D5: Random Signals

In this section, we will consider the problem of detecting a random signal with known statistics. Assume that the signal of interest yields data samples with a zero-mean Gaussian pdf that has a known covariance. Assume that the noise is WGN with known variance σ^2 and is independent from the signal. We have (~~with~~ for $s[n] \sim$ WGN with var σ_s^2):

$$\mathcal{H}_0: \; x[n] = w[n] \qquad n = 0, 1, \ldots, (N-1)$$
$$\mathcal{H}_1: \; x[n] = s[n] + w[n] \qquad n = 0, 1, \ldots, (N-1)$$

The NP detector decides $\mathcal{H}_1$ if $L(x) = \dfrac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} > \gamma$.

Here $x \sim \begin{cases} \mathcal{N}(0, \sigma^2 I) \text{ under } \mathcal{H}_0 \\ \mathcal{N}(0, (\sigma_s^2+\sigma^2) I) \text{ under } \mathcal{H}_1 \end{cases}$ and the ln-likelihood is:

$$\ell(x) = \frac{N}{2}\left(\frac{\sigma^2}{\sigma_s^2+\sigma^2}\right) + \frac{1}{2}\,\frac{\sigma_s^2}{\sigma^2(\sigma_s^2+\sigma^2)} \sum_{n=0}^{N-1} x^2[n]$$

which leads to deciding $\mathcal{H}_1$ if $T(x) = \sum_{n=0}^{N-1} x^2[n] > \gamma'$.

In this example, we have an <u>energy detector.</u>

Considering $T'(x) = \dfrac{T(x)}{N}$ as an estimator of the variance, and noticing that $\begin{cases} T(x)/\sigma^2 \sim \chi_N^2 \text{ under } \mathcal{H}_0 \\ T(x)/(\sigma_s^2+\sigma^2) \sim \chi_N^2 \text{ under } \mathcal{H}_1 \end{cases}$, we get

$$P_{FA} = Pr\left\{T(x) > \gamma' ; \mathcal{H}_0\right\} = Pr\left\{\frac{T(x)}{\sigma^2} > \frac{\gamma'}{\sigma^2} ; \mathcal{H}_0\right\} = Q_{\chi^2_N}\left(\frac{\gamma'}{\sigma^2}\right)$$

$$P_D = Pr\left\{T(x) > \gamma' ; \mathcal{H}_1\right\} = Q_{\chi^2_N}\left(\frac{\gamma'}{\sigma_s^2 + \sigma^2}\right) = Q_{\chi^2_N}\left(\frac{\gamma'/\sigma^2}{\sigma_s^2/\sigma^2 + 1}\right).$$

Note that as σ_s^2/σ^2 increases, P_D increases.

General Signal Covariance

Now consider the case of arbitrary signal covariance:

$$x \sim \begin{cases} \mathcal{N}(0, \sigma^2 I) & \text{under } \mathcal{H}_0 \\ \mathcal{N}(0, C_s + \sigma^2 I) & \text{under } \mathcal{H}_1 \end{cases}$$

Then $$L(x) = \frac{(2\pi)^{-N/2} \det^{-1/2}(C_s + \sigma^2 I)\, e^{-\frac{1}{2}x^T (C_s + \sigma^2 I)^{-1} x}}{(2\pi\sigma^2)^{-N/2}\, e^{-x^T x/(2\sigma^2)}} > \gamma$$

$\ln \downarrow$

or $$\ell(x) = -\frac{1}{2}\ln(C_s + \sigma^2 I) + N \ln \sigma - \frac{1}{2} x^T (C_s + \sigma^2 I)^{-1} x + \frac{1}{2\sigma^2} x^T x > \gamma$$

which reduces to $$-\frac{1}{2} x^T \left[(C_s + \sigma^2 I)^{-1} - \frac{1}{\sigma^2} I\right] x > \gamma'$$

or equivalently $$T(x) = \sigma^2 x^T \left[\frac{1}{\sigma^2} I - (C_s + \sigma^2 I)^{-1}\right] x > \cancel{2}\gamma' \sigma^2.$$

<u>Matrix Inversion Lemma</u>: $(A + BCD)^{-1} = A^{-1} - A^{-1}B(DA^{-1}B + C^{-1})^{-1} DA^{-1}$.

