電腦視覺與應用
Computer Vision and Applications

Lecture-05

Projective 3D geometry

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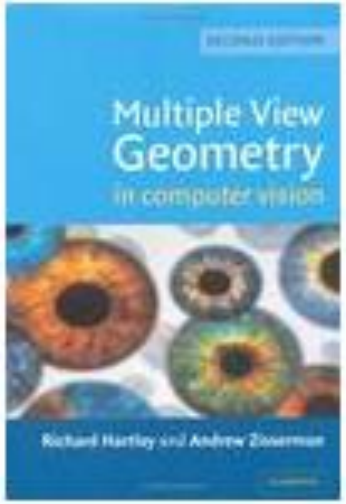
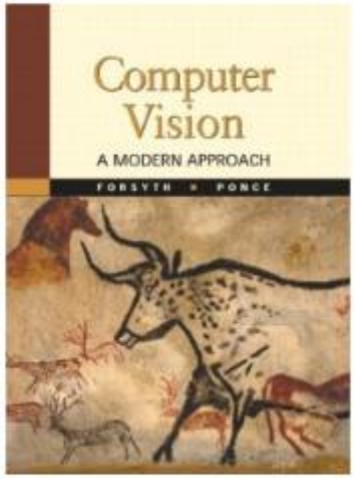
Projective 3D geometry

- Lecture Reference at:
  - Multiple View Geometry in Computer Vision, Chapter 3. (major)
  - Computer Vision A Modern Approach, (NA).





Keyword list

- 3D Homography (4x4 matrix)
- Rigid body motion
- ICP (Iterative closest point), registration
- Projective transformation, affine, similarity, Euclidean
- Screw decomposition

Notation description

- Capital character → for 3D (4 elements in homogenous)
- Low case character → for 2D (3 elements in homogenous)

- **Bold** → vector or matrix.
- *Italic* → real, scalar or variable.

NOTE:

Notation in this lecture may differ from those in reference/textbook.

3D point representation

- In general, 3D point is written as

$$(X, Y, Z)^{\mathsf{T}} \text{ in } \mathbf{R}^3$$

$$\mathbf{X} = (X_1, X_2, X_3, X_4)^{\mathsf{T}} \text{ in } \mathbf{P}^3$$

- Homogenous representation

$$\mathbf{X} = \left(\frac{X_1}{X_4}, \frac{X_2}{X_4}, \frac{X_3}{X_4}, 1\right)^{\mathsf{T}} = (X, Y, Z, 1)^{\mathsf{T}} \quad (X_4 \neq 0)$$

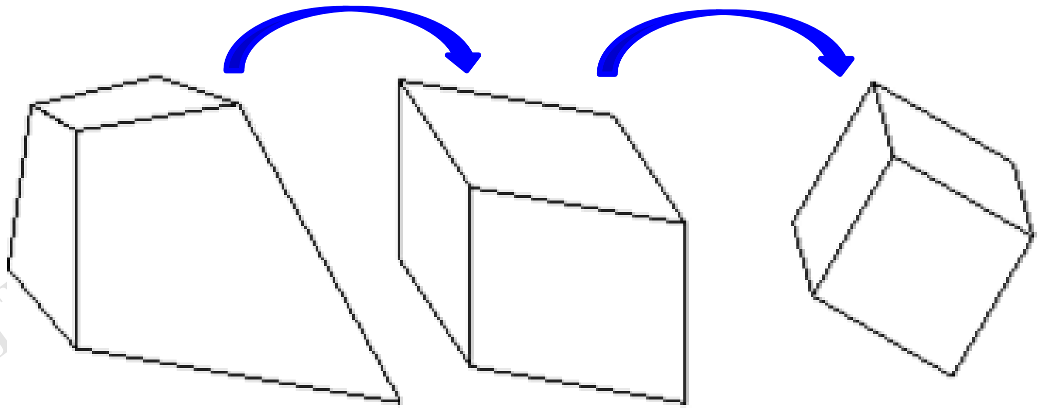
- 3D point at infinity

$$\mathbf{X} = (X, Y, Z, 0)^{\mathsf{T}}$$

3D point transformation

- 3D point transformation is similar to 2D, projective transformation (homography)

$$\mathbf{X}' = \mathbf{H}\mathbf{X} \quad \text{(4x4-1=15 DOF, upto scale)}$$



3D point transformation—cont.

$$\mathbf{X}' = \mathbf{HX}$$

$$\rightarrow \begin{bmatrix} X_1' \\ X_2' \\ X_3' \\ X_4' \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} X' \\ Y' \\ Z' \\ 1 \end{bmatrix} \sim \begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14} \\ H_{21} & H_{22} & H_{23} & H_{24} \\ H_{31} & H_{32} & H_{33} & H_{34} \\ H_{41} & H_{42} & H_{43} & H_{44} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \rightarrow \begin{aligned} X' &= \frac{H_{11}X + H_{12}Y + H_{13}Z + H_{14}}{H_{41}X + H_{42}Y + H_{43}Z + H_{44}} \\ Y' &= \frac{H_{21}X + H_{22}Y + H_{23}Z + H_{24}}{H_{41}X + H_{42}Y + H_{43}Z + H_{44}} \\ Z' &= \frac{H_{31}X + H_{32}Y + H_{33}Z + H_{34}}{H_{41}X + H_{42}Y + H_{43}Z + H_{44}} \end{aligned}$$

3D point transformation—cont.

