電腦視覺與應用
Computer Vision and Applications

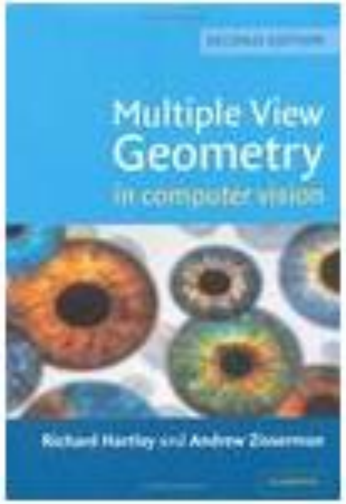
Lecture-03 Projective 2D geometry

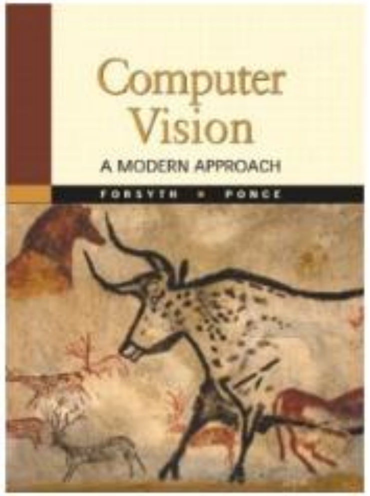
Tzung-Han Lin
National Taiwan University of Science and Technology
Graduate Institute of Color and Illumination Technology
e-mail: thl@mail.ntust.edu.tw

Projective 2D geometry

Lecture Reference at:

- Multiple View Geometry in Computer Vision, Chapter 2. (major)
- Computer Vision A Modern Approach, Chapter 10.





Keywords list

- Scalar, vector, matrix
- Line equation, conics
- Homogeneous, inhomogenous, homogeneous coordinates
- Homogenous represnetation for point, line.
- Projective invariants
- DOF: degree of freedom
- Homography (homographic transform)
- Point at infinity
- Dual, duality, envelopes

Notation remark

- **Bolod**: vector or matrix form, ex. $a\mathbf{x} + b\mathbf{y} = \mathbf{c}$
- *Italic*: scalar or variable, ex. $ax+by = 1$
- Upper case for 3D, ex. X
- Lower case for 2D, ex. x
- **X**→ a 3D vector, **x**→ a 2D vector
- $[X, Y, Z, 1]$ → a homogenous 3D vector (consisting of 3 scalars)

- Normal text is usually a list of text or character.

Projective 2D geometry

Topics

- Points, lines & conics
- Transformations & invariants (between images)
- 1D projective geometry and the Cross-ratio

Homogeneous coordinates

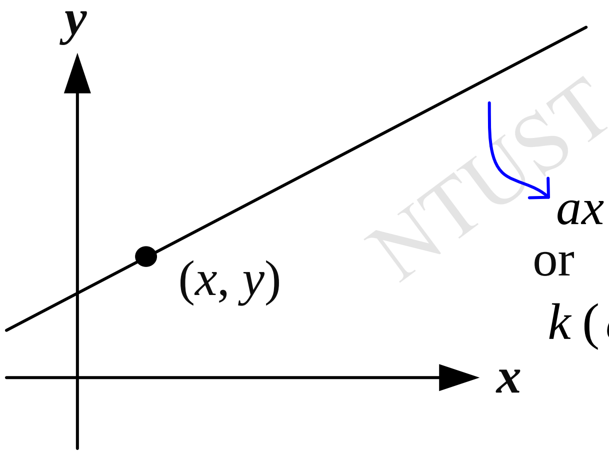
- Homogeneous representation of lines

$$ax + by + c = 0 \qquad (a,b,c)^{\mathsf{T}}$$

$$(ka)x + (kb)y + kc = 0, \forall k \neq 0 \qquad (a,b,c)^{\mathsf{T}} \sim k(a,b,c)^{\mathsf{T}}$$

equivalence class of vectors, any vector is representative

Set of all equivalence classes in $\mathbf{R}^3 - (0,0,0)^{\mathsf{T}}$ forms $\mathbf{P}^2$



$$ax + by + c = 0$$

or

$$k(ax + by + c) = 0$$

define one line as a vector format:

$$\mathbf{l} = (a,b,c)^{\mathsf{T}}$$

Homogeneous coordinates

- Homogeneous coordinates of points

$$\mathbf{x}=(x,y,1)^\mathsf{T} \text{ on } \mathbf{l}=(a,b,c)^\mathsf{T} \text{ if and only if } ax+by+c=0$$

$$(x,y,1)(a,b,c)^\mathsf{T}=(x,y,1)\mathbf{l}=0$$

$$(x,y,1)^\mathsf{T}\sim k(x,y,1)^\mathsf{T}, \forall k\neq 0$$

- The point **x** lies on the line **l** if and only if $\mathbf{x}^\mathsf{T}\mathbf{l}=\mathbf{l}^\mathsf{T}\mathbf{x}=0$

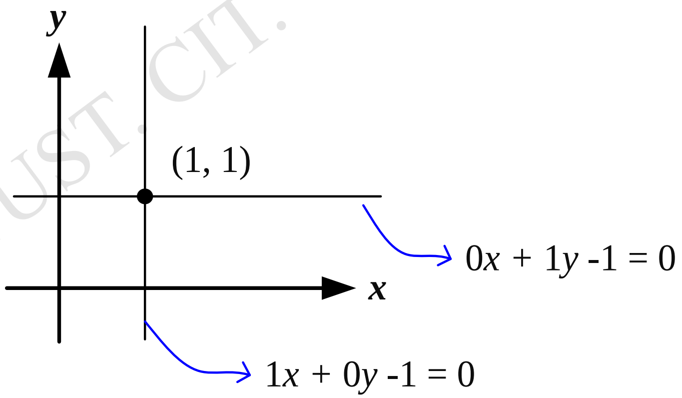
Homogeneous coordinates $(x_1,x_2,x_3)^\mathsf{T}$ but only 2DOF

Inhomogeneous coordinates $(x,y)^\mathsf{T}$

2D Points from lines and vice-versa

- Intersections of lines
  - The intersection of two lines $\mathbf{l}$ and $\mathbf{l}'$ is $\mathbf{x} = \mathbf{l} \times \mathbf{l}'$
- Line joining two points
  - The line through two points $\mathbf{x}$ and $\mathbf{x}'$ is $\mathbf{l} = \mathbf{x} \times \mathbf{x}'$

Example: intersections of lines



$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

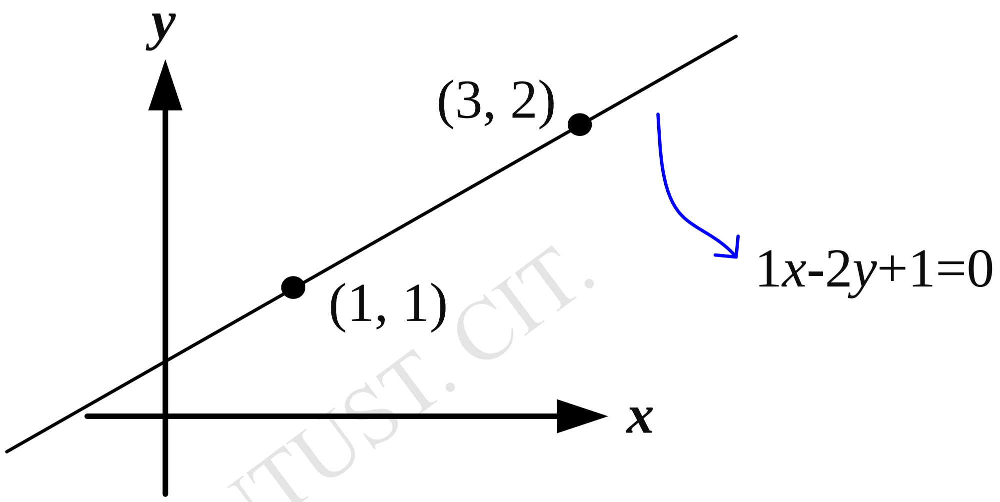
or

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\mathbf{l} \times \mathbf{l}' = \mathbf{x}$$

2D Points from lines and vice-versa

- Example: Line joining two points



$$\begin{bmatrix}3\\2\\1\end{bmatrix}\times\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

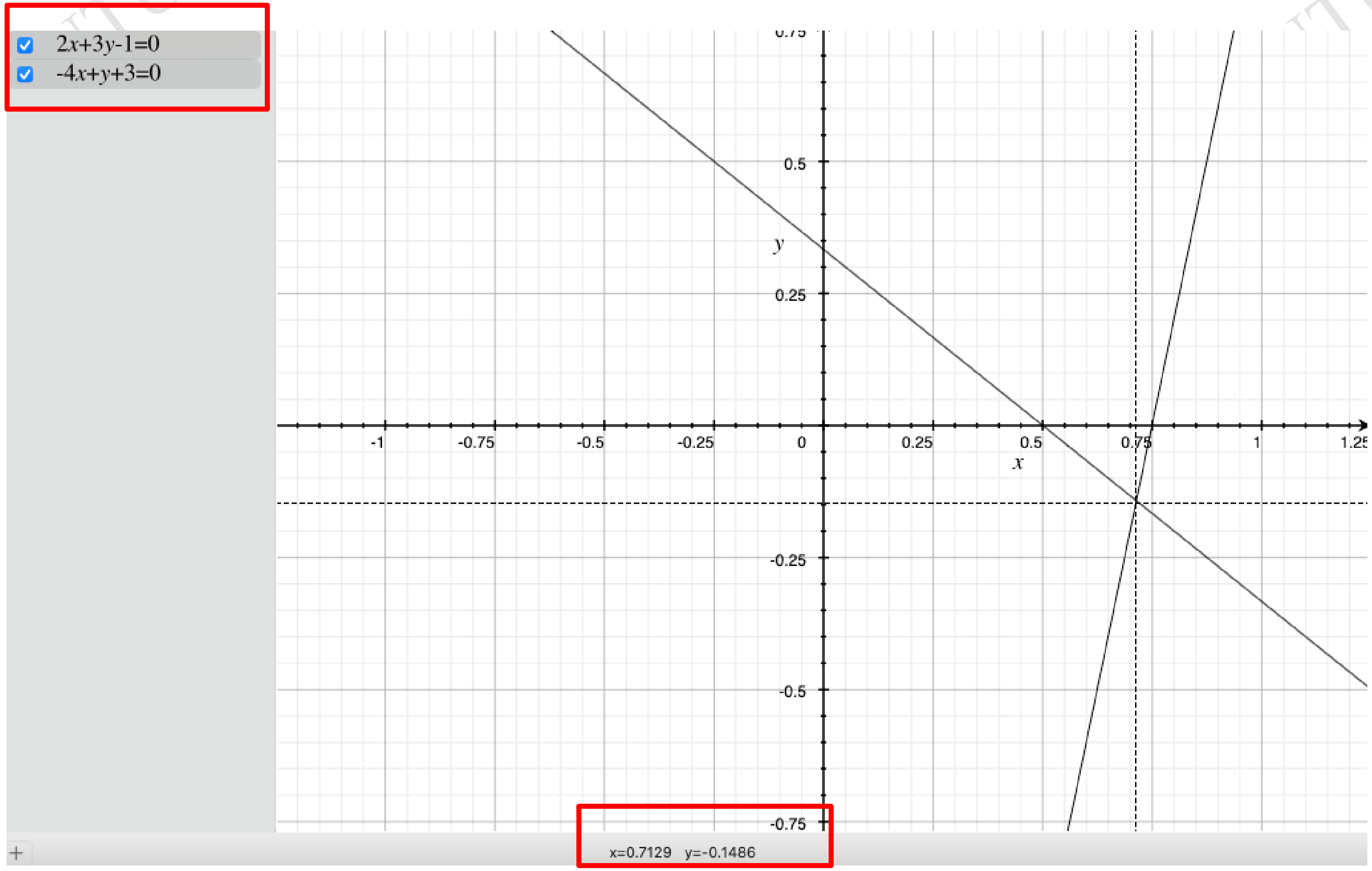
or

$$\begin{bmatrix}1\\1\\1\end{bmatrix}\times\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}-1\\2\\-1\end{bmatrix}$$

$$\mathbf{x}\times\mathbf{x}'=\mathbf{l}$$

2D Points from lines and vice-versa

- Example: line intersection example

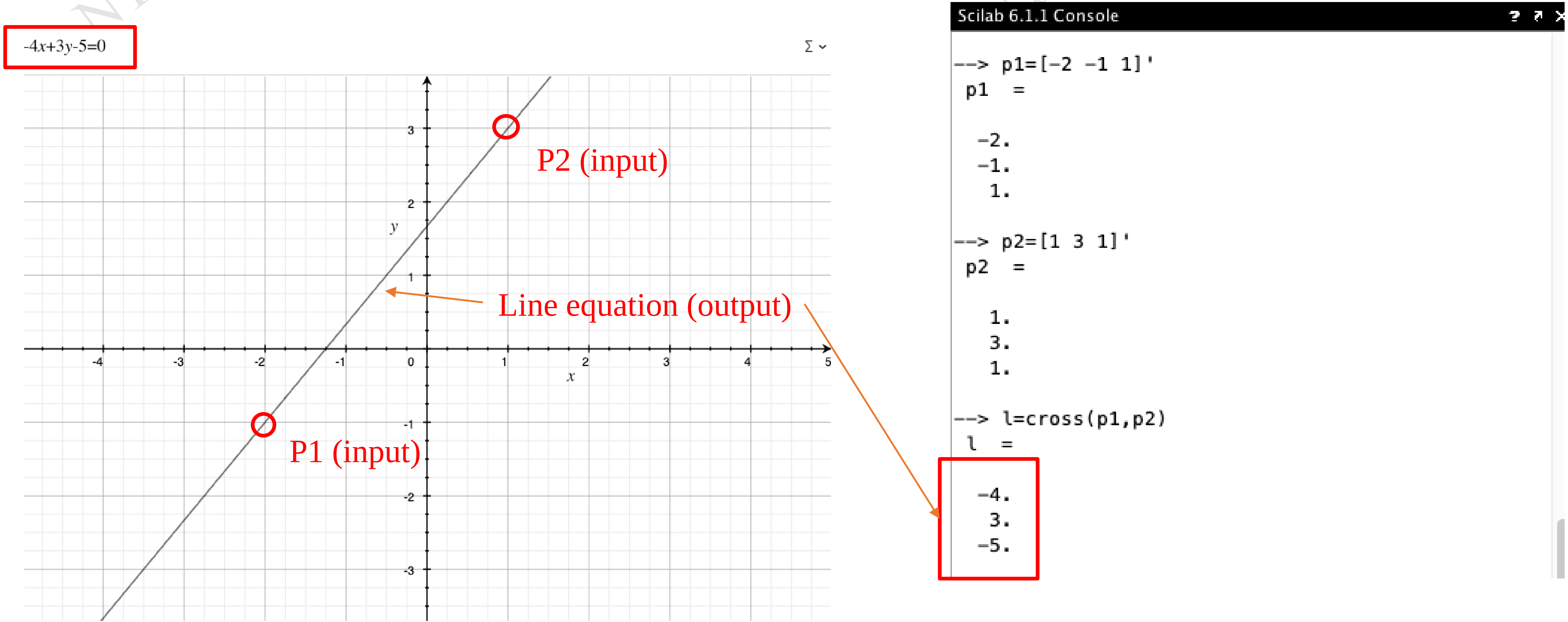


Scilab 6.1.1 Console

```
--> l1=[2 3 -1]'
 l1  =

   2.
   3.
  -1.

--> l2=[-4 1 3]'
 l2  =

  -4.
   1.
   3.

--> p=cross(l1,l2)
 p  =

   10.
  -2.
   14.

--> p./p(3)
 ans  =

   0.7142857
  -0.1428571
   1.
```

