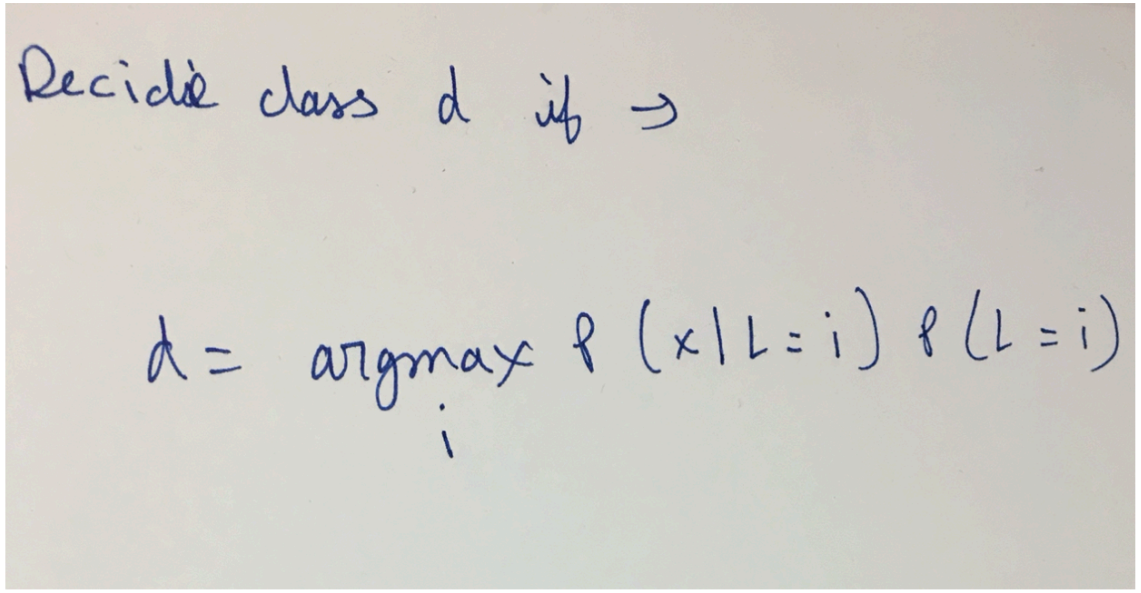
If Github links are not clicky, please enter the link into your browser. Or refer the comments section for the hyperlink. Or refer the code in ZIP file. Or the code in this PDF

Github repo for this exam- https://github.com/rohinarora/EECE5644-Machine_Learning/tree/master/Exam1

Additionally all the code and figures are present in this file, and also attached on blackboard as additional reference Please refer whatever you find convenient

Answer 1 **Github https://github.com/rohinarora/EECE5644-Machine_Learning/blob/master/Exam1/Q1.ipynb**

- 3 Gaussian classes are generated by applying linear transformations to a gassuian with zero mean and I (identity) covariance.
- Minimizing P(error) implies MAP estimate.
- For MAP, we need to select the class that maximizes un-normalized posterior-
  - decide class D = argmax (i) P(x|L=i)P(L=i)
  - 

- Number of samples generated from each class-
  - Class 1- 1492
  - Class 2- 3550
  - Class 3- 4958

In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
```

In [2]:
```
def multivariate_normal(x, d, mean, covariance):
    """pdf of the multivariate normal distribution."""
    x_m = x - mean
    return (1. / (np.sqrt((2 * np.pi)**d * np.linalg.det(covariance))) *
            np.exp(-(np.linalg.solve(covariance, x_m).T.dot(x_m)) / 2))
def highlight_max(data, color='yellow'):
    '''
    highlight the maximum in a Series or DataFrame
    '''
    attr = 'background-color: {}'.format(color)
    if data.ndim == 1:  # Series from .apply(axis=0) or axis=1
        is_max = data == data.max()
        return [attr if v else '' for v in is_max]
    else:  # from .apply(axis=None)
        is_max = data == data.max().max()
        return pd.DataFrame(np.where(is_max, attr, ''),
                            index=data.index, columns=data.columns)
```

In [187]:
```
def generate_GMM_samples(prior,number_of_samples,sig1,sig2,sig3,u1,u2,u3
):
    '''
    Args:
    prior of class 1= prior[0]
    prior of class 2 = prior[1]
    prior of class 3 = 1-prior[0]-prior[1]

    number_of_samples

    class 1- u_1, sig_1
    class 2- u_2, sig_2
    class 2- u_3, sig_3

    x is samples from zero-mean identity-covariance Gaussian sample gene
rators

    generating class 1- A1*x+b1
    generating class 2- A2*x+b2
    generating class 2- A3*x+b3

    Minimizing P(error) implies MAP estimate.

    For MAP, we need to find un-normalized posterior- P(x|L=i)P(L=i).  L
et us call it Check(i) for class i.

    Check(L=1)=N(u_1, sig_1)*Prior(L=1)
    Check(L=2)=N(u_2, sig_2)*Prior(L=2)
    Check(L=3)=N(u_3, sig_3)*Prior(L=3)

    Decide class i where Check(L=i) is greatest out of i=1 to 3

    Count the total misclassifications for class 1, class 2 and class 3

    P(error)= Total errors/Total samples
            = Total errors/10000

    '''

    from matplotlib.pyplot import figure
    txt="Plot of data sampled from 3 gaussians along with their MAP esti
mated class labels. "
    fig = plt.figure(figsize=(20,20));
    fig.text(.35,0.09,txt,fontsize=15);
    samples_class1=[]
    samples_class2=[]
    samples_class3=[]
    sig_1=np.matrix(sig1)
    sig_2=np.matrix(sig2)
    sig_3=np.matrix(sig3)
    u_1=np.matrix(u1).transpose()
    u_2=np.matrix(u2).transpose()
    u_3=np.matrix(u3).transpose()
    prior=prior
    A1=np.linalg.cholesky(sig_1)
    b1=u_1
```

```
    A2=np.linalg.cholesky(sig_2)
    b2=u_2

    A3=np.linalg.cholesky(sig_3)
    b3=u_3

    zero_mean=[0,0]
    cov=[[1,0],[0,1]]
    for i in range(number_of_samples):
        uniform_sample=np.random.uniform()

        sample_from_zero_mean_identity_covariance=np.random.multivariate
_normal(zero_mean,cov,[1]).transpose()

        if uniform_sample<prior[0]:
            '''sample from class class 1'''
            sample=A1.dot(sample_from_zero_mean_identity_covariance)+b1
            samples_class1.append(sample)
        elif uniform_sample>(prior[1]+prior[0]):
            '''sample from class class 3'''
            sample=A3.dot(sample_from_zero_mean_identity_covariance)+b3
            samples_class3.append(sample)
        else:
            '''sample from class class 2'''
            sample=A2.dot(sample_from_zero_mean_identity_covariance)+b2
            samples_class2.append(sample)

    samples_class1_final=np.hstack(samples_class1)
    samples_class2_final=np.hstack(samples_class2)
    samples_class3_final=np.hstack(samples_class3)
    a=np.squeeze(np.asarray(samples_class1_final.transpose()[:,1]))
    b=np.squeeze(np.asarray(samples_class1_final.transpose()[:,0]))

    c=np.squeeze(np.asarray(samples_class2_final.transpose()[:,1]))
    d=np.squeeze(np.asarray(samples_class2_final.transpose()[:,0]))

    e=np.squeeze(np.asarray(samples_class3_final.transpose()[:,1]))
    f=np.squeeze(np.asarray(samples_class3_final.transpose()[:,0]))


    # A list containing entries of class prediction for labels coming fr
om class i
    classify_1=[]
    classify_2=[]
    classify_3=[]

    for a in samples_class1_final.transpose():
        pred_class_1=multivariate_normal(a.transpose(), 2, u_1, sig_1)
        pred_class_2=multivariate_normal(a.transpose(), 2, u_2, sig_2)
        pred_class_3=multivariate_normal(a.transpose(), 2, u_3, sig_3)
        prediction=0
        if (pred_class_1> pred_class_2) and (pred_class_1> pred_class_3
):
            prediction=1
        elif(pred_class_2> pred_class_3):
            prediction=2
```

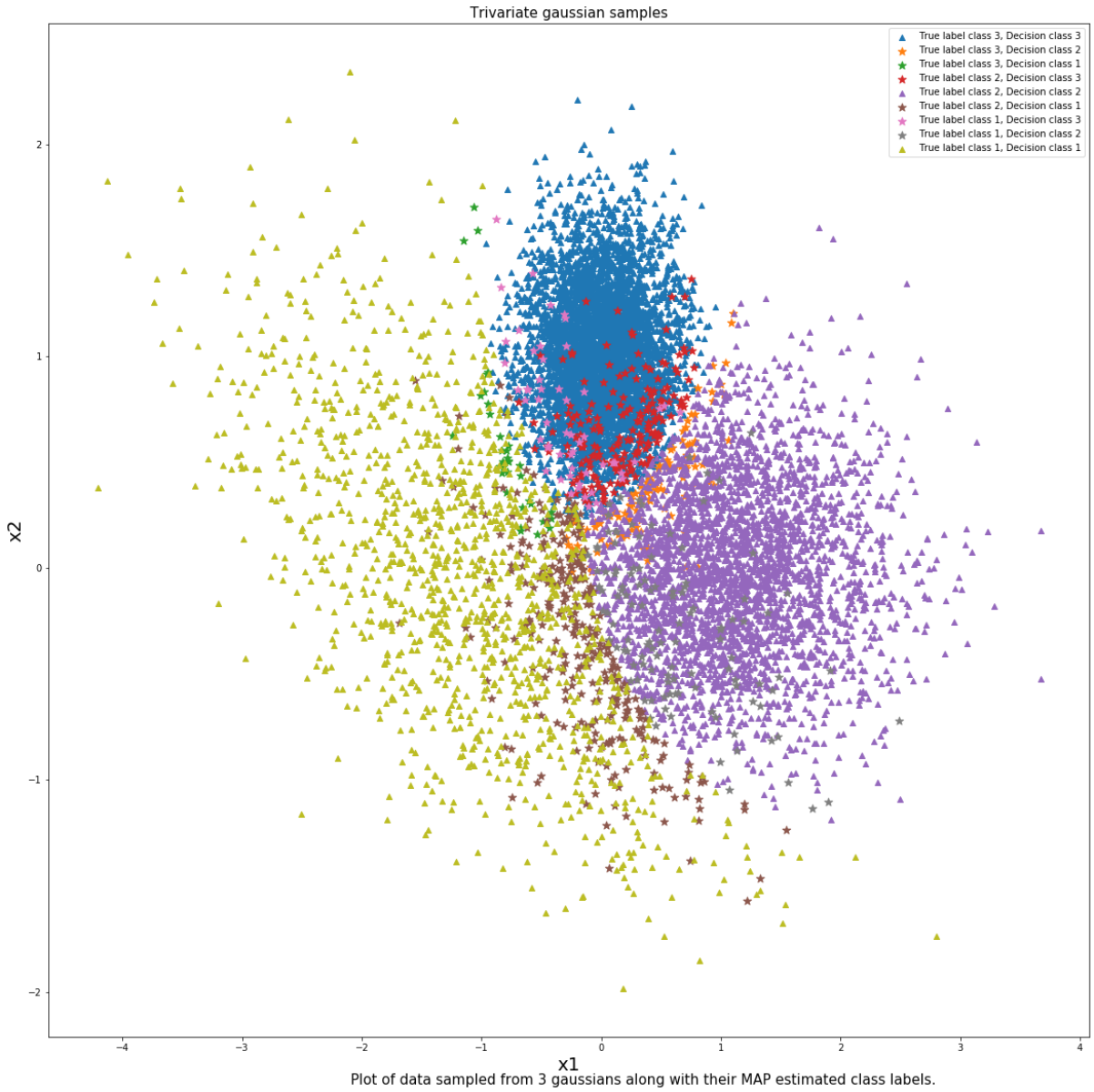
```
        else:
            prediction=3
        classify_1.append(prediction)


    for a in samples_class2_final.transpose():
        pred_class_1=multivariate_normal(a.transpose(), 2, u_1, sig_1)
        pred_class_2=multivariate_normal(a.transpose(), 2, u_2, sig_2)
        pred_class_3=multivariate_normal(a.transpose(), 2, u_3, sig_3)
        prediction=0
        if (pred_class_1> pred_class_2) and (pred_class_1> pred_class_3
):
            prediction=1
        elif(pred_class_2> pred_class_3):
            prediction=2
        else:
            prediction=3
        classify_2.append(prediction)

    for a in samples_class3_final.transpose():
        pred_class_1=(multivariate_normal(a.transpose(), 2, u_1, sig_1))
*prior[0]
        pred_class_2=(multivariate_normal(a.transpose(), 2, u_2, sig_2))
*prior[1]
        pred_class_3=(multivariate_normal(a.transpose(), 2, u_3, sig_3))
*(1-prior[0]-prior[1])
        prediction=0
        if (pred_class_1> pred_class_2) and (pred_class_1> pred_class_3
):
            prediction=1
        elif(pred_class_2> pred_class_3):
            prediction=2
        else:
            prediction=3
        classify_3.append(prediction)
    df_main=pd.DataFrame(data=[[np.sum(np.array(classify_1)==1),np.sum(n
p.array(classify_2)==1),np.sum(np.array(classify_3)==1)],\
                    [np.sum(np.array(classify_1)==2),np.sum(np.array(clas
sify_2)==2),np.sum(np.array(classify_3)==2)],
                    [np.sum(np.array(classify_1)==3),np.sum(np.array(clas
sify_2)==3),np.sum(np.array(classify_3)==3)]],columns=[1,2,3])
    df_main.index=[1,2,3]

    a=np.squeeze(np.asarray(samples_class1_final.transpose()[:,1]))
    b=np.squeeze(np.asarray(samples_class1_final.transpose()[:,0]))

    c=np.squeeze(np.asarray(samples_class2_final.transpose()[:,1]))
    d=np.squeeze(np.asarray(samples_class2_final.transpose()[:,0]))

    e=np.squeeze(np.asarray(samples_class3_final.transpose()[:,1]))
    f=np.squeeze(np.asarray(samples_class3_final.transpose()[:,0]))
    #plt.scatter(b,a,color='r',marker='*',label='class 1',s=50)
    #plt.scatter(d,c,color='g',marker='*',label='class 2',s=50)
    #plt.scatter(f,e,color='b',marker='*',label='class 3',s=50)


    df=pd.DataFrame(data= {'x': np.squeeze(np.asarray(samples_class3_fin
```

```
al.T[:,0])).tolist(),\
                    'y': np.squeeze(np.asarray(samples_class3_final.T[:,
1])).tolist(),\
                    'label': np.squeeze(np.asarray(classify_3))})
    plt.scatter((df[df['label']==3]['x']).values,(df[df['label']==3]['y'
]).values,label="True label class 3, Decision class 3",marker='^')
    plt.scatter((df[df['label']==2]['x']).values,(df[df['label']==2]['y'
]).values,label="True label class 3, Decision class 2",marker='*',s=80)
    plt.scatter((df[df['label']==1]['x']).values,(df[df['label']==1]['y'
]).values,label="True label class 3, Decision class 1",marker='*',s=80)