- Cont.

$$\rightarrow \begin{aligned} &H_{11}X + H_{12}Y + H_{13}Z + H_{14} - X'(H_{41}X + H_{42}Y + H_{43}Z + H_{44}) = 0 \\ &H_{21}X + H_{22}Y + H_{23}Z + H_{24} - Y'(H_{41}X + H_{42}Y + H_{43}Z + H_{44}) = 0 \\ &H_{31}X + H_{32}Y + H_{33}Z + H_{34} - Z'(H_{41}X + H_{42}Y + H_{43}Z + H_{44}) = 0 \end{aligned}$$

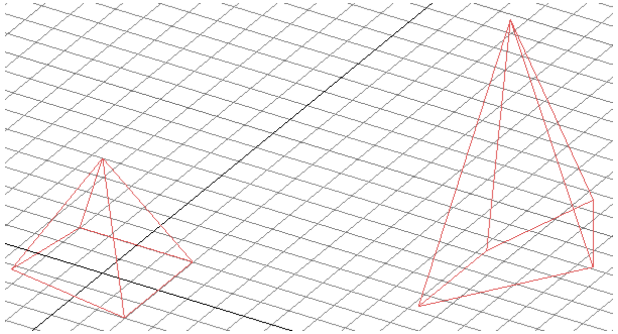
- For abbreviation, let

$$\boxed{\begin{aligned} &\tilde{\mathrm{H}}_1^{\mathrm{T}} = [H_{11} \quad H_{12} \quad H_{13} \quad H_{14}] \\ &\tilde{\mathrm{H}}_2^{\mathrm{T}} = [H_{21} \quad H_{22} \quad H_{23} \quad H_{24}] \\ &\tilde{\mathrm{H}}_3^{\mathrm{T}} = [H_{31} \quad H_{32} \quad H_{33} \quad H_{34}] \\ &\tilde{\mathrm{H}}_4^{\mathrm{T}} = [H_{41} \quad H_{42} \quad H_{43} \quad H_{44}] \\ &\mathbf{X}^{\mathrm{T}} = [X \quad Y \quad Z \quad 1] \end{aligned}}$$

$$\rightarrow \begin{bmatrix} \mathbf{X}^{\mathrm{T}} & 0^{\mathrm{T}} & 0^{\mathrm{T}} & -X'\mathbf{X}^{\mathrm{T}} \\ 0^{\mathrm{T}} & \mathbf{X}^{\mathrm{T}} & 0^{\mathrm{T}} & -Y'\mathbf{X}^{\mathrm{T}} \\ 0^{\mathrm{T}} & 0^{\mathrm{T}} & \mathbf{X}^{\mathrm{T}} & -Z'\mathbf{X}^{\mathrm{T}} \end{bmatrix}_{3\times 16} \begin{bmatrix} \tilde{\mathrm{H}}_1 \\ \tilde{\mathrm{H}}_2 \\ \tilde{\mathrm{H}}_3 \\ \tilde{\mathrm{H}}_4 \end{bmatrix}_{16\times 1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$$

3D point transformation—cont.

- Example



(Note: need to avoid degenerated points)

X1=[0,0,40,1]^T
X2=[-20,-20,5,1]^T
X3=[20,-20, 0,1]^T
X4=[20,20,0,1]^T
X5=[-20,20,0,1]^T

XP1=[90,90,71.4392,1]^T
XP2=[70,70,-24.9519,1]^T
XP3=[125.1275,80.8687,0.0,1]^T
XP4=[104.0309,116.0521,0.0,1]^T
XP5=[70,110,-24.9519,1]^T

solved by DLT method

H =

1.0463	0.8320	0.1572	90.9871
0.1791	2.0459	-0.0409	98.9097
0.7726	0	2.3166	-15.4527
-0.0001	0.0119	0.0020	1.0000

3D point transformation—cont.

- Example

(in Matlab)

