

The Math and Physics of Billiards

SHOT

For the purposes of my program, I am equating the motion of drawing the cue back to that of pulling on a spring. This assumes that the distance the cue is drawn away from the cue ball is directly proportional to the desired power of the shot, which is usually the case. Using *Hooke's Law*, which describes the motion of a spring with the equation $F=kx$, I can calculate the force applied to the cue ball using the distance back the cue is drawn.

Given the force applied to the cue ball, there are two quantities I must calculate in this stage of the simulation. The first is the initial velocity of the ball that results from the hit. To calculate this, I use Newton's Second Law, given by

$$\vec{F} = m\vec{a}$$

Using *Hooke's Law*, I use an adaptation of this equation derived below:

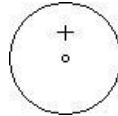
$$\begin{aligned} -k\vec{x} &= m\vec{a} = m \frac{\Delta\vec{v}}{\Delta t} \\ \Delta\vec{v} &= \frac{-k\vec{x}\Delta t}{m} \end{aligned}$$

Since, for the purposes of the shot, we can assume the initial velocity of the cue ball is zero, this equation becomes

$$\vec{v}' = \frac{-k\Delta t}{m} \vec{x}$$

The quantity Δt in this equation is the amount of time the cue is in contact with the ball, a quantity which I assume to be relatively constant and on the order of 10 milliseconds. The quantity m is the mass of the cue ball, usually around .5 kg; and the quantity k is the spring constant from *Hooke's Law*, which I arbitrarily assign a reasonable value (around 10 kg·m/s²). The vector \vec{x} is the difference between the cue ball and the tip of the cue after it is drawn back.

Besides the initial translational velocity of the cue ball, there is also a rotational aspect of the ball's movement that I must take into account. Anybody who has ever played a game of pool knows the importance of applying spin to the cue ball. Therefore, the second quantity I must calculate is the angular velocity of the cue ball. To do this, I need to know not only how far back the cue ball is drawn, but also where the cue ball is hit with respect to its center of mass. For example, assume the circle below is a lateral view of the cue ball.



If the small circle is the center of the cue ball (i.e. the center of mass), and the ball is struck where the cross is, the ball will have an initial angular velocity (top spin) that will affect the ball's movement across the table as well as its collision with any other ball.

To calculate the initial angular velocity, I use the concepts of rotational motion and another adaptation of *Newton's Second Law*, given by the equation:

$$\tau = I\alpha$$

In this equation, τ (torque) is the rotational equivalent to force, and is given by the equation

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Just as the cue must apply force to the ball to give it an initial velocity, the cue must apply torque to the ball to give it an initial rotation. The r in this equation represents the radius from the center that the cue ball was struck. Similarly, the *moment of inertia* of the cue ball is the rotational equivalent of its mass, and is given by the equation

$$I = \frac{2}{5} mR^2$$

Just as the force must overcome the mass of the ball in order to move it, the torque must overcome the rotational inertia of the ball to rotate it. The R in this equation represents the radius of the cue ball itself. These equations can be combined and manipulated as shown below:

$$\begin{aligned} \vec{r} \times (-k\vec{x}) &= \frac{2}{5} mR^2 \frac{\Delta\vec{\omega}}{\Delta t} \\ \Delta\vec{\omega} &= \frac{5}{2} \left(\frac{\Delta t}{mR^2} \right) (\vec{r} \times (-k\vec{x})) \end{aligned}$$

Since I can assume that the angular velocity is zero before the shot is made, this equation simplifies to

$$\vec{\omega}' = \frac{5}{2} \left(\frac{\Delta t}{mR^2} \right) (\vec{r} \times (-k\vec{x}))$$

The quantities in this equation are identical to those in the equation for linear velocity given above, with the exception of the r and R , which I explained above.

Using these equations, I can calculate the initial linear and angular velocities of the cue ball after the shot is made. This completes the first stage of my simulation.

BALL IN MOTION

Once the ball has an initial linear and angular velocities, it begins to move across the table. After it leaves the tip of the cue, the only force acting on the cue ball is the force of friction from the felt. The purpose of this stage of the simulation is to continually update those velocity vectors according to this frictional force.

There are a couple important concepts that must be realized at this stage in the simulation. The first is that the ball is not always rolling. Immediately after it is struck, it slides along the felt for some period of time. How far it slides depends on its initial velocity and spin. The second is that the force of friction is not being applied through the ball's center of mass. It is actually being applied along the ball's perimeter, and so it becomes necessary to calculate the ball's perimeter speed.

