MATHEMATICS PREREQUISITE

Constraint optimization, Covariance matrix, Calculus

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01

OPTIMIZATION

PROBLEM & SOLUTION

$$C = \frac{B^3 + C^2 + A}{3BA} = \frac{C^3 + 5CA}{2CA} = C^4 + 2 + D = 3C4$$

1) Bankruptcy problems ask for how to "fairly" distribute $E to honest claimants whose claims exceed the amount $E. For example, A claims $30, B claims $50 and C claims $120 and there is only $160 to distribute.

SIN°

Progress:

PROBLEM & SOLUTION

$$C = \frac{B^3 + C^2 + A}{3BA} = \frac{C^3 + 5CA}{2CA} = C^4 + 2 + D = 3C4$$

2) A gardener has 20 meters of fence-material and wants to fence a rectangle shaped garden with maximal area. Furthermore, the garden will be located next to the house such that the house will serve as one of the four walls.

SIN°

Progress:

PROBLEM & SOLUTION

$B^3 = CD + DA$

$B^3 = (D - C \sin B)$

$B^3 = D^2 - 3A \cos B^2 + A \sin B$

$B^3 = D^2 - 4A \cos B^3 + C \sin B$

$B^3 = C^3 - A^2 - 3 \cos B$

CONSTRAINT OPTIMIZATION

IS

THE SOLUTION .

PROBLEM & SOLUTION

$B^3 = CD + DA$
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$B^3 = D^2 - 4A \cos B^3 + C \sin B$
$B^3 = C^3 - A^2 - 3 \cos B$

For bankruptcy problem

Solution could be -

a. Try to equalize the amount given (gain) to each claimant but without giving the claimant more than the claimant asks for.

b. Try to equalize the loss to each claimant but without asking the claimant to subsidize the settlement by adding money to E to make this possible.

c. One could give each claimant an amount proportional to his/her claim.

PROBLEM & SOLUTION

$$B^3 = CD + DA$$
$$B^3 = (D - C \sin B)$$
$$B^3 = D^2 - 3A \cos B^2 + A \sin B$$
$$B^3 = D^2 - 4A \cos B^3 + C \sin B$$
$$B^3 = C^3 - A^2 - 3 \cos B$$

<u>For gardener fencing problem</u>

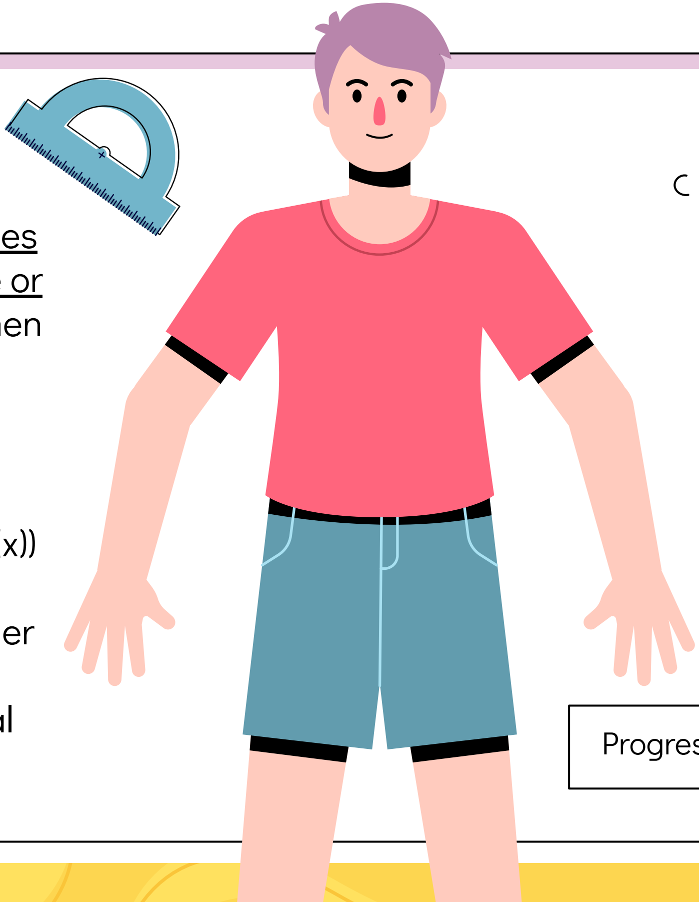
Solution could be -

The problem then reads:

Maximize f(a,b) = a*b subject to

2a+b=20

LET'S DISCUSS OPTIMIZATION !



According to **Wikipedia**, optimization is the process of methodically selecting input values from an authorised set in order to maximise or minimise the value of a real function. So, when we talk about optimization, we're constantly **looking for the greatest answer**.

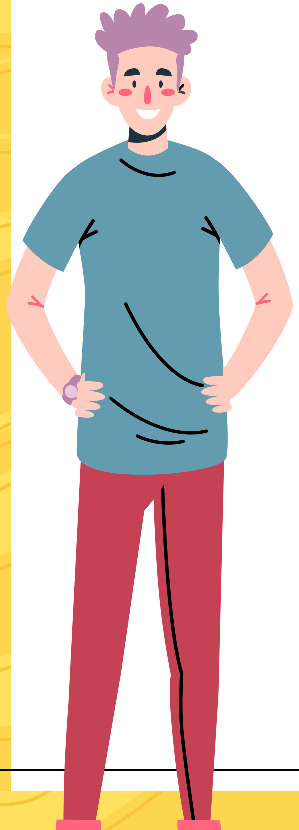
So, let's say someone is interested in some functional form (for example, in the form of f(x)) and is looking for the best way to solve it. What does "best" imply now? One could either say he is interested in **minimizing** this functional form or **maximizing** this functional form.

$$C = \frac{B^3 + C^2 + A}{3BA} = \frac{C^3 + 5CA}{2CA} = C^4 + 2 + D = 3C4$$

Progress:

WHY OPTIMIZATION ?

$$X_1 + X_3 = -C + \sqrt{B^2 + 2 - 4AC}$$
$$= -5 + \sqrt{3^2 + 2 - 4.32}$$
$$= 5 \pm \sqrt{3 - 4.}$$



In real life, optimization helps improve the efficiency of a system. It is used in a myriad of areas including medicine, manufacturing, transportation, supply chain, finance, government, physics, economics, artificial intelligence, etc.

In an optimization model, the goal can be to minimize cost in a production system.

In a hospital, the goal can be to minimize the wait time for patients in an emergency room before they are seen by a doctor.

In Marketing, the goal can be to maximize the profit obtained by targeting the right customers under budget and operational conditions.

In a Humanitarian Operation the goal would be to reach as many affected people as quickly as possible to distribute resources water, food, and medical services by designing optimal routes.

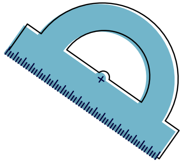
It also serves as a backbone for data science algorithms. It is at the heart of almost all machine learning and statistical techniques used in data science.

WHY OPTIMIZATION IN MACHINE LEARNING ?

- Optimization plays an important part in a machine learning project in addition to fitting the learning algorithm on the training dataset.

  The step of preparing the data prior to fitting the model and the step of tuning a chosen model also can be framed as an optimization problem. In fact, an entire predictive modeling project can be thought of as one large optimization problem.

$$C = \frac{B^3 + C^2 + A}{3BA} = \frac{C^3 + 5CA}{2CA} = C^4 + 2 + D = 3C4$$



Progress:

WHY OPTIMIZATION IN MACHINE LEARNING ?

Data Preparation as Optimization

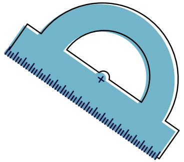
- Data preparation involves transforming raw data into a form that is most appropriate for the learning algorithms.

This might involve scaling values, handling missing values, and changing the probability distribution of variables.

Transforms can be made to change representation of the historical data to meet the expectations or requirements of specific learning algorithms. Yet, sometimes good or best results can be achieved when the expectations are violated or when an unrelated transform to the data is performed.

<u>We can think of choosing transforms to apply to the training data as a search or optimization problem of best exposing the unknown underlying structure of the data to the learning algorithm.</u>

$$C = \frac{B^3 + C^2 + A}{3BA} = \frac{C^3 + 5CA}{2CA} = C^4 + 2 + D = 3C4$$



Progress:

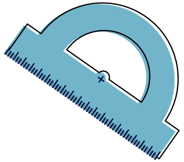
WHY OPTIMIZATION IN MACHINE LEARNING ?

Hyperparameter Tuning as Optimization

$$C = \frac{B^3 + C^2 + A}{3BA} = \frac{C^3 + 5CA}{2CA} = C^4 + 2 + D = 3C4$$

- Machine learning algorithms have hyperparameters that can be configured to tailor the algorithm to a specific dataset.

Although the dynamics of many hyperparameters are known, the specific effect they will have on the performance of the resulting model on a given dataset is not known. As such, it is a standard practice to test a suite of values for key algorithm hyperparameters for a chosen machine learning algorithm.



This is called hyperparameter tuning or hyperparameter optimization.

It is common to use a naive optimization algorithm for this purpose, such as a random search algorithm or a grid search algorithm.

Progress:

WHY OPTIMIZATION IN MACHINE LEARNING ?

Model Selection as Optimization

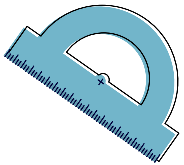
Model selection involves choosing one from among many candidate machine learning models for a predictive modeling problem.

Really, it involves choosing the machine learning algorithm or machine learning pipeline that produces a model. This is then used to train a final model that can then be used in the desired application to make predictions on new data.

This process of model selection is often a manual process performed by a machine learning practitioner involving tasks such as preparing data, evaluating candidate models, tuning well-performing models, and finally choosing the final model.

This can be framed as an optimization problem that subsumes part of or the entire predictive modeling project.

$$C = \frac{B^3 + C^2 + A}{3BA} = \frac{C^3 + 5CA}{2CA} = C^4 + 2 + D = 3C4$$



Progress:

COMPONENT OF OPTIMIZATION

OBJECTIVE FUNCTION

The first component is an objective function f(x) which we are trying to either maximize or minimize. In general, we minimization problems because if you have a maximization problem with f(x), it can be converted to a minimization problem with -f(x).

DECISION VARIABLE

Decision variables(x): The second component is the decision variables which we can choose to minimize the function. So, we usually write this as min f(x).

