Github repo for this exam-
https://github.com/rohinarora/EECE5644-Machine_Learning/Exam1/

Additionally all the code and figures are present in this file, and also attached on blackboard as additional reference

Answer 2

Answer 2

Let $\vec{\theta} = \begin{bmatrix} x \\ y \end{bmatrix}$

prior of $\vec{\theta} \rightarrow$

$$p\begin{bmatrix} x \\ y \end{bmatrix} = (2\pi \sigma_x \sigma_y)^{-1} e^{-\frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix}}$$

To find $\rightarrow \underbrace{\theta_{MAP}}_{\begin{bmatrix} x_{MAP} \\ y_{MAP} \end{bmatrix}} = \underset{\theta}{\operatorname{argmax}}\ p(\vec{\theta} \mid \vec{R})$

where $\vec{R} = (r_1, r_2, \dots, r_K) \rightarrow K$ measurements

$$r_i = d_{Ti} + n_i$$

$$d_{Ti} = \sqrt{(x_{Ti} - x_i)^2 + (y_{Ti} - y_i)}$$

where (x_i, y_i) is the sensor position (known to us)

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}}\; P(\vec{\theta} \mid \vec{R})$$

Bayes Rule

$$= \underset{\theta}{\operatorname{argmax}}\; \frac{P(\vec{R} \mid \vec{\theta})\, P(\vec{\theta})}{P(\vec{R})}$$

$$= \underset{\theta}{\operatorname{argmax}}\; P(\vec{R} \mid \vec{\theta})\; P(\theta)$$

we can take log of probability as log is increasing funct^{n}

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\log P(\underset{r_1, r_2, r_3 - - -, r_n}{\underset{\downarrow}{\vec{R}}} \mid \vec{\theta}) + \log P(\vec{\theta}) \right]$$

observations are independent

$$= \underset{\theta}{\operatorname{argmax}} \left[\log \prod_{i=1}^{K} P(r_i \mid \vec{\theta}) + \log P(\vec{\theta}) \right]$$

$$= \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} \log P(r_i \mid \vec{\theta}) + \log P(\vec{\theta}) \right]$$

$$= \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} \log P(r_i|\theta) + \log \left((2\pi \sigma_x \sigma_y)^{-1} e^{-\frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} \right. \right.$$

$$r_i|\theta = \sqrt{(x_i - x) + \underbrace{(y_i - y)^2}_{\text{constant}}} + n_i$$

$$n_i \sim N(0, \sigma_i^2)$$

$$\therefore \mu_{r_i|\theta} = \sqrt{(x_i - x) + (y_i - x)} = d_{i,xy} = d_i(x, y)$$

$$\sigma_{r_i|\theta} = \sigma_i^2$$

$$\therefore r_i|\theta \sim N(d_{i(x,y)}, \sigma_i^2)$$

$\therefore r_i|\theta$ is a gaussian with mean $d_i(x,y)$ and variance σ_i^2.

$$\Rightarrow \theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} \log \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(r_i - d_i(x,y))^2}{2\sigma_i^2}} + \underbrace{\log(2\pi\sigma_x\sigma_y)^{-1}}_{\text{independent of } (x, Y)} - \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \end{bmatrix} \right]$$

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} \left[\log \frac{1}{\sqrt{2\pi\sigma_i^2}} - \frac{(r_i - d_i(x,y))^2}{2\sigma_i^2} \right] - \frac{1}{2} \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1/\sigma_x^2 & 0 \\ 0 & 1/\sigma_y^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right]$$

independent of θ

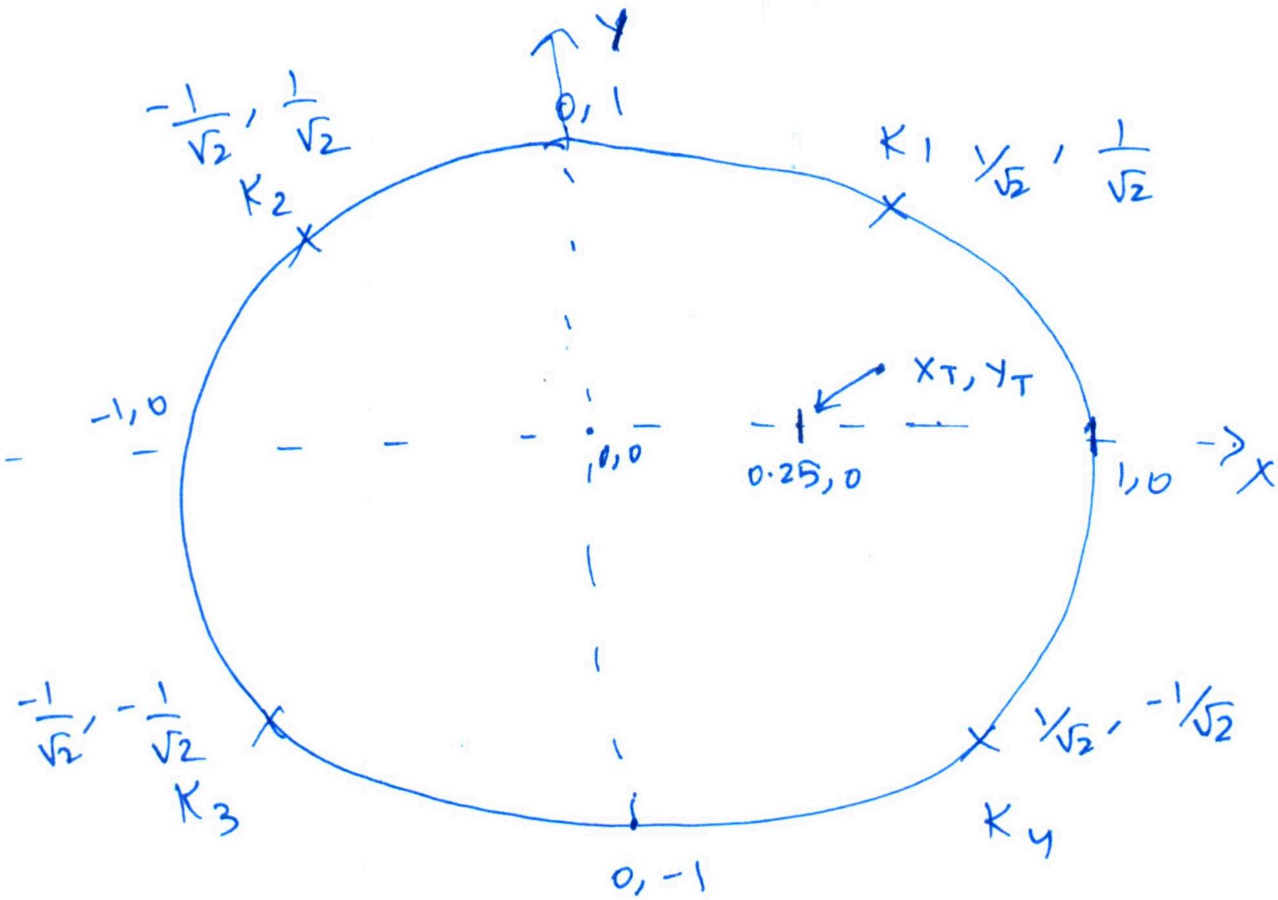
$$\theta_{MAP} = \underset{\theta}{\operatorname{argmax}} \left[\sum_{i=1}^{K} - \frac{\left(r_i - \left((x_i - x)^2 + (y_i - y)^2\right)^{1/2}\right)^2}{2\sigma_i^2} - \frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right] \right]$$

