Question 1

1-) $Var(x) = E[(x-\mu)^2] = E[x^2 - 2\mu x + \mu^2]$

$E[x]=\mu$

$= E[x^2] - 2\mu E[x] + \mu^2$

$= E[x^2] - 2\mu^2 + \mu^2$

$= E[x^2] - \mu^2.$

2-) $\Sigma_x = E[(x-\mu)(x-\mu)^T] = E[xx^T - x\mu^T - \mu x^T + \mu\mu^T]$

$E[x]=\mu$

$= E[xx^T] - \underbrace{E[x]}_{\mu}\mu^T - \mu \underbrace{E[x]^T}_{\mu^T} + \mu\mu^T$

$= E[xx^T] - \mu\mu^T - \mu\mu^T + \mu\mu^T$

$= E[xx^T] - \mu\mu^T.$

Question 2

1-) $p(x|L=\ell) = K.e^{-|x-a_\ell|/b_\ell}$ for $\ell \in 1,2$ and $b_\ell > 0$.

$$\int_{-\infty}^{\infty} p(x|L=\ell)\,dx = 1$$

$$\int_{-\infty}^{\infty} K.e^{\frac{-|x-a_\ell|}{b_\ell}}\,dx = 1$$

$$\int_{-\infty}^{a_\ell} K.e^{(x-a_\ell)/b_\ell}\,dx + \int_{a_\ell}^{\infty} K.e^{(a_\ell-x)/b_\ell}\,dx = 1.$$

$$K.b_\ell.e^{(x-a_\ell)/b_\ell}\Big|_{-\infty}^{a_\ell} + \left(-K.b_\ell.e^{(a_\ell-x)/b_\ell}\Big|_{a_\ell}^{\infty}\right) = 1.$$

$$\underbrace{\left[(K.b_\ell.\underset{1}{e^{0}}) - (K.b_\ell.\underset{0}{e^{-\infty}})\right] + \left[(-K.b_\ell.\underset{0}{e^{-\infty}}) - (-K.b_\ell.\underset{1}{e^{0}})\right]}_{2.K.b_\ell = 1} = 1.$$

$$\boxed{K = 1/2b_\ell}$$ Hence, $p(x|L=\ell) = \frac{1}{2b_\ell}.e^{-|x-a_\ell|/b_\ell}$.

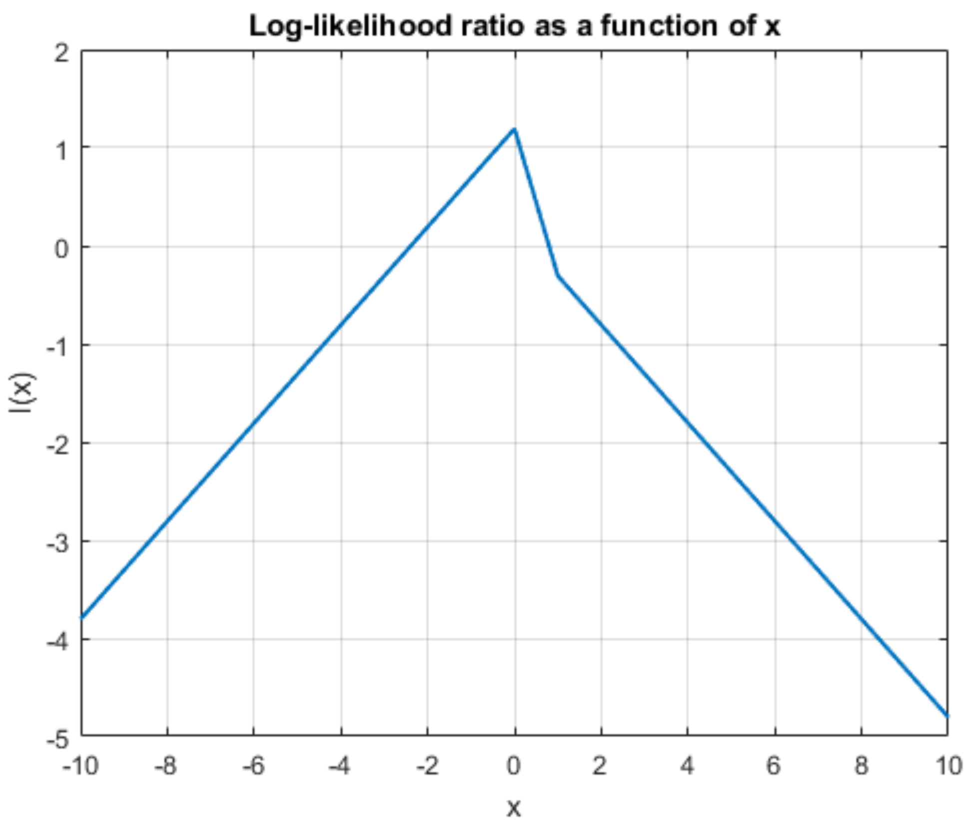
2-) $\ell(x) = \ln\left(\frac{p(x|L=1)}{p(x|L=2)}\right) = \ln p(x|L=1) - \ln p(x|L=2)$

$$\frac{p(x|L=1)}{p(x|L=2)} = \frac{(1/2b_1).e^{\frac{-|x-a_1|}{b_1}}}{(1/2b_2).e^{\frac{-|x-a_2|}{b_2}}} = \boxed{\frac{b_2}{b_1}.e^{\frac{|x-a_2|}{b_2} - \frac{|x-a_1|}{b_1}}}$$

$\Rightarrow$ Log-likelihood ratio: $\boxed{\ell(x) = \ln\frac{b_2}{b_1} + \frac{|x-a_2|}{b_2} - \frac{|x-a_1|}{b_1}}$.