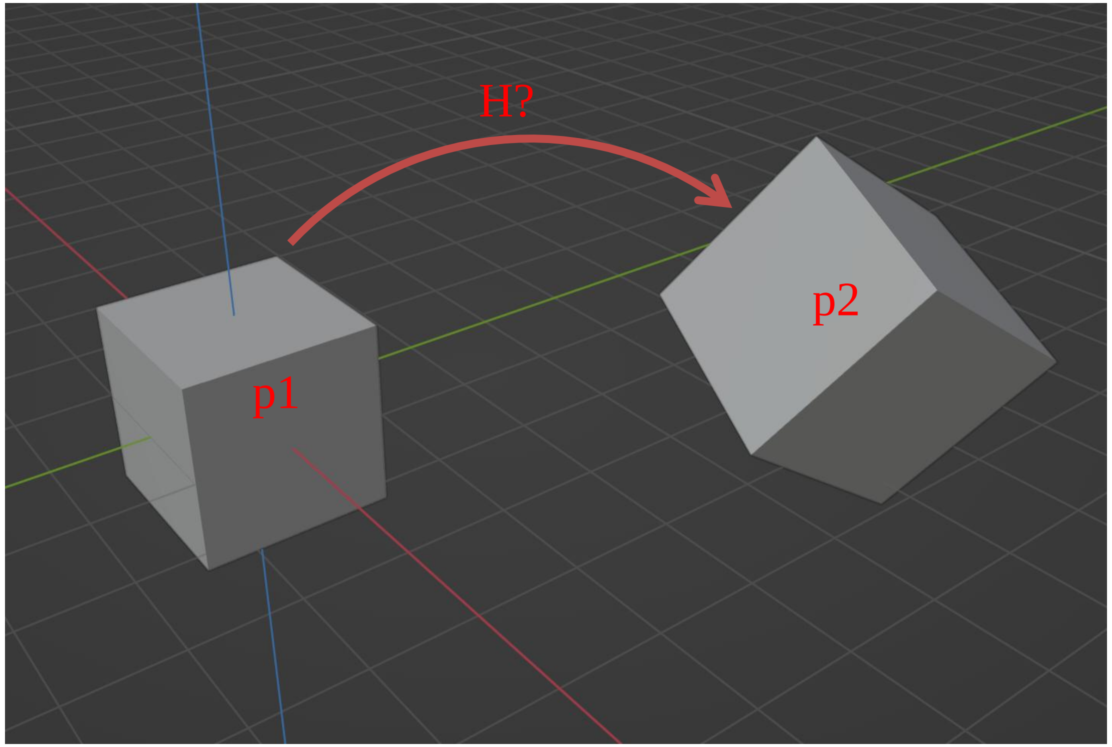
```
z=[0 0 0 0]'

A=[X1' z' z'  -XP1(1).*X1';
 z' X1' z'  -XP1(2).*X1';
 z'  z' X1' -XP1(3).*X1';
 X2' z' z'  -XP2(1).*X2';
 z' X2' z'  -XP2(2).*X2';
 z'  z' X2' -XP2(3).*X2';
 X3' z' z'  -XP3(1).*X3';
 z' X3' z'  -XP3(2).*X3';
 z'  z' X3' -XP3(3).*X3';
 X4' z' z'  -XP4(1).*X4';
 z' X4' z'  -XP4(2).*X4';
 z'  z' X4' -XP4(3).*X4';
 X5' z' z'  -XP5(1).*X5';
 z' X5' z'  -XP5(2).*X5';
 z'  z' X5' -XP5(3).*X5'];
```

$$
\mathrm{inv}\left(1.0e{+}003 * \begin{matrix}
0 & 0 & 0.0400 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3.6000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.0400 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & -3.6000 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0400 & 0.0010 & 0 & 0 & -2.8576 \\
-0.0200 & -0.0200 & 0.0050 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4000 & 1.4000 & -0.3500 \\
0 & 0 & 0 & 0 & -0.0200 & -0.0200 & 0.0050 & 0.0010 & 0 & 0 & 0 & 0 & 1.4000 & 1.4000 & -0.3500 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0200 & -0.0200 & 0.0050 & 0.0010 & -0.4990 & -0.4990 & 0.1248 \\
0.0200 & -0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.5026 & 2.5026 & 0 \\
0 & 0 & 0 & 0 & 0.0200 & -0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & -1.6174 & 1.6174 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0200 & -0.0200 & 0 & 0.0010 & 0 & 0 & 0 \\
0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2.0806 & -2.0806 & 0 \\
0 & 0 & 0 & 0 & 0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & -2.3210 & -2.3210 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 \\
-0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.4000 & -1.4000 & 0 \\
0 & 0 & 0 & 0 & -0.0200 & 0.0200 & 0 & 0.0010 & 0 & 0 & 0 & 0 & 2.2000 & -2.2000 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.0200 & 0.0200 & 0 & 0.0010 & -0.4990 & 0.4990 & 0
\end{matrix} \right) * \begin{matrix}
90.0000 \\ 90.0000 \\ 71.4392 \\ 70.0000 \\ 70.0000 \\ -24.9519 \\ 125.1275 \\ 80.8687 \\ 0 \\ 104.0309 \\ 116.0521 \\ 0 \\ 70.0000 \\ 110.0000 \\ -24.9519
\end{matrix}
$$

3D object transformation

Rigid body motion (6DOF) – solved by SVD

We collect 8 corresponding points (some of them are degenerated)




```
p1 = [-1.000000 -1.000000 -1.000000 1
-1.000000 -1.000000 1.000000 1
-1.000000 1.000000 1.000000 1
-1.000000 1.000000 -1.000000 1
1.000000 1.000000 1.000000 1
1.000000 1.000000 -1.000000 1
1.000000 -1.000000 1.000000 1
1.000000 -1.000000 -1.000000 1]'

p2 = [2.626868 3.096269 1.338255 1
3.456607 4.024694 2.903362 1
3.662979 5.685713 1.808634 1
2.833241 4.757287 0.243527 1
5.471002 5.070046 1.215330 1
4.641263 4.141621 -0.349777 1
5.264629 3.409028 2.310058 1
4.434891 2.480603 0.744951 1]'

zeroV = [0 0 0 0]'
```

3D object transformation

Rigid body motion (6DOF) – solved by SVD —cont.

