Global asymptotic stability of a general class of recurrent neural networks with time-varying delays

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Global Asymptotic Stability of a General Class of Recurrent Neural Networks With Time-Varying Delays

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***Abstract*—In this paper, the existence and uniqueness of the equilibrium point and its global asymptotic stability are discussed for a general class of recurrent neural networks with time-varying delays and Lipschitz continuous activation functions. The neural network model considered includes the delayed Hopfield neural networks, bidirectional associative memory networks, and delayed cellular neural networks as its special cases. Several new sufficient conditions for ascertaining the existence, uniqueness, and global asymptotic stability of the equilibrium point of such recurrent neural networks are obtained by using the theory of topological degree and properties of nonsingular M-matrix, and constructing suitable Lyapunov functionals. The new criteria do not require the activation functions to be differentiable, bounded or monotone nondecreasing and the connection weight matrices to be symmetric. Some stability results from previous works are extended and improved. Two illustrative examples are given to demonstrate the effectiveness of the obtained results.**

***Index Terms*—Equilibrium point, global asymptotic stability, Lyapunov functional, nonsingular M-matrix, recurrent neural networks, time-varying delays, topological degree.**

I. Introduction

IT IS well known that stability and convergence are prerequisites for designing neural networks. When designing a neural network to solve a problem such as linear program or pattern recognition, for example, we need foremost to guarantee that the neural network model is globally asymptotically stable. In practice, time delays, either constant or time varying, are often encountered in various engineering, biological, and economical systems. Due to the finite speed of information processing, the existence of time delays frequently causes oscillation, divergence, or instability in neural networks. In recent years, the stability of delayed neural networks has become a topic of great theoretic and practical importance. This issue has gained increasing interest in applications to signal and image processing, artificial intelligence, and industrial automation, to name a few. As a results, many criteria for testing the global stability of recurrent neural networks with constant delays and without delays have been derived (see, e.g., [2]–[8], [15]–[39]). To the best of our knowledge, recurrent neural networks with time-varying delays are seldom considered. However, time-varying delays in recurrent neural networks are more common in practice. The delays are known to be bounded but their values are unknown. Therefore, the studies of recurrent neural networks with time-varying delays are more important than those with constant delays.

Consider a class of recurrent neural networks with time-varying delays and Lipschitz continuous activation functions described by the following nonlinear differential equations:

$$\begin{aligned}\frac{dx_i(t)}{dt} =& -d_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &+ \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau_j(t))) + I_i, \\ & i = 1, 2, \ldots, n \end{aligned} \tag{1}$$

where n denotes the number of neurons in a neural network, $x_i(t)$ corresponds to the state of the ith neurons at time t, $f_j(x_j(t))$, $g_j(x_j(t))$ denote the activation functions of the jth neuron at time t, a_{ij} denotes the constant connection weight of the jth neuron on the ith neuron at time t, b_{ij} denotes the constant connection weight of the jth neuron on the ith neuron at time $t-\tau_j(t)$, I_i is the external bias on the ith neuron, $d_i > 0$ represents the rate with which the ith neuron will reset its potential to the resting state in isolation when disconnected from the network and external inputs. $\tau_i(t)$ is nonnegative, bounded, and differentiable with $0 \le \tau_i(t) \le \tau$, $i = 1, 2, \ldots, n$.

We can rewrite model (1) in the following matrix-vector form:

$$\frac{dx(t)}{dt} = -Dx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + I \tag{2}$$

where $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$, $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$, $I = (I_1, I_2, \ldots, I_n)^T$ and

$$\begin{aligned} f(x) =& (f_1(x_1), f_2(x_2), \ldots, f_n(x_n))^T \\ g(x(t-\tau(t))) =& (g_1(x_1(t-\tau_1(t))) \\ & g_2(x_2(t-\tau_2(t))), \ldots, g_n(x_n(t-\tau_n(t))))^T. \end{aligned}$$

To obtain our results, we give the following assumption.

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A_1 : f_i, g_i $(i = 1, 2, \ldots, n)$ are globally Lipschitz continuous; i.e., there exist positive constants k_i, l_i such that

$$|f_i(u) - f_i(v)| \le k_i|u - v|, |g_i(u) - g_i(v)| \le l_i|u - v|$$

for $\forall u, v \in R$, and $i = 1, 2, \ldots, n$.

It can be easily seen that the model (1) is a general recurrent neural network model which includes some well-known neural networks as its special cases.

i) Let $B = (b_{ij})_{n\times n} = 0$, and f is a sigmoid function, the model (1) becomes the continuous-time Hopfield neural network studied in [3], [5], [29], [30].

ii) Let

$$A = (a_{ij})_{n\times n} = 0, g_i(x_i) = \tanh(\lambda_i x_i)$$
$$B = (b_{ij})_{n\times n} = \begin{pmatrix} 0 & B_1 \\ B_2 & 0 \end{pmatrix}$$

and n be an even number, the model (1) turns into a biodirectional associative memory (BAM) network. Several stability conditions have been given in [6], [15], [16], [31].

iii) Let f_i, g_i be piecewise linear saturation functions, the model (1) is reduced into a delayed cellular neural network (DCNN) which was investigated in [4], [7], [8], [17]–[28].

For all these cases, A_1 is satisfied. Our methods in this paper are suitable to the Hopfield neural network, BAM networks and DCNNs. Of course, our results of this paper can also be applied to various neural networks with and without delays.

The aim of this paper is to study the global asymptotic stability of the recurrent neural network (1) with time-varying delays and Lipschitz continuous activation functions, and give a set of criteria ensuring the existence and uniqueness of the equilibrium point and its global asymptotic stability of such a recurrent neural network by constructing suitable Lyapunov functional, utilizing the property of nonsingular M-matrix and applying the theory of topological degree. The obtained sufficient conditions only require the activation functions to be Lipschitz continuous and do not require them to be differentiable, bounded, or monotone nondecreasing. Our results extend and improve some previous stability results for recurrent neural networks with and without delays.

II. Preliminaries

In order to simplify the proofs and compare our results, we give some notations and definitions as follows.

First, denote $K = \mathrm{diag}(k_1, k_2, \ldots, k_n)$, $L = \mathrm{diag}(l_1, l_2, \ldots, l_n)$, $|A| = (|a_{ij}|)_{n\times n}$, $|B| = (|b_{ij}|)_{n\times n}$.

Definition 1: An $n\times n$ matrix A is said to belong to the class $\mathcal{P}_0$ if A satisfies one of the following equivalent conditions.

i) All principal minors of A are nonnegative.

ii) For each $x \in R^n$, if $x \ne 0$, there exists an index $i \in \{1, 2, \ldots, n\}$ such that $x_i \ne 0$ and $x_i(Ax)_i \ge 0$, where $(Ax)_i$ denotes the ith component of the vector Ax.

iii) For each diagonal matrix $P = \mathrm{diag}(p_1, p_2, \ldots, p_n)$ with $p_i > 0$, $i = 1, 2, \ldots, n$, $\det(A + P) \ne 0$.

Definition 2 [13]: Let the $n\times n$ matrix A have nonpositive off-diagonal elements, then A is an M-matrix if one of the following conditions hold.

i) All principal minors of A are nonnegative; i.e., $A \in \mathcal{P}_0$.

ii) $A + P$ is nonsingular for any positive diagonal matrix P.

Definition 3 [35]: An $n\times n$ matrix $A = (a_{ij})$ is said to be an H-matrix if its *comparison matrix* $C = (c_{ij})$ is an M-matrix. where $C = (c_{ij})$ is defined as $c_{ii} = |a_{ii}|$, $c_{ij} = -|a_{ij}|$ $(i \ne j)$, $i, j = 1, 2, \ldots, n$.

