D2: Summary of Important PDFs

Gaussian (Normal): The multivariate Gaussian pdf of an $n\times 1$ (real) random vector x is given by

$$p(x) = (2\pi)^{-1/2} \det{}^{1/2}(C)\, e^{-\frac{1}{2}(x-\mu)^T C^{-1}(x-\mu)}, \quad x \in \mathbb{R}^n$$

where $\mu = E[x]$ and $C = E[(x-\mu)(x-\mu)^T]$.

For $\mu=0$, all odd-order joint moments are zero. Even order moments are combinations of 2^{nd}-order moments. For instance, $E[x_i x_j x_k x_l] = C_{ij}C_{kl} + C_{ik}C_{jl} + C_{il}C_{jk}$, where $C_{ij} = E[x_i x_j]$ since the mean is zero.

For scalar x, $p(x) = (2\pi\sigma^2)^{-1/2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$, $-\infty < x < \infty$, where $\mu = E[x]$ and $\sigma^2 = E[(x-\mu)^2]$. We have, for $\mu=0$,

$$(\mu=0 \Rightarrow) \quad E[x^k] = \begin{cases} 1\cdot 3\cdot 5 \cdot\ldots\cdot (k-1)\sigma^k & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}$$

In general, $E[(x+\mu)^k] = \sum_{l=0}^{k} \binom{k}{l} E[x^l]\mu^{k-l}$, for a nonzero mean random variable $x+\mu$ (with $E[x]=0$).

The cumulative distribution function (cdf) for $\mu=0$ & $\sigma^2=1$ is $\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$. The complementary cumulative distribution (or the right-tail probability) is $Q(x) = 1-\phi(x)$, or $Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,dt$ more explicitly. The following approximation might be useful: $Q(x) \approx (2\pi)^{-1/2} \times e^{-x^2/2}$.

Chi-Squared (Central): A chi-squared (χ^2_ν) pdf with ν degrees of freedom is defined as

$$p(x) = \begin{cases} \dfrac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Here $\nu \in \mathbb{Z}_+$ (an integer ≥ 1), $\Gamma(u)$ is the Gamma function, defined by

$$\Gamma(u) = \int_0^\infty t^{u-1} e^{-t}\, dt$$

which satisfies $\Gamma(u) = (u-1)\Gamma(u-1)$ $\forall u$, $\Gamma(\frac{1}{2}) = \pi$, and $\Gamma(n) = (n-1)!$ $\forall n \in \mathbb{Z}$. As $\nu \to \infty$, the χ^2_ν pdf approaches a Gaussian pdf. For $\nu = 1$, $p(0) = \infty$.

__Fact__: Let $x = \sum_{i=1}^{\nu} x_i^2$ be a scalar random variable where $x_i \sim \mathcal{N}(0,1)$ are iid. Then, $x \sim \chi^2_\nu$. [redacted]

Clearly, $E[x] = \nu$ and $\operatorname{var}(x) = 2\nu$. When $\nu = 2$,

$$p(x) = \begin{cases} \frac{1}{2} e^{-x/2} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$ is exponentially distributed.

[redacted] $Q_{\chi^2_\nu}(x) = \int_x^\infty p(t)\, dt$ for $x > 0$ can be shown to be

$$Q_{\chi^2_\nu}(x) = \begin{cases} e^{-x/2} \sum_{k=0}^{\nu/2-1} \dfrac{(\frac{x}{2})^k}{k!} & \nu \geq 2 \text{ and } \nu \text{ even} \\ 2Q(\sqrt{x}) & \nu = 1 \\ 2Q(\sqrt{x}) + \dfrac{e^{-x/2}}{\sqrt{\pi}} \sum_{k=1}^{(\frac{\nu-1}{2})} \dfrac{(k-1)!\,(2x)^{k-\frac{1}{2}}}{(2k-1)!} & \nu \geq 3 \text{ and } \nu \text{ odd} \end{cases} \quad x > 0$$