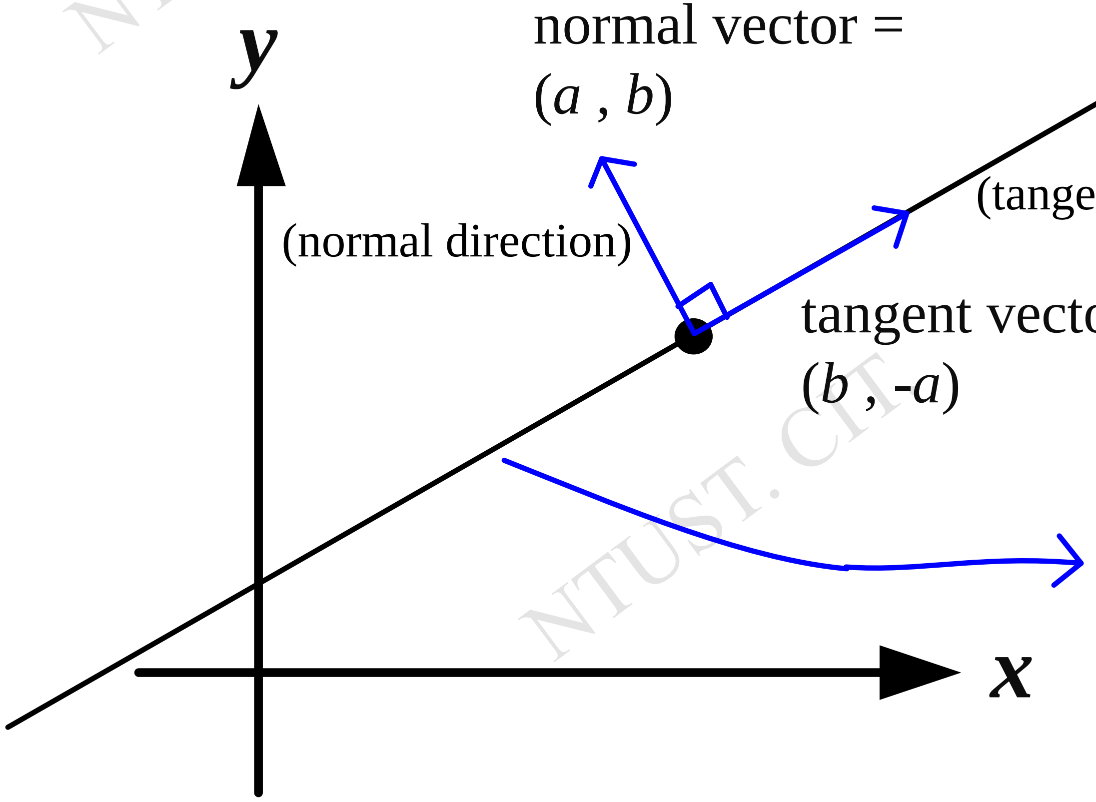
2D Points from lines and vice-versa

- Exampel: to form a line from two points



Points from lines and vice-versa

- Normal vector and tangent vector of one line:



$$ax + by + c = 0$$

(recommend: enforce the norm as 1)

Ideal points and the line at infinity

- Intersections of parallel lines

$$\mathbf{l}=(a,b,c)^{\mathsf{T}} \text{ and } \mathbf{l}'=(a,b,c')^{\mathsf{T}} \qquad \mathbf{x}=\mathbf{l}\times\mathbf{l}'=(b,-a,\underline{0})^{\mathsf{T}}$$ → point at infinity

Example

$(b,-a)$ tangent vector
(a,b) normal direction

Ideal points→ $(x_1,x_2,0)^{\mathsf{T}}$

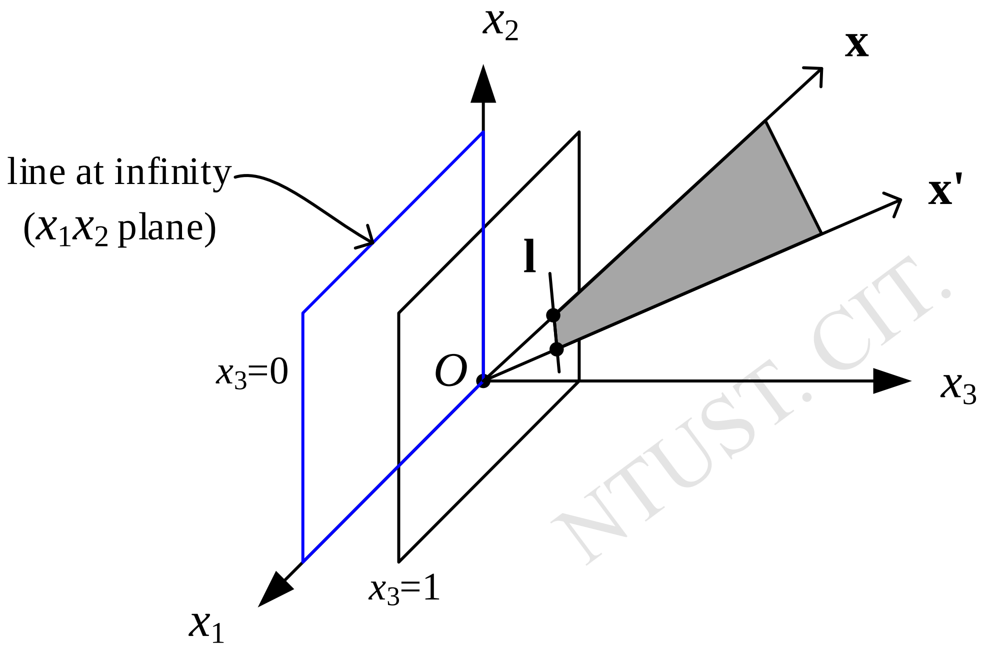
Line at infinity→ $\mathbf{l}_\infty=(0,0,1)^{\mathsf{T}}$

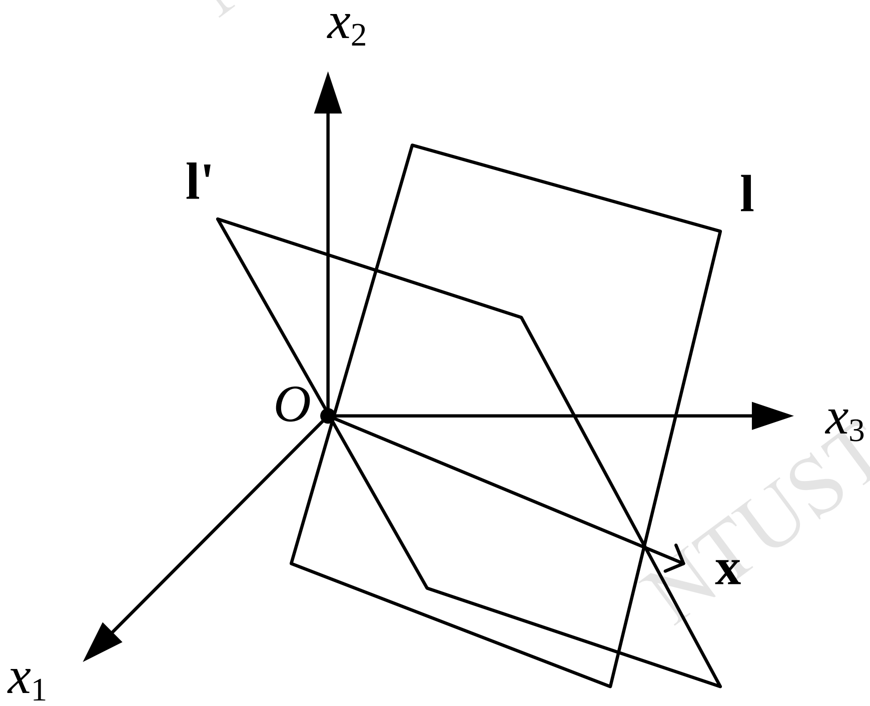
$$\mathbf{P}^2=\mathbf{R}^2\cup \mathbf{l}_\infty$$

Note that in $\mathbf{P}^2$ there is no distinction between ideal points and others

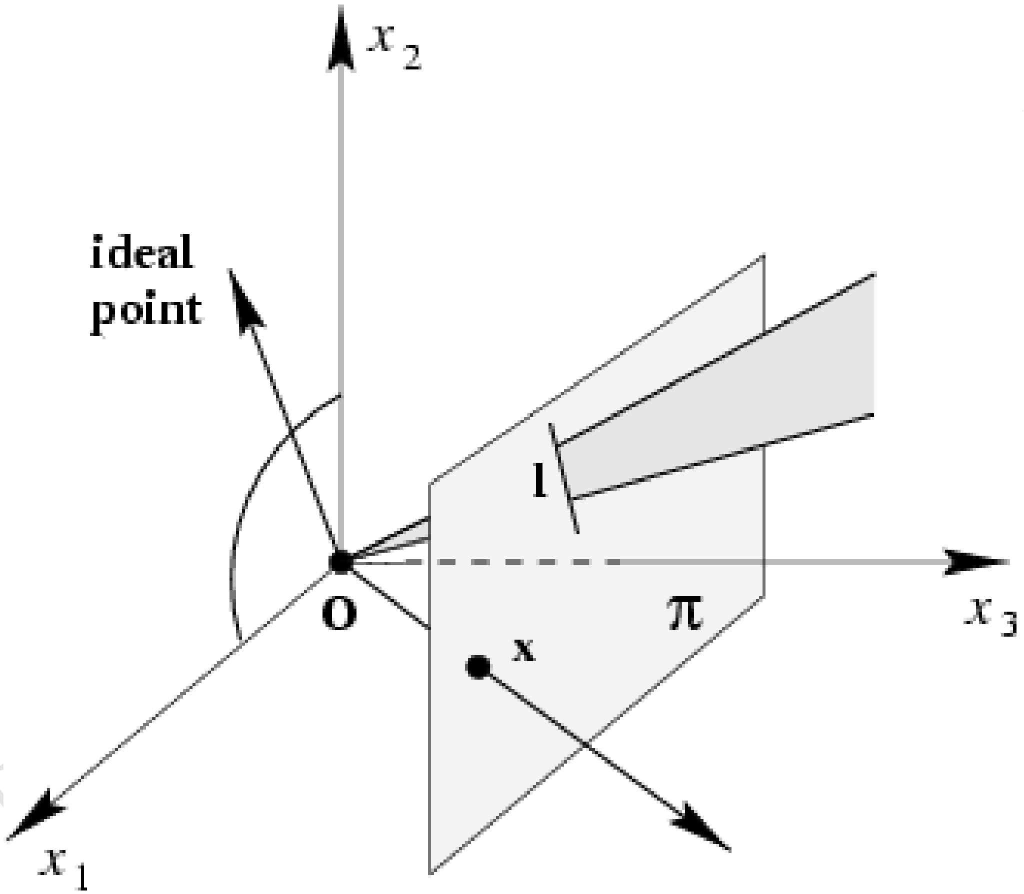
Ideal points and the line at infinity

- Schematic of homogenous coordinates:





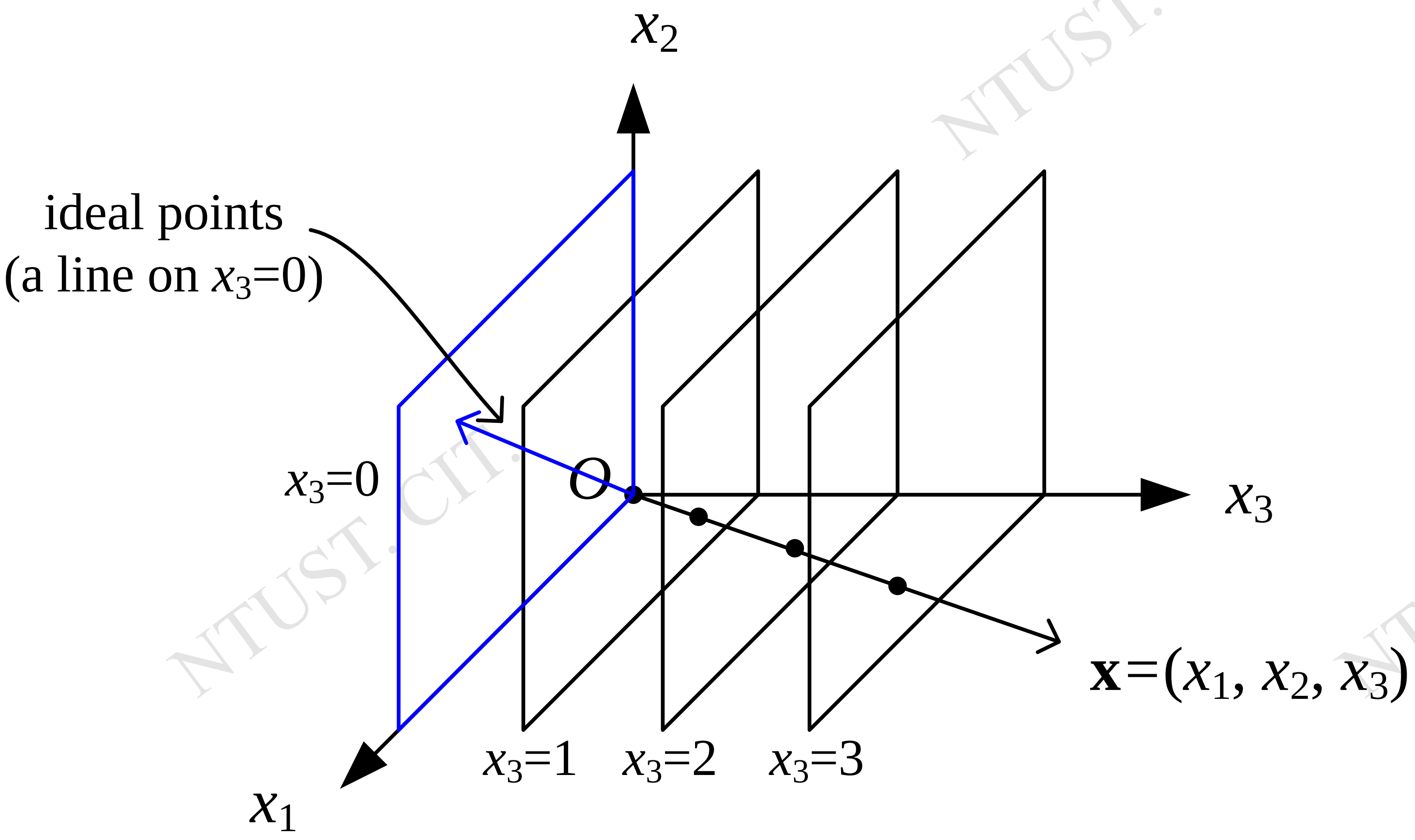
A model for the projective plane



exactly one line through two points

exactly one point at intersection of two lines

A model for the projective plane—cont.



Duality of 2D lines and points

Duality principle:

- To any theorem of 2-dimensional projective geometry there corresponds a dual theorem, which may be derived by interchanging the role of points and lines in the original theorem

$$\mathbf{x} \longleftrightarrow \mathbf{l}$$

$$\mathbf{x}^{\mathsf{T}}\mathbf{l}=0 \longleftrightarrow \mathbf{l}^{\mathsf{T}}\mathbf{x}=0$$

$$\mathbf{x}=\mathbf{l}\times\mathbf{l}' \longleftrightarrow \mathbf{l}=\mathbf{x}\times\mathbf{x}'$$

Conics

- Curve described by 2nd-degree equation in the plane

$$ax^2+bxy+cy^2+dx+ey+f=0$$

- or homogenized $x \mapsto x_1/x_3,\ y \mapsto x_2/x_3$

$$ax_1^2+bx_1x_2+cx_2^2+dx_1x_3+ex_2x_3+fx_3^2=0$$

- or in matrix form

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=0 \longrightarrow \mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x}=0$$

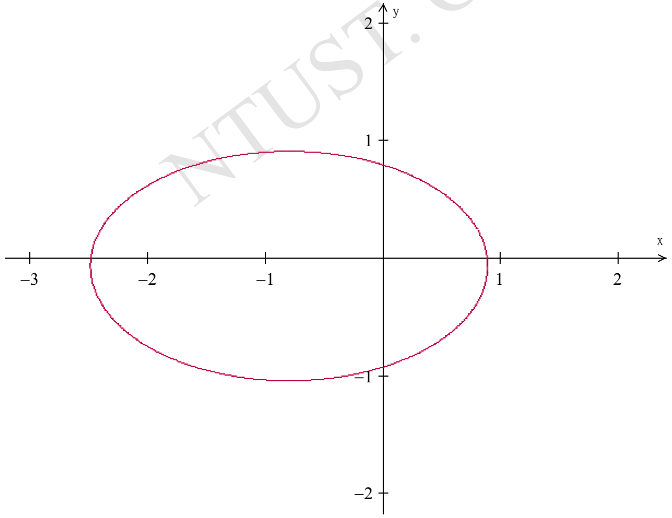
with $\mathbf{C}=\begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}$

5DOF: $\{a:b:c:d:e:f\}$

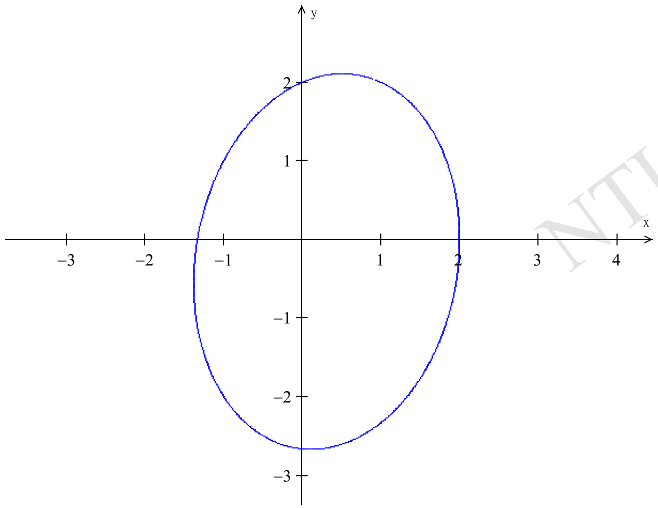
Conics (example)

- Example

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ 0 & 15 & 1 \\ 4 & 1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$



$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 6 & -0.5 & 2 \\ -0.5 & 3 & 1 \\ 2 & 1 & -16 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$



Conics

- Five points define a conic

For each point the conic passes through

$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or in matrix form

$$(x_i^2, x_iy_i, y_i^2, x_i, y_i, f)\cdot\mathbf{c} = 0 \qquad \mathbf{c} = (a,b,c,d,e,f)^\mathsf{T}$$

stacking constraints yields

$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}\mathbf{c} = 0 \qquad \begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

Conics

- Five points define a conic, an example:

5 points determine a conic:

(-6,1.6733,1)

(-3,2.8636,1)

(0,3.1623,1)

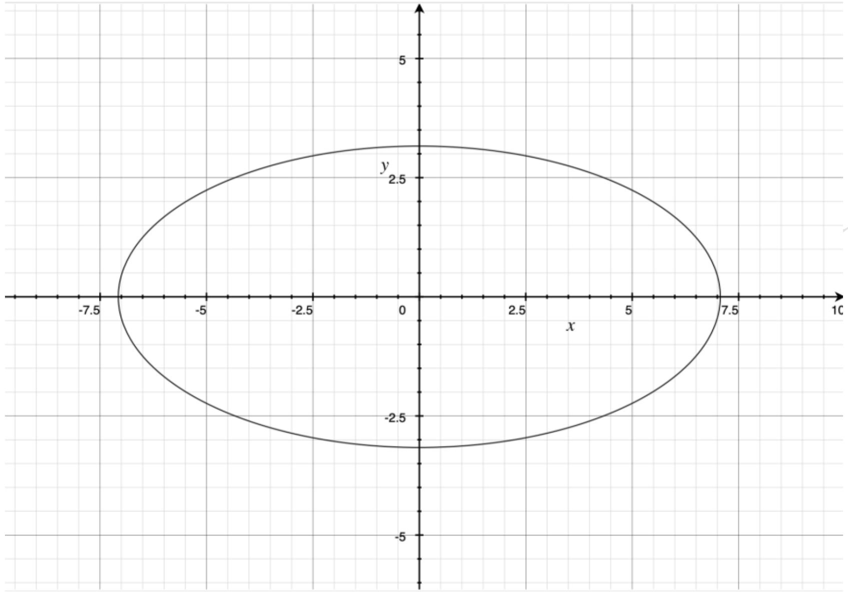
(3,2.8636,1)

(6,1.6733,1)

$$\begin{matrix} 36.0000 & -10.0399 & 2.8000 & -6.0000 & 1.6733 \\ 9.0000 & -8.5907 & 8.2000 & -3.0000 & 2.8636 \\ 0 & 0 & 10.0000 & 0 & 3.1623 \\ 9.0000 & 8.5907 & 8.2000 & 3.0000 & 2.8636 \\ 36.0000 & 10.0399 & 2.8000 & 6.0000 & 1.6733 \end{matrix} * \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

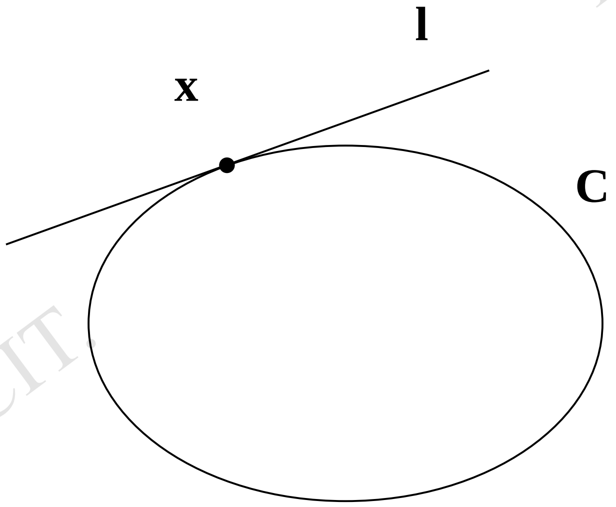
Solve it, then get $\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -0.02 \\ 0 \\ -0.1 \\ -0.000 \\ -0.000 \end{bmatrix} \rightarrow 0.2x^2 + y^2 - 10 = 0$

$$0.2x^2 + y^2 - 10 = 0$$



Tangent lines to conics

- The line **l** tangent to **C** at point **x** on **C** is given by **l=Cx**

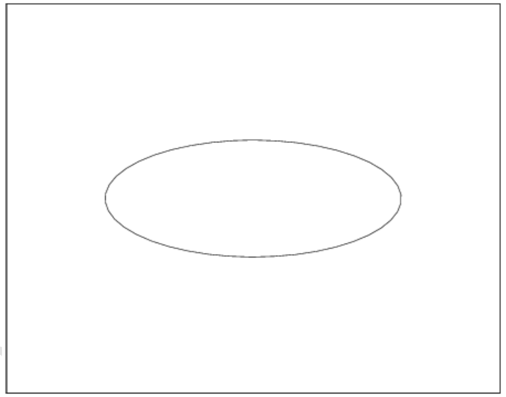


Since $\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x} = 0 = \mathbf{x}^{\mathsf{T}}\mathbf{l} = \mathbf{l}^{\mathsf{T}}\mathbf{x}$

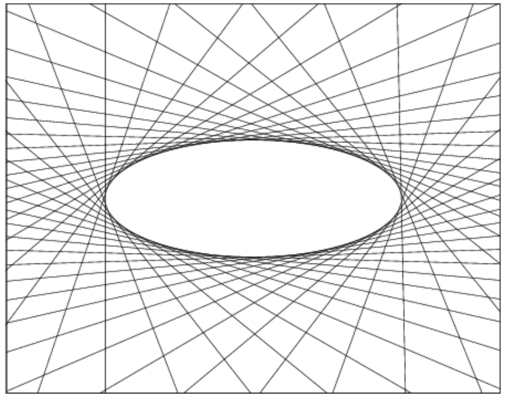
Dual conics

A line tangent to the conic C satisfies $\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$

- In general (**C** full rank): $\mathbf{C}^{*}=\mathbf{C}^{-1}$
- Dual conics = line conics = conic envelopes



$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x}=0$



$\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$

Dual conics

- A line tangent to the conic C satisfies

$$\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0 \rightarrow \mathbf{C}^{*}=\mathbf{C}^{-1}$$

- Proof:

Since $\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x}=0$

And line on the conic : $\mathbf{l}=\mathbf{C}\mathbf{x} \rightarrow \mathbf{x}=\mathbf{C}^{-1}\mathbf{l}$ (says tangent points)

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x}=0$$
$$(\mathbf{C}^{-1}\mathbf{l})^{\mathsf{T}}\mathbf{C}(\mathbf{C}^{-1}\mathbf{l})=0$$
$$\mathbf{l}^{\mathsf{T}}\mathbf{C}^{-\mathsf{T}}\mathbf{C}\mathbf{C}^{-1}\mathbf{l}=0$$
$$\mathbf{l}^{\mathsf{T}}\mathbf{C}^{-\mathsf{T}}\mathbf{l}=0$$
$$\mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0$$

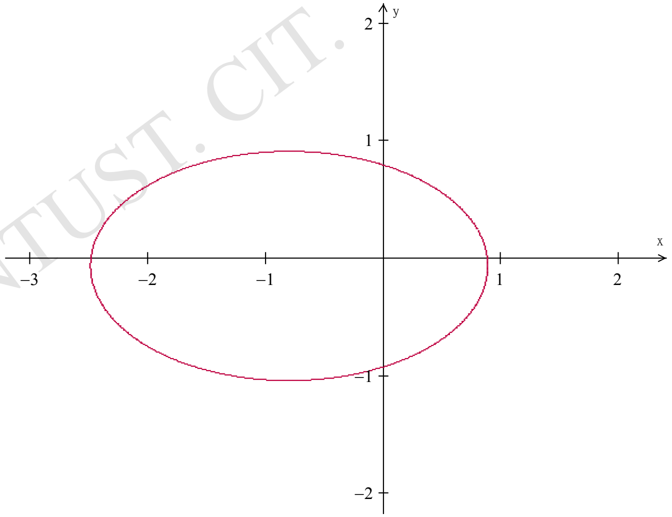
$$\therefore \mathbf{C}^{*}=\mathbf{C}^{-1}$$