CONSTRAINT

Constraints($a \leq x \leq b$): The third component is the constraint which constrains this x to a certain set of values.

Progress:

TYPES OF OPTIMIZATION

Depending on type of objective function, constraints and decision variables, optimization can be of various types.

1. IF THE DECISION VARIABLE(X) IS A CONTINUOUS VARIABLE:
2. IF THE DECISION VARIABLE(X) IS AN INTEGER VARIABLE:
3. IF THE DECISION VARIABLE(X) IS A MIXED VARIABLE:

Progress:

TYPES OF OPTIMIZATION

1. IF THE DECISION VARIABLE(X) IS A CONTINUOUS VARIABLE:

A variable x is said to be continuous if it takes an infinite number of values. In this case, x can take an infinite number of values between -2 to 2.

$$\min f(x),\ x \in (-2, 2)$$

- **Linear programming problem:** If the decision variable(x) is a continuous variable and if the objective function(f) is linear and all the constraints are also linear then this type of problem known as a linear programming problem. So, in this case, the decision variables are continuous, the objective function is linear and the constraints are also linear.

- **Nonlinear programming problem:** If the decision variable(x) remains continuous; however, if either the objective function(f) or the constraints are non-linear then this type of problem known as a non-linear programming problem. So, a programming problem becomes non-linear if either the objective or the constraints become non-linear.

TYPES OF OPTIMIZATION

2. IF THE DECISION VARIABLE(X) IS AN INTEGER VARIABLE:

All numbers whose fractional part is 0 (zero) like -3, -2, 1, 0, 10, 100 are integers.

$$\min f(x),\ x \in [0, 1, 2, 3]$$

- **Linear integer programming problem:** If the decision variable(x) is an integer variable and if the objective function(f) is linear and all the constraints are also linear then this type of problem known as a linear integer programming problem. So, in this case, the decision variables are integers, the objective function is linear and the constraints are also linear.

- **Nonlinear integer programming problem:** If the decision variable(x) remains integer; however, if either the objective function(f) or the constraints are non-linear then this type of problem known as a non-linear integer programming problem. So, a programming problem becomes non-linear if either the objective or the constraints become non-linear.

- **Binary integer programming problem:** If the decision variable(x) can take only binary values like 0 and 1 only then this type of problem known as a binary integer programming problem.

$$\min f(x),\ x \in [0, 1]$$

TYPES OF **OPTIMIZATION**

3. IF THE DECISION VARIABLE(X) IS A MIXED VARIABLE:

If we combine both continuous variable and integer variable then this decision variable known as a mixed variable.

$$\min f(x1, x2),\ x1 \in [0, 1, 2, 3] \text{ and } x2 \in (-2, 2)$$

- **Mixed-integer linear programming problem**: If the decision variable(x) is a mixed variable and if the objective function(f) is linear and all the constraints are also linear then this type of problem known as a mixed-integer linear programming problem. So, in this case, the decision variables are mixed, the objective function is linear and the constraints are also linear.
- **Mixed-integer non - linear programming problem**: If the decision variable(x) remains mixed; however, if either the objective function(f) or the constraints are non-linear then this type of problem known as a mixed-integer non - linear programming problem. So, a programming problem becomes non-linear if either the objective or the constraints become non-linear.

Mathematical Implementation For Constraint Optimization

LINEAR PROGRAMMING PROBLEM

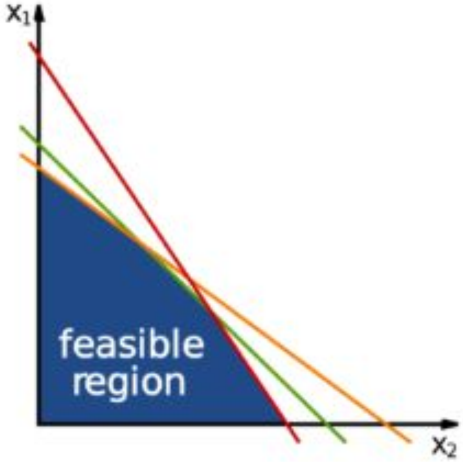
Sometimes one seeks to optimize (maximize or minimize) a known function (could be profit/loss or any output), subject to a set of linear constraints on the function. <u>Linear Programming Problems (LPP) provide the method of finding such an optimized function along with/or the values which would optimize the required function accordingly.</u>

It is one of the most important Operations Research tools. It is widely used as a decision making aid in almost all industries. There can be various fields of application of LPP, in the areas of Economics, Computer Sciences, Mathematics, etc.

LINEAR PROGRAMMING PROBLEM

Mathematical Formulation

Formulation of an LPP refers to translating the real-world problem into the form of mathematical equations which could be solved



LINEAR PROGRAMMING PROBLEM

Steps towards formulating a Linear Programming problem:

- **Step 1: Identify the 'n' number of decision variables which govern the behaviour of the objective function (which needs to be optimized).**
- **Step 2: Identify the set of constraints on the decision variables and express them in the form of linear equations /inequations. This will set up our region in the n-dimensional space within which the objective function needs to be optimized.** Don't forget to impose the condition of non-negativity on the decision variables i.e. all of them must be positive since the problem might represent a physical scenario, and such variables can't be negative.
- **Step 3: Express the objective function in the form of a linear equation in the decision variables.**
- **Step 4: Optimize the objective function either graphically or mathematically.**

TYPES OF LINEAR PROGRAMMING PROBLEM

Type of LPP	Constraints	Objective Function
Diet Problems	The specified nutritional requirement	The cost of food intake
Manufacturing Problems	Variables like work hour, material availability, production rate, etc.	The cost of production
Transportation Problems	The specific supply demand pattern	The cost of transportation
Optimal Assignment Problems	The number of employees, work hour of each employee, efficiency of employee etc.	The total assignment done

LINEAR PROGRAMMING PROBLEM

Problem Statement -

A calculator company produces a handheld calculator and a scientific calculator. Long-term projections indicate an expected demand of at least 150 scientific and 100 handheld calculators each day. Because of limitations on production capacity, no more than 250 scientific and 200 handheld calculators can be made daily.

To satisfy a shipping contract, a minimum of 250 calculators must be shipped each day. If each scientific calculator sold, results in a 20 rupees loss, but each handheld calculator produces a 50 rupees profit; then how many of each type should be manufactured daily to maximize the net profit?

LINEAR PROGRAMMING PROBLEM

Solution -

Let us approach it in a step by step manner.

Step 1: The decision variables: Since the question has asked for an optimum number of calculators, that's what our decision variables in this problem would be. Let,

x = number of scientific calculators produced

y = number of handheld calculators produced

Therefore, we have 2 decision variables in this problem, namely 'x' and 'y'.

Step 2: The constraints: Since the company can't produce a negative number of calculators in a day, a natural constraint would be:

$x \geq 0$

$y \geq 0$

LINEAR PROGRAMMING PROBLEM

However, a lower bound for the company to sell calculators is already supplied in the problem. We can note it down as:

$$x \geq 150$$
$$y \geq 100$$

We have also been given an upper bound for these variables, owing to the limitations on production by the company. We can write as follows:

$$x \leq 250$$
$$y \leq 200$$

Besides, we also have a joint constraint on the values of 'x' and 'y' due to the minimum order on a shipping consignment; given as:

$$x + y \geq 250$$

LINEAR PROGRAMMING PROBLEM

Solution -

Step 3: Objective Function: Clearly, here we need to optimize the Net Profit function. The Net Profit Function is given as:

$$P = -20x + 50y$$

Step 4: Solving the problem: the system here –

Maximization of $P = -20x + 50y$, subject to:

$$150 \le x \le 250$$
$$100 \le y \le 200$$
$$x + y \ge 250$$

We can solve it graphically or mathematically as per convenience.

LINEAR PROGRAMMING PROBLEM

Solution -

1. To draw constraint $x_1 + x_2 \geq 250 \rightarrow (1)$

Treat it as $x_1 + x_2 = 250$

When $x_1 = 0$ then $x_2 = ?$

$\Rightarrow (0) + x_2 = 250$

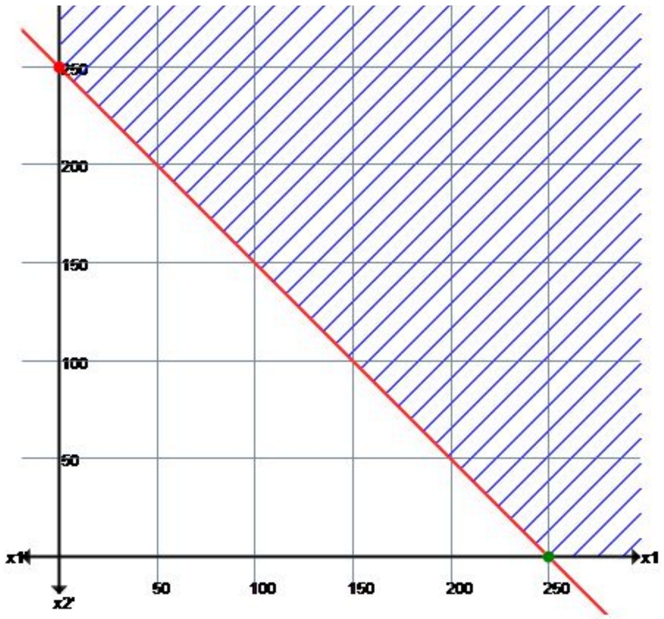
$\Rightarrow x_2 = 250$

When $x_2 = 0$ then $x_1 = ?$

$\Rightarrow x_1 + (0) = 250$

$\Rightarrow x_1 = 250$

x_1	0	250
x_2	250	0



LINEAR PROGRAMMING PROBLEM

The value of the objective function at each of these extreme points is as follows:

Extreme Point Coordinates (x_1, x_2)	Lines through Extreme Point	Objective function value $Z = -20x_1 + 50x_2$
$A(0, 250)$	$1 \rightarrow x_1 + x_2 \geq 250$ $2 \rightarrow x_1 \geq 0$	$-20(0) + 50(250) = 12500$
$B(250, 0)$	$1 \rightarrow x_1 + x_2 \geq 250$ $3 \rightarrow x_2 \geq 0$	$-20(250) + 50(0) = -5000$

Problem has an unbounded solution.

