Answer 1

Entire working program-

```
#include "iostream"
#include <chrono>

using namespace std;
using namespace std::chrono;

void print_array(int *A,int size_of_array){
  /*
  Helper function
  Takes in array A of size size_of_array and prints the contents
  */
  for (int i=0;i<size_of_array;i++){
    cout<<A[i]<<" ";
  }
  cout<<endl;
}

void insertion_sort(int *A,int size_of_array){
  /*
  Takes in array A of size size_of_array and sorts via insertion_sort
  */
  int i,key;
  for (int j=1;j<size_of_array;j++){
    i=j-1;
    key=A[j];
    while (key<A[i] & i>-1) {
      A[i+1]=A[i];
      i=i-1;
    }
    A[i+1]=key;
  }
}

void merge(int *A,int l,int mid, int r){
  /*
  A[l:mid] is sorted,
  A[mid+1:r] is sorted
  */
  int n1=mid-l+1;
  int n2=r-mid;
  int left[n1];
  int right[n2];

  for (int i=0;i<n1;i++){
    left[i]=A[l+i];
  }
```

```
  for (int j=0;j<n2;j++){
    right[j]=A[mid+j+1];
  }
  int i=0;
  int j=0;
  int k=l;
  while (i<n1 & j<n2){
    if (left[i]<right[j]){
      A[k]=left[i];
      i=i+1;
    }
    else{
      A[k]=right[j];
      j=j+1;
    }
    k=k+1;
  }

  while (i<n1){
    A[k]=left[i];
    k=k+1;
    i=i+1;
  }

  while (j<n2){
    A[k]=right[j];
    k=k+1;
    j=j+1;
  }
}

void merge_sort(int *A,int l,int r){
  /*
  Input: A[l...r].
  Initial call: l=0, r=len(A)-1.
  If len(A)==1, "if" condition at start would be false, and merge_sort would return A directly
  */
  int mid;
  if (l<r){
    mid=(int)(l+r-1)/2;
    merge_sort(A,l,mid);
    merge_sort(A,mid+1,r);
    merge(A,l,mid,r);
  }
}
```

```
auto time_insertion_sort(int size_of_array){
  /*
  create array of size size_of_array, sorted in descending order - which is the worst case for insertion_s
  call insertion_sort() and wrap the timing functions around it. return the time taken by insertion sort
  */
  int A[size_of_array];
  for (int j=size_of_array;j>0;j--){
    A[size_of_array-j] = j;
  }
  //print_array(A,size_of_array);
  //cout<<"Insertion Sort ";
  auto start = high_resolution_clock::now();
  insertion_sort(A,size_of_array);
  auto stop = high_resolution_clock::now();
  auto duration = duration_cast<nanoseconds>(stop - start);
  //cout << "time "<<duration.count() << " ns"<<endl;
  //print_array(A,size_of_array);
  return duration;
}

auto time_merge_sort(int size_of_array){
  /*
  create array of size size_of_array, sorted in descending order.
  call merge_sort() and wrap the timing functions around it. return the time taken by merge sort
  */

  int B[size_of_array];
  for (int j=size_of_array;j>0;j--){
    B[size_of_array-j] = j;
  }
  //print_array(B,size_of_array);
  //cout<<"Merge Sort ";
  auto start = high_resolution_clock::now();
  merge_sort(B,0,size_of_array-1);
  auto stop = high_resolution_clock::now();
  auto duration = duration_cast<nanoseconds>(stop - start);
  //cout << "time "<<duration.count() << " ns"<<endl;
  //print_array(B,size_of_array);
  return duration;
}
int main() {
  cout<<"size_of_array"<<"\t"<<"insert sort time"<<"\t"<<"merge sort time"<<endl;

  for (int size_of_array=2;size_of_array<50;size_of_array=size_of_array+1){
    auto insert_time=time_insertion_sort(size_of_array); // time taken by insertion sort
```

```
    auto merge_time=time_merge_sort(size_of_array); // time taken by merge sort
    cout<<size_of_array<<"\t"<<insert_time.count()<<"\t"<<merge_time.count()<<endl;
  }
  return 0;
}
```