Definition 4 [13], [35]: Let matrix $A = (a_{ij})_{n\times n}$ have nonpositive off-diagonal elements, then A is a nonsingular M-matrix if one of the following conditions holds:

i) All principal minors of A are positive;

ii) A have all positive diagonal elements and there exists a positive diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $A\Lambda$ is strictly diagonally dominant; that is

$$a_{ii}\lambda_i > \sum_{j\ne i} |a_{ij}|\lambda_j, \qquad i = 1, 2, \ldots, n$$

which can be rewritten as

$$\sum_{j=1}^{n} a_{ij}\lambda_j > 0, \qquad i = 1, 2, \ldots, n.$$

This class of matrices is denoted by $A \in \mathcal{K}$.

According to Definition 2 and Definition 4, a matrix A with nonpositive off-diagonal elements is an M-matrix or a nonsingular M-matrix if and only if its transposition A^T is also. That is, if a matrix A with nonpositive off-diagonal elements is a nonsingular M-matrix, then there exists a positive diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $A\Lambda$ or $A^T\Lambda$ is strictly diagonally dominant; that is

$$\sum_{j=1}^{n} a_{ij}\lambda_j > 0 \text{ or } \sum_{j=1}^{n} a_{ji}\lambda_j > 0, \qquad i = 1, 2, \ldots, n. \quad (3)$$

Definition 5 [12]: A matrix $A = (a_{ij})_{n\times n}$ is said to be quasidominant if there exists a positive diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $A\Lambda$ is strictly diagonally dominant; that is, there exists $\lambda_i > 0$, $i = 1, 2, \ldots, n$, such that $\lambda_i a_{ii} > \sum_{j\ne i} \lambda_j |a_{ij}|$.

Definition 6: An $n\times n$ matrix A is said to be a nonsingular H-matrix if its comparison matrix $C = (c_{ij}) \in \mathcal{K}$. This class of matrices is denoted by $A \in \mathcal{C}$.

Definition 7: An $n\times n$ matrix A is said to be a singular M-matrix if $a_{ii} \ge 0$, $a_{ij} \le 0$ and the real part of every eigenvalue of A is nonnegative. This class of matrices is denoted by $A \in \mathcal{K}_0$.

Definition 8: An $n\times n$ matrix A is said to be a singular H-matrix if its comparison matrix $C = (c_{ij}) \in \mathcal{K}_0$. This class of matrices is denoted by $A \in \mathcal{C}_0$.

Definition 9: An $n\times n$ matrix A is said to be *Lyapunov diagonally stable* (respectively, *Lyapunov diagonally semistable*) if there exists a positive diagonal matrix P such that $PA + A^TP < 0$ (respectively, $PA + A^TP \le 0$). This class of matrices is denoted by $A \in D$ (respectively, $A \in \mathcal{D}_0$).

It is clear that the negative semidefinite matrix (either is symmetric or asymmetric), and the antisymmetric matrix are both Lyapunov diagonally semistable.

Definition 10: An $n \times n$ matrix A is said to be *additively diagonally stable* if for any positive diagonal matrix D_1, there exists a positive diagonal matrix D_2 such that $D_2(A - D_1) + (A - D_1)^T D_2 < 0$. This class of matrices is denoted by $A \in \mathcal{M}_0$.

According to the results in [35], we know that the two matrix classes $\mathcal{C}_0$ and $\mathcal{D}_0$ are both *proper* subclasses of $\mathcal{M}_0$, while they are not included by each other.

Definition 11: Let $f(t) : R \to R$ be a continuous function. The upper right dini derivative $D^+ f(t)$ is defined as

$$D^+ f(t) = \overline{\lim_{h \to 0^+}} \frac{1}{h}(f(t+h) - f(t)).$$

Definition 12 [14]: Let $f(x) : \Omega \to R^n$ be a continuous and differentiable function, if $p \notin f(\partial\Omega)$ and $J_f(x) \neq 0, \forall x \in f^{-1}(p)$, then, the topological degree is defined by

$$\deg(f, \Omega, p) \triangleq \sum_{x \in f^{-1}(p)} \operatorname{sgn} J_f(x)$$

where $\Omega \subset R^n$ is bounded and open, $J_f(x) = \det(f_{i,j}(x))$, $f_{i,j}(x) = \partial f_i / \partial x_j$.

Let $f(x) : \Omega \to R^n$ be a continuous function and $g(x) : \Omega \to R^n$ be a continuous and differentiable function, if $p \notin f(\partial\Omega)$ and $\| f(x) - g(x) \| < \rho(p, f(\partial\Omega))$, then

$$\deg(f, \Omega, p) = \deg(g, \Omega, p).$$

For example, $\deg(i_d, \Omega, p) = 1$, if $p \in \Omega$, where i_d is identity mapping, i.e., $i_d(x) = x \quad \forall x \in \Omega$.

Definition 13 [9], [10]: The equilibrium point $x^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is said to be globally asymptotically stable if it is locally stable in the sense of Lyapunov and global attractive, where global attractivity means that every trajectory tends to x^* as $t \to +\infty$.

Let a vector norm $\|x\|_p$ ($p = 1, 2, \infty$) (simply denoted by $\|x\|$) for $x \in R^n$ be defined as

$$\|x\|_1 = \sum_{i=1}^n |x_i|, \|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}, \|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

For completeness we will introduce the initial function space, $C = C([-\tau, 0], R^n)$, the Banach space of continuous functions $\phi : [-\tau, 0] \to R^n$ with the topology of uniform convergence, and $x_t \in C$ is the map defined by $x_t(\theta) = x(t + \theta)$, for $t \in R$. Clearly, x_t is obtained by restricting the map x to the interval $[t - \tau, t]$ and then translating to $[-\tau, 0]$. To prove our main theorems, we give the following Lemma which is a restatement of Theorem 4.2.8 and in [9, Corollary, pp. 265, 266].

Lemma [9]: Given a functional differential equation(FDE) $dx(t)/dt = F(t, x_t)$, let $V, Z : R \times C \to R$ be nonnegative functionals and suppose that for some $m > 0$, V is locally Lipschitz in ϕ on $[\sigma, +\infty) \times B_m$, where $B_m = \{\phi \in C | \; \| \phi \| = \sup_{\theta \in [-\tau, 0]} \|\phi(\theta)\| < m\}$. Suppose for all bounded functions $x : [\sigma - \tau, +\infty)$, there exists $c_1 > 0$ such that $\sigma \le s < t$ implies

$$Z(t, x_t) - Z(s, x_s) \le c_1(t - s).$$

If there exist scalar functions $u, v, w : R^+ \to R^+$, satisfying $u(0) = v(0) = w(0)$, with $u(r) \to +\infty$, $r \to +\infty$ such that

$$\begin{aligned} u(\|\phi(0)\|) + Z(t, \phi) \le & V(t, \phi) \le v(\|\phi(0)\|) + Z(t, \phi) \\ D^+ V \le & -w(\|\phi(0)\|) \end{aligned}$$

then, all solutions are bounded and tend to the stable zero solution as $t \to +\infty$.

III. Existence and Uniqueness of Equilibrium Point

In this section, we will present two sufficient conditions for the existence and uniqueness of equilibrium point of the recurrent neural network (1) with time-varying delays and Lipschitz continuous activation functions. We have the following two results.

Theorem 1: Assume that the assumption A_1 is satisfied. the neural network model (1) has one unique equilibrium point if $D - |A|K - |B|L$ is a nonsingular M-matrix.