Note: In maximization problem, if shaded area is open-ended. This means that the maximization is not possible and the LPP has no finite solution. Hence the solution of the given problem is unbounded

NON LINEAR PROGRAMMING PROBLEM

Most mathematical techniques for solving nonlinear programming problems are very complex.Here we will discuss two of the more well known –**the substitution method** **and** **the method of Lagrange multipliers**.

SUBSTITUTION METHOD - Substitution method to solve constrained optimisation problem is used when constraint equation is simple and not too complex. For example substitution method to maximise or minimise the objective function is used when it is subject to only one constraint equation of a very simple nature.

In this method, we solve the constraint equation for one of the decision variables and substitute that variable in the objective function that is to be maximised or minimised. In this way this method converts the constrained optimisation problem into one of unconstrained optimisation problems of maximisation or minimisation.

NON LINEAR PROGRAMMING PROBLME

Problem Statement - Suppose a manager of a firm which is producing two products x and y, seeks to maximise total profits function which is given by the following equation

$$\pi = 50x - 2x2 - xy - 3y2 + 95y$$

Where x and y represent the quantities of the two products. The manager of the firm faces the constraints that the total output of the two products must be equal to 25. That is, according to the constraint, $x + y = 25$

Solution - To solve this constrained optimisation problem through substitution we first solve the constraint equation for x. Thus $x = 25 - y$

The next step in the substitution method is to substitute this value of $x = 25 - y$ in the objective function (i.e. the given profit function) which has to be maximised.

NON LINEAR PROGRAMMING PROBLME

Thus, substituting the constrained value of x in the profit function we have:

π = 50 (25 – y) – 2 (25 – y) 2 – (25 – y) y – 3y2 + 95y

= 1250 – 50y – 2 (625 / 50y + y2) – 25y + y2 – 3y2 + 95y

= 1250 – 50y – 1250 + 100y – 2y2 – 25y + y2 – 3y2 + 95y

= 12y – 4y2

To maximise the above profit function converted into the above unconstrained form we differentiate it with respect to y and set it equal to zero and solve for y.

Thus, dπ / dy =120 – 8y = 0 or, 8y = 120 , y = 15.

NON LINEAR PROGRAMMING PROBLME

Substituting y = 15 in the given constraint equation (x + y = 25) we have

X + 15 =25 => X = 25- 15 =10

Thus, given the constraint, profit will be maximised if the manager of the firm decides to produce 10 units of the product x and 15 units of the product y. We can find the total profits in this constrained optimum situation by substituting the values of x and y obtained in the given profit function. Thus,

π = 50x- 2x2 – xy – 3y2 + 95y

π = 50 x 10 – 2(10)2 – 10 x 15 – 3(15)2 + 95 x 15

= 500 – 200 – 150 – 675 + 1425

= 1925 – 1025 = 900

NON LINEAR PROGRAMMING PROBLME

LAGRANGE MULTIPLIER TECHNIQUE :

The substitution method for solving constrained optimisation problem cannot be used easily when the constraint equation is very complex and therefore cannot be solved for one of the decision variable. In such cases of constrained optimisation we employ the Lagrangian Multiplier technique.

In this Lagrangian technique of solving constrained optimisation problem, a combined equation called Lagrangian function is formed which incorporates both the original objective function and constraint equation.

NON LINEAR PROGRAMMING PROBLME

This Lagrangian function is formed in a way which ensures that when it is maximised or minimised, the original given objective function is also maximised or minimised and at the same time it fulfills all the constraint requirements. In creating this Lagrangian function, an artificial variable λ (Greek letter Lamda) is used and it is multiplied by the given constraint function having been set equal to zero. λ is known as Lagrangian multiplier.

Lagrangian Multiplier: Since Lagrangian function incorporates the constraint equation into the objective function, it can be considered as unconstrained optimisation problem and solved accordingly.

We will take an example constrained optimisation problem solved above by substitution method.

NON LINEAR PROGRAMMING PROBLME

Lagrange multiplier method, for solving constrained optimization problems:

Maximize (or minimize) : $f(x,y)$(or $f(x,y,z)$)

given : $g(x,y)=c$(or $g(x,y,z)=c$) for some constant c

Maximize (or minimize) : $f(x,y)$(or $f(x,y,z)$)given : $g(x,y)=c$(or $g(x,y,z)=c$) for some constant c

The equation $g(x,y)=c$ is called the constraint equation, and we say that x and y are constrained by $g(x,y)=c$. Points

(x,y) which are maxima or minima of $f(x,y)$ with the condition that they satisfy the constraint equation $g(x,y)=c$ are called constrained maximum or constrained minimum points, respectively.

NON LINEAR PROGRAMMING PROBLME

Problem Statement

For a rectangle whose perimeter is 20 m, use the Lagrange multiplier method to find the dimensions that will maximize the area.

Solution -

The area A of a rectangle with width x and height y is A=xy. The perimeter P of the rectangle is then given by the formula P=2x+2y. Since we are given that the perimeter P=20, this problem can be stated as:

$$\text{Maximize : } f(x,y)=xy$$

$$\text{given : } 2x+2y=20$$

NON LINEAR PROGRAMMING PROBLEM

As we saw with x and y representing the width and height, respectively, of the rectangle, this problem can be stated as:

Maximize : $f(x,y)=xy$

given : $g(x,y)=2x+2y=20$

Then solving the equation $\nabla f(x,y)=\lambda \nabla g(x,y)$

for some λ means solving the equations $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$, namely

$y=2\lambda$, $x=2\lambda$

NON LINEAR PROGRAMMING PROBLME

The general idea is to solve for λ in both equations, then set those expressions equal (since they both equal λ) to solve for x and y. Doing this we get

$$\frac{y}{2} = \lambda = \frac{x}{2} \Rightarrow x = y,$$

so now substitute either of the expressions for x or y into the constraint equation to solve for x and y:

$20=g(x,y)=2x+2y=2x+2x=4x \Rightarrow x=5 \Rightarrow y=5$

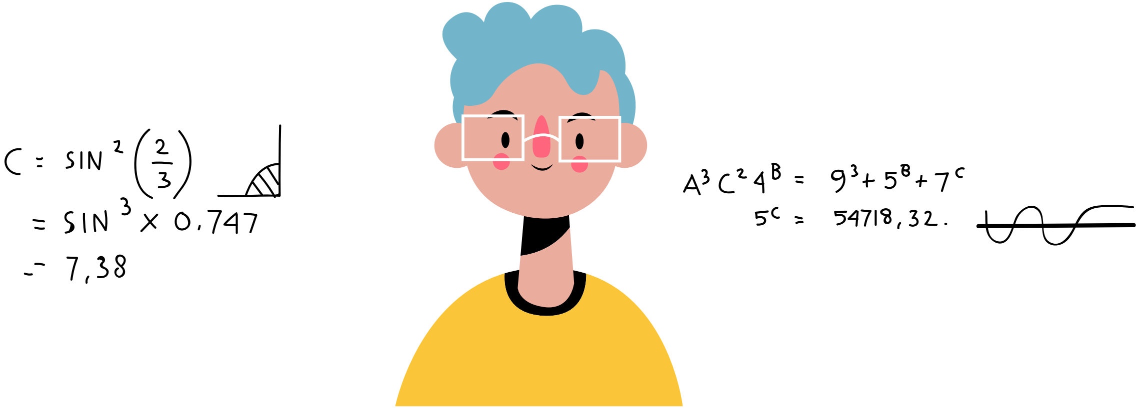
$20=g(x,y)=2x+2y=2x+2x=4x \Rightarrow x=5 \Rightarrow y=5$

There must be a maximum area, since the minimum area is 0 and $f(5,5)=25>0$, so the point (5,5) that we found (called a constrained critical point) must be the constrained maximum.

∴ The maximum area occurs for a rectangle whose width and height both are 5 m.

LINEAR VS NON-LINEAR PROGRAMMING PROBLME

LINEAR PROGRAMMING	NONLINEAR PROGRAMMING
A method to achieve the best outcome in a mathematical model whose requirements are represented by linear relationships	A process of solving an optimization problem where the constraints or the objective functions are nonlinear
Helps to find the best solution to a problem using constraints that are linear	Helps to find the best solution to a problem using constraints that are nonlinear



02 COVARIANCE MATRIX

WHAT IS COVARAINCE MATRIX?

$B^3 = CD + DA$
$B^3 = (D - C \sin B)$
$B^3 = D^2 - 3A \cos B^2 + A \sin B$
$B^3 = D^2 - 4A \cos B^3 + C \sin B$
$B^3 = C^3 - A^2 - 3 \cos B$

DEFINITION

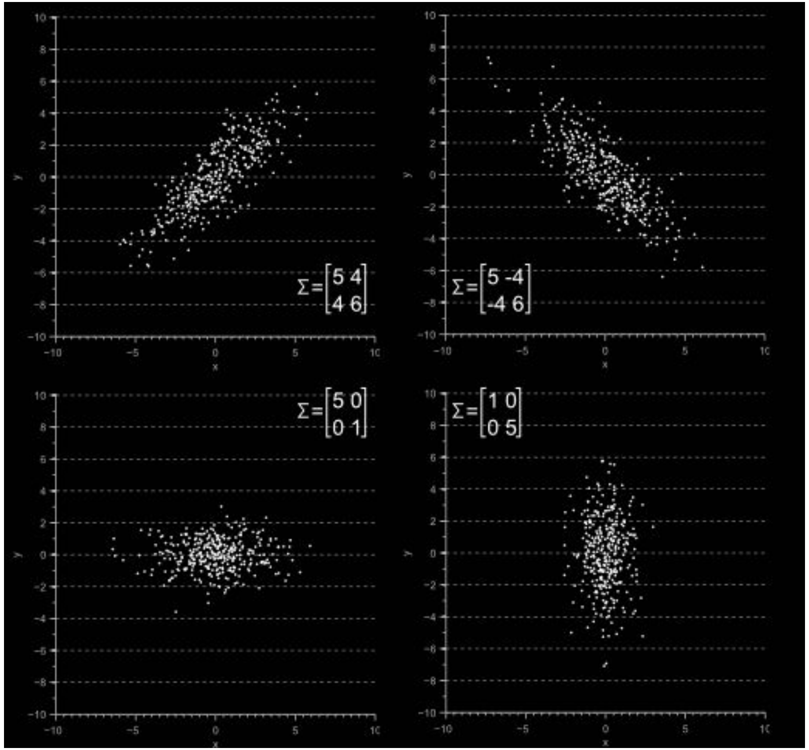
Covariance matrix is a type of matrix that is used to represent the covariance values between pairs of elements given in a random vector.