Insertion sort code-

```
void insertion_sort(int *A,int size_of_array){
  /*
  Takes in array A of size size_of_array and sorts via insertion_sort
  */
  int i,key;
  for (int j=1;j<size_of_array;j++){
    i=j-1;
    key=A[j];
    while (key<A[i] & i>-1) {
      A[i+1]=A[i];
      i=i-1;
    }
    A[i+1]=key;
  }
}
```

Merge sort code-

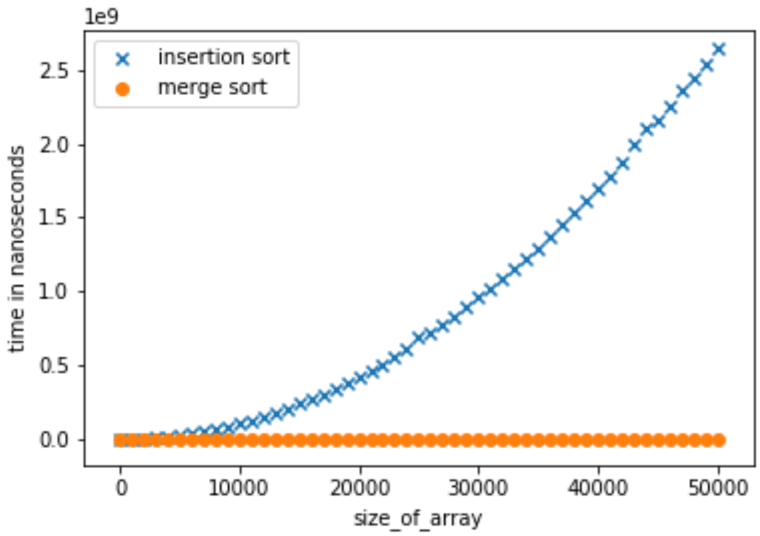
```
void merge(int *A,int l,int mid, int r){
  /*
  A[l:mid] is sorted,
  A[mid+1:r] is sorted
  */
  int n1=mid-l+1;
  int n2=r-mid;
  int left[n1];
  int right[n2];

  for (int i=0;i<n1;i++){
    left[i]=A[l+i];
  }

  for (int j=0;j<n2;j++){
    right[j]=A[mid+j+1];
  }
  int i=0;
  int j=0;
  int k=l;
  while (i<n1 & j<n2){
    if (left[i]<right[j]){
      A[k]=left[i];
      i=i+1;
    }
    else{
      A[k]=right[j];
      j=j+1;
    }
    k=k+1;
  }

  while (i<n1){
    A[k]=left[i];
    k=k+1;
    i=i+1;
  }

  while (j<n2){
    A[k]=right[j];
    k=k+1;
    j=j+1;
  }
}

void merge_sort(int *A,int l,int r){
```

```
  /*
  Input: A[l...r].
  Initial call: l=0, r=len(A)-1.
  If len(A)==1, "if" condition at start would be false, and merge_sort would return A directly
  */
  int mid;
  if (l<r){
    mid=(int)(l+r-1)/2;
    merge_sort(A,l,mid);
    merge_sort(A,mid+1,r);
    merge(A,l,mid,r);
  }
}
```

- Input has a been set a sequence in decreasing order, which is the worst case for insertion sort
- Input size n for which merge sort starts to beat insertion sort in terms of the worst-case running time-> n=31

Size_of_Array	insert_sort_time (ns)	merge_sort_time(ns)
2	145	167
3	109	234
4	117	233
5	123	275
6	169	305
7	176	364
8	188	390
9	203	502
10	256	407
11	287	487
12	314	534
13	329	573
14	384	570
15	416	668
16	419	634
17	487	716
18	524	713
19	587	857
20	628	829
21	689	898
22	734	880
23	797	1023
24	886	999
25	927	1050
26	967	1072
27	1030	1145
28	1105	1124
29	1204	1251
30	1269	1270
31	**1348**	**1312**
32	1404	1372
33	1535	1377
34	1609	1406
35	1683	1507
36	1786	1513
37	1931	1584
38	1965	1710
39	2026	1777
40	2139	1767