Letting $A = \sigma^2 I$, $B = D = I$, $C = C_s$, we get

$$(\sigma^2 I + C_s)^{-1} = \frac{1}{\sigma^2} I - \frac{1}{\sigma^4}\left(\frac{1}{\sigma^2} I + C_s^{-1}\right)^{-1}$$

and $$T(x) = x^T \left[\frac{1}{\sigma^2}\left(\frac{1}{\sigma^2} I + C_s^{-1}\right)^{-1}\right] x.$$

Let $$\hat{s} = \frac{1}{\sigma^2}\left[\frac{1}{\sigma^2} I + C_s^{-1}\right]^{-1} x = \frac{1}{\sigma^2}\left[\frac{1}{\sigma^2}(C_s + \sigma^2 I) C_s^{-1}\right]^{-1} x = C_s (C_s + \sigma^2 I)^{-1} x.$$

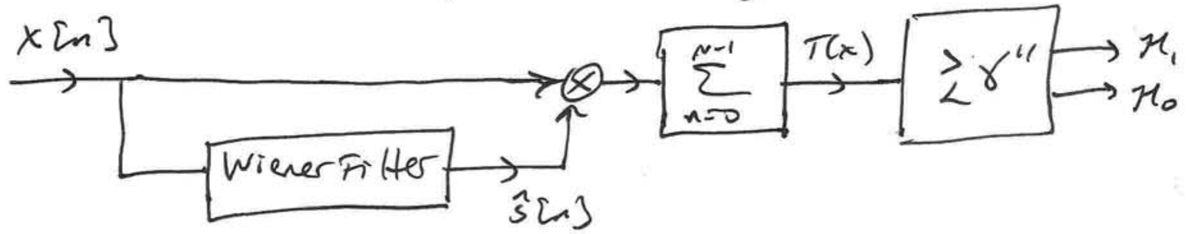
$\therefore$ We decide $\mathcal{H}_1$ if $T(x) = x^T \hat{s} > \gamma''$. This is an estimator-correlator.

(signal estimate)

In fact, $\hat{s}$ is the MMSE (Wiener) estimate of the signal. Recall that if θ and x are jointly Gaussian with zero-mean, $\hat{\theta}_{MMSE} = C_{\theta x} C_{xx}^{-1} x$, where $C_{\theta x} = E[\theta x^T]$ and $C_{xx} = E[x x^T]$. Here, we have $\theta = s$, $x = s + w$ with s & w uncorrelated. The MMSE estimator of s is:

$$\hat{s}_{MMSE} = E\left[s(s+w)^T \right] \left(E[(s+w)(s+w)^T] \right)^{-1} x$$
$$= C_s (C_s + \sigma^2 I)^{-1} x \quad \text{(after simplifications)}.$$

The estimator-correlator for detection of Gaussian signal in WGN:



Ex) If the signal is white, then $C_s = \sigma_s^2 I$ and

$$\hat{s} = \sigma_s^2 I (\sigma_s^2 I + \sigma^2 I)^{-1} x = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} x$$

$$\therefore T(x) = x^T \hat{s} = \frac{\sigma_s^2}{\sigma_s^2 + \sigma^2} x^T x > \gamma'' \iff x^T x > \gamma''(\sigma_s^2 + \sigma^2)/\sigma_s^2,$$

as before.

Ex) Correlated signal with $N = 2$, $C_s = \sigma_s^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$ where ρ is the correlation coefficient between $s[0]$ and $s[1]$.