The exact point at which a sliding ball becomes a rolling ball is given by the equation

$$|\vec{v}_p| = R \cdot |\vec{\omega}|$$

In this equation, v_p is the perimeter velocity of the ball at its contact point with the felt, R is the radius of the ball, and ω is the angular velocity of the ball. The *perimeter velocity* of the ball at the point of contact with the felt can be calculated using the equation

$$\vec{v}_p = (\vec{\omega} \times \vec{R}) + \vec{v}$$

where \vec{R} is the vector extending from the center of the ball to the point of contact with the table.

To deal with friction, I use *Newton's Second Law* again. This time, however, I am dealing with the force of friction instead of the force applied by the cue. The force of friction is given by the equation

$$\vec{F}_f = \mu_s m \vec{g}$$

In this equation, μ_s is the coefficient of sliding friction (on the order of .2), m is the mass of the ball, and g is the force of gravity. Since the direction of this force is opposite the perimeter velocity, we can modify this equation to represent the frictional force vector:

$$\vec{F}_f = -\mu_s m \vec{g} \frac{\vec{v}_p}{|\vec{v}_p|}$$

Combining this with *Newton's Second Law*, we can get an equation for the change in the linear and angular velocity of the ball. The change in linear velocity is derived below:

$$\begin{aligned} \vec{F}_f &= m \vec{a} \\ -\mu_s m \vec{g} \frac{\vec{v}_p}{|\vec{v}_p|} &= m \frac{\Delta \vec{v}}{\Delta t} \\ \Delta \vec{v} &= -\mu_s \vec{g} \frac{\vec{v}_p}{|\vec{v}_p|} \Delta t \end{aligned}$$

Similarly, we can derive a formula for the change in angular velocity:

$$\begin{aligned} \vec{\tau}_f &= I \vec{\alpha} \\ \left[\vec{r} \times \left(-\mu_s m \vec{g} R \frac{\vec{v}_p}{|\vec{v}_p|} \right) \right] &= \frac{2}{5} m R^2 \frac{\Delta \omega}{\Delta t} \\ \Delta \omega &= \frac{5}{2} \left[\vec{r} \times \left(-\mu_s m \vec{g} R \frac{\vec{v}_p}{|\vec{v}_p|} \right) \right] \frac{\Delta t}{m R^2} \end{aligned}$$

By combining these equations, I can calculate the velocity a certain time later using the initial linear and angular velocity vectors. However, these equations only apply to a sliding ball. A rolling ball behaves similarly, but with a different coefficient of friction: the coefficient of rolling friction, μ_s . The coefficient of rolling friction is significantly smaller than the coefficient of static friction, on the order of .01.

I know whether the ball is sliding or rolling using the condition stated above. Using this information, I can calculate velocity changes during a given time step. This completes the second stage of my simulation.

COLLISION DETECTION

To detect collisions between two balls or between a ball and a rail, I use the vector equation of a line to describe the position of a given ball at

a given time:

$$\vec{r} = \vec{r}_0 + \vec{v}t$$

In this equation, r represents the position of the ball, with r_0 being the initial position; and v represents the velocity of the ball. For collisions between two balls, I construct a position equation for each ball and calculate at what times they will be two radii apart. This uses the distance formula, and the math is shown below.

$$\sqrt{(\vec{r}_1 - \vec{r}_2) \cdot (\vec{r}_1 - \vec{r}_2)} = 2R$$

$$\sqrt{(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{r}_{01} - \vec{r}_{02}) + (\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{v}_1 - \vec{v}_2)t + (\vec{v}_1 - \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2)t^2} = 2R$$

To simplify things, I make the following assignments:

$$a = (\vec{v}_1 - \vec{v}_2) \cdot (\vec{v}_1 - \vec{v}_2)$$

$$b = 2(\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{v}_1 - \vec{v}_2)$$

$$c = (\vec{r}_{01} - \vec{r}_{02}) \cdot (\vec{r}_{01} - \vec{r}_{02})$$

The equation above now simplifies to:

$$a \cdot t^2 + b \cdot t + c = 4R^2$$

Using the quadratic formula, I can easily solve for t :

$$t = \frac{-b \pm \sqrt{b^2 - 4a(c - 4R^2)}}{2a}$$

This process is similar for collisions between balls and rails, but instead I calculate the difference between the ball and the rail and set it equal to R (not $2R$).

Using the times calculated, I can determine if a collision is going to occur within a given amount of time. If so, I move the balls to the time of that collision and recalculate their velocity vectors before continuing.