Proof: Let $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ denote an equilibrium point of the neural network model (1). Then x^* satisfies

$$Dx^* - Af(x^*) - Bg(x^*) - I = 0. \tag{4}$$

Let

$$\tilde{f}(x) = Dx - Af(x) - Bg(x) - I = 0. \tag{5}$$

Obviously, the solutions of (5) are the equilibrium point of model (1). Let us define homotopic mapping

$$F(x, \lambda) = \lambda \tilde{f}(x) + (1 - \lambda)x \tag{6}$$

where $\lambda \in [0, 1]$, $F(x, \lambda) = (F_1(x, \lambda), F_2(x, \lambda), \ldots, F_n(x, \lambda))^T$, then it follows from condition A_1 that for $1 \le i \le n$

$$\begin{aligned} |F_i(x, \lambda)| = & \left| \lambda \left[d_i x_i - \sum_{j=1}^n a_{ij} f_j(x_j) \right. \right. \\ & \left. \left. - \sum_{j=1}^n b_{ij} g_j(x_j) - I_i \right] + (1 - \lambda) x_i \right| \\ \ge & |\lambda d_i x_i + (1 - \lambda) x_i| - \lambda \sum_{j=1}^n |a_{ij}| |f_j(x_j)| \\ & - \lambda \sum_{j=1}^n |b_{ij}| |g_j(x_j)| - \lambda |I_i| \\ \ge & [1 + \lambda(d_i - 1)] |x_i| - \lambda \sum_{j=1}^n k_j |a_{ij}| |x_j| \\ & - \lambda \sum_{j=1}^n l_j |b_{ij}| |x_j| \\ & - \lambda \left[|I_i| + \sum_{j=1}^n |a_{ij}| |f_j(0)| + \sum_{j=1}^n |b_{ij}| |g_j(0)| \right]. \end{aligned}$$

Since $D-|A|K-|B|L$ is a nonsingular M-matrix, hence, there exist constants $r_i > 0$ such that

$$r_i d_i - \sum_{j=1}^{n} r_j k_j |a_{ij}| - \sum_{j=1}^{n} r_j l_j |b_{ij}| > 0, \qquad i = 1, 2, \ldots, n \quad (7)$$

then, we have

$$\begin{aligned}
&\sum_{i=1}^{n} r_i |F_i(x,\lambda)| \\
&\geq \sum_{i=1}^{n} r_i (1-\lambda)|x_i)| + \lambda \sum_{i=1}^{n} \Bigg[d_i r_i |x_i| \\
&\quad - r_i \sum_{j=1}^{n} k_j |a_{ij}||x_j| - r_i \sum_{j=1}^{n} l_j |b_{ij}||x_j| \Bigg] \\
&\quad - \lambda \sum_{i=1}^{n} r_i \left(|I_i| + \sum_{j=1}^{n} |a_{ij}||f_j(0)| + \sum_{j=1}^{n} |b_{ij}||g_j(0)| \right) \\
&\geq \lambda \sum_{i=1}^{n} \left[r_i d_i |x_i| - r_i \sum_{j=1}^{n} k_j |a_{ij}||x_j| - r_i \sum_{j=1}^{n} l_j |b_{ij}||x_j| \right] \\
&\quad - \lambda \sum_{i=1}^{n} r_i \left[|I_i| + \sum_{j=1}^{n} |a_{ij}||f_j(0)| + \sum_{j=1}^{n} |b_{ij}||g_j(0)| \right] \\
&= \lambda \sum_{i=1}^{n} \left[r_i d_i - \sum_{j=1}^{n} r_j |a_{ji}| k_i - \sum_{j=1}^{n} r_j |b_{ji}| l_i \right] |x_i| \\
&\quad - \lambda \sum_{i=1}^{n} r_i \left[|I_i| + \sum_{j=1}^{n} |a_{ij}||f_j(0)| + \sum_{j=1}^{n} |b_{ij}||g_j(0)| \right] \\
&\geq \lambda r_0 \|x\|_1 - \lambda n I_0.
\end{aligned}$$

Define

$$\begin{aligned}
r_0 &= \min_{1\leq i\leq n} \left\{ r_i d_i - \sum_{j=1}^{n} r_j |a_{ji}| k_i - \sum_{j=1}^{n} r_j |b_{ji}| l_i \right\} \\
I_0 &= \max_{1\leq i\leq n} \left\{ r_i \left(|I_i| + \sum_{j=1}^{n} |a_{ij}||f_j(0)| + \sum_{j=1}^{n} |b_{ij}||g_j(0)| \right) \right\}
\end{aligned}$$

and let

$$U(R_0) = \left\{ x : \|x\|_1 < R_0 = \frac{n(I_0+1)}{r_0} \right\} \quad (8)$$

then, it follows from (8) that $\|x\|_1 = R_0 = n(I_0+1)/r_0$ for any $x \in \partial U(R_0)$, so, we obtain

$$\sum_{i=1}^{n} r_i |F_i(x,\lambda)| \geq \lambda r_0 \frac{n(I_0+1)}{r_0} - \lambda n I_0 > 0 \qquad \forall \lambda \in (0,1]$$

that is $F(x,\lambda) \neq 0$, for any $x \in \partial U(R_0)$, $\lambda \in (0,1]$. Also, as $\lambda = 0$, $F(x,\lambda) = i_d(x) = x \neq 0$, for any $x \in \partial U(R_0)$, Here, i_d is identity mapping. Hence, we have $F(x,\lambda) \neq 0$, for any $x \in \partial U(R_0)$, $\lambda \in [0,1]$.

From A_1, it is easy to prove $\deg(i_d, U(R_0), 0) = 1$ thus we have from homotopy invariance theorem [14] that

$$\deg(\tilde{f}, U(R_0), 0) = \deg(i_d, U(R_0), 0) = 1.$$

By the topological degree theory, we can conclude that (5) has at least one solution in $U(R_0)$. That is, model (1) has at least an equilibrium point.

Suppose $y^* = (y_1^*, y_2^*, \ldots, y_n^*)^T$ is also an equilibrium point of model (1), then, we have

$$\begin{aligned}
-d_i x_i^* + \sum_{j=1}^{n} a_{ij} f_j(x_j^*) + \sum_{j=1}^{n} b_{ij} g_j(x_j^*) + I_i &= 0 \\
-d_i y_i^* + \sum_{j=1}^{n} a_{ij} f_j(y_j^*) + \sum_{j=1}^{n} b_{ij} g_j(y_j^*) + I_i &= 0
\end{aligned}$$

this implies that

$$\begin{aligned}
d_i(x_i^* - y_i^*) = &\sum_{j=1}^{n} a_{ij}(f_j(x_j^*) - f_j(y_j^*)) \\
&+ \sum_{j=1}^{n} b_{ij}(g_j(x_j^*) - g_j(y_j^*)), \qquad i = 1, 2, \ldots, n
\end{aligned}$$

and using A_1, we have

$$\begin{aligned}
&d_i |x_i^* - y_i^*| \leq \sum_{j=1}^{n} k_j |a_{ij}||x_j^* - y_j^*| + \sum_{j=1}^{n} l_j |b_{ij}||x_j^* - y_j^*|, \\
&\quad i = 1, 2, \ldots, n. \qquad (9)
\end{aligned}$$

Rewrite (9) as

$$(D-|A|K-|B|L)(|x_1^*-y_1^*|, |x_2^*-y_2^*|, \ldots, |x_n^*-y_n^*|)^T \leq 0.$$

Since $D-|A|K-|B|L$ is a nonsingular M-matrix, so $(D-|A|K-|B|L)^{-1}$ is a nonnegative matrix. Thus multiplying both sides of the above inequality by $(D-|A|K-|B|L)^{-1}$ does not change the inequality direction, it comes to

$$(|x_1^*-y_1^*|, |x_2^*-y_2^*|, \ldots, |x_n^*-y_n^*|)^T \leq 0.$$

This means that $x^* = y^*$. Hence, the neural network model (1) has one unique equilibrium point. □

Theorem 2: Assume that the assumption A_1 is satisfied. The neural network model (1) has one unique equilibrium point if $G + G^T$ is a nonsingular M-matrix, where $G \triangleq D - |A|K - |B|L$.

Proof: Similar to Theorem 1, we still consider the homotopic mapping

$$F(x,\lambda) = \lambda \tilde{f}(x) + (1-\lambda)x$$

where $\lambda \in [0,1]$, $F(x,\lambda) = (F_1(x,\lambda), F_2(x,\lambda), \ldots, F_n(x,\lambda))^T$..

