Thesis Title

THESIS

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by

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Abstract

Thesis Title

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The Thesis Abstract is written here (and usually kept to just this page). The page is kept centered vertically so can expand into the blank space above the title too…

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List of Abbreviations

AC	**A**lternating **C**urrent
FWHM	**F**ull **W**idth at **H**alf **M**aximum
TEM	**T**ransverse **E**lectric and **M**agnetic
TE	**T**ransverse **E**lectric
TM	**T**ransverse **M**agnetic
QND	**Q**uantum **N**on **D**emolition
SHO	**S**imple **H**armonic **O**scillator
SIS	**S**uperconductor-**I**nsulator-**S**uperconductor
VNA	**V**ector **N**etwork **A**nalyser

Physical Constants

Speed of Light $c_0 = 2.997\,924\,58 \times 10^8\,\mathrm{m\,s^{-1}}$ (exact)

List of Symbols

a	distance	m
P	power	W (J s^{-1})
ω	angular frequency	rad

For/Dedicated to/To my…

Chapter 1

Theory

1.1 Microwave Resonators

In most low frequency AC circuits, we are used to transmitting the signal in 2 conductors (or wires). We can do this because at these frequencies, the wavelength of the signal is very large compared to the length of the conductors. In reality, there will be a small phase shift between the signal at the signal generator and the other end of the "wires". This phase shift, along with other phenomena can be easily observed at high frequencies.

At high frequencies, the geometry and properties of the material plays an important role in the transmission. The replacement for what we knew as just "wires" are called *Transmission Lines* or *Waveguides*.

1.1.1 Waveguides

There are many different types of waveguides. Some of them are shown in Fig. 1.1. The case we will be dealing with in this thesis pertains to rectangular waveguide.

General Waveguide

Consider a general cross-section of a dielectric surrounded by conductor (can have one more conductor in the dielectric) which continues infinitely along the z axis. We can write down the electric and magnetic fields in the dielectric in phasor domain. We assume that the wave propagates in the z-axis and has an $e^{j\omega t}$ dependence.

$$\bar{E}(x, y, z) = [\hat{x}e_x(x, y) + \hat{y}e_y(x, y) + \hat{z}e_z(x, y)]e^{-j\beta z} \tag{1.1}$$

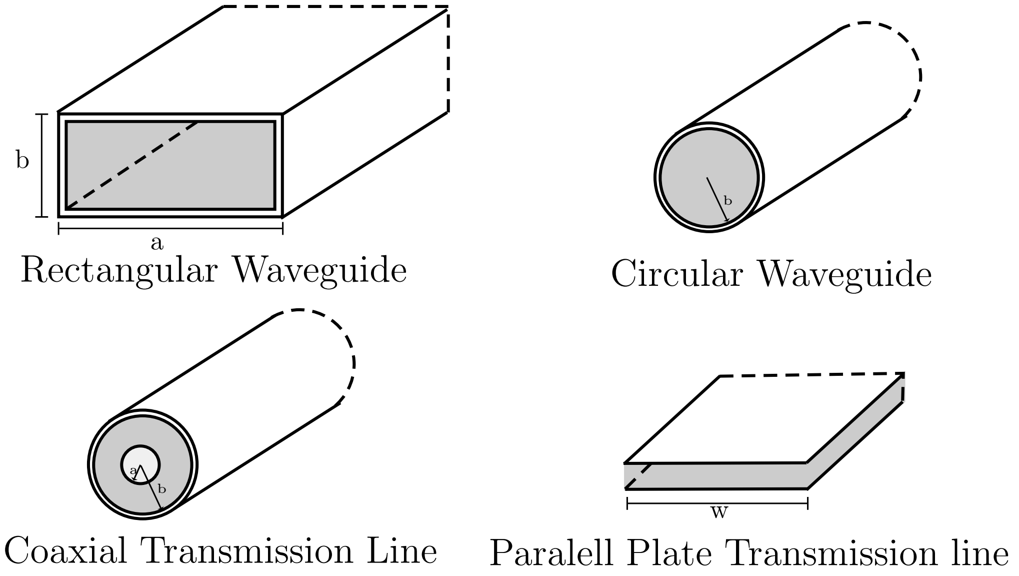

FIGURE 1.1: Types of Waveguides and Transmission Lines

$$\bar{H}(x, y, z) = [\hat{x}h_x(x, y) + \hat{y}h_y(x, y) + \hat{z}h_z(x, y)]e^{-j\beta z} \tag{1.2}$$

Here β, the propagation constant, is a real number. $j\beta$ must be replaced with $\gamma = \alpha + j\beta$ if attenuation is also to be considered.

Then, if the dielectric in the waveguide has no charges or currents, we can write Maxwell's equations as

$$\nabla \times \bar{E} = -j\omega\mu\bar{H} \tag{1.3a}$$

$$\nabla \times \bar{H} = j\omega\epsilon\bar{E} \tag{1.3b}$$

Taking the curl of 1.3a gives

$$\nabla \times \nabla \times \bar{E} = -j\omega\mu\nabla \times \bar{H} = \omega^2\mu\epsilon\bar{E} \tag{1.4}$$

Using the vector identity $\nabla \times \nabla \times \bar{A} = \nabla(\nabla.\bar{A}) - \nabla^2\bar{A}$ and $\nabla.\bar{E} = 0$ for a region with no sources ($\rho = 0$) we get

$$\nabla^2\bar{E} + \omega^2\mu\epsilon\bar{E} = 0 \tag{1.5}$$

Similarly, we can also take the curl of 1.3b to get

$$\nabla^2 \bar{H} + \omega^2 \mu \epsilon \bar{H} = 0 \tag{1.6}$$

For a z dependence of $e^{-j\beta z}$, E_z and H_z can be written as

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 - \beta^2 \right) E_z = 0 \tag{1.7}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 - \beta^2 \right) H_z = 0 \tag{1.8}$$

since $\frac{\partial^2}{\partial z^2} \left(A e^{-j\beta z} \right) = -\beta A e^{-j\beta z}$. Let us define $k_c^2 = k^2 - \beta^2$ for convenience.

After writing down the 6 equations that arise from 1.3 and eliminating variables, we can write E_x, E_y, H_x, H_y in terms of E_z and H_z as follows

$$E_x = \frac{-j}{k_c^2} \left(\beta \frac{\partial E_z}{\partial x} + \omega \mu \frac{\partial H_z}{\partial y} \right) \tag{1.9a}$$

$$E_y = \frac{j}{k_c^2} \left(-\beta \frac{\partial E_z}{\partial y} + \omega \mu \frac{\partial H_z}{\partial x} \right) \tag{1.9b}$$

$$H_x = \frac{j}{k_c^2} \left(\omega \epsilon \frac{\partial E_z}{\partial y} - \beta \frac{\partial H_z}{\partial x} \right) \tag{1.9c}$$

$$H_y = \frac{-j}{k_c^2} \left(\omega \epsilon \frac{\partial E_z}{\partial x} + \beta \frac{\partial H_z}{\partial y} \right) \tag{1.9d}$$

where

$$k_c^2 = k^2 - \beta^2 \tag{1.10}$$

$$k = \omega \sqrt{\mu \epsilon} = 2\pi / \lambda \tag{1.11}$$

These equations (1.7, 1.8 and 1.9) can be used for any waveguide. There are three types of waves that are possible in waveguides: Transverse Electric and Magnetic mode (TEM), Transverse Electric mode (TE) and Transverse Magnetic mode (TM).

1. **TEM modes**

   In this mode $E_z = H_z = 0$, meaning there are only transverse fields.

2. **TE modes**

   In this mode $E_z = 0$, meaning there are only transverse electric fields.

3. **TM modes**

   In this mode $H_z = 0$, meaning there are only transverse magnetic fields.

Rectangular Waveguide

Let us now concentrate on the fields in a rectangular waveguide. It can be shown that in the TEM mode, fields in the dielectric follow the same rules as electrostatics [12]. In a single conductor waveguide like the rectangular waveguide, the electrostatic potential is zero (or constant) which means that $E = 0$ and $H = 0$. This means we can only have TE and TM modes in the rectangular waveguide (or any single conductor waveguide).

1. **TE modes**

   Equation 1.8 has been rewritten below with $k^2 - \beta^2$ replaced with k_c and divided by $e^{-j\beta z}$.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2 \right) h_z = 0 \tag{1.12}$$

   Here, $H_z(x, y, z) = h_z(x, y)e^{-j\beta z}$.

   We can solve 1.12 using separation of variables. We assume

$$h_z(x, y) = F(x)G(y) \tag{1.13}$$

   Substituting this into 1.12 gives

$$\frac{1}{F}\frac{d^2F}{dx^2} + \frac{1}{G}\frac{d^2G}{dy^2} + k_c^2 = 0 \tag{1.14}$$

   Now, since each term is independent of each other, each term must be a constant. We define the first term to be k_x^2 and the second term to be k_y^2 to get

$$k_x^2 + k_y^2 + k_c^2 = 0 \tag{1.15}$$

Then we get 2 ordinary differential equations

$$\frac{d^2F}{dx^2} + k_x F = 0 \tag{1.16a}$$
$$\frac{d^2G}{dy^2} + k_y G = 0 \tag{1.16b}$$

The general solution to 1.16 is

$$F = A\cos(k_x x) + B\sin(k_x x) \tag{1.17a}$$
$$G = C\cos(k_y y) + D\sin(k_y y) \tag{1.17b}$$

Which gives

$$h_z = (A\cos(k_x x) + B\sin(k_x x))(C\cos(k_y y) + D\sin(k_y y)) \tag{1.18}$$

Since the boundary conditions we have are that the tangential electric field at the conductor is zero, i.e.

$$e_x(x, y) = 0 \qquad \text{at } y = 0 \text{ and } y = b \tag{1.19a}$$
$$e_y(x, y) = 0 \qquad \text{at } x = 0 \text{ and } x = a \tag{1.19b}$$

Substituting $E_z = 0$ in 1.9, we get

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y} \tag{1.20a}$$
$$E_y = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x} \tag{1.20b}$$
$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x} \tag{1.20c}$$
$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y} \tag{1.20d}$$

Now substituting $h_z(x, y)$ from 1.18 we get the following electric fields

$$e_x = \frac{-j\omega\mu}{k_c^2} k_y (A\cos(k_x x) + B\sin(k_x x))(-C\sin(k_y y) + D\cos(k_y y)) \tag{1.21a}$$
$$e_y = \frac{j\omega\mu}{k_c^2} k_x (-A\sin(k_x x) + B\cos(k_x x))(C\cos(k_y y) + D\sin(k_y y)) \tag{1.21b}$$

Now applying the boundary conditions,

from 1.19a we get $D = 0$ and $k_y = n\pi/b$ for $n = 0, 1, 2\ldots$,

and from 1.19b we get $B = 0$ and $k_x = m\pi/a$ for $m = 0, 1, 2\ldots$.