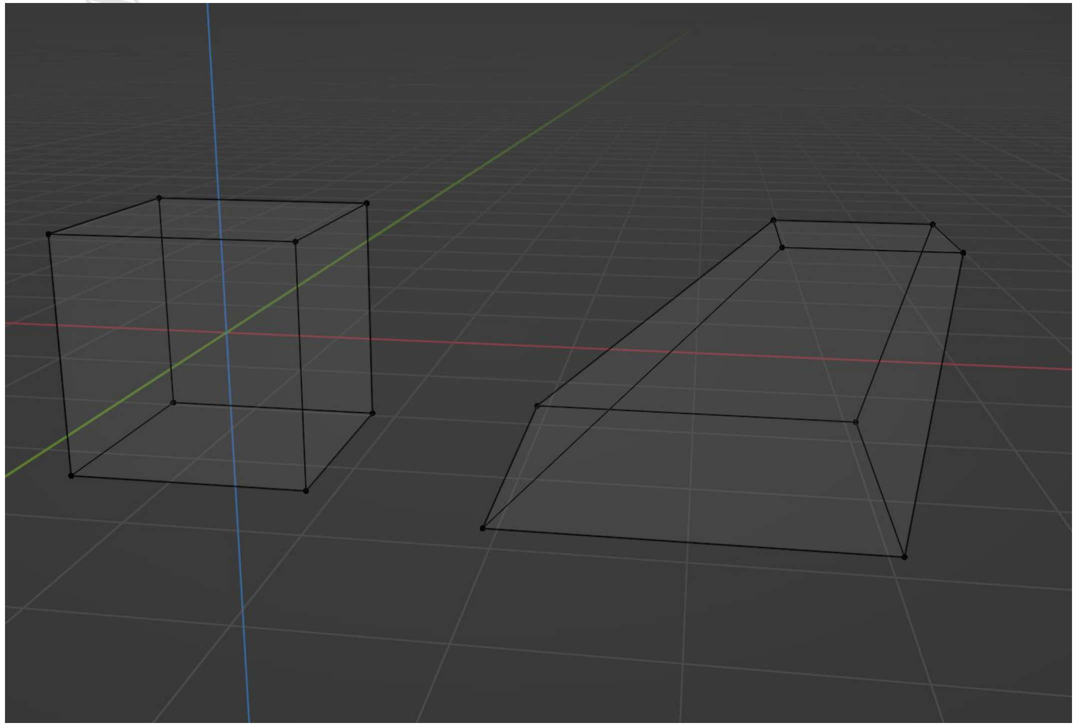
```
A =[
p1(1:4,1)' zeroV' zeroV' -p2(1,1)*p1(1:4,1)'
zeroV' p1(1:4,1)' zeroV' -p2(2,1)*p1(1:4,1)'
zeroV' zeroV' p1(1:4,1)' -p2(3,1)*p1(1:4,1)'
p1(1:4,2)' zeroV' zeroV' -p2(1,2)*p1(1:4,2)'
zeroV' p1(1:4,2)' zeroV' -p2(2,2)*p1(1:4,2)'
zeroV' zeroV' p1(1:4,2)' -p2(3,2)*p1(1:4,2)'
p1(1:4,3)' zeroV' zeroV' -p2(1,3)*p1(1:4,3)'
zeroV' p1(1:4,3)' zeroV' -p2(2,3)*p1(1:4,3)'
zeroV' zeroV' p1(1:4,3)' -p2(3,3)*p1(1:4,3)'
p1(1:4,4)' zeroV' zeroV' -p2(1,4)*p1(1:4,4)'
zeroV' p1(1:4,4)' zeroV' -p2(2,4)*p1(1:4,4)'
zeroV' zeroV' p1(1:4,4)' -p2(3,4)*p1(1:4,4)'
p1(1:4,5)' zeroV' zeroV' -p2(1,5)*p1(1:4,5)'
zeroV' p1(1:4,5)' zeroV' -p2(2,5)*p1(1:4,5)'
zeroV' zeroV' p1(1:4,5)' -p2(3,5)*p1(1:4,5)'
p1(1:4,6)' zeroV' zeroV' -p2(1,6)*p1(1:4,6)'
zeroV' p1(1:4,6)' zeroV' -p2(2,6)*p1(1:4,6)'
zeroV' zeroV' p1(1:4,6)' -p2(3,6)*p1(1:4,6)'
]

[U,S,V] = svd(A)

H = [V(1:4,16)'; V(5:8,16)' ; V(9:12,16)' ; V(13:16,16)']
H = H./H(4,4)
```

```
H  =

   0.9040113    0.1031862    0.4148689    4.0489349
  -0.3078332    0.8305092    0.4642121    4.0831574
  -0.296652    -0.547364     0.7825533    1.2767924
   1.234D-08   -1.234D-08   -0.0000001    1.
```