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmin}} \left[\sum_{i=1}^{K} \frac{\left(r_i - \sqrt{(x_i - x)^2 + (y_i - y)^2}\right)^2}{\sigma_i^2} + \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right) \right]$$

where $\theta = [x \;\; y]^T$

solving this optimization will give θ_{MAP} or $\begin{bmatrix} x_{MAP} \\ y_{MAP} \end{bmatrix}$

Visualization of the data-



The code is well commented explaining the steps

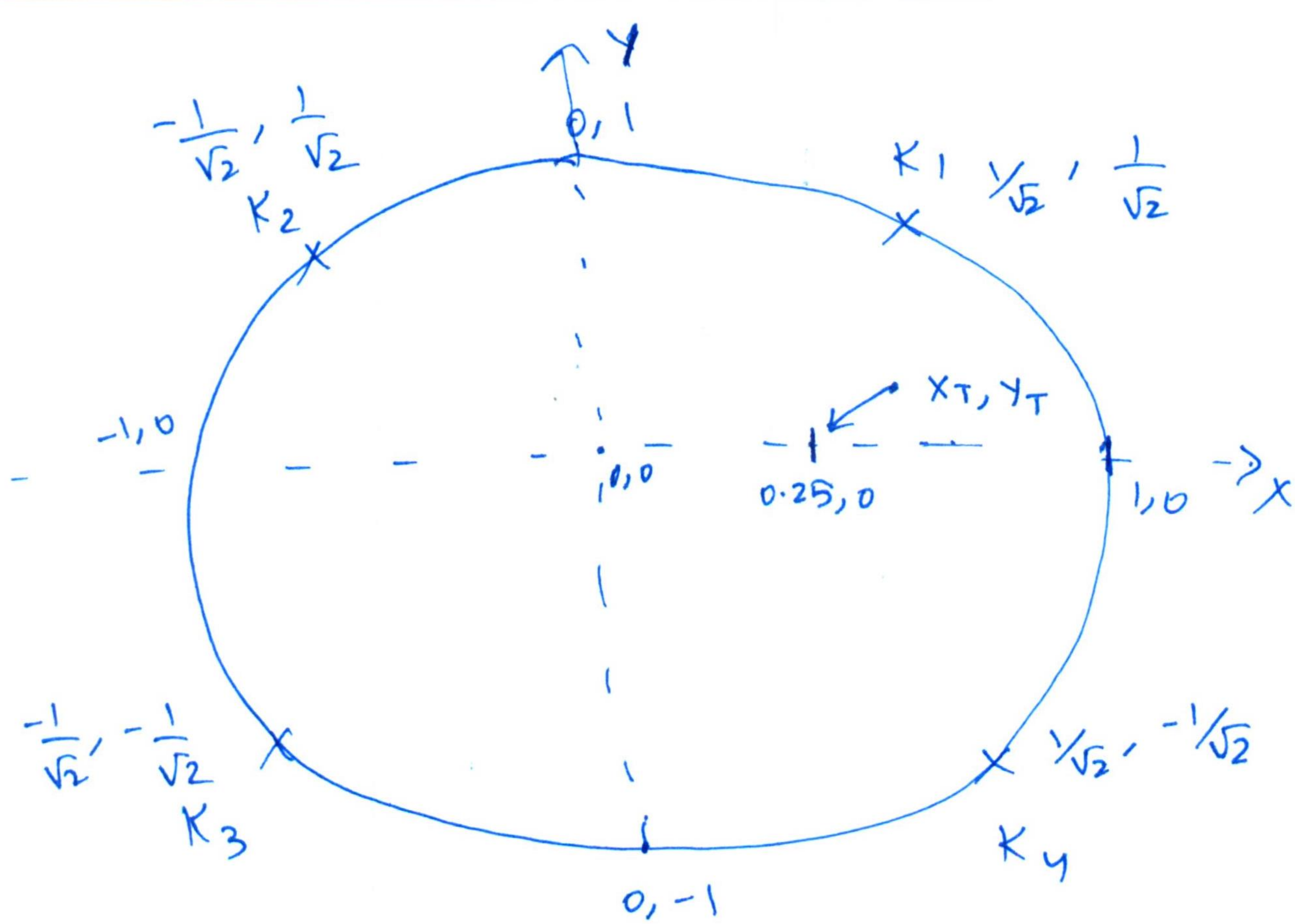
sigma_x=0.1

sigma_y=0.1

sigma_i=.01

In [2]:
```
'''
imports
'''

import matplotlib.pyplot as plt
%matplotlib inline
import numpy as np
import sympy as sym
```

Viz of the data-



In [3]:
```
'''
Position the sensor. k1-k4 represent the 4 sensors
'''

k1=(1/np.sqrt(2),1/np.sqrt(2))
k2=(-1/np.sqrt(2),1/np.sqrt(2))
k3=(-1/np.sqrt(2),-1/np.sqrt(2))
k4=(1/np.sqrt(2),-1/np.sqrt(2))

'''
True location of the object
'''

xy_true=(0.2,0.3)

sensors=[k1,k2,k3,k4] # make a list of sensors


# set the sigma_x, sigma_y, sigma_i
sigma_x=0.1
sigma_y=0.1
sigma_i=.01 #0.3 for submission
```

In [4]:
```
def dist_points(p1,p2):
    '''
    L2 distance between points p1 and p2
    Assumes p1 and p2 are both tuples (x,y)
    '''
    return ((p1[0]-p2[0])**2+(p1[1]-p2[1])**2)**(0.5)
```

In [5]:
```
def measure_value(k):
    '''
    measure the distance between sensor k and true position "xy_true". A
dd gaussian noise with std sigma_i to it
    '''
    measurement=dist_points(k,xy_true)+np.random.normal(scale=sigma_i)
    if measurement>0:
        return measurement
    else:
        measure_value(k)
```

In [6]:
```python
'''
Calculate the distance of sensor to true position "xy_true", by calling
 the function measure_value
Store the resulting values in a dictionary
'''


sensor_measurements=dict()
sensor_measurements[k1]=measure_value(k1)
sensor_measurements[k2]=measure_value(k2)
sensor_measurements[k3]=measure_value(k3)
sensor_measurements[k4]=measure_value(k4)
print ("sensor data is :\n", sensor_measurements)
print ('\n\n')
for j,i in enumerate(sensor_measurements):
    print ("Distance of sensor k"+str(j+1)+" from true positon",sensor_measurements[i])
```

```
sensor data is :
 {(0.7071067811865475, 0.7071067811865475): 0.6451761942076968, (-0.7071067811865475, 0.7071067811865475): 1.0024351553984014, (-0.7071067811865475, -0.7071067811865475): 1.3675174669665544, (0.7071067811865475, -0.7071067811865475): 1.1276801471635933}



Distance of sensor k1 from true positon 0.6451761942076968
Distance of sensor k2 from true positon 1.0024351553984014
Distance of sensor k3 from true positon 1.3675174669665544
Distance of sensor k4 from true positon 1.1276801471635933
```

$$\theta_{MAP} = \underset{\theta}{\operatorname{argmin}} \left[\sum_{i=1}^{K} \frac{\left(r_i - \sqrt{(x_i - x)^2 + (y_i - y)^2}\right)^2}{\sigma_i^2} + \left(\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right) \right]$$