The covariance matrix can also be referred to **as the variance covariance matrix**. This is because the variance of each element is represented along the **main diagonal of the matrix**.

Covariance measures <u>how much two random variables vary together in a population</u>. When the population contains higher dimensions or more random variables, a matrix is used to describe the relationship between different dimensions. In a more easy-to-understand way, **covariance matrix is to define the relationship in the entire dimensions as the relationships between every two random variables.**

Progress:

FIGURE SHOWS HOW THE OVERALL SHAPE OF THE DATA DEFINES THE COVARIANCE MATRIX:



The covariance matrix defines the shape of the data. Diagonal spread is captured by the covariance, while axis-aligned spread is captured by the variance.

WHAT DOES THE COVARIANCE MATRIX TELL YOU?

It is a symmetric matrix that shows covariances of each pair of variables. The values in the covariance matrix show the distribution magnitude and direction of multivariate data in multidimensional space.

By controlling those values we can have information about how data spread among two dimensions.

Progress:

COVARIANCE MATRIX FORMULA

To determine the covariance matrix, the formulas for variance and covariance are required. ***Depending upon the type of data available, the variance and covariance can be found for both sample data and population data.*** These formulas are given below.

$$\begin{bmatrix} Var(x_1) & \cdots\cdots & Cov(x_n, x_1) \\ \vdots & \ddots & \vdots \\ Cov(x_n, x_1) & \cdots\cdots & Var(x_n) \end{bmatrix}$$

Population Variance: $var(x) = \dfrac{\sum_1^n (x_i - \mu)^2}{n}$

Population Covariance: $cov(x, y) = \dfrac{\sum_1^n (x_i - \mu_x)(y_i - \mu_y)}{n}$

Sample Variance: $var(x) = \dfrac{\sum_1^n (x_i - \bar{x})^2}{n-1}$

Sample Covariance: $cov(x, y) = \dfrac{\sum_1^n (x_i - \bar{x})(y_i - \bar{y})}{n-1}$

μ = mean of population data.

$\bar{x}$ = mean of sample data.

n = number of observations in the dataset.

xi = observations in dataset x.

COVARIANCE MATRIX FORMULA

Using these formulas, the **general form of a variance covariance matrix** is given as follows:

$$\begin{bmatrix} Var(x_1) & \ldots & Cov(x_1, x_n) \\ \vdots & \ddots & \vdots \\ Cov(x_n, x_1) & \ldots & Var(x_n) \end{bmatrix}$$

Covariance Matrix 2 × 2
A 2 × 2 matrix is one which has 2 rows and 2 columns. The formula for a 2 × 2 covariance matrix is given as follows:

$$\begin{bmatrix} var(x) & cov(x, y) \\ cov(x, y) & var(y) \end{bmatrix}$$

COVARIANCE MATRIX FORMULA

Covariance Matrix 3 × 3

If there are 3 datasets, x, y, and z, then the formula to find the 3 × 3 covariance matrix is given below

$$\begin{bmatrix} var(x) & cov(x,y) & cov(x,z) \\ cov(x,y) & var(y) & cov(y,z) \\ cov(x,z) & cov(y,z) & var(z) \end{bmatrix}$$

REAL LIFE USE CASE OF COVARIANCE MATRIX

Financial economics - The covariance matrix plays a key role in financial economics, especially in portfolio theory and its mutual fund separation theorem and in the capital asset pricing model. The matrix of covariances among various assets' returns is used to determine, under certain assumptions, the relative amounts of different assets that investors should (in a normative analysis) or are predicted to (in a positive analysis) choose to hold in a context of diversification.

In optimization - The evolution strategy, a particular family of Randomized Search Heuristics, fundamentally relies on a covariance matrix in its mechanism. The characteristic mutation operator draws the update step from a multivariate normal distribution using an evolving covariance matrix.

PROPERTIES OF COVARIANCE MATRIX

- A covariance matrix is always a square matrix. This means that the number of rows of the matrix will be equal to the number of columns.
- The matrix is symmetric. Suppose M is the covariance matrix then $M^T = M$.
- It is positive semi-definite. Let u be a column vector, u^T is the transpose of that vector and M be the covariance matrix then $u^T M u \geq 0$.
- All eigenvalues of the variance covariance matrix are real and non-negative.

Progress:

MATHEMATICAL IMPLEMENTATION STEPWISE

SIN² + 2 COS

The number of variables determines the dimension of a variance-covariance matrix. For example, if there are two variables (or datasets) it indicates that the covariance matrix will be 2 dimensional. Suppose the math and science scores of 3 students are given as follows:

Student	Math (X)	Science (Y)
1	92	80
2	60	30
3	100	70

The steps to calculate the covariance matrix for the sample are given below:

Progress:

MATHEMATICAL IMPLEMENTATION STEPWISE

SIN² + 2 COS

- Step 1: Find the mean of one variable (X). This can be done by dividing the sum of all observations by the number of observations. Thus, $(92 + 60 + 100) / 3 = 84$
- Step 2: Subtract the mean from all observations; $(92 - 84)$, $(60 - 84)$, $(100 - 84)$
- Step 3: Take the sum of the squares of the differences obtained in the previous step. $(92 - 84)^2 + (60 - 84)^2 + (100 - 84)^2$.
- Step 4: Divide this value by 1 less than the total to get the sample variance of the first variable (X). $var(X) = [(92 - 84)^2 + (60 - 84)^2 + (100 - 84)^2] / (3 - 1) = 448$

Progress:

MATHEMATICAL IMPLEMENTATION STEPWISE

SIN² + 2 COS

- Step 5: Repeat steps 1 to 4 to find the variances of all variables. Using these steps, var(Y) = 700.
- Step 6: Choose a pair of variables (X and Y).
- Step 7: Subtract the mean of the first variable (X) from all observations; (92 - 84), (60 - 84), (100 - 84).
- Step 8: Repeat step 7 for the second variable (Y); (80 - 60), (30 - 60), (70 - 60).
- Step 9: Multiply the corresponding observations. (92 - 84)(80 - 60), (60 - 84)(30 - 60), (100 - 84)(70 - 60).
-

Progress:

MATHEMATICAL IMPLEMENTATION STEPWISE

- Step 10: Add these values and divide them by (n - 1) to get the covariance. cov(x, y) = cov(y, x) = [(92 - 84)(80 - 60) + (60 - 84)(30 - 60) + (100 - 84)(70 - 60)] / (3 - 1) = 520.
- Step 11: Repeat steps 6 to 10 for different pairs of variables.
- Step 12: Now using the general formula for covariance matrix arrange these values in matrix form. Thus, the variance covariance matrix for the example is given as

$$\begin{bmatrix} 448 & 520 \\ 520 & 700 \end{bmatrix}$$

Progress:

STATES OF COVARIANCE

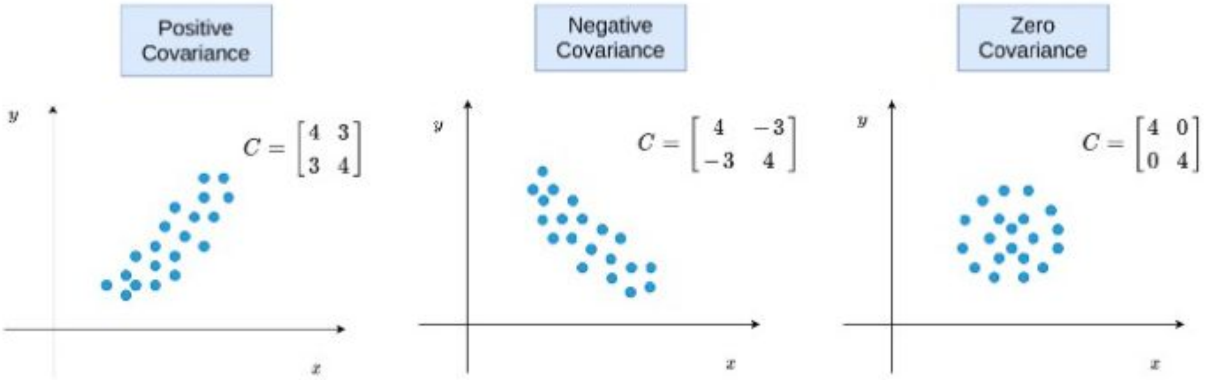
When ***Xi-Xmean*** and ***Yi- Ymean*** are both negative or positive at the same time, multiplication returns a positive value. If the sum of these values is positive, covariance gets found as positive.

It means ***variable X*** and ***variable Y*** variate in the same direction. In other words, if a value in variable X is higher, it is expected to be high in the corresponding value in variable Y too.

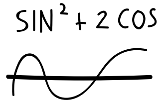
In short, there is a positive relationship between them. If there is a negative covariance, this is interpreted right as the opposite. That is, there is a negative relationship between the two variables.

STATES OF COVARIANCE

The covariance can only be zero when the sum of products of ***Xi-Xmean*** and ***Yi- Ymean*** is is zero. However, the products of ***Xi-Xmean*** and ***Yi- Ymean*** can be near-zero when one or both are zero. In such a scenario, there aren't any relations between variables.



IMPORTANT NOTES ON COVARIANCE MATRIX



- The covariance matrix depicts the variance of datasets and covariance of a pair of datasets in matrix format.
- The diagonal elements represent the variance of a dataset and the off-diagonal terms give the covariance between a pair of datasets.
- The variance covariance matrix is always square, symmetric, and positive semi-definite.
- The general formula to represent a covariance matrix is

$$\begin{bmatrix} Var(x_1) & \cdots & Cov(x_1, x_n) \\ \vdots & \ddots & \vdots \\ \vdots & & \vdots \\ Cov(x_n, x_1) & \cdots & Var(x_n) \end{bmatrix}.$$

Progress:

03 CALCULUS

$$x_1 + 2_A = 3\sqrt{5+2AB}$$
$$= 9\sqrt{12}$$

LIMIT...