(hint: since **C** is symmetric)

Dual conics (example)

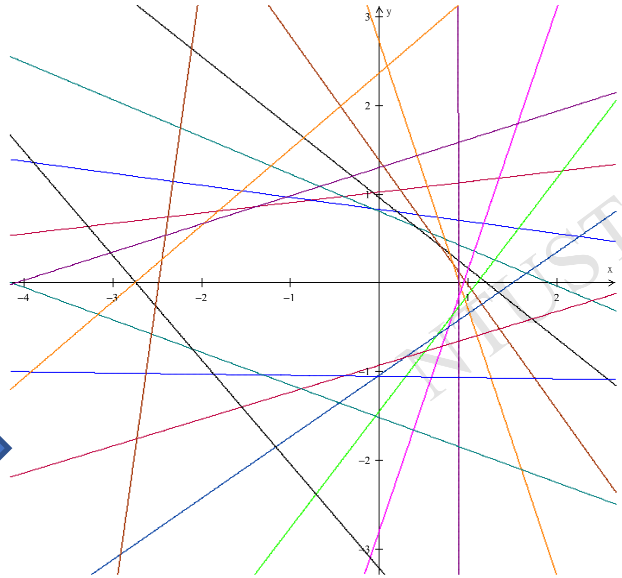
- Example

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 4 \\ 0 & 15 & 1 \\ 4 & 1 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$



$$\begin{bmatrix} 5 & 0 & 4 \\ 0 & 15 & 1 \\ 4 & 1 & -11 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1551 & -0.0037 & 0.0561 \\ -0.0037 & 0.0664 & 0.0047 \\ 0.0561 & 0.0047 & -0.0701 \end{bmatrix}$$

$$\mathbf{l}^{T} \begin{bmatrix} 0.1551 & -0.0037 & 0.0561 \\ -0.0037 & 0.0664 & 0.0047 \\ 0.0561 & 0.0047 & -0.0701 \end{bmatrix} \mathbf{l} = 0 \Rightarrow$$



Degenerate conics

- A conic is degenerate if matrix C is not of full rank

e.g. two lines (rank 2)

$$\mathbf{C} = \mathbf{l}\mathbf{m}^\top + \mathbf{m}\mathbf{l}^\top$$

e.g. repeated line (rank 1)

$$\mathbf{C} = \mathbf{l}\mathbf{l}^\top$$

Example:

$$\mathbf{x}^\top \mathbf{C}\mathbf{x} = 0$$

$$\mathbf{x}^\top (\mathrm{l}\mathrm{m}^\top + \mathrm{m}\mathrm{l}^\top)\mathbf{x} = (\mathbf{x}^\top \mathrm{l})(\mathrm{m}^\top \mathbf{x}) + (\mathbf{x}^\top \mathrm{m})(\mathrm{l}^\top \mathbf{x}) = 0$$

So, either $\mathbf{x}^\top \mathbf{l} = 0$, or $\mathbf{x}^\top \mathbf{m} = 0$ → two lines

Example:

$$\mathrm{l} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

Degenerate line conics: 2 points (rank 2), double point (rank1)

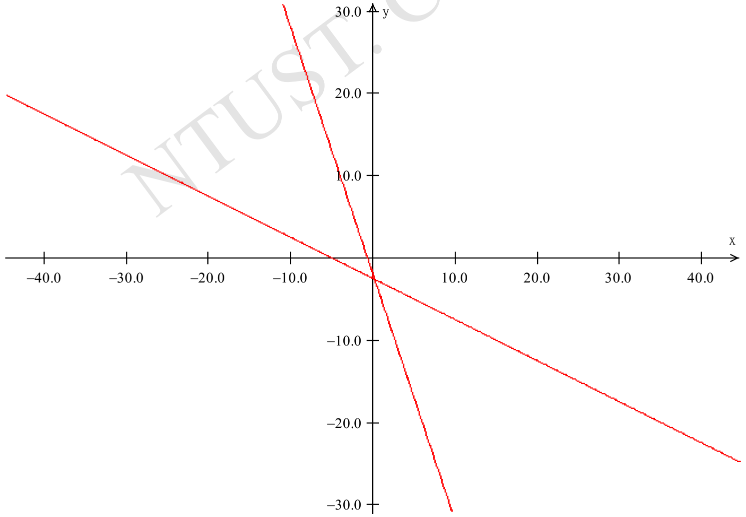
Note that for degenerate conics $\left(\mathbf{C}^*\right)^* \neq \mathbf{C}$

Degenerate conics (example)

- Example

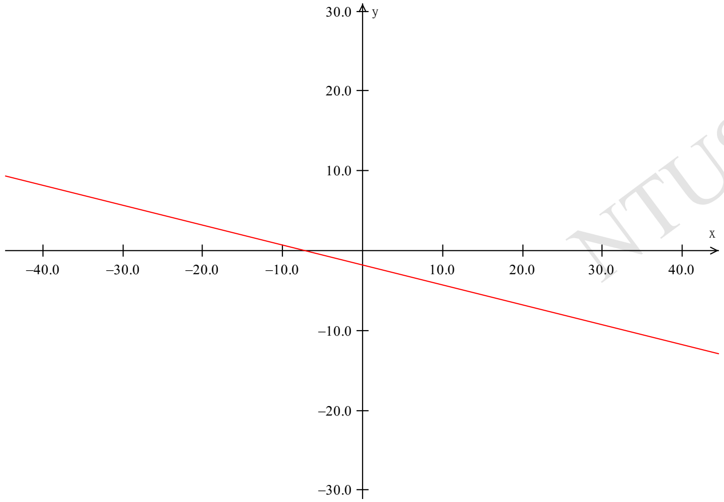
$$\mathbf{C} = \mathbf{l}\mathbf{m}^{\mathsf{T}} + \mathbf{m}\mathbf{l}^{\mathsf{T}}$$

$$\mathbf{l} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \mathbf{m} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 6 & 7 & 17 \\ 7 & 4 & 9 \\ 17 & 9 & 20 \end{bmatrix}$$



$$\mathbf{C} = \mathbf{l}\mathbf{l}^{\mathsf{T}}$$

$$\mathrm{l} = \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 16 & 28 \\ 7 & 28 & 49 \end{bmatrix}$$



Projective transformations

- Definition: A *projectivity* is an invertible mapping ***h*** from P^2 to itself such that three points x_1,x_2,x_3 lie on the same line if and only if $h(x_1),h(x_2),h(x_3)$ do.
- Theorem: A mapping $\boldsymbol{h}:P^2 \rightarrow P^2$ is a projectivity if and only if there exists a non-singular 3x3 matrix **H** such that for any point in P^2 represented by a vector **x** it is true that $\boldsymbol{h}(\mathbf{x})=\mathbf{Hx}$
- Definition: Projective transformation

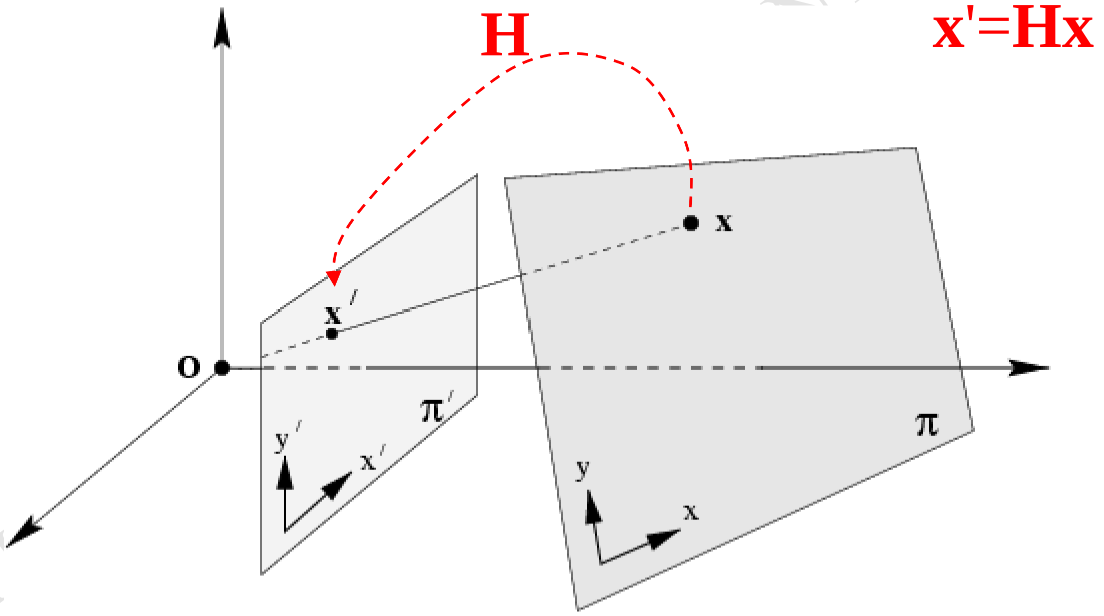
$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{or} \quad \mathbf{x}' = \mathbf{H}\mathbf{x}$$

8DOF

- Projectivity = Collineation = Projective Transformation = Homography

Application: mapping between planes

- Homography



central projection may be expressed by **x'=Hx**

Homography: to remove projective effect

select four points in a plane with know coordinates

$$x' = \frac{x'_1}{x'_3} = \frac{h_{11}x + h_{12}y + h_{13}}{h_{31}x + h_{32}y + h_{33}} \qquad y' = \frac{x'_2}{x'_3} = \frac{h_{21}x + h_{22}y + h_{23}}{h_{31}x + h_{32}y + h_{33}}$$

$$x'(h_{31}x + h_{32}y + h_{33}) = h_{11}x + h_{12}y + h_{13}$$
$$y'(h_{31}x + h_{32}y + h_{33}) = h_{21}x + h_{22}y + h_{23}$$
(linear in h_{ij})

(2 constraints/point, 8DOF $\Rightarrow$ 4 points needed)

Note! NO calibration at all necessary, better ways to compute

Homography: to remove projective effect

- Rewrite equation

$$xh_{11}+yh_{12}+h_{13}-x'xh_{31}-x'yh_{32}-x'h_{33}=0$$
$$xh_{21}+yh_{22}+h_{23}-y'xh_{31}-y'yh_{32}-y'h_{33}=0$$

- Normalize hij with h33, (replace h_{ij}/h_{33} with h_{ij} temporarily)

$$xh_{11}+yh_{12}+h_{13}-x'xh_{31}-x'yh_{32}=x'$$
$$xh_{21}+yh_{22}+h_{23}-y'xh_{31}-y'yh_{32}=y'$$

- In matrix form:

$$\begin{bmatrix} x & y & 1 & 0 & 0 & 0 & -x'x & -x'y \\ 0 & 0 & 0 & x & y & 1 & -y'x & -y'y \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

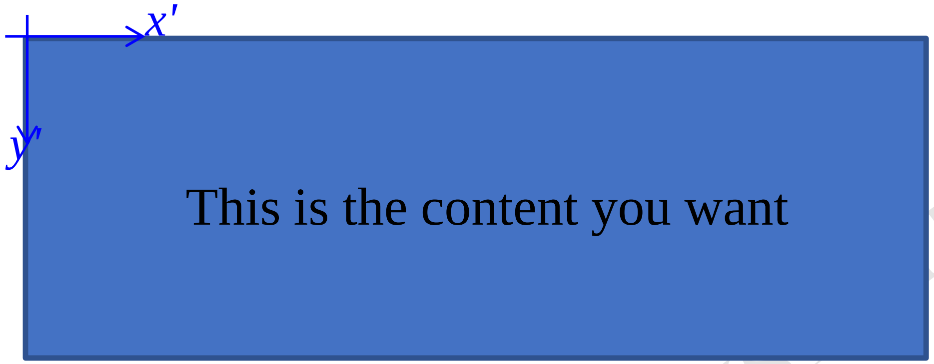
If you have one set of correspondence, you get (x, y) & (x', y'). So, you need at least four correspondences for solving 8 unknowns. Note: here, one correspondence forms two equations

8 unknowns to be solved

Homography: to remove projective effect

- For example: Take a picture, then remove the distortion. Someday…

A picture you took



The result you want
(This is normally defined in your application)

Define your problem, first!!!

Homography: to remove projective effect

- For example, —cont.