Chi-Squared (Noncentral): If $x \sim \sum_{i=1}^{\nu} x_i^2$ where for $\mu_i \neq 0$ $x_i \sim \mathcal{N}(\mu_i, 1)$ are independent, then x has a noncentral chi-squared pdf with ν degrees of freedom. Here,

$\lambda = \sum_{i=1}^{\nu} \mu_i^2$ is called the noncentrality parameter

and
$$p(x) = \begin{cases} \frac{1}{2}\left(\frac{x}{\lambda}\right)^{(\nu-2)/4} e^{-(x+\lambda)/2} I_{\frac{\nu}{2}-1}\left(\sqrt{\lambda x}\right) & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

where
$$I_r(u) = \frac{(u/2)^r}{\sqrt{\pi}\,\Gamma(r+\frac{1}{2})} \int_0^{\pi} e^{u\cos\theta} \sin^{2r}\theta \, d\theta$$
$$= \sum_{k=0}^{\infty} \frac{(u/2)^{2k+r}}{k!\,\Gamma(r+k+1)}$$

is the modified Bessel function of the first kind and order r.

As $\nu \to \infty$, the noncentral chi-squared pdf approaches a Gaussian and for $\lambda = 0$ it reduces to the central χ_ν^2. The noncentral chi-squared pdf with ν degrees of freedom and noncentrality parameter λ is denoted by $\chi_\nu'^2(\lambda)$.

$E[x] = \nu + \lambda$ and $\mathrm{var}(x) = 2\nu + 4\lambda$

F (Central): Let $x = \dfrac{(x_1/\nu_1)}{(x_2/\nu_2)}$ where $x_i \sim \chi^2_{\nu_i}$ $i=1,2.$ are independent. Then x has the (central) F pdf:

$$p(x) = \begin{cases} \dfrac{1}{B(\nu_1/2, \nu_2/2)\left(1+\frac{\nu_1}{\nu_2}x\right)^{(\nu_1+\nu_2)/2}} \left(\dfrac{\nu_1}{\nu_2}\right)^{\nu_1/2} x^{\frac{\nu_1}{2}-1} & \text{if } x>0 \\ 0 & \text{if } x<0 \end{cases}$$

denoted by $\overrightarrow{F_{\nu_1,\nu_2}}$

where $B(u,v) = \dfrac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ is the Beta function.

$$E[x] = \frac{\nu_2}{\nu_2-2} \text{ for } \nu_2>2 \ \& \ \operatorname{var}(x) = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}, \ (\text{for } \nu_2>4)$$

As $\nu_2 \to \infty$, $x_2/\nu_2 \to 1$; therefore $x \to x_1/\nu_1 \sim \chi^2_{\nu_1}/\nu_1$.

F (Noncentral): Let $x = \dfrac{(x_1/\nu_1)}{(x_2/\nu_2)}$ where $x_1 \sim \chi'^2_{\nu_1}(\lambda)$ and $x_2 \sim \chi^2_{\nu_2}$ are independent. Then $x \sim F'_{\nu_1,\nu_2}(\lambda)$ has

$$p(x) = e^{-\lambda/2} \sum_{k=0}^{\infty} \frac{(\lambda/2)^k}{k!} \frac{(\nu_1/\nu_2)^{\frac{\nu_1}{2}+k}}{B\left(\frac{\nu_1+2k}{2}, \frac{\nu_2}{2}\right)} x^{\frac{\nu_1}{2}+k-1} \left(1+\frac{\nu_1}{\nu_2}x\right)^{-\frac{1}{2}(\nu_1+\nu_2)-k} \quad \text{for } x>0,$$

Clearly, $F'_{\nu_1,\nu_2}(0) = F_{\nu_1,\nu_2}$ (only the $k=0$ term in the series is nonzero). Also as $\nu_2 \to \infty$, $F'_{\nu_1,\nu_2}(\lambda) \to \chi'^2_{\nu_1}(\lambda)$.

$$E[x] = \frac{\nu_2(\nu_1+\lambda)}{\nu_1(\nu_2-2)} \text{ for } \nu_2>2 \ \&$$

$$\operatorname{var}(x) = 2\left(\frac{\nu_2}{\nu_1}\right)^2 \frac{(\nu_1+\lambda)^2+(\nu_1+2\lambda)(\nu_2-2)}{(\nu_2-2)^2(\nu_2-4)} \quad \text{for } \nu_2>4$$