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \quad V^T = V^{-1}$$

$$\begin{aligned}
T(x) &= x^T C_s (C_s + \sigma^2 I)^{-1} x \\
&= x^T V V^T C_s V V^T (C_s + \sigma^2 I)^{-1} V V^T x \qquad y = V^T x \\
&= y^T (V^T C_s V)(V^T C_s V + \sigma^2 I)^{-1} y \qquad \text{but } V^T C_s V = \Lambda_s \text{ (diagonal)} = \sigma_s^2 \begin{bmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{bmatrix} \\
&= y^T \Lambda_s (\Lambda_s + \sigma^2 I)^{-1} y \\
&= y^T A y
\end{aligned}$$

where $A = \begin{bmatrix} \dfrac{\sigma_s^2(1+\rho)}{\sigma_s^2(1+\rho)+\sigma^2} & 0 \\ 0 & \dfrac{\sigma_s^2(1-\rho)}{\sigma_s^2(1-\rho)+\sigma^2} \end{bmatrix}$.

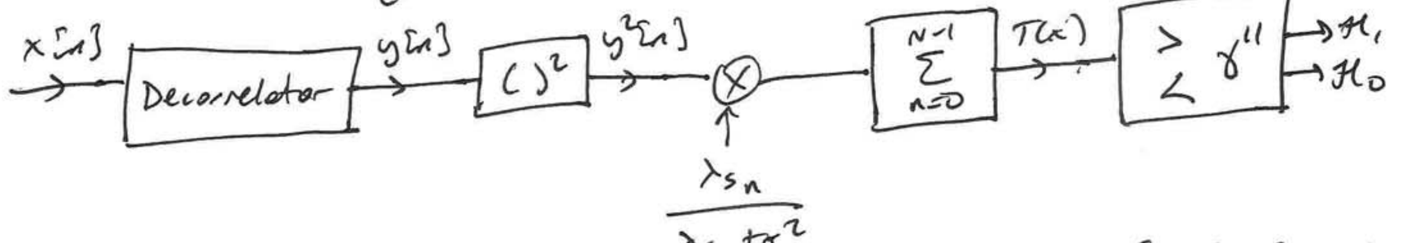
$$\therefore T(x) = \frac{\sigma_s^2(1+\rho)}{\sigma_s^2(1+\rho)+\sigma^2}\, y^2[0] + \frac{\sigma_s^2(1-\rho)}{\sigma_s^2(1-\rho)+\sigma^2}\, y^2[1].$$

In the general case of N samples and arbitrary C_s, if we determine $y = V^T x$ such that $V^T C_s V = \Lambda_s$ yielding

$$C_y = E[y y^T] = E[V^T x x^T V] = V^T C_x V = V^T (C_s + \sigma^2 I) V$$
$$= V^T C_s V + \sigma^2 I = \Lambda_s + \sigma^2 I,$$

we have
$$T(x) = x^T C_s (C_s + \sigma^2 I)^{-1} x = y^T \Lambda_s (\Lambda_s + \sigma^2 I)^{-1} y$$
$$= \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n} + \sigma^2}\, y^2[n].$$

Canonical detector of Gaussian random signal in WGN:



After the orthonormal transformation of data: $y \sim \begin{cases} \mathcal{N}(0, \sigma^2 I), & \mathcal{H}_0 \\ \mathcal{N}(0, \Lambda_s + \sigma^2 I), & \mathcal{H}_1 \end{cases}$.

$$P_{FA} = \Pr\{T(x) > \gamma''; \mathcal{H}_0\} = \Pr\left\{ \sum_{n=0}^{N-1} \frac{\lambda_{s_n}}{\lambda_{s_n}+\sigma^2}\, y^2[n] > \gamma''; \mathcal{H}_0 \right\}$$
$$= \Pr\left\{ \sum_{n=0}^{N-1} \frac{\lambda_{s_n}\sigma^2}{\lambda_{s_n}+\sigma^2}\, z^2[n] > \gamma''; \mathcal{H}_0 \right\} \quad \text{where } z[n] = y[n]/\sigma.$$

Now, $T(x) = \sum_{n=0}^{N-1} \alpha_n z^2[n]$ where $z[n]$ are iid with $\mathcal{N}(0,1)$ pdf.