Correspondence:
(54,45)→(0,0)
(58,196)→(0,100)
(332,172)→(400,100)
(329,91)→(400,0)
Then, find **H**

$$\begin{bmatrix} 54 & 45 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 54 & 45 & 1 & 0 & 0 \\ 58 & 196 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 58 & 196 & 1 & -5800 & -19600 \\ 332 & 172 & 1 & 0 & 0 & 0 & -132800 & -68800 \\ 0 & 0 & 0 & 332 & 172 & 1 & -33200 & -17200 \\ 329 & 91 & 1 & 0 & 0 & 0 & -131600 & -36400 \\ 0 & 0 & 0 & 329 & 91 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \\ 400 \\ 100 \\ 400 \\ 0 \end{bmatrix}$$

Homography: to remove projective effect

- For example, —cont.

$$\begin{bmatrix} h_{11} \\ h_{12} \\ h_{13} \\ h_{21} \\ h_{22} \\ h_{23} \\ h_{31} \\ h_{32} \end{bmatrix} = \begin{bmatrix} 54 & 45 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 54 & 45 & 1 & 0 & 0 \\ 58 & 196 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 58 & 196 & 1 & -5800 & -19600 \\ 332 & 172 & 1 & 0 & 0 & 0 & -132800 & -68800 \\ 0 & 0 & 0 & 332 & 172 & 1 & -33200 & -17200 \\ 329 & 91 & 1 & 0 & 0 & 0 & -131600 & -36400 \\ 0 & 0 & 0 & 329 & 91 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 100 \\ 400 \\ 100 \\ 400 \\ 0 \end{bmatrix}$$

$$= \begin{matrix} -0.0030 & -0.0086 & -0.0007 & 0.0085 & -0.0008 & -0.0160 & 0.0045 & 0.0160 \\ -0.0065 & 0.0002 & 0.0066 & -0.0002 & 0.0000 & 0.0004 & -0.0001 & -0.0004 \\ 1.4550 & 0.4537 & -0.2620 & -0.4511 & 0.0425 & 0.8424 & -0.2355 & -0.8450 \\ 0.0003 & -0.0024 & -0.0003 & -0.0012 & 0.0006 & 0.0002 & -0.0006 & 0.0035 \\ -0.0020 & -0.0072 & 0.0020 & 0.0073 & -0.0038 & -0.0012 & 0.0038 & 0.0011 \\ 0.0735 & 1.4550 & -0.0730 & -0.2620 & 0.1364 & 0.0425 & -0.1368 & -0.2355 \\ 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 \\ -0.0000 & 0.0000 & 0.0000 & -0.0000 & -0.0000 & 0.0000 & 0.0000 & -0.0000 \end{matrix} * \begin{matrix} 0 \\ 0 \\ 0 \\ 100 \\ 400 \\ 100 \\ 400 \\ 0 \end{matrix} = \begin{matrix} 0.7210 \\ -0.0191 \\ -38.0771 \\ -0.1029 \\ 0.6150 \\ -22.1199 \\ -0.0016 \\ 0.0001 \end{matrix} \Rightarrow \mathbf{H} = \begin{bmatrix} 0.721 & -0.0191 & -38.0771 \\ -0.1029 & 0.6150 & -22.1199 \\ -0.0016 & 0.0001 & 1 \end{bmatrix}$$

Homography: to remove projective effect

- For example, —cont.

	p1 =	p2 =	p3 =	p4 =
original points:	54 45 1	58 196 1	332 172 1	329 91 1
x'=Hx	>> H*p1 ans = 0 0.0000 0.9192	>> H*p2 ans = -0.0000 92.4536 0.9245	>> H*p3 ans = 198.0257 49.5064 0.4951	>> H*p4 ans = 197.4097 0.0000 0.4935
normalized	0 0.0000 1.0000	-0.0000 100.0000 1.0000	400.0000 100.0000 1.0000	400.0000 0.0000 1.0000
Desired points:	$(0,0,1)^T$	$(0,100,1)^T$	$(400,100,1)^T$	$(400, 0,1)^T$

Homography: to remove projective effect

- For example, —cont. (inverse mapping)

	pp1 =	pp2 =	pp3 =	pp4 =
desired points:	0 0 1	0 100 1	400 100 1	400 0 1
	>> inv(H)*pp1 ans =	>> inv(H)*pp2 ans =	>> inv(H)*pp3 ans =	>> inv(H)*pp4 ans =
$x=H^{-1}x'$	58.7480 48.9567 1.0879	62.7342 211.9982 1.0816	670.6200 347.4296 2.0199	666.6338 184.3881 2.0262
normalized	54.0000 45.0000 1.0000	58.0000 196.0000 1.0000	332.0000 172.0000 1.0000	329.0000 91.0000 1.0000
Original points:	$(54,45,1)^T$	$(58,196,1)^T$	$(332,172,1)^T$	$(329,91,1)^T$

Homography: to remove projective effect

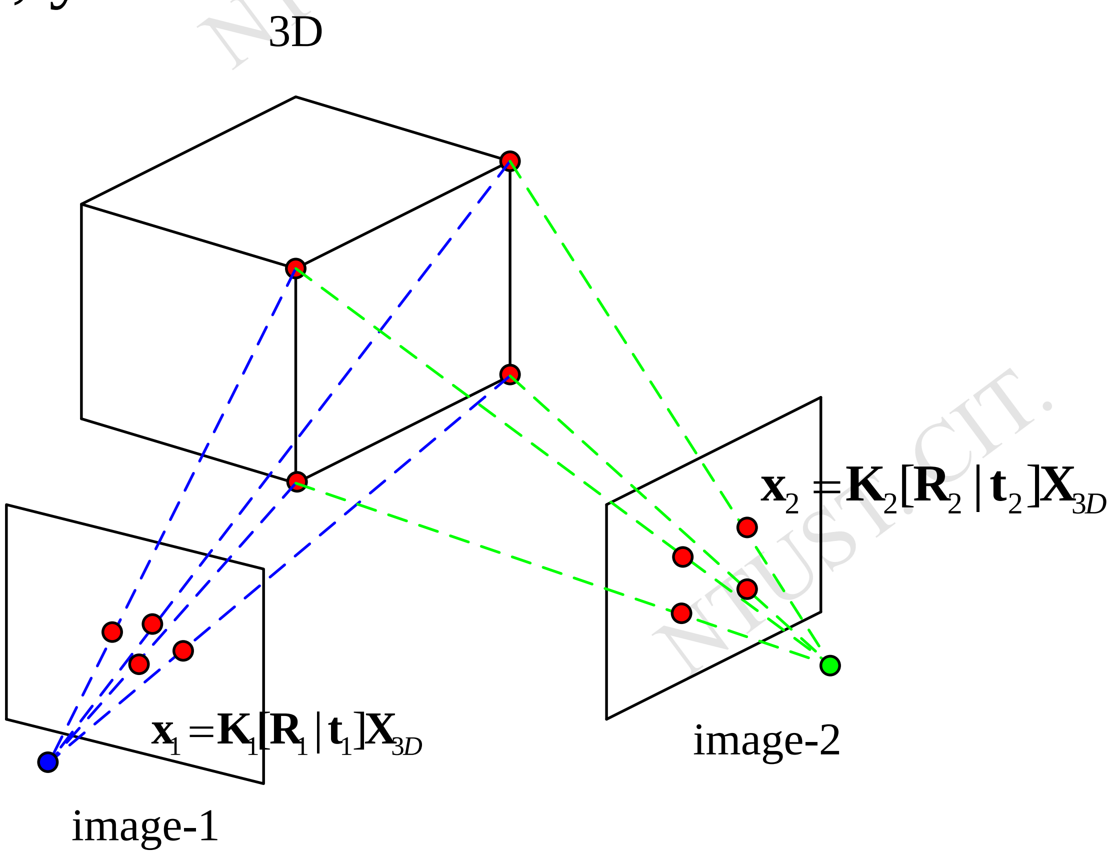
- For example, —cont. (inverse mapping)

Filling correct COLOR:
You are knowing to filling color in a “400x100” image. For each pixel, you need to calculate its color by applying H^{-1} to its coordinate.

Homography: What is it?

If you have at least 4 corresponding points, a homography can dominate the transformation between two images. So, you do NOT need to determine **K**[**R**|**t**] for 2 views

(Note: ONLY planar structure in 3D)



$$\mathbf{x}_2 = \mathbf{H}\mathbf{x}_1$$

Homography in OpenCV

- openCV provides various kinds of mapping operations in computer vision.

Correspondence:
(54,45)→(0,0)
(58,196)→(0,100)
(332,172)→(400,100)
(329,91)→(400,0)
Then, find **H**

Sample Code (in openCV):
findHomography

Source Points:
54.000000 45.000000
58.000000 196.000000
332.000000 172.000000
329.000000 91.000000

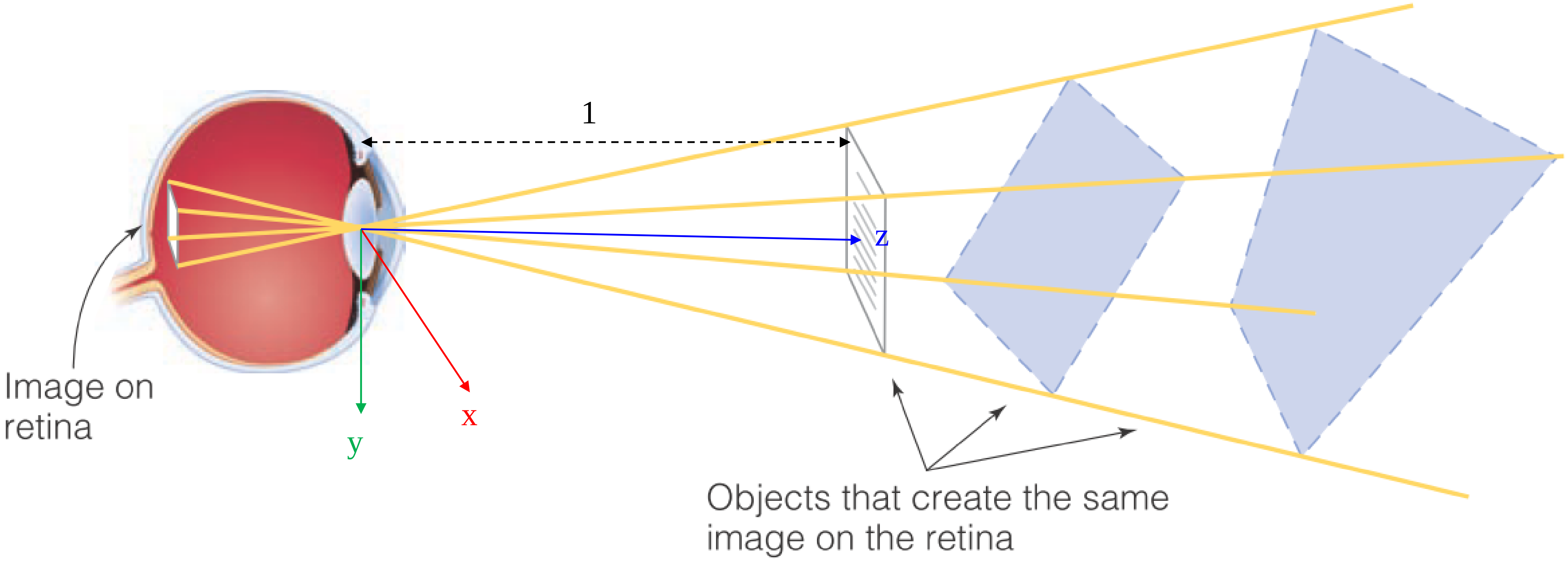
Destination Points:
0.000000 0.000000
0.000000 100.000000
400.000000 100.000000
400.000000 0.000000

Homography Matrix:
0.721049 -0.019101 -38.077095
-0.102873 0.615001 -22.119894
-0.001561 0.000077 1.000000

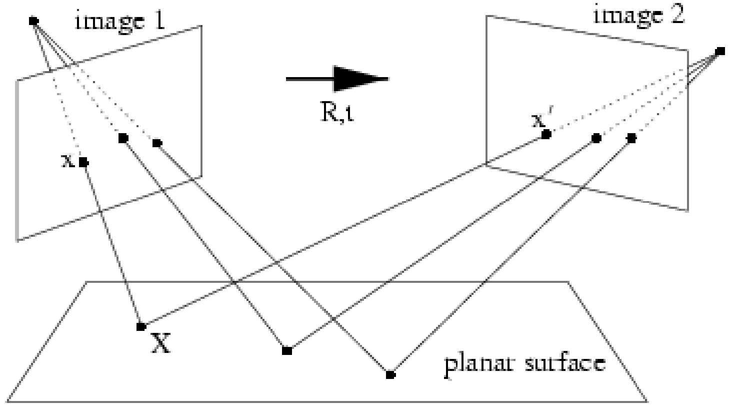
Homography: example: develop a program

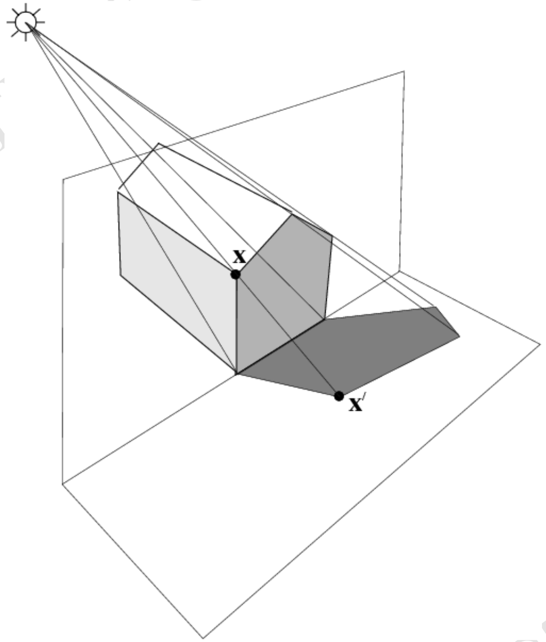
- Example

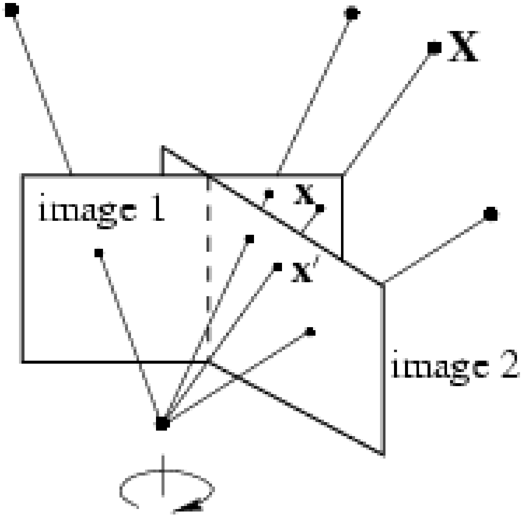
Homography: a projection conversion



Homography: more examples







Transformation of lines and conics

Homography for specific shape

- For a point transformation

$$\mathbf{x}' = \mathbf{H}\mathbf{x}$$

- Transformation for lines

$$\mathbf{l}' = \mathbf{H}^{-\mathsf{T}}\mathbf{l}$$

- Transformation for conics

$$\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$$

- Transformation for dual conics

$$\mathbf{C}'^{*} = \mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

Transformation of lines and conics

- Transformation for lines $\mathbf{l}' = \mathbf{H}^{-\mathsf{T}}\mathbf{l}$ (proof)

  If we have $\mathbf{x}' = \mathbf{H}\mathbf{x}$

  And **x**' on line **l**', and **x** on **l**.