<u>Rayleigh:</u> The pdf of $x = \sqrt{x_1^2 + x_2^2}$, where x_1 and x_2 are independent and both have pdfs $\mathcal{N}(0, \sigma^2)$, is Rayleigh, given by

$$p(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/(2\sigma^2)} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$E\{x\} = \sigma\sqrt{\frac{\pi}{2}}$ and $\mathrm{var}(x) = \left(2 - \frac{\pi}{2}\right)\sigma^2$

The right-tail probability is $\int_x^\infty p(t)\,dt = e^{-x^2/(2\sigma^2)}$.

If $y \sim \chi_2^2$ then $x = \sqrt{\sigma^2 y}$ is Rayleigh.

Consequently,
$$\begin{aligned} \Pr\{x > \sqrt{\gamma'}\} &= \Pr\left\{\frac{x}{\sigma} > \frac{\sqrt{\gamma'}}{\sigma}\right\} \\ &= \Pr\left\{\sqrt{y} > \frac{\sqrt{\gamma'}}{\sigma}\right\} \\ &= \Pr\{y > \gamma'/\sigma^2\} \end{aligned}$$

$$\Rightarrow \Pr\{x > \gamma\} = Q_{\chi_2^2}\left(\frac{\gamma'}{\sigma^2}\right)$$

(Recall that $Q_{\chi_2^2}(x) = e^{-x/2}$.)

Rician: The pdf of $x = \sqrt{x_1^2 + x_2^2}$, where $x_1 \sim \mathcal{N}(\mu_1, \sigma^2)$ and $x_2 \sim \mathcal{N}(\mu_2, \sigma^2)$ are independent is Rician, given by

$$p(x) = \begin{cases} \frac{x}{\sigma^2} e^{-(x^2+\alpha^2)/(2\sigma^2)} I_0\left(\frac{\alpha x}{\sigma^2}\right) & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$$

with $\alpha^2 = \mu_1^2 + \mu_2^2$ and $I_0(u)$ is the modified Bessel function of the first kind of order $r=0$.

$$I_0(u) = \frac{1}{\pi}\int_0^{\pi} e^{u\cos\theta}\, d\theta = \int_0^{2\pi} e^{u\cos\theta} \frac{d\theta}{2\pi}$$

Clearly Rician with $\alpha = 0$ reduces to Rayleigh.

$$Pr\{x > \sqrt{\gamma'}\} = Pr\left\{\sqrt{\frac{x_1^2+x_2^2}{\sigma^2}} > \sqrt{\frac{\gamma'}{\sigma^2}}\right\} = Pr\left\{\frac{x_1^2+x_2^2}{\sigma^2} > \frac{\gamma'}{\sigma^2}\right\}$$

$$= Q_{\chi_2'^2(\lambda)}\left(\frac{\gamma'}{\sigma^2}\right)$$

$$\Rightarrow Pr\{x > \gamma\} = Q_{\chi_2'^2(\lambda)}\left(\frac{\gamma^2}{\sigma^2}\right) \quad \text{where } \lambda = \frac{\mu_1^2 + \mu_2^2}{\sigma^2}.$$

Quadratic Forms of Gaussian Random Variables

Consider $y = x^T A x$ where A is a symmetric $n \times n$ matrix and x is an $n \times 1$ random vector $x \sim \mathcal{N}(\mu, \mathbf{C})$.

1) $A = C^{-1}$ and $\mu = 0 \Rightarrow x^T C^{-1} x \sim \chi_n^2$
2) $A = C^{-1}$ and $\mu \neq 0 \Rightarrow x^T C^{-1} x \sim \chi_n'^2(\lambda)$ where $\lambda = \mu^T C^{-1} \mu$
3) $A^2 = A$ (idempotent) and rank r, $C = I$, $\mu = 0 \Rightarrow x^T A x \sim \chi_r^2$

Asymptotic Gaussian PDF

We have seen in estimation that for large data lengths, the multivariate joint Gaussian pdf of the data and its statistics can be approximated using an asymptotic approximation. Let $x = [x[0], x[1], \ldots, x[N-1]]^T$ where $x[n]$ is a zero-mean WSS Gaussian random process.

