Maths Assignment - 5

Ques: Prove that intersection of two normal subgroups of G is again a normal subgroup of G.

Sol: Let H and K be normal subgroups of G.

Let $x \in H \cap K$
Then, $x \in H$ and $x \in K$
For any element $g \in G$

$gxg^{-1} \in H$ (since H is normal)
$gxg^{-1} \in K$ (since K is normal)

So, $gxg^{-1} \in H \cap K$

Thus $H \cap K$ is normal subgroup of G.

Ques: Prove that every subgroup of an abelia group is normal. The converse need [illegible] be true. Give an example of such gr

Sol: A group $H \leq G$ is a normal subgrou if for any $g \in G$, the set gH equals the set Hg.

$$gH = Hg$$
$$H = gHg^{-1}$$

Now, take a subgroup H of an abelian group G.

$$x \in gHg^{-1}$$

Thus,

$$x = g h g^{-1}$$

where $g \in G$ and $h \in H$

Hence H is a normal subgroup of G

Ques: Suppose N is the only subgroup of order $o(N)$ in the group G. Prove that N is a normal subgroup of G.

Sol: Assume the subgroup H has order n and pick $g \in G$. Then, for any $h \in H$:

$$(ghg^{-1})^n = ghg^{-1} \cdot ghg^{-1} \cdots ghg^{-1}$$

$$= gh^n g^{-1} = geg^{-1} = gg^{-1} = e$$

Since H has order n.
The above holds for all $h \in H$; so the subgroup gHg^{-1} has order n, so it is equal to H

Then every $h \in H$ has some $h' \in H$ such that $ghg^{-1} = h'$, implying

$$gh = h'g$$

$$gH = Hg$$

Ques: If G is an abelian group and N is normal subgroup of G, then prove that G/N is an abelian group.

Sol: Let G be an abelian group and let N be a normal subgroup of G.

Each element of G/N is a coset aN for some $a \in G$.

Let aN, bN be arbitrary elements of G/N, where $a, b \in G$

Then we have,

$$(aN)(bN) = (ab)N$$

$$= (ba)N \quad \text{(since } G \text{ is abelia}$$

$$= (bN)(aN).$$

Since N is a normal subgroup of G, the set of left cosets G/H becomes a group with group operation

$$(aN)(bN) = (ab)N$$

for any, $a, b \in G$

Thus, G/N is an abelian group.

Ques: If G is a cyclic group and N is a subgroup of G, then prove that G/N is a cyclic group.

Sol: N is normal since G being cyclic implies that G is Abelian.

Suppose that $\{a^i \mid i \in Z\} = \langle a \rangle = G$

Now,

$$G/N = \{bN \mid b \in G\} = \{a^i N \mid i \in Z\}$$
$$= \{(aN)^i \mid i \in Z\}$$
$$= \langle aN \rangle$$

This proves that G/N is cyclic generated by aN.

Ques: Prove that the centre $Z(G)$ of a group G is normal subgroup of G.

Sol: $Z(G)$ is the subgroup of G consisting of all elements that commute with every element of G. Let z be any element of the center, and let g be any element of G. Then, $gz = zg$. Since z is chosen arbitrarily this shows that $gZ(G) \subseteq Z(G)g$, for every g. It also demonstrates the reverse inclusion, so that $gZ(G) = Z(G)g$. Since this holds for every $g \in G$, $Z(G)$ is normal in G.

Ques: If $G/Z(G)$ is a cyclic group, then prove that G must be abelian.

Sol: Suppose that $G/Z(G)$ is a cyclic group. Then there exists an $gZ(G) \in G/Z(G)$ such that:

$$G/Z(G) = \langle gZ(G) \rangle$$

Let $h \in G$. Then $hZ(G) \in G/Z(G)$. Then there exists an $n \in \mathbb{Z}$ such that

$$hZ(G) = (gZ(G))^n$$
$$= \underbrace{(gZ(G))(gZ(G)) \cdots (gZ(G))}_{n \text{ times}}$$
$$= g^n Z(G)$$

$\therefore$ $(g^n)^{-1} h \in Z(G)$. So there exists an $i \in Z(G)$ such that:

$$i = (g^n)^{-1} h$$

So $h = g^n i$. So for every $h \in G$ there exists an $n \in \mathbb{Z}$ and an $i \in Z(G)$ such that $h = g^n i$. Let $h_1, h_2 \in G$ and write $h_1 = g^{n_1} i_1$, $h_2 = g^{n_2} i_2$ where $n_1, n_2 \in \mathbb{Z}$ and $i_1, i_2 \in Z(G)$.

Then:

$$h_1 h_2 = (g^{n_1} i_1)(g^{n_2} i_2).$$

Since $i_1 \in Z(G)$ we have that

$$i_1 g^{n_2} = g^{n_2} i_1 \text{, so:}$$

$$\begin{aligned} h_1 h_2 &= g^{n_1} g^{n_2} i_1 i_2 \\ &= g^{n_1+n_2} i_1 i_2 \\ &= g^{n_2+n_1} i_1 i_2 \\ &= g^{n_2} \cdot g^{n_1} i_1 \cdot i_2 \end{aligned}$$

Since $i_1, i_2 \in Z(G)$ we have that $i_1 i_2 = i_2 i_1$, so:

$$h_1 h_2 = g^{n_2} g^{n_1} i_2 i_1$$

And lastly, since, $i_2 \in Z(G)$ we have that $g^{n_1} i_2 = i_2 g^{n_1}$, so:

$$\begin{aligned} h_1 h_2 &= g^{n_2} i_2 g^{n_1} i_1 \\ &= h_2 h_1 \end{aligned}$$

Since this holds for all $h_1, h_2 \in G$ we have that G is abelian.