Since $(G+G^T)/2$ is a nonsingular M-matrix, hence there exist constants $r_i > 0$ such that

$$\begin{aligned}
d_i r_i - &\frac{1}{2}\sum_{j=1}^{n}(r_i|a_{ij}|k_j + r_j|a_{ji}|k_i) - \frac{1}{2}\sum_{j=1}^{n}(r_i|b_{ij}|l_j + r_j|b_{ji}|l_i) \\
&> 0, \qquad i = 1, 2, \ldots, n.
\end{aligned}$$

Using similar estimations to Theorem 1, it follows from condition A_1 that for $1 \leq i \leq n$

$$\begin{aligned}
&\sum_{i=1}^{n} r_i|x_i||F_i(x,\lambda)| \\
\geq&\lambda \sum_{i=1}^{n}|x_i|\Bigg[r_i d_i|x_i| \\
&-r_i\sum_{j=1}^{n}k_j|a_{ij}||x_j| - r_i\sum_{j=1}^{n}l_j|b_{ij}||x_j|\Bigg] \\
&-\lambda\sum_{i=1}^{n}r_i\left[|I_i|+\sum_{j=1}^{n}|a_{ij}||f_j(0)|+\sum_{j=1}^{n}|b_{ij}||g_j(0)|\right]|x_i| \\
=&\lambda \\
&\cdot\sum_{i=1}^{n}\left[r_i d_i|x_i|^2 - r_i\sum_{j=1}^{n}k_j|a_{ij}||x_i||x_j| - r_i\sum_{j=1}^{n}l_j|b_{ij}||x_i||x_j|\right] \\
&-\lambda\sum_{i=1}^{n}r_i\left[|I_i|+\sum_{j=1}^{n}|a_{ij}||f_j(0)|+\sum_{j=1}^{n}|b_{ij}||g_j(0)|\right]|x_i| \\
\geq&\lambda\sum_{i=1}^{n}\Bigg[r_i d_i|x_i|^2 - r_i\sum_{j=1}^{n}k_j|a_{ij}|\frac{1}{2}(|x_i|^2+|x_j|^2) \\
&-r_i\sum_{j=1}^{n}l_j|b_{ij}|\frac{1}{2}(|x_i|^2+|x_j|^2)\Bigg] \\
&-\lambda\sum_{i=1}^{n}r_i\left[|I_i|+\sum_{j=1}^{n}|a_{ij}||f_j(0)|+\sum_{j=1}^{n}|b_{ij}||g_j(0)|\right]|x_i| \\
=&\lambda\sum_{i=1}^{n}\Bigg[d_i r_i - \frac{1}{2}\sum_{j=1}^{n}(r_i|a_{ij}|k_j + r_j|a_{ji}|k_i) \\
&-\frac{1}{2}\sum_{j=1}^{n}(r_i|b_{ij}|l_j + r_j|b_{ji}|l_i)\Bigg]|x_i|^2 \\
&-\lambda\sum_{i=1}^{n}r_i\left[|I_i|+\sum_{j=1}^{n}|a_{ij}||f_j(0)|+\sum_{j=1}^{n}|b_{ij}||g_j(0)|\right]|x_i| \\
\geq&\lambda r_0^*\sum_{i=1}^{n}|x_i|^2 - \lambda I_0^*\sum_{i=1}^{n}|x_i| \\
\geq&\lambda r_0^*\sum_{i=1}^{n}|x_i|^2 - \lambda I_0^*\sqrt{n}\sqrt{\sum_{i=1}^{n}|x_i|^2} \\
=&\lambda r_0^*\|x\|_2^2 - \lambda I_0^*\sqrt{n}\|x\|_2.
\end{aligned}$$

Define

$$\begin{aligned}
r_0^* = \min_{1\leq i\leq n}\Bigg\{&d_i r_i - \frac{1}{2}\sum_{j=1}^{n}(r_i|a_{ij}|k_j + r_j|a_{ji}|k_i) \\
&-\frac{1}{2}\sum_{j=1}^{n}(r_i|b_{ij}|l_j + r_j|b_{ji}|l_i)\Bigg\}
\end{aligned}$$

$$I_0^* = \max_{1\leq i\leq n}\left\{r_i\left(|I_i|+\sum_{j=1}^{n}|a_{ij}||f_j(0)|+\sum_{j=1}^{n}|b_{ij}||g_j(0)|\right)\right\}$$

and let

$$U(R_0^*) = \left\{x : \|x\|_2 < R_0^* = \frac{\sqrt{n}(I_0^*+1)}{r_0^*}\right\}$$

then, it follows that for any $x \in \partial U(R_0^*)$, we have $\|x\|_2 = R_0^*$, hence

$$\begin{aligned}
\sum_{i=1}^{n} r_i|x_i||F_i(x,\lambda)| \geq&\lambda r_0^*[R_0^*]^2 - \lambda\sqrt{n}I_0^*R_0^* \\
=&\lambda r_0^*\frac{n(I_0^*+1)^2}{{r_0^*}^2} - \lambda\sqrt{n}I_0^*\frac{\sqrt{n}(I_0^*+1)}{r_0^*} \\
>&0 \qquad \forall\lambda\in(0,1]
\end{aligned}$$

this means $F(x,\lambda) \neq 0$, for any $x \in \partial U(R_0^*)$, $\lambda \in (0,1]$. Also, as $\lambda = 0$, $F(x,\lambda) = i_d(x) \neq 0$, for any $x \in \partial U(R_0^*)$. Hence, we have $F(x,\lambda) \neq 0$, for any $x \in \partial U(R_0^*)$, $\lambda \in [0,1]$.

Applying the same method as Theorem 1, we can also conclude that (5) has at least a solution in $U(R_0^*)$. This means that model (1) has at least an equilibrium point.

To prove the uniqueness of equilibrium point, let $y^* = (y_1^*, y_2^*, \ldots, y_n^*)^T$ be also an equilibrium point of model (1), similar to theorem 1, we also can easily derive that inequality (9) holds. Multiplying (9) by $r_i|x_i^* - y_i^*|$, we have

$$\begin{aligned}
r_i d_i|x_i^*-y_i^*|^2 \leq&\sum_{j=1}^{n}k_j r_i|a_{ij}||x_i^*-y_i^*||x_j^*-y_j^*| \\
&+\sum_{j=1}^{n}l_j r_i|b_{ij}||x_i^*-y_i^*||x_j^*-y_j^*| \\
\leq&\sum_{j=1}^{n}k_j r_i|a_{ij}|\frac{1}{2}(|x_i^*-y_i^*|^2+|x_j^*-y_j^*|^2) \\
&+\sum_{j=1}^{n}l_j r_i|b_{ij}|\frac{1}{2}(|x_i^*-y_i^*|^2+|x_j^*-y_j^*|^2).
\end{aligned}$$

It follows from the formula above that

$$\begin{aligned}
\sum_{i=1}^{n}\Bigg[&d_i r_i - \frac{1}{2}\sum_{j=1}^{n}(r_i|a_{ij}|k_j + r_j|a_{ji}|k_i) \\
&-\frac{1}{2}\sum_{j=1}^{n}(r_i|b_{ij}|l_j + r_j|b_{ji}|l_i)\Bigg]|x_i^*-y_i^*|^2 \leq 0.
\end{aligned}$$

This implies that $x^* = y^*$. Hence, the neural network model (1) has one unique equilibrium point. □

IV. Global Asymptotic Stability

In this section, we will give several sufficient conditions on the global asymptotic stability of equilibrium point for the recurrent neural network (1) with time-varying delays and Lipschitz continuous activation functions. We have the following main results.

Theorem 3: Assume that Assumption A_1 holds. The equilibrium point of the recurrent neural network model (1) is globally asymptotically stable if $D - |A|K - |B|L$ is a nonsingular M-matrix and $d\tau_i(t)/dt \leq 0$ $(i = 1, 2, \ldots, n)$.

