Characterizing Safety: Minimal Barrier Functions from Scalar Comparison Systems

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***Abstract*—Verifying set invariance, considered a fundamental problem in dynamical systems theory and practically motivated by problems in safety assurance, has classical solutions stemming from the seminal work by Nagumo. Defining sets via a smooth barrier function constraint inequality results in computable flow conditions for guaranteeing set invariance. While a majority of these historic results on set invariance consider flow conditions on the boundary, recent results on control barrier functions extended these conditions to the entire set—although they still reduced to the classic Nagumo conditions on the boundary and thus require regularity conditions on the barrier function. This paper fully characterizes set invariance through *minimal barrier functions* by directly appealing to a comparison result to define a novel flow condition over the entire domain of the system. A considerable benefit of this approach is the removal of regularity assumptions of the barrier function. This paper also outlines necessary and sufficient conditions for a valid differential inequality condition, giving the minimum conditions for this type of approach and allowing for the verification of the largest class of invariant sets. We also show when minimal barrier functions are necessary and sufficient for set invariance. This paper further discusses extensions into time varying and control formulations, and outlines the connections between the proposed minimal barrier function and the historic boundary-based conditions for set invariance.**

***Index Terms*—Barrier Functions, Set Invariance, Nonlinear Dynamics, Nonlinear Control, Safety**

I. Introduction

Safety is a major consideration in many engineered systems. As such systems become increasingly complex, new tools for provable verification of safety are required. In the context of dynamical systems, safety has become synonymous with *set invariance*, the property that state trajectories of a system are contained within a given subset of the state space. Specifically, if it can be established that some set containing the system's initial condition(s) is invariant, and further this invariant set is a subset of a given set of safe states, then the system is declared to be safe.

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Given a candidate set contained within a set of safe states, safety verification then reduces to certifying that this set is invariant, a now classical problem in dynamical systems theory and control; e.g., see the textbook [1]. Intuitively, invariance can be established by ensuring that a system's vector field evaluated on the boundary of the candidate invariant set is always sub-tangent to the set so that trajectories cannot escape the set. The main technical challenge of this approach is in defining an appropriate notion of sub-tangency applicable to general sets, and finding conditions that extend over the entire set so they can be used for controller synthesis. Recent work on (control) barrier functions provided necessary and sufficient conditions for set invariance [2], [3], subject to regularity conditions on the set, that therefore allow for the synthesis of safety critical controllers. The question this paper addresses is: *Are these the strongest possible conditions for set invariance?*

The main result of this paper is necessary and sufficient conditions on set invariance that are *minimal* in that they are the least restrictive conditions needed to ensure set invariance. To obtain this result, we begin considering comparison results for scalar systems which lead to a notion of a minimal solution. This motivates the introduction of a *minimal barrier function* which leverages a comparison result for scalar systems. Minimal barrier functions are necessary and sufficient for set invariance and, importantly, they do not require the regularity conditions imposed by the original formulation of barrier functions. To put this in context, we relate these results to historic contributions to set invariance, including Nagumo's theorem, and illustrate through examples. Finally, minimal control barrier functions are introduced wherein state dependent input constraints and controller synthesis are considered.

Conditions on set invariance, and hence safety, have the potential to positively impact a variety of safety-critical domains. Examples include automotive systems where semi-autonomous safety features like adaptive cruise control and lane keeping can be framed as set invariance problems; thus the ability to give certificates of invariance can be used to verify these safety features. This was the genesis for the modern version of control barrier functions, and has been studied in depth [4]–[7]. Another domain where safety is critical is robotic systems. For example, swarms of robots must achieve objectives without colliding into each other and the world around them. In this context, control barrier functions have been utilized to guarantee collision avoidance, even when the system is governed by an existing nominal controller; these ideas have been applied to both ground robots [8]–[10] and drones [11], [12]. These are a few examples among many where the characterization of safety plays an important role in the controller synthesis of safety-critical systems.

A. History of Set Invariance & Existing Results

There is a long and rich history establishing conditions for set invariance. These conditions often considered general sets, $\mathcal{S}$, for which the set is (positively) invariant with respect to a dynamical system $\dot{x} = f(x)$ if for all $x_0 \in \mathcal{S}$ solutions with this initial condition satisfy $x(t) \in \mathcal{S}$ for all $t \geq 0$. Conditions on set invariance can be traced back to the *tangent cone*, first introduced by Bouligand in 1932 [13], which characterized the dynamic's tangent vectors on the boundary of $\mathcal{S}$. These were utilized by Nagumo in 1942 to provide the first necessary and sufficient conditions on set invariance [14]—the set is invariant if and only if the vector field lies in the tangent cone. Nagumo's theorem still plays an essential role in results characterizing set invariance to this day.

In the late 1960's and early 1970's, Nagumo's theorem was independently rediscovered by Bony and Brezis [15], [16]. The results of Brezis again utilized the tangent cone to give conditions on set invariance, while Bony utilized a comparison theorem type result; these were generalized by Redheffer in [17] by leveraging uniqueness functions. These ideas were further generalized by Ladde and Lakshmikantham in [18] via Lyapunov-like conditions which again leverage uniqueness functions and comparison type theorems—it is shown that the results of Bony, Brezis and Redheffer (and thus Nagumo) all form special cases of this broader formulation. These historic results serve as motivation for the approach considered in this paper, which utilize minimal solutions and corresponding comparison theorems.

There was a modern resurgence of interest in set invariance from the dynamics and control community beginning in the 1990s. This is best captured by the survey paper [19], which includes a nice overview of set invariance. In the dynamics and control domain, the set $\mathcal{S}$ is often assumed to be *practical*, i.e., given as the superlevel set of a collection of functions [1] (these were also considered for control systems in [20]). For simplicity, we will consider the case in which the invariant set is defined as the superlevel set of a single scalar-valued (candidate) *barrier function*, h, of the state space, *i.e.*, $\mathcal{S} = \{x : h(x) \geq 0\}$ wherein $\mathcal{S}$ is an invariant set if and only if $h(x(t)) \geq 0$ for all $t \geq 0$ whenever $h(x_0) \geq 0$. The goal is to find conditions on h such that it becomes a barrier function, i.e., certifies the forward invariance of $\mathcal{S}$.

The modern literature has predominately focused on verifying flow conditions on the boundary of the set $\mathcal{S}$. The most visible example of this is *barrier certificates*, which were first introduced to verify safety properties of hybrid systems in [21], through the independent discovery of Nagumo's theorem [14], to get the familiar condition $\frac{\partial h}{\partial x} f(x) \geq 0 \quad \forall x \in \partial\mathcal{S}$. Extensions of barrier certificates have been plenty, see e.g. [22]–[24], but a major assumption is that the gradient $\frac{\partial h}{\partial x}$ on the boundary does not vanish and corresponds to the exterior normal vector of $\mathcal{S}$. This assumption is necessary, as a simple counterexample is given in Example 5 and also appears in [21]. A notable exception is in [25], [26], where they remove the assumption of a nonvanishing gradient in favor of a flow condition that holds in a neighborhood around the boundary $\partial\mathcal{S}$.

The alternative approach to considering conditions only on the boundary is to enforce a flow constraint over the entire domain —this has important ramifications for controller synthesis. For example, [21] proposes the convex condition $\frac{\partial h}{\partial x} f(x) \geq 0 \quad \forall x \in \mathcal{D}$ as a sufficient condition for ensuring invariance of $\mathcal{S} = \{x : h(x) \geq 0\} \subset \mathcal{D}$. However, this sufficient condition is quite strong as it requires invariance of all super-level sets of h that are contained in $\mathcal{D}$. Similar conditions were considered in [27]–[29] wherein Lyapunov-like functions are considered and combined with barrier functions. Less conservative sufficient conditions for invariance have also been proposed. For example, both [20] and [30] introduced the convex relaxation conditions of the form: $\frac{\partial h}{\partial x} f(x) \geq \lambda h(x) \quad \forall x \in \mathcal{D}$ for some $\lambda \in \mathbb{R}$ (in the case of [20], $\lambda = 1$ for the relaxation). Importantly, while these conditions can be enforced over the entire set, they are not necessary and sufficient and thus are overly conservative.

With a view toward obtaining the least restrictive conditions for set invariance that hold over the entire set $\mathcal{S}$, a new form of barrier functions was recently introduced in [3] (which details the original formulations from [2] and [4]). The set $\mathcal{S}$ is invariant if $\frac{\partial h}{\partial x} f(x) \geq -\alpha(h(x)) \quad \forall x \in \mathcal{D}$ for an extended class $\mathcal{K}$ function α. Here, the use of an extended class $\mathcal{K}$ function implies stability of the set $\mathcal{S}$ for initial conditions outside of the set. Importantly, these conditions are necessary and sufficient for set invariance in the case when $\mathcal{S}$ is compact and 0 is a regular value of h—sufficiency follows from Nagumo's theorem while necessity follows from constructing the function α. Thus they meet the requirement that they are the least restrictive conditions possible under the aforementioned assumptions. Yet the question remains: can these assumptions, especially with respect to the regularity of h, be relaxed further and still guarantee set invariance? Answering this question is important as it allows for the verification for a larger set of invariance specifications for a given system. This leads to the main contribution of paper: the largest possible set of continuous functions, μ, in which to lower bound the flow via $\frac{\partial h}{\partial x} f(x) \geq -\mu(h(x)) \quad \forall x \in \mathcal{D}$.

B. Overview of Contributions

In this paper, we introduce novel conditions for ensuring invariance of a set defined via a smooth barrier function. A major objective of this paper is to characterize conditions, that are, in a certain sense, the minimum conditions required on the resulting differential inequality to certify invariance for a smooth barrier function $h(x)$. The remainder of this paper is organized as follows:

<u>Section II:</u> introduces standard definitions associated with solutions of autonomous systems and positive invariance.

<u>Section III:</u> presents two main theorems for minimal barrier functions.

- Theorem 1 establishes that $\dot{h} = \frac{\partial h}{\partial x}(x) f(x) \geq \mu(h(x))$ globally for $x \in \mathcal{D}$ is sufficient for establishing invariance

of $\mathcal{S} = \{x : h(x) \geq 0\}$ when $\mu(\cdot)$ is a *minimal function*, that is, solutions of the initial value problem $\dot{w} = \mu(w)$, $w(0) = 0$ remain positive for all time. Notably, μ need not be Lipschitz continuous. The proof of Theorem 1 relies on a comparison result tailored for non-Lipschitz vector fields with potentially nonunique solutions.
- Theorem 2 presents necessary and sufficient conditions for verifying that a function μ is a minimal function. These conditions recover as a special case the instance when μ is locally Lipschitz.

We also give several examples on the application of minimal barrier functions, including Example 4, which delineates an instance when a Lipschitz vector field for the comparison system is impossible. Furthermore, connections of minimal barrier functions to Lyapunov theory regarding sets and extensions to nonautonomous systems are stated. Lastly, we compare minimal barrier functions to prior work in zeroing barrier functions.

Section IV: recalls several classic tangent conditions for verifying invariance and discusses their connection with minimal barrier functions. That is, we connect the results presented in this paper with the historic results discussed above. Thus, we are able to show that the results presented in this paper include classic results as a special case.

Section V: presents necessary conditions on the existence for minimal functions. More concretely, the main result is:

- Under regularity conditions on $\mathcal{S}$, namely, twice-differentiability of h, compactness assumptions of level sets of h, and the requirement that $\frac{\partial h}{\partial x}(x) \neq 0$ for all x such that $h(x) = 0$, Theorem 5 proves that a locally Lipschitz minimal function μ always exists satisfying the barrier function $\frac{\partial h}{\partial x}(x) \geq \mu(h(x))$ for all x if $\mathcal{S}$ is invariant.

Section VI: considers the case of control systems and controller synthesis through the introduction of minimal control barrier functions. Systems with state dependent input constraints are considered in this case, and several results are presented on guaranteeing continuity properties of a viable controller.

Section VII: presents concluding remarks.