From this we know the propagation constant is

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{1.22}$$

Since β is real, we now have a cut-off frequency for which $k^2 > k_c^2$. This means that if $a > b$, there will be a range of frequencies for which $TE_{mn} = TE_{10}$ will have propagation but TE_{01} will not.

The final solution for H_z is

$$H_z(x, y, z) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.23}$$

where $A_{mn} = AC$.

Now we can find E_x, E_y, H_x and H_y using 1.20

$$E_x(x, y, z) = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.24a}$$

$$E_y(x, y, z) = \frac{-j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.24b}$$

$$H_x(x, y, z) = \frac{j\beta m\pi}{k_c^2 a} A_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.24c}$$

$$H_y(x, y, z) = \frac{j\beta n\pi}{k_c^2 b} A_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.24d}$$

These equations are only for a wave propagating in one direction. The total electric field well have another term for the fields with a different constant. We can replace A_{mn} with A_{mn}^+ (for $+z$ direction propagation) and A_{mn}^- (for $-z$ direction propagation). Then the transverse fields for each mode (m, n) would take the form

$$\bar{E}_t(x, y, z) = [\hat{x}e_x(x, y) + \hat{y}e_y(x, y)](A^+e^{-j\beta z} + A^-e^{+j\beta z}) \tag{1.25a}$$

$$\bar{H}_t(x, y, z) = [\hat{x}h_x(x, y) + \hat{y}h_y(x, y)](A^+e^{-j\beta z} - A^-e^{+j\beta z}) \tag{1.25b}$$

The negative sign for A^- in the magnetic field is to ensure that the direction of propagation given by $\bar{E}_t \times \bar{H}_t$ is opposite.

2. **TM modes**

The TM modes can be derived in exactly the same way except that the boundary conditions will apply directly to E_z this time.

The fields for the TM modes are

$$E_z(x,y,z) = B_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.26a}$$

$$E_x(x,y,z) = \frac{-j\beta m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.26b}$$

$$E_y(x,y,z) = \frac{-j\beta n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.26c}$$

$$H_x(x,y,z) = \frac{j\omega\epsilon n\pi}{k_c^2 b} B_{mn} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.26d}$$

$$H_y(x,y,z) = \frac{-j\omega\epsilon m\pi}{k_c^2 a} B_{mn} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-j\beta z} \tag{1.26e}$$

Notice that if m or n is zero, then the fields all go to zero. So there is no TM_{10} or TM_{01} mode.

The propagation constant β is

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{1.27}$$

This means the cut-off frequencies are the same for the TE and TM modes. Now we can see that there is a range of frequencies where only the TE_{10} mode will propagate. This feature of waveguides is used extensively to avoid complications of other modes interfering with the signal.

1.1.2 Rectangular Waveguide Resonators

Now that we know what modes and what frequencies can propagate in a rectangular waveguide, we can convert the waveguide into a resonator by walling the 2 infinitely open faces with conducting surfaces to make a cuboid filled with dielectric. This structure is often called a rectangular cavity.

We can use the equations we derived in the previous section for fields and the propagation constant to see what effects the new conducting walls will have.

We can start by writing down the transverse electric field ($E_t = \hat{x}E_x + \hat{y}E_y$)

$$\bar{E}_t = \bar{e}(x,y)(A^+e^{-j\beta z} + A^-e^{+j\beta z}) \tag{1.28}$$

where $\bar{e}(x,y)$ is the variation in the transverse fields.

$$\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \tag{1.29}$$

The new boundary conditions added now are that $E_t = 0$ at the 2 new walls, $z = 0$ and $z = d$.

For $z = 0$, 1.28 gives

$$A^+ = -A^- \tag{1.30}$$

.

For $z = d$, 1.28 gives

$$-\bar{e}(x,y)A^+2j\sin(\beta_{mn}d) = 0 \tag{1.31}$$

The solution to this equation (other than $A^+ = 0$) is

$$\beta_{mn} = \frac{l\pi}{d} \text{ where } l = 1, 2, 3\ldots \tag{1.32}$$

This means that, given a frequency, propagation (or in this case resonance) occurs only for particular lengths. $\beta^2 = k^2 - k_c^2$ can be rearranged to get

$$k_{mnl} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2} \tag{1.33}$$

The resonant frequency is given by

$$f_{mnl} = \frac{ck_{mnl}}{2\pi\sqrt{\mu_r\epsilon_r}} = \frac{c}{2\pi\sqrt{\mu_r\epsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2} \tag{1.34}$$

Now let us restrict ourselves to the TE_{10l} mode of the resonator. Since $A^- = -A^+$, the fields for this mode are

$$E_y(x, y, z) = A^+ \sin\left(\frac{\pi x}{a}\right)(e^{-j\beta z} - e^{+j\beta z}) \tag{1.35a}$$

$$H_x(x, y, z) = \frac{-A^+}{Z_{TE}} \sin\left(\frac{\pi x}{a}\right)(e^{-j\beta z} + e^{+j\beta z}) \tag{1.35b}$$

$$H_z(x, y, z) = \frac{j\pi A^+}{k\eta a} \cos\left(\frac{\pi x}{a}\right)(e^{-j\beta z} - e^{+j\beta z}) \tag{1.35c}$$

where

$$A^+ = \frac{-j\omega\mu m\pi}{k_c^2 a}$$

$$Z_{TE} = \frac{\omega\mu}{\beta} \qquad k = \omega\sqrt{\mu\epsilon}$$

$$k_c^2 = \sqrt{\frac{\pi}{a}} \qquad \eta = \sqrt{\frac{\mu}{\epsilon}}$$

Using $-2jA^+ = E_0$, we can simplify the above equations to

$$E_y(x, y, z) = E_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{l\pi z}{d}\right) \tag{1.37a}$$

$$H_x(x, y, z) = \frac{-jE_0}{Z_{TE}} \sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{l\pi z}{d}\right) \tag{1.37b}$$

$$H_z(x, y, z) = \frac{j\pi E_0}{k\eta a} \cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{l\pi z}{d}\right) \tag{1.37c}$$

We can now calculate the *quality factor* Q by calculating the energy stored and power lost in the resonator. The stored electric energy is, from [12]

$$W_e = \frac{\epsilon}{4}\int_V E_y E_y^* dv = \frac{\epsilon abd}{16}E_0^2 \tag{1.38}$$

and the stored magnetic energy is

$$\begin{aligned} W_m &= \frac{\mu}{4}\int_V (H_x H_x^* + H_z H_z^*) dv \\ &= \frac{\mu abd}{16} E_0^2 \left(\frac{1}{Z_{TE}^2} + \frac{\pi^2}{k^2 \eta^2 a^2} \right) \\ &= \frac{\mu abd}{16} E_0^2 \left(\frac{\beta^2 + (\pi/a)^2}{k^2\eta^2} \right) \\ &= \frac{\mu abd}{16} E_0^2 \left(\frac{1}{\eta^2} \right) \\ &= \frac{\epsilon abd}{16} E_0^2 \end{aligned} \tag{1.39}$$

Note that $W_e = W_m$ at resonance.

The power lost by the conducting walls is

$$P_c = \frac{R_s}{2} \int_{walls} |H_t|^2 ds \tag{1.40}$$

where $R_s = \sqrt{\omega\mu_0/w\sigma}$ is the surface resistivity and H_t is the tangential magnetic field at the walls. This gives

$$P_c = \frac{R_s E_0^2 \lambda^2}{8\eta^2} \left(\frac{l^2 ab}{d^2} + \frac{bd}{a^2} + \frac{l^2 a}{2d} + \frac{d}{2a} \right) \tag{1.41}$$

The power dissipated from the lossy dielectric with $\epsilon = \epsilon' - j\epsilon''$ is

$$P_d = \frac{1}{2}\int_V \bar{J}.\bar{E} = \frac{\omega\epsilon''}{2}\int_V |\bar{E}|^2 dv = \frac{abd\omega\epsilon''|E_0|^2}{8} \tag{1.42}$$

The quality factor Q is defined as

$$\begin{aligned} Q &= \omega \frac{\text{average energy stored}}{\text{average power loss}} \\ &= \omega \frac{W_e + W_m}{P_{loss}} \\ &= \omega \frac{2W_e}{P_c + P_d} \end{aligned} \tag{1.43}$$

1.1.3 Coupling to an External Circuit

A resonator is useless if we cannot communicate with it in some way. There are many ways of interacting with a resonator. In the experiments to follow, the rectangular cavity resonator will be coupled to and external circuit through a probe of

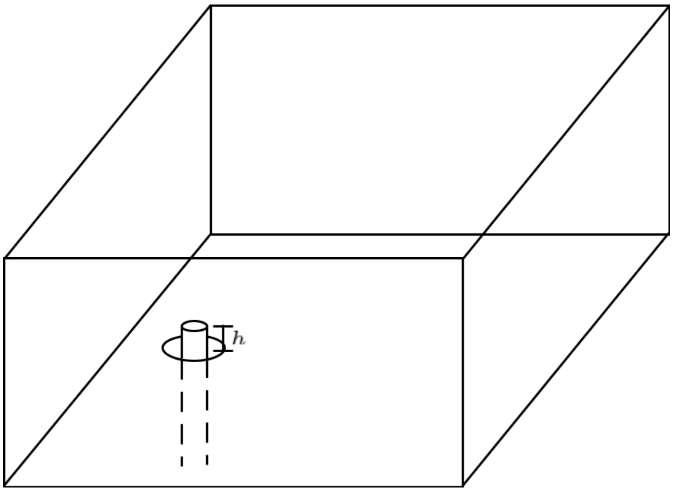

FIGURE 1.2: The probe is inserted up to a height h into the cavity. The other end of the probe is a coaxial cable whose outer terminal is connected to the cavity.

height h inserted at the $y = 0$ wall at $x = a/4$ and $z = z_0$. A figure of the probe inserted in the rectangular cavity is shown in Fig.1.3.

The equivalent circuit, shown in Fig.**??** for this setup would be a paralell RLC circuit capacitively coupled to the measurement device, which in this case is a VNA (Vector Network Analyzer).