where $\theta = [x \;\; y]^T$

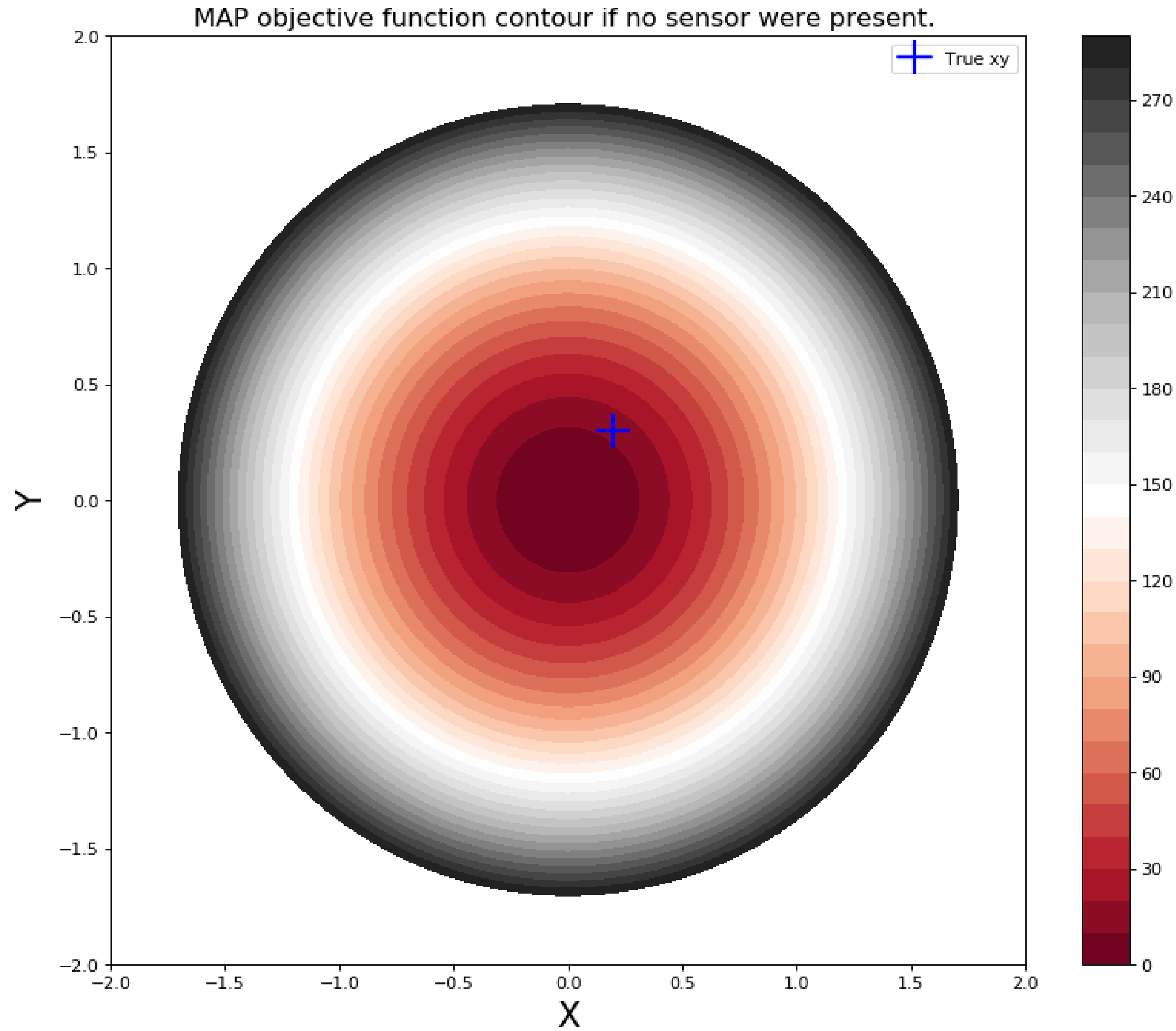
solving this optimization will give θ_{MAP} or $\begin{bmatrix} x_{MAP} \\ y_{MAP} \end{bmatrix}$

In [7]:
```
# Pre define the contour_level. This sets the contour levels for all the
plots.
contour_level=[]
for i in range(0,300,10):
    contour_level.append(i)


# Create a meshgrid of X-Y
x = np.linspace(-2, 2, 1000)
y = np.linspace(-2, 2, 1000)
X, Y = np.meshgrid(x, y)


# Below functions give the MAP objective values for different cases. Ref
er above image for the formaula.

def f0(x, y):
    '''
    MAP objective if no sensor were present.
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    return prior

def f1(x, y):
    '''
    MAP objective if 1 sensor was present.
    '''


    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    return first_senor+prior




def f2(x, y):
    '''
    two sensor
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    second_senor=(np.square(sensor_measurements[k3]-np.sqrt((k3[0]-x)**2
+(k3[1]-y)**2)))*(1/sigma_i**2)
    return first_senor+second_senor+prior

def f3(x, y):
    '''
    three sensor
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    second_senor=(np.square(sensor_measurements[k2]-np.sqrt((k2[0]-x)**2
+(k2[1]-y)**2)))*(1/sigma_i**2)
    third_senor=(np.square(sensor_measurements[k3]-np.sqrt((k3[0]-x)**2+
```