II. Background

We study the system

$$\dot{x} = f(x) \tag{1}$$

with state $x \in \mathcal{D}$ where $\mathcal{D} \subseteq \mathbb{R}^n$ is assumed to be an open set and $f : \mathcal{D} \to \mathbb{R}^n$ is assumed to be continuous. In general, continuity of f ensures solutions exist for (1), but they need not be unique. A solution $x(t)$ to (1) with the initial condition $x(0) = x_0 \in \mathcal{D}$ defined for $t \in [0, \tau)$ is called *maximal* if it cannot be extended for time beyond τ [31]. Given a maximal solution $x(t)$ defined on $[0, \tau)$, we write $\tau_{\max}[x(\cdot)]$ to denote the right (maximal) endpoint τ of its interval of existence and we write $I[x(\cdot)] = [0, \tau_{\max}[x(\cdot)])$ to indicate the (maximal) interval of existence of $x(t)$.

To avoid cumbersome notation, we often write simply $\tau_{\max}$ instead of $\tau_{\max}[x(\cdot)]$ when $x(t)$ is clear, *e.g.*, $I[x(\cdot)] = [0, \tau_{\max})$. Further, we write $t \geq 0$ instead of $t \in I[x(\cdot)]$ when clear from context. The system (1) is *forward complete* if $\tau_{\max}[x(\cdot)] = \infty$ for all maximal solutions $x(t)$.

We adapt the following definitions from [1]. A set $\mathcal{S} \subseteq \mathcal{D}$ is *positively invariant* for (1) if, for any $x_0 \in \mathcal{S}$, all corresponding maximal solutions $x(t)$ with $x(0) = x_0$ satisfy $x(t) \in \mathcal{S}$ for all $t \in I[x(\cdot)]$. A set $\mathcal{S} \subseteq \mathcal{D}$ is *weakly positive invariant* for (1) if, for any $x_0 \in \mathcal{S}$, there exists at least one maximal solution $x(t)$ with $x(0) = x_0$ satisfying $x(t) \in \mathcal{S}$ for all $t \in I[x(\cdot)]$.

If solutions are unique for (1) for every $x_0 \in \mathcal{D}$, then the definitions of positive invariance and weak positive invariance coincide. While we are almost exclusively interested in positive invariance in this paper, we will occasionally reference the weaker formulation.

Throughout this paper, we will study invariance of sets defined as $\mathcal{S} = \{x \in \mathcal{D} : h(x) \geq 0\}$ for a continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$. For a set $\mathcal{S} \subseteq \mathcal{D}$, we refer $\partial\mathcal{S}$ as the boundary of $\mathcal{S}$, $\mathcal{S}^\circ$ as the interior of $\mathcal{S}$, and $\overline{\mathcal{S}}$ as the closure of $\mathcal{S}$ with the standard topological definitions. The Lie derivative of h along the vector field f is denoted $L_f h : \mathcal{D} \to \mathbb{R}$ and defined by $L_f h(x) := \frac{\partial h}{\partial x}(x) f(x)$. We denote standard Euclidean norm by $\|\cdot\|$.

III. Minimal Barrier Functions

By studying sets $\mathcal{S}$ defined by inequality constraints of a smooth function h, we can develop Lyapunov-like conditions on the time evolution of h over the whole domain $\mathcal{D}$. In particular, we observe that if $h(x(t)) \geq 0$ can be assured for all $t \geq 0$ and for all initial conditions $x_0 \in \mathcal{S}$, then $\mathcal{S}$ is positively invariant. In this section, we make clear the connection between barrier functions and differential inequalities for scalar systems and propose a new barrier function condition. First we recall some standard notions of solutions for first order differential equations.

Consider the initial value problem

$$\dot{w} = g(w) \qquad w(0) = w_0 \tag{2}$$

where $g : W \to \mathbb{R}$ is a continuous function defined on an open set $W \subseteq \mathbb{R}$ and $w_0 \in W$. Again, continuity of g guarantees existence but not uniqueness of solutions to (2). The following definitions appear in, *e.g.*, [32, Section 2.2].

- A differentiable function $w(t)$ defined on some interval $[0, \tau)$ is a *solution* of (2) if $w(t) \in W$ for $t \in [0, \tau)$, $w(0) = w_0$, and $\dot{w}(t) = g(w(t))$ for all $t \in [0, \tau)$.
- A solution $\tilde{w}(t)$ is a *minimal solution* of (2) on $[0, \tau)$ if, for any other solution $w'(t)$ defined on $[0, \tau)$, $\tilde{w}(t) \leq w'(t)$ for all $t \in [0, \tau)$.

The existence of minimal solutions is guaranteed by the fact that $g(w)$ is continuous on the domain W [33, Thm 1.3.2], while uniqueness of minimal solutions is guaranteed by properties of the standard ordering on $\mathbb{R}$. Minimal solutions are fundamental for establishing comparison results in scalar differential inequalities. In particular, the following proposition presents a comparison result similar to the one outlined in [34, Thm 6.3] for which a solution to a differential inequality is bounded below by the minimal solution of the corresponding comparison system. For completeness, a proof is provided that follows closely to the proof of [34, Thm 6.3].

Proposition 1. *Let the minimal solution of the initial value problem* (2) *be* $\tilde{w}(t)$ *with domain* $[0,\tau)$. *If* $\eta(t)$ *is any differentiable function such that* $\eta(t) \in W$ *for all* $t \in [0,\tau)$ *and*

$$\dot{\eta}(t) \geq g(\eta(t)) \text{ for all } t \in [0,\tau), \quad \eta(0) \geq w_0, \tag{3}$$

then

$$\eta(t) \geq \tilde{w}(t) \text{ for all } t \in [0,\tau). \tag{4}$$

Proof. We initially show $\eta(t) \geq \tilde{w}(t)$ on any compact time interval $[0,\tau_f]$ with $\tau_f < \tau$. Let $\{\epsilon_n\}$ be a strictly monotonically decreasing sequence with $\lim_{n\to\infty} \epsilon_n = 0$ and $\epsilon_n > 0$ for all n. We have that for all n,

$$\dot{\eta}(t) \geq g(\eta(t)) > g(\eta(t)) - \epsilon_n \quad \forall t \in [0,\tau_f] \tag{5}$$

and

$$\eta(0) \geq w_0 > w_0 - \epsilon_n. \tag{6}$$

Let $\{r_n\}$ denote a sequence of solutions that satisfy the initial value problem

$$\dot{r}_n(t) = g(r_n(t)) - \epsilon_n \qquad r_n(0) = w_0 - \epsilon_n \tag{7}$$

for each n. By [34, Thm 6.2], the minimal solution for (7) exists on $[0,\tau_f]$ for large enough n, and to simplify notation, we assume this holds for all n.

We first claim that $\eta(t) > r_n(t)$ for all $t \in [0,\tau_f]$ for any n. Suppose for contradiction that $\eta(t) \leq r_n(t)$ for some $t \in [0,\tau_f]$ for some n. Then

$$T := \inf\{t \in (0,\tau_f] : \eta(t) \leq r_n(t)\} \tag{8}$$

is well defined and $T \in (0,\tau_f]$ since $\eta(0) > w_0 - \epsilon_n$. This fact coupled with continuity of $r_n(t)$ and $\eta(t)$ shows that $r_n(T) = \eta(T)$ and $\eta(t) > r_n(t)$ for $t \in [0,T)$. Moreover, $\dot{\eta}(T) > \dot{r}_n(T)$ and since $r_n(T) = \eta(T)$, this implies that $\eta(t) < r_n(t)$ on the interval $(T-\varepsilon, T)$ for some $\varepsilon > 0$. However, this contradicts the definition of T, proving the claim. Thus, letting $n \to \infty$, we have $\eta(t) \geq r(t)$ for $t \in [0,\tau_f]$ with $r(t) = \lim_{n\to\infty} r_n(t)$.

Now we claim that $\lim_{n\to\infty} r_n$ converges uniformly to some function r on $[0,\tau_f]$ and that r is the minimal solution to (2). Note that because $\{\epsilon_n\}$ is strictly monotone, $\dot{r}_{n+1}(t) > \dot{r}_n(t)$ for $t \in [0,\tau_f]$ and $r_{n+1}(0) > r_n(0)$, it holds that $r_{n+1}(t) > r_n(t)$ for $t \in [0,\tau_f]$. By similar reasoning, $r_n(t)$ is also bounded above by any solution of (2), and in particular by $\tilde{w}(t)$, so the limit $r(t)$ exists for all $t \in [0,\tau_f]$. Solving for the solution gives

$$r_n(t) = w_0 + \int_0^t g(r_n(s))ds - \epsilon_n(1+t). \tag{9}$$

Hence for $n < m$,

$$\begin{aligned} \|r_n(t) - r_m(t)\| \leq & (\epsilon_n - \epsilon_m)(1+\tau_f) + \\ & \int_0^{\tau_f} \|g(r_n(s)) - g(r_m(s))\| ds \end{aligned} \tag{10}$$

Because g is continuous, it is uniformly continuous on the compact set

$$\{z : \min_{t\in[0,\tau_f]} r_1(t) \leq z \leq \max_{t\in[0,\tau_f]} \tilde{w}(t)\}. \tag{11}$$

Together with the proposed Cauchy criterion (10), this implies uniform convergence of $\{r_n\}$ to r. Furthermore letting $n \to \infty$ gives

$$r(t) = w_0 + \int_0^t g(r_n(s))ds, \tag{12}$$

so r is a solution for (2). As established previously, r is upper bounded by any solution to (2), so $r($ is necessarily the minimal solution for (2), i.e. $r(t) = \tilde{w}(t)$ and $\eta(t) \geq r(t) = \tilde{w}(t)$ holds for all $t \in [0,\tau_f]$.

Finally we show the result holds over $[0,\tau)$ with τ potentially being ∞. Suppose for contradiction, $\eta(t) < \tilde{w}(t)$ for some $t \in [0,\tau)$ and let

$$\mathcal{T} := \inf\{t \in [0,\tau) : \eta(t) < \tilde{w}(t)\}. \tag{13}$$

Consider some τ_f such that $\mathcal{T} < \tau_f < \tau$. Since the minimal solution $\tilde{w}(t)$ also exists on $[0,\tau_f]$, the preceding argument implies $\eta(t) \geq \tilde{w}(t)$ on $[0,\tau_f]$, contradicting the definition of $\mathcal{T}$ and proving $\eta(t) \geq \tilde{w}(t)$ for all $t \in [0,\tau)$. □

Motivated by our interest in using scalar differential equations as barrier functions, we are especially interested in scalar systems for which minimal solutions remain nonnegative when initialized at the origin.

Definition 1. A continuous function $\mu : \mathbb{R} \to \mathbb{R}$ is a *minimal function* if the minimal solution $\tilde{w}(t)$ defined on $t \in [0,\tau)$ for the initial value problem $\dot{w} = \mu(w)$, $w(0) = 0$ satisfies $\tilde{w}(t) \geq 0$ for all $t \in [0,\tau)$.

We now present a formulation for barrier functions that directly utilizes minimal functions and the comparison result of Proposition 1. This notion will be a central building block for the results presented in this paper.

Definition 2. For the system in (1), a continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ is a *minimal barrier function* (MBF) if there exists a minimal function μ that satisfies

$$L_f h(x) \geq \mu(h(x)) \quad \forall x \in \mathcal{D}. \tag{14}$$

The notion of a minimal barrier function allows us to establish that, given such a function, it follows that $h(x(t))$ will be nonnegative for all time, i.e., the set $\mathcal{S}$ will be invariant.

Theorem 1. *Consider the system* (1) *and a nonempty* $\mathcal{S} = \{x \in \mathcal{D} : h(x) \geq 0\}$ *for some continuously differentiable* $h : \mathcal{D} \to \mathbb{R}$. *If* h *is a MBF as in Definition 2, then* $\mathcal{S}$ *is positively invariant.*

Proof. Consider some $x_0 \in \mathcal{S}$, and let $x(t)$ be a solution to (1) with the initial condition $x(0) = x_0$ with an interval of existence $I[x(\cdot)] = [0,\tau_{\max})$. Observe that $h(x(0)) \geq 0$. Now we formulate the comparison system

$$\dot{w} = \mu(w) \qquad w(0) = 0. \tag{15}$$

We first claim there exists a minimal solution $\tilde{w}(t)$ of the comparison system (15) defined on $I[x(\cdot)]$. To prove the claim, suppose for contradiction that the largest interval of existence for a minimal solution to the comparison system is $[0, \tau^*)$ with $\tau^* < \tau_{\max}$. Because μ is a minimal function, $\tilde{w}(t) \geq 0$ for $t \in [0, \tau^*)$. Moreover, it must be that $\lim_{t\to\tau^*} \tilde{w}(t) = \infty$ [33, Corollary 1.1.2]. By Proposition 1, $h(x(t)) \geq \tilde{w}(t)$ for $t \in [0, \tau^*)$, which implies $\lim_{t\to\tau^*} h(x(t)) = \infty$, but this is a contradiction since $h(x)$ is continuous on $\mathcal{D}$ and $x(t) \in \mathcal{D}$ for $t \in [0, \tau_{\max})$, and in particular for $t \in [\tau^*, \tau_{\max})$, and the claim is proved.