    df=pd.DataFrame(data= {'x': np.squeeze(np.asarray(samples_class2_fin
al.T[:,0])).tolist(),\
                    'y': np.squeeze(np.asarray(samples_class2_final.T[:,
1])).tolist(),\
                    'label': np.squeeze(np.asarray(classify_2))})
    plt.scatter((df[df['label']==3]['x']).values,(df[df['label']==3]['y'
]).values,label="True label class 2, Decision class 3",marker='*',s=80)
    plt.scatter((df[df['label']==2]['x']).values,(df[df['label']==2]['y'
]).values,label="True label class 2, Decision class 2",marker='^')
    plt.scatter((df[df['label']==1]['x']).values,(df[df['label']==1]['y'
]).values,label="True label class 2, Decision class 1",marker='*',s=80)


    df=pd.DataFrame(data= {'x': np.squeeze(np.asarray(samples_class1_fin
al.T[:,0])).tolist(),\
                    'y': np.squeeze(np.asarray(samples_class1_final.T[:,
1])).tolist(),\
                    'label': np.squeeze(np.asarray(classify_1))})
    plt.scatter((df[df['label']==3]['x']).values,(df[df['label']==3]['y'
]).values,label="True label class 1, Decision class 3",marker='*',s=80)
    plt.scatter((df[df['label']==2]['x']).values,(df[df['label']==2]['y'
]).values,label="True label class 1, Decision class 2",marker='*',s=80)
    plt.scatter((df[df['label']==1]['x']).values,(df[df['label']==1]['y'
]).values,label="True label class 1, Decision class 1",marker='^')



    plt.legend();
    plt.title('Trivariate gaussian samples',fontsize=15)
    plt.xlabel('x1',fontsize=20)
    plt.ylabel('x2',fontsize=20);
    plt.legend()
    plt.show();
    print ("Samples from class 1 - ", samples_class1_final.shape[1])
    print ("Samples from class 2 - ", samples_class2_final.shape[1])
    print ("Samples from class 3 - ", samples_class3_final.shape[1])
    from matplotlib.pyplot import figure
    txt="Plot of data sampled 1 gaussian along with their MAP estimates.
"
    fig = plt.figure(figsize=(20,20));
    fig.text(.35,0.09,txt,fontsize=15);
    df=pd.DataFrame(data= {'x': np.squeeze(np.asarray(samples_class1_fin
al.T[:,0])).tolist(),\
                    'y': np.squeeze(np.asarray(samples_class1_final.T[:,
1])).tolist(),\
```

```
                    'label': np.squeeze(np.asarray(classify_1))})
    plt.scatter((df[df['label']==3]['x']).values,(df[df['label']==3]['y'
]).values,label="True label class 1, Decision class 3",marker='*',s=80)
    plt.scatter((df[df['label']==2]['x']).values,(df[df['label']==2]['y'
]).values,label="True label class 1, Decision class 2",marker='*',s=80)
    plt.scatter((df[df['label']==1]['x']).values,(df[df['label']==1]['y'
]).values,label="True label class 1, Decision class 1",marker='^')


    plt.legend();
    plt.title('Gaussian samples from class 1, along with MAP decision la
bels',fontsize=15)
    plt.xlabel('x1',fontsize=20)
    plt.ylabel('x2',fontsize=20);
    plt.legend()
    plt.show();

    print ('\n\n\\n')

    from matplotlib.pyplot import figure
    txt="Plot of data sampled 1 gaussian along with their MAP estimates.
"
    fig = plt.figure(figsize=(20,20));
    fig.text(.35,0.09,txt,fontsize=15);
    df=pd.DataFrame(data= {'x': np.squeeze(np.asarray(samples_class2_fin
al.T[:,0])).tolist(),\
                    'y': np.squeeze(np.asarray(samples_class2_final.T[:,
1])).tolist(),\
                    'label': np.squeeze(np.asarray(classify_2))})
    plt.scatter((df[df['label']==3]['x']).values,(df[df['label']==3]['y'
]).values,label="True label class 2, Decision class 3",marker='*',s=80)
    plt.scatter((df[df['label']==2]['x']).values,(df[df['label']==2]['y'
]).values,label="True label class 2, Decision class 2",marker='^')
    plt.scatter((df[df['label']==1]['x']).values,(df[df['label']==1]['y'
]).values,label="True label class 2, Decision class 1",marker='*',s=80)


    plt.legend();
    plt.title('Gaussian samples from class 2, along with MAP decision la
bels',fontsize=15)
    plt.xlabel('x1',fontsize=20)
    plt.ylabel('x2',fontsize=20);
    plt.legend()
    plt.show();

    print ('\n\n\\n')
    from matplotlib.pyplot import figure
    txt="Plot of data sampled 1 gaussian along with their MAP estimates.
"
    fig = plt.figure(figsize=(20,20));
    fig.text(.35,0.09,txt,fontsize=15);
    df=pd.DataFrame(data= {'x': np.squeeze(np.asarray(samples_class3_fin
al.T[:,0])).tolist(),\
                    'y': np.squeeze(np.asarray(samples_class3_final.T[:,
1])).tolist(),\
                    'label': np.squeeze(np.asarray(classify_3))})
    plt.scatter((df[df['label']==3]['x']).values,(df[df['label']==3]['y'
```

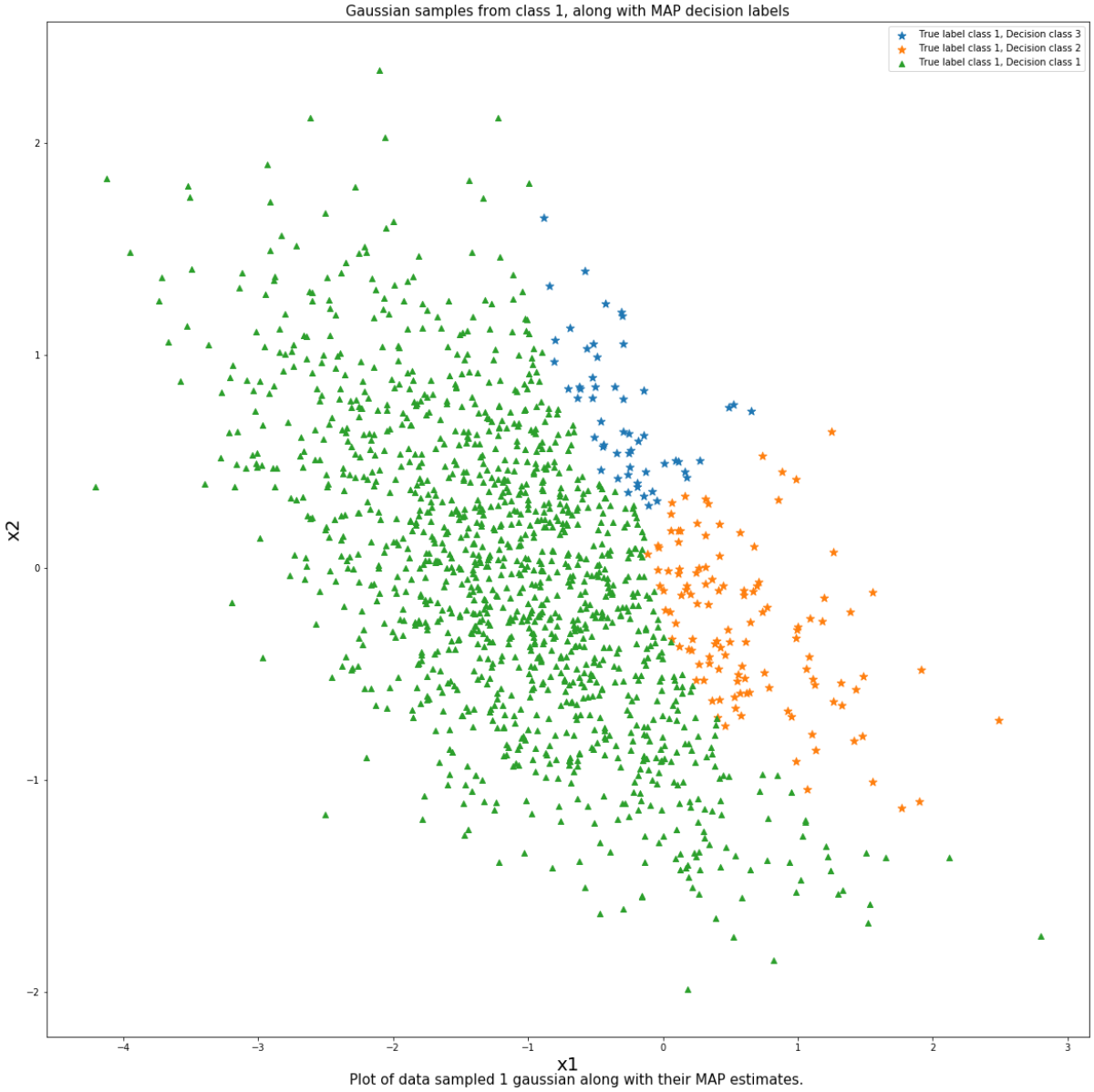
```
]).values,label="True label class 3, Decision class 3",marker='^')
    plt.scatter((df[df['label']==2]['x']).values,(df[df['label']==2]['y'
]).values,label="True label class 3, Decision class 2",marker='*',s=80)
    plt.scatter((df[df['label']==1]['x']).values,(df[df['label']==1]['y'
]).values,label="True label class 3, Decision class 1",marker='*',s=80)



    plt.legend();
    plt.title('Gaussian samples from class 3, along with MAP decision la
bels',fontsize=15)
    plt.xlabel('x1',fontsize=20)
    plt.ylabel('x2',fontsize=20);
    plt.legend()
    plt.show();
    return df_main
'''
    if len(mis_classify)==0:
        pass
    else:
        class1_mis_classify=np.vstack(mis_classify).transpose()
        a=np.squeeze(np.asarray(class1_mis_classify.transpose()[:,1]))
        b=np.squeeze(np.asarray(class1_mis_classify.transpose()[:,0]))
        plt.scatter(a,b,color='b',marker='x',label='class 1 labels miscl
assified',s=100)


    mis_classify=[]
    for a in samples_class2_final.transpose():
        b= multivariate_normal(a.transpose(), 2, u_1, sig_1)*prior > mul
tivariate_normal(a.transpose(), 2, u_2, sig_2)*(1-prior)
        if b==True:
            mis_classify.append(a)
    num_class2_mis_classify= (len(mis_classify))

    if len(mis_classify)==0:
        pass
    else:
        class2_mis_classify=np.vstack(mis_classify).transpose()
        a=np.squeeze(np.asarray(class2_mis_classify.transpose()[:,1]))
        b=np.squeeze(np.asarray(class2_mis_classify.transpose()[:,0]))
        plt.scatter(a,b,color='r',marker='H',label='class 2 labels miscl
assified',s=100);
    errors= num_class1_mis_classify+num_class2_mis_classify
    print ("P(error) = ",errors/400)
'''    ;
```

In [188]:
```
df=generate_GMM_samples(prior=[0.15,0.35],number_of_samples=10000,sig1=
[[1,-0.4],[-0.4,0.5]],sig2=[[0.5,0],[0,0.2]],sig3=[[0.1,0],[0,0.1]],u1=[
-1,0],u2=[1,0],u3=[0,1]);
```



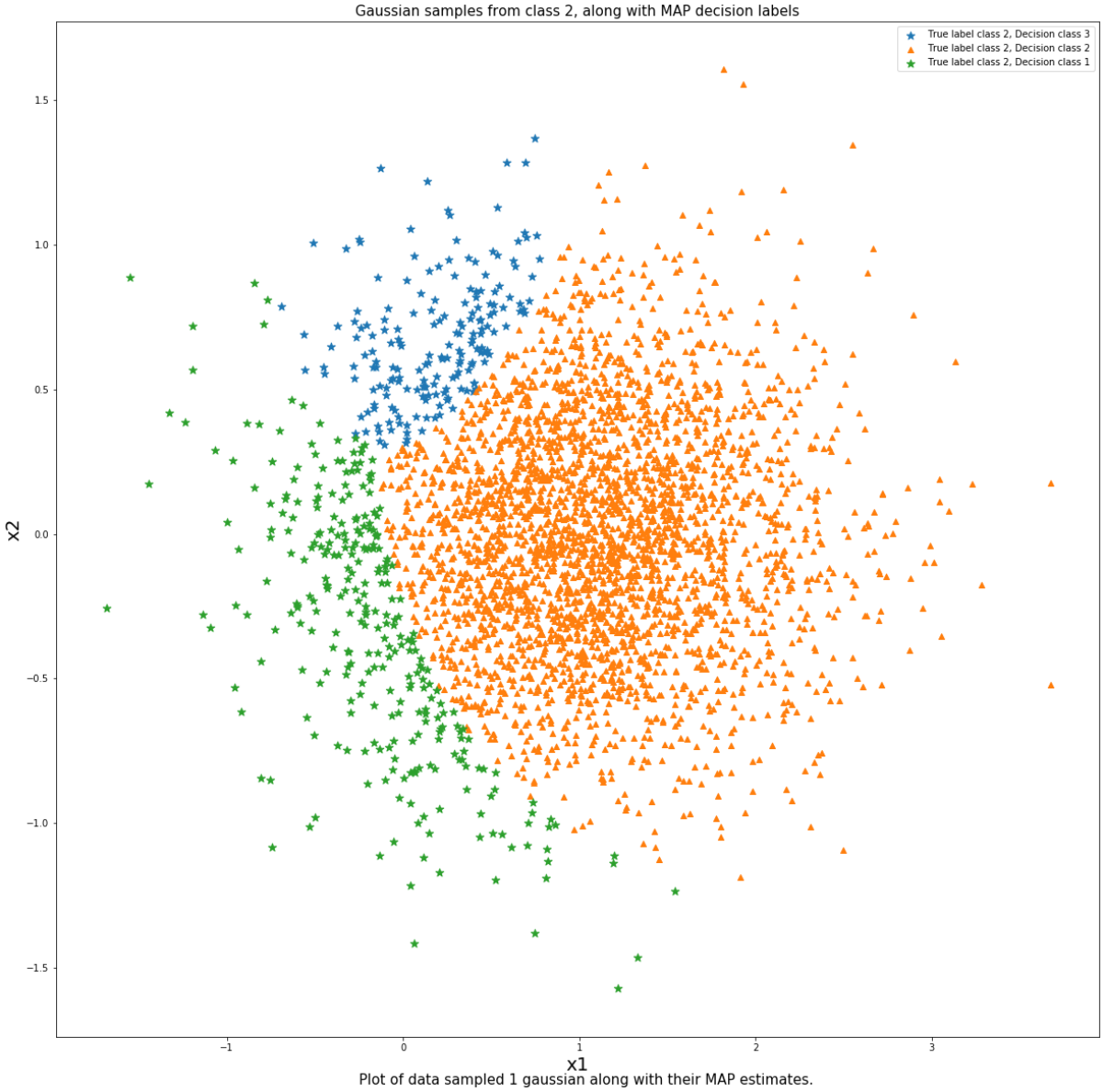
Plot of data sampled from 3 gaussians along with their MAP estimated class labels.