3-) MATLAB code and plot on the next page.

```matlab
%% EECE5644 - Homework 1 - Question 2 - Part 3
clear all; close all; clc;
x = linspace(-10,10,1000);
a1 = 0; b1 = 1;
a2 = 1; b2 = 2;
f = log(b2/b1) + abs(x-a2)/b2 - abs(x-a1)/b1;
figure;
plot(x,f,'LineWidth',1.5); grid on;
title('Log-likelihood ratio as a function of x','FontSize',16);
xlabel('x','FontSize',14);
ylabel('l(x)','FontSize',14);
```



Question 3

$$p(x|L=1) = \frac{1}{b-a} \quad , \quad p(x|L=2) = \frac{1}{t-r}.$$

* Minimum probability of error classification rule:

$$p(L=1|x) \underset{L=2}{\overset{L=1}{\gtrless}} p(L=2|x)$$

Since class priors are equal; it's simplified into: $\dfrac{p(x|L=1)}{p(x|L=2)} \underset{L=2}{\overset{L=1}{\gtrless}} 1.$

Classification rule:

$$\boxed{\begin{array}{l} \dfrac{t-r}{b-a} \underset{L=2}{\overset{L=1}{\gtrless}} 1, \quad \text{for } r < x < b. \\ \text{decide } L=1, \quad \text{for } x \leq r. \\ \text{decide } L=2, \quad \text{for } x \geq b. \end{array}}$$

Question 4

1-) Since the priors are equal;

$$\underbrace{p(x|w_1)}_{N(0,1)} \underset{w_2}{\overset{w_1}{\gtrless}} \underbrace{p(x|w_2)}_{N(\mu,\sigma^2)} \quad \text{for classes } w_1 \text{ and } w_2.$$

$$\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \underset{w_2}{\overset{w_1}{\gtrless}} \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$\frac{-x^2}{2} \underset{w_2}{\overset{w_1}{\gtrless}} -\ln\sigma - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\frac{-x^2}{2} \underset{w_2}{\overset{w_1}{\gtrless}} -\ln\sigma - \frac{(x^2-2x\mu+\mu^2)}{2\sigma^2}$$

$$2\sigma^2\ln\sigma - x^2\sigma^2 + x^2 - 2x\mu + \mu^2 \underset{w_2}{\overset{w_1}{\gtrless}} 0$$

$$\boxed{x^2(1-\sigma^2) - 2\mu x + \mu^2 + 2\sigma^2\ln\sigma \underset{w_2}{\overset{w_1}{\gtrless}} 0}$$

$\cancel{x} \sim N(0,1)$ for w_1,
$x \sim N(\mu,\sigma^2)$ for w_2.

↓

with roots at the bandory of decision where;

$$x_0 = \frac{-\mu \pm \sqrt{\mu^2 + (\sigma^2-1)(\mu^2+2\sigma^2\ln\sigma)}}{\sigma^2-1}.$$

2-) MATLAB code and plots are on the next page.

3-) For minimum probability of error, we basically estimate: $\int_{-\infty}^{\infty} \min[p(w_1|x), p(w_2|x)]\,.\,p(x)\,dx$

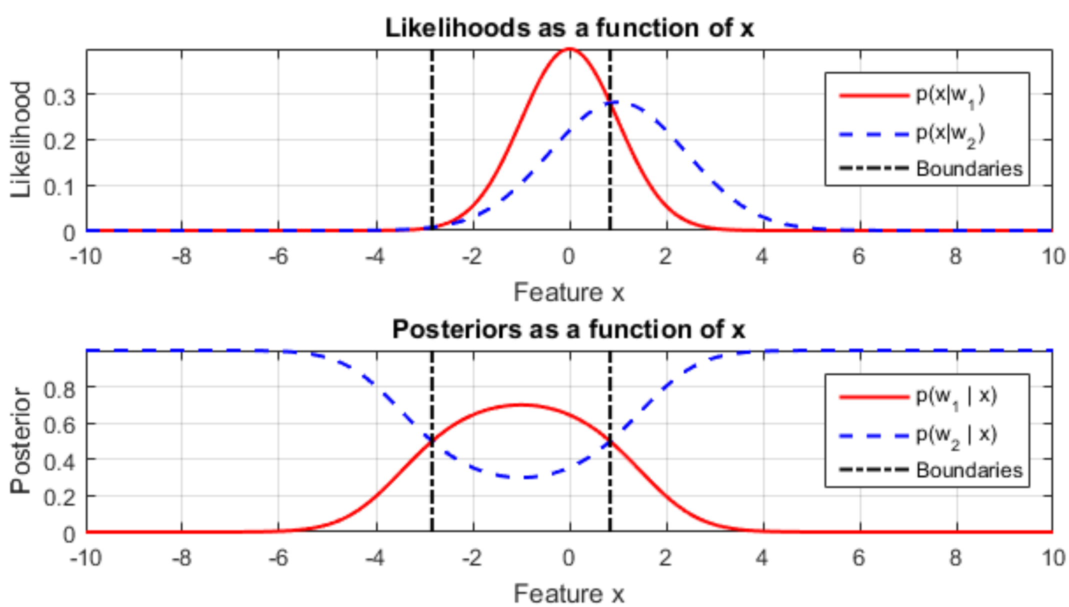
This can be calculated with MATLAB, yielding $\boxed{P_e \approx 0{,}32718.}$

