TOPIC 3 DYNAMIC PROGRAMMING

TEXAS
The University of Texas at Austin

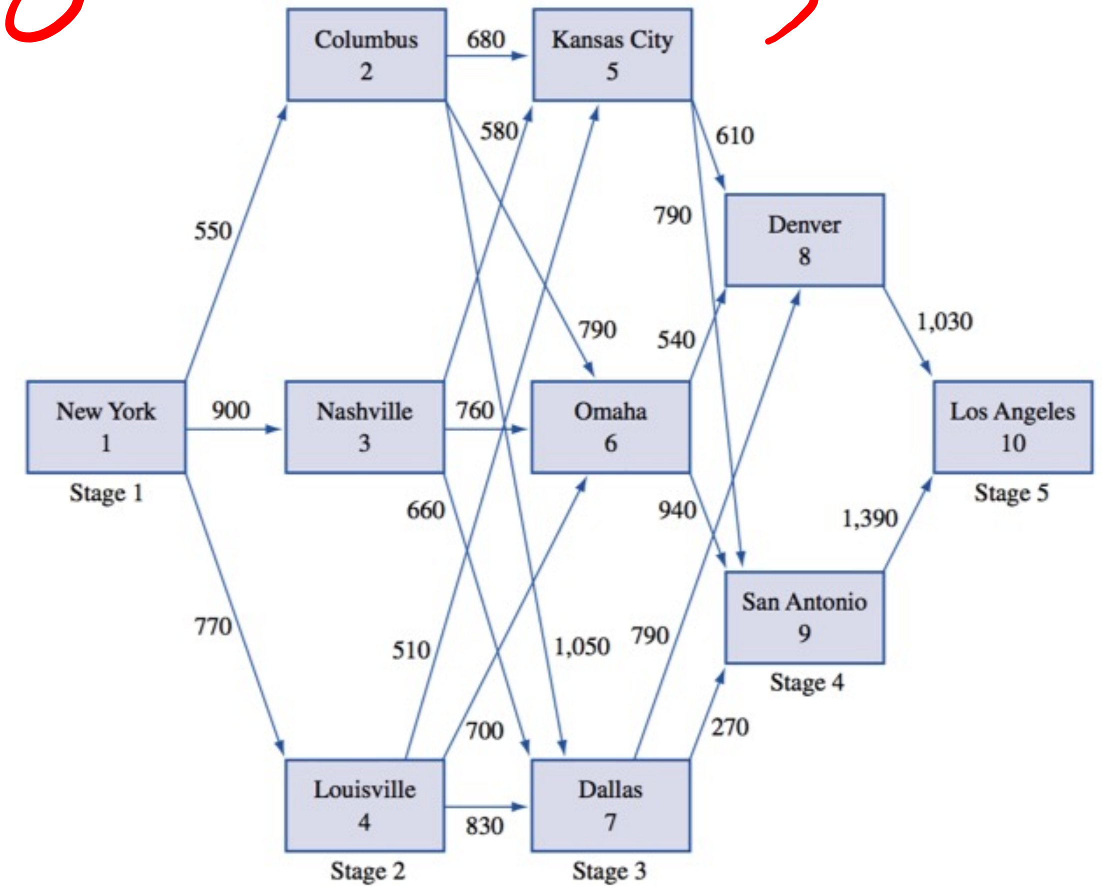
Shortest Path Problem

Joe Cougar lives in New York City, but he plans to drive to Los Angeles to seek fame and fortune. Joe's funds are limited, so he has decided to spend each night on his trip at a friend's house. Joe has friends in Columbus, Nashville, Louisville, Kansas City, Omaha, Dallas, San Antonio, and Denver. Joe knows that after one day's drive he can reach Columbus, Nashville, or Louisville. After two days of driving, he can reach Kansas City, Omaha, or Dallas. After three days of driving, he can reach San Antonio or Denver. Finally, after four days of driving, he can reach Los Angeles. To minimize the number of miles traveled, where should Joe spend each night of the trip? The actual road mileages between cities are given in Figure 1.