Note the final row

3D object deformation

Project transformation (Full Rank)



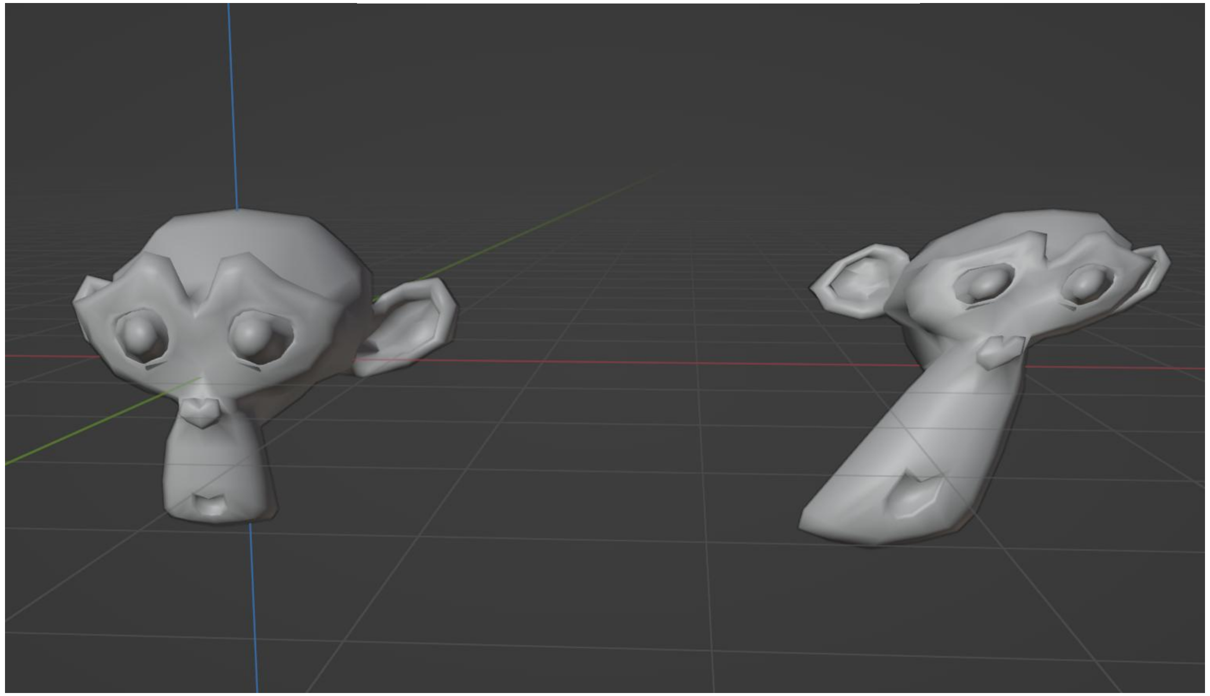
H =

0.8885907	-1.770D-15	2.4710969	4.8569564
-1.408D-16	0.8885907	-7.743D-16	8.730D-16
-1.716D-15	-6.605D-16	1.	0.4035351
-2.671D-15	7.139D-16	0.4035351	1.

3D object deformation

Project transformation (Full Rank) —cont.

```
H  =

  0.8885907  -1.770D-15   2.4710969   4.8569564
 -1.408D-16   0.8885907  -7.743D-16   8.730D-16
 -1.716D-15  -6.605D-16   1.          0.4035351
 -2.671D-15   7.139D-16   0.4035351   1.
```



Planes

- 3D plane

$$\pi_1 X + \pi_2 Y + \pi_3 Z + \pi_4 1 = 0$$

$$\pi_1 X_1 + \pi_2 X_2 + \pi_3 X_3 + \pi_4 X_4 = 0$$ →homogenous

$$\boldsymbol{\pi}^{\mathsf{T}} \mathbf{X} = 0$$ →3D points **X** on a plane **π**

Note: **π** is a plane equation equals to (π_1, π_2, π_3, π_4)

X denotes 3D points equal to (X_1, X_2, X_3, X_4)

- Transformation

$$\mathbf{X}' = \mathbf{H}\mathbf{X}$$ →3D points mapping to another 3D points

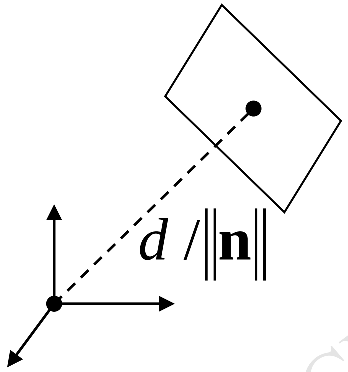
$$\boldsymbol{\pi}' = \mathbf{H}^{-\mathsf{T}} \boldsymbol{\pi}$$ →3D planes mapping to another 3D planes