```
Samples from class 1 -  1445
Samples from class 2 -  3504
Samples from class 3 -  5051
```



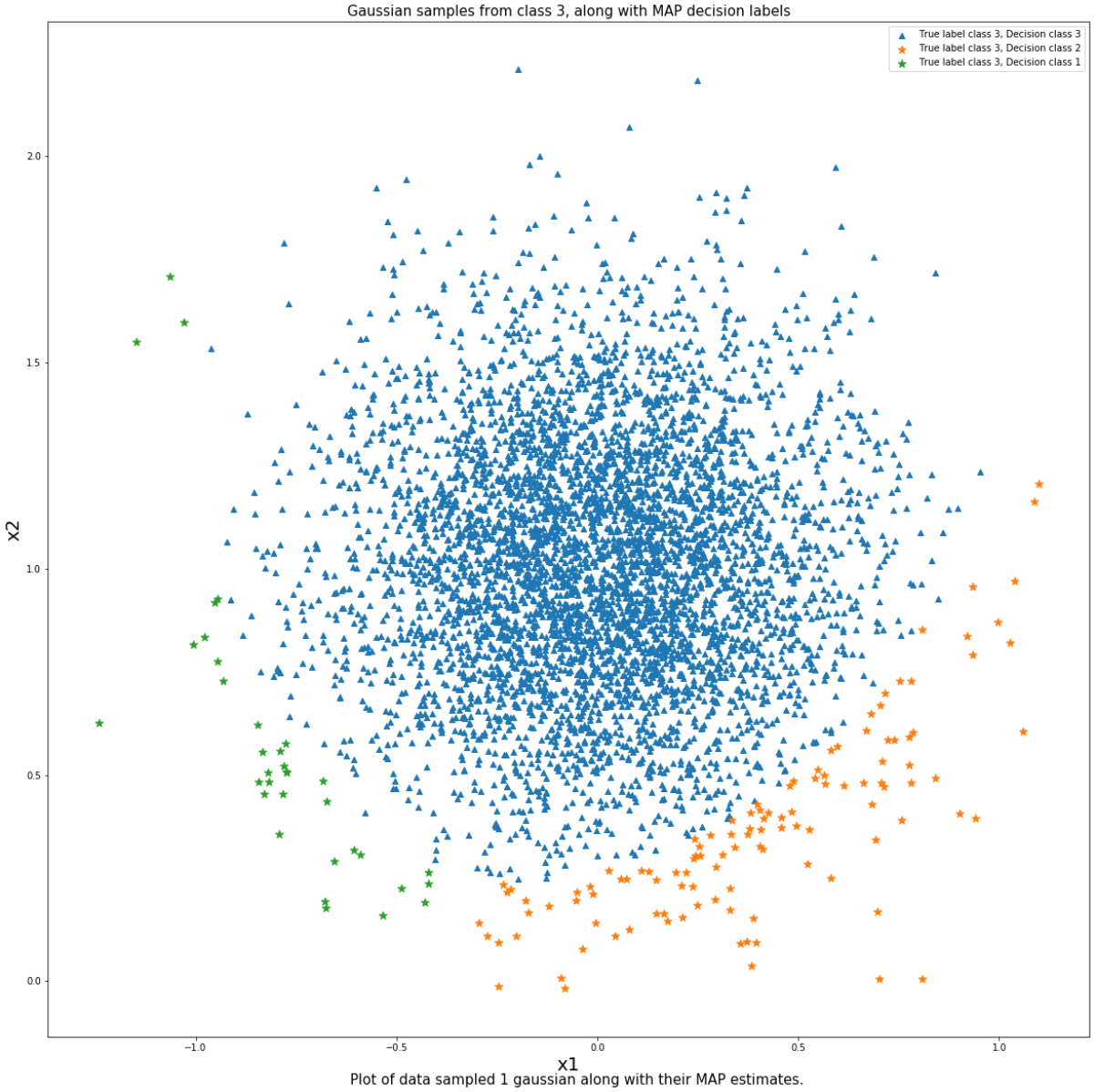
Plot of data sampled 1 gaussian along with their MAP estimates.

\n



Plot of data sampled 1 gaussian along with their MAP estimates.

\n



Plot of data sampled 1 gaussian along with their MAP estimates.

In [189]:
```
print ("Columns have the True class; \nand Rows have decided class. \n C
onfusion matrix below")
df.style.apply(highlight_max)
```

```
Columns have the True class;
and Rows have decided class.
 Confusion matrix below
```

Out[189]:

	1	2	3
1	1272	300	35
2	118	3016	116
3	55	188	4900

In [190]:
```
print("Total samples misclassified : ",df.mask(np.eye(3, dtype = bool)).
sum().sum())
print("P(error) : ",(df.mask(np.eye(3, dtype = bool)).sum().sum())/100,
"%")
```

```
Total samples misclassified :  812.0
P(error)%:  8.12 %
```
or P(error) = 0.0812

With just 8% P(error), MAP classifier correctly classifies majority of the points Samples where class conditional of true class label is high, are correctly classified. As the points are fairly distributed, mojority of the points are that way.

For eg. consider class i.

Posterior(i)=P(x|class =i)* P(i).

This is compared against Posterior of all classes j,k.a, j!=i

The class conditional (P(x|class =i)) is gaussian. Exponential decaying function. As seperation of classes increases, "class conditional" is the main dominating term. As the exponentials evaluated at large distances decay way faster than difference in class priors of class i and j. Class priors are fixed, and roughly on the same order for all classes. Since the points are fairly scattered, MAP classifier gives just 8% P(error)

In [ ]:

Answer 2

Github- https://github.com/rohinarora/EECE5644-Machine_Learning/blob/master/Exam1/Q2.ipynb

Answer 2

Let $\vec{\theta} = \begin{bmatrix} x \\ y \end{bmatrix}$

prior of $\vec{\theta}$ →

$$P\begin{bmatrix} x \\ y \end{bmatrix} = (2\pi\sigma_x\sigma_y)^{-1} \; e^{-\frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}}$$

To find → $\theta_{MAP} = \underset{\theta}{\operatorname{argmax}}\, P(\vec{\theta} \mid \vec{R})$

↳ $\begin{bmatrix} x_{MAP} \\ y_{MAP} \end{bmatrix}$

where $\vec{R} = (r_1, r_2 - - - , r_K)$ → K measurements

$$r_i = d_{Ti} + n_i$$

$$d_{Ti} = \sqrt{(x_{Ti} - x_i)^2 + (y_{Ti} - y_i)}$$

where (x_i, y_i) is the sensor position (known to us)

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}}\ P(\vec{\theta} \mid \vec{R})$$

$$= \underset{\theta}{\operatorname{argmax}}\ \frac{P(\vec{R} \mid \vec{\theta})\, P(\vec{\theta})}{P(\vec{R})}$$ ⟩ Bayes Rule

$$= \underset{\theta}{\operatorname{argmax}}\ P(\vec{R} \mid \vec{\theta})\ P(\theta)$$

we can take log of probability as log is increasing funct.

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\log P(\vec{R} \mid \vec{\theta}) + \log P(\vec{\theta}) \right]$$

$\vec{R} \rightarrow r_1, r_2, r_3 - - -, r_n$

observations are independent

$$= \underset{\theta}{\operatorname{argmax}} \left[\log \prod_{i=1}^{k} P(r_i \mid \vec{\theta}) + \log P(\vec{\theta}) \right]$$

$$= \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{k} \log P(r_i \mid \vec{\theta}) + \log P(\vec{\theta}) \right]$$

$$= \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} \log P(r_i|\theta) + \log \left((2\pi\sigma_x\sigma_y)^{-1} e^{-\frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} \right) \right.$$

$$r_i|\theta = \sqrt{(x_i - x) + (y_i - y)^2} + n_i$$

(constant)

$$n_i \sim N(0, \sigma_i^2)$$

$$\therefore M_{r_i|\theta} = \sqrt{(x_i - x) + (y_i - x)} = d_{i,xy} = d_i(x,y)$$

$$\sigma_{r_i|\theta} = \sigma_i^2$$

$$\therefore r_i|\theta \sim N(d_{ixy}, \sigma_i^2)$$

$\therefore r_i|\theta$ is a gaussian with mean $d_i(x,y)$ and variance σ_i^2.

$$\Rightarrow \theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} \log \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(r_i - d_i(x,y))^2}{2\sigma_i^2}} + \right.$$

$$\log(2\pi\sigma_x\sigma_y)^{-1} - \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}^{-1} \left. \begin{bmatrix} x \\ y \end{bmatrix} \right]$$

independent of (x,y)

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} \left[\log \frac{1}{\sqrt{2\pi\sigma_i^2}} - \frac{\left(r_i - d_i(x,y)\right)^2}{2\sigma_i^2} \right] - \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1/\sigma_x^2 & 0 \\ 0 & 1/\sigma_y^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right]$$

independent of θ

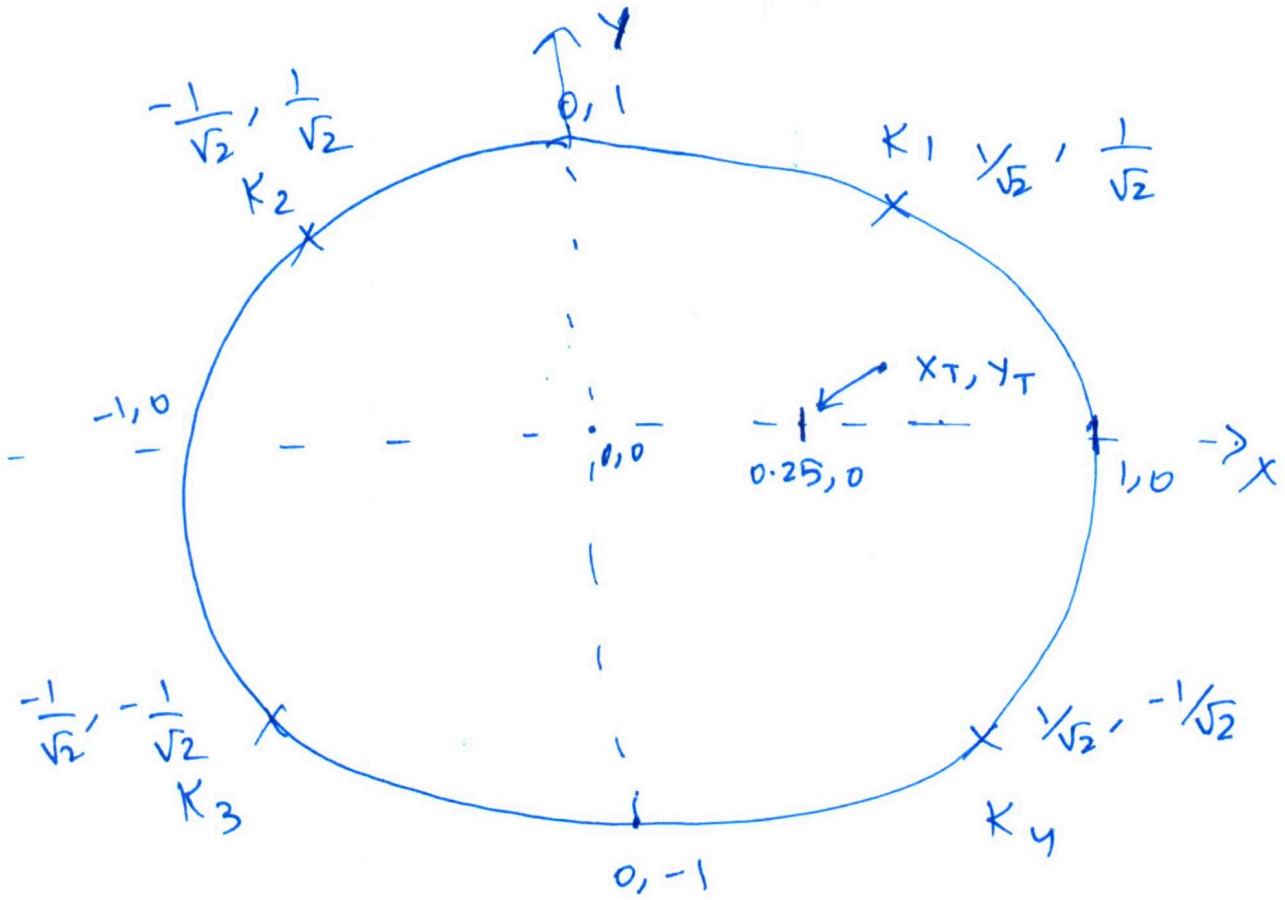
$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} - \frac{\left(r_i - \left((x_i - x)^2 + (y_i - y)^2\right)^{1/2}\right)^2}{2\sigma_i^2} - \frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right] \right]$$

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmin}} \left[\sum_{i=1}^{K} \frac{\left(r_i - \sqrt{(x_i - x)^2 + (y_i - y)^2}\right)^2}{\sigma_i^2} + \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right) \right]$$

where $\theta = [x \; y]^T$

solving this optimization will give θ_{MAP} or $\begin{bmatrix} x_{MAP} \\ y_{MAP} \end{bmatrix}$

Visualization of the data-



The code is well commented explaining the steps

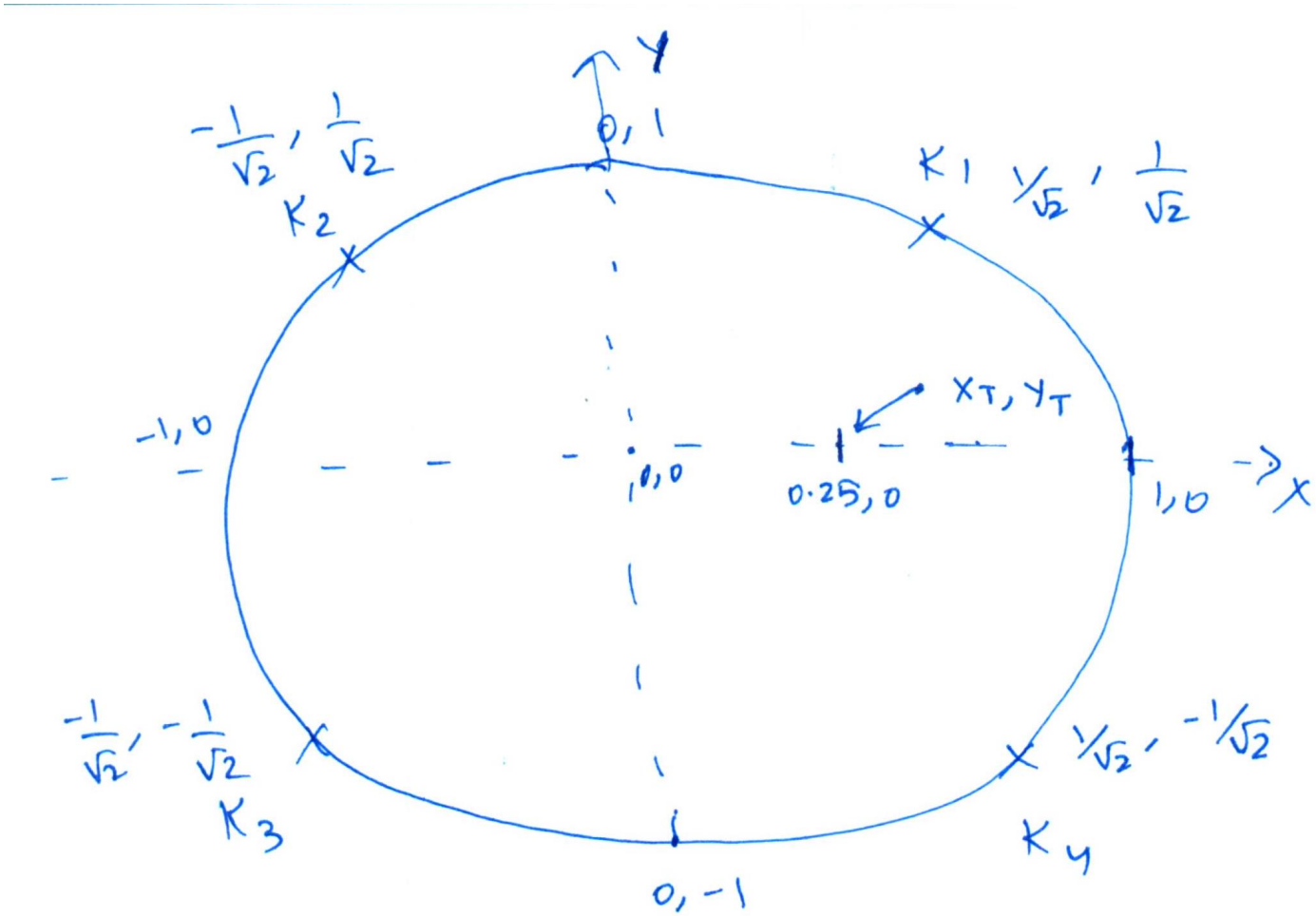
sigma_x=0.1

sigma_y=0.1

sigma_i=.01

In [2]:
```
'''
imports
'''