Recall the characteristic functions: $\phi_x(\omega) = E[e^{j\omega x}]$.

$$\phi_T(\omega) = E\left[e^{j\omega T}\right] = E\left[e^{j\omega \sum_{n=0}^{N-1} \alpha_n z^2[n]}\right]$$

$$= \prod_{n=0}^{N-1} E\left[e^{j\omega \alpha_n z^2[n]}\right] = \prod_{n=0}^{N-1} \phi_{z^2}(\alpha_n \omega)$$

$z[n]$'s are independent

Then $P_T(t) = \begin{cases} \int_{-\infty}^{\infty} \phi_T(\omega) e^{-j\omega t} \frac{d\omega}{2\pi} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$.

We have $z^2[n] \sim \chi_1^2$ and $\phi_{\chi_1^2}(\omega) = \frac{1}{\sqrt{1-2j\omega}}$. Thus

$\phi_T(\omega) = \prod_{n=0}^{N-1} \frac{1}{\sqrt{1-2j\alpha_n \omega}}$, where $\alpha_n = \frac{\lambda_{s_n} \sigma^2}{\lambda_{s_n} + \sigma^2}$. This yields:

$$P_{FA} = \int_{\gamma''}^{\infty} \int_{-\infty}^{\infty} \prod_{n=0}^{N-1} \frac{1}{\sqrt{1-2j\alpha_n \omega}} e^{-j\omega t} \frac{d\omega}{2\pi} dt ,$$

and similarly $P_D = \int_{\gamma''}^{\infty} \int_{-\infty}^{\infty} \prod_{n=0}^{N-1} \frac{1}{\sqrt{1-2j\lambda_{s_n}\omega}} e^{-j\omega t} \frac{d\omega}{2\pi} dt$.

P_{FA} and P_D could be approximated numerically using these.

When the observation noise $w[n]$ is Gaussian, but not white, the estimator-correlator is generalized as follows:

$$T(x) = x^T C_w^{-1} \hat{s} , \text{ where } \hat{s} = C_s (C_s + C_w)^{-1} x$$

with C_w denoting the covariance matrix for noise.

Linear Model

Assume that $x = H\theta + w$ where $x = [x[0], \ldots, x[N-1]]^T$, $H \in \mathbb{R}^{N \times p}$ is known, $\theta \sim N(0, C_\theta)$ and $w \sim N(0, \sigma^2 I)$ is independent of θ.

Then $\mathcal{H}_0: x = w$ and $s = H\theta \sim \mathcal{N}(0, HC_\theta H^T)$.
$\mathcal{H}_1: x = H\theta + w$

Thus, the estimator-correlator decides $\mathcal{H}_1$ if

$$T(x) = x^T C_s (C_s + \sigma^2 I)^{-1} x > \gamma'' \qquad \ni C_s = HC_\theta H^T$$

$$\text{or } T(x) = x^T HC_\theta H^T (HC_\theta H^T + \sigma^2 I)^{-1} x > \gamma''$$

$$\text{or } T(x) = x^T \hat{s} = x^T H \hat{\theta}_{MMSE}$$

Ex) Rayleigh Fading Sinusoid

$$x[n] = A\cos(2\pi f_0 n + \phi) + w[n] \quad n = 0, \dots, (N-1) \quad 0 < f_0 < \tfrac{1}{2}$$

and $w[n] \sim$ WGN with var σ^2. Note that

$$s[n] = A\cos(2\pi f_0 n + \phi) = a\cos(2\pi f_0 n) + b\sin(2\pi f_0 n)$$

where $a = A\cos\phi$, $b = -A\sin\phi$. Let $\theta = \begin{bmatrix} a \\ b \end{bmatrix} \sim \mathcal{N}(0, \sigma_s^2 I)$, and assume $\theta \perp\!\!\!\perp w[n]$. Then $s[n] \sim$ WSS Gaussian since $E[s[n]] = E[a]\cos(2\pi f_0 n) + E[b]\sin(2\pi f_0 n) = 0$ and $E[s[n]s[n+k]] = \dots = \sigma_s^2 \cos(2\pi f_0 k)$ (after some effort).