Ques: Give an example of group G and subgroups H, K such that H is normal in K and K is normal in G, but H is not normal in G.

Sol: Let $G = S_4$

$H = \{I, (12)(34)\}$

$K = \{I, (12)(34), (13)(24), (14)(23)\}$

here, H is normal in K

$$\frac{|K|}{|H|} = \frac{4}{2} = 2$$

$\Rightarrow H \trianglelefteq K$

and also $K \trianglelefteq G$

But $H \ntrianglelefteq G$

Ques: Let N be a normal subgroup of G and $g \in G$. If g is of order K in G and Ng is of order r in G/N, then prove that r is a divisor of K.

Sol: $(gN)^K = g^K = eN = N$ but is the least power such that $(Ng)^r = N$. Assume that r will have to divide K which is apparently wrong.

$(Ng)^K = (Ng)^{rq+m}$ $0 \leq m < r$

$\Rightarrow ((Ng)^r)^{-q} (Ng)^K = (Ng)^m$

$$\Rightarrow N = (Ng)^m$$

but $m < r$

then $m = 0$, hence $rq = K$

Thus r is a divisor of K

Ques: Let N, H be two subgroups of a group G. Recall that $NH = \{nh : n \in N, h \in H\}$ and $HN = \{hn : h \in H, n \in N\}$. Prove that

(a) If N is normal in G, then NH and HN are subgroups of G.

Since N normal, then $K \cap N$ is also normal subgroup of K, K is a subgroup of G.

$HN = \{hn : h \in H, n \in N\}$ and
$NH = \{nh : h \in H, n \in N\}$

Now,

$$HN = \{h(H \cap N) : h \in H\} \cup \{h(H^c \cap N) : h \in H\}$$
$$= \{h(H \cap N) : h \in H\} \cup \{h(\cup_{a \notin H} (aH) \cap N) : h \in H\}$$

$= \{h(H \cap N) : h \in H\} \cup$
$\{\cup_{a \notin H} h(aH \cap N) : h \in H\}$
$= \{(H \cap N)h : h \in H\} \cup$
$\{\cup_{a \notin H} (aH \cap N)h : h \in H\}$
$= NH$

$HN \subset G$

$h_1, h_2 \in H, n_1, n_2 \in N$, then
$h_1 n_1 h_2 n_2 = h_1 h_2 n_3 n_2 \in HN$,
where $n_3 = h_2^{-1} n_1 h_2 \in N$, $1 \in HN$

$h \in H$, $n \in N$, then $hn \in HN$
$(hn)^{-1} = n^{-1} h^{-1} \in NH$
$= HN$.

$\therefore$ NH and HN are subgroups of G

(b) If N and H are normal in G, then NH and HN are normal in G.

There is a property that says:
$N \leq G$ Normal and $H \leq G \Rightarrow H \vee N$
$= HN = NH$
And if H & N are both normal, then
$HN \leq G$ is also normal.

$\therefore$ NH & HN are normal in G

(c) If N and H are normal in G such that $N \cap M = \{e\}$, then $nh = hn$ for every $n \in N$ and $h \in H$.

Let $h \in H$ and $n \in N$, then we have $n^{-1}hn \in H$ and $hnh^{-1} \in N$
so, we get $n^{-1}hnh^{-1}$ is in both N and H

As, $H \cap N = \{e\}$
we have,

$$n^{-1}hnh^{-1} = e \quad so,$$

$$hn = nh$$

Ques: Show that a non-empty finite set with an associative binary operation satisfying the cancellation laws is a group.

Sol: Let G be a non-empty finite set with an associative binary operation so that cancellation law holds, i.e. $ab = ac$ or $ba = ca$ $\Rightarrow b = c$, for any choices a, b, c in G.

To show G is a group, conditions must hold:

Suppose that $ab = ac$, then

$b = eb = (a^{-1}a)b = a^{-1}(ab)$
$= a^{-1}(ac) = (a^{-1}a)c$
$= ec = c$ (So left cancella-
-tion holds in G

Suppose $ba = ca$, then

$b = be = b(aa^{-1}) = (ba)a^{-1}$
$= (ca)a^{-1} = c(aa^{-1})$
$= ce = c$ (so right cancellation holds in G)

If a exists in G then $aa^{-1} = a^{-1}a$
$= e$,
where a is an inverse of a^{-1}
$(a^{-1})^{-1} = a$

Let x be the inverse of ab. Then $(ab)x = e$. By associativity we have $a(bx) = aa^{-1}$. Through left cancellation we have
$bx = a^{-1}bx = ea^{-1} = b(b^{-1}a^{-1})$ and
$x = b^{-1}a^{-1}$

Thus, $(ab)^{-1} = b^{-1}a^{-1}$. So all cancellation hold, G is a group.

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Ques: Let $(Cr, *)$ be a group in which the square of every element is the identity. show that Cr is abelian

Sol: We need to prove that $(Cr, *)$ a group, if for every $a \in Cr$

$$a * a = e$$

where e is identity element of that group, then the group is abelian group.

We know that:

$$a * e = a = e * a$$
$$6 * e = b = e * 6$$
$$6 * 6 = e = a * a \text{, so if}$$
$$a * a = 6 * b \text{, then}$$
$$a * (e * a) = (e * 6) * b$$
$$a * (6 * 6 * a) = (a * a * 6) * 6$$

$$(a * 6) * (6 * a) = (a * a) * (6 * 6)$$
$$= e * e = e$$
$$(a * 6) * (6 * a) = e$$
$$\Rightarrow a * 6 = 6 * a$$

$\therefore$ G is abelian.