The S_{11} response for this circuit is given by [1]

$$S_{11}(\omega) = 1 - \frac{\kappa_e}{\kappa/2 - j(\omega - \omega_0)} \tag{1.44}$$

where

ω_0 is the resonant frequency of the cavity,

$\kappa = \Delta\omega$ is the FWHM of the resonance peak which represents total losses,

κ_e is the part of κ which represents the losses to the external circuit through the capacitor C_c.

The quality factor can be calculated using $Q = \omega_0/\text{FWHM}$. The internal, external

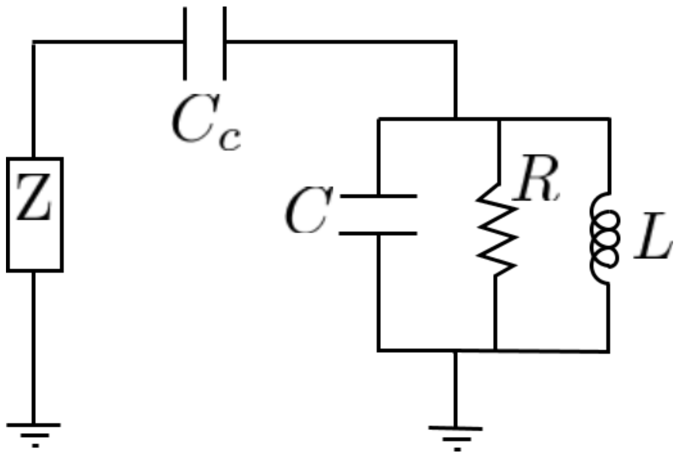

FIGURE 1.3: The equivalent circuit for the resonator is a parallel RLC circuit.The external losses are in the external inpedence Z of the meaasurement device. The energy in the resonator leaks to the external circuit at a rate $\kappa_e \propto 1/C_c$. The internal losses are due to the resistance R. The rate of instrinsic energy loss in the resonator is $\kappa_i = \kappa - \kappa_e$ where κ is the FWHM of the S_{11} response of the resonator.

and total quality factors can be calculated using

$$Q_{total} = \frac{\omega_0}{\kappa} \tag{1.45}$$

$$Q_{external} = \frac{\omega_0}{\kappa_e} \tag{1.46}$$

$$Q_{internal} = \left(\frac{1}{Q_{total}} - \frac{1}{Q_{external}}\right)^{-1} \tag{1.47}$$

$$\tag{1.48}$$

The S_{11} response of the cavity is asymmetrical, as shown in Fig. **??**, due to the finite cable length and finite isolation of the directional coupler.

- The effect of the finite cable length is an added frequency dependent phase factor $\exp(2j\omega l/v_p)$ where l is the cable length and $v_p = c/\sqrt{\epsilon_r}$ is the phase velocity. The '2' in the expression is because the cable length is traversed twice. This factor is multiplied to the original expression for S_{11} in **??**.

- The effect of the finite isolation of the directional coupler is modelled by considering a part of the signal as a complex background ($\alpha e^{j\phi}$) and the rest of the signal ($(1-\alpha)S_{11}$) as the S_{11} response.

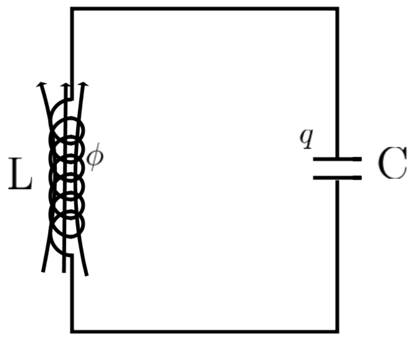

FIGURE 1.4: LC oscillator circuit

So the final function used to fit the data and get the parameters ω_0,κ and κ is

$$S_{11}(\omega) = \alpha e^{j\phi} + (1-\alpha)\left(1 - \frac{\kappa_e}{\kappa/2 - j(\omega-\omega_0)}\right) e^{2j\omega l/v_p} \tag{1.49}$$

1.2 Superconducting Qubits

1.2.1 Classical LC oscillator

The LC oscillator, shown in Fig.1.4, when treated classically has a charge q on the capacitor, and a flux ϕ in the inductor. The flux is related to the charge via the inductance as $\phi = L\frac{dq}{dt}$. The Hamiltonian for this circuit is

$$\mathcal{H} = \frac{q^2}{2C} + \frac{\phi^2}{2L} \tag{1.50}$$

1.2.2 Quantum Electrical Circuits

Quantum LC oscillator

Observe that the 2 variables involved in the LC oscillator, q and $\phi = L\frac{dq}{dt}$, are similar in form to the position and momentum operators in quantum mechanics, $\hat{x}$ and $\hat{p} = -j\hbar\frac{\partial}{\partial x}$. Even the Hamiltonian is of the same form.[5]

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2} \tag{1.51}$$

Because of this, we can treat this circuit like the simple harmonic oscillator and introduce the annihilation and creation operators to define $\hat{q}$, $\hat{\phi}$ and Hamiltonian operators as

$$\hat{\phi} = \frac{1}{j}\sqrt{\frac{\hbar}{2Z_0}}(a - a^\dagger) \tag{1.52a}$$

$$\hat{q} = \sqrt{\frac{\hbar Z_0}{2}}(a + a^\dagger) \tag{1.52b}$$

$$\hat{\mathcal{H}} = \frac{\hbar\omega_0}{2}(a^\dagger a + aa^\dagger) = \hbar\omega_0\left(a^\dagger a + \frac{1}{2}\right) \tag{1.52c}$$

where

$$[\hat{\phi}, \hat{q}] = -j\hbar \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad Z_0 = \sqrt{\frac{L}{C}}$$

ω_0 and C in the LC oscillator is analogous to ω and m in the harmonic oscillator.

We can write the wave-functions of the energy eigenstates of the LC oscillator as

$$\langle x|0\rangle = \psi_0 = \left(\frac{C\omega_0}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\left(\frac{C\omega_0}{2\hbar}\right)x^2} \tag{1.53}$$

This solution can be obtained using $a\,|0\rangle = 0$

The rest of the eigenstates can be obtained by using the creation operator $a^\dagger$ since

$$a^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{1.54}$$

which gives

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle \tag{1.55}$$

The energy corresponding to these states are

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_0 \tag{1.56}$$

The general solution to the Schrödinger equation $\hat{\mathcal{H}}\,|\psi\rangle = E\,|\psi\rangle$ is

$$|\psi\rangle = \sum_n c_n\,|n\rangle \tag{1.57}$$

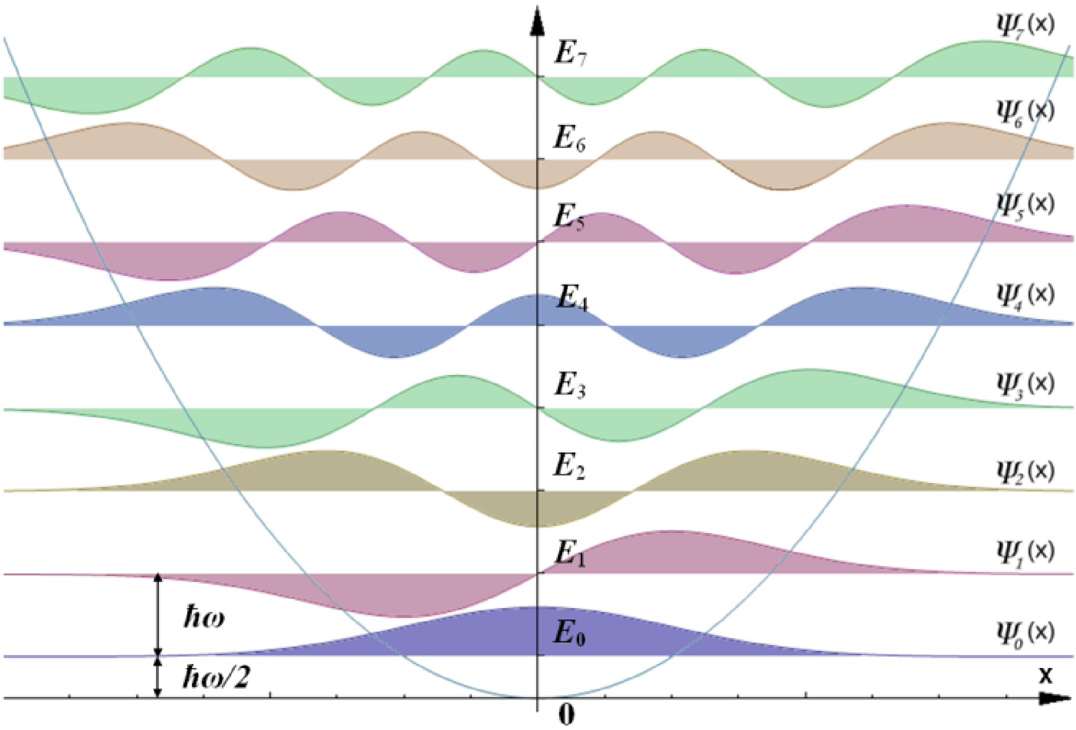

FIGURE 1.5: The harmonic oscillator potential showing the Energy Levels $E_1, E_2, \ldots$ along with the corresponding wavefunctions $\psi_1, \psi_2, \ldots$. Replacing the position coordinate here with charge would give us the Energy levels for an LC oscillator.Taken from © User:Tomasz59 / Wikimedia Commons / CC-BY-SA-3.0

The first few energy levels along with the corresponding wavefunctions are shown in Fig.1.5.

Coherent states of a harmonic oscillator are defined as the eigenstates of the amplitude operator, or annhilation operator a, such that

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{1.58}$$

where $\alpha = |\alpha|e^{j\varphi}$ is a complex number and corresponds to the complex wave amplitude in classical optics. Thus coherent states are wave-like states of the electromagnetic oscillator is a complex number [8].

The coherent state $|\alpha\rangle$ can be represented in terms of the number states $|n\rangle$ as

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-|\alpha|^2/2}\,|n\rangle \tag{1.59}$$

Nonlinear Harmonic Oscillator

Note that the energy levels in the Simple Harmonic Oscillator (SHO) or the LC oscillator are equispaced. This means that if we supply a photon of energy $\hbar\omega_0$ to the LC oscillator, we can change the state from any $|n\rangle$ to $|n+1\rangle$, and all photons emitted due to the transition from $|n\rangle$ to $|n-1\rangle$ will have the same energy.