Then, by Proposition 1,

$$h(x(t)) \geq \tilde{w}(t) \geq 0 \text{ for all } t \in I[x(\cdot)] \tag{16}$$

and equivalently, $x(t) \in \mathcal{S}$ for all $t \in I[x(\cdot)]$. Therefore $\mathcal{S}$ is positively invariant. □

Remark 1. Theorem 1 is a direct application of Proposition 1 and highlights the strong connection between minimal solutions and differential inequalities and the invariance of sets parameterized by a barrier function. In this way, by interpreting h as a scalar-valued output system, we see that verifying invariance of a set $\mathcal{S} \subset \mathbb{R}^n$ parameterized by a barrier function h is converted to the problem of verifying invariance of $\{h \in \mathbb{R} : h \geq 0\}$ in the scalar output system, which is accomplished using differential inequalities.

It can be seen that the flow constraint in condition (14) is similar to the standard Lyapunov flow constraint. And indeed, as h is a scalar function defining a positively invariant set $\mathcal{S}$, we can also utilize h to verify stability properties of $\mathcal{S}$ as well. We recall the following definitions of stability for closed sets in $\mathbb{R}^n$.

Let $\rho(x, \mathcal{S}) = \inf\{\|y - x\| : y \in \mathcal{S}\}$ denote the distance from x to a set $\mathcal{S}$. A closed, invariant set $\mathcal{S} \subset \mathcal{D}$ is *uniformly stable* if for any $\delta > 0$, there exists $\epsilon > 0$ such that for any x_0 that satisfies $\rho(x_0, \mathcal{S}) < \epsilon$, the solution $x(t)$ satisfies $\rho(x(t), \mathcal{S}) < \delta$ for $t \in I[x(\cdot)]$ [35, Def 1, Sec. 1.9].

A set $\mathcal{S} \subset \mathcal{D}$ is *asymptotically stable* if it is uniformly stable and there exists a $\delta > 0$ such that for all x_0 with $\rho(x_0, \mathcal{S}) < \delta$, $\rho(x(t), \mathcal{S}) \to 0$ as $t \to \infty$ [36, Def 1.6.26].

A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$.

Proposition 2. *Let h be a minimal barrier function for* (1) *and $\mathcal{S} = \{x \in \mathcal{D} : h(x) \geq 0\}$. Assume there exists class $\mathcal{K}$ functions α, β and a constant $\delta > 0$ such that*

$$-\beta(\rho(x, \mathcal{S})) \leq h(x) \leq -\alpha(\rho(x, \mathcal{S})) \;\forall x \in \mathcal{S}_\delta \setminus \mathcal{S} \tag{17}$$

where $\mathcal{S}_\delta = \{x \in \mathcal{D} : h(x) \geq -\delta\}$. If there exists a minimal function μ for h such that (14) *is satisfied and $\mu(w) \geq 0$ for all $-\delta \leq w < 0$, then $\mathcal{S}$ is uniformly stable. Furthermore if the system* (1) *is forward complete and $\mu(w) > 0$ for all $-\delta \leq w < 0$, then $\mathcal{S}$ is asymptotically stable.*

Proof. Define a Lyapunov function candidate

$$\mathcal{V}_h(x) = \begin{cases} 0 & \text{if } x \in \mathcal{S} \\ -h(x) & \text{if } x \in \mathcal{D} \setminus \mathcal{S}. \end{cases} \tag{18}$$

Notice that for all $x \in \mathcal{S}_\delta$,

$$\alpha(\rho(x, \mathcal{S})) \leq \mathcal{V}_h(x) \leq \beta(\rho(x, \mathcal{S})). \tag{19}$$

Additionally, since $\mathcal{V}_h(x)$ is continuous, $\mathcal{V}_h(x)$ is upper bounded by $\beta(\rho(x, \mathcal{S}))$, and for $\delta > 0$, there exists $\gamma > 0$ such that the set

$$\mathcal{S}_\gamma = \{x \in \mathcal{D} : \rho(x, \mathcal{S}) < \gamma\} \tag{20}$$

is a subset of $\mathcal{S}_\delta$.

We first prove that $\mathcal{V}_h(x)$ satisfies the hypotheses of [36, Corollary 1.7.5], namely, that $\mathcal{V}_h(x)$ is upper and lower bounded by class $\mathcal{K}$ functions as in (19) over $\mathcal{S}_\gamma$ and that $\mathcal{V}_h(x(t)) \leq \mathcal{V}_h(x_0)$ for any $x_0 \in \mathcal{S}_\gamma$ and all $t \in I[x(\cdot)]$, which implies that $\mathcal{S}$ is uniformly stable. If $x_0 \in \mathcal{S}$, then $\mathcal{V}_h(x(t)) = \mathcal{V}_h(x_0) = 0$ for all $t \in I[x(\cdot)]$, as $x(t) \in \mathcal{S}$ for all $t \in I[x(\cdot)]$, which is verified by $h(x)$ being a barrier function for $\mathcal{S}$ with the minimal function $\mu(w)$. If $x(t) \in \mathcal{S}_\gamma \setminus \mathcal{S}$ for all $t \in I[x(\cdot)]$, then $\mathcal{V}_h(x(t)) \leq \mathcal{V}_h(x_0)$ for all $t \in I[x(\cdot)]$ because

$$\dot{\mathcal{V}_h} \leq -\mu(-\mathcal{V}_h) \leq 0, \tag{21}$$

where the first inequality holds since $\mathcal{V}_h = -h$ on $\mathcal{S}_\gamma \setminus \mathcal{S}$ and the second follows by hypothesis. Finally, if $x(\tau) \in \mathcal{S}$ at some time $\tau > 0$, then $\mathcal{V}_h(x(t)) = 0$ for all $t > \tau$, so $\mathcal{V}_h(x(t)) \leq \mathcal{V}_h(x_0)$ for all $t \in I[x(\cdot)]$ as well. So for any x_0 in $\mathcal{S}_\gamma$, $\mathcal{V}_h(x(t)) \leq \mathcal{V}_h(x_0)$ for all $t \in I[x(\cdot)]$. Therefore $\mathcal{V}_h$ satisfies all the required hypotheses and $\mathcal{S}$ is uniformly stable.

Second, under the further assumptions of the proposition regarding asymptotic stability, we claim that $\mathcal{V}_h$ satisfies all of the hypotheses of [36, Theorem 1.7.8], proving that $\mathcal{S}$ is indeed asymptotically stable. In particular, these hypotheses are that there exists an open invariant set $\mathcal{B}$ with $\mathcal{S}_\gamma \subset \mathcal{B}$, where $\mathcal{S}_\gamma$ is as defined in (20) and $\gamma > 0$, such that: H1) $\lim_{t\to\infty} \mathcal{V}_h(x(t)) = 0$ for any $x_0 \in \mathcal{B}$; H2) $\mathcal{V}_h(x(t)) < \mathcal{V}_h(x_0)$ for all $t \geq 0$ and any $x_0 \in \mathcal{B} \setminus \mathcal{S}$; H3) $\mathcal{V}_h(x)$ is upper and lower bounded by class $\mathcal{K}$ functions as in (19) over $\mathcal{B}$.

We first claim $\mathcal{S}^\circ_\delta = \{x \in \mathcal{D} : h(x) > -\delta\}$ is an open invariant set and observe that there exists a $\gamma > 0$ where $\mathcal{S}_\gamma \subset \mathcal{S}^\circ_\delta$. By assumption, we have $L_f h(x) \geq \mu(h(x)) > 0$ for $x \in \mathcal{D}$ such that $-\delta \leq h(x) < 0$. Invoking Case 1 in Theorem 2 below, $\mathcal{S}_\delta$ is invariant, and in fact, $\mathcal{S}^\circ_\delta$ is invariant as well, due to the fact that $L_f h(x) > 0$ on $h(x) \in [-\delta, 0)$, showing the claim.

If $x_0 \in \mathcal{S}^\circ_\delta \setminus \mathcal{S}$, then $\mathcal{V}_h(x(t)) < \mathcal{V}_h(x_0)$ for all $t > 0$, since

$$\dot{\mathcal{V}_h} \leq -\mu(-\mathcal{V}_h) < 0, \tag{22}$$

where the first inequality holds since $\mathcal{V}_h = -h$ on $\mathcal{S}^\circ_\delta \setminus \mathcal{S}$ and the second follows by hypothesis, and thus H2) holds. Moreover, we claim that $\lim_{t\to\infty} \mathcal{V}_h(x(t)) = 0$ for all $x \in \mathcal{S}^\circ_\delta$. If $x_0 \in \mathcal{S}$, then $\mathcal{V}_h(x(t)) = 0$ for all $t \geq 0$ because $\mathcal{S}$ is invariant. If $x_0 \in \mathcal{S}^\circ_\delta \setminus \mathcal{S}$, using Proposition 1 with the comparison system $\dot{w} = \mu(w)$ and the initial condition $w_0 = h(x_0)$, $h(t) \geq \tilde{w}(t)$, for the minimal solution $\tilde{w}(t)$. In addition, as the system (1) is assumed to be forward complete, both solutions $\tilde{w}(t)$ and $h(x(t))$ are defined for all $t \geq 0$. Since $w_0 < 0$ and $\mu(w) > 0$ on $[w_0, 0)$, $\lim_{t\to\infty} \tilde{w}(t)$ must be non-negative. Thus $\lim_{t\to\infty} h(x(t)) \geq 0$ and $\lim_{t\to\infty} \mathcal{V}_h(x(t)) = 0$ as well and H1) holds. Hypothesis H3) holds by the assumption (19), and therefore $\mathcal{V}_h$ satisfies all the necessary hypotheses and $\mathcal{S}$ is asymptotically stable. □

Remark 2. The explicit requirement of forward completeness of solutions is only needed if $\mathcal{S}$ is not compact. If $\mathcal{S}$ is compact, this necessarily means that there exists a $\delta > 0$ such that $\mathcal{S}_\delta^\circ$ is compact as well, since its assumed that $h(x) < -\alpha(\rho(x, \mathcal{S}))$ outside of $\mathcal{S}$. Because it can be shown that $\mathcal{S}_\delta^\circ$ is invariant, the system (1) is automatically forward complete in the domain $\mathcal{S}_\delta^\circ$.

Remark 3. The barrier function h can also define sets $\mathcal{S}$ in which the closure of the interior of $\mathcal{S}$ doesn't necessarily equal the set $\mathcal{S}$. This can happen, for example, if the interior of $\mathcal{S}$ is empty. For the special case that $\mathcal{S}$ is a point, the utility of h is analogous to a standard Lyapunov function.

A. Necessary and sufficient conditions for minimal functions

In this section, we present verifiable conditions on μ to ensure that it is a minimal function. Because these results are necessary and sufficient, we note that there does not exist a larger class of continuous comparison functions that can be used for a barrier function. The backbone of the following theorem comes from uniqueness results in [37].

Theorem 2. *A continuous function $\mu : \mathbb{R} \to \mathbb{R}$ is a minimal function if and only if any of the following cases are satisfied:*

1) $\mu(0) > 0$
2) $\mu(0) = 0$ *and there exists $\varepsilon > 0$ such that $\mu(w) \geq 0$ for all $w \in [-\varepsilon, 0)$*
3) $\mu(0) = 0$ *and for every $\varepsilon > 0$, there exists some w', w'' in $[-\varepsilon, 0]$ such that $\mu(w') > 0$ and $\mu(w'') < 0$*
4) $\mu(0) = 0$ *and there exists $k > 0$ such that for all ε with $0 < \varepsilon < k$, $\mu(w) \leq 0$ on $[-\varepsilon, 0]$ and $1/\mu(w)$ is not integrable on $[-\varepsilon, 0]$, i.e. $\int_0^{-\varepsilon} \frac{dw}{\mu(w)}$ is divergent*

Proof. Let the system be $\dot{w} = \mu(w)$ and the corresponding minimal solution be $\tilde{w}(t)$ for the initial condition $w(0) = 0$. We use $a.e.$ for abbreviation for almost everywhere. First we show the sufficient direction.

