Time Series Analytics Notes
Random Walks

A Random Walk (RW) is a special kind of time series model. The Random Walk is used as a model for stock prices and other financial time series and is very successful in that application. It has other uses as well. Many other time series data are "almost" random walks. The Random Walk Model can sometimes be used as a starting point in developing a model for other time series.

In this document, I define a Random Walk, I show how to tell whether a time series is a RW, I show how to forecast a RW, and I discuss the implications of stock prices being RWs. I note the relationship between the RW and the RS – how they are at opposite ends of a spectrum of ways that the future can depend upon the past. I then discuss an extension of the RW that is more suitable for long duration time series, the Geometric Random Walk, and a continuous time version, Geometric Brownian Motion. Let us begin with the …

Definition: A **Random Walk** is a time series in which the period-to-period changes are a Random Sample.

It is very important to note that the changes in a RW are an old friend – a Random Sample. A Random Sample is a set of independent draws from the same uncertainty distribution. Random Sample time series were discussed in detail in a previous set of notes in this course. You may wish to review that material if you have not retained a good working notion of what random sampling means. Since you now know (hopefully) a great deal about Random Samples, you also know a great deal about the changes in a RW – and by extension, a great deal about Random Walks. In particular, you know how to test whether a time series is a RW: Just test whether its changes are a RS! You would determine if the *changes* are level, homoscedastic, and have essentially zero autocorrelation. In this set of notes, I will illustrate how to do that with Dell stock prices.

It is also important to note that the Random Walk shifts the focus from the original primary time series to a secondary time series. The period-to-period changes in the original time series are a time series also. But the uncertainty distribution of daily stock price changes is different from the uncertainty distribution of stock prices. In a RW, we are concerned with the change distribution, for reasons that will become clear momentarily. However, we give the name Random Walk to the original time series, not to the changes.

For the sake of specificity, let me discuss RWs as a model for stock prices. Figure 1 shows the daily closing prices of Dell Computer stock over a three-month period when the stock was still publicly traded. The most common reason for wanting a good statistical model for stock prices is to forecast future prices and thereby make money. To be sure, we are uncertain about future prices of Dell Computer stock. But for a RW, we shift our focus from the uncertainty about the stock prices to uncertainty about *changes* in the stock price.

It is easy to forecast the price of Dell stock if you can forecast changes in Dell's price. How? Just add the forecasted change to the current price! The justification is very simple: Since

Future price = Current price + Future change in price

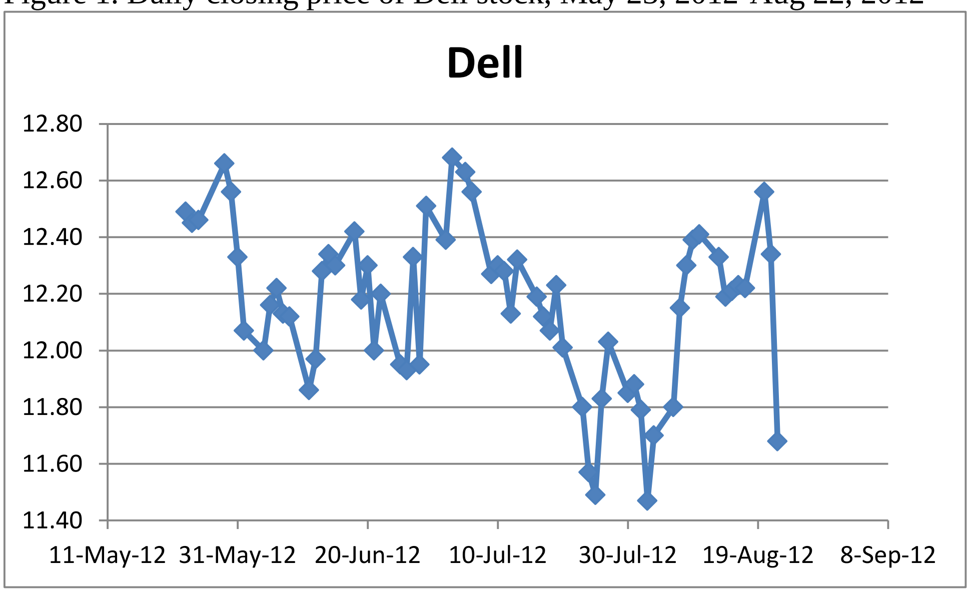
then …

***Forecast* of future price = Current price + *Forecast* of future change in price**.

We know the current price. We do not know the future change in price. But we can forecast the future change in price because it is the next value in a Random Sample, which we know how to forecast. The forecast is the mean of the past values, i.e., the mean of the past changes.