A Nonlinear Harmonic Oscillator is one where the energy levels do not increase linearly. We can create a nonlinear oscillator by adding a perturbation to the Hamiltonian. The new Hamiltonian will be of the form

$$\hat{\mathcal{H}} = \frac{q^2}{2C} + \frac{\phi^2}{2L} + \mathcal{H}' \tag{1.60}$$

where H' is the perturbation term. The energy levels for this new Hamiltonian can be written in terms of the unperturbed Hamiltonian using perturbation theory as

$$E_n = E_n^{(0)} + \left\langle n^{(0)} \middle| \mathcal{H}' \middle| n^{(0)} \right\rangle + \sum_{k \neq n} \frac{\left| \left\langle k^{(0)} \middle| \mathcal{H}' \middle| n^{(0)} \right\rangle \right|^2}{E_n^{(0)} - E_k^{(0)}} + \ldots \tag{1.61}$$

where,
E_n is the new energy for the nth eigenstate,
$E_n^{(0)}$ is the energy for the nth eigenstate of the unperturbed Hamiltonian,
$\left\langle n^{(0)} \middle| \mathcal{H}' \middle| n^{(0)} \right\rangle$ is the first order correction to the energy and
$\sum_{k \neq n} \frac{\left| \left\langle k^{(0)} \middle| \mathcal{H}' \middle| n^{(0)} \right\rangle \right|^2}{E_n^{(0)} - E_k^{(0)}}$ is the second order correction to the energy.

If the perturbation $\mathcal{H}'$ is not a constant (i.e $\mathcal{H}'(n) \neq \mathcal{H}'(m)$ where $n \neq m$), then we can access only the ground state and the first exited state with one frequency of photons. This is because if the particle is in the first excited state, and another photon of the same energy ($E_{10} = E_1 - E_0$) is supplied to the system, it will not excite the particle further.

This means that we can selectively access only 2 states. If we can manipulate such a 2 level system and it's interactions, we have a qubit!

1.2.3 The Josephson Junction

The Josephson Junction is the nonlinear element used in the described experiments due to it's negligible dissipation rate which is essential for working in the quantum

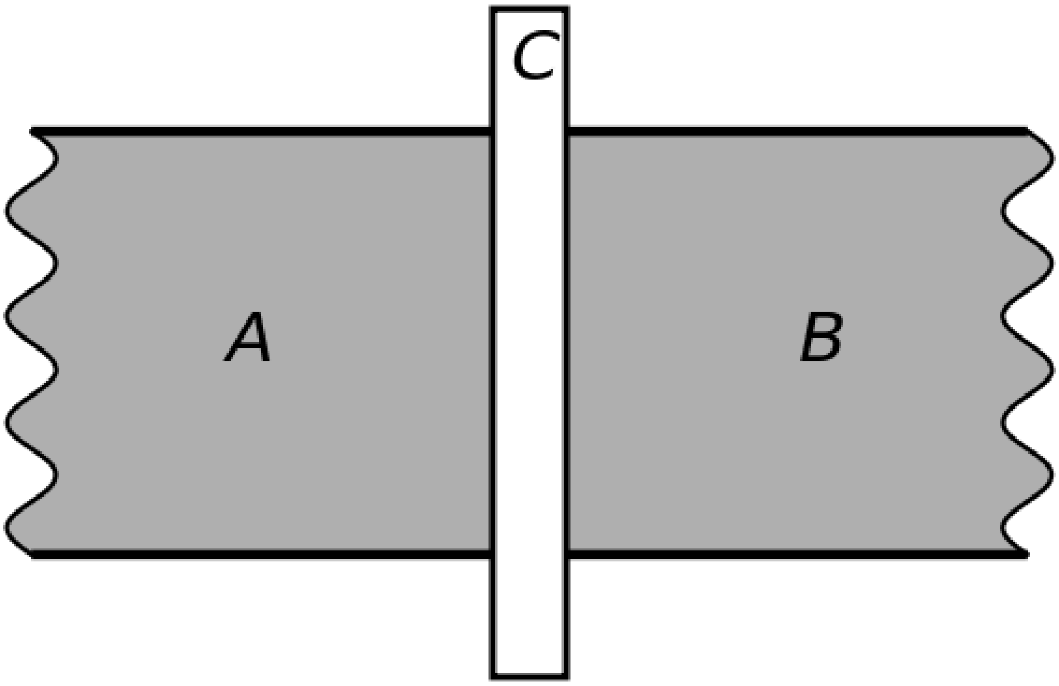

FIGURE 1.6: Simple geometric structure of a Josephson Junction with A and B being superconducting regions and C being the thin insulating layer.

regime.

The Josephson Junction is made of 2 superconductors coupled by a weak link. In our case the junction is an S-I-S (Superconductor-Insulator-Superconductor) junction as shown in Fig.1.6.

Electrons in a normal metal behave like fermions. But at very low temperatures, they form cooper pairs that act like bosons. Nearly all the bosons will be at the lowest energy in exactly the same state [6]. This means that the superconductor will have a macroscopic wavefunction with a single homogeneous amplitude and phase.

In the Josephson Junction , there are two superconductors, and so we can define 2 amplitudes and 2 phases corresponding to each superconductor.

$$\psi_1 = \sqrt{\rho_1} e^{j\theta_1}$$
$$\psi_2 = \sqrt{\rho_2} e^{j\theta_2}$$

Then, the current and voltage characteristics are given by [9]

$$I_s = I_0 \sin\delta \tag{1.62}$$

$$V = \frac{\Phi_0}{2\pi}\dot{\delta} \tag{1.63}$$

where $\delta = \theta_2 - \theta_1$ is the superconducting phase difference[1] associated with the Josephson Junction , $\Phi_0 = h/2e$ is the flux quantum for a cooper pair and I_0 is the critical current of the junction.

We can view the Josephson Junction as a nonlinear inductor and find the inductance simply by using $V = L_J \frac{dI_s}{dt}$ which gives L_J, the Josephson Inductance to be

$$L_J = \frac{\Phi_0}{2\pi}\frac{1}{I_0 \cos\delta} \tag{1.64}$$

In addition to this we can represent a real Josephson Junction using the RCSJ model with a shunting capacitance (C) and resistance (R) along with the bare Josephson Junction [9]. Then the current through the circuit is

$$I = I_s + \frac{V}{R} + C\frac{dV}{dt} \tag{1.65}$$

by using 1.62 and 1.63 we get

$$C\left(\frac{\hbar}{2e}\right)^2 \frac{d^2\delta}{dt^2} + \frac{1}{R}\left(\frac{\hbar}{2e}\right)^2 \frac{d\delta}{dt} + \frac{\hbar}{2e}(I_0 \sin\delta - I) = 0 \tag{1.66}$$

We can see that this is the equation of motion of a particle moving along the δ coordinate with

an acceleration $\frac{d^2\delta}{dt^2}$,

drag force proportional to velocity $\frac{d\delta}{dt}$ and

force due to the gradient of the potential energy as the last term.

[1] This phase difference δ is the generalized phase difference $\delta = \Delta\theta - \frac{2\pi}{\Phi_0}\int A \cdot dl$

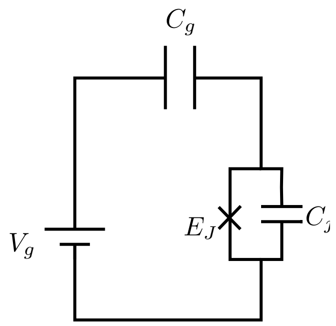

FIGURE 1.7: The Cooper Pair Box circuit with the Josephson Junction represented as a bare Josephson Junction with Josephson Energy E_J and a capacitance C_j.

This leads to the "particle mass" given by

$$M = C\left(\frac{\hbar}{2e}\right)^2 \qquad \text{"particle mass"} \tag{1.67}$$

$$U(I,\delta) = -E_J\cos\delta - \left(\frac{\hbar}{2e}\right)I\delta \qquad \text{potential energy} \tag{1.68}$$

where E_J, the Josephson Energy is given by

$$E_J = \frac{\hbar}{2e}I_0 = \frac{\Phi_0}{2\pi}I_0 \tag{1.69}$$

The current I is usually so small that we can ignore that term making the potential energy

$$U = -E_J\cos\delta \tag{1.70}$$

The electrical energy stored in the capacitance is analogous to kinetic energy and can be calculated as

$$E_{kin} = \frac{1}{2}Mv^2 = \frac{1}{2}C\left(\frac{\hbar}{2e}\right)^2\left(\frac{d\delta}{dt}\right)^2 \tag{1.71}$$

1.2.4 The Cooper Pair Box

The Cooper Pair Box circuit is shown in Fig.1.7. It consists of a superconducting island capacitively coupled (with capacitance C_g) to a voltage source (V_g) connected to ground, and a Josephson Junction connected to the ground. The Josephson Junction can be represented by a capacitance (C_j) and the bare Josephson Junction (represented by E_j) as shown in the figure.

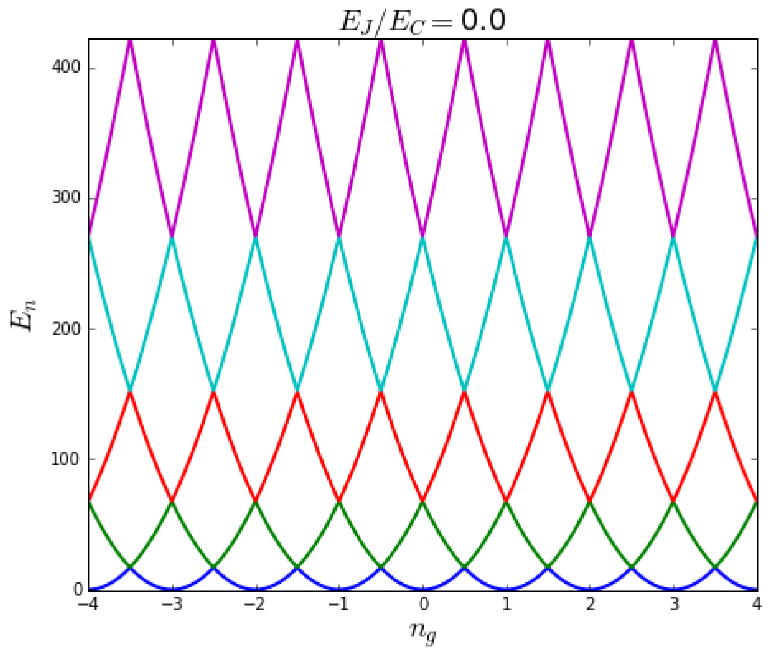

FIGURE 1.8: Energy levels for $E_J = 0$ or no tunneling. This figure was generated using [10].