4-) This means same means but different variances. Case: Classifying objects according to their size with respect to a conveyor belt. Boundaries would shift to wherever the likelihoods intersect.

```
%% EECE5644 - Homework 1 - Question 4 - Part 2
clear all; close all; clc;
mu1 = 0; mu2 = 1; sigma1 = 1; sigma2 = sqrt(2);
x = linspace(-10,10,1000);
xopt1 = (mu2 + sigma2 * sqrt(mu2^2 + 2*(1-sigma2^2)*log(1/sigma2)))/(1-
sigma2^2);
xopt2 = (mu2 - sigma2 * sqrt(mu2^2 + 2*(1-sigma2^2)*log(1/sigma2)))/(1-
sigma2^2);
pxw1 = normpdf(x,mu1,sigma1); pxw2 = normpdf(x,mu2,sigma2);
pw1 = 0.5; pw2 = 0.5;
px = pw1*pxw1 + pw2*pxw2;
pw1x = pw1*pxw1./px;
pw2x = pw2*pxw2./px;
perrorx = min([pw1x; pw2x]).*px;
disp(['P_e = ' num2str(trapz(x,perrorx))]);

figure; subplot(2,1,1);
plot(x,pxw1,'r-','linewidth',1.5); hold on
plot(x,pxw2,'b--','linewidth',1.5);
plot([xopt1 xopt1],[-1000 1000],'k-.','linewidth',1.5)
plot([xopt2 xopt2],[-1000 1000],'k-.','linewidth',1.5)
axis([min(x), max(x), 0, max([pxw1 pxw2])]);
grid on; legend('p(x|w_1)','p(x|w_2)','Boundaries');
title('Likelihoods as a function of x','FontSize', 16);
xlabel('Feature x','FontSize', 13); ylabel('Likelihood','FontSize', 13);

subplot(2,1,2);
plot(x,pw1x,'r-','linewidth',1.5); hold on
plot(x,pw2x,'b--','linewidth',1.5);
plot([xopt1 xopt1],[-1000 1000],'k-.','linewidth',1.5)
plot([xopt2 xopt2],[-1000 1000],'k-.','linewidth',1.5)
axis([min(x), max(x), 0, max([pw1x pw2x])]);
grid on; legend('p(w_1 | x)','p(w_2 | x)','Boundaries');
title('Posteriors as a function of x','FontSize',16);
xlabel('Feature x','FontSize',13); ylabel('Posterior','FontSize',13);
```



Question 5

1-) For $x = Az + b$ where $z \sim \mathcal{N}(0, I)$, we know that x will follow Normal distribution with specific mean and covariance.

Mean: $E[x] = E[Az+b] = A\underbrace{E[z]}_{0} + b = b.$

Covariance Matrix: $\Sigma_x = E\left[(x - E[x])(x - E[x])^T\right]$

$$= E\left[(Az+b-b)(Az+b-b)^T\right]$$

$$= E\left[Azz^TA^T\right]$$

$$= A\underbrace{E\left[zz^T\right]}_{I}A^T$$

$$= AA^T.$$

* Here $E[zz^T]$ is in fact the covariance matrix of z, due to z being zero-mean.
$\Sigma_z = E[(z-0)(z-0)^T] = E[zz^T] = I.$

$$\therefore \boxed{x \sim N(b, AA^T)}$$

2-) In order to achieve $\mathcal{N}(\mu, \Sigma)$ for a random vector drawn by the linear transformation technique above;

$$\mu = b \quad \text{and} \quad \Sigma = AA^T$$

should be satisfied. Hence; $\boxed{b = \mu}$ and $\boxed{A = \Sigma^{1/2}}$.

Note: Since AA^T is symmetric and positive semi-definite, the square root of Σ exists.

3-) MATLAB code and explanations are on the next page.

```matlab
%% EECE5644 - Homework 1 - Question 5 - Part 3
%
function x = sample_generator(N,n,mu,sigma)
    % Assumes that mu is a column vector and sigma is a square matrix.
    % Output x will be of dimensions n by N.
    z = normrnd(0,1,n,N);
    x = sigma^(0.5) * z + repmat(mu,[1,N]);
end
```