import matplotlib.pyplot as plt
%matplotlib inline
import numpy as np
import sympy as sym
```

Viz of the data-



In [3]:
```
'''
Position the sensor. k1-k4 represent the 4 sensors
'''

k1=(1/np.sqrt(2),1/np.sqrt(2))
k2=(-1/np.sqrt(2),1/np.sqrt(2))
k3=(-1/np.sqrt(2),-1/np.sqrt(2))
k4=(1/np.sqrt(2),-1/np.sqrt(2))

'''
True location of the object
'''

xy_true=(0.2,0.3)

sensors=[k1,k2,k3,k4] # make a list of sensors


# set the sigma_x, sigma_y, sigma_i
sigma_x=0.1
sigma_y=0.1
sigma_i=.01 #0.3 for submission
```

In [4]:
```
def dist_points(p1,p2):
    '''
    L2 distance between points p1 and p2
    Assumes p1 and p2 are both tuples (x,y)
    '''
    return ((p1[0]-p2[0])**2+(p1[1]-p2[1])**2)**(0.5)
```

In [5]:
```
def measure_value(k):
    '''
    measure the distance between sensor k and true position "xy_true". A
dd gaussian noise with std sigma_i to it
    '''
    measurement=dist_points(k,xy_true)+np.random.normal(scale=sigma_i)
    if measurement>0:
        return measurement
    else:
        measure_value(k)
```

In [6]:
```
'''
Calculate the distance of sensor to true position "xy_true", by calling
 the function measure_value
Store the resulting values in a dictionary
'''


sensor_measurements=dict()
sensor_measurements[k1]=measure_value(k1)
sensor_measurements[k2]=measure_value(k2)
sensor_measurements[k3]=measure_value(k3)
sensor_measurements[k4]=measure_value(k4)
print ("sensor data is :\n", sensor_measurements)
print ('\n\n')
for j,i in enumerate(sensor_measurements):
    print ("Distance of sensor k"+str(j+1)+" from true positon",sensor_m
easurements[i])
```

```
sensor data is :
 {(0.7071067811865475, 0.7071067811865475): 0.6451761942076968, (-0.707
1067811865475, 0.7071067811865475): 1.0024351553984014, (-0.70710678118
65475, -0.7071067811865475): 1.3675174669665544, (0.7071067811865475, -
0.7071067811865475): 1.1276801471635933}


Distance of sensor k1 from true positon 0.6451761942076968
Distance of sensor k2 from true positon 1.0024351553984014
Distance of sensor k3 from true positon 1.3675174669665544
Distance of sensor k4 from true positon 1.1276801471635933
```

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmin}} \left[\sum_{i=1}^{K} \frac{\left(r_i - \sqrt{(x_i - x)^2 + (y_i - y)^2} \right)^2}{\sigma_i^2} + \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} \right) \right]$$

where $\theta = [x \;\; y]^T$

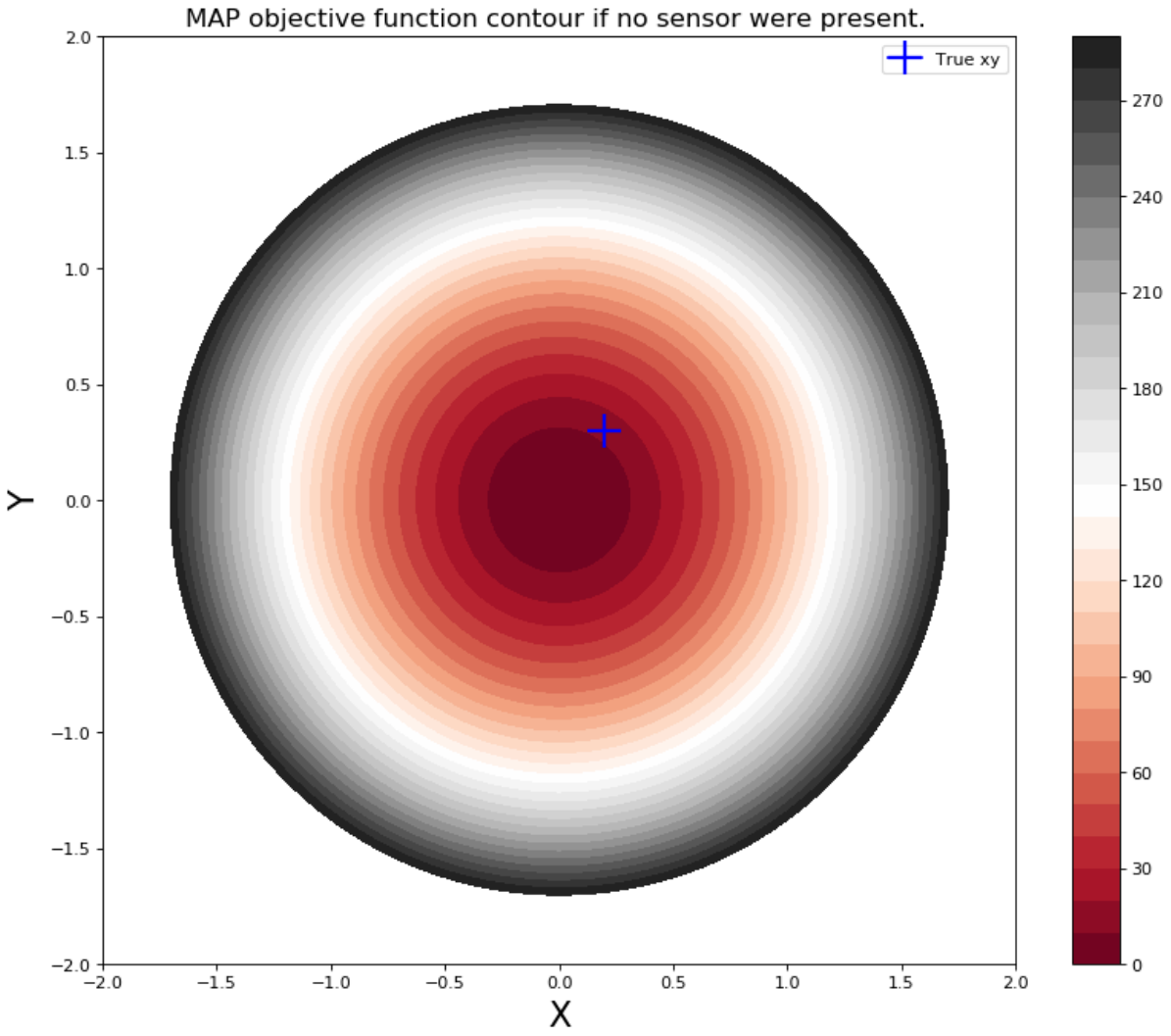
solving this optimization will give θ_{MAP} or $\begin{bmatrix} x_{MAP} \\ y_{MAP} \end{bmatrix}$

In [7]:
```
# Pre define the contour_level. This sets the contour levels for all the
plots.
contour_level=[]
for i in range(0,300,10):
    contour_level.append(i)


# Create a meshgrid of X-Y
x = np.linspace(-2, 2, 1000)
y = np.linspace(-2, 2, 1000)
X, Y = np.meshgrid(x, y)


# Below functions give the MAP objective values for different cases. Ref
er above image for the formaula.

def f0(x, y):
    '''
    MAP objective if no sensor were present.
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    return prior

def f1(x, y):
    '''
    MAP objective if 1 sensor was present.
    '''

    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    return first_senor+prior



def f2(x, y):
    '''
    two sensor
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    second_senor=(np.square(sensor_measurements[k3]-np.sqrt((k3[0]-x)**2
+(k3[1]-y)**2)))*(1/sigma_i**2)
    return first_senor+second_senor+prior

def f3(x, y):
    '''
    three sensor
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    second_senor=(np.square(sensor_measurements[k2]-np.sqrt((k2[0]-x)**2
+(k2[1]-y)**2)))*(1/sigma_i**2)
    third_senor=(np.square(sensor_measurements[k3]-np.sqrt((k3[0]-x)**2+
```

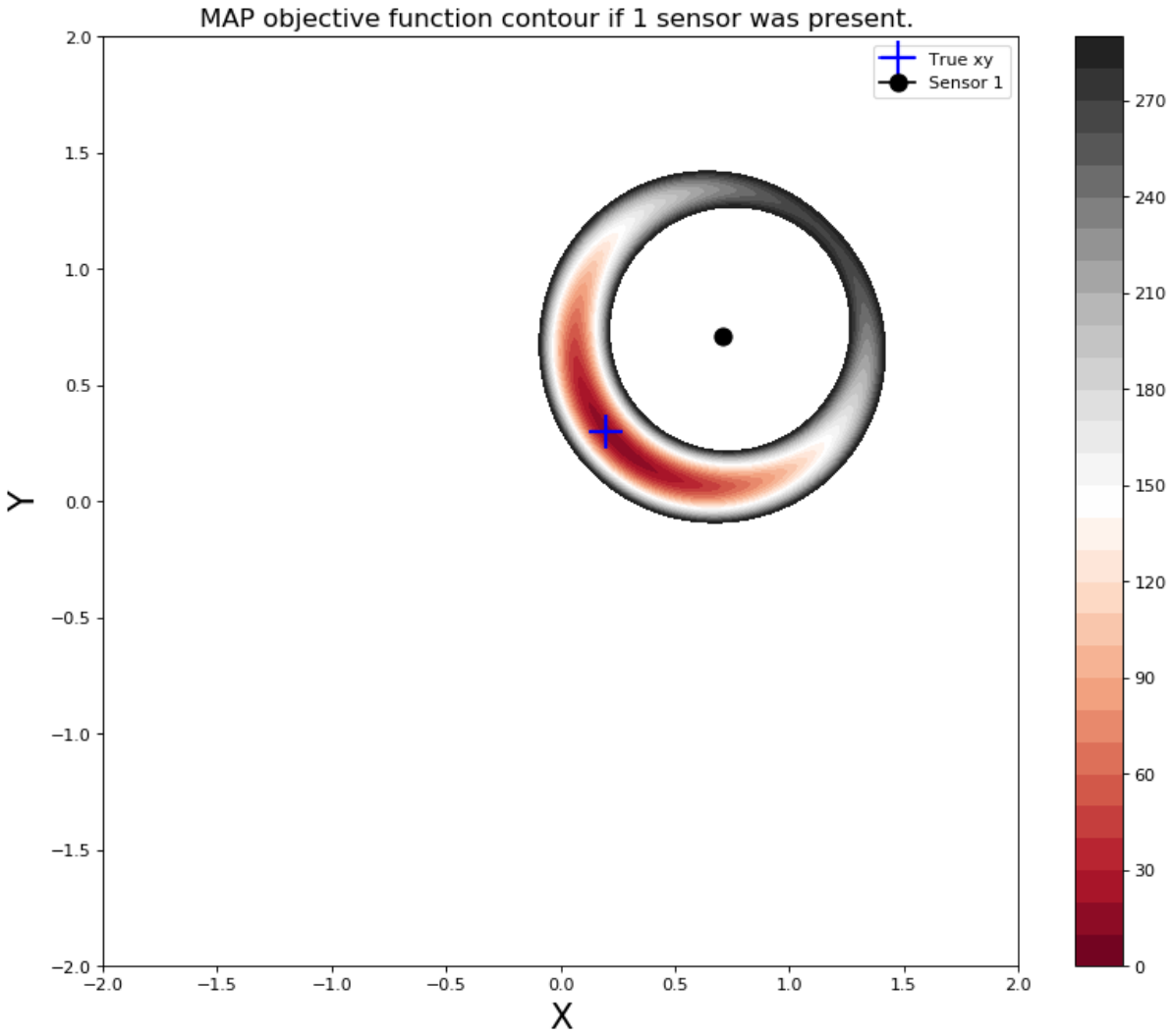
```
(k3[1]-y)**2)))*(1/sigma_i**2)
    return first_senor+second_senor+third_senor+prior

def f4(x, y):
    '''
    four sensor
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    second_senor=(np.square(sensor_measurements[k2]-np.sqrt((k2[0]-x)**2
+(k2[1]-y)**2)))*(1/sigma_i**2)
    third_senor=(np.square(sensor_measurements[k3]-np.sqrt((k3[0]-x)**2+
(k3[1]-y)**2)))*(1/sigma_i**2)
    forth_senor=(np.square(sensor_measurements[k4]-np.sqrt((k4[0]-x)**2+
(k4[1]-y)**2)))*(1/sigma_i**2)
    return first_senor+second_senor+third_senor+forth_senor+prior



'''

Plotting function.