$\therefore r_{ss}[k] = \sigma_s^2 \cos(2\pi f_0 k)$. Here, $A = \sqrt{a^2 + b^2}$ is Rayleigh distributed: $p(A) = \begin{cases} \frac{A}{\sigma_s^2} e^{-A^2/(2\sigma_s^2)}, & A > 0 \\ 0, & A < 0 \end{cases}$ and $\phi = \arctan(-b/a)$ is $\mathcal{U}[0, 2\pi)$ distributed. $A \perp\!\!\!\perp \phi$. This situation is referred to as "Rayleigh fading".

The signal model is $x = H\theta + w$, where

$$H = \begin{bmatrix} \cos(2\pi f_0) & \sin(2\pi f_0) \\ \vdots & \vdots \\ \cos(2\pi f_0(N-1)) & \sin(2\pi f_0(N-1)) \end{bmatrix}, \quad \theta \sim \mathcal{N}(0, \sigma_s^2 I) \quad \theta \perp\!\!\!\perp w.$$
$$w \sim \mathcal{N}(0, \sigma^2 I)$$

The NP detector is $T(x) = x^T H C_\theta H^T (H C_\theta H^T + \sigma^2 I)^{-1} x$,

$$= \sigma_s^2 x^T H H^T (\sigma_s^2 H H^T + \sigma^2 I)^{-1} x,$$

matrix inv. lemma

$$= \sigma_s^2 x^T H H^T \left[\frac{1}{\sigma^2} I - \frac{1}{\sigma^4}\sigma_s^2 H \left(\frac{\sigma_s^2 H^T H}{\sigma^2} + I \right)^{-1} H^T \right] x$$

Note that $H^T H = \begin{bmatrix} \sum_{n=0}^{N-1} \cos^2(2\pi f_0 n) & \sum_{n=0}^{N-1} \cos(2\pi f_0 n)\sin(2\pi f_0 n) \\ \sum_{n=0}^{N-1} \cos(2\pi f_0 n)\sin(2\pi f_0 n) & \sum_{n=0}^{N-1} \sin^2(2\pi f_0 n) \end{bmatrix}$

(we have seen this in estimation) $\approx \frac{N}{2} I$.

$$\therefore T(x) \approx \sigma_s^2 x^T H H^T \left(\frac{1}{\sigma^2} I - \frac{\sigma_s^2}{\sigma^2} H \frac{1}{\left(\frac{N\sigma_s^2}{2\sigma^2} + 1\right)} I H^T \right) x$$

$$= \frac{\sigma_s^2}{\sigma^2} x^T H H^T x - \frac{(N\sigma_s^4)/(2\sigma^4)}{(N\sigma_s^2)/(2\sigma^2)+1} x^T H H^T x = \frac{c}{N} x^T H H^T x$$

with $c = \dfrac{N\sigma_s^2}{\left(\frac{N\sigma_s^2}{2} + \sigma^2\right)} > 0$. Equivalently, $T'(x) = \frac{1}{N} x^T H H^T x = \frac{1}{N} \|H^T x\|_2^2$

$$\therefore T'(x) = \frac{1}{N} \left\| \begin{bmatrix} \sum_{n=0}^{N-1} x[n]\cos(2\pi f_0 n) \\ \sum_{n=0}^{N-1} x[n]\sin(2\pi f_0 n) \end{bmatrix} \right\|^2 = \frac{1}{N} \left| \sum_{n=0}^{N-1} x[n] e^{-j2\pi f_0 n} \right|^2 > \frac{\gamma''}{c}$$

to decide $\mathcal{H}_1$.