Asymptotically, merge sort beats insertion sort-



Answer 2

$$A = [10, 5, 7, 9, 8, 3]$$

<u>Arrangement of array for iterations of insertion sort</u>

$$A = [10, 5, 7, 9, 8, 3]$$
$$A = [5, 10, 7, 9, 8, 3]$$
$$A = [5, 7, 10, 9, 8, 3]$$
$$A = [5, 7, 9, 10, 8, 3]$$
$$A = [5, 7, 8, 9, 10, 3]$$
$$A = [3, 5, 7, 8, 9, 10]$$

<u>Arrangement of array for iterations of partition</u>

First element of the array has been chosen as pivot in every iteration here. The algorithm can be easily modified to randomized quicksort by choosing the partition element randomly in each iteration

$$[10, 5, 7, 9, 8, 3] \quad i = 0 \quad j = 1$$
$$[10, 5, 7, 9, 8, 3] \quad i = 1 \quad j = 2$$
$$[10, 5, 7, 9, 8, 3] \quad i = 2 \quad j = 3$$
$$[10, 5, 7, 9, 8, 3] \quad i = 3 \quad j = 4$$
$$[10, 5, 7, 9, 8, 3] \quad i = 4 \quad j = 5$$
$$[3, 5, 7, 9, 8, 10] \quad i = 0 \quad j = 1$$
$$[3, 5, 7, 9, 8, 10] \quad i = 0 \quad j = 2$$
$$[3, 5, 7, 9, 8, 10] \quad i = 0 \quad j = 3$$

$[3, 5, 7, 9, 8, 10] \quad i = 0 \quad j = 4$
$[3, 5, 7, 9, 8, 10] \quad i = 1 \quad j = 2$
$[3, 5, 7, 9, 8, 10] \quad i = 1 \quad j = 3$
$[3, 5, 7, 9, 8, 10] \quad i = 1 \quad j = 4$
$[3, 5, 7, 9, 8, 10] \quad i = 2 \quad j = 3$
$[3, 5, 7, 9, 8, 10] \quad i = 2 \quad j = 4$
$[3, 5, 7, 9, 8, 10] \quad i = 3 \quad j = 4$

Answer 3

$n + 3 \in \Omega(n)$ **True**
$n + 3 \in O(n^2)$ **True**
$n + 3 \in \theta(n^2)$ **False**
$2^{n+1} \in O(n + 1)$ **False**
$2^{n+1} \in \theta(2^n)$ **True**