Proof: First, make a transformation for model (1): $z_i(t) = x_i(t) - x_i^*$, $i = 1, 2, \ldots, n$, we have

$$\frac{dz_i(t)}{dt} = -d_i z_i(t) + \sum_{j=1}^{n} a_{ij}[f_j(z_j(t) + x_j^*) - f_j(x_j^*)] + \sum_{j=1}^{n} b_{ij}[g_j(z_j(t-\tau_j(t)) + x_j^*) - g_j(x_j^*)], \qquad i = 1, 2, \ldots, n. \tag{10}$$

We know from Theorem 1 that $z = 0$ is a uniqueness equilibrium point of model (10).

Since $D - |A|K - |B|L$ is a nonsingular M-matrix, hence, there exist constants $r_i > 0$ such that

$$r_i d_i - \sum_{j=1}^{n} r_j k_i |a_{ji}| - \sum_{j=1}^{n} r_j l_i |b_{ji}| > 0, \qquad i = 1, 2, \ldots, n$$

that is

$$r_j d_j - \sum_{i=1}^{n} r_i k_j |a_{ij}| - \sum_{i=1}^{n} r_i l_j |b_{ij}| > 0, \qquad j = 1, 2, \ldots, n.$$

Consider the Lyapunov functional

$$V(t, z) = \sum_{i=1}^{n} r_i \left(|z_i(t)| + \sum_{j=1}^{n} |b_{ij}| l_j \int_{t-\tau_j(t)}^{t} |z_j(s)| ds\right). \tag{11}$$

that is

$$V(t, \phi) = \sum_{i=1}^{n} r_i \left(|\phi_i(0)| + \sum_{j=1}^{n} |b_{ij}| l_j \int_{-\tau_j(t)}^{0} |\phi_j(\theta)| d\theta \right).$$

Calculating the upper right derivative D^+V of V along the solutions of (10), we have

$$\begin{aligned}
&D^+V \\
&= \sum_{i=1}^{n} r_i \Bigg[\mathrm{sgn}(z_i(t)) \frac{dz_i(t)}{dt} \\
&\quad + \sum_{j=1}^{n} |b_{ij}| l_j \left(|z_j(t)| - |z_j(t - \tau_j(t))| \left(1 - \frac{d\tau_j(t)}{dt}\right)\right) \Bigg] \\
&\le \sum_{i=1}^{n} r_i \left[-d_i |z_i(t)| + \sum_{j=1}^{n} |a_{ij}| k_j |z_j(t)| + \sum_{j=1}^{n} |b_{ij}| l_j |z_j(t)| \right] \\
&= \sum_{j=1}^{n} \left(-r_j d_j + \sum_{i=1}^{n} r_i |a_{ij}| k_j + \sum_{i=1}^{n} r_i |b_{ij}| l_j \right) |z_j(t)| \\
&\le -e_0 \sum_{j=1}^{n} |z_j(t)|
\end{aligned}$$

where

$$e_0 = \min_{1 \le j \le n} \left(r_j d_j - \sum_{i=1}^{n} r_i |a_{ij}| k_j - \sum_{i=1}^{n} r_i |b_{ij}| l_j \right) > 0.$$

Define

$$Z(t, z_t) = \sum_{i=1}^{n} r_i \sum_{j=1}^{n} |b_{ij}| l_j \int_{t-\tau_j(t)}^{t} |z_j(\xi)| d\xi$$

one can see that $Z(t, z_t)$ is the second term of V. If $z_i(t)$ is bounded and $d\tau_j(t)/dt \le 0$, so for $0 \le s < t$ we have

$$Z(t, z_t) - Z(s, z_s) \le \sum_{i=1}^{n} r_i \sum_{j=1}^{n} |b_{ij}| l_j \int_{s}^{t} |z_j(\xi)| d\xi \le c_0(t - s).$$

Take

$$\begin{aligned}
u(\|\phi(0)\|) &= r_{\min} \|\phi(0)\| = r_{\min} \sum_{i=1}^{n} |z_i(t)| \\
v(\|\phi(0)\|) &= r_{\max} \|\phi(0)\| = r_{\max} \sum_{i=1}^{n} |z_i(t)| \\
w(\|\phi(0)\|) &= e_0 \|\phi(0)\| = e_0 \sum_{i=1}^{n} |z_i(t)|
\end{aligned}$$

where $r_{\min} = \min_{1 \le i \le n}\{r_i\}$, $r_{\max} = \max_{i \le i \le n}\{r_i\}$. Clearly, all conditions of Lemma are satisfied, so $z = 0$ is globally asymptotically stable. This implies that the equilibrium point x^* of the recurrent neural network (1) is globally asymptotically stable. $\square$

One can easily prove the following two corollaries.

Corollary 1: Assume that the assumption A_1 is satisfied and $\tau_j(t) = \tau$ is a positive constant. The equilibrium point of the neural network model (1) is globally asymptotically stable if $D - |A|K - |B|L$ is a nonsingular M-matrix.

Corollary 2: Assume that the assumption A_1 is satisfied and $\tau_j(t) = \tau$ is a positive constant. The equilibrium point of the neural network model (1) is globally asymptotically stable if $D - |A|K - |B|L$ is quasi-dominant.

Their proofs are straightforward and here omitted. Specially, as $\tau_j(t) \equiv 0$, one can see that the neural network model (1) becomes the recurrent neural networks without any delays.

We may further give another assumption:

A_2 : $\tau_j(t)$ $(j = 1, 2, \ldots, n)$ are differential functions with $\tau_j(0) = 0$ and $0 \le d\tau_j(t)/dt \le \mu < 1$ $(j = 1, 2, \ldots, n)$.

Clearly, the assumption A_2 can ensure the $t - \tau_j(t)$ $(j = 1, 2, \ldots, n)$ have a differential inverse function, denoted by $\sigma_j(t)$ $(j = 1, 2, \ldots, n)$. In addition, the assumption of $0 \le d\tau_j(t)/dt \le \mu < 1$ $(j = 1, 2, \ldots, n)$ stems from the need to bound the growth of variations in the delay factor as a function of time.

Theorem 4: Assume that Assumptions A_1 and A_2 hold. The equilibrium point of the the neural network model (1) is globally asymptotically stable if there exist $r_i > 0 (i = 1, 2, \ldots, n)$ such that

$$\min_{1 \le j \le n} \left(r_j d_j - \sum_{i=1}^{n} r_i |a_{ij}| k_j - \sum_{i=1}^{n} r_i \frac{1}{1-\mu} |b_{ij}| l_j \right) > 0.$$

Proof: Since there exist $r_i > 0 \quad (i = 1, 2, \ldots, n)$ such that

$$r_j d_j - \sum_{i=1}^{n} r_i |a_{ij}| k_j - \sum_{i=1}^{n} r_i \frac{1}{1-\mu} |b_{ij}| l_j > 0, \quad j = 1, 2, \ldots, n.$$

Clearly, we follow from $0 \le \mu < 1$ that the following inequality holds:

$$\begin{aligned} r_j d_j - &\sum_{i=1}^n r_i |a_{ij}| k_j - \sum_{i=1}^n r_i |b_{ij}| l_j \\ \ge r_j d_j - &\sum_{i=1}^n r_i |a_{ij}| k_j - \sum_{i=1}^n r_i \frac{1}{1-\mu} |b_{ij}| l_j > 0, \\ &j = 1, 2, \ldots, n. \end{aligned}$$

This implies that $D - |A|K - |B|L$ is a nonsingular M-matrix, then we know from Theorem 1 that $z = 0$ is a uniqueness equilibrium point of model (10).