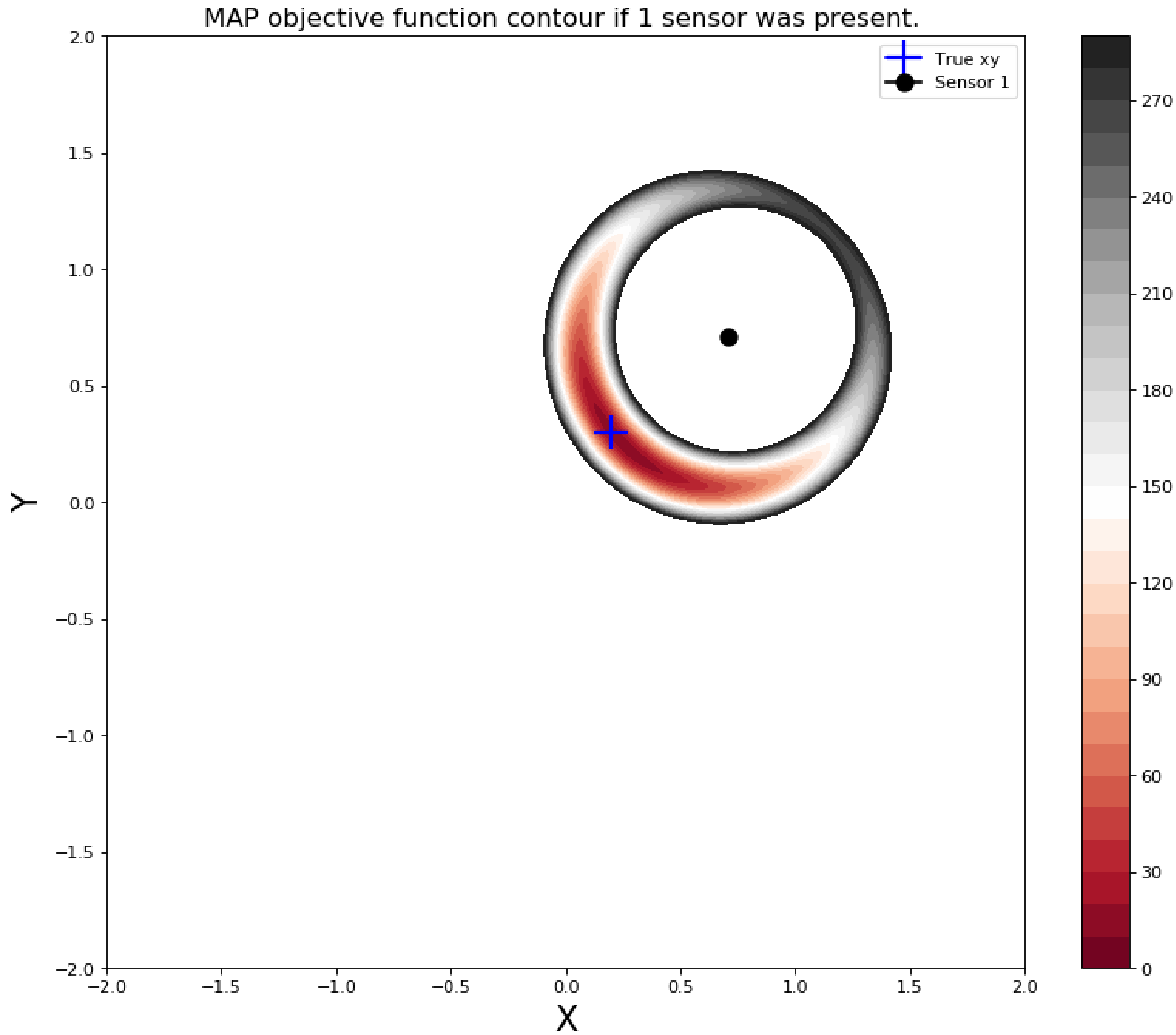
```
(k3[1]-y)**2)))*(1/sigma_i**2)
    return first_senor+second_senor+third_senor+prior

def f4(x, y):
    '''
    four sensor
    '''
    prior=(x**2)/(sigma_x**2)+(y**2)/(sigma_y**2)
    first_senor=(np.square(sensor_measurements[k1]-np.sqrt((k1[0]-x)**2+
(k1[1]-y)**2)))*(1/sigma_i**2)
    second_senor=(np.square(sensor_measurements[k2]-np.sqrt((k2[0]-x)**2
+(k2[1]-y)**2)))*(1/sigma_i**2)
    third_senor=(np.square(sensor_measurements[k3]-np.sqrt((k3[0]-x)**2+
(k3[1]-y)**2)))*(1/sigma_i**2)
    forth_senor=(np.square(sensor_measurements[k4]-np.sqrt((k4[0]-x)**2+
(k4[1]-y)**2)))*(1/sigma_i**2)
    return first_senor+second_senor+third_senor+forth_senor+prior



'''


Plotting function.

Takes in argument objective function and number of sensors
Objective function can be one of functions defined above- f0,f1,f2,f3,f4
num_sensors can be 0-4
'''



def _plot_(f,num_sensors):
    Z = f(X, Y)   # compute Z for objective function f
    from matplotlib.pyplot import figure
    fig=figure(num=None, figsize=(12, 10), dpi=80, facecolor='w', edgeco
lor='k')

    # plot contour
    plt.contourf(X, Y, Z, 20, cmap='RdGy',levels=contour_level);

    # set true labels and sensor positions
    plt.plot([xy_true[0]], [xy_true[1]], marker='+', markersize=20, colo
r="blue",label="True xy",mew=2)
    if num_sensors==1:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
    elif num_sensors==2:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
        plt.plot([k3[0]], [k3[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 2")
    elif num_sensors==3:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
        plt.plot([k2[0]], [k2[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 2")
        plt.plot([k3[0]], [k3[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 3")
```

```
    elif num_sensors==4:
        plt.plot([k1[0]], [k1[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 1")
        plt.plot([k2[0]], [k2[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 2")
        plt.plot([k3[0]], [k3[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 3")
        plt.plot([k4[0]], [k4[1]], marker='o', markersize=10, color="bla
ck",label="Sensor 4")


    plt.xlim(-2,2)
    plt.ylim(-2,2)
    plt.colorbar();
    plt.legend();
    return fig
```

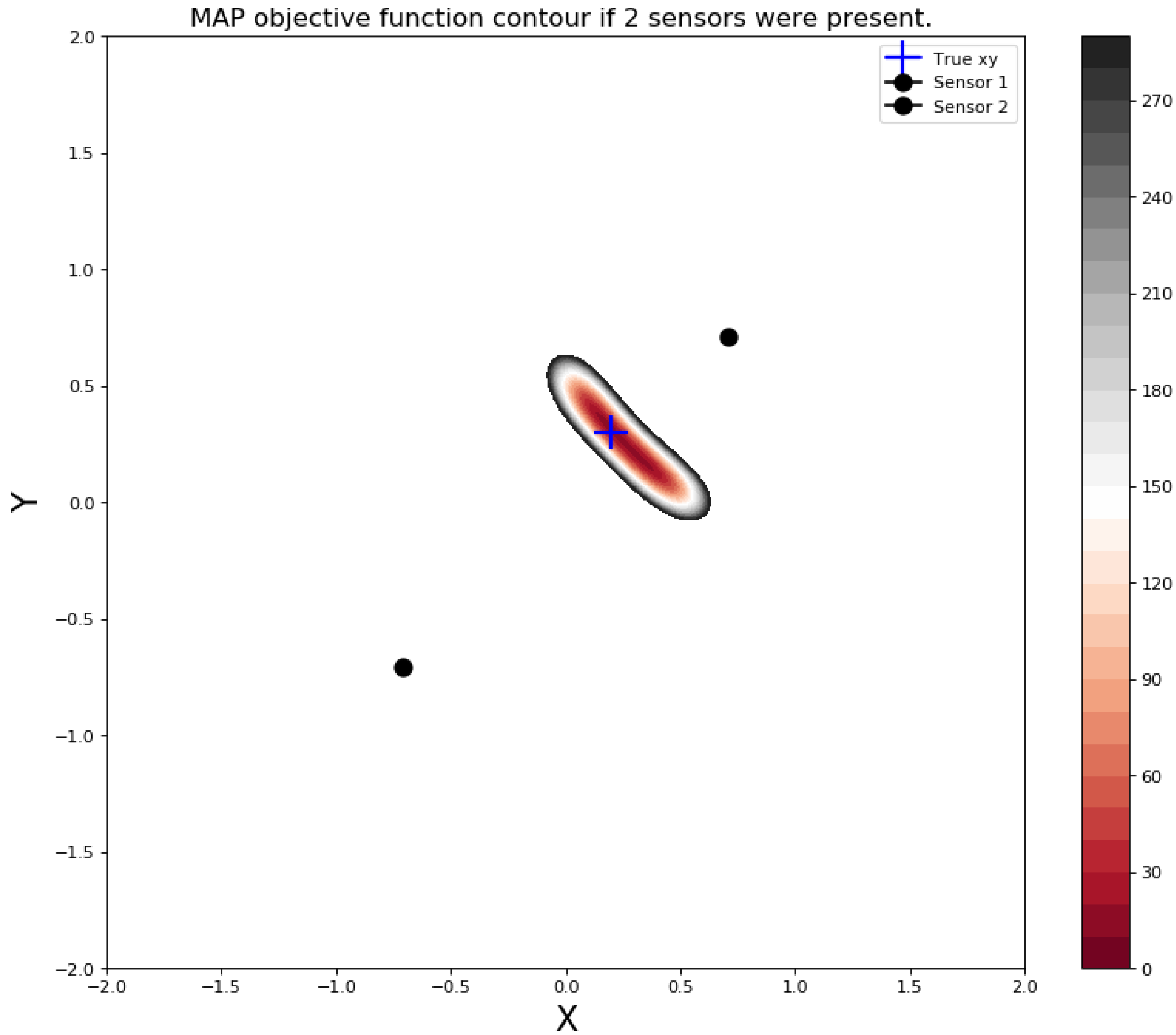
In [8]:
```
fig=_plot_(f0,0)
# MAP estimate matches with prior value.
plt.title('MAP objective function contour if no sensor were present. ',fontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective is centered at origin 0,0 -> which is the prior value',fontsize=15);

fig=_plot_(f1,1)
plt.title('MAP objective function contour if 1 sensor was present. ',fontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center moves closer to the the true value',fontsize=15);

fig=_plot_(f2,2)
plt.title('MAP objective function contour if 2 sensors were present. ',fontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center moves further closer to the the true value',fontsize=15);

fig=_plot_(f3,3)
plt.title('MAP objective function contour if 3 sensors were present. ',fontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center moves very close to the true value',fontsize=15);

fig=_plot_(f4,4)
plt.title('MAP objective function contour if 4 sensors were present. ',fontsize=15)
plt.xlabel('X',fontsize=20)
plt.ylabel('Y',fontsize=20);
fig.text(.15,0.025,'The objective center almost coincidences to the true value',fontsize=15);
```