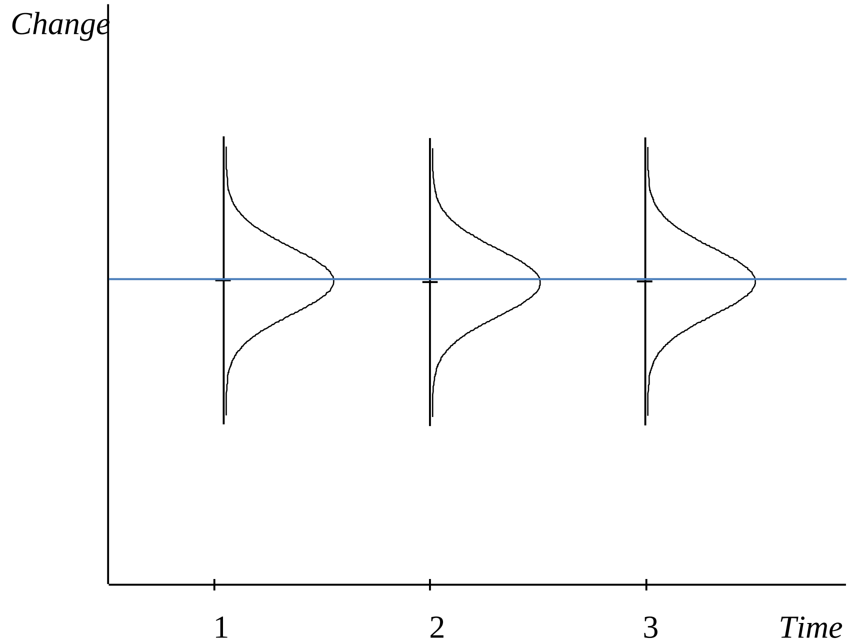
If the forecasting of the next value of a Random Sample is unfamiliar, you may wish to review the relevant section of the topic notes on Random Samples. Here is a quick explanation as applied to changes in a RW: Because we are uncertain about the changes, there is an uncertainty distribution of changes. The change distribution has potential outcomes and probabilities for those outcomes. Since the definition says that changes in a RW are a Random Sample, then the same change distribution applies to all time periods. Every change in stock prices is a random draw from that same distribution of changes. So future changes can be expected to be like past changes, and past changes – on average – can be expected to be close to the mean of the change distribution. The mean of past changes is therefore an excellent estimate not only of the mean of the change distribution but also of individual draws from that change distribution – in particular, a future draw.

Figure 1. Daily closing price of Dell stock, May 23, 2012-Aug 22, 2012



The following Figure 2 shows the statistical model for the changes. At every time t = 1, 2, 3, etc., there is a distribution of changes, shown in the (hypothetical) normal distributions. The normal distributions are the same: same mean, same standard deviation. So their means lie on a flat line, and their shapes do not change from one time to the next. Moreover, a draw from the distribution at time 1 does not affect the outcome drawn at time 2, etc.

Figure 2. Statistical model for the changes of a Random Walk



The ideas of the preceding paragraph and Figure 2 suggest what we should see if we plot the observed outcomes of the changes for a RW:

- Since every change is drawn from the same distribution, then the observed changes should cluster around a flat line at the mean of the change distribution, regardless of the time period – early, middle, or late. The mean of the change distribution should remain stable over time.
- In addition, the up and down variability of the changes should remain stable over time.
- And past change outcomes should not affect future outcomes.

If you have a sense of *déjà vu* about these three bullet points, you would be correct. They are just the L, H, and I assumptions of the RS model.

