TOPIC 2 STOCHASTIC PROGRAMMING

TEXAS
The University of Texas at Austin

Stochastic Programming

- We have solved problems like
  - $max_x\, f(x)$
  - Where we know f exactly

- When we don’t know the outcome, we need to introduce a notion of randomness

Stochastic Programming

- Let Y be a random variable, maybe unaffected by your choice, x
- Y reveals itself after you pick x
- We call the payoff, that we want to make big, $f(x, Y)$
- If we don't know what Y will be then we can't max f!
- How do we resolve this issue?

Stochastic Programming

- Let’s maximize the expected value of the payoff!
- $\max_x E_Y[f(x, Y)]$
- We call this stochastic programming
- This is something we can at least attempt to do!
- It is VERY important to note that this is NOT the same as
  - $\max_x f(x, E[Y])$

Why not?

- Let Y be a standard normal
- Let f(y) = y^2
- E[Y] = 0 because it's a standard normal
  - $f(E[Y]) = f(0) = 0$
- $f(Y)$ is a chi-square random variable!
  - $E_Y[f(Y)] = 1$

- $E_Y[f(Y)] \neq f(E[Y])$!!!!!!!!!!!!!!!!!!

Stochastic Programming

- In some situations, solving the expectation is reasonable
  - Simple functions with simple random variables
- But remember an expectation is an integral
- If Y is a multi-dimensional random variable, then the expectation is a multi-dimensional integral
  - Demand of multiple products in a store
  - These can be very hard to solve!

Stochastic Programming

- $\max_x E_Y[f(x,Y)] = \max_x \int_y f(x,y)pdf(y)dy$
- This can be some very difficult calculus
- Even if we can solve this integral, how do we know the pdf of y?
  - Do we estimate it from data?
  - This isn't always a very good idea!

Stochastic Programming

- Let's examine the problem without a model
- Remember stats 101
  - How do we estimate an expectation?
  - A sample average!!!
- If we have data on Y, then we can reformulate the problem as
  - $\max_x \frac{1}{n} \sum_{i=1}^{n} f(x, Y_i)$
- In stochastic programming this is called the sample average approximation (SAA)
- This is now a completely deterministic problem!

Stochastic Programming

- Y is some random quantity
- We want to solve
  - $\max_x E_Y[f(x,Y)]$
- We will approximate this using data from Y
  - $\max_x \frac{1}{n}\sum_{i=1}^{n} f(x,Y_i)$
- We interpret this as though old data on Y approximates all the possible values Y could take in the future

Newsvendor Problem

- You are the printer of the Daily Texan
- You don't know how many people will want newspapers tomorrow
- You must decide how many to print tonight
- You have data on demand from the past 25 days, D_i
- It costs \$c to print a newspaper
- You can sell each one for \$p
- What quantity, q, should you print tonight to maximize tomorrow's expected profit?

Newsvendor Problem

- Let's look at a simplification where we know the demand will exactly be D_i
- What is the profit?

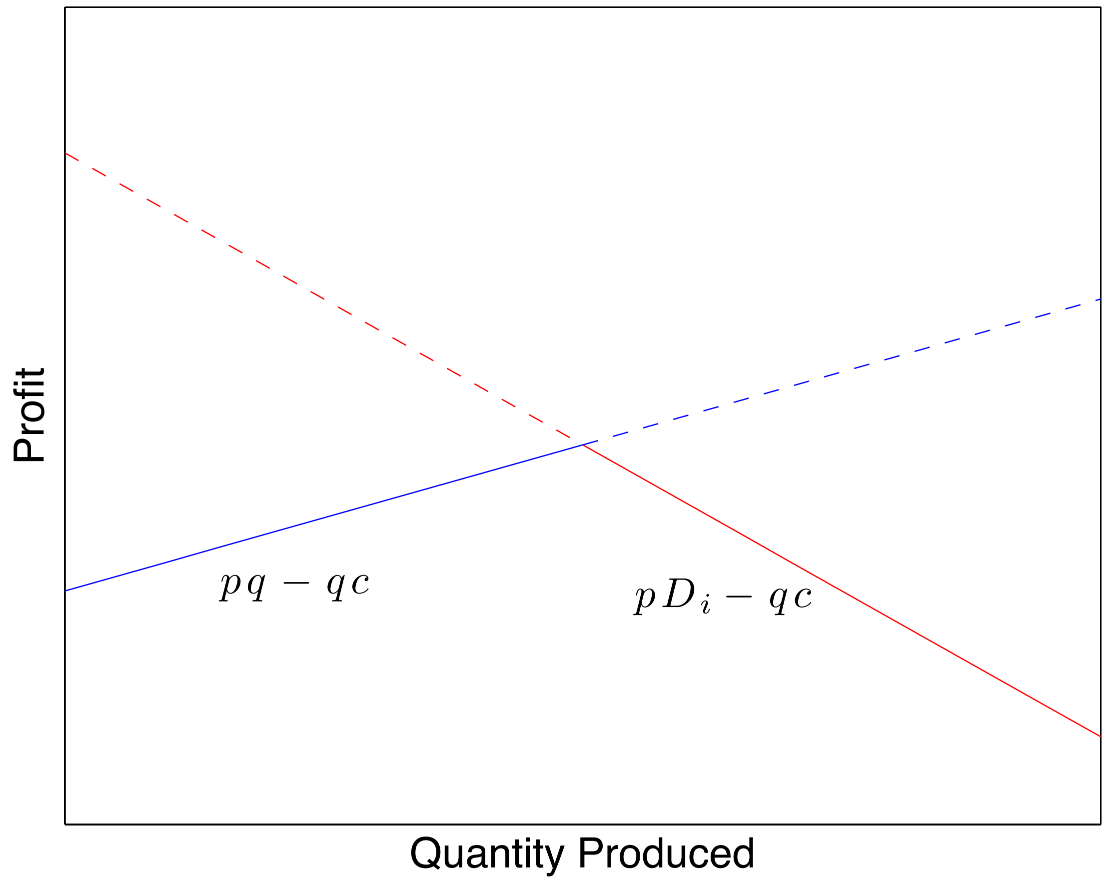
- $\max_q \{p \min(q, D_i) - qc\}$
  - This is an NLP; NLPs are hard to solve!