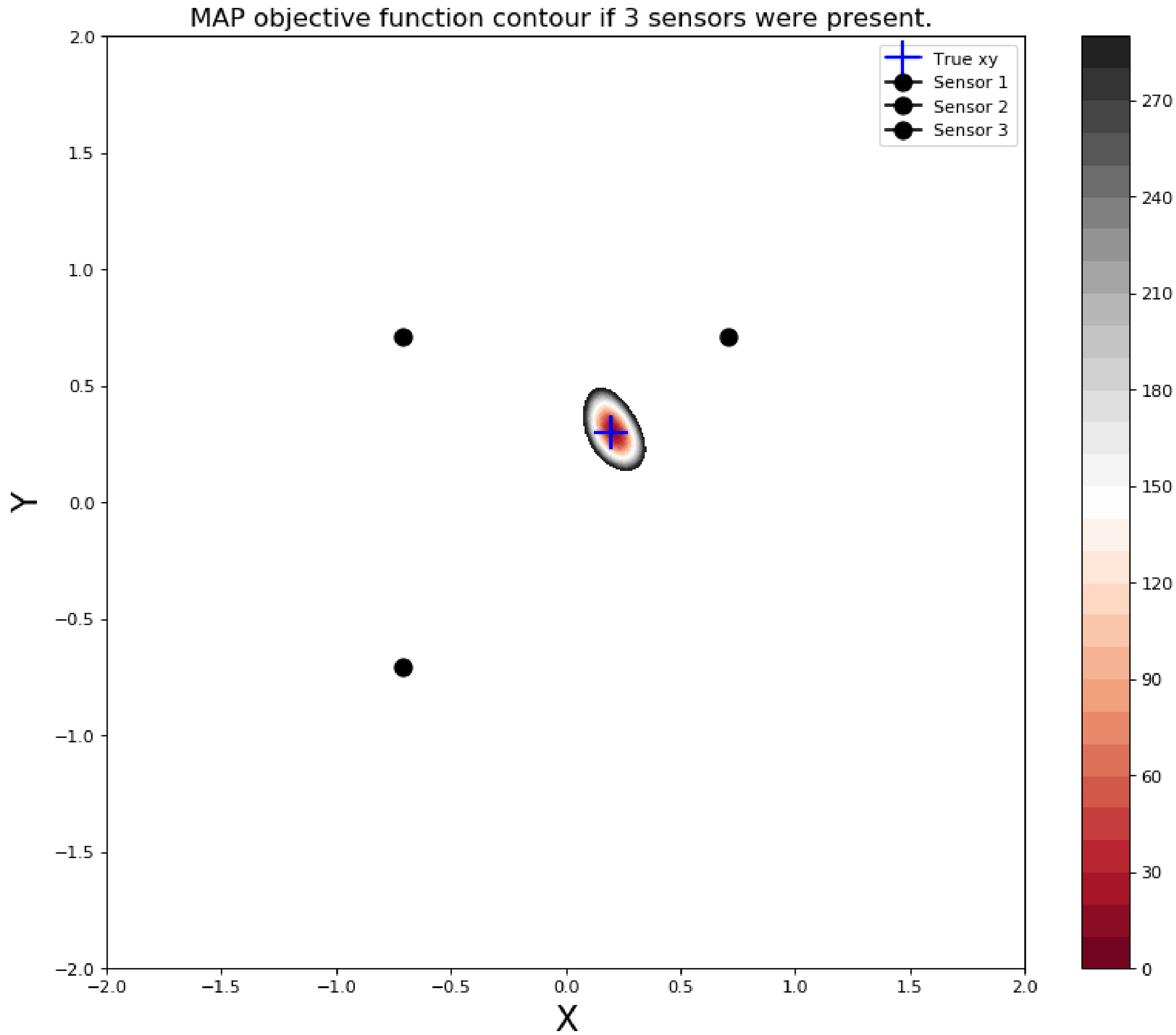
The objective is centered at origin 0,0 -> which is the prior value



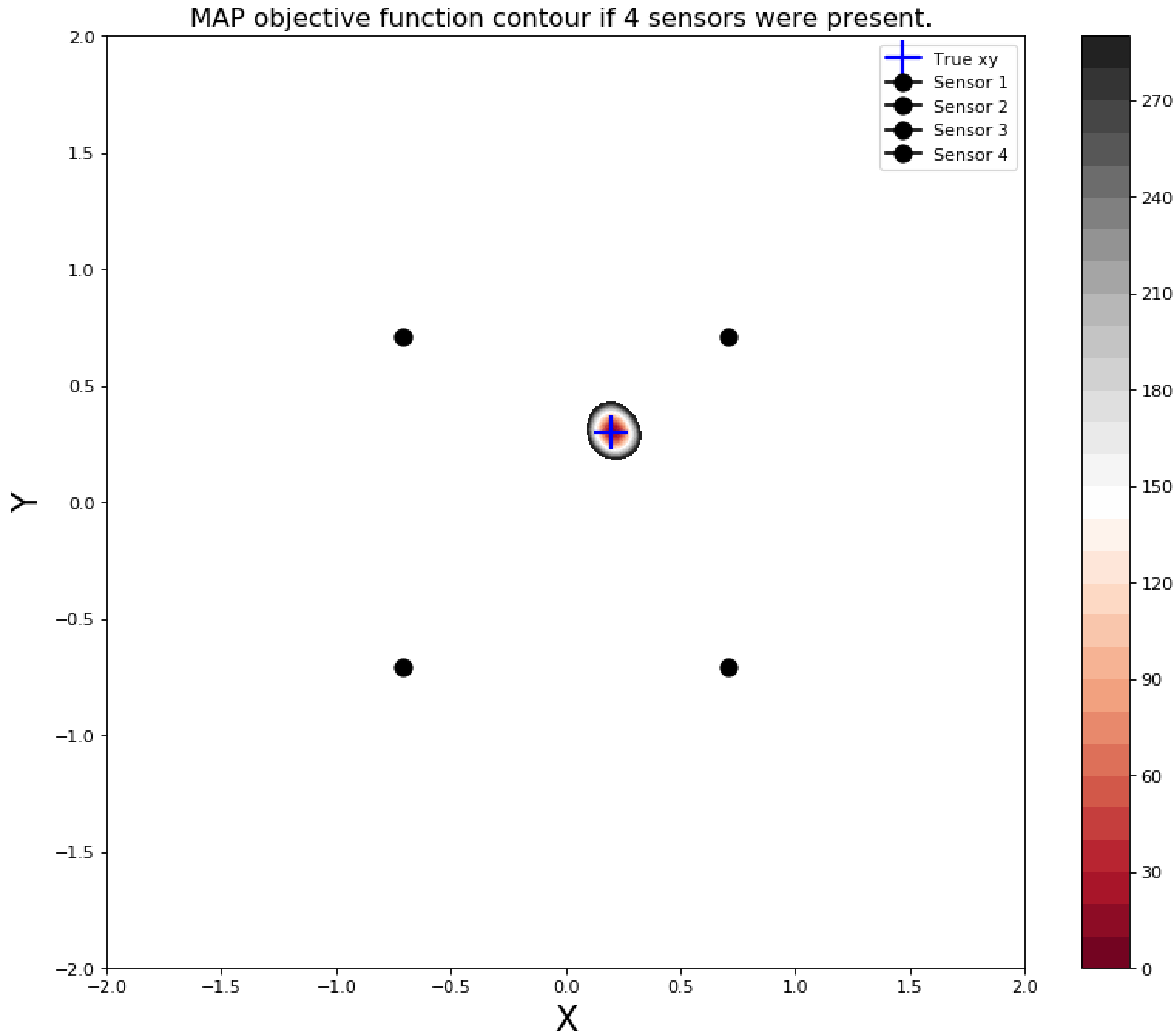
The objective center moves closer to the the true value