Planes—cont.: in Euclidean case

- Euclidean representation

$$\mathbf{n} \cdot \tilde{\mathbf{X}} + d = 0 \qquad \mathbf{n} = (\pi_1, \pi_2, \pi_3)^{\mathsf{T}}$$ →normal of this plane

$$\pi_4 = d$$



$$\tilde{\mathrm{X}} = (X, Y, Z)^{\mathsf{T}} \qquad X_4 = 1$$

Dual: points ↔ planes, lines ↔ lines

Planes from points

Solve $\boldsymbol{\pi}$ from $\mathrm{X}_1^\top \pi = 0$, $\mathrm{X}_2^\top \pi = 0$ and $\mathrm{X}_3^\top \pi = 0$

$$\begin{bmatrix} \mathbf{X}_1^\top \\ \mathbf{X}_2^\top \\ \mathbf{X}_3^\top \end{bmatrix} \pi = 0 \quad \text{(solve } \pi \text{ as right nullspace of } \begin{bmatrix} \mathrm{X}_1^\top \\ \mathrm{X}_2^\top \\ \mathrm{X}_3^\top \end{bmatrix} \text{)}$$

Given 3 3D-points to determine a plane in 3D space.

Known: $\mathbf{X}_1^\top = [(X_1)_1 \quad (X_1)_2 \quad (X_1)_3 \quad (X_1)_4]$ $\mathbf{X}_2^\top = [(X_2)_1 \quad (X_2)_2 \quad (X_2)_3 \quad (X_2)_4]$ $\mathbf{X}_3^\top = [(X_3)_1 \quad (X_3)_2 \quad (X_3)_3 \quad (X_3)_4]$

Unknown: $\boldsymbol{\pi} = \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix}$

$$\begin{bmatrix} (X_1)_1 & (X_1)_2 & (X_1)_3 & (X_1)_4 \\ (X_2)_1 & (X_2)_2 & (X_2)_3 & (X_2)_4 \\ (X_3)_1 & (X_3)_2 & (X_3)_3 & (X_3)_4 \end{bmatrix} \begin{bmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \\ \pi_4 \end{bmatrix} = 0$$

Planes from points

- Or implicitly from coplanarity condition

$$\det\begin{bmatrix} \boxed{X_1} & (X_1)_1 & (X_2)_1 & (X_3)_1 \\ X_2 & (X_1)_2 & (X_2)_2 & (X_3)_2 \\ X_3 & (X_1)_3 & (X_2)_3 & (X_3)_3 \\ X_4 & (X_1)_4 & (X_2)_4 & (X_3)_4 \end{bmatrix} = 0$$

determinant of sub-matrix

$$\det[\mathbf{X}\,\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3] = 0$$

$$X_1 D_{234} - X_2 D_{134} + X_3 D_{124} - X_4 D_{123} = 0$$

$$\boldsymbol{\pi} = (D_{234}, -D_{134}, D_{124}, -D_{123})^{\mathsf{T}}$$

Determinant of matrix—review

- 3x3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - bdi - afh.$$

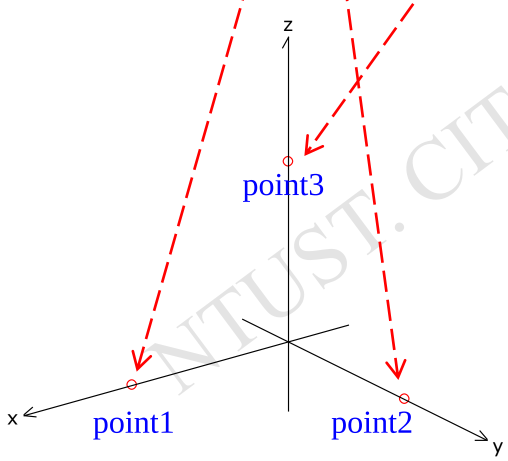
- higher order matrix

(decomposition from row or column)

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix}$$

Planes from points—example

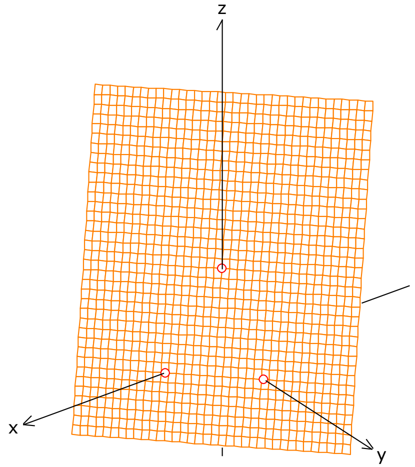
$$\det\begin{bmatrix} X_1 & 1 & 0 & 0 \\ X_2 & 0 & 1 & 0 \\ X_3 & 0 & 0 & 1 \\ X_4 & 1 & 1 & 1 \end{bmatrix} = 0$$



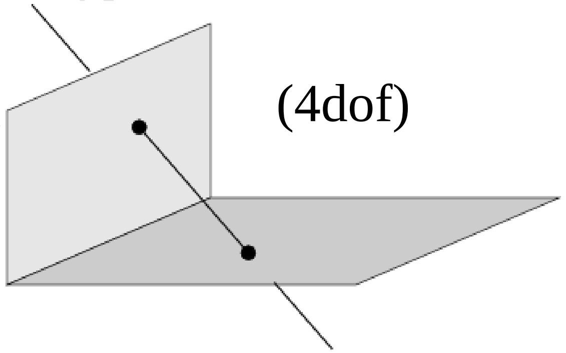
$$D_{234} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1 \qquad D_{124} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$D_{134} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -1 \qquad D_{123} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\boldsymbol{\pi} = \left(D_{234}, -D_{134}, D_{124}, -D_{123}\right)^{\mathsf{T}}$$
$$= (1,1,1,-1)^{\mathsf{T}}$$