Newsvendor Problem

- Therefore, the full objective is
  - $\max_{q} \frac{1}{n} \sum_{i=1}^{n} (p \min(q, D_i) - qc)$
- This is an NLP
- It only has 1 decision variable
- But it has lots of corners in the objective…
- Also, we may want to extend this problem to multiple products

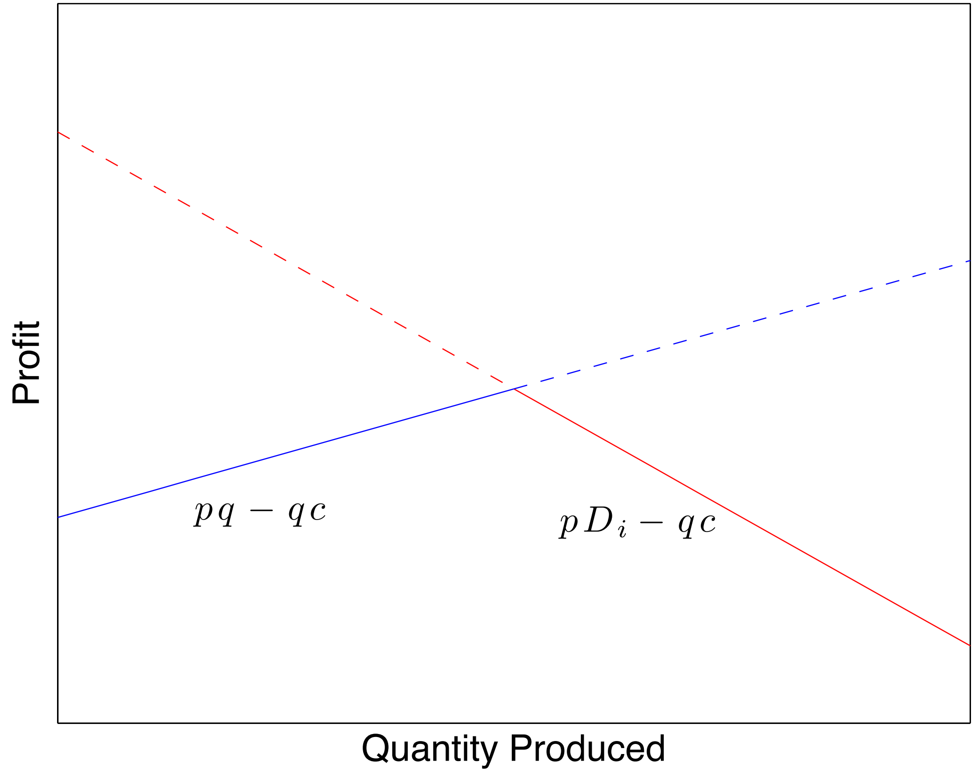
Newsvendor Problem

- Can we reformulate?
- The profit is
  - $pD_i - cq \;\; if\ q > D_i$
  - $pq - cq \;\; if\ q \leq D_i$
- The profit is always under the blue AND red lines!
- Note that sometimes the profit could be negative



Newsvendor Problem

- Define a new variable, h, as the profit for a given q
- We can therefore repose this as
  - $\max_{h,q} h$
  - s.t.
    - $h \leq pD_i - cq$
    - $h \leq pq - cq$
    - $h \geq -\infty$
    - $q \geq 0$
- BOOM! This is an LP!
- We can use the same trick on the big problem!



Class Participation

- Discuss how you think we'll continue this to include every day's demand data
- Hint: This is similar to the absolute value problem you did in project 2 last semester!

Newsvendor Problem

- $\max_{h_1,\ldots,h_n,q} \frac{1}{n}\sum_{i=1}^{n} h_i$
- s.t.
  - $h_i \leq p\, D_i - cq \quad \forall i$
  - $h_i \leq p\, q \quad - cq \quad \forall i$
  - $h_i \geq -\infty \qquad\qquad \forall i$
  - $q \geq 0$

- The complicated SP has become an LP!

Newsvendor Problem

- Let’s solve it!
- Suppose demand over the last 25 days was observed in the csv file on canvas
- It costs $0.50 to print a newspaper
- Each newspaper sells for $1.25
- How many should we print tomorrow?

Sample Average Approximation

- One very important fact to know is that SAA typically overestimates how well you can do!!!
- SAA is biased!
- $E\left[\max_x \frac{1}{n}\sum_{i=1}^{n} f(x, Y_i)\right] \geq \max_x E[f(x, Y)]$

Sample Average Approximation

- Why?
- $\max_x \frac{1}{n} \sum_{i=1}^{n} f(x, Y_i) \rightarrow x_n^*, f_n^*$
- $\max_x E[f(x, Y)] \rightarrow x^*, f^*$
- Plug in $E\left[\frac{1}{n} \sum_{i=1}^{n} f(x^*, Y_i)\right] = f^*$
- But x^* is not the optimal choice of x for the SAA
- $\frac{1}{n} \sum_{i=1}^{n} f(x_n^*, Y_i) \geq \frac{1}{n} \sum_{i=1}^{n} f(x^*, Y_i)$

Sample Average Approximation

- SAA is still a very powerful tool
- It is just important to remember its limits
- The optimal value function is biased
- But that doesn't mean the optimal x isn't useful!