Case 1) Since $\mu(0) \neq 0$, the minimal solution $\tilde{w}(t)$ is unique by [37, Thm 1.2.7]. Because $\mu(0) > 0$ and μ is continuous, there exists $\varepsilon > 0$ such that $\mu(w) > 0$ everywhere on $[-\varepsilon, 0]$ Suppose for contradiction, there exists $\tau > 0$ where $w(\tau) = b < 0$. Integration gives

$$\int_{w(0)=0}^{w(\tau)=b} \frac{dw}{\mu(w)} = \tau - 0. \tag{23}$$

The integral on the left is negative since $b < 0$ and $\mu(w) > 0$ on $[-\varepsilon, 0]$, but $\tau > 0$, giving a contradiction.

Case 2) [37, Thm 2.2.2] First we show that any solution $w(t)$ is monotone. Suppose for contradiction, that $w(t)$ is not monotone. Then there exists two times $t_1 \neq t_2 \in [0, \tau]$ such that $w(t_1) = w(t_2)$ and $\dot{w}(t_1) > 0$ and $\dot{w}(t_2) < 0$. But this is a contradiction, since $\dot{w}(t) = \mu(w(t))$ is a function of $w(t)$.

Since any solution $w(t)$ is monotone, and we assume that $\mu(w) \geq 0$ for all $w \in [-\varepsilon, 0]$, $w(t)$ is a non-decreasing function. Therefore, there does not exist a time τ in which $w(t) < w(0) = 0$, and any solution $w(t) \geq 0$ for all $t \geq 0$, including specifically $\tilde{w}(t)$.

Case 3) [37, Thm 2.2.2] Suppose for contradiction, there exists $\tau > 0$, where $w(\tau) = b < 0$. By assumption, $\mu(w(t))$ switches sign on $[b, 0]$, and thus $w(t)$ is not monotone on $[0, \tau]$. However, solutions must be monotone, giving a contradiction.

Case 4) [37, Thm 2.2.2] Suppose for contradiction, there exists a $\tau > 0$ where $w(\tau) = b \in [-\varepsilon, 0)$ for some $\varepsilon < k$. Let $\bar{\tau} < \tau$ be the greatest point such that $w(\bar{\tau}) = 0$. Take a monotonic sequence $\bar{\tau} < t^k < \tau$ converging down to $\bar{\tau}$. Integration of the ODE gives

$$\lim_{t^k \to \bar{\tau}^+} \int_{w(t^k)}^{b} \frac{dw}{\mu(w)} = \lim_{t^k \to \bar{\tau}^+} \tau - t. \tag{24}$$

Since $w(t)$ is monotone, the set of w in $[b, 0]$ in which $\dot{w} = \mu(w) = 0$ is a measure zero set. Thus, it must be that $\mu(w) < 0$ $a.e.$ on $w \in [b, 0]$ and there exists a set $G = [b, 0] \setminus Z$ such that $\mu(w) < 0$ for all $w \in G$, where Z is a measure zero set. Let $G_k = [b, w(t^k)] \cap G$. Then the integral $\int_{G_k} \frac{dw}{\mu(w)}$ converges to the improper integral $\int_0^b \frac{dw}{\mu(w)}$ via the monotone convergence theorem. By assumption, $1/\mu(w)$ is not integrable on $[b, 0]$, since $-b < k$ and $\int_0^b \frac{dw}{\mu(w)}$ diverges. However, $\lim_{t^k \to \bar{\tau}^+} \tau - t$ is bounded above by τ and therefore converges, giving a contradiction.

Now we consider the necessary direction. Assume all conditions do not hold. Then either $\mu(0) < 0$ or $\mu(0) = 0$, $1/\mu(w)$ is integrable on $[-\varepsilon, 0]$, and $\mu(w) < 0$ on $[-\varepsilon, 0] \setminus Z$ for some $\varepsilon > 0$ and some measure zero set Z. If $\mu(0) < 0$, there exists a $[-\varepsilon, 0]$ where $\mu(w) < 0$. Note also the minimal solution $\tilde{w}(t)$ is unique by [37, Thm 1.2.7]. Choose a point $b \in U_\varepsilon^-$ and integrate to get $\int_0^b \frac{dw}{\mu(w)} = \tau$. Because $\mu(w) < 0$ on $[b, 0]$, we can set τ to the value of the integral. Therefore this equation defines the solution where $w(\tau) = b < 0$.

Now consider the second condition, which follows from [37, Thm 1.4.3]. Let $G^t = [w(t), 0] \setminus Z$. The integral of the ODE is $\int_{G^t} \frac{dw}{\mu(w)} = t$ for $-\varepsilon \leq w(t) < 0$. Since $1/\mu(w)$ is integrable on $[-\varepsilon, 0]$ by assumption, the integral converges and defines a family of solutions $w_c(t)$ satisfying

$$\begin{cases} w_c(t) = 0 & t \leq c \\ \int_{G^t} \frac{dw}{\mu(w)} = t - c & t > c \end{cases} \tag{25}$$

for $c \in \mathbb{R}^+ \cup \{\infty\}$. Since $\mu(w) < 0$ on G^t, taking $c = 0$ gives the minimal solution in which $w_c(t) < 0$ for $t > 0$. $\square$

Remark 4. Case 1 and Case 2 are similar to standard Lyapunov conditions, as $L_f h \geq \mu(h) \geq 0$ on $h \in [-\varepsilon, 0]$ for some $\varepsilon > 0$. Case 3 considers the case when μ changes sign infinitely often. Case 4 relaxes the usual locally Lipschitz condition to a one-sided nonintegrability condition to handle a more general class of comparison functions.

Remark 5. Minimal function are akin to *uniqueness functions* in [38], as $-u(-w)$ can be extended to be a minimal function for a uniqueness function $u(w)$. However, in this paper, we use minimal functions directly in a comparison framework, instead of using it to define unique solutions. We note that comparison systems defined with minimal functions have one-sided uniqueness type behavior around 0.

Remark 6. Additionally, Osgood functions [37, Lemma 1.4.1], which are traditionally used to relax locally Lipschitz assumptions for unique solutions, share similarities with minimal functions, namely that $-o(-w)$ for any Osgood function $o(w)$ can be extended to satisfy the conditions outlined in Case 4.

If a minimal function is assumed to induce unique solutions, then only the condition that $\mu(0) \geq 0$ needs to be checked. More specifically, it can be shown that all locally Lipschitz minimal functions with $\mu(0) \geq 0$ satisfy the hypotheses of Theorem 2, as seen in the following Proposition.

Proposition 3. *Any locally Lipschitz continuous function $\mu_L : \mathbb{R} \to \mathbb{R}$ with $\mu_L(0) \geq 0$ satisfies the hypotheses of Theorem 2 and therefore is a minimal function.*

Proof. Notice that if, for any $\varepsilon > 0$, there exists a positive measure set $P \subset [-\varepsilon, 0]$ where $\mu_L(w) \geq 0$ for all $w \in P$, then μ_L has to satisfy one of the cases 1–4 of Theorem 2 and is necessarily a minimal function. Thus, assume μ_L is not a minimal function so that $\mu_L(0) = 0$ and there exists a constant $a > 0$ such that $\mu_L(w) < 0$ *a.e.* for $w \in [-a, 0]$. Since μ_L is locally Lipschitz, there exists a neighborhood U around 0 such that μ_L is Lipschitz on U. Choose $0 < k \leq a$ such that

$$[-k, 0] \subset [-a, 0] \ \cap \ U. \tag{26}$$

Since μ_L is Lipschitz on $[-k, 0]$,

$$\|\mu_L(w)\| \leq L\|w\| \tag{27}$$

for all $w \leq k$ for some Lipschitz constant L. Because $\mu_L < 0$ *a.e.* on $[-k, 0]$, it follows that $1/\mu_L(w) < 1/(Lw)$ *a.e.* on $[-k, 0]$. Then $1/\mu_L(w)$ is not integrable since $1/(Lw)$ is not integrable on any $[-\varepsilon, 0]$ for $\varepsilon \leq k$ and μ_L then satisfies Case 4 of Theorem 2. Therefore one of the cases of Theorem 2 must hold. □

Furthermore, if $\mu_L(0) < 0$, then μ_L is necessarily not a minimal function, which comes as a direct corollary of the first necessary condition in the proof of Theorem 2.

B. Examples

The following examples and anti-examples demonstrate the utility of the proposed formulation of minimal barrier functions.

This first example introduces a simple barrier function for a scalar system that results in a linear minimal function.

Example 1. Consider $\dot{x} = f(x) = -x$ for $x \in \mathbb{R}$ and let $h(x) = x^3$. We have $L_f h(x) = -3x^3 = -3h(x)$ so h satisfies (14) with $\mu(w) = -3w$, and μ is a minimal function by Proposition 3 since it is locally Lipschitz and $\mu(0) \geq 0$. Indeed, $\mathcal{S} = \{x : h(x) \geq 0\} = \{x : x \geq 0\}$ is positively invariant.

The next anti-example highlights the importance of considering minimal solutions to differential inequalities when constructing comparison systems.

Example 2. Consider $\dot{x} = f(x) = -1$ for $x \in \mathbb{R}$ and let $h(x) = x^3$. Take $\mu(w) = -3w^{2/3}$ where $(\cdot)^{2/3}$ is interpreted as $((\cdot)^{1/3})^2$. Then $L_f h(x) = \mu(h(x))$ for all $x \in \mathbb{R}$.

Although the function μ satisfies $\mu(0) \geq 0$, it is not locally Lipschitz, so Proposition 3 does not apply. Moreover, μ does not satisfy any of the conditions of Theorem 2. Indeed, $\mathcal{S}$ is not positively invariant on $\mathbb{R}$.

Further, the comparison system $\dot{w} = \mu(w)$ with initial condition $w(0) = 0$ has the minimal solution $\tilde{w}(t) = -t^3$. Observe that the comparison system with the initial condition $w(0) = 0$ also has as a solution $w(t) = 0$ for all $t \geq 0$. However, considering $x(t)$, the solution to $\dot{x} = f(x)$ with $x(0) = 0$, we see that $h(x(t)) = \tilde{w}(t)$, *i.e.*, the barrier h evaluated along solutions of the system $\dot{x} = f(x)$ match the minimal solution of the comparison system.

The following example examines the case where the set $\mathcal{S}$ has corners, but still can be verified using a minimal barrier function.

Example 3. Consider the system

$$\dot{x}_1 = -ax_1 + bx_2 \tag{28}$$
$$\dot{x}_2 = cx_1 - dx_2 \tag{29}$$

where $a, b, c, d \geq 0$. Let a barrier function be $h(x) = x_1 x_2$ so that $\mathcal{S}$ is the union of the first and third quadrants of the plane. $L_f h(x) = -ax_1x_2 + bx_2^2 + cx_1^2 - dx_1x_2 \geq -ax_1x_2 - dx_1x_2 = (-a-d)h(x)$ so that (14) is satisfied with $\mu(w) = -(a+d)w$ and $\mathcal{S}$ is positively invariant.

In the next example, it is necessary to consider a non-Lipschitz minimal function to establish forward invariance with a given barrier function. Even though the vector field of the system is Lipschitz, and the barrier function h is smooth, the resulting dynamics for $L_f h$, as a function of h, may not be Lipschitz.