  So, we have $\mathbf{l}'^{\mathsf{T}}\mathbf{x}' = 0 \quad \mathbf{l}^{\mathsf{T}}\mathbf{x} = 0$

  Rewrite $\mathbf{l}'^{\mathsf{T}}\mathbf{x}' = 0 = \mathbf{l}^{\mathsf{T}}\mathbf{H}^{-1}\mathbf{H}\mathbf{x}$

  Then, $\mathbf{l}'^{\mathsf{T}} = \mathbf{l}^{\mathsf{T}}\mathbf{H}^{-1} \rightarrow \mathbf{l}' = (\mathbf{l}'^{\mathsf{T}})^{\mathsf{T}} = (\mathbf{l}^{\mathsf{T}}\mathbf{H}^{-1})^{\mathsf{T}} = \mathbf{H}^{-\mathsf{T}}\mathbf{l}$

  Get: $\mathbf{l}' = \mathbf{H}^{-\mathsf{T}}\mathbf{l}$

Transformation of lines and conics

- Transformation for conics $\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$ (proof)

  If we have $\mathbf{x}' = \mathbf{H}\mathbf{x}$

  And know a conic equation: $\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x} = 0$

  So, we have $\mathbf{x} = \mathbf{H}^{-1}\mathbf{x}'$

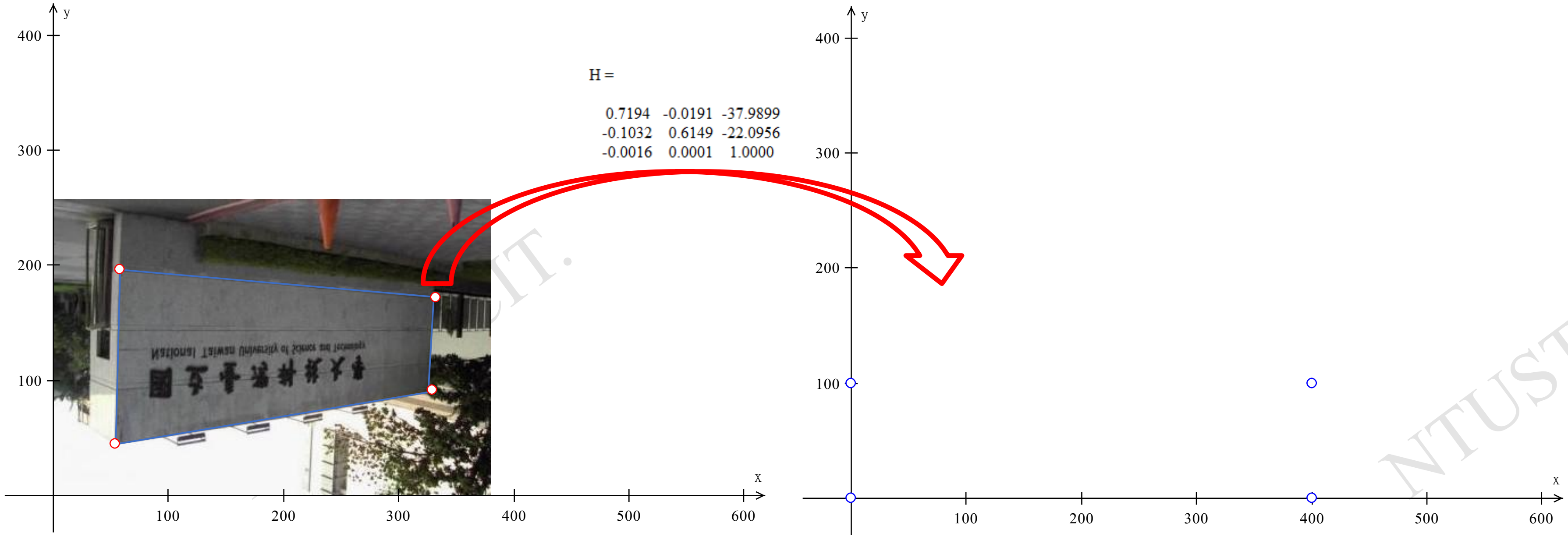
  Rewrite equation: $\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x} = 0 \rightarrow (\mathbf{H}^{-1}\mathbf{x}')^{\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}\mathbf{x}' = 0$

  Then, $\mathbf{x}'^{\mathsf{T}}(\mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1})\mathbf{x}' = 0 = \mathbf{x}'^{\mathsf{T}}\mathbf{C}'\mathbf{x}'$

  Get: $\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$

Transformation of lines and conics

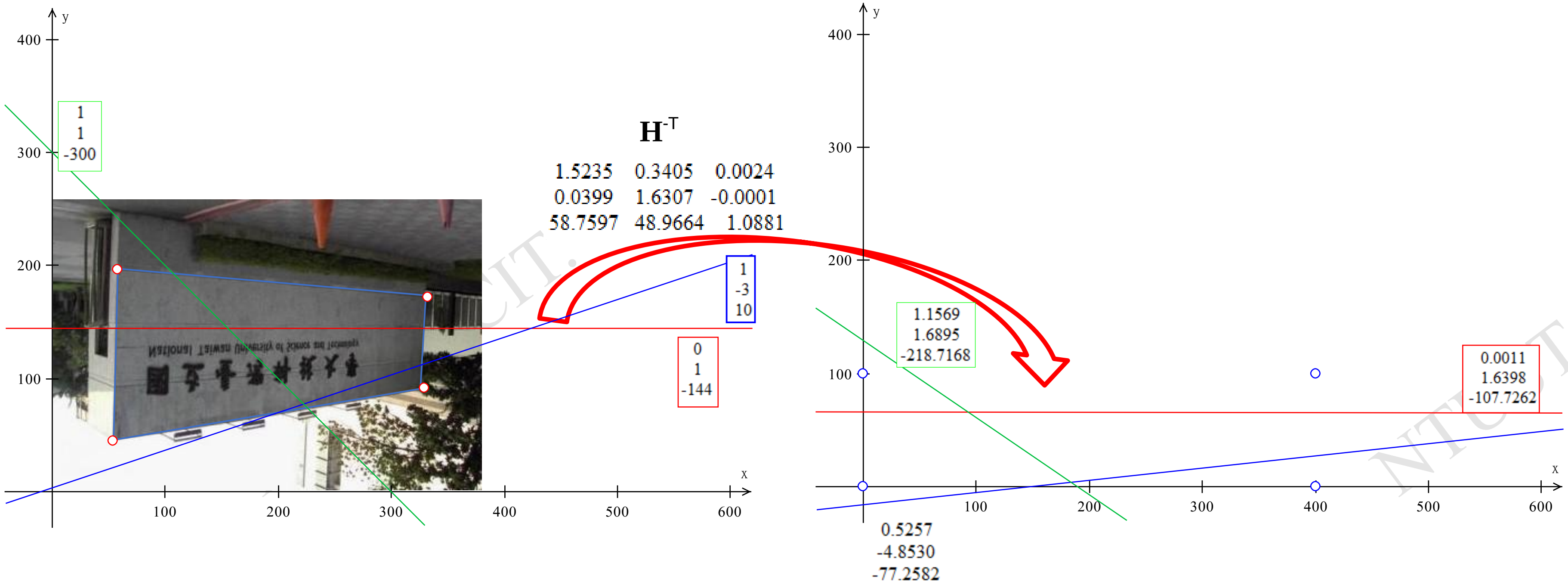
- Example: (the same with previous, but mirror for convenience)



Transformation of lines and conics

- Example: lines transformation

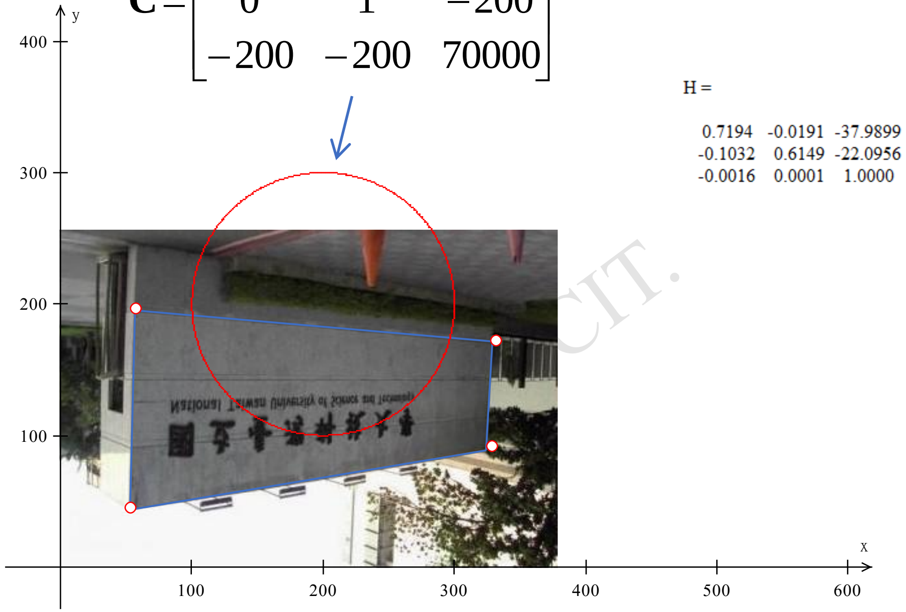
$$\mathbf{l}' = \mathbf{H}^{-T}\mathbf{l}$$



Transformation of lines and conics

- Example: conics transformation $\mathbf{C}' = \mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1}$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & -200 \\ 0 & 1 & -200 \\ -200 & -200 & 70000 \end{bmatrix}$$

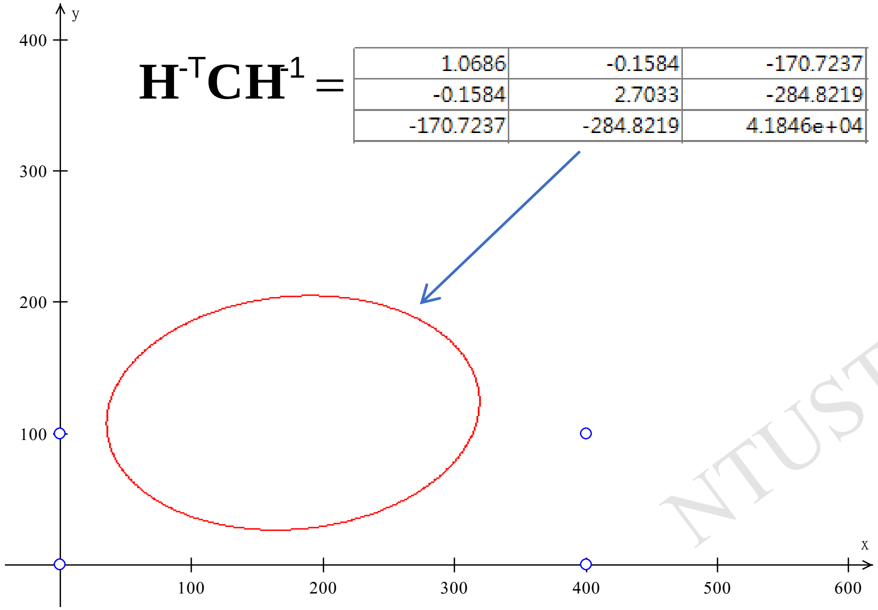


H =

$$\begin{matrix} 0.7194 & -0.0191 & -37.9899 \\ -0.1032 & 0.6149 & -22.0956 \\ -0.0016 & 0.0001 & 1.0000 \end{matrix}$$

$\mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1} =$

1.0686	-0.1584	-170.7237
-0.1584	2.7033	-284.8219
-170.7237	-284.8219	4.1846e+04



A hierarchy of transformations

- Projective linear group
  - Affine group (last row (0,0,1))
    - Euclidean group (upper left 2x2 orthogonal)
      - Oriented Euclidean group (upper left 2x2 det 1)

Alternative, characterize transformation in terms of elements or quantities that are preserved or *invariant*

e.g. Euclidean transformations leave distances unchanged

Four classic types of transformation

- Isometrics
- Similarities
- Affine mapping
- Projective mapping

Four classic types of transformation—cont.

- Class I: Isometries (*iso*=same, *metric*=measure)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} \varepsilon\cos\theta & -\sin\theta & t_x \\ \varepsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad \varepsilon = \pm 1$$

orientation preserving: $\varepsilon = 1$
orientation reversing: $\varepsilon = -1$

$$\mathrm{x}' = \mathbf{H}_E\,\mathrm{x} = \begin{bmatrix} \mathbf{R} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$$

3DOF (1 rotation, 2 translation)

special cases: pure rotation, pure translation

Invariants: length, angle, area

Four classic types of transformation—cont.

- Class II: Similarities (*isometry* + *scale*)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} s\cos\theta & -s\sin\theta & t_x \\ s\sin\theta & s\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

$$\mathrm{x}' = \mathbf{H}_S\, \mathrm{x} = \begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathsf{T}} & 1 \end{bmatrix} \mathrm{x} \qquad \mathbf{R}^{\mathsf{T}}\mathbf{R} = \mathbf{I}$$

4DOF (1 scale, 1 rotation, 2 translation)

also know as *equi-form* (shape preserving)

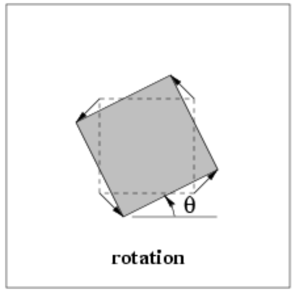
metric structure = structure up to similarity (in literature)

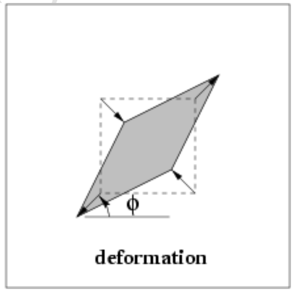
Invariants: ratios of length, angle, ratios of areas, parallel lines

Four classic types of transformation—cont.