The electrical energy of this circuit is the energy stored in the 2 capacitors, C_g and C_j. If the total charge of the superconducting island is $-n|e|$, then the electrical energy ($\mathcal{H}_{el}$) is given by [14]

$$\hat{\mathcal{H}_{el}} = 4E_C(\hat{n} - n_g)^2 \tag{1.72}$$

where $\hat{n}\,|n\rangle = n\,|n\rangle$, $|n\rangle$ is the charge state with n cooper pairs.
$E_C = e^2/2C_\Sigma = e^2/2(Cj + Cg)$ is the energy required to add one electron to the island and $n_g = C_g V_g/e$.
The energy levels are shown in Fig.1.8 for $E_J = 0$. The Josephson Junction would allow charge to tunnel through at $n_g = 0.5, -0.5, 1.5, -1.5\ldots$ in order to maintain the lowest energy.

However, to calculate the complete Hamiltonian, we must also take into account the energy of the bare Josephson Junction (H_J) which is given by.

$$\hat{\mathcal{H}_J} = -E_J \cos\hat{\delta} \tag{1.73}$$

This is the tunnelling energy in the phase basis. To find the expression in the charge basis, we start with the commutation relation between charge and phase (See Appendix 1-A-2 from [4]).

$$\left[\hat{n}, \hat{\delta}\right] = -j \tag{1.74}$$

Using 1.74 and the commutator identity 1.75, we get

$$\left[\hat{n}, \hat{\delta}^m\right] = \hat{\delta}^{m-1}\left[\hat{n}, \hat{\delta}\right] + \hat{\delta}\left[\hat{n}, \hat{\delta}^{m-1}\right] \tag{1.75}$$

$$\tag{1.76}$$

we can recursively use this relation to get

$$\left[\hat{n}, \hat{\delta}^m\right] = -jm(\hat{\delta})^{m-1} \tag{1.77}$$

$$\hat{n}\hat{\delta}^m = -jm\hat{\delta}^{m-1} + \hat{\delta}^m\hat{n} \tag{1.78}$$

The operator $\hat{n}e^{jp\hat{\delta}}$ after expansion is

$$\begin{aligned}
\hat{n}e^{jp\hat{\delta}} &= \hat{n}\sum_{m=0}^{\infty}\frac{(jp\hat{\delta})^m}{m!} = \sum_{m=0}^{\infty}\frac{(jp)^m\hat{n}\hat{\delta}^m}{m!} \\
&= \sum_{m=0}^{\infty}\frac{(jp)^m\hat{n}(\hat{\delta})^m}{m!} \\
&= \sum_{m=0}^{\infty}\frac{(jp)^m(-jm\hat{\delta}^{m-1} + \hat{\delta}^m\hat{n})}{m!} \qquad (1.79)\\
&= \sum_{m=0}^{\infty}\frac{(jp)^m(-jm\hat{\delta}^{m-1})}{m!} + \sum_{m=0}^{\infty}\frac{(jp\hat{\delta})^m\hat{n}}{m!} \\
&= p\sum_{m=1}^{\infty}\frac{(jp\hat{\delta})^{m-1}}{(m-1)!} + \sum_{m=0}^{\infty}\frac{(jp\hat{\delta})^m\hat{n}}{m!} \\
\hat{n}e^{jp\hat{\delta}} &= pe^{jp\hat{\delta}} + e^{jp\hat{\delta}}\hat{n} \qquad (1.80)
\end{aligned}$$

Using this operator on the charge state $|n\rangle$ gives

$$\hat{n}\{e^{jp\hat{\delta}}|n\rangle\} = pe^{jp\hat{\delta}}|n\rangle + e^{jp\hat{\delta}}\hat{n}|n\rangle \tag{1.81}$$

$$= (n+p)\{e^{jp\hat{\delta}}|n\rangle\} \tag{1.82}$$

$$\implies e^{jp\hat{\delta}}|n\rangle = |n+p\rangle \tag{1.83}$$

So, we can see that

$$e^{ip\hat{\delta}} = \sum_{m=-\infty}^{\infty} |m+p\rangle \langle m| \tag{1.84}$$

$$\begin{aligned} \cos\hat{\delta} = \frac{1}{2}\left(e^{i\hat{\delta}} + e^{-i\hat{\delta}}\right) &= \frac{1}{2}\left(\sum_{m=-\infty}^{\infty} |m+1\rangle \langle m| + \sum_{m=-\infty}^{\infty} |m-1\rangle \langle m|\right) \\ &= \frac{1}{2}\sum_{n=-\infty}^{+\infty} |n\rangle \langle n+1| + |n+1\rangle \langle n| \end{aligned} \tag{1.85}$$

$$\hat{\mathcal{H}}_J = -\frac{E_J}{2}\left(\sum_{n=-\infty}^{+\infty} |n\rangle \langle n+1| + |n+1\rangle \langle n|\right) \tag{1.86}$$

So the complete Hamiltonian in the charge basis is the sum of these

$$\hat{\mathcal{H}} = \sum_{n=-\infty}^{+\infty} \left(4E_C(\hat{n}-n_g)^2 |n\rangle \langle n| - \frac{E_J}{2} |n\rangle \langle n+1| + |n+1\rangle \langle n|\right) \tag{1.87}$$

In the phase basis we can replace $\hat{n}$ with $-i\frac{\partial}{\partial\delta}$ to get

$$\hat{\mathcal{H}} = 4E_C\left(-i\frac{\partial}{\partial\delta} - n_g\right)^2 - E_J\cos\delta \tag{1.88}$$

The energy eigenstates $|k\rangle$ are given by the schrödinger equation

$$\hat{\mathcal{H}}(n_g)|k\rangle = E_k(n_g)|k\rangle \tag{1.89}$$

These eigenenergies can be solved analytically in the phase basis in terms of Mathieu functions. The eigenenergies are given by

$$E_k(n_g) = E_C a_{2[n_g+k(m,n_g)]}(-E_J/2E_C) \tag{1.90}$$

where $a_\nu(q)$ denotes Mathieu's characteristic value, and $k(m,n_g)$ is a function appropriately sorting the eigenvalues [11].

1.2.5 The 3D Superconducting Transmon

A Transmon is basically a Cooper Pair Box in which the Josephson Junction is shunted with a large capacitance in order to decrease E_C and so increase E_J/E_C.

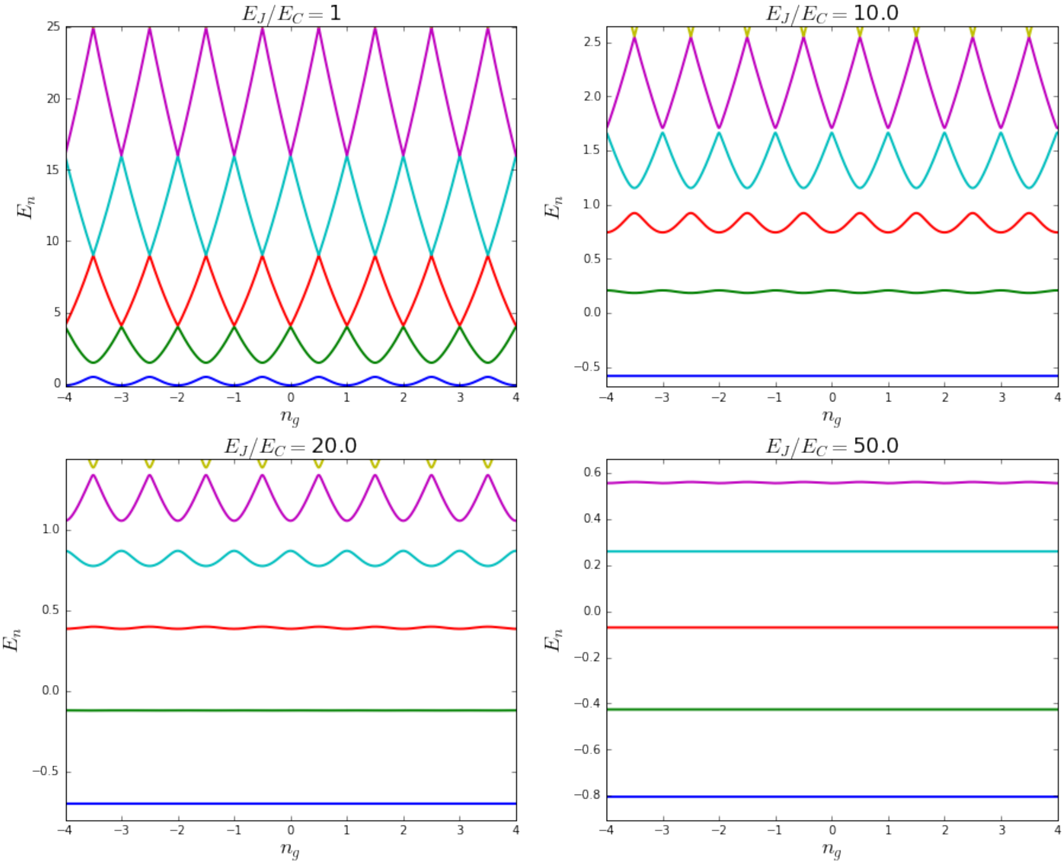

FIGURE 1.9: Energy levels for different E_J/E_C values. Charge noise goes to zero as E_J/E_C increases. Anharmonicity is also low but non-zero. This figure was generated using [10].

The energy levels (E_k) plotted against gate charge (n_g) for different E_J/E_C values are shown in Fig.1.9.

As we can see the charge noise is very low for the case of high E_J/E_C. Also, anharmonicity, which is defined as $\alpha = E_{21} - E_{10}$, is reduced but not zero. In-fact, charge noise reduces exponentially while anharmonicity reduces only algebraically as E_J/E_C is increased. This is the basis on which the transmon qubit is realised.