Similar to the proof of Theorem 3, consider another Lyapunov functional

$$V_1(t,z) = \sum_{i=1}^n r_i \left(|z_i(t)| + \sum_{j=1}^n |b_{ij}| l_j \int_t^{\sigma_j(t)} |z_j(s - \tau_j(s))| ds \right). \tag{12}$$

which we can also rewrite as

$$\begin{aligned} V_1(t,\phi) = &\sum_{i=1}^n r_i \Bigg(|\phi_i(0)| \\ &+ \sum_{j=1}^n |b_{ij}| l_j \int_{-\tau_j(t)}^0 \sigma_j'(t+\theta) |\phi_j(\theta)| d\theta \Bigg). \end{aligned}$$

Calculating the upper right derivative D^+V_1 of V_1 along the solutions of (10), we have

$$\begin{aligned} &D^+V_1 \\ &= \sum_{i=1}^n r_i \Bigg[\operatorname{sgn}(z_i(t)) \frac{dz_i(t)}{dt} \\ &\quad + \sum_{j=1}^n |b_{ij}| l_j (|z_j(t)| \sigma_j'(t) - |z_j(t - \tau_j(t))|) \Bigg] \\ &= \sum_{i=1}^n r_i \Bigg[\operatorname{sgn}(z_i(t)) \frac{dz_i(t)}{dt} \\ &\quad + \sum_{j=1}^n |b_{ij}| l_j \left(|z_j(t)| \frac{1}{1 - \frac{d\tau_j(\zeta)}{d\zeta}} - |z_j(t - \tau_j(t))| \right) \Bigg] \\ &\le \sum_{i=1}^n r_i \Bigg[-d_i |z_i(t)| + \sum_{j=1}^n |a_{ij}| k_j |z_j(t)| \\ &\quad + \sum_{j=1}^n |b_{ij}| l_j \frac{1}{1 - \frac{d\tau_j(\zeta)}{d\zeta}} |z_j(t)| \Bigg] \\ &\le \sum_{j=1}^n \left(-r_j d_j + \sum_{i=1}^n r_i |a_{ij}| k_j + \sum_{i=1}^n r_i \frac{1}{1-\mu} |b_{ij}| l_j \right) |z_j(t)| \\ &\le -e_1 \sum_{j=1}^n |z_j(t)| \end{aligned}$$

where $\zeta = \sigma_j(t)$, that is, ζ and t have the relation $t = \zeta - \tau_j(\zeta)$, and

$$e_1 = \min_{1 \le j \le n} (r_j d_j - \sum_{i=1}^n r_i |a_{ij}| k_j - \sum_{i=1}^n r_i \frac{1}{1-\mu} |b_{ij}| l_j) > 0.$$

Let

$$Z_1(t, z_t) = \sum_{i=1}^n r_i \sum_{j=1}^n |b_{ij}| l_j \int_t^{\sigma_j(t)} |z_j(\xi - \tau_j(\xi))| d\xi$$

it is easily to seen that $Z_1(t, z_t)$ is also the second term of V_1. If $z_i(t)$ is bounded and $d\tau_j(t)/dt \le \mu < 1$, so for $0 \le s < t$ we have

$$\begin{aligned} &Z_1(t, z_t) - Z_1(s, z_s) \\ &\le \sum_{i=1}^n r_i \sum_{j=1}^n |b_{ij}| l_j \int_s^t |z_j(\xi - \tau_j(\xi))| d\xi \le c_1(t-s). \end{aligned}$$

Let

$$\begin{aligned} u_1(\|\phi(0)\|) &= r_{\min} \|\phi(0)\| = r_{\min} \sum_{i=1}^n |z_i(t)| \\ v_1(\|\phi(0)\|) &= r_{\max} \|\phi(0)\| = r_{\max} \sum_{i=1}^n |z_i(t)| \\ w_1(\|\phi(0)\|) &= e_1 \|\phi(0)\| = e_1 \sum_{i=1}^n |z_i(t)| \end{aligned}$$

where $r_{\min} = \min_{1 \le i \le n}\{r_i\}$, $r_{\max} = \max_{i \le i \le n}\{r_i\}$. Thus, all conditions of Lemma are satisfied, so $z = 0$ is globally asymptotically stable. This means that the equilibrium point x^* of the recurrent neural network (1) is globally asymptotically stable. □

Theorem 5: Assume that the assumptions A_1 and A_2 hold. The equilibrium point of the neural network model (1) is globally asymptotically stable if there exist $r_i > 0 (i = 1, 2, \ldots, n)$ such that

$$\begin{aligned} \min_{1 \le i \le n} \Bigg[&d_i r_i - \frac{1}{2} \sum_{j=1}^n (r_i |a_{ij}| k_j + r_j |a_{ji}| k_i) \\ &- \frac{1}{2} \sum_{j=1}^n \left(r_i |b_{ij}| l_j + \frac{1}{1-\mu} r_j |b_{ji}| l_i \right) \Bigg] > 0. \end{aligned}$$

Proof: Since there exist $r_i > 0$ such that for $i = 1, 2, \ldots, n$

$$\begin{aligned} &d_i r_i - \frac{1}{2} \sum_{j=1}^n (r_i |a_{ij}| k_j + r_j |a_{ji}| k_i) \\ &\quad - \frac{1}{2} \sum_{j=1}^n \left(r_i |b_{ij}| l_j + \frac{1}{1-\mu} r_j |b_{ji}| l_i \right) > 0. \end{aligned}$$

Clearly, we follow from $0 \le \mu < 1$ that the following inequality holds:

$$\begin{aligned} &d_i r_i - \frac{1}{2} \sum_{j=1}^n (r_i |a_{ij}| k_j + r_j |a_{ji}| k_i) \\ &- \frac{1}{2} \sum_{j=1}^n (r_i |b_{ij}| l_j + r_j |b_{ji}| l_i) \\ &\ge d_i r_i - \frac{1}{2} \sum_{j=1}^n (r_i |a_{ij}| k_j + r_j |a_{ji}| k_i) \\ &- \frac{1}{2} \sum_{j=1}^n (r_i |b_{ij}| l_j + \frac{1}{1-\mu} r_j |b_{ji}| l_i) > 0, \\ &i = 1, 2, \ldots, n. \end{aligned}$$

This implies that $(G+G^T)/2$ is a nonsingular M-matrix. Thus we know from Theorem 2 that $z=0$ is a uniqueness equilibrium point of system (10).

Similar to the proof of Theorem 3 or Theorem 4, consider the following Lyapunov functional:

$$V_2(t,z)=\frac{1}{2}\sum_{i=1}^{n}r_i\Big(z_i^2(t)+\sum_{j=1}^{n}|b_{ij}|l_j\int_t^{\sigma_j(t)}z_j^2(s-\tau_j(s))ds\Big). \quad (13)$$

which we can rewrite as

$$V_2(t,\phi)=\frac{1}{2}\sum_{i=1}^{n}r_i(\phi_i^2(0)+\sum_{j=1}^{n}|b_{ij}|l_j\int_{-\tau_j(t)}^{0}\sigma_j'(t+\theta)\phi_j^2(\theta)d\theta).$$

Calculating the derivative of V_2 along the solutions of (10), we have

$$\begin{aligned}\frac{dV_2}{dt}=&\sum_{i=1}^{n}r_i\Bigg[z_i(t)\frac{dz_i(t)}{dt}+\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|l_j(z_j^2(t)\sigma_j'(t)-z_j^2(t-\tau_j(t)))\Bigg]\\
\le&\sum_{i=1}^{n}r_i\Bigg[-d_iz_i^2(t)+\sum_{j=1}^{n}|a_{ij}|k_j|z_i(t)||z_j(t)|+\sum_{j=1}^{n}|b_{ij}|l_j|z_i(t)||z_j(t-\tau_j(t))|\\
&+\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|l_j\left(z_j^2(t)\frac{1}{1-\frac{d\tau_j(\zeta)}{d\zeta}}-z_j^2(t-\tau_j(t))\right)\Bigg]\\
\le&\sum_{i=1}^{n}r_i\Bigg[-d_iz_i^2(t)+\frac{1}{2}\sum_{j=1}^{n}|a_{ij}|k_j(z_i^2(t)+z_j^2(t))+\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|l_j\left(z_i^2(t)+z_j^2(t-\tau_j(t))\right)\\
&+\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|l_j\left(z_j^2(t)\frac{1}{1-\mu}-z_j^2(t-\tau_j(t))\right)\Bigg]\\
=&\sum_{i=1}^{n}r_i\Bigg[-d_iz_i^2(t)+\frac{1}{2}\sum_{j=1}^{n}|a_{ij}|k_j(z_i^2(t)+z_j^2(t))+\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|l_jz_i^2(t)+\frac{1}{2}\sum_{j=1}^{n}|b_{ij}|l_j|z_j(t)|\frac{1}{1-\mu}\Bigg]\\
=&\sum_{i=1}^{n}\Bigg[-d_ir_i+\frac{1}{2}\sum_{j=1}^{n}(r_i|a_{ij}|k_j+r_j|a_{ji}|k_i)+\frac{1}{2}\sum_{j=1}^{n}\left(r_i|b_{ij}|l_j+\frac{1}{1-\mu}r_j|b_{ji}|l_i\right)\Bigg]z_i^2(t)\\
\le&-e_2\sum_{i=1}^{n}z_i^2(t)\end{aligned}$$

