Best of Both Worlds: Ex-Ante and Ex-Post Fairness in Resource Allocation

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Abstract

The authors study a resource allocation problem where indivisible goods must be allocated among agents with additive valuations, while striving to achieve fairness. With randomized allocations, it is possible to achieve *envy-freeness* (a notion of fairness), which states that no agent should prefer any other agent's allocation to her own. Whereas, with deterministic allocations, achieving exact fairness like envy-freeness is impossible, but approximate fairness such as envy-freeness up to one good can be guaranteed. The key contribution of this work is to prove there always exists a randomized allocation such that ex-ante envy-freeness can be achieved with ex-post envy-freeness up to one good and also compute one such allocation through the proposed algorithm called *Recursive Probabilistic Serial*. The authors also show an impossibility result in the case an additional requirement of economic efficiency is imposed. However, economic efficiency can be achieved along with ex-ante fairness if ex-post fairness guarantees are slightly relaxed. Finally, they also characterize the Maximum Nash Welfare allocation rule in terms of a recently introduced fairness guarantee.

1 Introduction

The central problem in fair allocation is to fairly allocate a set of goods among a set of agents who have different preferences over the goods. An agent's preferences are typically expressed through an *additive* valuation function. A desirable property for a fair allocation is *envy-freeness*, where no agent values the share of another agent more than the share allocated to herself. Ex-ante fairness means that fairness must hold in expectation before the actual allocation. Whereas, ex-post fairness means that fairness must hold regardless of the actual allocation.

The allocation problem setup consists of a set of indivisible goods M that must be allocated among a set of agents N, where an agent $i \in N$ values a good $j \in M$ at $v_{i,j}$. Here $M = \{1, 2, .., m\}$ and $N = \{1, 2, ..n\}$. An allocation is expressed as a matrix A of size $|N| \times |M|$, where the matrix element $A_{i,j}$ denotes the fraction of good j allocated to agent i.

An allocation is called *fractional* if $A_{i,j} \leq 1 \; \forall j \in M$. A fractional allocation is called *integral* if $A_{i,j} = 1 \; \forall i \in N, j \in M$. For an integral allocation, A_i is the set of items assigned to agent i as is defined as $A_i = \{j \in M : A_{i,j} = 1\}$.

A *randomized allocation* is a probability distribution over integral allocations, and naturally induces a fractional allocation. A randomized allocation is ex-ante fair if the induced fractional allocation is fair, and is ex-post fair if every integral allocation in its support is fair.

An allocation A is said to be *envy-free (EF)* [Foley, 1967] if $\forall i, h \in N \quad v_i(A_i) \geq v_i(A_h)$.

An allocation A is *Envy-free upto one good (EF1)* [Lipton et al., 2004, Budish, 2011] if $\forall i, h \in N$ with $A_h \neq \phi$, $\exists j \in A_j$ s.t. $v_i(A_i) \geq v_i(A_h \backslash j)$.

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An integral allocation A is *envy-free up to one good more-and-less* (EF_1^1) [Barman and Krishnamurthy, 2019] if for every pair of agents $i, h \in N$ such that $A_h \neq \phi$, we have $v_i(A_i \cup \{j_i\} \geq v_i(A_h \backslash \{j_h\})$.

An allocation A is said to be Prop1 if it is *proportional up to one good* [Conitzer et al., 2017].

The authors also prove the following proposition: *If a randomized allocation is ex-ante Pareto Optimal (PO), then it is also ex-post fractional Pareto Optimal (fPO) [Barman et al., 2018].*

The key question addressed in this work is that *does there always exist a randomized allocation that is ex-ante EF and ex-post EF1?*

2 Review of relevant work

A large amount of literature focuses on finding exact ex-ante fair randomized allocations and finding approximately ex-post fair allocations separately. However, little work focuses on combining the two problems. Aleksandrov et al. [2015] consider randomized allocation mechanisms for an online fair division problem and analyze their ex-ante and ex-post fairness guarantees but restrict their analysis to binary utilities. Budish et al. [2013] study the problem of implementing a general class of random allocation mechanisms subject to ex-post constraints, but do not consider ex-post axiomatic guarantees. Hylland and Zeckhauser [1979] introduced the idea of constructing a fractional assignment and implementing it as a lottery over pure assignments. Both ex-ante and ex-post fairness and efficiency guarantees provided by mechanisms in this setting have been addressed extensively, but most of this work studies ordinal utilities and does not consider approximate notions of ex-post fairness.