Takes in argument objective function and number of sensors
Objective function can be one of functions defined above- f0,f1,f2,f3,f4
num_sensors can be 0-4
'''


def _plot_(f,num_sensors):
    Z = f(X, Y)   # compute Z for objective function f
    from matplotlib.pyplot import figure
    fig=figure(num=None, figsize=(12, 10), dpi=80, facecolor='w', edgeco
lor='k')

    # plot contour
    plt.contourf(X, Y, Z, 20, cmap='RdGy',levels=contour_level);

    # set true labels and sensor positions
    plt.plot([xy_true[0]], [xy_true[1]], marker='+', markersize=20, colo
r="blue",label="True xy",mew=2)
    if num_sensors==1:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
    elif num_sensors==2:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
        plt.plot([k3[0]], [k3[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 2")
    elif num_sensors==3:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
        plt.plot([k2[0]], [k2[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 2")
        plt.plot([k3[0]], [k3[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 3")
```

```
    elif num_sensors==4:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
        plt.plot([k2[0]], [k2[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 2")
        plt.plot([k3[0]], [k3[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 3")
        plt.plot([k4[0]], [k4[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 4")


    plt.xlim(-2,2)
    plt.ylim(-2,2)
    plt.colorbar();
    plt.legend();
    return fig
```

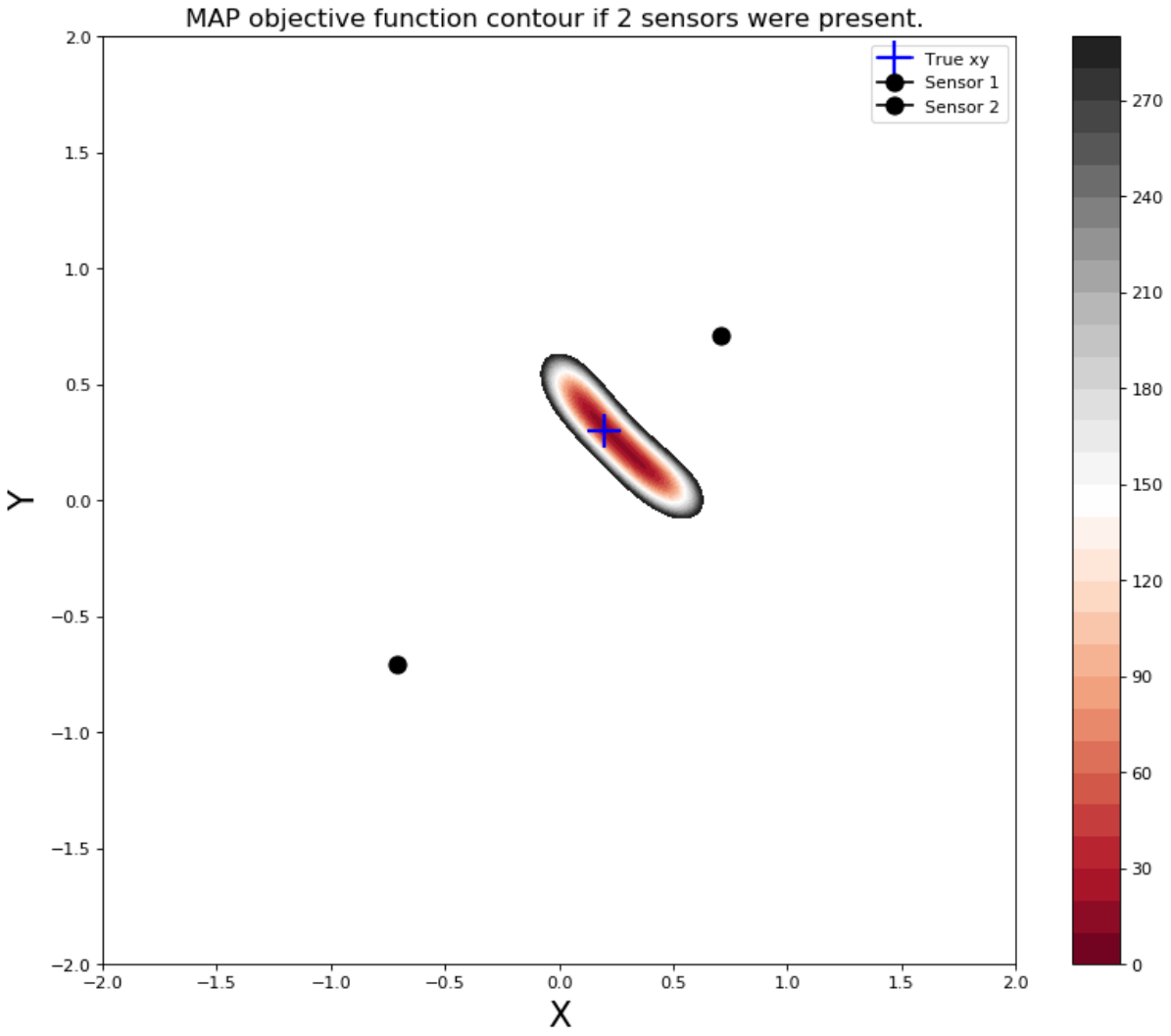
In [8]:
```
fig=_plot_(f0,0)
# MAP estimate matches with prior value.
plt.title('MAP objective function contour if no sensor were present. ',f
ontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective is centered at origin 0,0 -> which is
 the prior value',fontsize=15);

fig=_plot_(f1,1)
plt.title('MAP objective function contour if 1 sensor was present. ',fon
tsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center moves closer to the the true va
lue',fontsize=15);

fig=_plot_(f2,2)
plt.title('MAP objective function contour if 2 sensors were present. ',f
ontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center moves further closer to the the
true value',fontsize=15);

fig=_plot_(f3,3)
plt.title('MAP objective function contour if 3 sensors were present. ',f
ontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center moves very close to the true va
lue',fontsize=15);