where $\zeta=\sigma_j(t)$, that is, ζ and t have the relation $t=\zeta-\tau_j(\zeta)$, and

$$e_2=\min_{1\le i\le n}\left[d_ir_i-\frac{1}{2}\sum_{j=1}^{n}(r_i|a_{ij}|k_j+r_j|a_{ji}|k_i)-\frac{1}{2}\sum_{j=1}^{n}\left(r_i|b_{ij}|l_j+\frac{1}{1-\mu}r_j|b_{ji}|l_i\right)\right]>0.$$

Let

$$Z_2(t,z_t)=\sum_{i=1}^{n}r_i\sum_{j=1}^{n}|b_{ij}|l_j\int_t^{\sigma_j(t)}z_j^2(\xi-\tau_j(\xi))d\xi$$

clearly, $Z_2(t,z_t)$ is the second term of V_2. If $z_i(t)$ is bounded and $d\tau_j(t)/dt\le\mu<1$, so for $0\le s<t$ we have

$$\begin{aligned}&Z_2(t,z_t)-Z_2(s,z_s)\\&\le\sum_{i=1}^{n}r_i\sum_{j=1}^{n}|b_{ij}|l_j\int_s^t z_j^2(\xi-\tau_j(\xi))d\xi\le c_2(t-s).\end{aligned}$$

Take

$$\begin{aligned}u_2(\|\phi(0)\|)=&\frac{1}{2}r_{\min}\|\phi(0)\|^2=\frac{1}{2}r_{\min}\sum_{i=1}^{n}z_i^2(t)\\
v_2(\|\phi(0)\|)=&\frac{1}{2}r_{\max}\|\phi(0)\|^2=\frac{1}{2}r_{\max}\sum_{i=1}^{n}|z_i^2(t)|\\
w_2(\|\phi(0)\|)=&e_2\|\phi(0)\|^2=e_2\sum_{i=1}^{n}|z_i^2(t)|\end{aligned}$$

where $r_{\min}=\min_{1\le i\le n}\{r_i\}$, $r_{\max}=\max_{i\le i\le n}\{r_i\}$. Thus, all conditions of Lemma are satisfied, so $z=0$ is globally asymptotically stable. This means that the equilibrium point x^* of the neural network model (1) is globally asymptotically stable. □

We can easily derive the following corollary.

Corollary 3: Assume that the assumption A_1 is satisfied and $\tau_j(t)=\tau$ is a positive constant. The equilibrium point of the neural network model (1) is globally asymptotically stable if $G+G^T$ is a nonsingular M-matrix, where $G\triangleq D-|A|K-|B|L$.

The proof is straightforward and omitted here.

V. Comparisons and Examples

If we let $B=(b_{ij})_{n\times n}=0$, then the neural network model (1) becomes a neural network in the form

$$\frac{dx(t)}{dt}=-Dx(t)+Af(x(t))+I \quad (14)$$

which is studied in many references, e.g., [32]–[39] and their references therein.

Many stability results of neural network model (14) above have been obtained in the literature. In [32], a *necessary and sufficient* condition for absolute stability (ABST) is given for the neural network model (14) under $-A \in \mathcal{P}_0$ with symmetric ($a_{ij} = a_{ji}, \forall i \neq j$) and cooperative ($a_{ij} \geq 0, \forall i \neq j$) connection weights. The ABST result of symmetric neural networks also obtained in [32].

An ABST result is also given in [33] under the condition $A \in \mathcal{D}_0$. This condition and $A \in \mathcal{C}_0$ cannot be included each other, as shown in [34], [35], [39].

In [36], the ABST result is presented for neural networks under the condition that $-A$ is quasidiagonally row-sum or column-sum dominant, which is obviously a special case of the condition $A \in \mathcal{C}_0$.

Another absolute exponential stability (AEST) result is presented in [37] under the condition $A \in \mathcal{C}_0$, which extends the existing ABST results in [36].

For the proposed condition $A \in \mathcal{M}_0$, in [38], the AEST result is obtained for neural networks under the condition $A \in \mathcal{M}_0$, which may be viewed as a generalization of the known ABST and AEST results in the sense that the ABST conditions of additive diagonal stability [35] is the milder one among the existing criteria.

For neural network model (14) above, applying Corollary 1 and Corollary 3 we have the following corollaries.

Corollary 4: Assume that f_i ($i = 1, 2, \ldots, n$) are Lipschitz continuous with Lipschitz constant k_i. The equilibrium point of the neural network model (14) is globally asymptotically stable if $D - |A|K$ is a nonsingular M-matrix, where $K = \mathrm{diag}(k_1, k_2, \ldots, k_n)$.

Corollary 5: Assume that f_i ($i = 1, 2, \ldots, n$) are Lipschitz continuous with Lipschitz constant k_i. The equilibrium point of the neural network model (14) is globally asymptotically stable if $2D - |A|K - K|A^T|$ is a nonsingular M-matrix, where $K = \mathrm{diag}(k_1, k_2, \ldots, k_n)$.

We can easily see that, for the obtained results, A may not be belong to $\mathcal{M}_0$, $\mathcal{C}_0$, $-A$ may not be also belong to $\mathcal{P}_0$, and the activation functions may not be bounded, monotone nondecreasing and differentiable, see following given examples. This implies that our results also are an extension of references in [32]–[39] and their references therein even if the considered neural network models are degenerated as a neural network without any delay. Of course, for a neural network with constant delays, there exist many significant results, e.g., [2]–[8], [15]–[28]. However, the most existing results of neural networks with constant delays in the literature were obtained within the class of bounded, monotone nondecreasing and differentiable activation functions. In practical applications, it is not uncommon that the activation functions in optimization neural networks and cellular neural networks are nondifferentiable or unbounded, as demonstrated in previous work.

The following two illustrative examples will demonstrate the effectiveness of the obtained results.