To determine P_{FA} and P_D, consider

$$C_s = H C_\theta H^T = H \sigma_s^2 I H^T = \sigma_s^2 [h_0 \; h_1] \begin{bmatrix} h_0^T \\ h_1^T \end{bmatrix} = \frac{N\sigma_s^2}{2} \frac{h_0}{\sqrt{N/2}} \frac{h_0^T}{\sqrt{N/2}} + \frac{N\sigma_s^2}{2} \frac{h_1}{\sqrt{N/2}} \frac{h_1^T}{\sqrt{N/2}}$$

With $\lambda_{s_0} = \lambda_{s_1} = N\sigma_s^2/2$, $v_0 = \frac{h_0}{\sqrt{N/2}}$, $v_1 = \frac{h_1}{\sqrt{N/2}}$:

$C_s = \lambda_{s_0} v_0 v_0^T + \lambda_{s_1} v_1 v_1^T$ where $v_0^T v_1 \approx 0$ for large N.

$\left(v_0^T v_1 = \frac{2}{N} \sum_{n=0}^{N-1} \cos(2\pi f_0 n) \sin(2\pi f_0 n) \right)$. With this approximate eigendecomposition of C_s, $\lambda_{s_2} = \cdots = \lambda_{s_{N-1}} = 0$ (rank $C_s = 2$)

and $P_{FA} = Pr\{T'(x) > \gamma''' ; \mathcal{H}_0\} = Pr\{T(x) > \gamma'' ; \mathcal{H}_0\} = e^{-\frac{\gamma''}{2\alpha_0}}$

where $\alpha_0 = \dfrac{\lambda_{s_0}\sigma^2}{\lambda_{s_0} + \sigma^2} = \dfrac{N\sigma_s^2\sigma^2/2}{N\sigma_s^2/2 + \sigma^2} = \dfrac{c\sigma^2}{2}$. But $\gamma'' = c\gamma'''$

so $P_{FA} = e^{-\gamma'''/\sigma^2}$. Also $P_D = e^{-\frac{\gamma''}{2\lambda_{s_0}}} = e^{-\frac{\gamma''}{N\sigma_s^2}} = e^{-\frac{\gamma'''}{N\sigma_s^2/2+\sigma^2}}$

(see the book for details).

Defining $\bar{\eta} = N E[A^2/2]/\sigma^2 = N\sigma_s^2/\sigma^2$ (average energy-noise ratio)

we can show that $P_D = P_{FA}^{\frac{1}{1+\bar{\eta}/2}}$.

Estimator-Correlator for Large Data Records

Recall that for zero-mean WSS Gaussian $x[n]$ with PSD $P_{xx}(f)$, we have

$$\ln p(x) \approx -\frac{N}{2}\ln(2\pi) - \frac{N}{2}\int_{-1/2}^{1/2} \ln P_{xx}(f)\,df - \frac{N}{2}\int_{-1/2}^{1/2} \frac{I(f)}{P_{xx}(f)}\,df$$

where $I(f) = \frac{1}{N}\left| \sum_{n=0}^{N-1} x[n] e^{-j2\pi f n} \right|^2$ is the periodogram.

The NP detector decides $\mathcal{H}_1$ if

$$l(x) = \ln p(x; \mathcal{H}_1) - \ln p(x; \mathcal{H}_0) > \gamma'.$$

Under $\mathcal{H}_0$, $x[n] = w[n]$, $P_{xx}(f) = \sigma^2$ and under $\mathcal{H}_1$, $x[n] = s[n] + w[n]$, $P_{xx}(f) = P_{ss}(f) + \sigma^2$. Then

$$l(x) \approx -\frac{N}{2}\int_{-1/2}^{1/2} \ln(P_{ss}(f)+\sigma^2)\,df - \frac{N}{2}\int_{-1/2}^{1/2} \frac{I(f)}{P_{ss}(f)+\sigma^2}\,df$$
$$+ \frac{N}{2}\int_{-1/2}^{1/2} \ln\sigma^2\,df + \frac{N}{2}\int_{-1/2}^{1/2} \frac{I(f)}{\sigma^2}\,df$$
$$= -\frac{N}{2}\int_{-1/2}^{1/2} \ln\left(\frac{P_{ss}(f)}{\sigma^2}+1\right)df + \frac{N}{2}\int_{-1/2}^{1/2} I(f)\frac{P_{ss}(f)}{(P_{ss}(f)+\sigma^2)\sigma^2}\,df$$

$\therefore$ we decide $\mathcal{H}_1$ if $T(x) = N\int_{-1/2}^{1/2} \frac{P_{ss}(f)}{P_{ss}(f)+\sigma^2} I(f)\,df > \gamma''$.