Lines representation



(4dof)

$$W = \begin{bmatrix} A^\top \\ B^\top \end{bmatrix} \qquad \lambda A + \mu B$$

→one point & one direction
(two point representation)

or

$$W^* = \begin{bmatrix} P^\top \\ Q^\top \end{bmatrix} \qquad \lambda P + \mu Q$$

→intersection of two planes

$$W^* W^\top = W W^{*\top} = 0_{2\times 2}$$

Example: *X*-axis

$$W = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

→ the origin

→ ideal point on *x* axis

$$W^* = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

→*z* plane

→*y* plane

Other lines representation

- Plücker matrices (4x4 skew symmetric homogenous)
- Plücker line coordinates

Quadrics and dual quadrics

$$\mathbf{X}^{\mathsf{T}}\mathbf{Q}\mathbf{X} = 0 \qquad (\mathbf{Q} : \text{4x4 symmetric matrix})$$

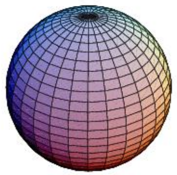
$$Q = \begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \bullet & \bullet \\ \circ & \circ & \bullet & \bullet \\ \circ & \circ & \circ & \bullet \end{bmatrix}$$

1. 9 DOF
2. in general 9 points define quadric
3. det|**Q**|=0 ↔ degenerate quadric
4. pole – polar $\boldsymbol{\pi} = \mathbf{QX}$
5. (plane ∩ quadric)=conic $\mathbf{C} = \mathbf{M}^{\mathsf{T}}\mathbf{Q}\mathbf{M}$ $\quad \pi : X = Mx$
6. transformation (under $\mathbf{X}' = \mathbf{HX}$)

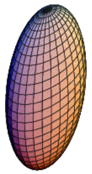
$$\mathbf{Q}' = \mathbf{H}^{-\mathsf{T}}\mathbf{Q}\mathbf{H}^{-1}$$

Quadric classification

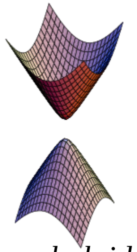
- Projective equivalent to sphere:



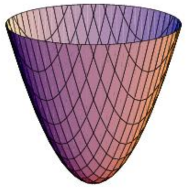
sphere



ellipsoid

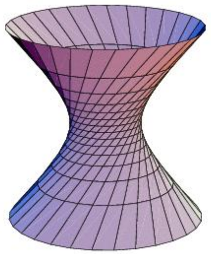


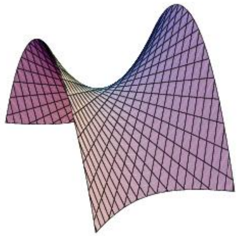
hyperboloid of two sheets



paraboloid

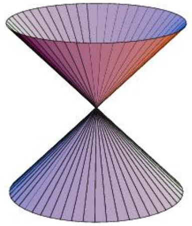
- Ruled quadrics:



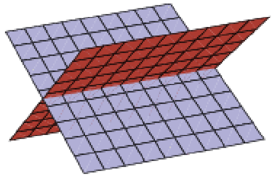


hyperboloids of one sheet

- Degenerate ruled quadrics:

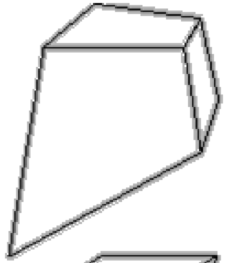
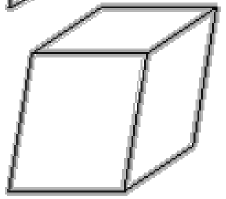
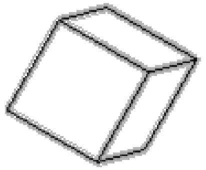
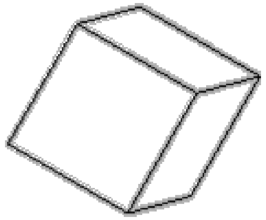