fig=_plot_(f4,4)
plt.title('MAP objective function contour if 4 sensors were present. ',f
ontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center almost coincidences to the true
value',fontsize=15);
```



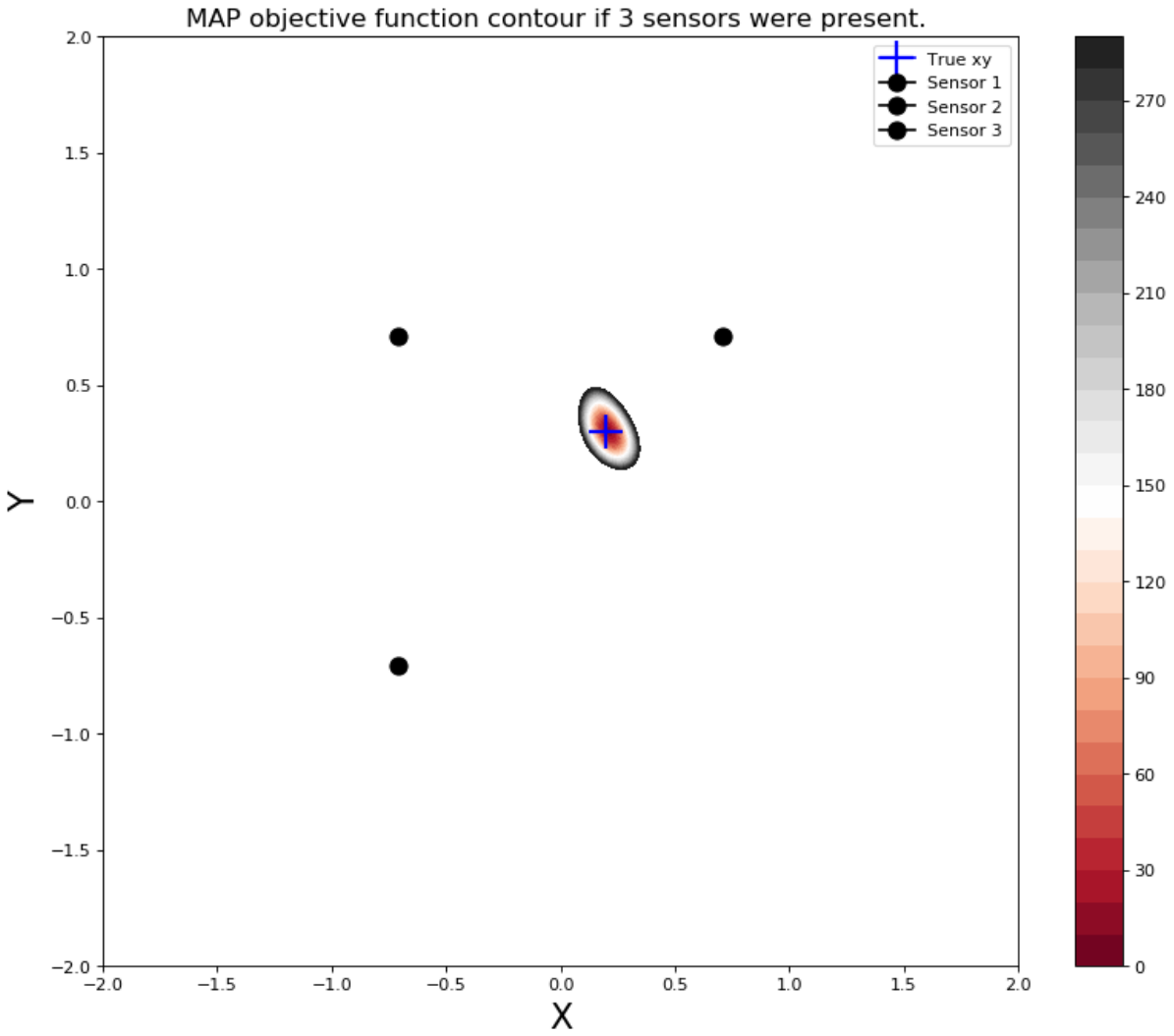
The objective is centered at origin 0,0 -> which is the prior value



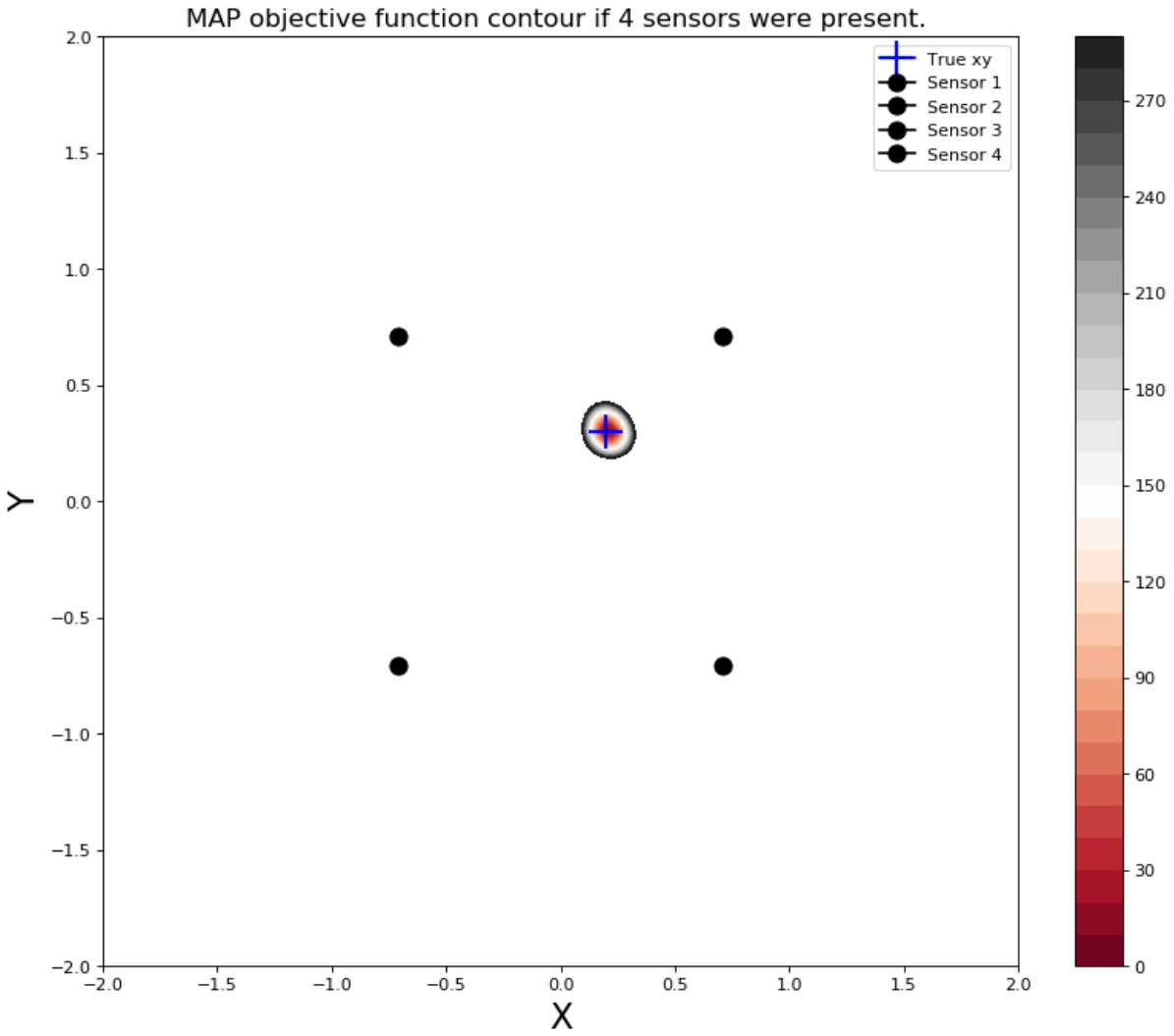
The objective center moves closer to the the true value



The objective center moves further closer to the the true value



The objective center moves very close to the true value

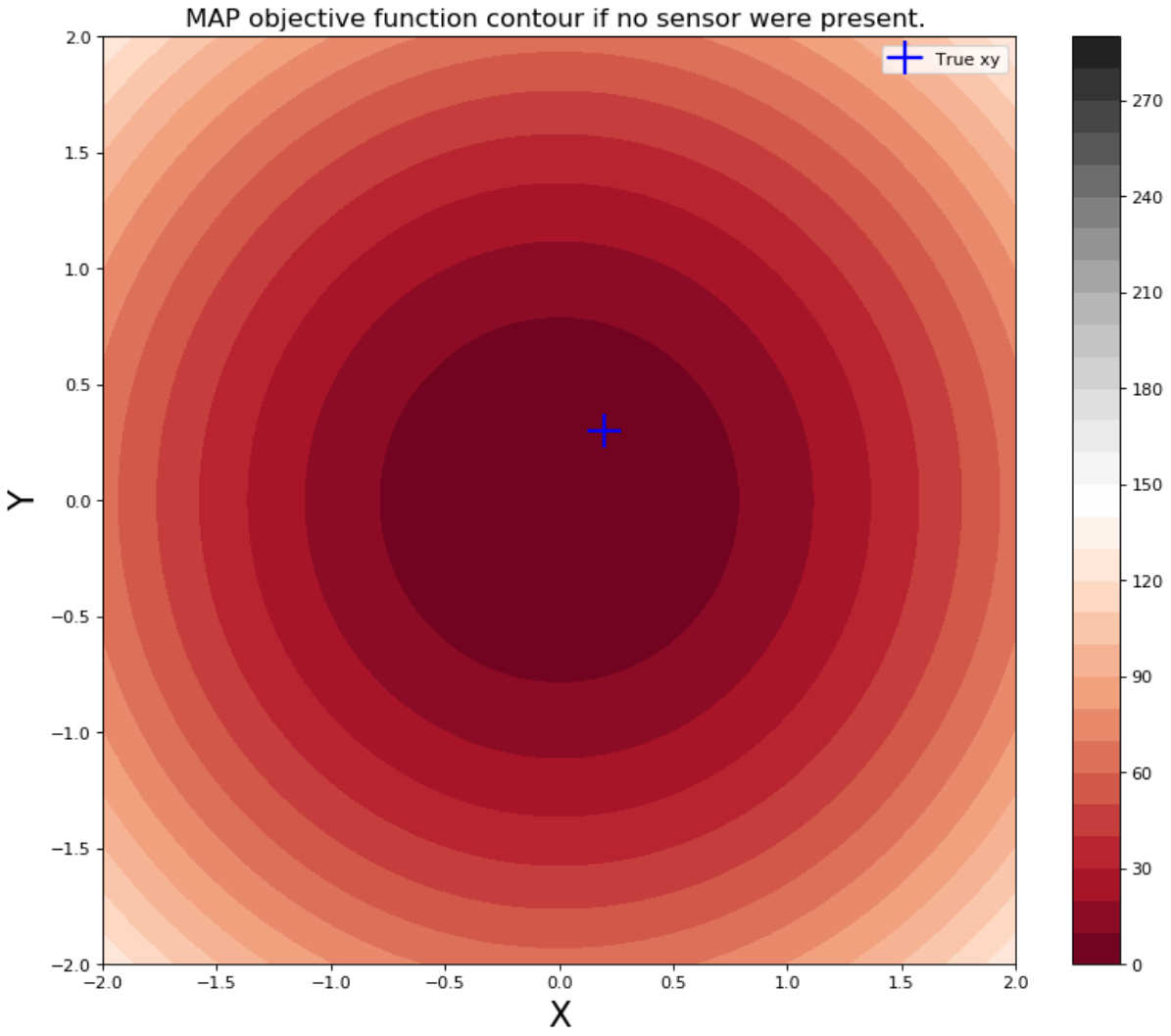


The objective center almost coincidences to the true value

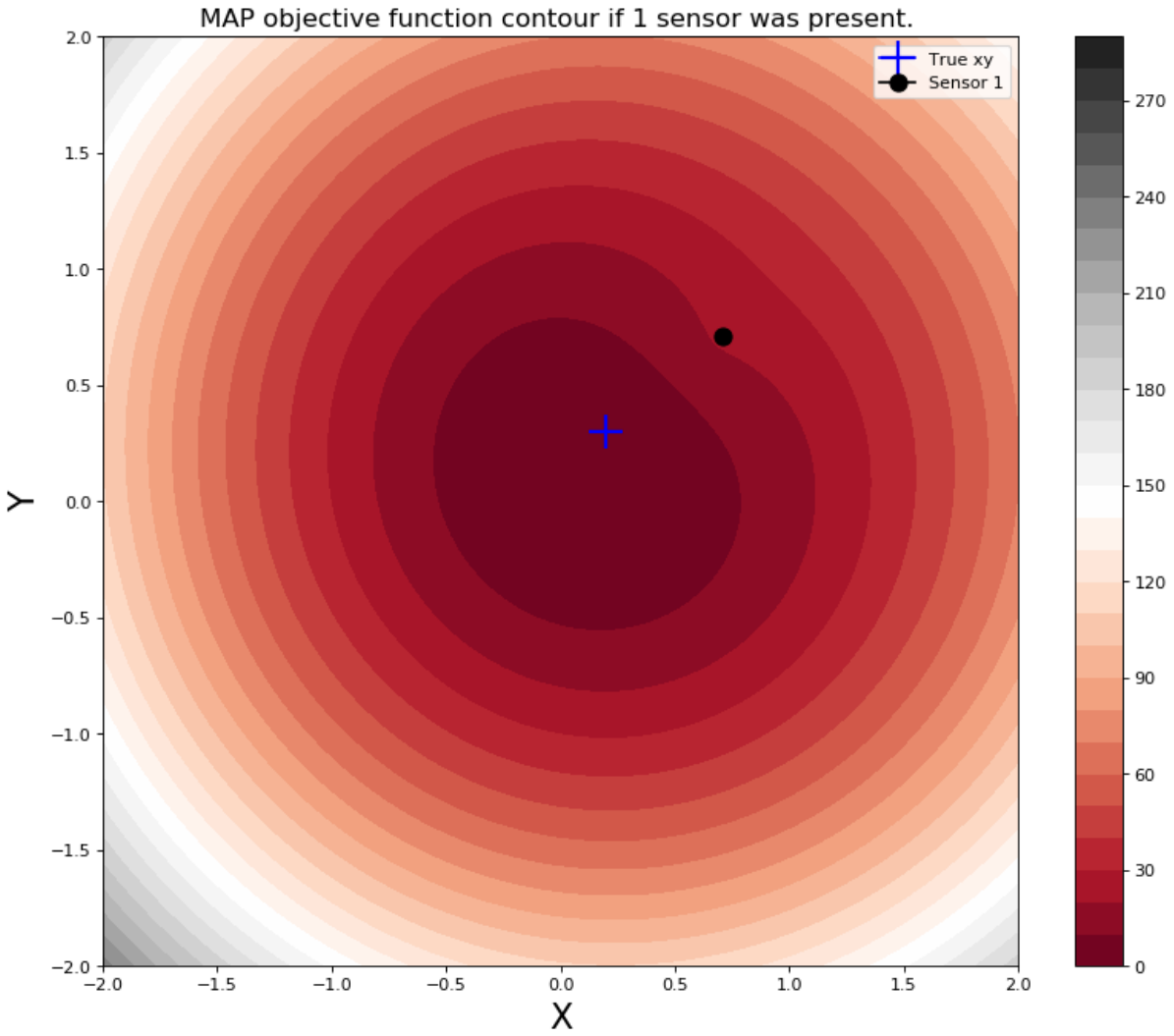
In [ ]:

Changing the sigmas-

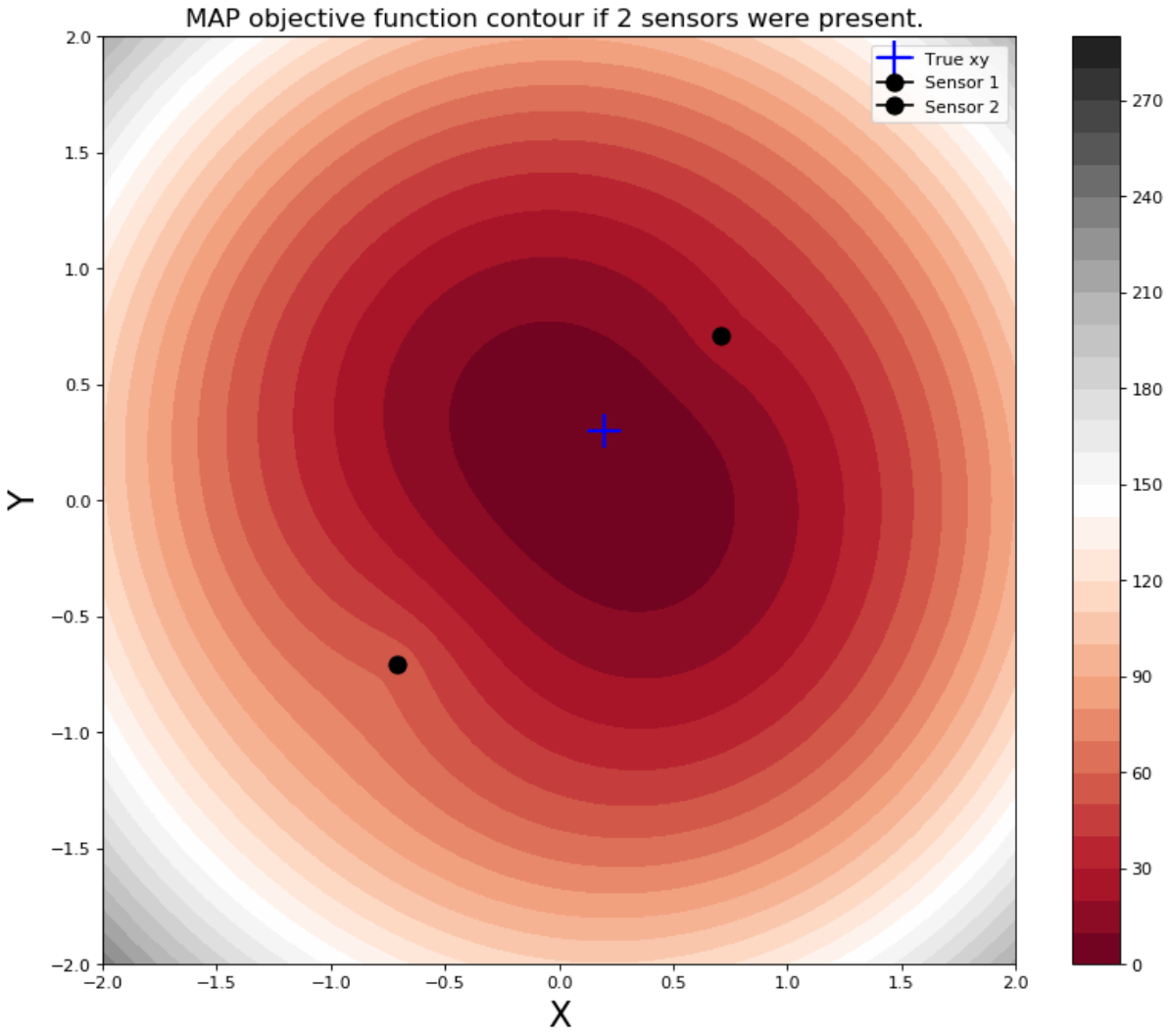
sigma_x=0.25
sigma_y=0.25
sigma_i=0.3



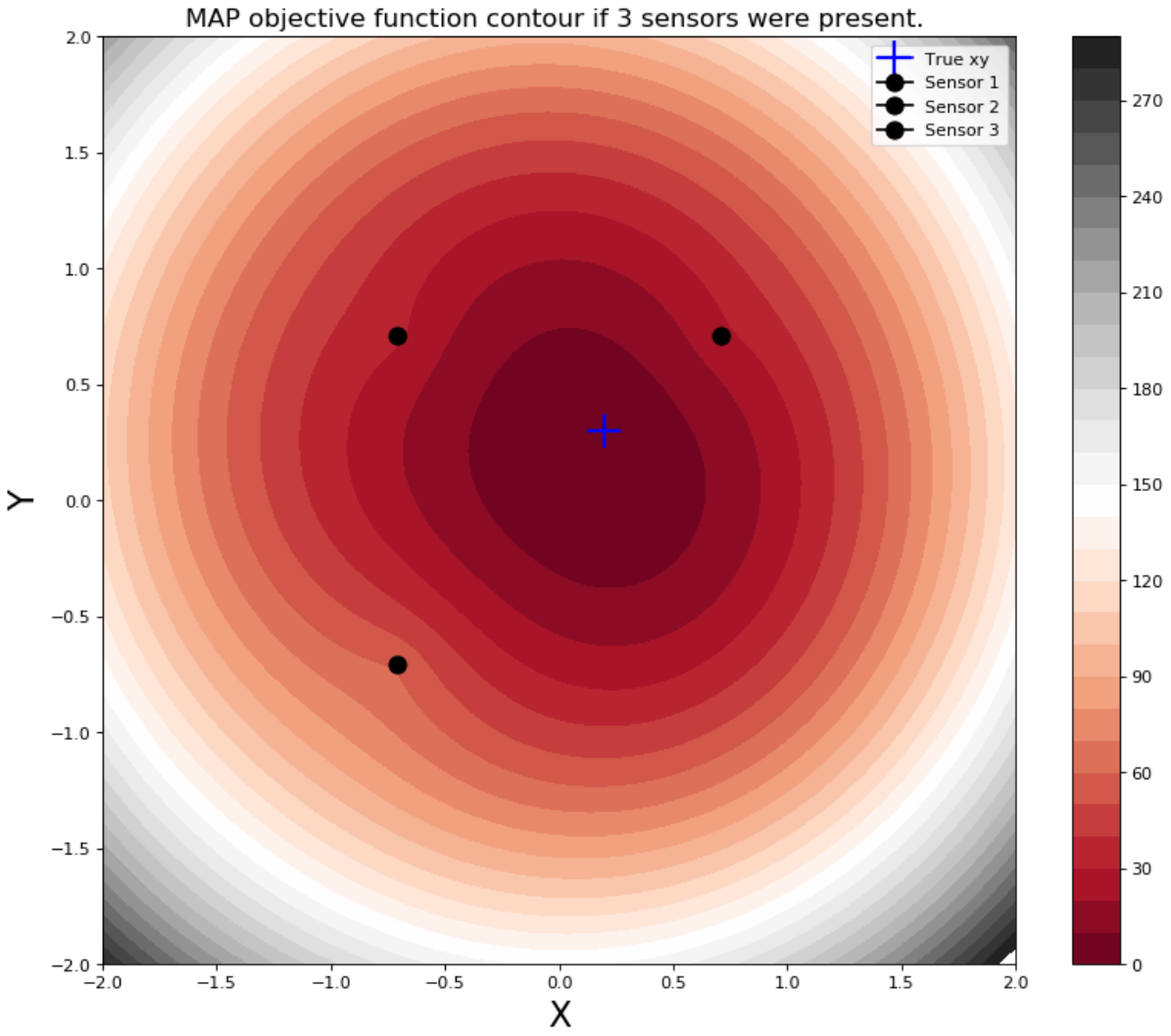
The objective is centered at origin 0,0 -> which is the prior value



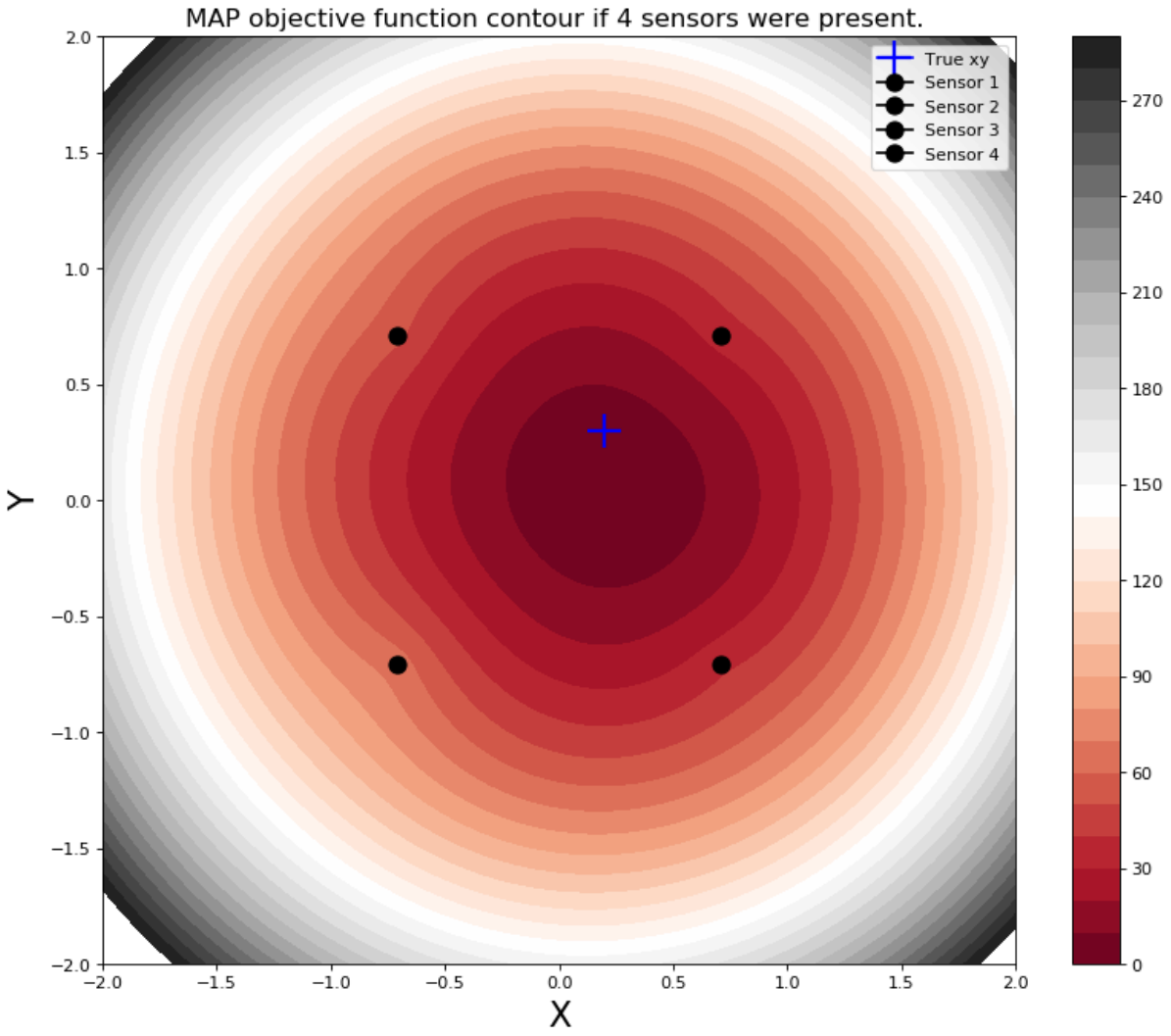
The objective center moves closer to the the true value



The objective center moves further closer to the the true value



The objective center moves very close to the true value



The objective center almost coincidences to the true value

In [ ]:

- Behavior of the MAP estimate based on centre of contour-

> The MAP estimate moves closer to the true position of object as we get data from more sensors. In the contour plots, black regions are peaks and red regions are the valleys. valleys represent minima. In 1st plot, we don't even see black contours. The objective is extremely uncertain about the true location of object. As number of sensors increase, we see red region getting concentrated in a region, and black circles coming in the view. This represents deeper valleys, and more confidence about the true location. Further, the red circles are very close to true location in last image (4 sensors). MAP objective is also exactly correct about the true location in this case.

Answer 3 Github- https://github.com/rohinarora/EECE5644-Machine_Learning/blob/master/Exam1/Q3.ipynb

Answer 3

given → $w \in \mathbb{R}^4$

$\Sigma \rightarrow$ covariance matrix

$w \sim N(0, \gamma^2 I)$

$\vec{D} = (x_1, y_1), (x_2, y_2) - - - , (x_n, y_n)$

$y_i = ax_i^3 + bx_i^2 + cx_i + d + v$

$v \sim N(0, \sigma^2)$

To find → $\underset{w}{\text{argmax}} \; p(\vec{w} \mid \vec{D})$

$$w_{MAP} = \underset{w}{\text{argmax}} \frac{p(\vec{D} \mid \vec{w}) p(\vec{w})}{p(\vec{D})}$$

