Day 7: Eigen Decomposition & Spectral Analysis

Math for AI Fundamentals (30-Day Series)

1. The Core Idea

For a square matrix $A \in \mathbb{R}^{n \times n}$, an eigenvector v and eigenvalue λ satisfy:

$$Av = \lambda v$$

This means: applying A to v only stretches or shrinks it, without changing its direction.

Eigen decomposition expresses A as:

$$A = Q \Lambda Q^{-1}$$

where Q is the matrix of eigenvectors and Λ is a diagonal matrix of eigenvalues.

This turns a complicated transformation into a set of independent scalings.

2. Geometric Intuition

- Each eigenvector is a **special direction** where the transformation acts as pure scaling. - Each eigenvalue tells **how much stretching or compressing** happens along that direction. - If an eigenvalue is 0, the transformation collapses space onto a lower-dimensional subspace. - Negative eigenvalues indicate a flip (reflection) along the eigenvector direction.

3. Worked Example: Decomposing a Matrix

Consider the matrix:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Step 1: Characteristic Equation

Solve $\det(A - \lambda I) = 0$:

$$\det \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2$$

$$= \lambda^2 - 7\lambda + 10 = 0$$

Thus, $\lambda_1 = 5$, $\lambda_2 = 2$.

Step 2: Eigenvectors

For $\lambda_1 = 5$:

$$(A - 5I)v = 0 \quad \Rightarrow \quad \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow y = x$. So one eigenvector is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For $\lambda_2 = 2$:

$$(A - 2I)v = 0 \quad \Rightarrow \quad \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$\Rightarrow y = -2x$. So one eigenvector is $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Step 3: Construct Decomposition

$$Q = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

Then:

$$A = Q\Lambda Q^{-1}$$

So, A is fully described by its eigenvalues and eigenvectors: scaling by 5 along $[1, 1]$ and scaling by 2 along $[1, -2]$.

4. Applications in AI

- **Dimensionality Reduction (PCA):** Covariance matrix eigenvectors = directions of maximum variance.
- **Optimization:** Eigenvalues of the Hessian classify minima, maxima, and saddle points.
- **Graph Learning:** Laplacian eigenvalues uncover community structure.
- **Neural Network Stability:** Spectral radius (largest eigenvalue) controls exploding/vanishing gradients in RNNs.

5. Why This Matters

Eigen decomposition transforms abstract linear algebra into practical tools for diagnosing, simplifying, and stabilizing AI systems.