The covariance matrix of x is, elementwise

$$[C]_{ij} = E[x[i]x[j]] = r_{xx}[i-j]$$

Since (for real-valued $x[n]$) $r_{xx}[-k] = r_{xx}[k]$:

$$\underset{\text{covariance matrix}}{C} = \begin{bmatrix} r_{xx}[0] & r_{xx}[-1] & \cdots & r_{xx}[-N+1] \\ r_{xx}[1] & \ddots & & \vdots \\ \vdots & & \ddots & r_{xx}[-1] \\ r_{xx}[N-1] & \cdots & r_{xx}[1] & r_{xx}[0] \end{bmatrix} = \begin{bmatrix} r_{xx}[0] & \cdots & r_{xx}[N-1] \\ \vdots & \ddots & \vdots \\ r_{xx}[N-1] & \cdots & r_{xx}[0] \end{bmatrix} = \underset{\text{autocorrelation matrix}}{R}$$

$C = R$ is a symmetric Toeplitz matrix.

Let $P_{xx}(f)$ denote the power spectral density (PSD) of $x[n]$. As $N \to \infty$, we have eigenvalues and eigenvectors

$$\lambda_i \to P_{xx}(f_i)$$

$$v_i \to \underbrace{N^{-1/2}\left[1 \quad e^{j2\pi f_i} \quad e^{j4\pi f_i} \quad \cdots \quad e^{j2\pi(N-1)f_i}\right]^T}_{\text{DFT vectors}}$$

for $i = 0, 1, \ldots, (N-1)$ and $f_i = i/N$.

This approximation is good when $N \gg M$, where $r_{xx}[k] \approx 0 \;\; \forall k > M$ (M is the correlation length of $x[n]$).

With this approximation, we have

$$R \approx \sum_{i=0}^{N-1} \lambda_i v_i v_i^H = \sum_{i=0}^{N-1} P_{xx}(f_i) v_i v_i^H$$

equally spaced samples on f

$$\det(R) = \prod_{i=0}^{N-1} \lambda_i = \prod_{i=0}^{N-1} P_{xx}(f_i)$$

$$R^{-1} \approx \sum_{i=0}^{N-1} \frac{1}{\lambda_i} v_i v_i^H = \sum_{i=0}^{N-1} \frac{1}{P_{xx}(f_i)} v_i v_i^H$$

We also have $\ln p(x) = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln\det(R) - \frac{1}{2}x^T R^{-1} x$

which can be approximated as

$$\ln p(x) \approx -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\ln \prod_{i=0}^{N-1} P_{xx}(f_i) - \frac{1}{2}x^T \sum_{i=0}^{N-1} \frac{1}{P_{xx}(f_i)} v_i v_i^H x$$

$$= -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=0}^{N-1} \ln P_{xx}(f_i) - \frac{1}{2}\sum_{i=0}^{N-1} \frac{|v_i^H x|^2}{P_{xx}(f_i)}$$

$$= -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=0}^{N-1}\left(\ln P_{xx}(f_i) + \frac{I(f_i)}{P_{xx}(f_i)}\right)$$

since $|v_i^H x|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x[n] e^{-j2\pi f_i n}\right|^2 = I(f_i)$ is the periodogram (estimate of PSD). Multiplying and dividing the last term by N, then taking the limit as $N \to \infty$.

$$\ln p(x) = -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\sum_{i=0}^{N-1}\left(\ln P_{xx}(f_i) + \frac{I(f_i)}{P_{xx}(f_i)}\right)\frac{1}{N}$$

$$\xrightarrow{N\to\infty} -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\int_{-1/2}^{1/2}\left(\ln P_{xx}(f) + \frac{I(f)}{P_{xx}(f)}\right)df$$

Suggested Problems: 4, 7, 8, 12, 14

Note: Read the Appendices of this chapter.