Example 4. Consider $\dot{x} = -|x|$ for $x \in \mathbb{R}$ and let

$$h(x) = \begin{cases} \exp(-1/x) & \text{if } x \geq 0 \\ -\exp(1/x) & \text{if } x < 0 \end{cases} \tag{30}$$

so that $\mathcal{S} = \{x : h(x) \geq 0\} = \{x : x \geq 0\}$ is indeed invariant. Now we calculate

$$L_f h(x) = \begin{cases} -\exp(-1/x)/x & \text{if } x \geq 0 \\ \exp(1/x)/x & \text{if } x < 0. \end{cases} \tag{31}$$

The function h is invertible with inverse

$$h^{-1}(w) = \begin{cases} \frac{-1}{\ln(w)} & \text{if } 0 \leq w < 1 \\ \frac{1}{\ln(-w)} & \text{if } -1 < w < 0. \end{cases} \tag{32}$$

Define a minimal function candidate

$$\mu(w) = \begin{cases} w\ln(w) & \text{if } 0 \leq w < 1 \\ -w\ln(-w) & \text{if } -1 < w < 0 \end{cases} \tag{33}$$

and observe that $\mu(h(x)) = L_f h(x)$. We check that μ is a minimal function. Indeed, $\mu(h)$ is continuous with $\mu(0) = 0$ and $\mu(h) < 0$ over $\mathcal{D} \setminus \{0\}$. Corresponding to Case 4 in Theorem 2, we check that the improper integral

$$\int_0^{-a} 1/\mu(w)dw = \int_0^{-a} \frac{-1}{w\ln(-w)}dw \tag{34}$$
$$= -\ln(\|\ln(-w)\|)|_0^{-a} \tag{35}$$

diverges to ∞ for any $a \in (0,1)$. Therefore μ is a valid minimal function. Observe that μ is not locally Lipschitz at 0. Indeed, it is easy to establish that there exists no locally Lipschitz minimal function satisfying (14) since any such function must be upperbounded by μ constructed above and be nonnegative at the origin.

C. Comparing to Zeroing Barrier Functions

In this section, we compare MBFs to *zeroing barrier functions* (ZBFs) in [39], which use extended class $\mathcal{K}$ functions for the class of comparison systems, and is a major inspiration for the work in this paper. We remark that if α is an extended class $\mathcal{K}$ function, then $-\alpha$ is a minimal function, as guaranteed by Case 2 in Theorem 2.

In the development of ZBFs in [39], the construction utilizes Nagumo's Theorem and therefore requires the assumption that $\frac{\partial h}{\partial x}$ does not degenerate to a zero vector on the boundary of the set. By directly invoking a comparison theorem argument, however, we can dispense with this assumption. Enforcing a minimal function to be of class $\mathcal{K}$ is not a restrictive assumption on $\mathbb{R}^+$, since any continuous function can be lower bounded on $\mathbb{R}^+$ by some $-\alpha$ with α being an extended class $\mathcal{K}$ function.

However, as seen through all the cases in Theorem 2, the matter of guaranteeing strong invariance is tightly related to the behavior of the comparison system on $[-\epsilon, 0]$ for some $\epsilon > 0$. In particular, observe that (14) in Theorem 1 is required to hold for all $x \in \mathcal{D}$. In general, this cannot be relaxed to just all $x \in \mathcal{S}$, as seen in the next example.

Example 5. Consider again Example 2, and take

$$\alpha(w) = \begin{cases} 3w^{2/3} & \text{if } w \geq 0 \\ -3w^{2/3} & \text{if } w < 0 \end{cases} \tag{36}$$

so that $L_f h(x) = -\alpha(h(x))$ for all $x \in \mathcal{S}$, although notably the equality does not hold for $x \in \mathbb{R} \backslash \mathcal{S}$ and thus Theorem 1 is not applicable since it requires (14) to hold for all $x \in \mathcal{D}$. Notice that α is an extended class $\mathcal{K}$ function on $\mathbb{R}$ and that $-\alpha$ is a minimal function. Then, for the dynamics $\dot{w} = -\alpha(w)$, the set $\{w : w \geq 0\}$ is positively invariant. It is tempting to use $\dot{w} = -\alpha(w)$ as a comparison system with $w(0) = h(x(0))$ to obtain the false conclusion that $\mathcal{S}$ is positively invariant.

A considerable benefit of using the more general class of minimal functions over extended class $\mathcal{K}$ functions for a comparison system is that using an extended class $\mathcal{K}$ function necessitates that $L_f h > 0$ on $\mathcal{D} \setminus \mathcal{S}$. While this type of robustness is sometimes desirable, it does not hold in general.

Example 6. Consider $\dot{x} = x$ for $x \in \mathbb{R}$ and $h(x) = x$, with the corresponding $\mathcal{S} = \{x : x \geq 0\}$. Therefore, $L_f h(x) = x = h(x)$. In particular, $L_f h(x) < 0$ whenever $h(x) < 0$, and thus there does not exist an extended class $\mathcal{K}$ function α satisfying $L_f h(x) \geq -\alpha(h(x))$ for all $x \in \mathbb{R}$. However, $\mu(w) = w$ is a minimal function satisfying $L_f h(x) \geq \mu(h(x))$ for all $x \in \mathbb{R}$, thus proving invariance of $\mathcal{S}$.

D. Extending to nonautonomous barrier functions

In this section, we extend the above results to nonautonomous systems and/or nonautonomous barrier functions. This is crucial when either the vector field of the system or the set under inspection is a function of time. In this section, we study the time varying system

$$\dot{x} = f(t, x) \tag{37}$$

where $f : [0, \infty) \times \mathcal{D} \to \mathbb{R}^n$ is continuous in t and in x. A solution $x(t)$ is defined on a maximum time interval $I[x(\cdot)] = [t_0, \tau_{\max})$ such that $x(t) \in \mathcal{D}$ and (37) is satisfied for $t \in I[x(\cdot)]$ with an initial condition $x(t_0) = x_0 \in \mathcal{D}$ for $t_0 \geq 0$, and the solution cannot be extended for time beyond $\tau_{\max}$.

Consider $\mathcal{S} \subseteq [0, \infty) \times \mathcal{D}$ so that $\mathcal{S}_t := \{x \in \mathcal{D} : (t, x) \in \mathcal{S}\}$ is nonempty for all $t \geq 0$. $\mathcal{S}$ is *positively invariant* for (37) if for all $t_0 \geq 0$ the condition $x(t_0) \in \mathcal{S}_{t_0}$ implies all corresponding solutions $x(t)$ satisfy $x(t) \in \mathcal{S}_t$ for all $t \in [t_0, \tau_{\max})$ [1]. $\mathcal{S}$ is *weakly positively invariant* for (37) if for all $t_0 \geq 0$ the condition $x(t_0) \in \mathcal{S}_{t_0}$ implies the existence of a solution $x(t)$ that satisfies $x(t) \in \mathcal{S}_t$ for all $t \in [t_0, \tau_{\max})$. We assume that $\mathcal{S} = \{(t, x) \in [0, \infty) \times \mathcal{D} : h(t, x) \geq 0\}$ for a function h that is continuously differentiable in both arguments.

Given a scalar initial value problem $\dot{w} = g(t, w)$, $w(t_0) = w_0$ with $g : [0, \infty) \times W \to \mathbb{R}$ continuous in t and w and for open set $W \subseteq \mathbb{R}$, $t_0 \geq 0$, and $w_0 \in W$, solutions and minimal solutions are defined analogously to the definitions in Section III.

We slightly modify the definition of minimal barrier functions for time varying formulations.

Definition 3. A continuous function $\mu : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a *time varying minimal function* if any minimal solution $\tilde{w}(t)$ defined on $t \in [t_0, \tau)$ to the initial value problem $\dot{w} = \mu(t, w)$, $w(t_0) = 0$ satisfies $w(t) \geq 0$ for all $t \in [t_0, \tau)$ for any $t_0 \in [0, \tau)$.

A sufficient condition for time varying minimal functions is that solutions to $\dot{w} = \mu(t, w)$ are unique for any initial conditions and that $\mu(t, 0) \geq 0 \; \forall t \geq 0$. Finding necessary and sufficient conditions analogous to Theorem 2 is challenging since separation of variables is not possible to get an explicit integral expression.

The following definition parallels Definition 2 for the time varying case.

Definition 4. For the system in (37), a continuously differentiable function $h : [0, \infty) \times \mathcal{D} \to \mathbb{R}$ is a *time varying minimal barrier function* (TMBF) if there exists a time varying minimal function μ that satisfies

$$\frac{\partial h}{\partial t}(t, x) + L_f h(t, x) \geq \mu(t, h(t, x)) \quad \forall (t, x) \in [0, \infty) \times \mathcal{D}. \tag{38}$$

Theorem 3. *Consider the system* (37) *and a nonempty* $\mathcal{S} = \{(t, x) \in [0, \infty) \times \mathcal{D} : h(t, x) \geq 0\}$ *for some continuously differentiable* $h : [0, \infty) \times \mathcal{D} \to \mathbb{R}$. *If* h *is a TMBF as in Definition 4, then* $\mathcal{S}$ *is positively invariant.*

Proof. Analogous to the proof of Theorem 1. $\square$

Remark 7. Time varying systems and/or barrier functions can also be handled by transforming to a time invariant system by appending an indicator state $\dot{\theta} = 1$ for time. Then it is possible to apply the standard formulations of minimal barrier function. However, for certain specifications, it is not possible to find a time invariant minimal function and a time varying minimal function is necessary.

The following examples illustrate scenarios in which a time varying formulation is necessary. We first look at a system with a time varying vector field and a time invariant barrier function.

Example 7. Consider $\dot{x} = f(t,x) = xt$ for $x \in \mathbb{R}$ and let $h(x) = x$ so that $\mathcal{S} = \{x : x \geq 0\}$. Along solutions, $L_f h(x) = xt = h(x)t$. Take $\mu(t,w) = wt$ so that $L_f h(x) = \mu(t, h(x))$. The function μ is smooth in both t and w, so $\dot{w} = \mu(t,w)$ has unique solutions. Since $\mu(t,0) = 0$, μ is a time varying minimal function, and h is a time varying minimal barrier function by Theorem 3 so that $\mathcal{S}$ is positively invariant.

It can be seen that a time invariant minimal function does not exist for this system and barrier. For $\mu(w)$ to be a minimal function, $\frac{\partial h}{\partial x}(x) f(t,x) = xt \geq \mu(h(x)) = \mu(x)$ for all $t \geq 0$ and all $x \in \mathbb{R}$, but there does not exist a smooth function μ satisfying $wt \geq \mu(w)$ for all $t \geq 0$. Specifically, the inequality does not hold for any $w \in \mathbb{R}^-$.

Now we consider a case where the barrier function is time varying but the vector field is time invariant. These formulations are important, for example, when considering barrier functions as an approach to verify reachability [40].

Example 8. Consider $\dot{x} = f(x) = x^3 + x$ for $x \in \mathbb{R}$ and let $h(t,x) = e^{-t}x - e^{-2t}$. At $t = 0$, $\mathcal{S}_0 = \{x : x \geq 1\}$, and $\mathcal{S}_t$ increases to $\{x : x \geq 0\}$ as $t \to \infty$. Along solutions, $L_f h(t,x) = e^{-t}x^3 + 2e^{-2t} = e^{-t}(h(x)e^t + e^{-t})^3 + 2e^{-2t} = h^3 e^{2t} + 3h^2 + 3he^{-2t} + e^{-4t} + 2e^{-2t}$. Let $\mu(t,h) = L_f h(t,x)$ where $L_f h(t,x)$ is understood to be a function of t and h as just computed. Then $\mu(t,h)$ can be verified to be a valid minimal function. However, like Example 7, no time-invariant minimal function exists for this system and barrier function.

IV. Comparison Between Boundary Conditions

Arguably, the most common approach for establishing positive invariance of a set $\mathcal{S}$ is to verify, in some appropriate sense, that the velocity field of the system points inwards to $\mathcal{S}$ at each point on the boundary of $\mathcal{S}$. First formalized by Nagumo in [14] and independently discovered by others, there has since been a volume of work dedicated to making this basic approach precise in various contexts, e.g. [38], [1], [20]. In particular, care must be taken to appropriately define normal and tangent directions to arbitrary sets and to accommodate vector fields with potentially nonunique solutions or differential inclusions. In this section, we recall a few versions of these results, and we then compare minimal barrier functions with these alternative approaches.

The following result by Nagumo (and independently discovered by Brezis) gives a tangent condition on the flow of the system relative to the set $\mathcal{S}$ that is necessary and sufficient for positive invariance.

