D7: Deterministic Signals with Unknown Parameters

Consider the problem of detecting a deterministic signal known except for amplitude in WGN. In this motivating example, we have

$$\mathcal{H}_0 : x[n] = w[n] \qquad n = 0, \ldots, N-1$$
$$\mathcal{H}_1 : x[n] = A\,s[n] + w[n] \qquad n = 0, \ldots, N-1$$

where $s[n]$ is known, A is unknown, $w[n] \sim$ WGN with var. σ^2. If $s[n] = 1$, this is the same problem as DC level of unknown amplitude in WGN.

To determine if a UMP test exists in this case, assume temporarily that A is known. Then LRT decides $\mathcal{H}_1$ if

$\dfrac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} > \gamma$ or equivalently, after simplifications,

if $A>0$: $\displaystyle\sum_{n=0}^{N-1} x[n]s[n] > \frac{\gamma'}{A} = \gamma''$ (as in the matched filter).

if $A<0$: $\displaystyle\sum_{n=0}^{N-1} x[n]s[n] < \frac{\gamma'}{A} = \gamma''$.

Clearly, if A is unknown, we cannot construct a unique test. $\therefore$ There is no UMP test.

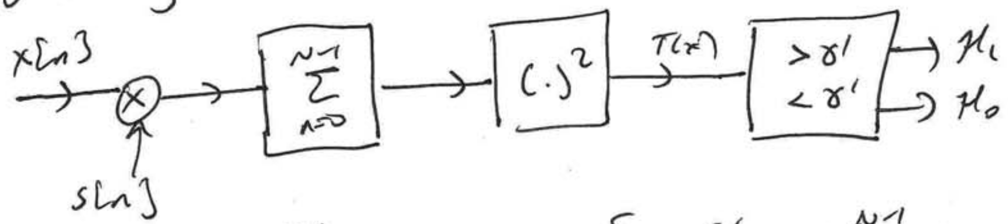
GLRT: We decide $\mathcal{H}_1$ if $\dfrac{p(x;\hat{A},\mathcal{H}_1)}{p(x;\mathcal{H}_0)} > \gamma$ where $\hat{A}$ is the MLE of A under $\mathcal{H}_1$. Specifically, $\hat{A} = \dfrac{x^T s}{s^T s}$, $x = \begin{bmatrix} x[0] \\ \vdots \\ x[N-1] \end{bmatrix}$ $s = \begin{bmatrix} s[0] \\ \vdots \\ s[N-1] \end{bmatrix}$

Substituting the Gaussian pdfs and simplifying as usual, we get

$$T(x) = \left(\sum_{n=0}^{N-1} x[n]s[n] \right)^2 > 2\sigma^2 \ln\gamma \sum_{n=0}^{N-1} s^2[n] = \gamma'$$

or $$\left| \sum_{n=0}^{N-1} x[n]s[n] \right| > \sqrt{2\sigma^2 \ln\gamma \sum_{n=0}^{N-1} s^2[n]} = \sqrt{\gamma'}$$

This test statistic accounts for the unknown sign of A by taking the absolute value.



We have $$u(x) = \sum_{n=0}^{N-1} x[n]s[n] \sim \begin{cases} \mathcal{N}\left(0, \sigma^2 \sum_{n=0}^{N-1} s^2[n]\right) & \text{under } \mathcal{H}_0 \\ \mathcal{N}\left(A \sum_{n=0}^{N-1} s^2[n], \sigma^2 \sum_{n=0}^{N-1} s^2[n]\right) & \text{under } \mathcal{H}_1. \end{cases}$$

$$\therefore P_{FA} = \Pr\{|u(x)| > \sqrt{\gamma'}; \mathcal{H}_0\} = \Pr\{u(x) > \sqrt{\gamma'}; \mathcal{H}_0\} + \Pr\{u(x) < -\sqrt{\gamma'}; \mathcal{H}_0\}$$

$$= Q\left(\frac{\sqrt{\gamma'}}{\sigma\sqrt{s^T s}}\right) + 1 - Q\left(\frac{-\sqrt{\gamma'}}{\sigma\sqrt{s^T s}}\right) = 2Q\left(\frac{\sqrt{\gamma'}}{\sigma\sqrt{s^T s}}\right)$$

$$P_D = \Pr\{|u(x)| > \gamma'; \mathcal{H}_1\}$$

$$= Q\left(\frac{\sqrt{\gamma'} - A s^T s}{\sigma\sqrt{s^T s}}\right) + Q\left(\frac{\sqrt{\gamma'} + A s^T s}{\sigma\sqrt{s^T s}}\right)$$

We have $\dfrac{\sqrt{\gamma'}}{\sigma\sqrt{s^T s}} = Q^{-1}\left(\dfrac{P_{FA}}{2}\right)$, so

$$P_D = Q\left(Q^{-1}(P_{FA}/2) - \frac{\sqrt{\mathcal{E}}}{\sigma}\right) + Q\left(Q^{-1}(P_{FA}/2) + \frac{\sqrt{\mathcal{E}}}{\sigma}\right)$$

$$= Q\left(Q^{-1}(P_{FA}/2) - d\right) + Q\left(Q^{-1}(P_{FA}/2) + d\right)$$

where $d^2 = (A^2 s^T s)/\sigma^2 = \mathcal{E}/\sigma^2$ is the deflection coefficient.