$C = \sin^2\left(\frac{2}{3}\right)$

$= \sin^3 \times 0{,}747$

$= 7{,}38$

LIMIT INTRODUCTION

Definition - A number A is said to be limit of a function f(x) at x= a if for any arbitrarily chosen positive integer e, however small but not zero there exist a corresponding number & greater than zero such that: $|f(x) - A| < \in$ for all values of x for which $0 < |x-a| < \delta$ where $|x-a|$ means the absolute value of (x-a) without any regard to sign.

Example - Let us understand the definition of "limit" with the help of an example, consider the function f(x). Let the independent variable x take values near a given constant 'a'. Then, f(x) takes a corresponding set of values. Suppose that when x is close to 'a', the values of f(x) are close to some constant. Let us say f(x) can be made to differ arbitrarily small from 'a' by taking values of x that are sufficiently close to 'a' but not equal to 'a' and that is true for all such values of x. Then, f(x) is said to approach limit A as x approaches 'a'.

RIGHT HAND & LEFT HAND LIMIT

Limits & Functions

A function may approach two different limits. One where the variable approaches its limit through values larger than the limit and the other where the variable approaches its limit through values smaller than the limit. In such a case, the limit is not defined but the right and left-hand limit exist.

The left-hand limit of a function is the value of the function approaches when the variable approaches its limit from the left.

This can be written as - $\lim_{x \to a} f(x) = A^-$

The right-hand limit of a function is the value of the function approaches when the variable approaches its limit from the right.

This can be written as - $\lim_{x \to a} f(x) = A^+$

Progress:

RIGHT HAND & LEFT HAND LIMIT

The limit of a function exists if and only if the left-hand limit is equal to the right-hand limit.

$$\lim_{x \to a^{-1}} f(x) = \lim_{x \to a^{+}} f(x) = L$$

Progress:

PROPERTIES OF LIMIT

If limits

$\lim_{x \to a}$ f(x) and

$\lim_{x \to a}$ g(x) exists, then,

Law of Addition	$\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
Law of Subtraction	$\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$
Law of Multiplication	$\lim\limits_{x \to a} [f(x).\, g(x)] = \lim\limits_{x \to a} f(x).\, \lim\limits_{x \to a} g(x)$

PROPERTIES OF LIMIT

Law of Division	$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, where $\lim_{x \to a} g(x) \neq 0$
Law of Constant	$\lim_{x \to a} c = c$
Law of Root	$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$
Law of Power	$\lim_{x \to a} (f(x)^n = (\lim_{x \to a} f(x))^n$

Where, n is an integer.

Standard Results of Limits

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$

(ii) $\lim_{x \to 0} \frac{\sin x}{x} = 1$

(iii) $\lim_{x \to 0} \frac{\tan x}{x} = 1$

(iv) $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

(v) $\lim_{x \to \infty} \frac{1}{x^p} = 0, p \in (0, \infty)$

(vi) $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$

(vii) $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$

(viii) $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1$

(x) $\lim_{x \to 0} (1+x)^{1/x} = e$

(xi) $\lim_{x \to \infty} \frac{\sin x}{x} = 0$

(xii) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$

(xiii) $\lim_{x \to \infty} \sin x = \lim_{x \to \infty} \cos x =$ lies between -1 to 1.

EXAMPLE OF LIMIT

Q1) Find the limit. lim_(x-> 0) (tan x)/(sin x)

Solution: lim_(x-> 0) (tan x)/(sin x)

We know, tan x = sin x / cos x

= lim_(x-> 0) (1/cos x)

= 1

=> lim_(x-> 0) (tan x)/(sin x) = 1

Progress:

EXAMPLE OF LIMIT

Q2)

Let's try to find the limit:

$$\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h}$$

We rationalize the expression.
In this case by multiplying and dividing by the conjugate.

$$\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{a+h}\right)^2-\left(\sqrt{a}\right)^2}{h\left(\sqrt{a+h}+\sqrt{a}\right)} = \lim_{h\to 0} \frac{\cancel{a}+h-\cancel{a}}{h\left(\sqrt{a+h}+\sqrt{a}\right)}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}\left(\sqrt{a+h}+\sqrt{a}\right)} = \lim_{h\to 0} \frac{1}{\cancel{\sqrt{a+h}}^{\sqrt{a}}+\sqrt{a}} = \boxed{\frac{1}{2\sqrt{a}}}$$

Progress:

EXAMPLE OF LIMIT

Q3)

Calculate $\lim_{x\to 0} |x|$.

Solution

Recall from the earlier chapter 1

that $|x| = \begin{cases} -x & \text{if } x<0 \\ 0 & \text{if } x=0 \\ x & \text{if } x>0 \end{cases}$

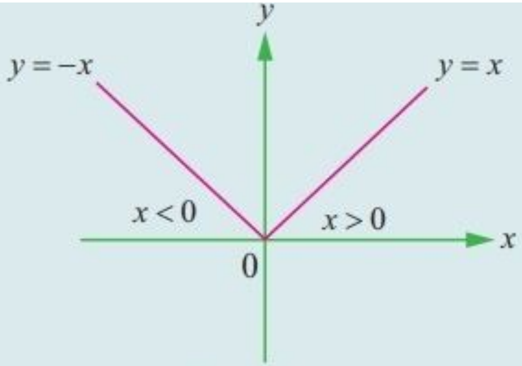


Fig. 9.10

If $x > 0$, then $|x| = x$, which tends to 0 as

$x \to 0$ from the right of 0. That is, $\lim_{x\to 0^+} |x| = 0$

If $x < 0$, then $|x| = -x$ which again tends to 0 as $x \to 0$ from the left of 0. That is, $\lim_{x\to 0^-} |x| = 0$.

Thus, $\lim_{x\to 0^-} |x| = 0 = \lim_{x\to 0^+} |x|$.

Hence $\lim_{x\to 0} |x| = 0$.

REAL LIFE APPLICATION OF LIMIT

$A^3 C^2 4^B = 9^3 + 5^8 + 7^c$

$5^c = 54718,32.$

If you drop an ice cube in a glass of warm water and measure the temperature with time, the temperature eventually approaches the room temperature where the glass is stored. Measuring the temperature is a limit again as time approaches infinity.

Progress:

CONTINUITY & DIFFERENTIABILITY

04

Continuity

Continuity at a Point: A function f(x) is said to be continuous at a point x = a, if
Left hand limit of f(x) at(x = a) = Right hand limit of f(x) at (x = a) = Value of f(x) at (x = a)
i.e. if at x = a, LHL = RHL = f(a)
where, LHL = $\lim_{x\to a^-} f(x)$ and RHL = $\lim_{x\to a^+} f(x)$

Note: To evaluate LHL of a function f(x) at (x = o), put x = a – h and to find RHL, put x = a + h.

Continuity in an Interval: A function y = f(x) is said to be continuous in an interval (a, b), where a < b if and only if f(x) is continuous at every point in that interval.

- Every identity function is continuous.
- Every constant function is continuous.
- Every polynomial function is continuous.
- Every rational function is continuous.
- All trigonometric functions are continuous in their domain.

Algebra of Continuous Functions

Suppose f and g are two real functions, continuous at real number c. Then,

- $f + g$ is continuous at $x = c$.
- $f - g$ is continuous at $x = c$.
- $f.g$ is continuous at $x = c$.
- cf is continuous, where c is any constant.
- (f/g) is continuous at $x = c$, [provide $g(c) \neq 0$]

Suppose f and g are two real valued functions such that (fog) is defined at c. If g is continuous at c and f is continuous at g (c), then (fog) is continuous at c.

If f is continuous, then $|f|$ is also continuous.

Differentiability

Differentiability: A function f(x) is said to be differentiable at a point x = a, if
Left hand derivative at (x = a) = Right hand derivative at (x = a)
i.e. LHD at (x = a) = RHD (at x = a), where Right hand derivative, where

$$\textbf{Right hand derivative, } Rf'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$\textbf{Left hand derivative, } Lf'(a) = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

Note: Every differentiable function is continuous but every continuous function is not differentiable.

Differentiation

Differentiation: The process of finding a derivative of a function is called differentiation.

<u>**Rules of Differentiation**</u>

Sum and Difference Rule: Let y = f(x) ± g(x).Then, by using sum and difference rule, it's derivative is written as

$$\frac{dy}{dx} = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x).$$

Rules of Differentiation

Product Rule: Let y = f(x) g(x). Then, by using product rule, it's derivative is written as

$$\frac{dy}{dx} = \left[\frac{d}{dx}(f(x))\right] g(x) + \left[\frac{d}{dx}(g(x))\right] f(x).$$

Quotient Rule: Let y = f(x)/g(x); g(x) ≠ 0, then by using quotient rule, it's derivative is written as

$$\frac{dy}{dx} = \frac{g(x) \times \frac{d}{dx}[f(x)] - f(x) \times \frac{d}{dx}[g(x)]}{[g(x)]^2}.$$

Rules of Differentiation

Chain Rule: Let y = f(u) and u = f(x), then by using chain rule, we may write

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \text{ when } \frac{dy}{du} \text{ and } \frac{du}{dx} \text{ both exist.}$$

Logarithmic Differentiation: Let y = $[f(x)]^{g(x)}$ ..(i)
So by taking log (to base e) we can write Eq. (i) as log y = g(x) log f(x). Then, by using chain rule

$$\frac{dy}{dx} = [f(x)]^{g(x)}\left[\frac{g(x)}{f(x)} f'(x) + g'(x) \log f(x)\right]$$

Differentiation

Differentiation of Functions in Parametric Form: A relation expressed between two variables x and y in the form x = f(t), y = g(t) is said to be parametric form with t as a parameter, when

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$$

(whenever $dx/dt \neq 0$)

Note: dy/dx is expressed in terms of parameter only without directly involving the main variables x and y.

Derivative

Second order Derivative: It is the derivative of the first order derivative.