- Class III: Affine transformations

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$





$$\mathrm{x}' = \mathbf{H}_A\,\mathrm{x} = \begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^\mathsf{T} & 1 \end{bmatrix} \mathrm{x}$$

$$\mathbf{A} = \mathbf{R}(\theta)\mathbf{R}(-\phi)\mathbf{D}\mathbf{R}(\phi) \qquad \mathbf{D} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

6DOF (2 scale, 2 rotation, 2 translation)

non-isotropic scaling! (2DOF: scale ratio and orientation)

Invariants: parallel lines, ratios of parallel lengths, ratios of areas

Four classic types of transformation—cont.

- Class IV: Projective transformations

$$x' = \mathbf{H}_P \, x = \begin{bmatrix} \mathbf{A} & t \\ v^{\mathsf{T}} & v \end{bmatrix} x \qquad v = (v_1, v_2)^{\mathsf{T}}$$

8DOF (2 scale, 2 rotation, 2 translation, 2 line at infinity)

Action non-homogeneous over the plane

Invariants: cross-ratio of four points on a line (ratio of ratio)

Four classic types of transformation—cont.

- Action of affinities and projectivities on line at infinity

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ 0^{\mathrm{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ 0 \end{pmatrix}$$

Line at infinity stays at infinity,
but points move along line

$$\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & v \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ v_1 x_1 + v_2 x_2 \end{pmatrix}$$

Line at infinity becomes finite,
allows to observe vanishing points, horizon,

Four classic types of transformation—cont.

- Decomposition of projective transformations

$$\mathbf{H}=\mathbf{H}_S\mathbf{H}_A\mathbf{H}_P=\begin{bmatrix} s\mathbf{R} & \mathrm{t} \\ 0^{\mathrm{T}} & 1\end{bmatrix}\begin{bmatrix} \mathbf{K} & 0 \\ 0^{\mathrm{T}} & 1\end{bmatrix}\begin{bmatrix} \mathbf{I} & 0 \\ \mathrm{v}^{\mathrm{T}} & v\end{bmatrix}=\begin{bmatrix} \mathbf{A} & \mathrm{t} \\ \mathrm{v}^{\mathrm{T}} & v\end{bmatrix}$$

Similarity Affine Projective

decomposition unique (if chosen s>0)

$$\mathbf{A}=s\mathbf{R}\mathbf{K}+\mathrm{t}\mathrm{v}^{\mathrm{T}}$$

$\mathbf{K}$ upper-triangular, $\det\mathbf{K}=1$

Step: 1. Determine $\mathbf{v}^{\mathrm{T}}$

Step: 2. Find $\mathbf{K}$

Step: 3. then s and $\mathbf{R}$

Example:

$$\mathbf{H}=\begin{bmatrix} 1.707 & 0.586 & 1.0 \\ 2.707 & 8.242 & 2.0 \\ 1.0 & 2.0 & 1.0 \end{bmatrix}$$

$$\mathbf{H}=\begin{bmatrix} 2\cos 45^\circ & -2\sin 45^\circ & 1.0 \\ 2\sin 45^\circ & 2\cos 45^\circ & 2.0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

Four classic types of transformation—cont.

- Decomposition of projective transformations
  - Example

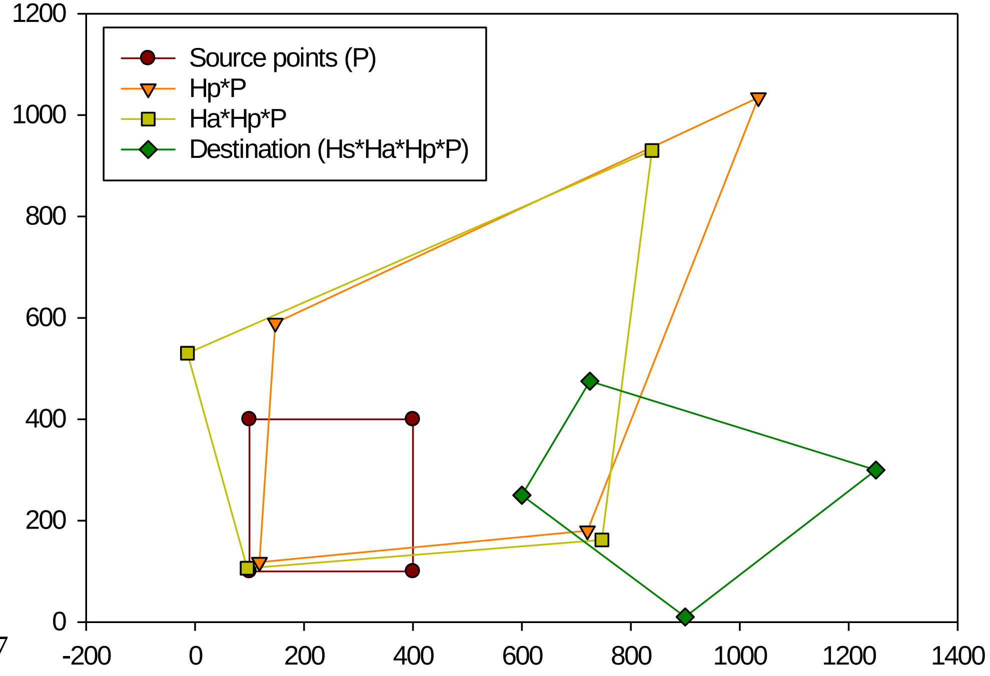
Source Points	Destination Points
(100,100)	(600,250)
(400,100)	(900,10)
(400,400)	(1250,300)
(100,400)	(725,475)

H=
-0.027759 -0.054527 516.226013
-0.687046 0.368143 243.555847
-0.000972 -0.000562 1.000000

Hp=
1.000000 0.000000 0.000000
0.000000 1.000000 0.000000
-0.000972 -0.000562 1.000000

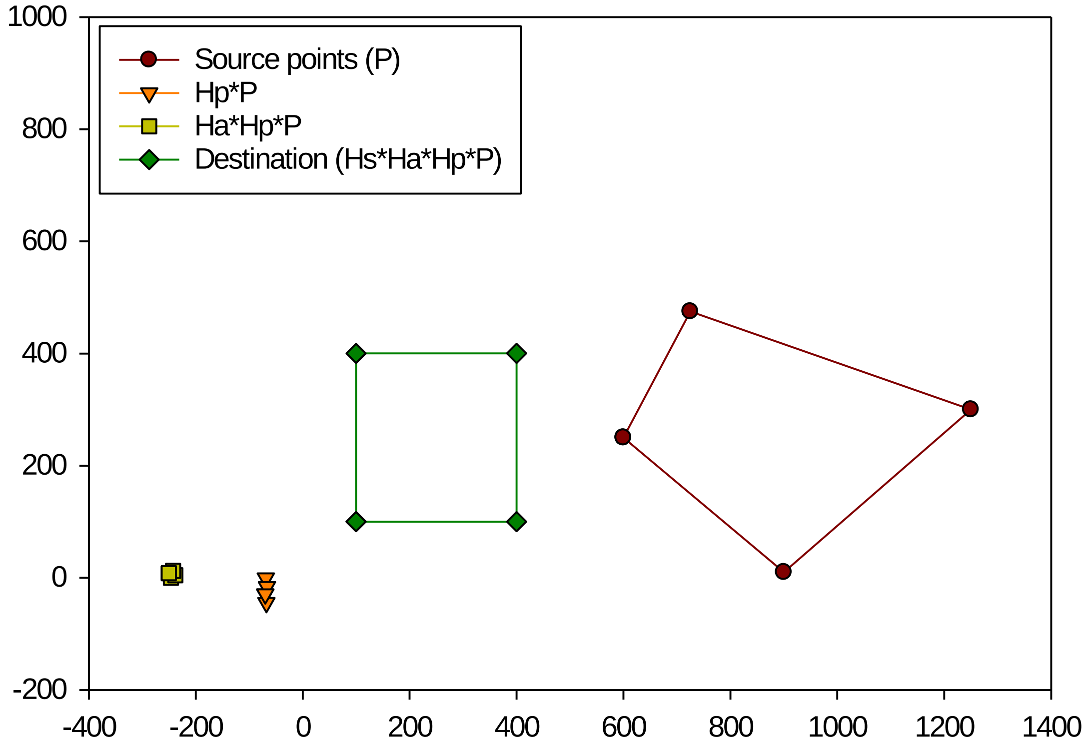
Ha=
1.112456 -0.301632 0.000000
-0.000000 0.898912 0.000000
0.000000 0.000000 1.000000

Hs=
0.425900 0.404881 516.226013
-0.404881 0.425900 243.555847
0.000000 0.000000 1.000000



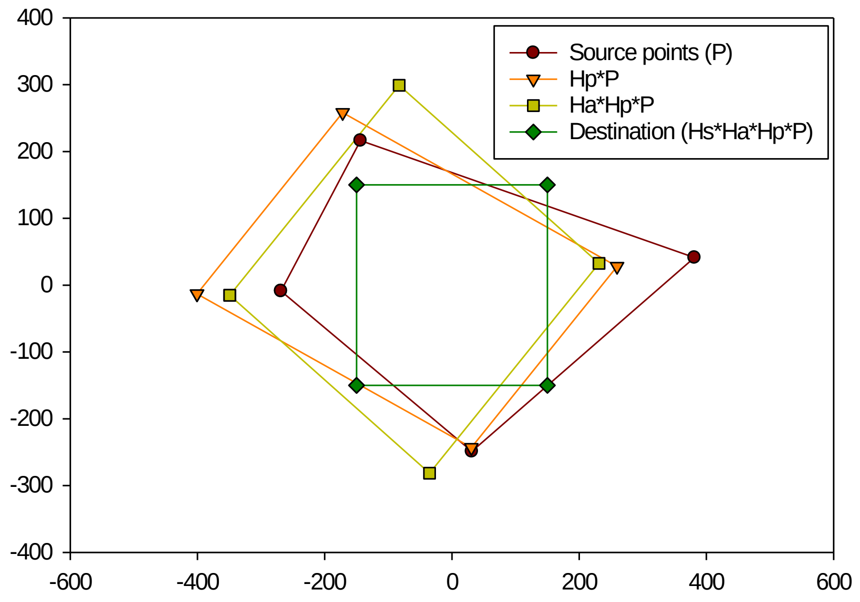
Four classic types of transformation—cont.

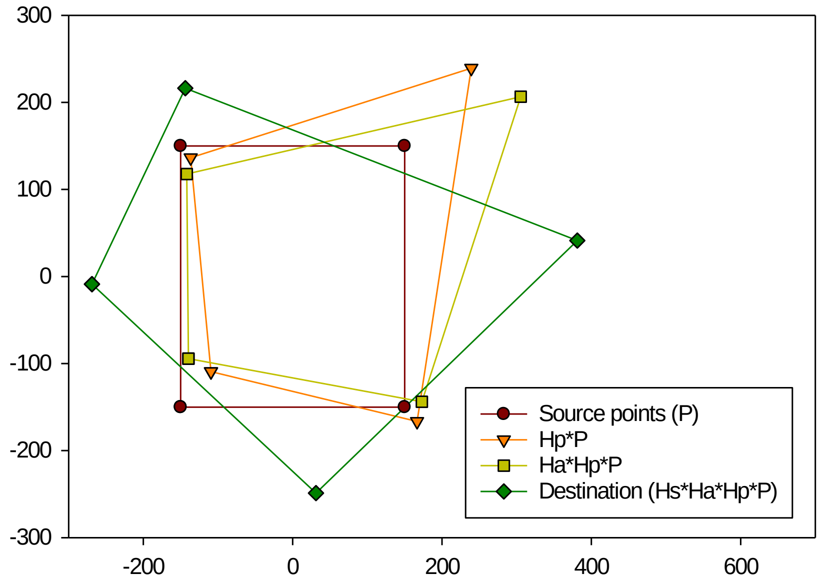
- Decomposition of projective transformations
  - (Inverse) Example



Four classic types of transformation—cont.

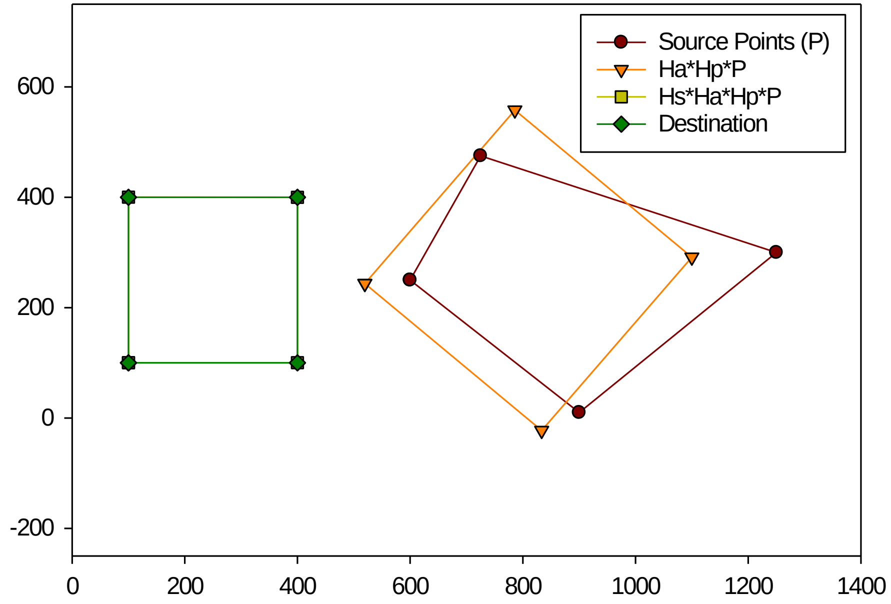
- Decomposition of projective transformations
  - Example—cont.