COLLISIONS

If a collision is detected, I must recalculate the velocity vectors of the balls involved. This stage of the simulation is the most complex, and requires concepts such as *conservation of energy*, *conservation of linear momentum*, and *conservation of angular momentum*.

For collisions between balls, the first step is to calculate the normal to the collision plane. This is along the line between the centers of the two balls, so the easiest way to calculate it is to normalize the difference between the position vectors of the two balls.

$$\vec{n} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|}$$

The next step is to divide each velocity vector into a normal component and a tangential component. The normal component for ball 2 will be in the direction of the normal vector, and the normal component for ball 1 will be in the direction opposite the normal vector. The magnitude of these normal vectors can be calculated using the dot product, as shown below.

$$\vec{v}_{n1} = [\vec{v}_1 \cdot (-\vec{n})](-\vec{n})$$

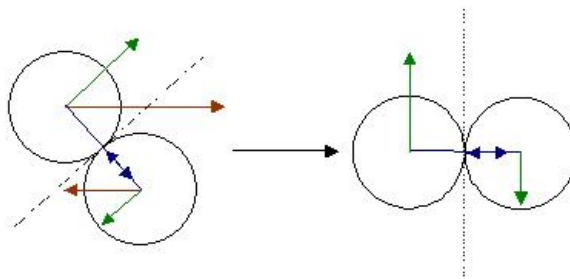
$$\vec{v}_{n2} = [\vec{v}_2 \cdot \vec{n}]\vec{n}$$

Once I've calculated the normal components of their velocity vectors, I can calculate the tangential components using vector subtraction:

$$\vec{v}_{t1} = \vec{v}_{n1} - \vec{v}_1$$

$$\vec{v}_{t2} = \vec{v}_{n2} - \vec{v}_2$$

During the collision, the tangential velocity components do not change. Using the normal components, I can treat the collision as 1-dimensional. This is depicted in the example below. The red arrows represent the velocity vectors immediately before the collision. The blue arrows represent the normal components of those vectors and the green arrows represent the tangential components of those vectors. The black dotted line is the collision plane.



To deal with the one-dimensional collision, I once again use *Newton's Second Law* in yet a different form, derived below:

$$\vec{F} = m\vec{a} = \frac{m\Delta\vec{v}}{\Delta t}$$

$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$$

The quantity mv is the linear momentum of the ball, denoted by the letter p . Since there is no net linear force on the two balls during the collision, Δp is 0. This is the concept known as *conservation of linear momentum*, which states that the total momentum of a system will remain constant in the absence of an outside force. From this, I obtain the following equation:

$$\begin{aligned}\vec{p}_i &= \vec{p}_f \\ m_1\vec{v}_1 + m_2\vec{v}_2 &= m_1\vec{v}_1' + m_2\vec{v}_2' \\ \vec{v}_1 + \vec{v}_2 &= \vec{v}_1' + \vec{v}_2'\end{aligned}$$

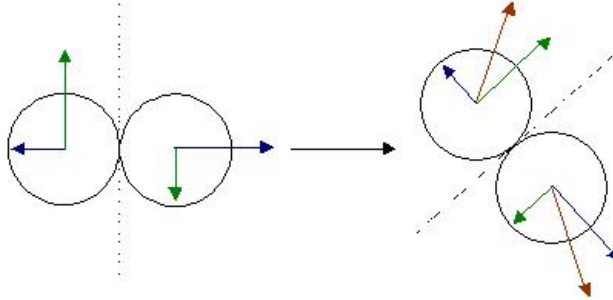
Since the masses of all the balls are identical, we can cancel out the mass terms, yielding the equation above. Using the *conservation of kinetic energy*, we can solve for \vec{v}_1' and \vec{v}_2' .

$$\begin{aligned}\frac{1}{2}m_1\vec{v}_1^2 + \frac{1}{2}m_2\vec{v}_2^2 &= \frac{1}{2}m_1\vec{v}_1'^2 + \frac{1}{2}m_2\vec{v}_2'^2 \\ \vec{v}_1^2 + \vec{v}_2^2 &= (\vec{v}_1 + \vec{v}_2 - \vec{v}_2')^2 + \vec{v}_2'^2 \\ \vec{v}_1^2 + \vec{v}_2^2 &= \vec{v}_1^2 + \vec{v}_2^2 + \vec{v}_2'^2 + 2\vec{v}_1\vec{v}_2 - 2\vec{v}_1\vec{v}_2' - 2\vec{v}_2\vec{v}_2' + \vec{v}_2'^2 \\ \vec{v}_2'^2 - \vec{v}_1\vec{v}_2' - \vec{v}_2\vec{v}_2' + \vec{v}_1\vec{v}_2 &= 0 \\ (\vec{v}_2' - \vec{v}_1)(\vec{v}_2' - \vec{v}_2) &= 0\end{aligned}$$