The above three bullet points suggest a simple procedure to test empirically whether the observed changes are a Random Sample, and hence, whether the original time series is a Random Walk:

- Compute the observed changes in the putative Random Walk.
- Plot the observed changes over time.
- Visually judge whether the plotted changes remain clustered around a flat line.
- Visually judge whether the up-and-down variability of the changes remains the same over time.
- Calculate the autocorrelation of the changes. Judge whether the autocorrelation is "reasonably" close to zero. Zero is good – it indicates that past changes do not affect current changes.

If these tests are all passed, then the changes are probably a Random Sample – so, by definition, the original time series is probably a Random Walk.

The reason that it is important to perform these tests is to enable us to rest easy about forecasting future prices with the formula

Forecast of future price = Current price + Forecast of future change in price.

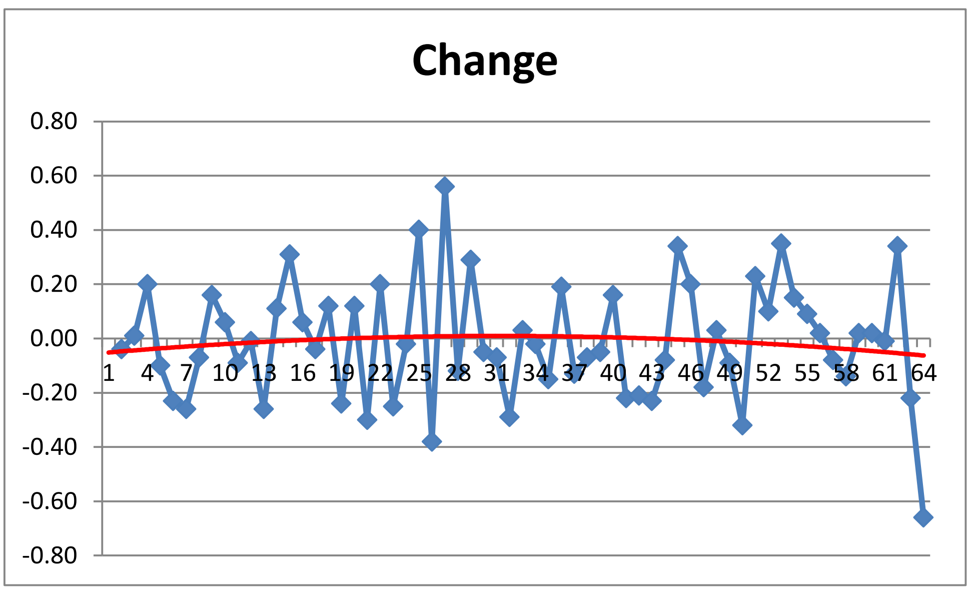
This formula is a valid forecasting method if the time series is a Random Walk (equivalently: if the changes are a Random Sample). The formula may not be valid otherwise.

Here is how these tests work out for Dell Computer stock for the time period May 23, 2012 to August 22, 2012. Changes in the closing price of Dell stock are plotted over a three-month period in Figure 3, below.

- Visually, the changes appear to cluster loosely around a flat line. I have added a quadratic trendline in order to see if there is any apparent trend or curvature in the changes. The quadratic appears to be virtually flat.
- Looking at the up-and-down variation in the changes from the left to the right in the plot, I see no strong evidence that the variability in the change distribution is any different at the beginning than in the middle than at the end.[1]
- The calculated autocorrelation of the changes is -0.1490. This is satisfactorily close to zero.[2]

Thus the changes qualify as a Random Sample. So Dell is a Random Walk.

Figure 3. Daily changes in the closing price of Dell stock, May 23, 2012-Aug 22, 2012



Change statistics

mean	stdev	autocorr
-0.0129	0.2154	-0.1490

[1] In fact, the standard deviation of the first third of the changes is 0.174, of the second third is 0.235, of the final third is 0.240, and the three are not statistically distinguishable.

[2] You may ask, "How close is close to zero?" A good question. See the next "Sidebar" coming up.

Sidebar. When is a sample autocorrelation close to zero?