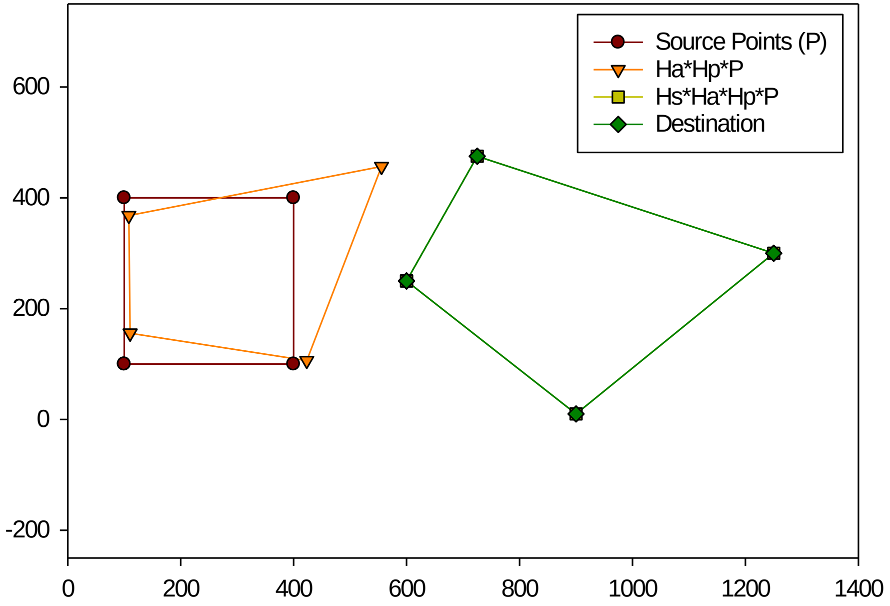




Four classic types of transformation—cont.

- Decomposition of projective transformations
  - Example—cont.

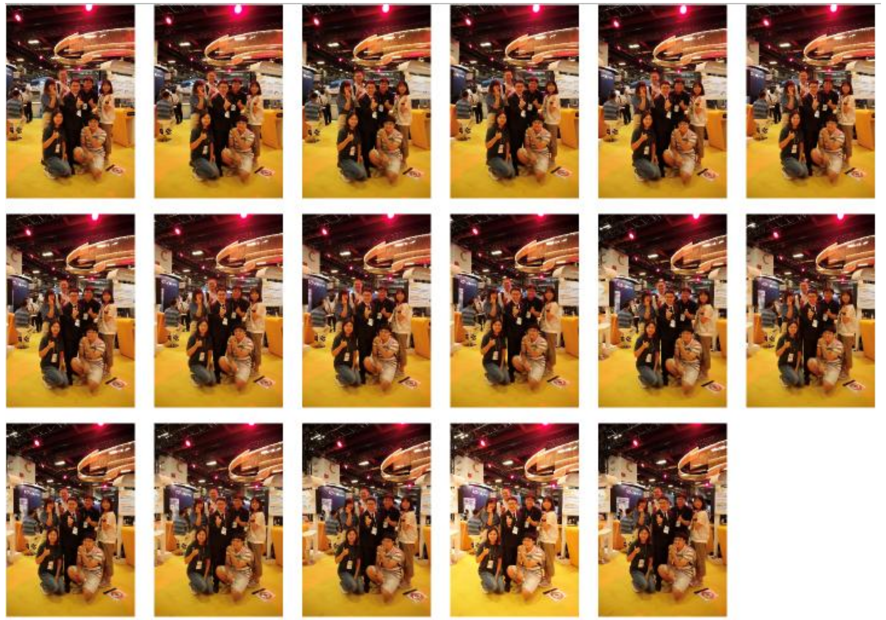




Application in homography decomposition

Multiple view images

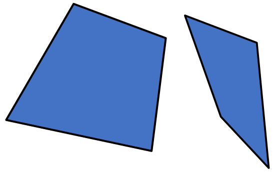
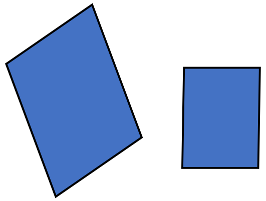
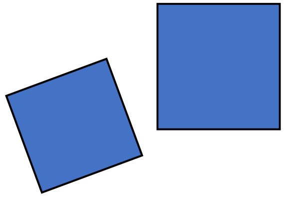
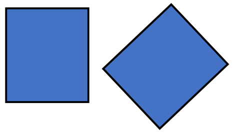
- Perform quality alignment
- Preseve property of “perspective effect”



Before

After

Overview transformations

Projective 8dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, order of contact (intersection, tangency, inflection, etc.), cross ratio
Affine 6dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallellism, ratio of areas, ratio of lengths on parallel lines (e.g midpoints), linear combinations of vectors (centroids). **The line at infinity $\mathbf{l}_\infty$**
Similarity 4dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratios of lengths, angles. **The circular points I,J**
Euclidean 3dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		lengths, areas.

Number of invariants?

- The number of functional invariants is equal to, or greater than, the number of degrees of freedom of the configuration less the number of degrees of freedom of the transformation
- e.g. configuration of 4 points in general position has 8 DOF (2/pt) and so 4 similarity, 2 affinity and zero projective invariants

Short summary

- Points and lines

$$\mathbf{l}^{\mathsf{T}}\mathbf{x}=0 \qquad \mathbf{x}=\mathbf{l}\times\mathbf{l}' \qquad \mathbf{l}=\mathbf{x}\times\mathbf{x}' \qquad \mathbf{l}_{\infty}=(0,0,1)^{\mathsf{T}}$$

- Conics and dual conics

$$\mathbf{x}^{\mathsf{T}}\mathbf{C}\mathbf{x}=0 \quad \mathbf{l}^{\mathsf{T}}\mathbf{C}^{*}\mathbf{l}=0 \quad \mathbf{C}^{*}=\mathbf{C}^{-1} \quad \mathbf{l}=\mathbf{C}\mathbf{x}$$

- Projective transformations

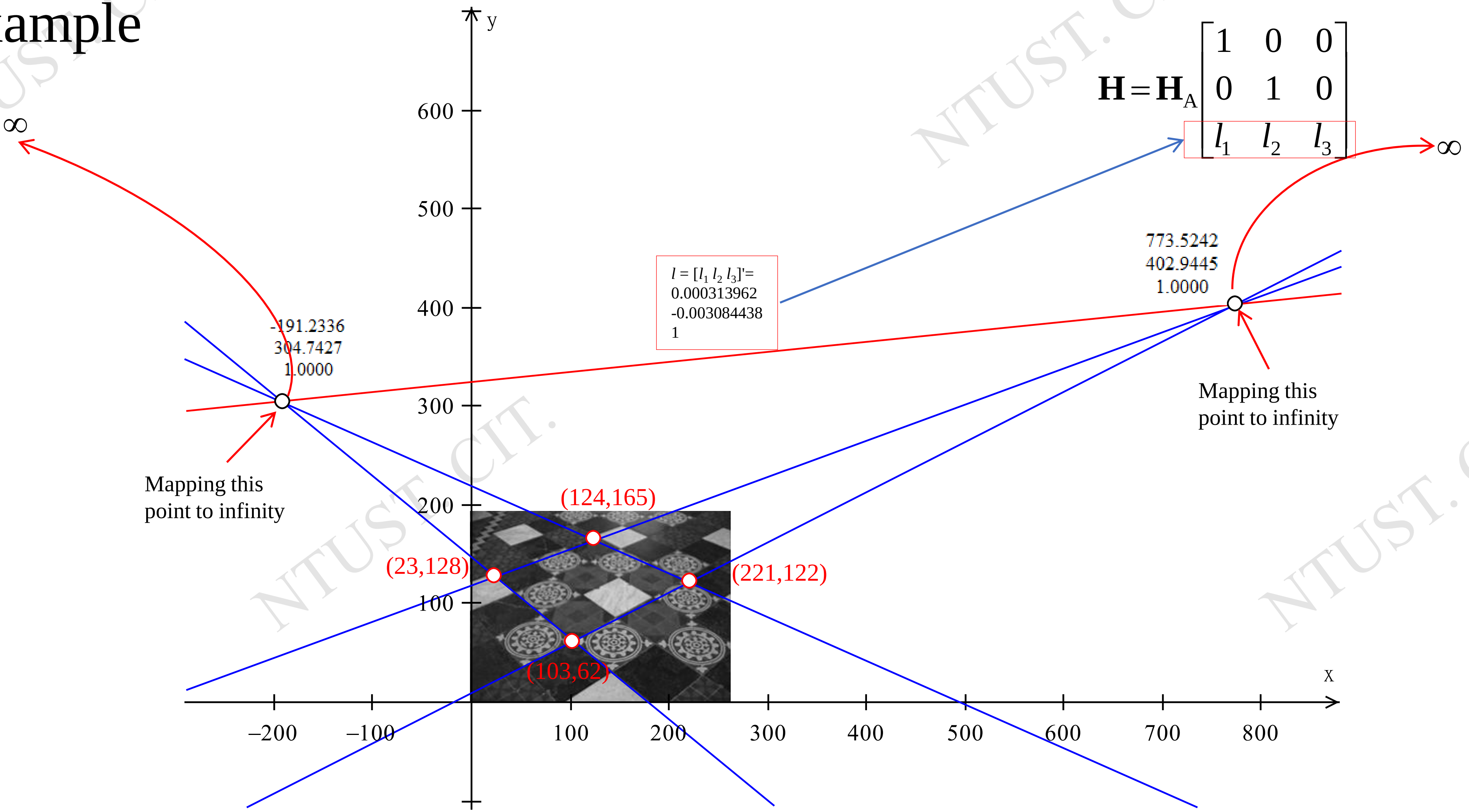
$$\mathbf{x}'=\mathbf{H}\mathbf{x} \qquad \mathbf{l}'=\mathbf{H}^{-\mathsf{T}}\mathbf{l}$$

$$\mathbf{C}'=\mathbf{H}^{-\mathsf{T}}\mathbf{C}\mathbf{H}^{-1} \qquad \mathbf{C}'^{*}=\mathbf{H}\mathbf{C}^{*}\mathbf{H}^{\mathsf{T}}$$

Affine rectification via the vanishing line

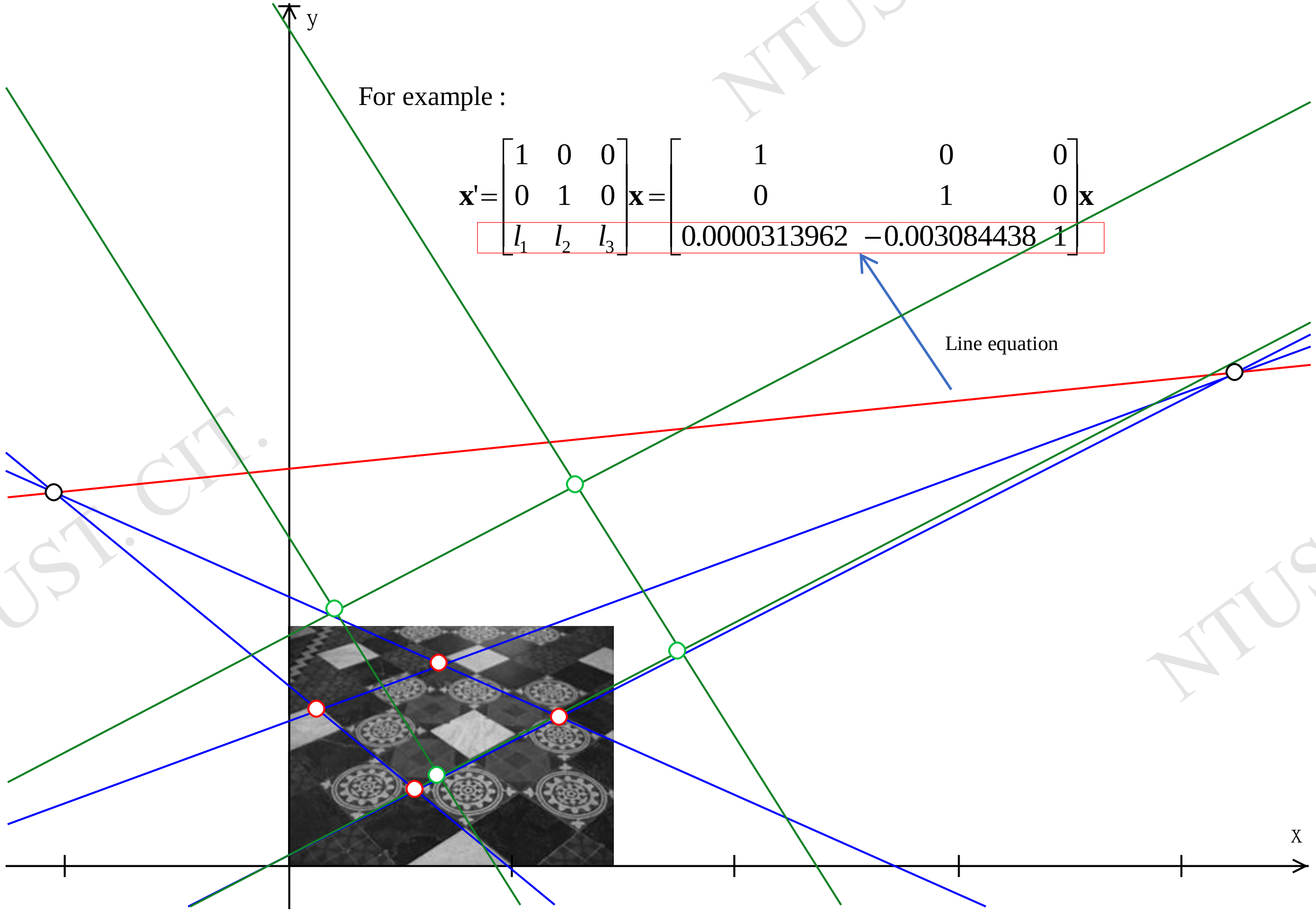
- Example

$$\mathbf{H} = \mathbf{H}_A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ l_1 & l_2 & l_3 \end{bmatrix}$$



Affine rectification via the vanishing line

- Example—cont.



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