Theorem 4 (Nagumo's Theorem [16])**.** *Given the system* (1) *under the further condition that $f(x)$ is locally Lipschitz, consider a nonempty $\mathcal{S} \subseteq \mathcal{D}$ assumed to be closed relative to $\mathcal{D}$. Then $\mathcal{S}$ is positively invariant if and only if*

$$\lim_{\epsilon \downarrow 0} \frac{\rho(x + \epsilon f(x), \mathcal{S})}{\epsilon} = 0 \text{ for all } x \in \mathcal{S} \tag{39}$$

where $\rho(x, \mathcal{S}) = \inf\{||y - x|| : y \in \mathcal{S}\}$ for $x \in \mathcal{D}$.

Another condition which uses normal vectors instead of tangent spaces is given below. A vector n is an *outer normal* to $\mathcal{S}$ at x if $n \neq 0$ and if the closed ball with the center $x + n$ and radius $\|n\|$ has exactly one point in common with $\mathcal{S}$ which is x.

Proposition 4 ([15])**.** *Given the system* (1) *under the further condition that f is locally Lipschitz, consider a nonempty $\mathcal{S} \subseteq \mathcal{D}$ assumed to be closed relative to $\mathcal{D}$. Then $\mathcal{S}$ is positively invariant if and only if*

$$n \cdot f(x) \leq 0 \quad \forall x \in \mathcal{S},\ \forall n \in N(x) \tag{40}$$

where $N(x)$ is the set of outer normal vectors to $\mathcal{S}$ at x.

In Theorem 4 and Proposition 4, it is assumed that the vector fields are locally Lipschitz. Interestingly, uniqueness functions are also used in [38], [17] to relax locally Lipschitz assumptions for the flow of the system for unique solutions, and in [26] to relax an inequality similar to (40).

An important specialization of Theorem 4 is to the case where $\mathcal{S} = \{x : h(x) \geq 0\}$ for a smooth function $h : \mathcal{D} \to \mathbb{R}$, the situation that is the focus of this paper. When $\mathcal{S}$ is defined this way, then the gradient of h corresponds to the outer normal, provided that the gradient is nonzero. Proving invariance of $\mathcal{S}$ then requires showing that the vector field $f(x)$ always has a nonnegative inner product with the gradient, as formalized next.

For a continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ for an open set $\mathcal{D} \subseteq \mathbb{R}^n$, $\lambda \in \mathbb{R}$ is a *regular value* of h if $\frac{\partial h}{\partial x}(x) \neq 0$ for all $x \in \{x \in \mathcal{D} : h(x) = \lambda\}$.

Corollary 1 ([1, Sec 4.2.1])**.** *Given the system* (1) *under the further condition that $f(x)$ is locally Lipschitz, consider a nonempty set $\mathcal{S} = \{x \in \mathcal{D} : h(x) \geq 0\}$ for some continuously differentiable $h : \mathcal{D} \to \mathbb{R}$. Further assume that 0 is a regular value of h. Then $\mathcal{S}$ is positively invariant if and only if*

$$L_f h(x) \geq 0 \tag{41}$$

for all $x \in \{x \in \mathcal{D} : h(x) = 0\}$.

Remark 8. Corollary 1 provides a powerful result for establishing invariance of $\mathcal{S}$ provided that 0 is a regular value of h. When 0 is a regular value of the barrier function h, then the only property of μ satisfying (14) that is relevant for establishing invariance of $\mathcal{S}$ is that $\mu(0) \geq 0$ since $L_f h(x) \geq \mu(0) \geq 0$ whenever $h(x) = 0$ and Corollary 1 then applies. In other words, when 0 is a regular value of h, there is no need to invoke a comparison result to prove invariance and μ does not need to be a minimal function or even be continuous or well-behaved, so long as $\mu(0) \geq 0$.

Remark 9. In the case where 0 is not a regular value of the barrier, however, the structure of μ defined in Theorem 2 is necessary for guaranteeing invariance, as demonstrated in Example 2. If the assumption that 0 is a regular value is lifted, Examples 1, 3, and 4 are then cases that can be considered.

Remark 10. An important case when 0 is not a regular value is if $\mathcal{S}$ denotes a set with no interior. For example, if the set $\mathcal{S}$ corresponds to a fixed point or a limit cycle, it is not possible to verify invariance using Corollary 1.

Extensions of invariance results from the standard Nagumo's theorem usually relax three conditions, specifically a smoothness assumption of the boundary of $\mathcal{S}$, a tangent condition on the flow of the system at the boundary, and a uniqueness assumption on the dynamics of the system (1) [38]. By adding a smoothness condition for differentiability of the minimal barrier function, a comparison argument can be utilized to relax some of the uniqueness assumptions for the resulting comparison system.

V. Necessary Conditions for Minimal Barrier Functions

In this section, conditions for existence of minimal functions are detailed. We motivate this discussion with the following simple example.

Example 9. Consider

$$\dot{x}_1 = x_1 x_2 \tag{42}$$

$$\dot{x}_2 = 0 \tag{43}$$

and $h(x) = x_1$ so that $\mathcal{S} = \{x : h(x) \geq 0\} = \{x : x_1 \geq 0\}$. Then $\frac{\partial h}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, and thus any $\lambda \in \mathbb{R}$ is a regular value of h, so Corollary 1 is applicable and $\mathcal{S}$ is invariant. However, there does not exist a minimal function that allows for verifying invariance using Theorem 1. To see this, notice that for such a minimal function μ to exist, $\mu(w) \leq L_f h(x)$ must hold for every x such that $h(x) = w$. But $L_f h(x) = x_1 x_2$ so that for any fixed w, all x satisfying $h(x) = w$ have the form $x = \begin{bmatrix} w & x_2 \end{bmatrix}^T$ and x_2 can be chosen to make $L_f h(x)$ an arbitrarily large negative number for $x_1 \neq 0$.

In this example, we see that even though $\mathcal{S}$ is positively invariant, there is no guarantee that a minimal function will exist, *i.e.*, minimal barrier functions are only sufficient for establishing invariance in general.

However, under certain structural conditions on the barrier function and the system (1), it is possible to guarantee the existence of a minimal function when the set $\mathcal{S} = \{x \in \mathcal{D} : h(x) \geq 0\}$ is known to be invariant. In this section, we provide these sufficient conditions to ensure that a valid minimal function can be found based on observations of an explicit construction.

We first provide a construction for a minimal function given a candidate barrier function, provided one exists.

Lemma 1. *Given a system of the form* (1) *and a candidate barrier function $h(x)$, let $\Gamma : W \to \mathbb{R}$ be defined as*

$$\Gamma(w) = \inf_{x : h(x) = w} L_f h(x) \tag{44}$$

where $W = \{h(x) : x \in \mathcal{D}\}$ is the range of h. If there exists some minimal function that satisfies condition (14)*, and if Γ is continuous, then Γ is also a minimal function that satisfies condition* (14).

Proof. Let μ be a minimal function satisfying (14) so that $\mu(h(x)) \leq L_f h(x)\ \forall x \in \mathcal{D}$. It follows from the definition of Γ that

$$\mu(w) \leq \Gamma(w) \tag{45}$$

for all $w \in W$. Since μ is a minimal function, μ satisfies one of the cases in Theorem 2.

If, for every $\varepsilon > 0$, there exists a positive measure set $P \subset [-\varepsilon, 0]$ where $\mu(w) \geq 0$ for $w \in P$, then $\Gamma(w) \geq \mu(w) \geq 0$ on P and therefore Γ has to satisfy one of the cases 1–4 of Theorem 2 so that Γ is a minimal function. We can then consider the alternative condition that $\Gamma(w)$ and $\mu(w) < 0$ *a.e.* on $[-k, 0]$ for some k. Because $\mu(h(x)) \leq \Gamma(h(x))$ for all $x \in \mathcal{D}$ and $\Gamma \neq 0$ *a.e.*, then it must be that $1/\mu(h(x)) \geq 1/\Gamma(h(x))$ *a.e.* Because $1/\mu(h(x))$ is nonintegrable and negative a.e., $1/\Gamma(h(x))$ is nonintegrable as well. So Γ must necessarily satisfy one of the cases if there exists a minimal function that does. □

Therefore, showing that $\Gamma(w)$ is a minimal function is equivalent to the existence of a minimal function under the assumption of continuity of $\Gamma(w)$. Thus, we will only focus our attention on $\Gamma(w)$.

To analyze what conditions on the barrier function are necessary for the continuity properties of Γ, we introduce some tools from topology and optimization.

First we describe a generalized inverse of $h(x)$ as a point-to-set mapping $h^{-1} : W \rightrightarrows \mathcal{D}$, where $h^{-1}(w) = \{x \in \mathcal{D} : h(x) = w\}$ and $W \subset \mathbb{R}$ is the range of h, *i.e.*, $W = \{h(x) : x \in \mathcal{D}\}$. We use $W \rightrightarrows \mathcal{D}$ in place of $W \to 2^{\mathcal{D}}$ for ease of notation. The function $\Gamma(w)$ can now be defined as $\Gamma(w) = \inf\{L_f h(x) : x \in h^{-1}(w)\}$.

We now introduce some necessary definitions for h^{-1} to be continuous as a point-to-set map:

- h^{-1} is *lower semicontinuous* (l.s.c) at w_0 if for each open set G s.t. $G \cap h^{-1}(w_0) \neq \varnothing$, there exists a neighborhood $U(w_0)$ s.t. $w \in U(w_0) \implies h^{-1}(w) \cap G \neq \varnothing$ [41].
- h^{-1} is *upper semicontinuous* (u.s.c) at w_0 if for each open set G s.t. $h^{-1}(w_0) \subset G$, there exists a neighborhood $U(w_0)$ s.t. $w \in U(w_0) \implies h^{-1}(w) \subset G$ [41].
- h^{-1} is *continuous* at w_0 if it is both upper and lower semicontinuous at w_0.

If h^{-1} always maps to a single point, then definitions of lower and upper semicontinuity coincide with the standard definitions of continuity for functions [41].

Since all cases in Theorem 2 depend only on some neighborhood U around 0, we are only concerned with showing continuity of Γ on some U around 0. The next proposition gives sufficient conditions on when this is the case.

Proposition 5. *Let $h : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function. If* 0 *is a regular value of h and there exists $\delta > 0$ such that $\Lambda := \{x \in \mathcal{D} : -\delta \le h(x) \le \delta\}$ is compact, then $\Gamma(w)$ defined in* (44) *is continuous on some neighborhood of* 0.

Proof. We first claim that h^{-1} is a continuous point-to-set map on some open set U containing 0. Let $W = \{h(x) : x \in \mathcal{D}\}$ and $\mathcal{S} = \{x \in \mathcal{D} : h(x) \ge 0\}$. Define two point-to-set maps $h^+, h^- : W \rightrightarrows \mathcal{D}$ according to

$$h^+(w) = \{x \in \mathcal{D} : -h(x) + w \le 0\}, \tag{46}$$

$$h^-(w) = \{x \in \mathcal{D} : h(x) - w \le 0\}, \tag{47}$$

and observe that $h^{-1}(w) = h^+(w) \cap h^-(w)$ for all $w \in W$. Because 0 is assumed to be a regular value of h and $\partial\mathcal{S}$ is nonempty, $h^+(w)$ and $h^-(w)$ are both nonempty for all w in some neighborhood V of 0. The functions $-h(x) + w$ and $h(x) - w$ are both continuous on $W \times \mathcal{D}$, so h^+ and h^- are closed on V [42, Thm 10], and so is h^{-1} due to [41, 6.1 Thm 5]. Since $h^{-1}(w)$ is assumed to map into the compact set Λ for $w \in [-\delta, \delta]$, $h^{-1}(w)$ is also u.s.c on $V \cap [-\delta, \delta]$ [41, 6.1 Corollary to Thm 7].