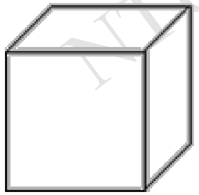
cone



two planes

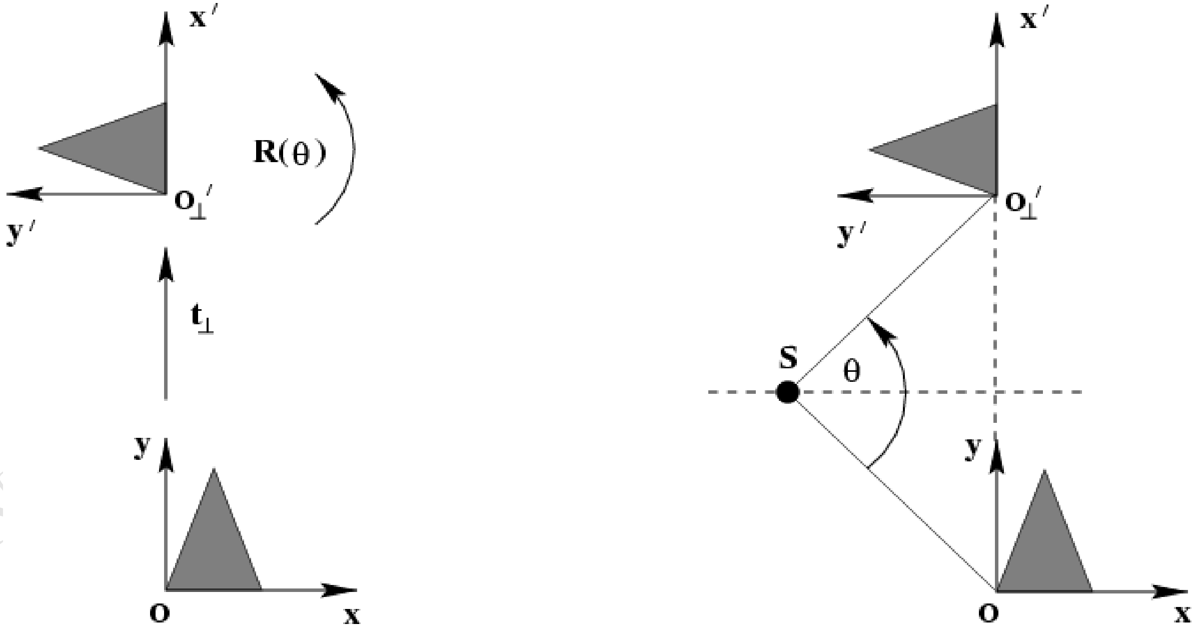
Hierarchy of transformations

Projective 15dof	$\begin{bmatrix} A & t \\ v^{\mathsf{T}} & v \end{bmatrix}$		Intersection and tangency
Affine 12dof	$\begin{bmatrix} A & t \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		Parallellism of planes, Volume ratios, centroids, **The plane at infinity $\boldsymbol{\pi}_\infty$**
Similarity 7dof	$\begin{bmatrix} sR & t \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		**The absolute conic Ω_∞**
Euclidean 6dof	$\begin{bmatrix} R & t \\ 0^{\mathsf{T}} & 1 \end{bmatrix}$		Volume



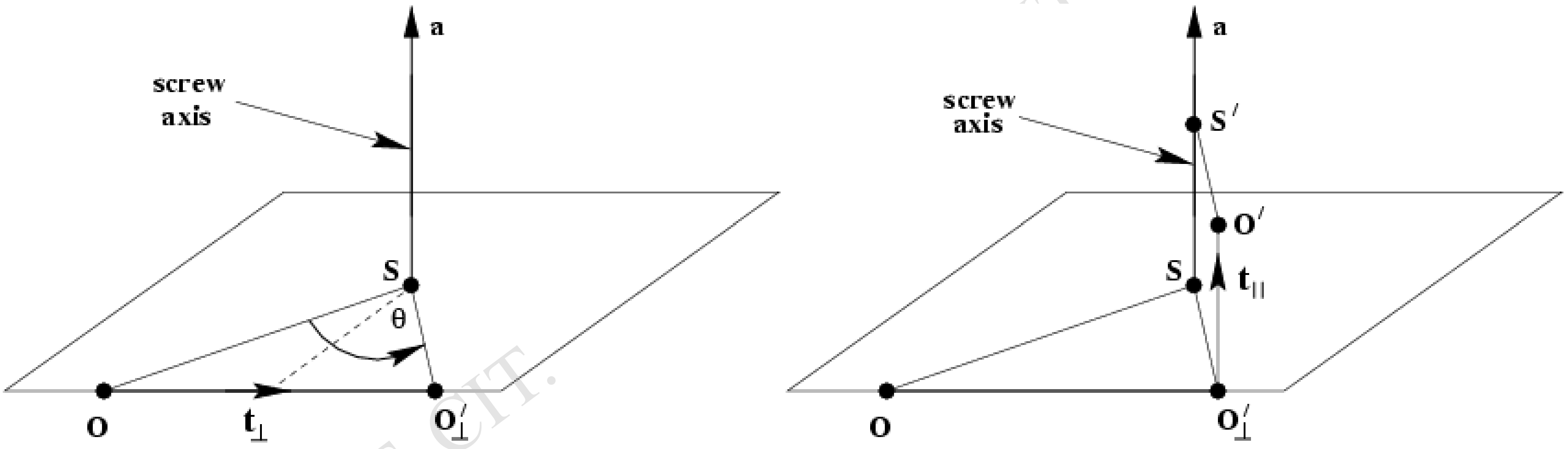
Screw decomposition

- Any particular translation and rotation is equivalent to a rotation about a screw axis and a translation along the screw axis.
- 2D case:



Screw decomposition

- 3D case:



screw axis // rotation axis

$$t = t_{//} + t_{\perp}$$

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