The objective center moves further closer to the the true value



The objective center moves very close to the true value

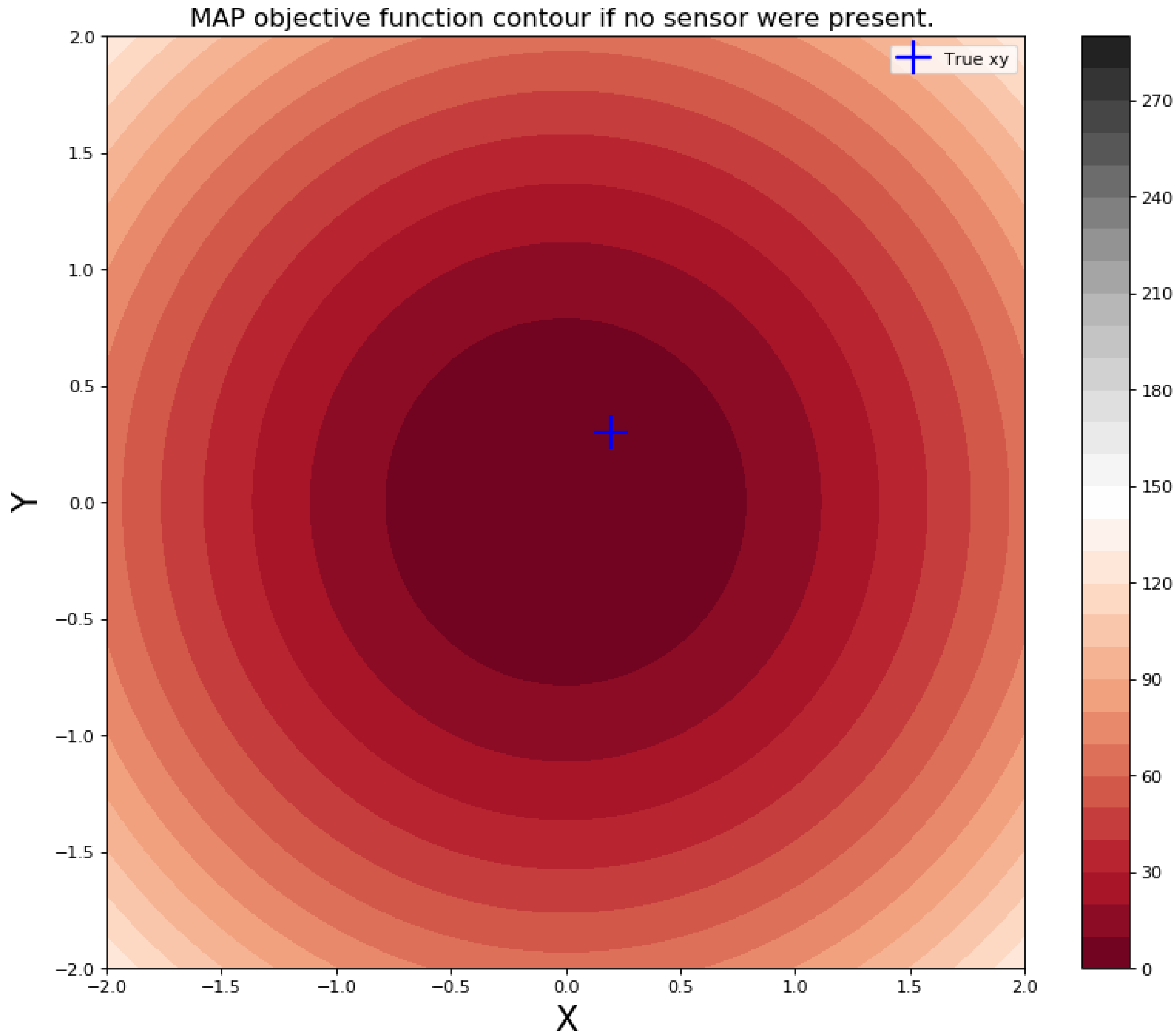


The objective center almost coincidences to the true value

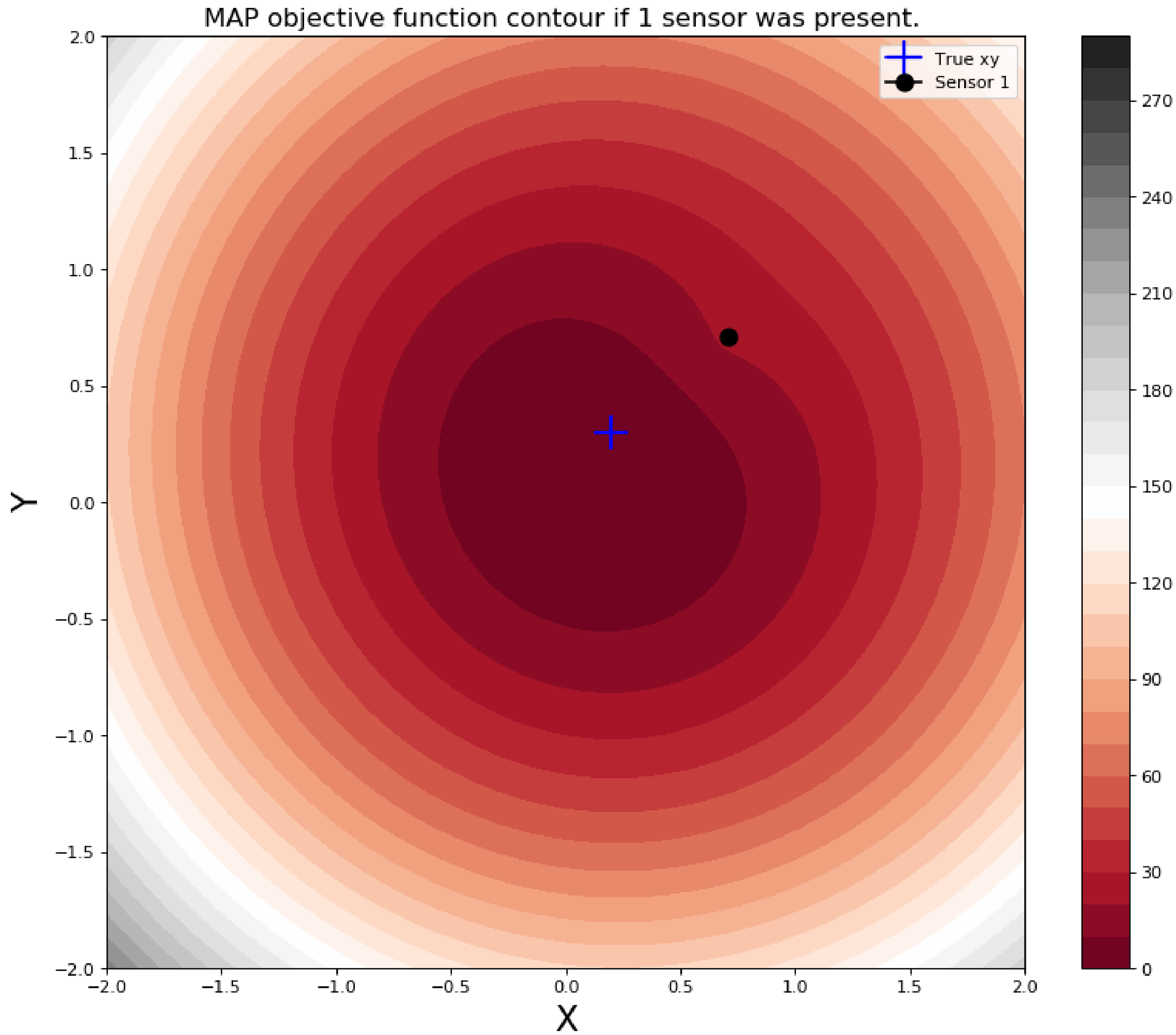
In [ ]:

Choosing-

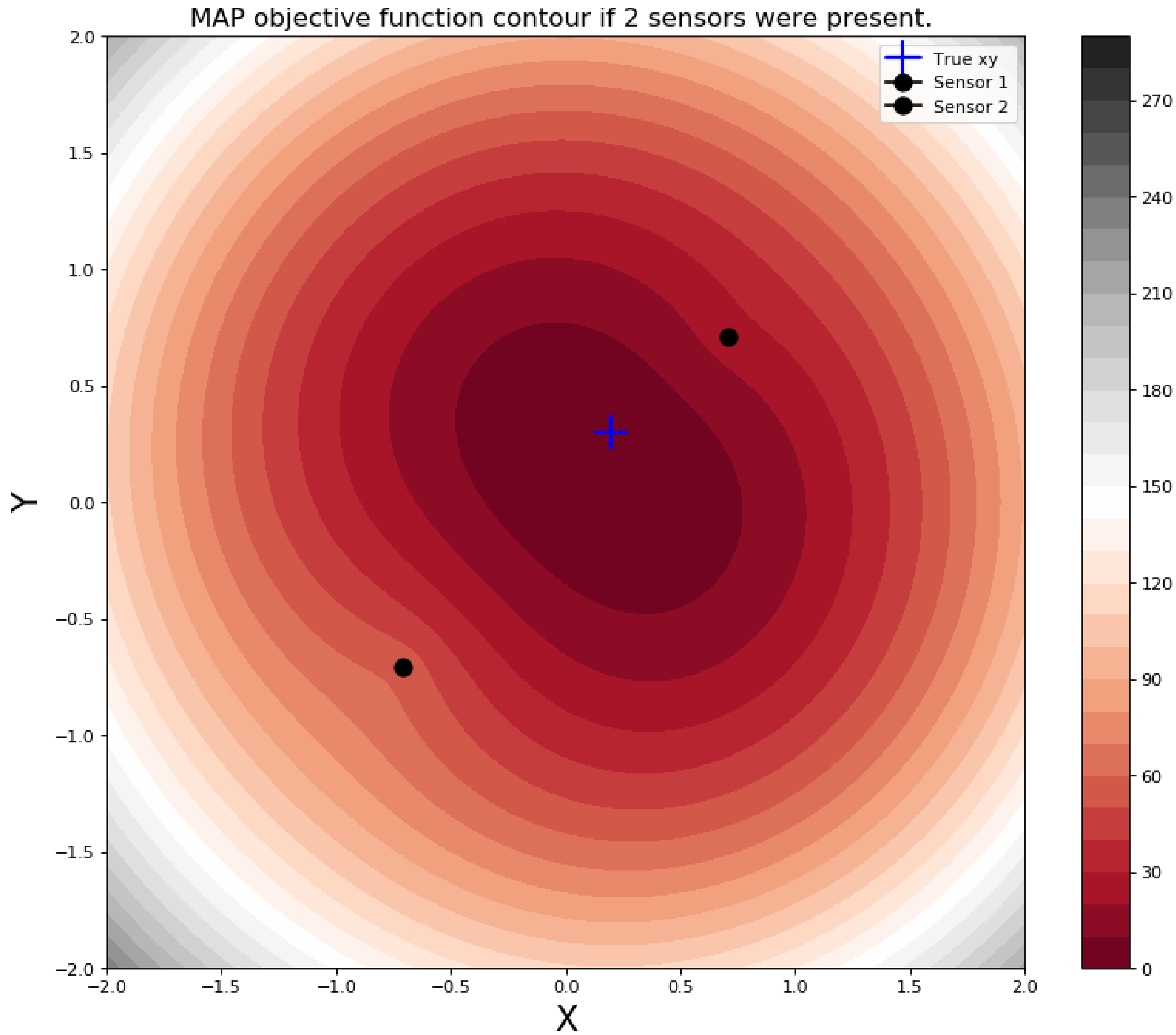
sigma_x=0.25
sigma_y=0.25
sigma_i=0.3



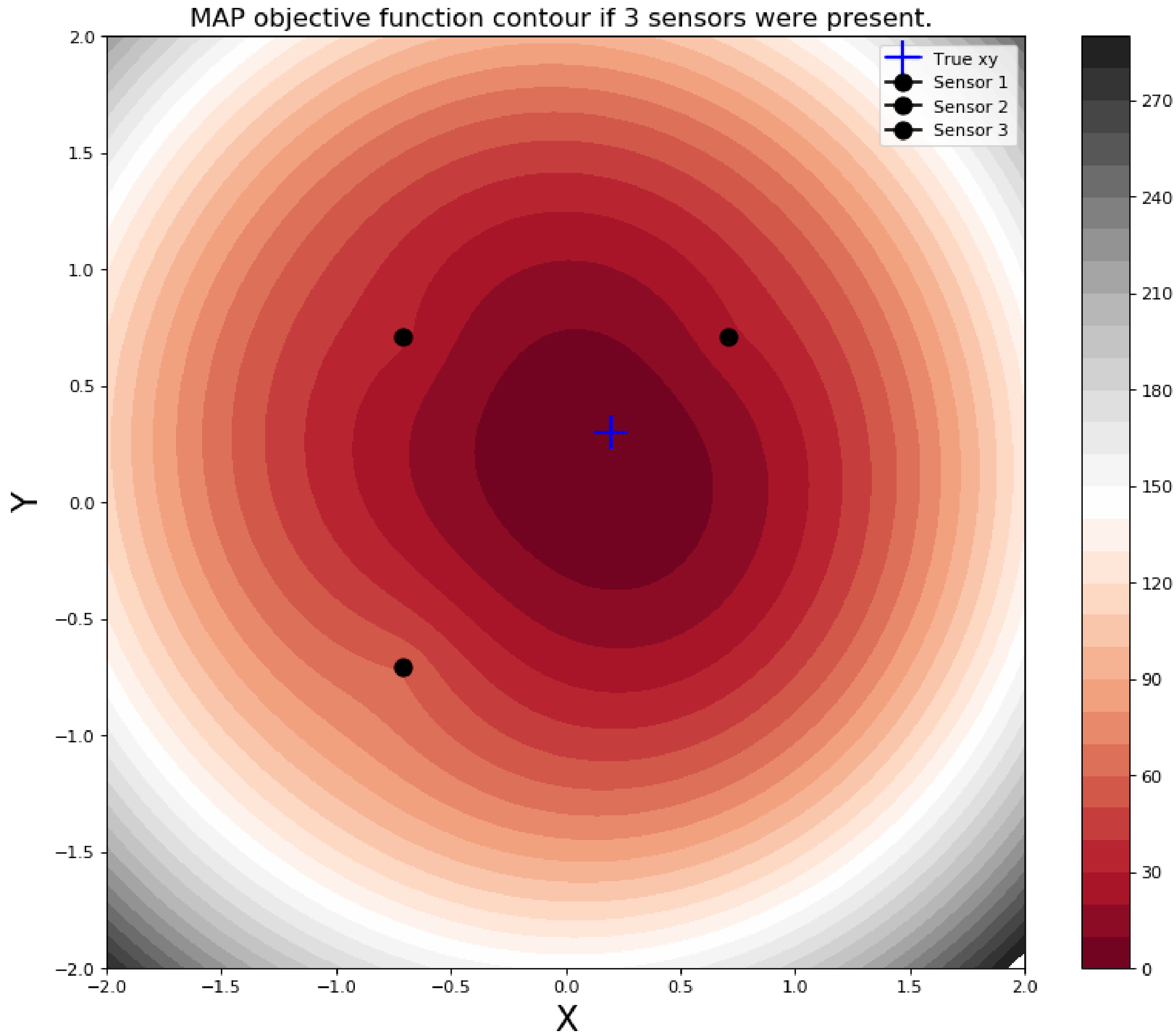
The objective is centered at origin 0,0 -> which is the prior value



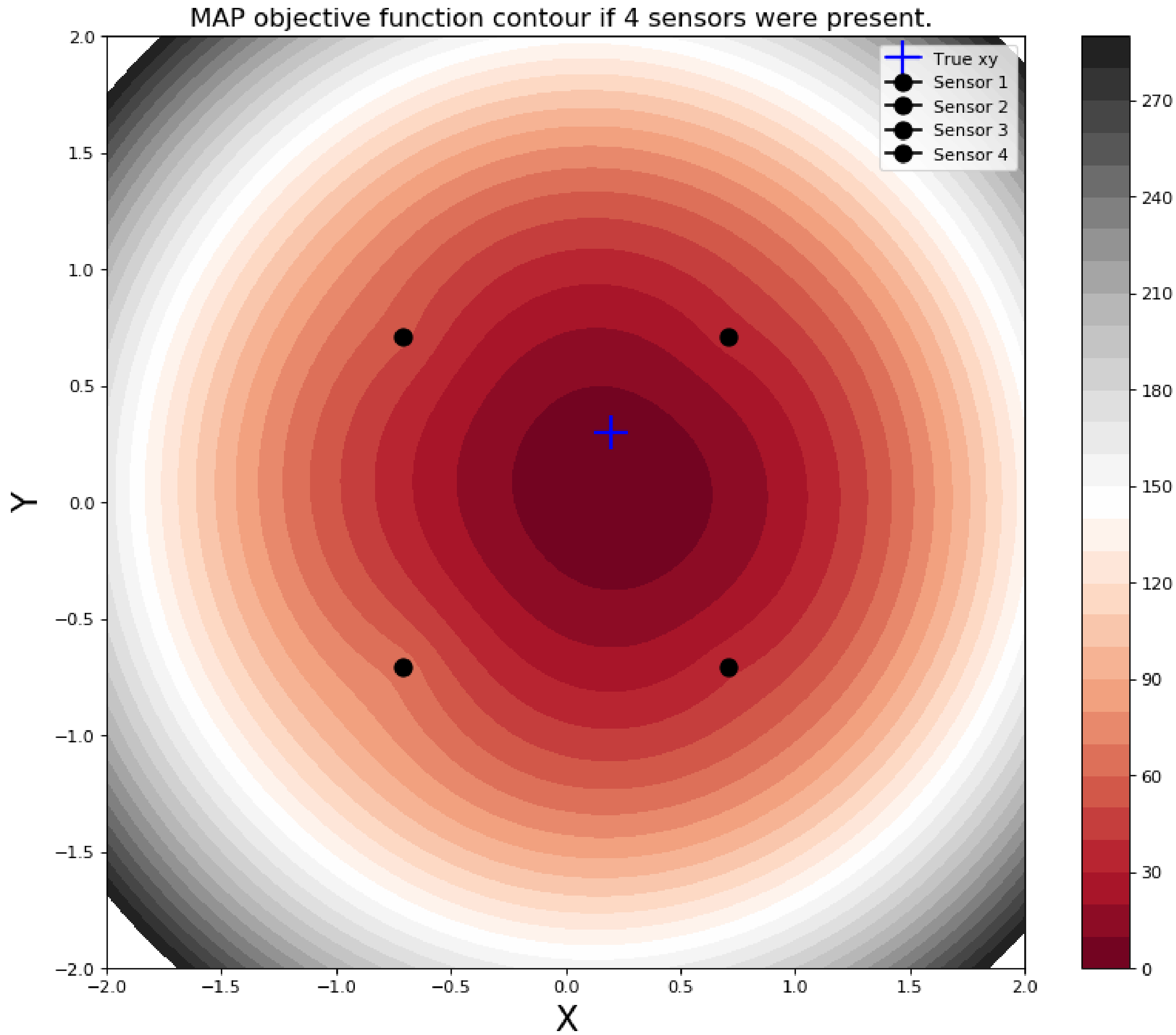
The objective center moves closer to the the true value



The objective center moves further closer to the the true value



The objective center moves very close to the true value



The objective center almost coincidences to the true value

```
In [ ]:
```

- Behavior of the MAP estimate based on centre of contour-

> The MAP estimate moves closer to the true position of object as we get data from more sensors. In the contour plots, black regions are peaks and red regions are the valleys. valleys represent minima. In 1st plot, we don't even see black contours. The objective is extremely uncertain about the true location of object. As number of sensors increase, we see red region getting concentrated in a region, and black circles coming in the view. This represents deeper valleys, and more confidence about the true location. Further, the red circles are very close to true location in last image (4 sensors). MAP objective is also exactly correct about the true location in this case.

Github- https://github.com/rohinarora/EECE5644-Machine_Learning/Exam1/Question-2.ipynb