Bogomolnaia and Moulin [2001] devised the Probabilistic Serial algorithm which provides a systematic method to find fractional allocation that satisfies many properties, the key property being that resulting allocation is envy-free. The algorithm has the following key steps:

- Convert agents' valuations into ordinal preferences over goods.
- Agents simultaneously start "eating" or "consuming" their most good at the rate of one good/unit time.
- As soon as a good is exhausted, all agents who were eating it switch to their next most preferred good that is not yet exhausted.

The mechanism results in an ex-ante EF allocation because at every point of time, each agent is eating its most preferable good and hence she does not envy other agents as those good are either equally or less preferable. This mechanism only gives a fractional allocation but does provide a method to "implement" this fractional allocation using randomization over integral allocations. Moreover, the allocation is not guaranteed to be ex-post EF1 as well, which is a challenge that is overcome by the work presented in this paper.

3 Key Contributions

The key result of this work is the development of a novel algorithm, named *Recursive Probabilistic Serial*, which produces an ex-ante EF distribution over deterministic allocations that each satisfy EF1. It is an adaptation of the Probabilistic Serial algorithm discussed in the previous section. The key steps in the vanilla version of Recursive Probabilistic Serial algorithm are as follows:

- Run simultaneous eating procedure for 1 unit of time such that each agent eats a total of one good.
- Use Birkhoff-von Neumann theorem [Birkhoff, 1946, Von Neumann, 1953] to implement this fractional allocation as a distribution over integral allocations in which each agent gets one good.
- Sample an integral allocation from this distribution, remove the allocated goods, and repeat 1-2 for the remaining goods.

Since the third step involves sampling, this is a randomized algorithm. It runs in polynomial time and returns an EF1 allocation (but not a distribution of allocations). The distribution implemented by the algorithm is EF in expectation. However, using Caratheodory's theorem, it is possible to slightly modify the above vanilla version to efficiently compute a distribution over integral allocations with polynomial support. Hence the authors show that there always exists a randomized allocation that is ex-ante EF and ex-post EF1 and one such allocation can be computed in polynomial time.

The authors also consider different combinations of other notions of ex-ante fairness/efficiency and ex-post fairness/efficiency. For example, they show that it is impossible to achieve ex-ante PO in addition to ex-ante EF and ex-post EF1 simultaneously. Similarly analysis with other types of ex-ante and ex-post fairness and efficiency guarantees are shown. The details are not mentioned here for brevity.

Finally, the authors obtain a characterization of the Maximum Nash Welfare (MNW) rule. It is already known that fractional MNW allocation is Ex-ante Group Fair and can be computed in strongly polynomial time . The authors start with this fractional MNW allocation and show that the fractional allocation can be decomposed over integral allocations that are Prop1 + EF_1^1 by using a generalization of the Birkhoff-von Neumann theorem due to Budish [2011].

4 Conclusion

In conclusion, the authors developed a novel algorithm called Recursive Probabilistic Serial which produces an ex-ante EF distribution over deterministic allocations that each satisfy EF1. They analyse whether guarantees can hold with different combinations of fairness and efficiency as well. Finally they obtain a characterization of the MNW allocation rule in terms of Prop1 and EF_1^1 guarantees.

The authors also address some open questions: Does there always exist a randomized allocation that is ex-ante EF, ex-post EF1, and ex-post PO? What about ex-ante Prop, ex-post EF1, and ex-post PO?

The framework presented in this work can be extended to other problems like chore division, fair public decision-making, non-additive fair division, etc.

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