Shortest Path Problem

- Work on this for class participation

0 1 2 3 4



$$v(LA) = 0$$

$$v(SA) = 1390 + v(\cancel{LA})$$

$$v(Den) = 1030 + v(\cancel{LA})$$

$$v(KC) = \min\left(\underline{610 + v(Den)},\ 790 + v(SA)\right) = 1640$$

$$v(OM) = \min\left(\underline{540 + v(Den)},\ 940 + v(SA)\right) = 1570$$

$$v(Dal) = \min\left(790 + v(Den),\ \underline{270 + v(SA)}\right) = 1660$$

$$v(Col) = \min\left(680 + v(KC),\ 790 + v(OM),\ 1050 + v(DAL)\right)$$

Dynamic Programming with Randomness

- What if the dynamics include some randomness
  - The fish repopulation is random
  - The amount of ore you extract is random
  - The travel time between cities is random
- We must now try to maximize ***expected*** payoffs

Dynamic Programming with Randomness

- We still need the same 6 components of the DP
- The dynamics now incorporate randomness!
  - $S_t \rightarrow S_{t+1}(S_t, x_t, R_{t+1})$
  - R_{t+1} is something random
- The Bellman equation needs expectations
  - $v(S_t, t) = \max_x E[r_t + \delta\, v(S_{t+1}, t+1)]$
  - $v(s, t) = \max_x \{E[r_t] + \delta\, E[v(S_{t+1}, t+1)]\}$

$E[X+Y] = E[X] + E[Y]$

Expectations

- Recall stats 101
- Let all possible outcomes be x_i
- Let the probability of each of those outcomes be p_i

- $E[X] = \sum x_i p_i$

Airline Ticket Pricing

- You oversee pricing seats of a passenger jet that has 100 seats
- Two possible prices: Low price of $180, High price of $300
- There are 365 days left until the departure time
- Based on historical data, the distribution of daily demand for the low and high price are

	0	1
$180	0.3	0.7
$300	0.6	0.4

- When should you quote the high or low price?
- Discounting is 0.98 per day

Class Participation

- What are the state variables, choice variables and dynamics?

Airline Ticket Pricing

- What are the state variables, choice variables, dynamics, value function, Bellman equation and terminal condition?

State

(s, t)

seats left

time

Choice

$p = H / L$

dynamics

$$(s,t) \xrightarrow{p=H} \begin{cases} (s, t+1) & \text{wp } 0.6 \\ (s-1, t+1) & \text{wp } 0.4 \end{cases}$$

$$(s,t) \xrightarrow{p=L} \begin{cases} (s, t+1) & \text{wp } 0.3 \\ (s-1, t+1) & \text{wp } 0.7 \end{cases}$$

value function

$$V(s,t) = \max_{p} E\left[\sum_{i=0}^{T-t} (\text{rev} @ t+i)\, \delta^{i} \right]$$

Bellman

$$V(S,t) = \max_{p} E[\text{rev today}] + \delta E[V \text{ tomorrow}]$$

$$V(S,t) = \max\left[\begin{array}{l} 180 * 0.7 + 0 * 0.3 + \delta\left(0.7V(S-1,t+1) + 0.3V(S,t+1)\right), \\ 300 * 0.4 + \delta\left(0.4V(S-1,t+1) + 0.6V(S,t+1)\right) \end{array}\right]$$

Terminal / Boundary

$$V(S,T) = 0$$

$$V(0,t) = 0$$

$V[:,T] = 0 \qquad V[0,:] = 0$

for $t = T-1, T-2, \ldots, 0$:

for $s = 0, 1, 2, \ldots, 100$

$VL = 180 * 0.7 + 0.98(0.7\, V(s-1, t+1) + 0.3\, V(s, t+1))$

$VH = 300 * 0.4 + 0.98(0.4\, V(s-1, t+1) + 0.6\, V(s, t+1))$

$V[s,t] = \max(VL, VH)$

$U(s,t) = \mathrm{argmax}(VL, VH) + 1$