The Hamiltonian can also be solved using perturbation theory in the limit $E_J/E_C \gg 1$ by expanding the cosine in the tunnelling energy up to fourth order and treating the fourth order term as a perturbation. There is no dependence on n_g because the system is charge insensitive at high E_J/E_C. The energy levels are [11]

$$E_k \approx -E_J + \sqrt{8E_C E_J}\left(k + \frac{1}{2}\right) - \frac{E_C}{12}(6k^2 + 6k + 3) \tag{1.91}$$

From 1.91 we can see that the qubit transition frequency (ω_{10}) is

$$\omega_{10} = \frac{E_1 - E_0}{\hbar} = \frac{\sqrt{8E_J E_C} - E_C}{\hbar} \tag{1.92}$$

and the anharmonicity

$$\alpha = (E_2 - E_1) - (E_1 - E_0) = -E_C \tag{1.93}$$

1.2.6 Coupling the Transmon to a Resonator

For qubit readout and control, we will couple the qubit to a microwave cavity resonator (A rectangular waveguide resonator in this case). The lumped element circuit model of the qubit and cavity resonator is shown in Fig.**??**.

We can approximate the transmon as a 2 level system with a ground state $|g\rangle$ and excited state $|e\rangle$ for large anharmonicity. In this approximation, the Hamiltonian of the system is discussed below [13, 14].

- The Hamiltonian of the transmon can be expressed as the following if we choose the coordinate system appropriately and the zero energy as the mean energy of the transmon
$$\hat{\mathcal{H}}_{qubit} = -\frac{\hbar\omega_q}{2}\sigma_z \tag{1.94}$$

- The Hamiltonian of the resonator is the same as the one for an LC oscillator 1.52c
$$\hat{\mathcal{H}}_{cavity} = \hbar\omega_r a^\dagger a \tag{1.95}$$
The ground state energy of $\hbar\omega_r/2$ has not been shown as it does not have any significance in qubit dynamics.

- The interaction hamiltonian is given by
$$\hat{\mathcal{H}}_{int} = \hbar g(a^\dagger \sigma_- + a\sigma_+) \tag{1.96}$$
where g is the coupling constant, proportional to the amplitude of the signal. The rotating wave approximation is made here, ignoring terms with $a^\dagger\sigma_+$ and $a\sigma_-$.

This gives us the Jaynes-Cummings Hamiltonian

$$\hat{\mathcal{H}} = \hbar\omega_r a^\dagger a - \frac{\hbar\omega_q}{2}\sigma_z + \hbar g(a^\dagger\sigma_- + a\sigma_+) \tag{1.97}$$

The eigenstates for the coupled system are

$$|n,+\rangle = \cos\frac{\theta_n}{2}|e\rangle|n\rangle + \sin\frac{\theta_n}{2}|g\rangle|n+1\rangle \tag{1.98a}$$
$$|n,-\rangle = -\sin\frac{\theta_n}{2}|e\rangle|n\rangle + \cos\frac{\theta_n}{2}|g\rangle|n+1\rangle \tag{1.98b}$$

with eigenenergies

$$E_{n\pm} = \hbar\omega_r n \pm \frac{\sqrt{\hbar^2\Delta^2 + 4g^2(n+1)}}{2} \tag{1.99}$$

where $\Delta = \omega_r - \omega_q$ and

$$\theta_n = \tan^{-1}\left(\frac{2g\sqrt{n+1}}{\hbar\Delta}\right) \tag{1.100}$$

1.2.7 The Bloch Sphere

A general pure state for a two level system can represented as

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1.101}$$

where $|0\rangle$ and $|1\rangle$ form an orthonormal basis. $\alpha = |\alpha|e^{j\phi_1}$ and $\beta = |\beta|e^{jphi_2}$ are complex numbers with a constraint that $|\alpha|^2 + |\beta|^2 = 1$ due to normalization. Since one can only measure the phase difference $\phi = \phi_2 - \phi_1$, the state can be represented as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{j\phi}\sin\frac{\theta}{2}|1\rangle \tag{1.102}$$

with no loss of generality where $\theta = 2\cos^{-1}|\alpha|$.

This state can be represented on a sphere with spherical coordinated with $\rho = 1$. The $|0\rangle$,$|1\rangle$ and an arbitrary state is shown in the Bloch Sphere in Fig.**??**.

1.2.8 Dynamics of the Jaynes-Cummings system

Let us consider the dynamics of the system in three different cases that are relevant to qubit manipulation and readout.

- **Zero Coupling** ($g = 0$)

  If there is no coupling (no photons in the cavity), then the qubit follows the free Hamiltonian given by 1.94. The energy eigenstates of this system are $|g\rangle$ and $|e\rangle$ with energies $-\hbar\omega_q/2$ and $\hbar\omega_q/2$ respectively. In this basis, the Hamiltonian of the qubit can be expressed as

$$\hat{\mathcal{H}}_{qubit} = -\hbar\omega_q/2\,|e\rangle\,\langle e| + \hbar\omega_q/2\,|g\rangle\,\langle g| \tag{1.103}$$

  Applying the unitary operator $e^{i\hat{\mathcal{H}}t/\hbar}$ to an initial superposition state $|\psi(0)\rangle = \alpha\,|e\rangle + \beta\,|g\rangle$ gives the state at a time t

$$\begin{aligned} |\psi(t)\rangle &= e^{i\hat{\mathcal{H}}t/\hbar}\,|\psi(0)\rangle \\ &= e^{-i\omega_q t/2}\left(\alpha\,|e\rangle + e^{i\omega_q t}\beta g\right) \end{aligned} \tag{1.104}$$

  We can see that if the qubit starts in an eigenstate, it remains in the same state (only a phase factor is added), but if it starts in a superposition, it's phase oscillates as a function of time with a frequency equal to the qubit frequency. On the bloch sphere this can be represented as a precession about the z-axis.

  We can consider the dynamics of the coupled system in a similar way, once we change the basis to the new energy eigenstates in 1.98.

- **On Resonance** ($\Delta \ll g$)

  If the drive signal or photons are at the same frequency as the qubit, then $\Delta = 0$, which implies $\theta_n = \pi/2$. Then the eigenstates are

$$|n,+\rangle = \frac{1}{\sqrt{2}}\,|e\rangle\,|n\rangle + \frac{1}{\sqrt{2}}\,|g\rangle\,|n+1\rangle \tag{1.105a}$$

$$|n,-\rangle = -\frac{1}{\sqrt{2}}\,|e\rangle\,|n\rangle + \frac{1}{\sqrt{2}}\,|g\rangle\,|n+1\rangle \tag{1.105b}$$

  With the new eigenstates, the qubit will oscillate about the x-axis. This means that if the initial state of the qubit is $|g\rangle$, it will oscillate to $|e\rangle$ in time $t = \pi g\sqrt{n+1}$. The excitation number of the cavity will also oscillate from $|n+1\rangle$ to $|n\rangle$ with the same frequency.

It is worth noting that if the resonant frequency of the cavity is far from that of the qubit, and if the drive signal is of qubit frequency, then the qubit will still oscillate because it takes finite time for the photons to decay in the cavity.

- **Dispersive limit** ($\Delta \gg g$)

  In the dispersive limit, we can rewrite the Hamiltonian by considering g/Δ as a perturbation [14] to get

$$\hat{\mathcal{H}} = \hbar\left(\omega_r + \frac{g^2}{\Delta}\sigma_z\right)a^\dagger a + \frac{\hbar}{2}\left(\omega_q + \frac{g^2}{\Delta}\right)\sigma_z \tag{1.106}$$

We can see from this Hamiltonian that dispersive coupling causes a shift of $\pm g^2/\Delta$ in the resonance frequency of the cavity depending on the qubit state. We can see the different S parameter responses for the ground and excited states of the qubit in Fig.**??**. This shift is crucial to measuring the state of the qubit. We can also rearrange the equation as follows

$$\hat{\mathcal{H}} = \hbar\omega_r a^\dagger a + \frac{\hbar}{2}\left(\omega_q + \frac{g^2}{\Delta}a^\dagger a + \frac{g^2}{\Delta}\right)\sigma_z \tag{1.107}$$

to see that the frequency of the qubit now has an added Lamb and ac-Stark shift of g^2/Δ and $g^2 n/\Delta$ respectively [3].

The Hamiltonian in 1.106 can be written as

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{int} \tag{1.108}$$

where

$$\hat{\mathcal{H}}_0 = \hbar\omega_r a^\dagger a + \frac{\hbar\omega_q}{2}\sigma_z \tag{1.109}$$

is the uncoupled Hamiltonian of the qubit and cavity and

$$\hat{\mathcal{H}}_{int} = \frac{\hbar g^2 \sigma_z}{\Delta}\left(a^\dagger a + \frac{1}{2}\right) \tag{1.110}$$

is the interaction Hamiltonian.

The unitary time evolution operator given by $\hat{U}(t) = e^{-j\hat{\mathcal{H}}t/\hbar}$ is

$$\begin{aligned}\hat{U}(t) = \exp\left(-j\hat{\mathcal{H}}_0 t/\hbar\right)\Bigg[&\exp\left(-j\frac{g^2 t}{\Delta}\left(\hat{n}+\frac{1}{2}\right)\right)|g\rangle\langle g| \\ &+\exp\left(j\frac{g^2 t}{\Delta}\left(\hat{n}+\frac{1}{2}\right)\right)|e\rangle\langle e|\Bigg]\end{aligned} \tag{1.111}$$

If the initial state is such that the cavity is in a coherent state $|\alpha\rangle$ and the qubit is in a superposition state $(|g\rangle + |e\rangle)/\sqrt{2}$, i.e

$$|\psi(0)\rangle = \frac{(|g\rangle + |e\rangle)}{\sqrt{2}}|\alpha\rangle \tag{1.112}$$

we can apply the unitary time evolution operator on this state and use the following relation

$$e^{i\varphi\hat{n}}|\alpha\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n e^{i\varphi n}}{\sqrt{n!}}e^{-\frac{|\alpha|^2}{2}}|n\rangle = \left|\alpha e^{i\varphi}\right\rangle \tag{1.113}$$

to get

$$\begin{aligned}|\psi(t)\rangle &= \hat{U}(t)|\psi(0)\rangle \\ &= \exp\left(-j\hat{\mathcal{H}}_0 t/\hbar\right)\frac{1}{\sqrt{2}}\left[e^{-j\varphi/2}|g\rangle\left|\alpha e^{-j\varphi}\right\rangle + e^{j\varphi/2}|e\rangle\left|\alpha e^{j\varphi}\right\rangle\right]\end{aligned} \tag{1.114}$$

where $\varphi = g^2 t/\Delta$. We can see that the cavity and qubit states are now entangled. The qubit ground state is entangled with the coherent state rotated by an angle $-\varphi$ and the excited state is entangled with the coherent state rotated by an angle φ.

At low powers, dispersive measurement is QND or Quantum Non-Demolition, which means that the qubit will remain in the eigenstate that was measured after a measurement. This means that repeated measurements will yield the same results.