$$= \underset{w}{\text{argmax}} \left[\ln p(\vec{D} \mid \vec{w}) + \ln p(\vec{w}) \right]$$

$\vec{D} = D_1, D_2, - - - - -, D_N$

↓ independent samples

$$= \underset{w}{\text{argmax}} \left[\ln \prod_{i=1}^{N} p(\vec{D}_i \mid \vec{w}) + \ln p(\vec{w}) \right]$$

$$= \underset{w}{\text{argmax}} \left[\sum_{i=1}^{N} \ln p(\vec{D}_i \mid \vec{w}) + \ln p(\vec{w}) \right]$$

$p_i | \vec{w} \equiv (y_i, x_i) | \vec{w}$

$P(y_i | x_i, w) = a x_i^3 + b x_i^2 + c x_i + d + V$

↳ gaussian with mean $w^T \vec{x}$

↳ and standard deviation σ^2

where $w^T x = a x_i^3 + b x_i^2 + c x_i + d$

$\therefore P(y_i | x_i, w) = N(w^T x_i, \sigma^2)$

$$\ln(y_i | x_i, w) = \ln\left[\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i - w^T x_i)^2}{2\sigma^2}}\right]$$

$\frac{1}{\sqrt{2\pi\sigma^2}}$ → independent of $\vec{w}$

$$w_{MAP} = \underset{w}{\operatorname{argmax}} \left[\sum_{i=1}^{N} -\frac{(y_i - w^T x_i)^2}{2\sigma^2} + \ln\left(\frac{1}{(2\pi)^4 |\Sigma|)^{-1/2}} \exp\left(-\frac{1}{2}(\vec{w})^T \Sigma^{-1} w\right)\right)\right]$$

$(2\pi)^4 |\Sigma|)^{-1/2}$ → independent of $\vec{w}$

$$W_{MAP} = \underset{W}{\operatorname{argmax}} \sum_{i=1}^{N} -\frac{(y_i - W^T x_i)^2}{2\sigma^2} - \frac{1}{2}(W^T \Sigma^{-1} W)$$

$$W_{MAP} = \underset{W}{\operatorname{argmin}} \sum_{i=1}^{N} \frac{(y_i - W^T x_i)^2}{2\sigma^2} + \frac{1}{2}(W^T \Sigma^{-1} W)$$

where $\Sigma = \gamma^2 I$

$$W_{MAP} = \underset{w}{\operatorname{argmin}} \sum_{i=1}^{W} \frac{(y_i - w^T x_i)^2}{2\sigma^2} + \frac{1}{2}(w^T \Sigma^{-1} w)$$

where $\Sigma = \gamma^2 I$

$$W_{map} = \underset{w}{\operatorname{argmin}} \left(\frac{1}{\sigma^2}\right)\left(\frac{1}{2}\right)(xw - y)^T(xw - y) + \frac{1}{2}(w^T \Sigma^{-1} w)$$

$$= \underset{w}{\operatorname{argmin}} \left(\frac{1}{2\sigma^2}\right)(xw - y)^T(xw - y) + \frac{1}{2\gamma^2} w^T w$$

Take derivative and set to 0 →

$$\nabla_w b = \frac{1}{\sigma^2}(x^T x w - x^T y) + \frac{1}{\gamma^2} w = 0$$

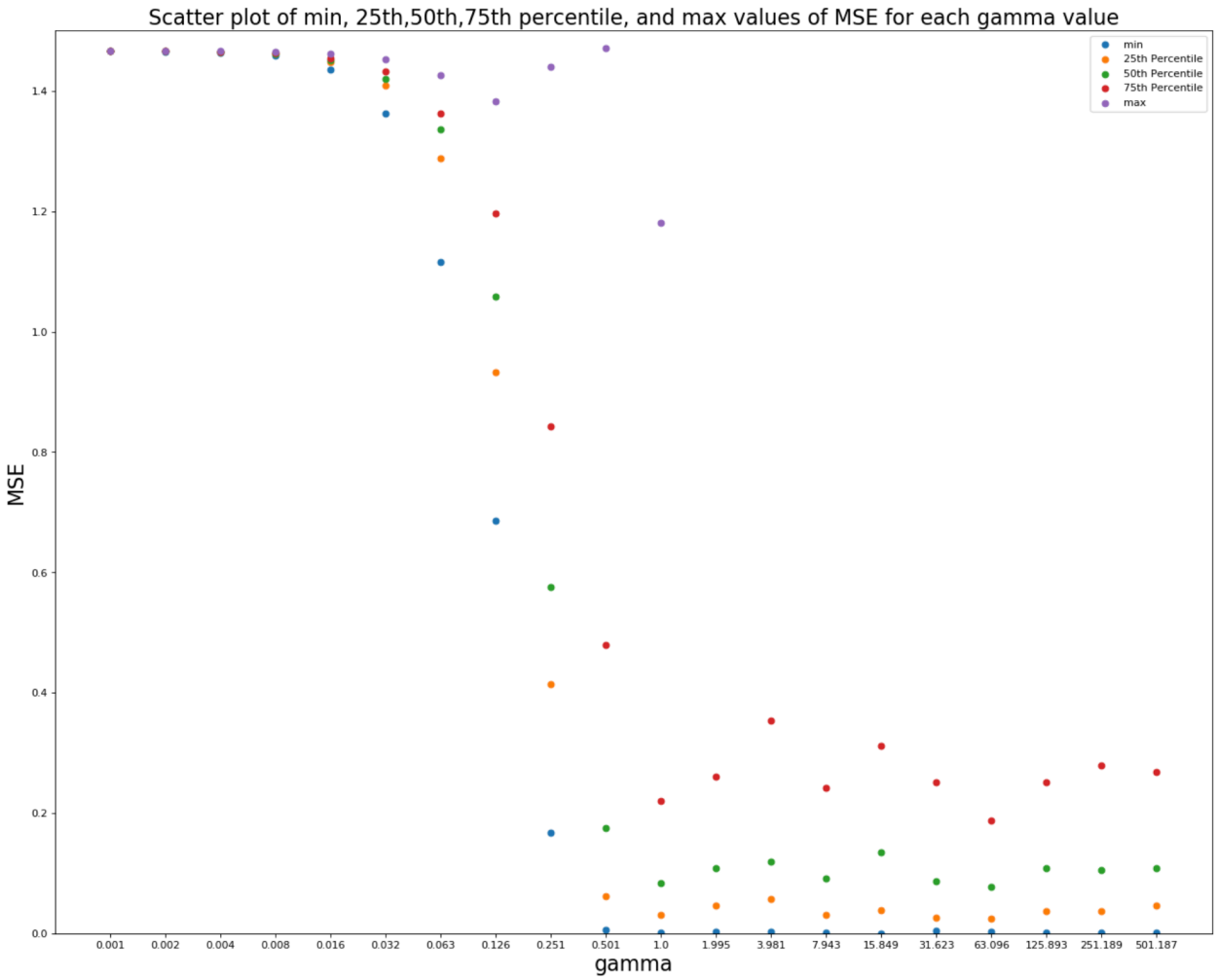
$$\left[\frac{x^T x}{\sigma^2} + \frac{1}{\gamma^2}\right] w = \frac{x^T y}{\sigma^2}$$

$$w = (\gamma^2 x^T x + \sigma^2) x^T y$$

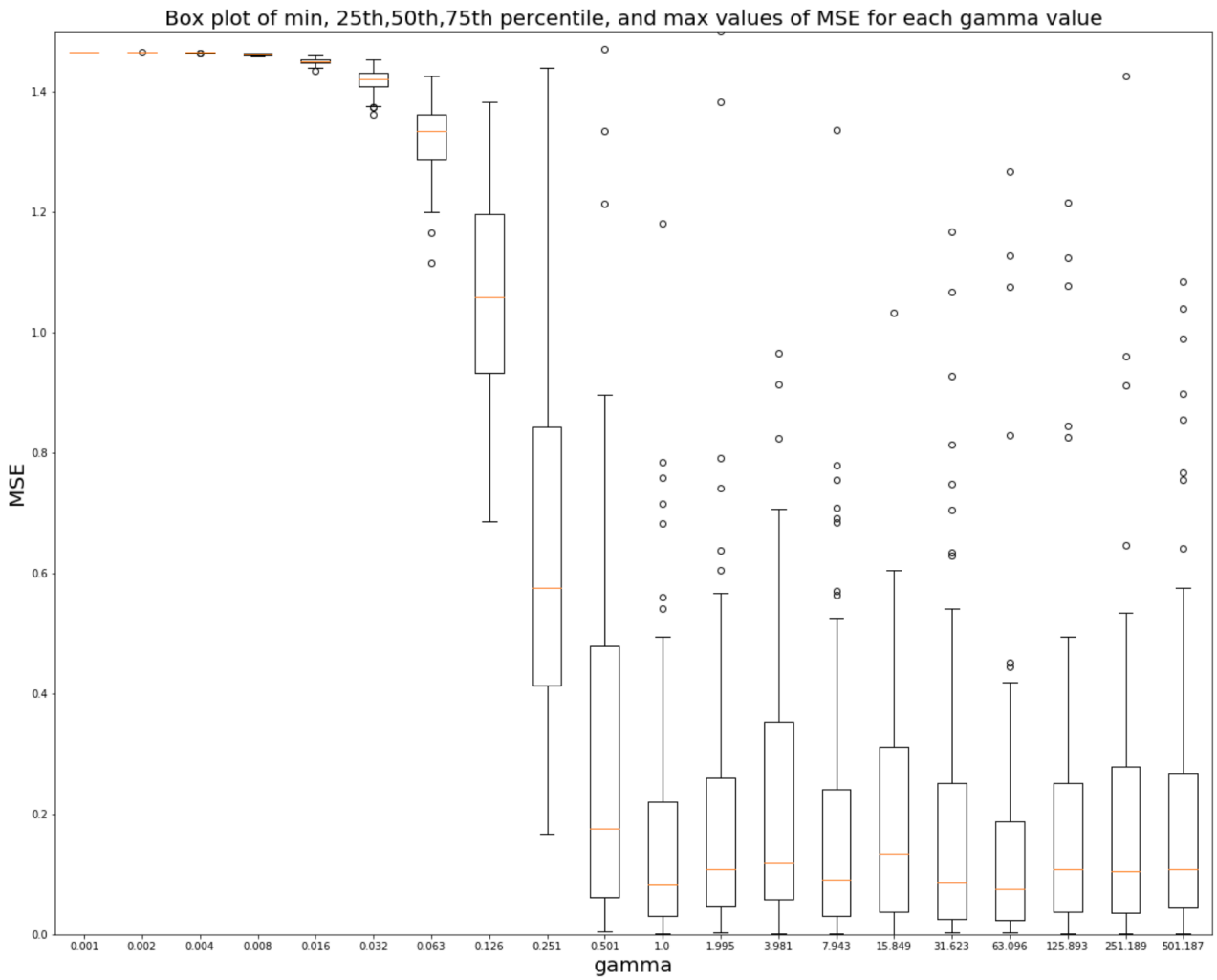
$$\left[X^T X + \frac{\sigma^2}{\gamma^2} \right] w = X^T y$$

★

$$w = \left(X^T X + \frac{\sigma^2 I}{\gamma^2} \right)^{-1} \left(X^T y \right)$$



MSE falls as Gamma increases



MSE falls as Gamma increases

In [182]:

```
'''
ML estimate
'''

C=np.matmul((X_stack.T),X_stack)
A=np.linalg.inv(C)
B=np.matmul(X_stack.T,Y)
w_ml=np.matmul(A,B)
print ("W_ML is \n",w_ml)

print ("\n\nWith Gamma :",gamma,", W_MAP is \n",w_map)
print ("\nAs gamma increases, w_MAP indeed approaches to w_ml")
```

```
W_ML is
 [[ 1.00279534]
 [-0.19552492]
 [-0.65031524]
 [ 0.13020576]]


With Gamma : 100000.0 , W_MAP is
 [[ 1.00279534]
 [-0.19552492]
 [-0.65031524]
 [ 0.13020576]]

As gamma increases, w_MAP indeed approaches to w_ml
```

Appendix

Consider least square expression →

$$\underset{w}{\operatorname{argmin}} \frac{1}{2\sigma^2} \sum_{i=1}^{N} (y_i - xw)^2$$

σ^2 → independent of w

in vectorized form →

$$\underset{w}{\operatorname{argmin}} f(w) = \frac{1}{2} (Xw - y)^T (Xw - y)$$

(Xw: 1000, 4 · w: 4, 1 · y: 1000, 1)

$$= \frac{1}{2} (xw - y)^T (xw - y)$$

→ assume w is feature vector of size 4; and 1000 samples

$$\underset{w}{\operatorname{argmin}} \frac{1}{2} (xw - y)^T (xw - y)$$

$$= \frac{1}{2} \underset{w}{\operatorname{argmin}} (w^T x^T x w - w^T x^T y - y^T x w + y^T y)$$

$$= \frac{1}{2} \text{tr} \left(w^T x^T x w - w^T x^T y - y^T x w + y^T y \right)$$

$\frac{1}{2}$ ↓ trace

$$= \frac{1}{2} \left(\text{tr}\, w^T x^T x w - 2 \,\text{tr}\, y^T x w \right)$$

$$\nabla_w f = \frac{1}{2} \left(x^T x w + x^T x w - 2 x^T y \right)$$

$$= x^T x w - x^T y$$

↓ formula used for derivative

where $X \in \mathbb{R}^{1000 \times 4}$

$X^T \in \mathbb{R}^{4 \times 1000}$

$Y \in \mathbb{R}^{1000 \times 1}$

$W \in \mathbb{R}^{4 \times 1}$

assuming 1000 samples.

For 1 sample → $\vec{x}_i = [x_i^3, x_i^2, x_i, x_i^0]$

$\vec{w} = [a, b, c, d]$

In [1]:
```
import sympy as sym
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
```

In [2]:
```
'''
true roots
'''
x=sym.symbols('x')
r1=-0.8
r2=0.2
r3=0.8
y=(x-r1)*(x-r2)*(x-r3)
z=sym.expand(y.simplify())
z
```

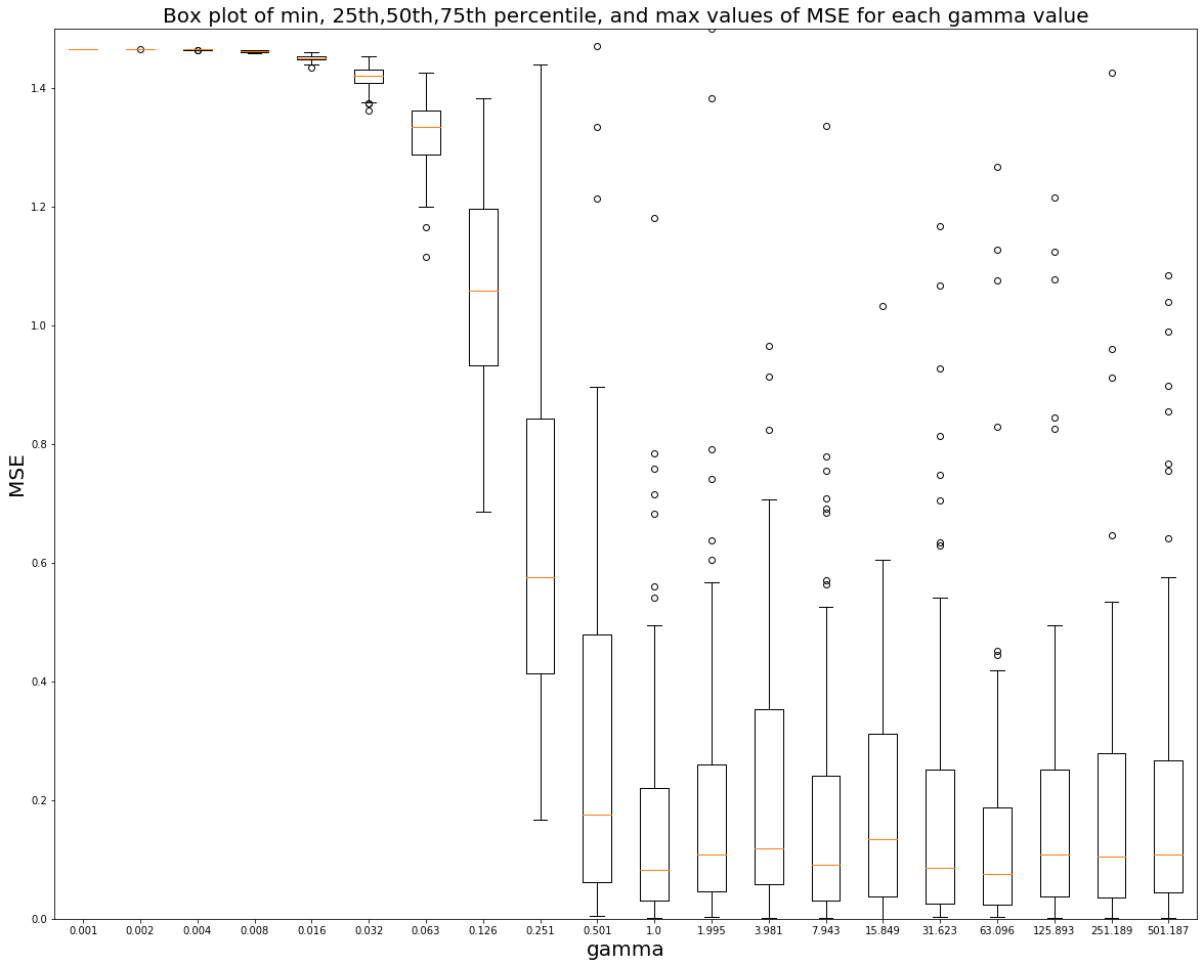
Out[2]: $x^3 - 0.2x^2 - 0.64x + 0.128$

In [3]:
```
'''
true values of W parameter
'''
a=(z.coeff(x**3))
b=(z.coeff(x**2))
c=(z.coeff(x))
d=z.coeff(x,n=0)
w=np.array([a,b,c,d])
w=np.array((w.reshape(-1,1)),dtype=float)
w
```

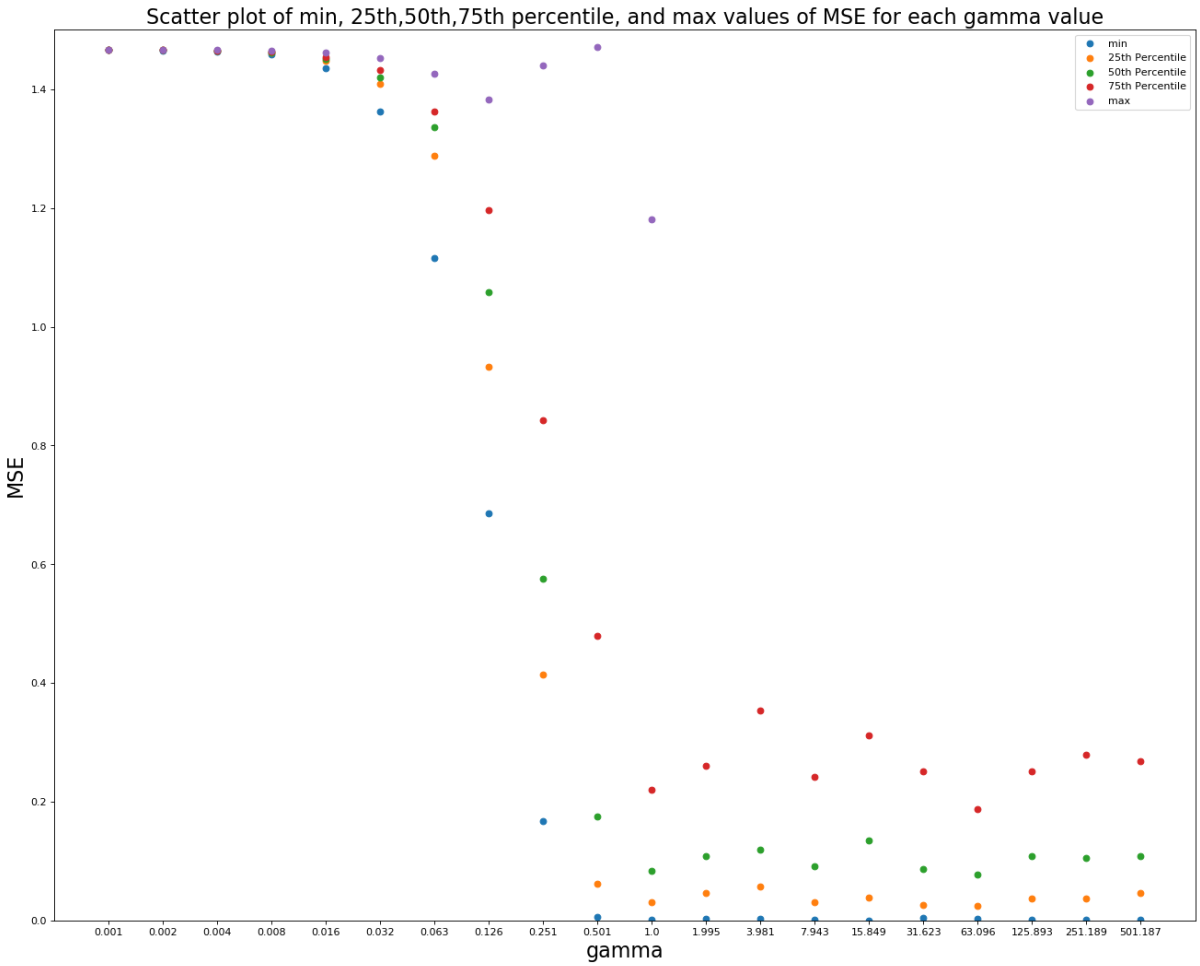
Out[3]:
```
array([[ 1.   ],
       [-0.2  ],
       [-0.64 ],
       [ 0.128]])
```

In [225]:
```
'''
X_stack.shape=(samples,4)
'''

num_samples=10
result={}
sigma=.1 # (Standard deviation of noise) (.01 seems decent) # fixed
gamma_low=-3
gamma_high=3
num_points=20
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    result[gamma]=[]
    for i in range(100):
        Y=[]
        X=[]
        for i in range(num_samples):
            x=np.random.uniform(-1,1)
            y=w[0]*x**3+w[1]*x**2+w[2]*x**1+w[3]*x**0+np.random.normal(s
cale=sigma)
            Y.append(y)
            X.append(x)
        Y=np.array(np.array(Y).reshape(-1,1),dtype=float)
        X=np.array(X).reshape(-1,1)
        X_stack=np.hstack([X**3,X**2,X**1,X**0])
        C=np.matmul((X_stack.T),X_stack)+((sigma**2)/(gamma**2))*np.iden
tity(4)
        A=np.linalg.inv(C)
        B=np.matmul(X_stack.T,Y)
        w_map=np.matmul(A,B)
        result[gamma].append(np.sum(np.square(w_map-w)))
label=[]
result_plot=[]
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    label.append(np.round(gamma,3))
    result_plot.append(result[gamma])
fig1, ax1 = plt.subplots(figsize=(20,16))
from matplotlib.pyplot import figure
plt.ylim(0,1.5)
fig1.text(.35,0.06,'MSE falls as Gamma increases',fontsize=25);
plt.title('Box plot of min, 25th,50th,75th percentile, and max values of
MSE for each gamma value',fontsize=20)
plt.ylabel('MSE',fontsize=20)
plt.xlabel('gamma',fontsize=20)
ax1.boxplot(result_plot,labels=label);

plt.show()
print ("\n\n\n")
from matplotlib.pyplot import figure
fig=figure(num=None, figsize=(20, 16), dpi=80, facecolor='w', edgecolor=
'k')
X=[]
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    X.append(str(np.round(gamma,3)))
```

```
result_min=[]
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    result_min.append((np.array(result[gamma])).min())

result_max=[]
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    result_max.append((np.array(result[gamma])).max())

result_25=[]
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    result_25.append(np.percentile(np.array(result[gamma]),25))
result_50=[]
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    result_50.append(np.percentile(np.array(result[gamma]),50))

result_75=[]
for gamma in np.logspace(gamma_low,gamma_high,num=num_points, endpoint=F
alse):
    result_75.append(np.percentile(np.array(result[gamma]),75))
plt.ylim(0,1.5)
plt.title('Scatter plot of min, 25th,50th,75th percentile, and max value
s of MSE for each gamma value',fontsize=20)
plt.scatter(X,result_min,label='min')
plt.scatter(X,result_25,label='25th Percentile')
plt.scatter(X,result_50,label='50th Percentile')
plt.scatter(X,result_75,label='75th Percentile')
plt.scatter(X,result_max,label='max')
plt.ylabel('MSE',fontsize=20)
plt.xlabel('gamma',fontsize=20)
fig.text(.35,0.06,'MSE falls as Gamma increases',fontsize=25);
plt.legend();
```



MSE falls as Gamma increases



MSE falls as Gamma increases

In [182]:
```
'''
ML estimate
'''

C=np.matmul((X_stack.T),X_stack)
A=np.linalg.inv(C)
B=np.matmul(X_stack.T,Y)
w_ml=np.matmul(A,B)
print ("W_ML is \n",w_ml)

print ("\n\nWith Gamma :",gamma,", W_MAP is \n",w_map)
print ("\nAs gamma increases, w_MAP indeed approaches to w_ml")
```

```
W_ML is
 [[ 1.00279534]
 [-0.19552492]
 [-0.65031524]
 [ 0.13020576]]


With Gamma : 100000.0 , W_MAP is
 [[ 1.00279534]
 [-0.19552492]
 [-0.65031524]
 [ 0.13020576]]

As gamma increases, w_MAP indeed approaches to w_ml
```