Because 0 is a regular value of h and h is continuously differentiable, $\frac{\partial h}{\partial x}$ is constant rank in a neighborhood $\mathcal{N}(x)$ of each $x \in h^{-1}(0)$. Defining $G = \bigcup_{x\in h^{-1}(0)} \mathcal{N}(x)$ gives an open cover of $h^{-1}(0) \subset G$ and since h^{-1} was shown to be u.s.c at 0, there exists a neighborhood V' of 0 where $h^{-1}(w) \subset G$ for $w \in V'$ by definition of u.s.c. Therefore for any $w \in V'$, w is a regular value of h. Now define

$$h_I^+(w) = \{x \in \mathcal{D} : -h(x) + w < 0\}, \tag{48}$$

$$h_I^-(w) = \{x \in \mathcal{D} : h(x) - w < 0\}. \tag{49}$$

We now show $\overline{h_I^+(w)} \supseteq h^+(w)$ for all $w \in V'$. Let $w \in V'$ and let $x^* \in h^+(w)$. If $h(x^*) > w$, then by definition, $x^* \in h_I^+(w)$. If $h(x^*) = w$ and $w \in V'$, then w is a regular value of h and there exists a direction $d \in \mathbb{R}^n$ satisfying $\frac{\partial h}{\partial x}(x^*)d > 0$. Let

$$F(a) = h(x^* + ad) \tag{50}$$

and notice that F is continuous, so $\lim_{a\to 0^+} F(a) = w$ and for $a > 0$ sufficiently close to 0, $F(a) > w$. Therefore, x^* is a limit point of $h_I^+(w)$, and thus $x^* \in \overline{h_I^+(w)}$. A symmetric argument implies $\overline{h_I^-(w)} \supseteq h^-(w)$ for all $w \in V'$. Then, since $-h(x) + w$ and $h(x) - w$ are both continuous on $W \times \mathcal{D}$, $\overline{h_I^+(w)} \supseteq h^+(w)$, and $\overline{h_I^-(w)} \supseteq h^-(w)$, it holds that h^+ and h^- are both l.s.c on V' [42, Thm 13]. Because h^+ and h^- are l.s.c, $h_I^+(w)$ and $h_I^-(w)$ are nonempty on V, and $h^{-1}(w)$ is assumed to be compact for $w \in [-\delta, \delta]$, h is a l.s.c. point-to-set map on $V \cap V' \cap [-\delta, \delta]$ [43, Thm 3], [44]. Finally, we have h^{-1} is both u.s.c and l.s.c on any open set $U \subset V \cap V' \cap [-\delta, \delta]$ and is therefore continuous on U.

Now we prove Γ is continuous on U. Since h is assumed to be continuously differentiable, and f in (1) is assumed to be continuous, $L_f h$ is also continuous. Because h^{-1} is a continuous point-to-set map on U, and $L_f h$ is continuous everywhere,

$$\Gamma(w) = -\sup\{-L_f h(x) : x \in h^{-1}(w)\} \tag{51}$$

is continuous on U [42, Thm 7]. □

Since Γ is continuous on a neighborhood of 0, and if Γ is a bounded function, then Γ only needs to be checked to satisfy one of the cases in Theorem 2 in a neighborhood U around 0. If Γ satisfies one of the cases on U, then it can be continuously extended to a minimal function over $\mathbb{R}$. However, it is easier to show conditions in which Γ is Lipschitz continuous around 0, which, as seen in Proposition 3, allows for only checking the condition that $\Gamma(0) \ge 0$ to show existence of a minimal function.

The next theorem gives the necessary conditions for a locally Lipschitz minimal function to exist.

Theorem 5. *Let $h : \mathcal{D} \to \mathbb{R}$ be a twice continuously differentiable function and assume f is locally Lipschitz. Assume further that* 0 *is a regular value of h, and $\Lambda_\delta := \{x \in \mathcal{D} : -\delta \le h(x) \le \delta\}$ is compact for all $\delta \ge 0$. If $\mathcal{S} = \{x : h(x) \ge 0\}$ is a positively invariant set, then h is a minimal barrier function for* (1) *with a locally Lipschitz minimal function μ.*

Proof. Let $\rho(x, A) = \inf\{\|x - y\| : y \in A\}$ denote the point-to-set distance from $x \in \mathcal{D}$ to some set $A \subseteq \mathbb{R}^n$. Because 0 is a regular value of h, Lyusternik's Theorem [45] applies, so that for all $x \in h^{-1}(0)$, there exists a neighborhood $\mathcal{N}_1(x) \subset \mathbb{R}^n$ of x, a neighborhood $\mathcal{N}_2(x) \subset \mathbb{R}$ of 0, and a constant $K(x) > 0$ such that

$$\rho(x', h^{-1}(w)) \le K(x)\|h(x') - w\| \tag{52}$$

for all $x' \in \mathcal{N}_1(x)$ and all $w \in \mathcal{N}_2(x)$.

Next, notice that $\bigcup_{x\in h^{-1}(0)} \mathcal{N}_1(x)$ is an open cover of $h^{-1}(0)$, and $h^{-1}(0) = \Lambda_0$ is assumed to be compact. By compactness of $h^{-1}(0)$, there exists a finite subcover, *i.e.*, a finite set of points $\{x_i\}_{i=1}^N \subset h^{-1}(0)$ such that

$$h^{-1}(0) \subset \bigcup_{i=1}^N \mathcal{N}_1(x_i) =: \mathcal{C}. \tag{53}$$

It was shown in Proposition 5 that h^{-1} is u.s.c at 0 under the assumptions of the theorem statement. By definition of upper semicontinuity, since $\mathcal{C}$ is an open set that covers $h^{-1}(0)$, there exists a neighborhood V of 0 such that

$$w \in V \implies h^{-1}(w) \subset \mathcal{C}. \tag{54}$$

For some $i \in \{1, \ldots, N\}$, there also exists a neighborhood X of the point x_i and a neighborhood W of 0 such that the solution for $h(x) = w$ exists for any $w \in W$ and some $x \in X$, due to 0 being a regular value [46]. Then, for all $w \in W$, $h^{-1}(w)$ is nonempty. Let

$$L_1 = \max_{i\in\{1,\ldots,N\}} K(x_i) \tag{55}$$

and let

$$U = \bigcap_{i=1}^N \mathcal{N}_2(x_i) \cap V \cap W. \tag{56}$$

Observe that U is a neighborhood of 0 since it is a finite intersection of neighborhoods of 0. Therefore,

$$\rho(x', h^{-1}(w)) \le L_1\|h(x') - w\| \tag{57}$$

for all $x' \in \mathcal{C}$ and all $w \in U$.

Because f and $\frac{\partial h}{\partial x}$ are locally Lipschitz, so is $L_f h$, and thus $L_f h$ is Lipschitz on some compact set $U_c \supset U$ with some Lipschitz constant L_2. Now we show that Γ defined in (44) is Lipschitz on U. Choose $w_1, w_2 \in U$. Now choose $x_1 \in h^{-1}(w_1)$ such that $L_f(x_1) = \Gamma(w_1)$. This is possible since $L_f h$ is continuous and $h^{-1}(w_1) \subset \Lambda_{\|w_1\|}$ is compact, so an extremal point exists. Next, choose $x_2 \in h^{-1}(w_2)$ such that

$$\|x_1 - x_2\| = \rho(x_1, h^{-1}(w_2)) \tag{58}$$

which is also possible since $h^{-1}(w_2)$ is also compact and $\|x_1 - x_2\|$ is continuous in x_2 for a fixed x_1. Note that

$$\begin{aligned}
\Gamma(w_2) - \Gamma(w_1) &\leq L_f h(x_2) - L_f h(x_1) && (59)\\
&\leq \|L_f h(x_2) - L_f h(x_1)\| && (60)\\
&\leq L_2 \|x_1 - x_2\| && (61)\\
&\leq L_2 \rho(x_1, h^{-1}(w_2)) && (62)\\
&\leq L_1 L_2 \|h(x_1) - w_2\| && (63)\\
&\leq L_1 L_2 \|w_1 - w_2\|. && (64)
\end{aligned}$$

The inequality (59) holds from properties of $\inf$ and (61) is due to $L_f h$ being Lipschitz with a Lipschitz constant L_2 on $U_c \supset U$. Note that we previously chose x_2 to give the inequality in (62). Finally, since w_1 and w_2 are chosen from U, we apply Lyusternik's theorem for the inequality in (63). A similar argument establishes that $\Gamma(w_1) - \Gamma(w_2) \leq L_1 L_2 \|w_1 - w_2\|$ i.e,

$$\|\Gamma(w_1) - \Gamma(w_2)\| \leq L_1 L_2 \|w_1 - w_2\|, \tag{65}$$

and thus Γ is Lipschitz on U with a Lipschitz constant $L_1 L_2$.

Because Λ_δ is assumed to be compact for all $\delta \geq 0$ and $L_f h$ is continuous, Γ is bounded on $[-\delta, \delta]$ for all $\delta \geq 0$ as well. Therefore, there exists a locally Lipschitz function $\mu : \mathbb{R} \to \mathbb{R}$ such that $\mu(w) \leq \Gamma(w)$ for all $w \in W$ and, for some neighborhood $U' \subset U$ of 0, μ restricted to U' is equal to Γ. Furthermore, $L_f h(x) \geq \Gamma(h(x)) \geq \mu(h(x))$ for all $x \in \mathcal{D}$, so the barrier condition (14) is satisfied. Since $\mathcal{S}$ is assumed to be invariant, $L_f h(x) \geq 0$ for all $x \in h^{-1}(0)$, so $\mu(0) = \Gamma(0) \geq 0$. Therefore μ is locally Lipschitz and $\mu(0) \geq 0$, and by Proposition 3, μ is a valid locally Lipschitz minimal function. □

Remark 11. In Section IV, it was noted that under the satisfaction of the hypotheses of Corollary 1, any function μ with $\mu(0) \geq 0$ can be used in a flow inequality, without having to invoke a comparison system. However, Theorem 5 shows that under some reasonable additional assumptions, specifically the smoothness properties of the barrier function and compactness properties of the level sets, a locally Lipschitz minimal function exists.

Remark 12. Under the hypotheses of Theorem 5, it can be seen that minimal barrier functions in Definition 2 are not only equivalent to the conditions in Corollary 1, but also to the positive invariance of $\mathcal{S}$.

VI. Minimal Control Barrier Functions

A major benefit for using differential inequalities defined over the whole domain rather than just a boundary-type condition is that it is more amenable to controlled invariance. In constraint-based control, it is desirable to have constraints on the controller that are applied at every point on the domain rather than just a condition on the boundary $\partial\mathcal{S}$. If a boundary condition is directly applied for controlled invariance, the constraints are only active on $\partial\mathcal{S}$. This may introduce discontinuities in the controller and render it sensitive to model and sensor noise.

Extensions of minimal barrier functions to control formulations is direct. In this section, we instead consider a control affine system of the form

$$\dot{x} = f(x) + g(x)k(x) \tag{66}$$

with state $x \in \mathcal{D}$, where $\mathcal{D} \subseteq \mathbb{R}^n$ is assumed to be an open set, a feedback controller $k : \mathcal{D} \to \mathbb{R}^m$, and $f : \mathcal{D} \to \mathbb{R}^n$ and $g : \mathcal{D} \to \mathbb{R}^{n \times m}$, both assumed to be continuous. We also assume that $k(x) \in U(x)$ for all $x \in \mathcal{D}$, where $U : \mathcal{D} \rightrightarrows \mathbb{R}^m$ is a point-to-set map defining state-based input constraints and define $\mathcal{U}$ as the viable set of continuous controllers

$$\mathcal{U} = \{k \in \mathcal{C}^0 : k(x) \in U(x) \ \forall x \in \mathcal{D}\}. \tag{67}$$

A practical way of representing U is through a set of q inequalities

$$U(x) = \{u \in \mathbb{R}^m : e_i(x, u) \leq 0 \quad i = 1, \ldots, q\} \tag{68}$$

where $e_i(x, u) : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ are scalar-valued functions that define state based input constraints. We assume U can be written in this form for the rest of the section.

A set $\mathcal{S} \subseteq \mathcal{D}$ is *positively controlled invariant* if there exists a continuous controller k within the possible class of controllers $\mathcal{U}$ such that $\mathcal{S}$ is positively invariant with respect to the closed loop system $\dot{x} = f(x) + g(x)k(x)$ [1, Def 4.4].

We now state the corresponding definition of minimal barrier functions for control affine systems.

Definition 5. For the control affine system in (66), a continuously differentiable function $h : \mathcal{D} \to \mathbb{R}$ is a *minimal control barrier function* (MCBF) if there exists a minimal function μ such that for all $x \in \mathcal{D}$,

$$\sup_{u \in U(x)} [L_f h(x) + L_g h(x) u] \geq \mu(h(x)) \tag{69}$$

where $L_f h(x) = \frac{\partial h}{\partial x} f(x)$ and $L_g h(x) = \frac{\partial h}{\partial x} g(x)$ denote corresponding Lie derivatives.