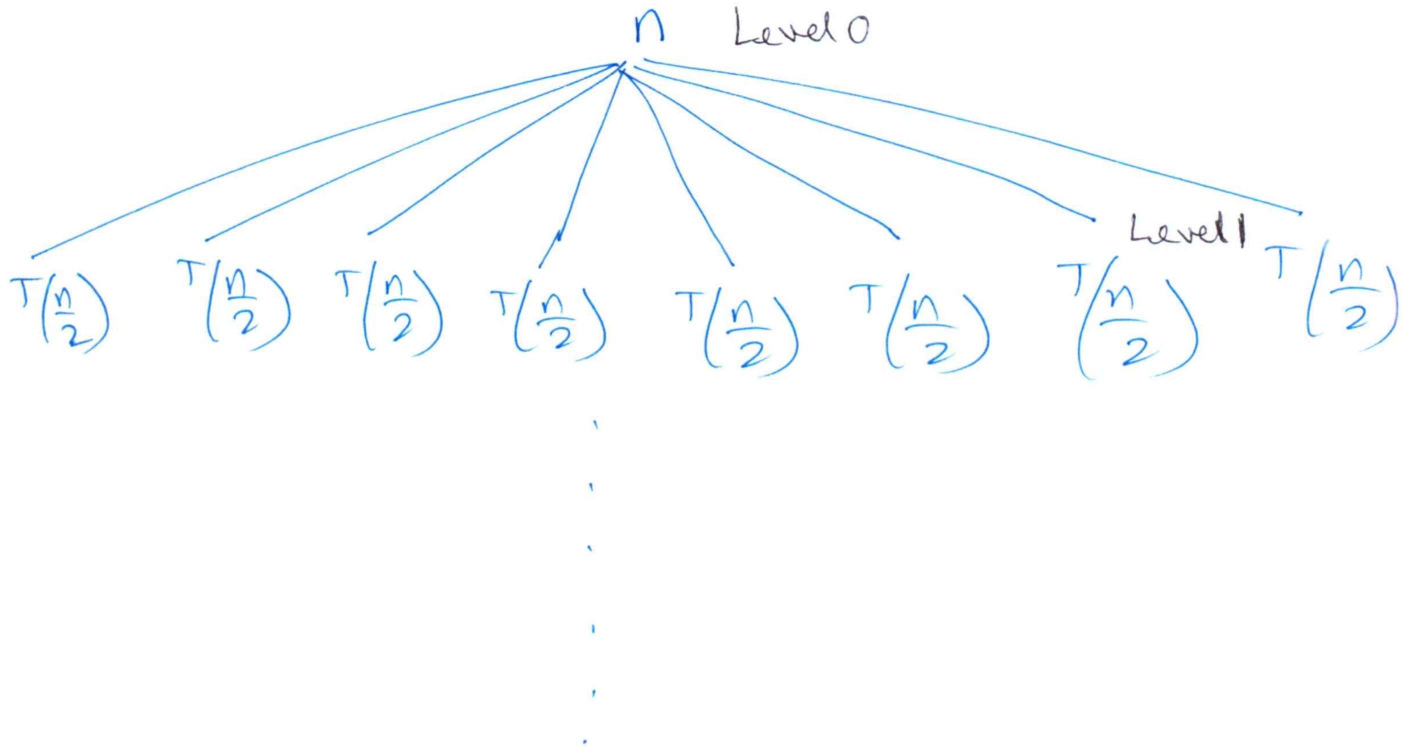
Answer 4

$T(n) = 8 * T(\frac{n}{2}) + n = \theta(n^3)$
$T(n) = 8 * T(\frac{n}{2}) + n^2 = \theta(n^3)$
$T(n) = 8 * T(\frac{n}{2}) + n^3 = \theta(n^3 * \log n)$
$T(n) = 8 * T(\frac{n}{2}) + n^4 = \theta(n^4)$

Answer 5

<u>Recursion tree</u>

Q5 $T(n) = 8T\left(\frac{n}{2}\right) + n$



- Total levels = $\log_2 n + 1$
- At level j -> 8^j subproblems each of size $n/2^j$
- Computation at level j= $c*(8^j)*(n/2^j)$
  = $c*(n)*(4^j)$

- Total computation across all levels= $\sum_{j=0}^{log_2 n} c*(n)*(4^j)$

  = $c*(n)*\sum_{j=0}^{log_2 n} *(4^j)$(This is a geometric progression)

  = $c*(n)*(4^{(log_2 n+1)}-1)/(4-1)$

  = $c1*(n*(4^{(log_2 n+1)}-1))$
  = $c1*(n*(4*4^{(log_2 n)})-n))$
  = $c1*(4*(n*n^{(log_2 4)})-n))$
  = $c1*(4*(n^3)-n))$
  = $O(n^3)$

Substitution method

- Given: $T(n) = 8 * T(\frac{n}{2}) + n$
  Let the guess for $T(n)$ be $T(n) \leq c1 * n^3 - c2 * n$
  - that is $O(n^3)$
- The substitution method requires us to prove that $T(n) \leq c1 * n^3 - c2 * n$ for an appropriate choice of the constant c1 > 0 and c2>0 valid $\forall \ n > n_0$
- We start by assuming that this bound holds for all positive m < n, in particular for m=n/2, yielding $T(\frac{n}{2}) \leq c1 * (\frac{n}{2})^3 - c2 * \frac{n}{2}$.
- Substituting into the recurrence yields
  $T(n) \leq 8 * c1 * (\frac{n}{2})^3 - 8 * c2 * \frac{n}{2} + n$
  $T(n) \leq c1 * (n)^3 - 4 * c2 * n + n$
  $T(n) \leq c1 * (n)^3 - c2 * n - (3 * c2 * n - n)$
  Residue=$(3 * c2 * n - n)$; must be positive
  True for $c2 > 1/3$
- Base case
  T(1)=1
  T(2)=10
  Also, $T(2) <= c1 * 8 - c2 * 2$
  True if c1>>c2. Example. c2=1, c1=10

Therefore, $T(n) = O(n)^3$

QED