1.2.9 Decoherence

A qubit which is initially in a superposition state $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle$ will not stay in the superposition forever. It will lose its quantum information stochastically at a rate described below. The time for which the qubit retains its quantum information is called "coherence time", and is denoted by T_2. There are 2 processes which cause decoherence [7].

- **Relaxation** The decay of the qubit to the ground state $|g\rangle$ due to spontaneous emission is referred to as relaxation. It does this at a rate $\Gamma_{\downarrow}$. If the qubit is in contact with a thermal bath at non zero temperature, there will also be excitation processes at a rate $\Gamma_{\uparrow}$. The combined effect gives the relaxation time $T_1 = (\Gamma_{\downarrow} + \Gamma_{\uparrow})^{-1}$. Relaxation processes can be attributed to phenomena that cause energy fluctuations.
- **Dephasing** Dephasing is the process by which the qubit loses its phase information due to fluctuations is qubit frequency. These fluctuations cause the qubit to gain or lose phase and on average, lead to a diffused phase.

1.2.10 Measurement Theory

The data presented in this thesis uses a low power, dispersive measurement technique described in [2].

Chapter 2

Chapter Title Here

2.1 Welcome and Thank You

Welcome to this LaTeX Thesis Template, a beautiful and easy to use template for writing a thesis using the LaTeX typesetting system.

If you are writing a thesis (or will be in the future) and its subject is technical or mathematical (though it doesn't have to be), then creating it in LaTeX is highly recommended as a way to make sure you can just get down to the essential writing without having to worry over formatting or wasting time arguing with your word processor.

LaTeX is easily able to professionally typeset documents that run to hundreds or thousands of pages long. With simple mark-up commands, it automatically sets out the table of contents, margins, page headers and footers and keeps the formatting consistent and beautiful. One of its main strengths is the way it can easily typeset mathematics, even *heavy* mathematics. Even if those equations are the most horribly twisted and most difficult mathematical problems that can only be solved on a super-computer, you can at least count on LaTeX to make them look stunning.

2.2 Learning LaTeX

LaTeX is not a WYSIWYG (What You See is What You Get) program, unlike word processors such as Microsoft Word or Apple's Pages. Instead, a document written for LaTeX is actually a simple, plain text file that contains *no formatting*. You tell LaTeX how you want the formatting in the finished document by writing in simple commands amongst the text, for example, if I want to use *italic text for emphasis*, I write the `\emph{text}` command and put the text I want in italics in between the curly braces. This means that LaTeX is a "mark-up" language, very much like HTML.

2.2.1 A (not so short) Introduction to LaTeX

If you are new to LaTeX, there is a very good eBook – freely available online as a PDF file – called, "The Not So Short Introduction to LaTeX". The book's title is typically shortened to just *lshort*. You can download the latest version (as it is occasionally updated) from here: http://www.ctan.org/tex-archive/info/lshort/english/lshort.pdf

It is also available in several other languages. Find yours from the list on this page: http://www.ctan.org/tex-archive/info/lshort/

It is recommended to take a little time out to learn how to use LaTeX by creating several, small 'test' documents, or having a close look at several templates on:
http://www.LaTeXTemplates.com
Making the effort now means you're not stuck learning the system when what you *really* need to be doing is writing your thesis.

2.2.2 A Short Math Guide for LaTeX

If you are writing a technical or mathematical thesis, then you may want to read the document by the AMS (American Mathematical Society) called, "A Short Math Guide for LaTeX". It can be found online here: http://www.ams.org/tex/amslatex.html under the "Additional Documentation" section towards the bottom of the page.

2.2.3 Common LaTeX Math Symbols

There are a multitude of mathematical symbols available for LaTeX and it would take a great effort to learn the commands for them all. The most common ones you are likely to use are shown on this page: http://www.sunilpatel.co.uk/latex-type/latex-math-symbols/

You can use this page as a reference or crib sheet, the symbols are rendered as large, high quality images so you can quickly find the LaTeX command for the symbol you need.

2.2.4 LaTeX on a Mac

The LaTeX distribution is available for many systems including Windows, Linux and Mac OS X. The package for OS X is called MacTeX and it contains all the applications you need – bundled together and pre-customized – for a fully working LaTeX environment and work flow.

MacTeX includes a custom dedicated LaTeX editor called TeXShop for writing your '**`.tex`**' files and BibDesk: a program to manage your references and create your bibliography section just as easily as managing songs and creating playlists in iTunes.

2.3 Getting Started with this Template

If you are familiar with LaTeX, then you should explore the directory structure of the template and then proceed to place your own information into the *THESIS INFORMATION* block of the **`main.tex`** file. You can then modify the rest of this file to your unique specifications based on your degree/university. Section 2.5 on page 35 will help you do this. Make sure you also read section 2.7 about thesis conventions to get the most out of this template.

If you are new to LaTeX it is recommended that you carry on reading through the rest of the information in this document.

Before you begin using this template you should ensure that its style complies with the thesis style guidelines imposed by your institution. In most cases this template style and layout will be suitable. If it is not, it may only require a small change to bring the template in line with your institution's recommendations. These modifications will need to be done on the **`MastersDoctoralThesis.cls`** file.

2.3.1 About this Template

This LaTeX Thesis Template is originally based and created around a LaTeX style file created by Steve R. Gunn from the University of Southampton (UK), department of Electronics and Computer Science. You can find his original thesis style file at his site, here: http://www.ecs.soton.ac.uk/~srg/softwaretools/document/templates/

Steve's **ecsthesis.cls** was then taken by Sunil Patel who modified it by creating a skeleton framework and folder structure to place the thesis files in. The resulting template can be found on Sunil's site here: http://www.sunilpatel.co.uk/thesis-template

Sunil's template was made available through http://www.LaTeXTemplates.com where it was modified many times based on user requests and questions. Version 2.0 and onwards of this template represents a major modification to Sunil's template and is, in fact, hardly recognisable. The work to make version 2.0 possible was carried out by Vel and Johannes Böttcher.

2.4 What this Template Includes

2.4.1 Folders

This template comes as a single zip file that expands out to several files and folders. The folder names are mostly self-explanatory:

Appendices – this is the folder where you put the appendices. Each appendix should go into its own separate **.tex** file. An example and template are included in the directory.

Chapters – this is the folder where you put the thesis chapters. A thesis usually has about six chapters, though there is no hard rule on this. Each chapter should go in its own separate **.tex** file and they can be split as:

- Chapter 1: Introduction to the thesis topic
- Chapter 2: Background information and theory
- Chapter 3: (Laboratory) experimental setup

- Chapter 4: Details of experiment 1
- Chapter 5: Details of experiment 2
- Chapter 6: Discussion of the experimental results
- Chapter 7: Conclusion and future directions

This chapter layout is specialised for the experimental sciences, your discipline may be different.

Figures – this folder contains all figures for the thesis. These are the final images that will go into the thesis document.

2.4.2 Files

Included are also several files, most of them are plain text and you can see their contents in a text editor. After initial compilation, you will see that more auxiliary files are created by LaTeX or BibTeX and which you don't need to delete or worry about:

example.bib – this is an important file that contains all the bibliographic information and references that you will be citing in the thesis for use with BibTeX. You can write it manually, but there are reference manager programs available that will create and manage it for you. Bibliographies in LaTeX are a large subject and you may need to read about BibTeX before starting with this. Many modern reference managers will allow you to export your references in BibTeX format which greatly eases the amount of work you have to do.

MastersDoctoralThesis.cls – this is an important file. It is the class file that tells LaTeX how to format the thesis.

main.pdf – this is your beautifully typeset thesis (in the PDF file format) created by LaTeX. It is supplied in the PDF with the template and after you compile the template you should get an identical version.

main.tex – this is an important file. This is the file that you tell LaTeX to compile to produce your thesis as a PDF file. It contains the framework and constructs that tell LaTeX how to layout the thesis. It is heavily commented so you can read exactly what each line of code does and why it is there. After you put your own information into the *THESIS INFORMATION* block – you have now started your thesis!

Files that are *not* included, but are created by LaTeX as auxiliary files include:

main.aux – this is an auxiliary file generated by LaTeX, if it is deleted LaTeX simply regenerates it when you run the main `.tex` file.

main.bbl – this is an auxiliary file generated by BibTeX, if it is deleted, BibTeX simply regenerates it when you run the `main.aux` file. Whereas the `.bib` file contains all the references you have, this `.bbl` file contains the references you have actually cited in the thesis and is used to build the bibliography section of the thesis.

main.blg – this is an auxiliary file generated by BibTeX, if it is deleted BibTeX simply regenerates it when you run the main `.aux` file.

main.lof – this is an auxiliary file generated by LaTeX, if it is deleted LaTeX simply regenerates it when you run the main `.tex` file. It tells LaTeX how to build the *List of Figures* section.

main.log – this is an auxiliary file generated by LaTeX, if it is deleted LaTeX simply regenerates it when you run the main `.tex` file. It contains messages from LaTeX, if you receive errors and warnings from LaTeX, they will be in this `.log` file.

main.lot – this is an auxiliary file generated by LaTeX, if it is deleted LaTeX simply regenerates it when you run the main `.tex` file. It tells LaTeX how to build the *List of Tables* section.

main.out – this is an auxiliary file generated by LaTeX, if it is deleted LaTeX simply regenerates it when you run the main `.tex` file.

So from this long list, only the files with the `.bib`, `.cls` and `.tex` extensions are the most important ones. The other auxiliary files can be ignored or deleted as LaTeX and BibTeX will regenerate them.

2.5 Filling in Your Information in the `main.tex` File

You will need to personalise the thesis template and make it your own by filling in your own information. This is done by editing the `main.tex` file in a text editor or your favourite LaTeX environment.

Open the file and scroll down to the third large block titled *THESIS INFORMATION* where you can see the entries for *University Name*, *Department Name*, etc …

Fill out the information about yourself, your group and institution. You can also insert web links, if you do, make sure you use the full URL, including the `http://` for this. If you don't want these to be linked, simply remove the `\href{url}{name}` and only leave the name.

When you have done this, save the file and recompile `main.tex`. All the information you filled in should now be in the PDF, complete with web links. You can now begin your thesis proper!

2.6 The `main.tex` File Explained

The **`main.tex`** file contains the structure of the thesis. There are plenty of written comments that explain what pages, sections and formatting the LaTeX code is creating. Each major document element is divided into commented blocks with titles in all capitals to make it obvious what the following bit of code is doing. Initially there seems to be a lot of LaTeX code, but this is all formatting, and it has all been taken care of so you don't have to do it.