This equation yields two possible solutions, only one of which is practical. Therefore, assuming total elasticity in the balls,

$$\vec{v}_2' = \vec{v}_1, \text{ and } \vec{v}_1' = \vec{v}_2$$

Now I return to the 2-dimensional collision problem above. Since the normal velocity vectors can simply be exchanged, the velocity of each ball immediately after the collision can be obtained by adding the tangential velocity to the normal velocity of the ball it collides with. This is shown below (the red arrows now represent the velocity vectors immediately after the collision).



The red vectors in the diagram on the right can be calculated using the equations below:

$$\begin{aligned}\vec{v}_1' &= \vec{v}_{t1} + \vec{v}_{n2} \\ \vec{v}_2' &= \vec{v}_{t2} + \vec{v}_{n1}\end{aligned}$$

Now that I've handled the linear aspect of the collision, I have to consider the rotational aspect of it. The concepts are similar, but must now concern angular momentum and angular velocity instead of linear momentum and linear velocity. Angular momentum is denoted by L and defined below:

$$\vec{L} = \vec{r} \times \vec{p}$$

A rotational equivalent to the version of *Newton's Second Law* given above yields the equation:

$$\vec{\tau} = \frac{\Delta\vec{L}}{\Delta t}$$

However, the net torque during the collision is not zero because of the friction between the balls. Calculating the torque applied by this frictional force requires calculating the perimeter velocities of each of the balls at the point of collision. This is similar to the way I calculated the frictional force applied by the felt in stage 2, with some slight changes. There is a different coefficient of friction, μ_b , which I will assign to be the coefficient of friction between the two balls (on the order of 0.1). There is also extra calculation required to determine the direction of the frictional force. Since we are dealing with two perimeter velocities, we have to find the relative velocity of one with respect to the other. For example, the perimeter velocity of ball 2 with respect to ball 1 would be given by the equation

$$\begin{aligned}\vec{v}_{pR} &= \vec{v}_{p2} - \vec{v}_{p1} \\ \vec{v}_{pR} &= (\vec{r}_2 \times \vec{\omega}_2) - (\vec{r}_1 \times \vec{\omega}_1)\end{aligned}$$

In this equation, vectors \vec{r}_1 and \vec{r}_2 are the vectors extending from the centers of ball 1 and ball 2, respectively, to the point of contact between them. To get the direction of the frictional force applied to ball 2 by ball 1, I must add this relative perimeter velocity to the tangential velocity of ball 2. To get the magnitude of the frictional force, I must know the normal force applied to ball 2 by ball 1, which is proportional to the normal change in momentum of ball 1 (using the impulse-momentum theory given above). All of this yields the following equation for the frictional

force applied to ball 2:

$$\vec{F}_f = -\mu_b \frac{m\Delta v_{n1}}{\Delta t} \cdot \frac{\vec{v}_{pR} + \vec{v}_{t2}}{|\vec{v}_{pR} + \vec{v}_{t2}|}$$

Using this frictional force, I can calculate the change in angular velocity of ball 2 in the same way I did during stage 2, substituting the proper frictional force. This is shown below.

$$\begin{aligned}\vec{\tau}_f &= I_2 \vec{\alpha}_2 \\ \left[\vec{r}_2 \times \left(-\mu_b \frac{m\Delta v_{n1}}{\Delta t} \cdot \frac{\vec{v}_{pR} + \vec{v}_{t2}}{|\vec{v}_{pR} + \vec{v}_{t2}|} \right) \right] &= \frac{2}{5} m R^2 \frac{\Delta \omega_2}{\Delta t} \\ \Delta \omega_2 &= \frac{5}{2} \left[\vec{r}_2 \times \left(-\mu_b \frac{m\Delta v_{n1}}{\Delta t} \cdot \frac{\vec{v}_{pR} + \vec{v}_{t2}}{|\vec{v}_{pR} + \vec{v}_{t2}|} \right) \right] \frac{\Delta t}{m R^2}\end{aligned}$$

The exact same calculations can be made for ball 1, allowing me to obtain the change in angular velocity for ball 1 as well.

For collisions between a ball and a rail, a similar process is necessary.

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