Bayesian Approach

Assume $A \sim \mathcal{N}(\mu_A, \sigma_A^2)$, $A \perp\!\!\!\perp w[n]$. Then under $\mathcal{H}_1$ $x = sA + w$ where $s = [s[0], \ldots, s[N-1]]^T$. This is the Bayesian linear model with $H = s$, $\theta = A$. In this case, the NP test decides $\mathcal{H}_1$ if

$$T'(x) = x^T (H C_\theta H^T + C_w)^{-1} H \mu_\theta$$

$$+ \frac{1}{2} x^T C_w^{-1} H C_\theta H^T (H C_\theta H^T + C_w)^{-1} x > \gamma'$$

$H = s$
$\theta = A$
$\mu_\theta = \mu_A$
$C_\theta = \sigma_A^2$
$C_w = \sigma^2 I$

$$= x^T (\sigma_A^2 s s^T + \sigma^2 I)^{-1} s \mu_A$$

$$+ \frac{1}{2\sigma^2} x^T \sigma_A^2 s s^T (\sigma_A^2 s s^T + \sigma^2 I)^{-1} x$$

$$= x^T (\sigma_A^2 s s^T + \sigma^2 I)^{-1} s \mu_A + \frac{\sigma_A^2}{2\sigma^2} x^T s x^T (\sigma_A^2 s s^T + \sigma^2 I)^{-1} s$$

using the matrix inversion lemma

$$= \frac{\mu_A}{\sigma^2 + \sigma_A^2 s^T s} x^T s + \frac{\sigma_A^2}{2\sigma^2(\sigma^2 + \sigma_A^2 s^T s)} (x^T s)^2$$

Unknown Arrival Time

Consider the detection problem

$$\begin{aligned} \mathcal{H}_0 &: x[n] = w[n] & n = 0, \ldots, N-1 \\ \mathcal{H}_1 &: x[n] = s[n - n_0] + w[n] & \text{"} \end{aligned}$$

where $s[n]$ is known and nonzero over the interval $[0, M-1]$. Here, n_0 is the unknown delay, $w[n]$ is WGN with var. σ^2. Assume that $[0, N-1]$ includes the entire signal for all possible delays n_0.

A GLRT would decide $\mathcal{H}_1$ if $\dfrac{p(x; \hat{n}_0, \mathcal{H}_1)}{p(x; \mathcal{H}_0)} > \gamma$, $\hat{n}_0$ is the MLE of n_0.

The MLE $\hat{n}_0$ is found by maximizing $\sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0]$ over all possible n_0. Note that

$$p(x; n_0, \mathcal{H}_1) = \prod_{n=0}^{n_0-1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} x^2[n]}$$

$$\cdot \prod_{n=n_0}^{n_0+M-1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x[n]-s[n-n_0])^2}$$

$$\cdot \prod_{n=n_0+M}^{N-1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2} x^2[n]}$$

$$= \prod_{n=0}^{N-1} (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}x^2[n]} \prod_{n=n_0}^{n_0+M-1} e^{-\frac{1}{2\sigma^2}(-2x[n]s[n-n_0]+s^2[n-n_0])}$$

where $0 \le n_0 \le N-M$.

$$\therefore \frac{p(x; \hat{n}_0, \mathcal{H}_1)}{p(x; \mathcal{H}_0)} = \prod_{n=\hat{n}_0}^{\hat{n}_0+M-1} e^{-\frac{1}{2\sigma^2}(-2x[n]s[n-\hat{n}_0]+s^2[n-n_0])}$$

Taking the ln and using $\sum_{n=\hat{n}_0}^{\hat{n}_0+M-1} s^2[n-\hat{n}_0] = \mathcal{E}$, we decide $\mathcal{H}_1$ if

$$T(x) = \sum_{n=\hat{n}_0}^{\hat{n}_0+M-1} x[n]s[n-\hat{n}_0] > \frac{\mathcal{E}}{2} + \sigma^2 \ln\gamma = \gamma'$$

$$= \max_{n_0 \in [0, N-M]} \sum_{n=n_0}^{n_0+M-1} x[n]s[n-n_0]$$

Classical Linear Model

Under $\mathcal{H}_1$, we assume $x = H\theta + w$ with $w \sim \mathcal{N}(0, \sigma^2 I)$ and σ^2 known. We wish to test if θ satisfies $A\theta = b$, where A is known. So $\mathcal{H}_0: A\theta = b$, $\mathcal{H}_1: A\theta \ne b$.