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Some Standard Derivatives

(i) $\frac{d}{dx}(\sin x) = \cos x$

(ii) $\frac{d}{dx}(\cos x) = -\sin x$

(iii) $\frac{d}{dx}(\tan x) = \sec^2 x$

(iv) $\frac{d}{dx}(\sec x) = \sec x \tan x$

(v) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

(vi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

Some Standard Derivatives

(vii) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(viii) $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$

(ix) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(x) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

(xi) $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$

(xii) $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$

(xiii) $\frac{d}{dx}(x^n) = nx^{n-1}$

(xiv) $\frac{d}{dx}(\text{constant}) = 0$

(xv) $\frac{d}{dx}(e^x) = e^x$

(xvi) $\frac{d}{dx}(\log_e x) = \frac{1}{x}, x > 0$

(xvii) $\frac{d}{dx}(a^x) = a^x \log_e a, a > 0$

Rolle's Theorem: Let f : [a, b] → R be continuous on [a, b] and differentiable on (a, b) such that f(a) = f(b), where a and b are some real numbers. Then, there exists at least one number c in (a, b) such that f'(c) = 0.

Mean Value Theorem: Let f : [a, b] → R be continuous function on [a, b]and differentiable on (a, b). Then, there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Note: Mean value theorem is an expansion of Rolle's theorem.

Progressive and Regressive derivatives

Example -

Consider the derivability of the function f(x) = | x | at the origin.

Left hand derivative =

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^-} \frac{|x|}{x}$$

$$= \lim_{x \to 0^-} \frac{-x}{x}$$

$$= -1$$

Progressive and Regressive derivatives

Right hand derivative $= \lim_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0}$

$$= \lim_{x \to 0^+} \frac{|x|}{x}$$

$$= \lim_{x \to 0^+} \frac{-x}{x}$$

$$= +1$$

Thus $f'(0^-) \neq f'(0^+)$. Hence the function is not derivable at $x = 0$

Progressive and Regressive derivatives

The progressive derivative of ‘f ’ at x = a is given by $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Where h > 0 i.e. h -> 0^+ or x -> a^+ and it is denoted by R f ’ (a) or by f ’(a + 0) or by f ’ (a^+) . It is also known as the Right hand derivative of ‘f ’ at x = a.

The regressive derivative of ’ f ’ at x = a, is given by $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Where h > 0 i.e. h -> 0^- or x -> a^- and it is denoted by Lf ’ (a) or by f ’ (a - 0) or f " (a^-). It is also known as the Left hand derivative of ‘ f ’ at x = a

It is easy to see that f ’ (a) exists if and only if Rf ’ (a) and Lf ’ (a) exist and are equal.

Differentiable Examples

Example 1 - Use the differentiability rules to determine the derivative of $f(x) = (2x + 1)/x^{3.}$

Solution: We will use the quotient rule for differentiable functions to determine the derivative of f(x).

$df/dx = d((2x + 1)/x^3)/dx$

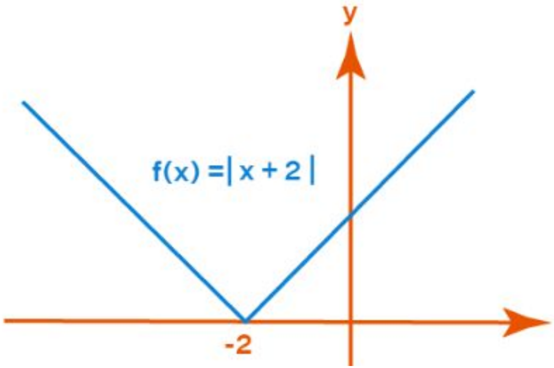
$= [2x^3 - 3x^2.(2x + 1)]/x^6$

$= [2x^3 - 6x^3 - 3x^2]/x^6$

$= -(4x^3 + 3x^2)/x^6$

Differentiable Examples

Example 2: Find out where the given function $f(x) = |x + 2|$ is not differentiable using graph and limit definition.



Solution:

Clearly, there is a sharp corner at point $x = -2$. The function is not differentiable at $x = -2$. Now, let's use the limit definition of differentiable functions.

Differentiable Examples

We already observed that the limits are different for absolute value of function.

$$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h \to 0} \frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h \to 0} \frac{|h|-0}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

This means that the limit $\lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$

does not exists at c = -2 for f(x) = |x+2|.

Tips and Tricks for Differentiable Functions

- If a graph has a sharp corner at a point, then the function is not differentiable at that point.
- If a graph has a break at a point, then the function is not differentiable at that point.
- If a graph has a vertical tangent line at a point, then the function is not differentiable at that point.

Important Notes on Differentiable

- Differentiable functions are those functions whose derivatives exist.
- If a function is differentiable, then it is continuous.
- If a function is continuous, then it is not necessarily differentiable.
- The graph of a differentiable function does not have breaks, corners, or cusps.

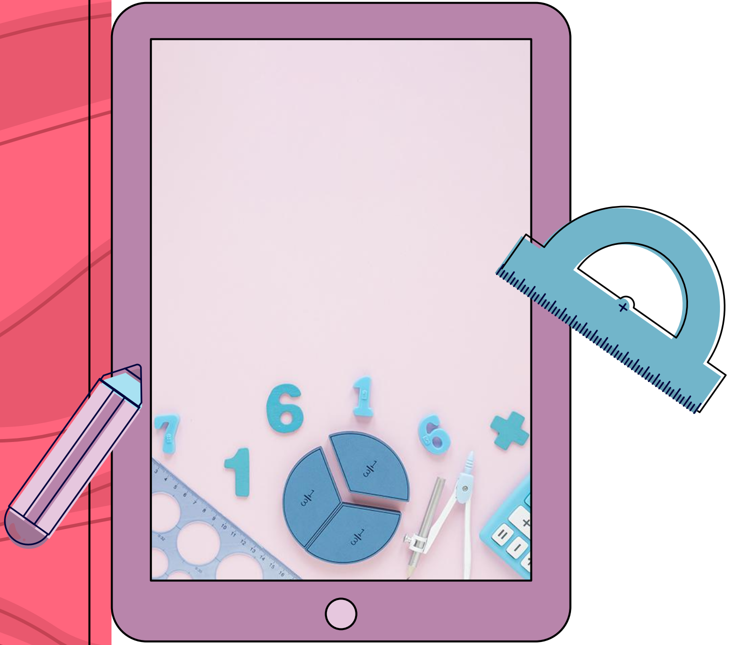
REAL LIFE APPLICATION OF DIFFERENTIATION

$A^3 C^2 4^B = 9^3 + 5^8 + 7^c$

$5^c = 54718,32.$

- To calculate the profit and loss in business using graphs.
- To check the temperature variation.
- To determine the speed or distance covered such as miles per hour, kilometre per hour etc.
- Derivatives are used to derive many equations in Physics.
- In the study of Seismology like to find the range of magnitudes of the earthquake.

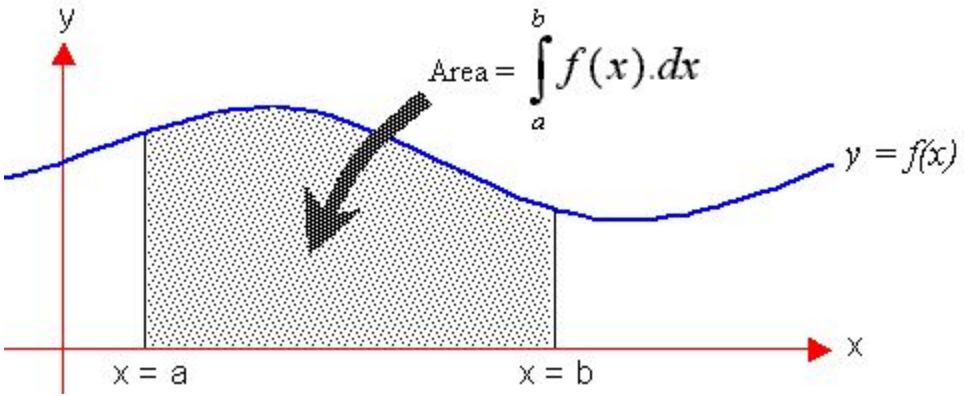
Progress:



INTEGRAL

Integration

Integration can be considered the reverse process of differentiation or called Inverse Differentiation. Integration is the process of finding a function with its derivative.



List of Integral Formulas

The list of basic integral formulas are -

- $\int 1 \, dx = x + C$
- $\int a \, dx = ax + C$
- $\int x^n \, dx = ((x^{n+1})/(n+1)) + C \; ; n \neq 1$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$
- $\int \sec^2 x \, dx = \tan x + C$
- $\int \csc^2 x \, dx = -\cot x + C$
- $\int \sec x (\tan x) \, dx = \sec x + C$
- $\int \csc x (\cot x) \, dx = -\csc x + C$
- $\int (1/x) \, dx = \ln |x| + C$
- $\int e^x \, dx = e^x + C$
- $\int a^x \, dx = (a^x/\ln a) + C \; ; a>0, \; a \neq 1$

These integral formulas are equally important as differentiation formulas. Some other important integration formulas are -

- $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$
- $\int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$
- $\int \frac{1}{|x|\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$
- $\int \sin^n(x)dx = \frac{-1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)dx$
- $\int \cos^n(x)dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)dx$
- $\int \tan^n(x)dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)dx$
- $\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx$
- $\int \csc^n(x)dx = \frac{-1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)dx$

Some special functions listed below

- $\int \frac{1}{(x^2-a^2)}dx = \frac{1}{2a}.\log\left|\frac{(x-a)}{(x+a)}\right| + C$
- $\int \frac{1}{(a^2-x^2)}dx = \frac{1}{2a}.\log\left|\frac{(a+x)}{(a-x)}\right| + C$
- $\int \frac{1}{(x^2+a^2)}dx = \frac{1}{a}\tan^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{x^2-a^2}}dx = \log|x + \sqrt{x^2-a^2}| + C$
- $\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}(\frac{x}{a}) + C$
- $\int \frac{1}{\sqrt{x^2+a^2}}dx = \log|x + \sqrt{x^2+a^2}| + C$

Indefinite Integral

In calculus, Integration is defined as the inverse process of differentiation and hence the evaluation of an integral is called as anti derivative. It is a process of the summation of a product. If we have not said the summation is to be done from which point to which point.

In other words, the interval of summation is indefinite and hence these types of integrals are known as indefinite integrals. Integration is an important part of calculus. It includes single integral, double integral, and multiple integrals. Various types of integral are used to find surface area and the volume of geometric solids

What is Infinite Integral in Calculus?