Begin by checking that your information on the title page is correct. For the thesis declaration, your institution may insist on something different than the text given. If this is the case, just replace what you see with what is required in the *DECLARATION PAGE* block.

Then comes a page which contains a funny quote. You can put your own, or quote your favourite scientist, author, person, and so on. Make sure to put the name of the person who you took the quote from.

Following this is the abstract page which summarises your work in a condensed way and can almost be used as a standalone document to describe what you have done. The text you write will cause the heading to move up so don't worry about running out of space.

Next come the acknowledgements. On this page, write about all the people who you wish to thank (not forgetting parents, partners and your advisor/supervisor).

The contents pages, list of figures and tables are all taken care of for you and do not need to be manually created or edited. The next set of pages are more likely to be optional and can be deleted since they are for a more technical thesis: insert a list of abbreviations you have used in the thesis, then a list of the physical constants and numbers you refer to and finally, a list of mathematical symbols used in any formulae. Making the effort to fill these tables means the reader has a one-stop place to refer to instead of searching the internet and references to try and find out what you meant by certain abbreviations or symbols.

The list of symbols is split into the Roman and Greek alphabets. Whereas the abbreviations and symbols ought to be listed in alphabetical order (and this is *not* done automatically for you) the list of physical constants should be grouped into similar themes.

The next page contains a one line dedication. Who will you dedicate your thesis to?

Finally, there is the block where the chapters are included. Uncomment the lines (delete the `%` character) as you write the chapters. Each chapter should be written in its own file and put into the *Chapters* folder and named **`Chapter1`**, **`Chapter2`**, etc... Similarly for the appendices, uncomment the lines as you need them. Each appendix should go into its own file and placed in the *Appendices* folder.

After the preamble, chapters and appendices finally comes the bibliography. The bibliography style (called *`authoryear`*) is used for the bibliography and is a fully featured style that will even include links to where the referenced paper can be found online. Do not underestimate how grateful your reader will be to find that a reference to a paper is just a click away. Of course, this relies on you putting the URL information into the BibTeX file in the first place.

2.7 Thesis Features and Conventions

To get the best out of this template, there are a few conventions that you may want to follow.

One of the most important (and most difficult) things to keep track of in such a long document as a thesis is consistency. Using certain conventions and ways of doing things (such as using a Todo list) makes the job easier. Of course, all of these are optional and you can adopt your own method.

2.7.1 Printing Format

This thesis template is designed for double sided printing (i.e. content on the front and back of pages) as most theses are printed and bound this way. Switching to one sided printing is as simple as uncommenting the *`oneside`* option of the `documentclass` command at the top of the **`main.tex`** file. You may then wish to adjust the margins to suit specifications from your institution.

The headers for the pages contain the page number on the outer side (so it is easy to flick through to the page you want) and the chapter name on the inner side.

The text is set to 11 point by default with single line spacing, again, you can tune the text size and spacing should you want or need to using the options at the very start of **`main.tex`**. The spacing can be changed similarly by replacing the *`singlespacing`* with *`onehalfspacing`* or *`doublespacing`*.

2.7.2 Using US Letter Paper

The paper size used in the template is A4, which is the standard size in Europe. If you are using this thesis template elsewhere and particularly in the United States, then you may have to change the A4 paper size to the US Letter size. This can be done in the margins settings section in **`main.tex`**.

Due to the differences in the paper size, the resulting margins may be different to what you like or require (as it is common for institutions to dictate certain margin sizes). If this is the case, then the margin sizes can be tweaked by modifying the values in the same block as where you set the paper size. Now your document should be set up for US Letter paper size with suitable margins.

2.7.3 References

The `biblatex` package is used to format the bibliography and inserts references such as this one [**Reference1**]. The options used in the **`main.tex`** file mean that the in-text citations of references are formatted with the author(s) listed with the date of the publication. Multiple references are separated by semicolons (e.g. [**Reference2**, **Reference1**]) and references with more than three authors only show the first author with *et al.* indicating there are more authors (e.g. [**Reference3**]). This is done automatically for you. To see how you use references, have a look at the **`Chapter1.tex`** source file. Many reference managers allow you to simply drag the reference into the document as you type.

Scientific references should come *before* the punctuation mark if there is one (such as a comma or period). The same goes for footnotes[1]. You can change this but the most important thing is to keep the convention consistent throughout the thesis. Footnotes themselves should be full, descriptive sentences (beginning with a capital letter and ending with a full stop). The APA6 states: "Footnote numbers should be superscripted, [...], following any punctuation mark except a dash." The Chicago manual of style states: "A note number should be placed at the end of a sentence or clause. The number follows any punctuation mark except the dash, which it precedes. It follows a closing parenthesis."

The bibliography is typeset with references listed in alphabetical order by the first author's last name. This is similar to the APA referencing style. To see how LaTeX typesets the bibliography, have a look at the very end of this document (or just click on the reference number links in in-text citations).

A Note on bibtex

The bibtex backend used in the template by default does not correctly handle unicode character encoding (i.e. "international" characters). You may see a warning about this in the compilation log and, if your references contain unicode characters, they may not show up correctly or at all. The solution to this is to use the biber backend instead of the outdated bibtex backend. This is done by finding this in **`main.tex`**: *`backend=bibtex`* and changing it to *`backend=biber`*. You will then need to delete all auxiliary BibTeX files and navigate to the template directory in your terminal (command prompt). Once there, simply type `biber main` and biber will compile your bibliography. You can then compile **`main.tex`** as normal and

[1] Such as this footnote, here down at the bottom of the page.

TABLE 2.1: The effects of treatments X and Y on the four groups studied.

Groups	**Treatment X**	**Treatment Y**
1	0.2	0.8
2	0.17	0.7
3	0.24	0.75
4	0.68	0.3

your bibliography will be updated. An alternative is to set up your LaTeX editor to compile with biber instead of bibtex, see here for how to do this for various editors.

2.7.4 Tables

Tables are an important way of displaying your results, below is an example table which was generated with this code:

```
\begin{table}
\caption{The effects of treatments X and Y on the four groups studied.}
\label{tab:treatments}
\centering
\begin{tabular}{l l l}
\toprule
\tabhead{Groups} & \tabhead{Treatment X} & \tabhead{Treatment Y} \\
\midrule
1 & 0.2 & 0.8\\
2 & 0.17 & 0.7\\
3 & 0.24 & 0.75\\
4 & 0.68 & 0.3\\
\bottomrule\\
\end{tabular}
\end{table}
```

You can reference tables with `\ref{<label>}` where the label is defined within the table environment. See **`Chapter1.tex`** for an example of the label and citation (e.g. Table 2.1).

2.7.5 Figures

There will hopefully be many figures in your thesis (that should be placed in the *Figures* folder). The way to insert figures into your thesis is to use a code template like this:

```
\begin{figure}
\centering
\includegraphics{Figures/Electron}
\decoRule
\caption[An Electron]{An electron (artist's impression).}
\label{fig:Electron}
\end{figure}
```

Also look in the source file. Putting this code into the source file produces the picture of the electron that you can see in the figure below.

FIGURE 2.1: An electron (artist's impression).

Sometimes figures don't always appear where you write them in the source. The placement depends on how much space there is on the page for the figure. Sometimes there is not enough room to fit a figure directly where it should go (in relation to the text) and so LaTeX puts it at the top of the next page. Positioning figures is the job of LaTeX and so you should only worry about making them look good!

Figures usually should have captions just in case you need to refer to them (such as in Figure 2.1). The `\caption` command contains two parts, the first part, inside the square brackets is the title that will appear in the *List of Figures*, and so should be short. The second part in the curly brackets should contain the longer and more descriptive caption text.

The `\decoRule` command is optional and simply puts an aesthetic horizontal line below the image. If you do this for one image, do it for all of them.

LaTeX is capable of using images in pdf, jpg and png format.

2.7.6 Typesetting mathematics

If your thesis is going to contain heavy mathematical content, be sure that LaTeX will make it look beautiful, even though it won't be able to solve the equations for you.

The "Not So Short Introduction to LaTeX" (available on CTAN) should tell you everything you need to know for most cases of typesetting mathematics. If you need more information, a much more thorough mathematical guide is available from the AMS called, "A Short Math Guide to LaTeX" and can be downloaded from: `ftp://ftp.ams.org/pub/tex/doc/amsmath/short-math-guide.pdf`

There are many different LaTeX symbols to remember, luckily you can find the most common symbols in The Comprehensive LaTeX Symbol List.

You can write an equation, which is automatically given an equation number by LaTeX like this:

```
\begin{equation}
E = mc^{2}
\label{eqn:Einstein}
\end{equation}
```

This will produce Einstein's famous energy-matter equivalence equation:

$$E = mc^2 \tag{2.1}$$

All equations you write (which are not in the middle of paragraph text) are automatically given equation numbers by LaTeX. If you don't want a particular equation numbered, use the unnumbered form:

```
\[ a^{2}=4 \]
```

2.8 Sectioning and Subsectioning

You should break your thesis up into nice, bite-sized sections and subsections. LaTeX automatically builds a table of Contents by looking at all the `\chapter{}`, `\section{}` and `\subsection{}` commands you write in the source.

The Table of Contents should only list the sections to three (3) levels. A `chapter{}` is level zero (0). A `\section{}` is level one (1) and so a `\subsection{}` is level two (2). In your thesis it is likely that you will even use a `subsubsection{}`, which is level three (3). The depth to which the Table of Contents is formatted is set within **`MastersDoctoralThesis.cls`**. If you need this changed, you can do it in **`main.tex`**.

2.9 In Closing

You have reached the end of this mini-guide. You can now rename or overwrite this pdf file and begin writing your own **`Chapter1.tex`** and the rest of your thesis. The easy work of setting up the structure and framework has been taken care of for you. It's now your job to fill it out!

Good luck and have lots of fun!

Guide written by —

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Vel: LaTeXTemplates.com

Appendix A

Frequently Asked Questions

A.1 How do I change the colors of links?

The color of links can be changed to your liking using:

`\hypersetup{urlcolor=red}`, or

`\hypersetup{citecolor=green}`, or

`\hypersetup{allcolor=blue}`.

If you want to completely hide the links, you can use:

`\hypersetup{allcolors=.}`, or even better:

`\hypersetup{hidelinks}`.

If you want to have obvious links in the PDF but not the printed text, use:

`\hypersetup{colorlinks=false}`.

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