Ex) Unknown Amplitude Signal in WGN

$$\mathcal{H}_0 : x[n] = w[n] \qquad n = 0, \ldots, N-1 \qquad \equiv \qquad x[n] = As[n] + w[n],\ A = 0$$
$$\mathcal{H}_1 : x[n] = As[n] + w[n] \qquad \text{"} \qquad\qquad x[n] = As[n] + w[n],\ A \neq 0$$

or in matrix-vector form $x = H\theta + w$ with $H = s$, $\theta = A$,

$\mathcal{H}_0 : \theta = 0$; $\mathcal{H}_1 : \theta \neq 0$. In this example, $A = I$, $b = 0$.

Ex) Sinusoid with unknown amplitude and phase in WGN

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n] \quad \text{under } \mathcal{H}_1$$
$$= \alpha_1 \cos(2\pi f_0 n) + \alpha_2 \sin(2\pi f_0 n) + w[n], \quad A = (\alpha_1^2 + \alpha_2^2)^{1/2}$$

Here $H = \begin{bmatrix} 1 & 1 \\ \cos(2\pi f_0) & \sin(2\pi f_0) \\ \vdots & \vdots \\ \cos(2\pi f_0(N-1)) & \sin(2\pi f_0(N-1)) \end{bmatrix}$ $\theta = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \Rightarrow \begin{array}{l} \mathcal{H}_0 : \theta = 0 \\ \mathcal{H}_1 : \theta \neq 0 . \end{array}$

Thm 7.1 GLRT for Classical Linear Model (App 7B)

Assume that $x = H\theta + w$, where H is known, $w \sim \mathcal{N}(0, \sigma^2 I)$. σ^2 known

The GLRT for the hypothesis testing problem

$$\mathcal{H}_0 : A\theta = b \quad , \quad \mathcal{H}_1 : A\theta \neq b$$

where A is $r \times p$ (rank $r \leq p$), b is $r \times 1$ and $A\theta = b$ is a consistent set of linear equations, is to decide $\mathcal{H}_1$ if

$$T(x) = 2\ln L_G(x) = \frac{1}{\sigma^2}(A\hat{\theta}_1 - b)^T \left[A(H^T H)^{-1} A^T\right]^{-1} (A\hat{\theta}_1 - b) > \gamma'$$

where $\hat{\theta}_1 = (H^T H)^{-1} H^T x$ is the MLE of θ under $\mathcal{H}_1$. The exact detection performance is given by

$$P_{FA} = Q_{\chi_r^2}(\gamma') \quad , \quad P_D = Q_{\chi_r'^2(\lambda)}(\gamma')$$

where $\lambda = \frac{1}{\sigma^2}(A\theta_1 - b)^T \left[A(H^TH)^{-1}A^T\right]^{-1}(A\theta_1 - b)$

Ex] Unknown Amplitude Signal in WGN

$H = s$, $\theta = A$, $\mathcal{H}_0: A = 0$; $\mathcal{H}_1: A \neq 0$, $r = p = 1$, $A = 1$, $b = 0$.

From the theorem $T(x) = \frac{1}{\sigma^2}\hat{\theta}_1 (H^TH)\hat{\theta}_1 = \frac{1}{\sigma^2} H^TH\,\hat{\theta}_1^2$

where $\hat{\theta}_1 = \hat{A} = (H^TH)^{-1}H^Tx = \frac{x^Ts}{s^Ts}$.

We decide $\mathcal{H}_1$ if $T(x) = \frac{(s^Tx)^2}{\sigma^2(s^Ts)} > \gamma' = 2\ln\gamma$ with

$P_{FA} = Q_{\chi_1^2}(\gamma')$ and $P_D = Q_{\chi_1'^2(\lambda)}(\gamma')$ where

$$\lambda = \frac{\theta_1^2(H^TH)}{\sigma^2} = \frac{A^2 s^Ts}{\sigma^2} = \frac{\mathcal{E}}{\sigma^2}$$

After some simplifications, $P_D = Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) - \frac{\sqrt{\mathcal{E}}}{\sigma}\right) + Q\left(Q^{-1}\left(\frac{P_{FA}}{2}\right) + \frac{\sqrt{\mathcal{E}}}{\sigma}\right)$

Suggested Problems: 8, 16, 17, 20, 21, 23, 25