The indefinite integral is defined as a function that will describe an area under the function's curve from an undefined point to another arbitrary point. The absence of a specified first point leads to an arbitrary constant most often denoted as C and is always considered a part of indefinite integral.
In other words, the indefinite integral is the family of all functions whose derivative is f(x) but with a possibly finite set of exceptions.

$$\frac{d}{dx}\int f(x)dx = f(x)$$

Indefinite integrals are integrals without limits. The technique of integration is very useful in two ways.

- To find the function whose derivative is given.
- To find the area bounded by a curve given by the function under certain conditions.

Rules of Integration

The anti derivative of a definite integral is only implicit, that is, the solution will only be in a functional form. That is, $\int f(x)dx = g(x) + C$, where g(x) is another function of x and C is an arbitrary constant. However, there are many integrals which are to be integrated within a given interval.

They are denoted in general as $\int_a^b f(x)dx$

where, a and b are the limits of the interval. Such types of integrals are known as definite integrals.

The solution of a definite integral is unique and the solution to $\int_a^b f(x)dx = F(b) - F(a)$,

where F(x) is the anti derivative of the given integral.

Few important rules for integration

Integration of some functions may be readily done for functions whose derivatives are known. But, in many cases, the integrant (the function to be integrated) may not be that simple. It may be a sum, difference, product or a quotient of two functions. To perform the integration of such functions we need to follow the basic integration rules. Some basic integration rules as given below.

Rule 1: The integration of a sum or difference of two functions is the sum or difference (respectively) of the integration of the individual functions. That is,
$\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
Rule 2: The integration of a product of a function is **[First function * the integration of the second function – integration of { the integration of the second function*derivative of the first function}]**
That is, if u and v are two functions, $\int udv = uv - \int vdu$

There are many ways to find the integration of a given function, such as:

- Integration by Parts
- Integration by Substitution Method or Change of Variable
- Directly use the formula
- Integration by Partial Fraction Method

Examples

Q1) Solve

$$\int \frac{1}{1+cos6x}\, dx$$

Solution -

$$\int \frac{1}{1+cos6x}\, dx$$

$$= \int \frac{1}{2cos^2 3x}\, dx$$

$$= \int \frac{1}{2} sec^2 3x\, dx$$

= tan(3x)/6 + C

Example

Q2) Solve

$$\int \frac{x^4}{1+x^{10}}\, dx$$

Solution -

$$\int \frac{x^4}{1+x^{10}}\, dx$$
$$= \int \frac{x^4}{1+(x^5)^2}\, dx$$

Let us use substitution method to solve the problem.
Put $x^5 = t$, then
Differentiating it we have
$\Rightarrow 5x^4\, dx = dt$
$\Rightarrow x^4\, dx = dt/5$
Putting this value in equation (1) we have

$$\Rightarrow \int \frac{1}{5(1+t^2)} = 1/5 \times tan^{-1} t + C$$
$\Rightarrow (1/5) \tan^{-1} x^5 + C$

Example

Q3) Solve

$$\int \frac{6x+8}{3x^2+6x+2}\,dx$$

Solution:

$$\int \frac{6x+8}{3x^2+6x+2}$$

dx

$$= \int \frac{AD'(3x^2+6x+2)}{3x^2+6x+2}dx + \int \frac{B}{3x^2+6x+2}dx$$

$\Rightarrow D'(3x^2+6x+2) = 6x + 6$

So, $6x + 8 = A(6x + 6) + B$

By comparing coefficients, we have

$\Rightarrow A = 1$ and $B = 2$

So we have

$$\int \frac{6x+8}{3x^2+6x+2} = \int \frac{6x+6}{3x^2+6x+2}dx + \int \frac{2}{3x^2+6x+2}dx$$

$= I_1 + I_2$ ...(1)

Where

$$I_1 = \int \frac{6x+6}{3x^2+6x+2}dx \ldots . (2)$$

and

$$I_2 = \frac{2}{3x^2+6x+2}dx$$

Solve for I_1

Let $3x^2 + 6x + 2 = t$

Differentiating it, we have

⇒ (6x + 6) dx = dt

Substituting this value in equation (2), we have

$$I_1 = \int \frac{1}{t} dt = ln(t) + c = ln(3x^2 + 6x + 2) + c_1$$

And

$$I_2 = \int \frac{2}{3x^2+6x+2} dx = \int \frac{2}{3(x^2+2x+\frac{2}{3})} dx$$

$$= \int \frac{2}{3(x+1)^2-1+\frac{2}{3}} dx$$

$$= \int \frac{2}{3(x+1)^2-\frac{1}{3}} dx$$

$$= \frac{2}{3} ln\left|\frac{x+1-\frac{1}{\sqrt{3}}}{x+1+\frac{1}{\sqrt{3}}}\right| + c_2$$

Substituting the value of I_1 and I_2 in equation (1) we have

$$\Rightarrow \int \frac{6x+8}{3x^2+6x+2}dx = I_1 + I_2$$

$$\Rightarrow \int \frac{6x+8}{3x^2+6x+2}dx = \ln(3x^2+6x+2) + c_1 + \frac{2}{3}ln\left|\frac{x+1-\frac{1}{\sqrt{3}}}{x+1+\frac{1}{\sqrt{3}}}\right| + c_2$$

$$\Rightarrow \int \frac{6x+8}{3x^2+6x+2}dx = \ln(3x^2+6x+2) + \frac{2}{3}ln\left|\frac{x+1-\frac{1}{\sqrt{3}}}{x+1+\frac{1}{\sqrt{3}}}\right| + c \quad [c_1 + c_2 = c]$$

Definite Integral

Given a function f(x) that is continuous on the interval [a,b] we divide the interval into n subintervals of equal width Δx and from each interval choose a point $x^{i}{}_{*}$.
Then the definite integral of f(x) from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

Where a is lower limit , and b is upper limit.

A definite integral refers to an integral with upper and lower limits. If it is restricted to exist on the real line, the definite integral is called by the name of Riemann integral.

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ … [Also, $\int_a^a f(x)\,dx = 0$]

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4. $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

5. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \ldots$

6. $\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$

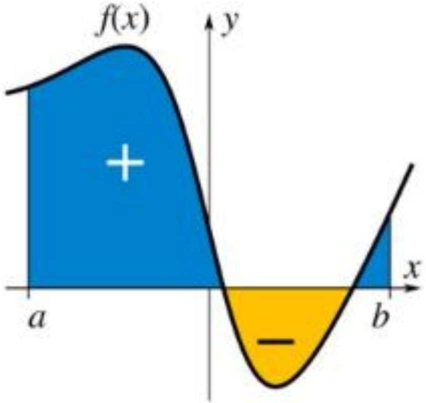
7. Two parts

1. $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \ldots$ if $f(2a-x) = f(x)$.
2. $\int_0^{2a} f(x)\,dx = 0 \ldots$ if $f(2a-x) = -f(x)$

8. Two parts

1. $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ … if $f(-x) = f(x)$ or it is an even function

2. $\int_{-a}^{a} f(x)\,dx = 0$ … if $f(-x) = -f(x)$ or it is an odd function



Example of Definite Integrals

Question 1: Evaluate $\int_{-1}^{2} |x^3 - x|\, dx$

Answer : Observe that, $(x^3 - x) \geq 0$ on $[-1, 0]$, $(x^3 - x) \leq 0$ on $[0, 1]$ and $(x^3 - x) \geq 0$ on $[1, 2]$

Hence, using Property 3, we can write

$\int_{-1}^{2} |x^3 - x|\, dx = \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} -(x^3 - x)\, dx + \int_{1}^{2} (x^3 - x)\, dx$

$= \int_{-1}^{0} (x^3 - x)\, dx + \int_{0}^{1} (x - x^3)\, dx + \int_{1}^{2} (x^3 - x)\, dx$

Example of Definite Integrals

Solving the integrals, we get

$\int_{-1}^{2} |x^3 - x|\, dx = [(x^4/4 - (x^2/2)]_{-1}^{0} + [(x^2/2 - (x^4/4)]_0^1 + [(x^4/4 - (x^2/2)]_1^2$

$= -[1/4 - ½] + [1/2 - ¼] + [4 - 2] - [1/4 - ½] = 11/4.$

Example of Definite Integrals

Example 1:

Evaluate the value of $\int_2^3 x^2\,dx$.

Solution:

Let $I = \int_2^3 x^2\,dx$

Now, $\int x^2\,dx = (x^3)/3$

Now, $I = \int_2^3 x^2\,dx = [(x^3)/3]_2^3$

$= (3^3)/3 - (2^3)/3$

$= (27/3) - (8/3)$

$= (27 - 8)/3$

$= 19/3$

Therefore, $\int_2^3 x^2\,dx = 19/3$

Example 2:

Calculate: $\int_0^{\pi/4} \sin 2x\,dx$

Solution:

Let $I = \int_0^{\pi/4} \sin 2x\,dx$

Now, $\int \sin 2x\,dx = -(½) \cos 2x$

$I = \int_0^{\pi/4} \sin 2x\,dx$

$= [-(½) \cos 2x]_0^{\pi/4}$

$= -(½) \cos 2(\pi/4) - \{-(½) \cos 2(0)\}$

$= -(½) \cos \pi/2 + (½) \cos 0$

$= -(½)(0) + (½)$

$= 1/2$

Therefore, $\int_0^{\pi/4} \sin 2x\,dx = 1/2$

REAL LIFE APPLICATION OF INTEGRATION

$A^3 C^2 4^B = 9^3 + 5^8 + 7^c$

$5^c = 54718,32.$

1. Biologists use differential calculus to determine the exact rate of growth in a bacterial culture when different variables such as temperature and food source are changed.

2. In Electrical Engineering, Calculus (Integration) is used to determine the exact length of power cable needed to connect two substations, which are miles away from each other.

Progress:

APPLICATION OF DERIVATIVES

APPLICATION OF DERIVATIVES

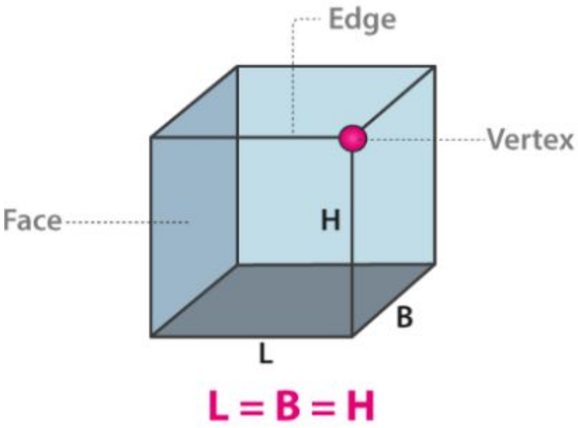
The derivative is defined as the rate of change of one quantity with respect to another. In terms of functions, the rate of change of function is defined as dy/dx = f(x) = y'.

