# Bayesian Learning and Inference in Recurrent Switching Linear Dynamical Systems

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May 1, 2023

UNIVERSITY *of* WASHINGTON | Applied Mathematics

# Introduction and Motivation

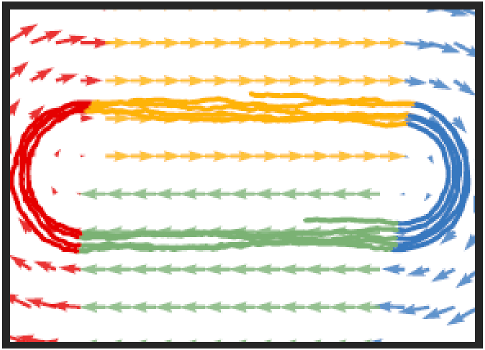
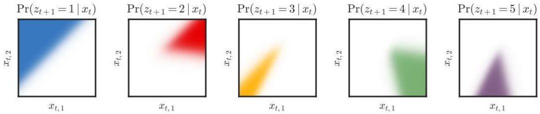


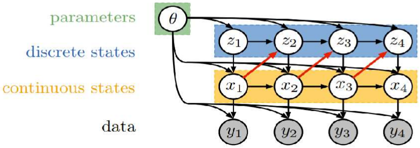
Figure: NASCAR Dynamical System

# Model

## SLDS and rSLDS

- Observation $y_t = Cx_t + d + w_t,\ w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, S)$
- Continous latent state $x_{t+1} = A_{z_{t+1}} x_t + b_{z_{t+1}} + \nu_t,\ \nu_t \overset{\text{iid}}{\sim} \mathcal{N}(0, Q_{z_{t+1}})$
- Discrete latent state $z_t \in \{1, \ldots, K\}$
  - SLDS [1] $z_{t+1} | z_t \sim \pi_{z_t}$
  - rSLDS [2] $z_{t+1} | z_t, x_t \sim \pi_{SB}(\nu_{t+1}),\ \nu_{t+1} = R_{z_t} x_t + r_{z_t}$



# Model

## Stick Breaking Logitstic Regression [3]

- $p(z|x) \sim \pi_{SB}(\nu),\ \nu = Rx + r$
- Link function: $\pi_{SB}(\nu) = \left(\pi_{SB}^{(1)}(\nu), \dots, \pi_{SB}^{(K)}(\nu)\right)$ (with $\sigma(x) = \frac{e^x}{1+e^x}$)

$$\pi_{SB}^{(k)}(\nu) = \begin{cases} \sigma(\nu_k) \prod_{j<k} \sigma(-\nu_j), & \text{if } k = 1, \dots, K-1 \\ \prod_1^K \sigma(-\nu_j), & \text{if} k = K \end{cases}$$

- $p(z|x) \sim \prod_{k=1}^K \sigma(\nu_k)^{\mathbb{1}[z=k]} \sigma(-\nu_k)^{\mathbb{1}[z>k]}$ (Likelihood)

- With a Gaussian Prior $p(x)$, the posterior $p(x|z) \propto p(x) * p(z|x)$ is non-Gaussian
- Bayesian updating is not efficient
  - Gibbs Sampler: sampling $x^{(i+1)} \sim p(x|z = z^{(i)})$, sampling $z^{(i+1)} \sim p(z|x = x^{(i+1)})$
  - Message Passing $m_{t \to (t+1)}(x_{t+1}) = \int \psi(x_t, y_t) \psi(x_t, z_{t+1}) \psi(x_t, x_{t+1}, z_{t+1}) m_{(t-1) \to t}(x_t) \, \mathrm{d}x_t$

# Model

## Polya-gamma augmentation [3]

$$\frac{(e^{\nu})^{a}}{(1+e^{\nu})^{b}} = 2^{-b}\int_0^{\infty} \boldsymbol{e}^{\kappa\nu}\boldsymbol{e}^{-\omega\nu^2/2} p_{\mathrm{PG}}\left(\omega \mid b, 0\right)\mathrm{d}\omega \ (\kappa = a - \frac{b}{2}) \tag{2.1}$$

$$p(x_t|z_{t+1}) \propto \prod_{k=1}^{K-1} \frac{(e^{\nu_{t+1,k}})^{\mathbb{I}[z_{t+1}=k]}}{(1+e^{\nu_{t+1,k}})^{\mathbb{I}[z_{t+1}\geq k]}} \tag{2.2}$$

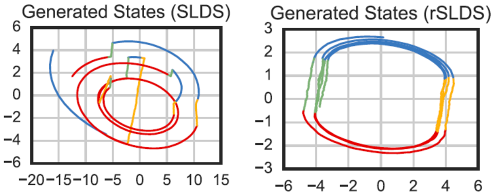
$$p(x_t|z_{t+1}) = \int p(x_t, \omega_t|z_{t+1})\mathrm{d}\omega_t = \int \boldsymbol{p(x_t|z_{t+1}, \omega_t)}\, p(\omega_t)\mathrm{d}\omega_t \tag{2.3}$$

$$\omega_{t,k} \mid x_t, z_{t+1} \sim \mathrm{PG}\left(\mathbb{I}\left[z_{t+1} \geq k\right], \nu_{t+1,k}\right) \tag{2.4}$$

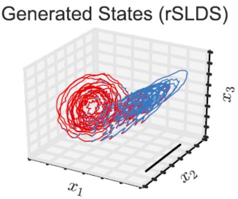
- $p(x_t|z_{t+1}, \omega_t)$ is Gaussian
- Message Passing $\psi(x_t, z_{t+1}, \omega_t) \propto \mathcal{N}(\nu_{t+1}|\Omega_t^{-1}\kappa_{t+1}, \Omega_t^{-1})$
- Thus instantiating these auxiliary variables in a Gibbs sampler enables efficient block updates

# Conclusions

- SLDS vs rSLDS




- Canonical Dynamical System - Lorenz Attractor



# Next Steps

- Van der Pol Oscillator
  - Canonical dynamical system with a stable limit cycle and bifurcations
- Double Pendulum
  - System with chaotic dynamics and multiple states (depending on the relative positions of the weights)

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Dependent multinomial models made easy: Stick-breaking with the polya-gamma augmentation.