C + \sigma^2 I)^{-1} x$$

Note that $\dfrac{\partial \ln\det C(\theta)}{\partial\theta} = \mathrm{tr}\left(C^{-1}(\theta)\dfrac{\partial C(\theta)}{\partial\theta}\right)$

$$\frac{\partial C^{-1}(\theta)}{\partial\theta} = -C^{-1}(\theta)\frac{\partial C(\theta)}{\partial\theta}C^{-1}(\theta)$$

Letting $C(P_0) = P_0 C + \sigma^2 I$, and noting that $\partial C(P_0)/\partial P_0 = C$:

$$\frac{\partial \ln p(x; P_0, \mathcal{H}_1)}{\partial P_0} = -\frac{1}{2}\mathrm{tr}\left(C^{-1}(P_0)\frac{\partial C(P_0)}{\partial P_0}\right) + \frac{1}{2}x^T C^{-1}(P_0)\frac{\partial C(P_0)}{\partial P_0}C^{-1}(P_0)x$$
$$= -\frac{1}{2}\mathrm{tr}\left((P_0 C + \sigma^2 I)^{-1} C\right) + \frac{1}{2}x^T (P_0 C + \sigma^2 I)^{-1} C (P_0 C + \sigma^2 I)^{-1} x$$

Evaluating at $P_0 = 0$, yields

$$T(x) = -\frac{1}{2\sigma^2}\mathrm{tr}(C) + \frac{1}{2}\frac{x^T C x}{\sigma^4}$$

so we decide $\mathcal{H}_1$ if

$$x^T C x > 2\sigma^4\left(\gamma + \frac{1}{2\sigma^2}\mathrm{tr}(C)\right) = \gamma'.$$

Suggested Problems: 1, 4, 6, 8, 12, 13