The concept of derivatives has been used in small scale and large scale. The concept of derivatives used in many ways such as change of temperature or rate of change of shapes and sizes of an object depending on the conditions etc.

Progress:

RATE OF CHANGE OF A QUANTITY

This is the general and most important application of derivative. For example, to check the rate of change of the volume of a cube with respect to its decreasing sides, we can use the derivative form as dy/dx. Where dy represents the rate of change of volume of cube and dx represents the change of sides of the cube.



Progress:

INCREASING & DESCRESING FUNCTION

To find that a given function is increasing or decreasing or constant, say in a graph, we use derivatives. If f is a function which is continuous in [p, q] and differentiable in the open interval (p, q), then,

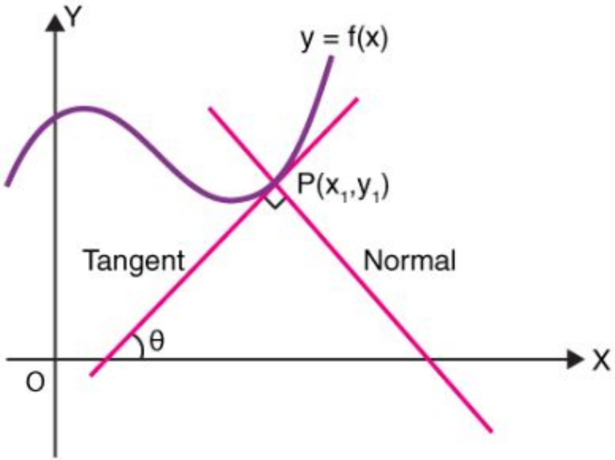
- f is increasing at [p, q] if $f'(x) > 0$ for each $x \in (p, q)$
- f is decreasing at [p, q] if $f'(x) < 0$ for each $x \in (p, q)$
- f is constant function in [p, q], if $f'(x)=0$ for each $x \in (p, q)$

$$x_1 + 2_A = 3\sqrt{5+2AB}$$
$$= 9\sqrt{12}$$

Progress:

TANGENT AND NORMAL TO A CURVE

A tangent is a line that touches the curve at a point and doesn't cross it, whereas normal is perpendicular to that tangent.
Let the tangent meet the curve at $P(x_1, y_1)$.



Progress:

TANGENT AND NORMAL TO A CURVE

Now the straight-line equation which passes through a point having slope m could be written as;

$$y - y_1 = m(x - x_1)$$

We can see from the above equation, the slope of the tangent to the curve y = f(x) and at the point P(x_1, y_1),it is given as dy/dx at P(x_1, y_1)= f'(x).

Therefore,

Equation of the tangent to the curve at P(x_1, y_1) can be written as:

$$y - y_1 = f'(x_1)(x - x_1)$$

Equation of normal to the curve is given by;

$$y - y_1 = [-1/ f'(x_1)] (x - x_1)$$

Or

$$(y - y_1) f'(x_1) + (x-x_1) = 0$$

Progress:

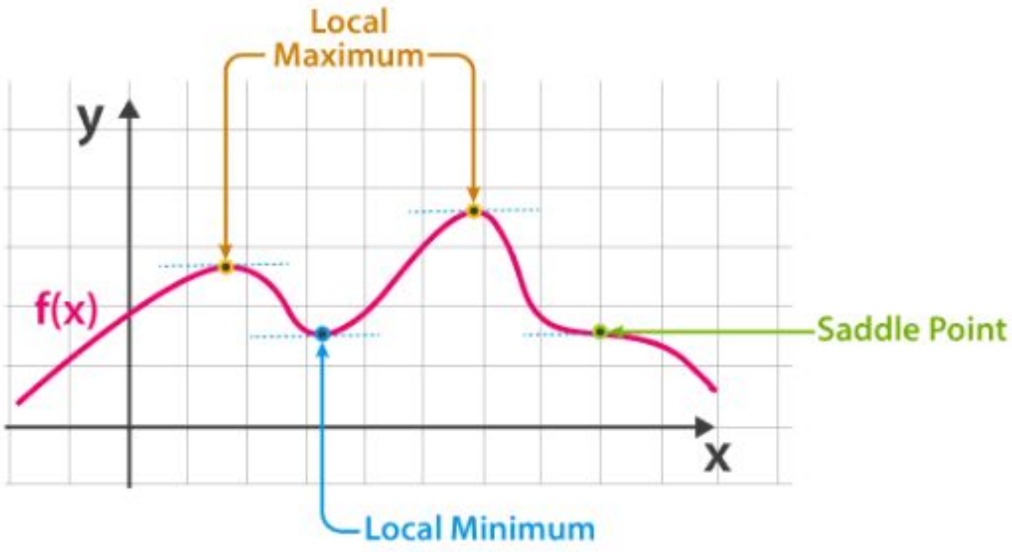
MAXIMA & MINIMA

To calculate the highest and lowest point of the curve in a graph or to know its turning point, the derivative function is used.

- When $x = a$, if $f(x) \le f(a)$ for every x in the domain, then $f(x)$ has an Absolute Maximum value and the point a is the point of the maximum value of f.
- When $x = a$, if $f(x) \le f(a)$ for every x in some open interval (p, q) then $f(x)$ has a Relative Maximum value.
- When $x= a$, if $f(x) \ge f(a)$ for every x in the domain then $f(x)$ has an Absolute Minimum value and the point a is the point of the minimum value of f.
- When $x = a$, if $f(x) \ge f(a)$ for every x in some open interval (p, q) then $f(x)$ has a Relative Minimum value.

Progress:

MAXIMA AND MINIMA



$x_1 + 2_A = 3\sqrt{5+2AB}$
$= 9\sqrt{12}$

Progress:

MONOTONICITY

Functions are said to be monotonic if they are either increasing or decreasing in their entire domain. f(x) = ex, f(x) = nx, f(x) = 2x + 3 are some examples.

Functions which are increasing and decreasing in their domain are said to be non-monotonic

For example: $f(x) = \sin x$, $f(x) = x^2$

Monotonicity Of A function At A Point

A function is said to be monotonically decreasing at x = a if f(x) satisfy;

$f(x + h) < f(a)$ for a small positive h

- $f'(x)$ will be positive if the function is increasing
- $f'(x)$ will be negative if the function is decreasing
- $f'(x)$ will be zero when the function is at its maxima or minima

Progress:

APPROXIMATION OR FINDING APPROXIMATE VALUE

To find a very small change or variation of a quantity, we can use derivatives to give the approximate value of it. The approximate value is represented by delta Δ.

Suppose change in the value of x, dx = x then,

$dy/dx = \Delta x = x$.

Since the change in x, $dx \approx x$ therefore, $dy \approx y$.

$$x_1 + 2_A = 3\sqrt{5+2AB}$$
$$= 9\sqrt{12}$$

Progress:

POINT OF INFLECTION

For continuous function f(x), if $f'(x0) = 0$ or $f'''(x0)$ does not exist at points where $f'(x0)$ exists and if $f''(x)$ changes sign when passing through $x = x0$ then x0 is called the point of inflection.

(a) If $f''(x) < 0$, $x \in (a, b)$ then the curve $y = f(x)$ in concave downward

(b) if $f''(x) > 0$, $x \in (a, b)$ then the curve $y = f(x)$ is concave upwards in (a, b)

For example: $f(x) = \sin x$

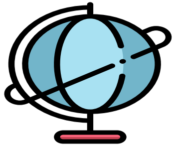
Solution: $f'(x) = \cos x$

$f''(x) = \sin x = 0$ $x = n\pi, n \in z$

Progress:

APPLICATION OF DERIVATIVES IN REAL LIFE

- To calculate the profit and loss in business using graphs.
- To check the temperature variation.
- To determine the speed or distance covered such as miles per hour, kilometre per hour etc.
- Derivatives are used to derive many equations in Physics.
- In the study of Seismology like to find the range of magnitudes of the earthquake.



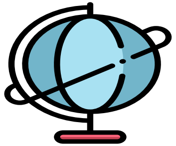
EXAMPLE APPLICATION OF DERIVATIVES

Question 1: A city's population is modelled as *$P(t) = 2t^2 + 10t + 200$ persons(t is the number of years since 2000).* What would be the average rate at which the population is changing in 2005?

Solution- For the average rate, we need the information at the beginning point and the endpoint of our domain. Clearly, the beginning point would be the year 2000 corresponding to which, we have $t_1 = 0$ and $P(t = 0) = 200$. The endpoint: $t_2 = (2005\text{-}2000) = 5$ corresponding to which, $P(t = 5) = 300$. The average rate of change of population in 2005:

$$\frac{\Delta P}{\Delta t} = \frac{P_2 - P_1}{t_2 - t_1}$$

$$= \frac{300 - 200}{5 - 0} = 20 \text{ persons per year}$$



EXAMPLE APPLICATION OF DERIVATIVES

Question: A train moves along the railway track in such a way that the distance it covers, starting from the station A, is given by the equation x = 5t2 + t where x is in meters and t is in seconds. What would be the velocity and the acceleration of the train at time t = 25 seconds?

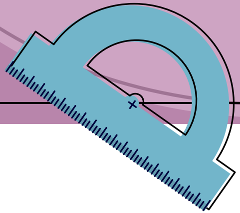
$$v = \frac{dx}{dt} = 10t + 1$$

Solution: Given x(t) = 5t2 + t
To find: v(t = 25 sec) and a(t = 25 sec)

From the definitions of velocity and acceleration, we have

$$a = \frac{dv}{dt} = \frac{d^2x}{d^2t} = 10$$

Now putting t = 25sec in the equations, we get v = 251 meters and a = 10 m/s2. Note that in this problem, the acceleration is independent of time. Such motion is said to be uniformly accelerated motion. This concludes the discussion on the Rate of Change of quantities.



THANK YOU

$$C = \sin^2\left(\frac{2}{3}\right)$$

$$= \sin^3 \times 0{,}747$$

$$\approx 7{,}38$$