Example 1: Consider a recurrent neural network in the form

$$\frac{dx(t)}{dt} = -Dx(t) + Af(x(t)) + I \tag{15}$$

where

$$D = \begin{pmatrix} 7 & 0 \\ 0 & 11 \end{pmatrix} \quad A = \begin{pmatrix} 2 & 8 \\ 2 & -5 \end{pmatrix} \tag{16}$$

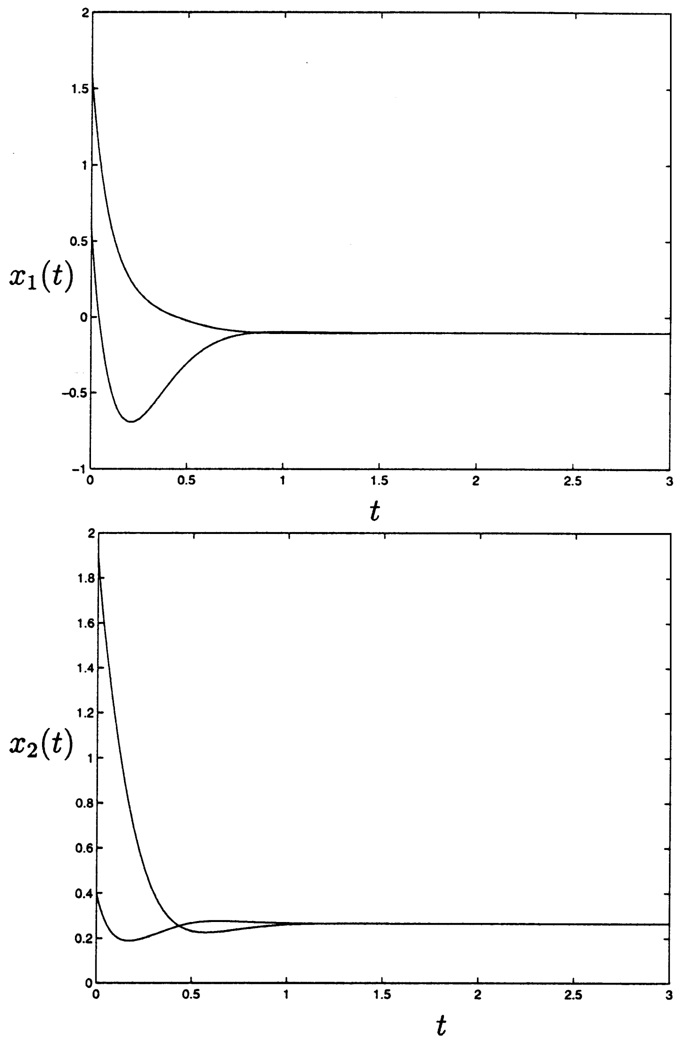

Fig. 1. Transient response of state variables in Example 1.

and $f_i(x) = -|x_i|$ ($i = 1,2$). Clearly, f_i is unbounded and Lipschitz continuous with the Lipschitz constant $k_i = 1$. We can check that

$$D - |A|K = D - |A| = \begin{pmatrix} 5 & -8 \\ -2 & 6 \end{pmatrix}$$

is a nonsingular M-matrix, where $K = \mathrm{diag}(k_1, k_2) = \mathrm{diag}(1, 1)$. Hence, the equilibrium point of neural networks (15) is globally asymptotically stable. However, at this time, $A \notin \mathcal{M}_0$, $A \notin \mathcal{C}_0$ and $-A \notin \mathcal{P}_0$, is not a H-matrix or quasi-diagonally dominant. It is worth nothing that the approaches in [32]–[39] and the references therein are not applicable to ascertain the stability of such a neural network.

For numerical simulation, let $I = (I_1, I_2)^T = (1.6, 1.8)^T$. The following two cases are given: case 1 with the initial state $x(0) = (0.6, 1.9)^T$; case 2 with the initial state $x(0) = (1.6, 0.4)^T$. Fig. 1 depicts the time responses of state variables of $x_1(t)$ and $x_2(t)$ for the above two cases. it confirms that the proposed condition leads to the uniqueness and global stability of the equilibrium point for the model (15).

Example 2: Consider a delayed recurrent neural network

$$\frac{dx(t)}{dt} = -Dx(t) + Af(x(t)) + Bg(x(t-\tau)) + I \tag{17}$$

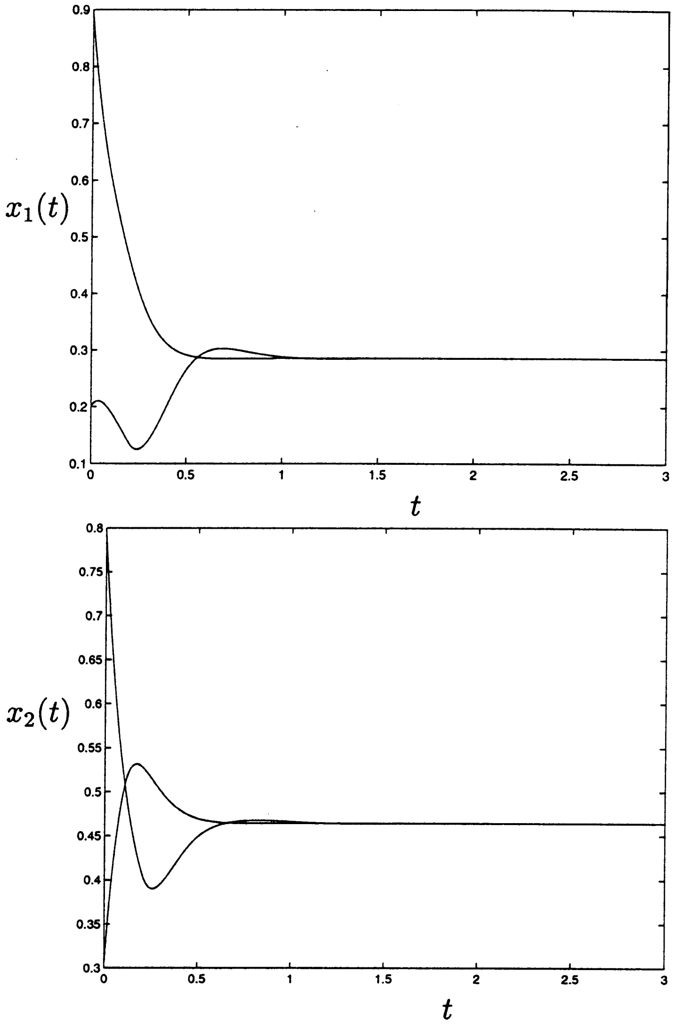

Fig. 2. Transient response of state variables in Example 2.

where

$$D = \begin{pmatrix} 7 & 0 \\ 0 & 11 \end{pmatrix}$$
$$A = \begin{pmatrix} -1 & -4 \\ -1 & -2 \end{pmatrix}$$
$$B = \begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix} \tag{18}$$

and $f_i(x) = g_i(x) = -|x_i|$ ($i = 1,2$). Clearly, f_i is unbounded and Lipschitz continuous with the Lipschitz constant $k_i = 1$, $l_i = 1$. One can check that

$$D - |A|K - |B|L = D - |A| - |B| = \begin{pmatrix} 5 & -8 \\ -2 & 6 \end{pmatrix}$$

is a nonsingular M-matrix, where $K = \mathrm{diag}(k_1, k_2) = \mathrm{diag}(1,1)$, $L = \mathrm{diag}(l_1, l_2) = \mathrm{diag}(1,1)$. Hence the equilibrium point of the neural network model (17) is globally asymptotically stable.

It is worth nothing that the approaches in [2]–[8], [15]–[28] and the references therein are not applicable to ascertain the stability of such a neural network due to the most existing results of neural networks with constant delays in the literature were obtained within the class of bounded and differentiable activation functions. Of course, our results in this paper also include the applications to recurrent neural networks with time-varying delays.

For numerical simulation, let $I = (I_1, I_2)^T = (2, 5)^T$. The following two cases are given: Case 1 with the delay parameter $\tau = 0.2$ and the initial state $x(t) = (0.2, 0.8)^T$ for $t \in [-0.2, 0]$; Case 2 with the delay parameter $\tau = 0.1$ and the initial state $x(t) = (0.9, 0.3)^T$ for $t \in [-0.1, 0]$. Fig. 2 depicts the time responses of state variables of $x_1(t)$ and $x_2(t)$ for the above two cases. It confirms that the proposed condition leads to the unique and globally asymptotically stable equilibrium point for the model (17). In addition, the result is also shown to be independent of the initial state.

VI. Conclusion

New criteria are derived for ascertaining uniqueness for the equilibrium point and its global asymptotic stability of neural networks with time-varying delays and Lipschitz continuous activation functions. These stability conditions do not require the activation functions to be differential, bounded, or monotone nondecreasing and also not need the connection weight matrices to be symmetric. In addition, these criteria can be easily checked in practice, which is important in the design and applications of recurrent neural networks. Also, our methods of stability analysis can be applied to some more complex systems.

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