The set of viable controls is described by the point-to-set map $K : \mathcal{D} \rightrightarrows \mathbb{R}^m$ given by

$$K(x) = \{u \in U(x) : L_f h(x) + L_g h(x) u \geq \mu(h(x))\}. \tag{70}$$

Verification of the existence of controllers with certain properties can be treated as a selection problem, which has been extensively studied in topology [47]. Specifically, the feedback controller k is a *selection* of K if $k(x) \in K(x)$ for all $x \in \mathcal{D}$. Note that for a controller k to render the set $\mathcal{S}$ invariant, it must necessarily be a selection from the point-to-set map K.

Additionally, the controller k must come from the set of continuous viable controllers $\mathcal{U}$ in order for solutions of the closed loop system to exist. Furthermore, continuity of k is also necessary to apply the differential inequality in Proposition 1 and to satisfy the proposed definition of positive controlled invariance.

Theorem 6. *Given the control affine system* (66)*, consider a nonempty* $\mathcal{S} = \{x \in \mathcal{D} : h(x) \geq 0\}$ *for some continuously differentiable* $h : \mathcal{D} \to \mathbb{R}$*. If* h *is a MCBF as in Definition 5 and there exists a continuous controller* $k \in \mathcal{U}$ *such that* k *is a selection of* K*, then* $\mathcal{S}$ *is positively controlled invariant.*

Proof. Analogous to the proof of Theorem 1. □

To guarantee existence of a continuous controller k, K being nonempty is not sufficient, and additional conditions on K must be assumed. The next theorem gives sufficient conditions on the existence of a continuous controller k that is a selection of K and therefore can be used to satisfy Theorem 6 to render $\mathcal{S}$ positively invariant.

We introduce *strictly quasiconvex* [44] functions as functions $e : \mathcal{D} \to \mathbb{R}$ that satisfy

$$e(u_1) < e(u_2) \implies e(\theta u_1 + (1-\theta)u_2) < e(u_2) \tag{71}$$

for $\theta \in (0,1)$. We use strictly quasiconvex functions to generalize linear input constraints to a certain class of convex input constraints.

With U characterized as in (68), we also denote the strict interior $K_I : \mathcal{D} \rightrightarrows \mathbb{R}^m$ as

$$\begin{aligned} K_I(x) = \{u \in \mathbb{R}^m : e_i(x,u) < 0 \quad i = 1, \ldots, q, \\ L_f h(x) + L_g h(x) u > \mu(h(x))\}. \end{aligned} \tag{72}$$

using $e_i(x,u)$ directly in lieu of the admissible input set $U(x)$.

Proposition 6. *Given* U *defined as in* (68)*, assume the constraint functions* e_i *are all continuous in* x *and* u *and strictly quasiconvex in* u *for each fixed* x*. If* $K_I(x)$ *is nonempty for each* x*, then there exists a continuous controller* k *that is a selection of* K*.*

Proof. Define $e_0 : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}$ according to

$$e_0(x,u) = -L_f h(x) - L_g h(x) u + \mu(h(x)). \tag{73}$$

Notice that e_0 is also continuous in x and u and strictly quasiconvex in u for each fixed x. The viable control map K defined in (70) can then be described as

$$K(x) = \{u \in \mathbb{R}^m : e_i(x,u) \leq 0 \text{ for all } i = 0, \ldots, q\}. \tag{74}$$

Because $K_I(x)$ is assumed to be nonempty for each x, and all e_i are continuous and strictly quasiconvex in u for each fixed x, the closure $\overline{K_I(x)} = K(x)$ for all $x \in \mathcal{D}$ [44, Lemma 5], [42], and therefore K is a l.s.c map [42, Thm 13]. Furthermore, since all e_i are continuous in x and u and strictly quasiconvex in u for each fixed x, it holds that $K(x)$ is a closed, convex set in $\mathbb{R}^m$ for all $x \in \mathcal{D}$. Because K is a l.s.c point-to-set map that maps to closed, convex subsets, there exists a continuous controller k that is a selection of K [48, 1.11 Thm 1]. □

Proposition 6 is based on the well known Michael's selection theorem [47]. Other selection theorems exist, e.g. see [48], which can give existence conditions for different types of controllers.

Usually, a controller k is selected from K based on some optimality criteria. A common approach for safety based control is to first obtain a *nominal controller* $k_{nom} : \mathcal{D} \to \mathbb{R}^m$ that is not verified for either guaranteeing invariance or satisfying input constraints. The nominal controller is then used within an optimization program in which $\hat{k}$ is selected from K, while minimizing loss from $\hat{k}(x)$ and $k_{nom}(x)$ at each x, that is,

$$\hat{k}(x) = \underset{u \in K(x)}{\operatorname{argmin}}\, \ell(u, k_{nom}(x)) \tag{75}$$

where $\ell : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^+$ is some loss function.

Properties of the controller $\hat{k}$ can be analyzed as a selection of K, and conditions on K can be formulated to guarantee continuity of $\hat{k}$. In [4] and [49], Lipschitz continuity of controllers specifically for quadratic programs regarding a minimum norm controller of $k_{nom} \equiv 0$ was explored. In [49], continuity of a controller defined by a quadratic program was also explored with linear input constraints. In this paper, however, continuity of a controller is defined by a general nonlinear optimization program, with a quadratic cost as a special case.

The next proposition shows continuity of a pointwise optimal controller defined by a general nonlinear program.

Proposition 7. *Consider* $\hat{k}$ *defined by the optimization problem* (75)*. If* K *defined in* (70) *is a continuous, nonempty point-to-set-map, the nominal controller* k_{nom} *is continuous,* $\ell(u_1, u_2)$ *is continuous in* u_1 *and* u_2*, and there exists a unique minimizer* $u \in K(x)$ *of* $\ell(\cdot, k_{nom}(x))$ *for each* $x \in \mathcal{D}$*, then the controller* $\hat{k}$ *is continuous.*

Proof. Since k_{nom} is a continuous function, it trivially induces the continuous singleton point-to-set map $k'_{nom}(x) = \{k_{nom}(x)\}$. Since k'_{nom} and K are continuous point-to-set maps, the Cartesian product

$$K \times k'_{nom} : \mathcal{D} \rightrightarrows \mathbb{R}^m \times \mathbb{R}^m \tag{76}$$

is a continuous point-to-set map as well [41, Sec 6.4, Thm 4, 4']. Define

$$V(x) = \inf\{\ell(u_1, u_2) : (u_1, u_2) \in K(x) \times k'_{nom}(x)\} \tag{77}$$

and let the optimal selection function be

$$\Phi(x) = \{(u_1, u_2) \in K(x) \times k'_{nom}(x) : \ell(u_1, u_2) = V(x)\}. \tag{78}$$

By assumption of a unique minimizer u_1 for a fixed $u_2 = k_{nom}(x)$, Φ is a singleton point-to-set map. We show that Φ is a u.s.c map. Since $K \times k'_{nom}$ is a continuous point-to-set map and ℓ is a continuous function, V is a continuous function [41, Max Thm 4.2]. The point-to-set map

$$\begin{aligned} \Delta(x) = \{(u_1, u_2) \in K(x) \times k'_{nom}(x) : \\ V(x) - \ell(u_1, u_2) \leq 0\} \end{aligned} \tag{79}$$

is a closed point-to-set map, since V is continuous in x and ℓ is continuous in u_1 and u_2 [42, Thm 10]. Notice that

$$\Phi(x) = (K(x) \times k'_{nom}(x)) \cap \Delta(x). \tag{80}$$

Because K is assumed to be continuous map and Δ is a closed map, Φ is an u.s.c map [41, Sec 6.1 Thm 7]. Since Φ is an u.s.c singleton point-to-set map, it directly induces a continuous function Φ' defined by $\Phi(x) = \{\Phi'(x)\}$ [41]. Take

$$\hat{k}(x) = (p \circ \Phi')(x) \tag{81}$$

where p is the projection function from (u_1, u_2) to u_1. It follows that the optimal controller $\hat{k}$ is a continuous function since Φ' is continuous. □

A point-to-set map formulation allows for analyzing properties regarding a general pointwise optimization of a safe controller. However, in real time applications, a quadratic program can be utilized to minimize the Euclidean distance between the optimal and nominal controller [2]. The next proposition gives practical conditions on when the quadratic program gives a continuous controller.

Proposition 8. *Given U as defined in (68), assume that $\bigcup_{x\in\mathcal{D}} U(x)$ is compact, e_i are all continuous and strictly quasiconvex functions in u for each fixed x, and k_{nom} is continuous in x. If $K_I(x)$ as defined in (72) is nonempty for each x, the controller $\hat{k}$ defined by the quadratic program*

$$\hat{k}(x) = \underset{u\in K(x)}{\operatorname{argmin}} \; \left\|u - k_{nom}(x)\right\|^2 \tag{82}$$

is continuous.

Proof. It is shown in the proof of Proposition 6 that K is an l.s.c point-to-set map that maps to convex sets in $\mathbb{R}^m$. As all e_i are continuous, K is a closed mapping [42, Thm 10], and since K is assumed to map into the compact set $\bigcup_{x\in\mathcal{D}} U(x)$, K is u.s.c [41, Sec 6.1 Thm 7]. Therefore K is a continuous point-to-set map. Since K maps to a convex set and $\ell(u, k_{nom}(x)) = \left\|u - k_{nom}(x)\right\|^2$ is strictly convex in u for a fixed $k_{nom}(x)$ for each $x \in \mathcal{D}$, there exists at most one solution to the quadratic program. Because K is a continuous, nonempty point-to-set map, k_{nom} is continuous in x, and ℓ is continuous in u_1 and u_2 and there is a unique minimizer of $\ell(\cdot, k_{nom}(x))$ for each $x \in \mathcal{D}$, by Proposition 7, $\hat{k}$ is a continuous controller that is a selection of K. □

The next example verifies that a program with a quadratic cost and linear constraints returns a continuous controller.

Example 10. Consider a system $\dot{x} = u$ for the domain $\mathcal{D} = (-1, \infty)$, a barrier function $h(x) = x$, and a minimal function $\mu(w) = -w$. Furthermore, let the input constraints be a state-independent box constraint $U(x) = \{u \in \mathbb{R} : -1 \le u \le 1\}$ and the desired nominal controller be $k_{nom}(x) \equiv 0$. Then we can define the following quadratic program to generate an optimal controller

$$\begin{aligned}\hat{k}(x) = \underset{u\in\mathbb{R}^m}{\operatorname{argmin}} \; & \left\|u\right\|^2 \\ \text{s.t.} \quad & u \ge \frac{-L_f h(x) + \mu(h(x))}{L_g h(x)} = -x \\ & u \ge -1 \\ & -u \ge -1.\end{aligned}$$

Solving for the explicit controller gives

$$\hat{k}(x) = \begin{cases} -x & 0 \ge x > -1 \\ 0 & x \ge 0 \end{cases} \tag{83}$$

Notice all of the assumptions for Proposition 8 are satisfied, and indeed the resulting controller k is continuous in x. We also observe that there is no feasible solution on the interval $\{x < -1\}$.

VII. Conclusion

This paper presents minimal barrier functions, which stem from scalar differential inequalities, to give the minimum assumptions for utilizing a continuously differentiable barrier function. We have characterized a class of comparison systems viable for verifying invariance of sets defined via a barrier function inequality and have proposed equivalent computable conditions. This paper also exemplifies the connection between barrier flow constraints and ones stemming from set-based Lyapunov theory. We then further extend these minimal barrier functions to time varying and control formulations. We also propose relevant conditions for the existence of a continuous controller that can be used for a minimal control barrier function.

While directly associated with Lyapunov theory, with a comparable domain-wide flow inequality, this paper also explores the relationship of minimal barrier functions and the historic theory for set-invariance verification developed by Nagumo, Bony, Brezis and others. By formulating necessary conditions for the existence of minimal barrier functions, the relation between the assumptions of both approaches is also elucidated. By characterizing this relationship with the classical approach of verifying set invariance and rooting the proposed formulation directly in differential inequalities, this paper aims to provide a theoretical foundation for minimal barrier functions.

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