Note that $H(f) = \dfrac{P_{ss}(f)}{P_{ss}(f)+\sigma^2}$ is the noncausal IIR Wiener filter for our signal model. Hence

$$T(x) = \int_{-1/2}^{1/2} H(f)\,X(f)\,X^*(f)\,df = \int_{-1/2}^{1/2} X(f)\left[H(f)X(f)\right]^* df$$
$$= \int_{-1/2}^{1/2} X(f)\,\hat{S}^*(f)\,df = \langle X, \hat{S}\rangle \text{ in } f\text{-domain.}$$

estimated signal

General Gaussian Detection

For $\mathcal{H}_0: x = w$, $\mathcal{H}_1: x = s + w$ where $w \sim \mathcal{N}(0, C_w)$, $s \sim \mathcal{N}(\mu_s, C_s)$ and $s \perp\!\!\!\perp w$, the NP detector will decide $\mathcal{H}_1$ if $\dfrac{p(x;\mathcal{H}_1)}{p(x;\mathcal{H}_0)} > \gamma$, or (after simplification)

$$T(x) = x^T C_w^{-1} x - (x-\mu_s)^T (C_s + C_w)^{-1}(x-\mu_s) > \gamma'$$

From the matrix inversion lemma

$C_w^{-1} - (C_s + C_w)^{-1} = C_w^{-1} C_s (C_s + C_w)^{-1}$, so

$$T'(x) = x^T (C_s + C_w)^{-1} \mu_s + \frac{1}{2} x^T C_w^{-1} C_s (C_s + C_w)^{-1} x.$$

Special cases: $C_s = 0$ ($s = \mu_s$ deterministic) $\Rightarrow T'(x) = x^T C_w^{-1} \mu_s$

$\mu_s = 0$ ($s \sim \mathcal{N}(0, C_s)$) $\Rightarrow T'(x) = \frac{1}{2} x^T C_w^{-1} \hat{s}$,

where $\hat{s} = C_s (C_s + C_w)^{-1} x$ is the MMSE estimator.

Ex) Assume $\mathcal{H}_0: x[n] = w[n]$ where $w[n] \sim$ WGN and

$\mathcal{H}_1: x[n] = s[n] + w[n]$ where $s[n] \sim \mathcal{N}(A, \sigma_s^2)$

with $s[n]$ iid and $\perp\!\!\!\perp$ of $w[n]$. Then, $C_w = \sigma^2 I$, $\mu_s = A\mathbf{1}$,

$C_s = \sigma_s^2 I$; so

$$T'(x) = x^T (\sigma_s^2 I + \sigma^2 I)^{-1} A\mathbf{1} + \frac{1}{2} x^T \frac{1}{\sigma^2} I \sigma_s^2 I (\sigma_s^2 I + \sigma^2 I)^{-1} x$$

$$= \frac{NA}{\sigma_s^2 + \sigma^2} \bar{x} + \frac{1}{2} \frac{\sigma_s^2 / \sigma^2}{\sigma_s^2 + \sigma^2} \sum_{n=0}^{N-1} x^2[n]$$

For the Bayesian linear model in which $s = H\theta$, $\theta \sim \mathcal{N}(\mu_\theta, C_\theta)$, and $\theta \perp\!\!\!\perp w$, we have $\mu_s = H\mu_\theta$ and $C_s = H C_\theta H^T$:

$$T'(x) = x^T (H C_\theta H^T + C_w)^{-1} H \mu_\theta + \frac{1}{2} x^T C_w^{-1} H C_\theta H^T (H C_\theta H^T + C_w)^{-1} x$$

Suggested Problems: 2, 5, 11, 24 (check out sec 5-7 first)...