Since the price of Dell each day is a draw from a distribution, then the actual prices could have been different. Therefore, the changes could have been different. Therefore, the autocorrelation of the changes could have been different. That is, the sample autocorrelation has an uncertainty distribution, with alternative potential outcomes and with associated probabilities. This is a sampling distribution. Here are three properties of the sampling distribution of autocorrelations.[3] These properties parallel those of the sample mean and other statistics:

1. The sampling distribution of autocorrelations is approximately normal if the sample size is sufficiently large.
2. The mean of the sampling distribution of autocorrelations equals the true autocorrelation.
3. The standard deviation of the distribution of autocorrelations is approximately $1/\sqrt{n}$.

There are 63 Dell changes. So the approximate standard deviation (standard error) of the distribution of autocorrelations is $1/\sqrt{63}$ = 0.1260. Thus the sample autocorrelation of -0.1490 is only a little more than one standard deviation from zero – right about where it should be if the true autocorrelation is zero. As a rule of thumb, we should consider the autocorrelation "close to zero" if it is not more than two standard errors from zero. More than two standard errors of deviation correspond to an approximate *p*-value of less than 0.05 in a formal test of the hypothesis that the true autocorrelation is zero.

End Sidebar.

Forecasting a Random Walk

How can we forecast Dell's stock price? In general, how can we forecast a RW? To answer this question, observe the truism:

Future price = Current price + Change in price

The current price is known. No uncertainty there. But the change in price is unknown. However, since Dell's price is a RW, then its changes are a Random Sample. We have a random sample of 63 of its changes. The best forecast of the next value of a RS is the mean of the RS. So the best forecast of the next change is the mean of the past changes. So

Forecast of Future price = Current price + Mean change in past prices

Dell's price on August 22, 2012 was 11.68. The mean change in past prices is -0.0129. So

Forecast of Aug 23 price = 11.68 + (-0.0129) = 11.6671

We do not expect the next change to be exactly -0.0129. We do expect the next change to differ from our estimate by ±stdev of changes = ±0.2154. This is our uncertainty about the next change, which is the only uncertain part of the model. So we can say

Forecast of Aug 23 price = 11.68 + (-0.0129) ± 0.2154 = 11.6671 ± 0.2154

The interval has 68% chance of including the true Aug 23 price if the distribution of changes is normal.[4]

[3] You may accept these properties as facts. They can be proven mathematically (not your responsibility).

[4] You should not assume the distribution of changes is normal. Absent normality, an empirical confidence estimate can be given by tallying the number of past changes in the ±0.2154 range and dividing by 63 (the number of changes).

<u>Sidebar.</u> How to forecast more than one time period in the future?

Suppose that "*t*" denotes the current time period. Then the mathematical truism

$$Y_{t+1} = Y_t + (Y_{t+1} - Y_t)$$

provides the insight necessary to forecast one period ahead. The symbols say

Future price = Current price + Change in price

So you can estimate the next price Y_{t+1} by estimating the change in the price $(Y_{t+1} - Y_t)$ and adding it to the current price Y_t. The uncertainty is measured by the standard deviation of the changes.

This idea can be extended to more than one time period ahead:

$$Y_{t+2} = Y_t + (Y_{t+1} - Y_t) + (Y_{t+2} - Y_{t+1})$$

So you can forecast the price two periods into the future by starting with the current price and adding two estimated changes:

$$\hat{Y}_{t+2} = Y_t + 2 \cdot mean\ change$$

What is the standard error of this estimate? The uncertainty comes from the sum of the two changes $(Y_{t+1} - Y_t) + (Y_{t+2} - Y_{t+1})$. This is the sum of two independent identically distributed random variables (since the changes are a RS). Mathematical statisticians know that the standard deviation of the sum of two independent identically distributed random variables is the common standard deviation times $\sqrt{2}$. So the forecast and margin of error are

$$\hat{Y}_{t+2} = Y_t + 2 \cdot mean\ change \pm stdev\ changes * \sqrt{2}$$

The idea can be extended even further into the future. In general the forecast and margin of error for the price *k* time periods in the future is

$$\hat{Y}_{t+2} = Y_t + k \cdot mean\ change \pm stdev\ changes * \sqrt{k}$$

However, you should take this formula with a grain of salt for large *k*. The reason is that over long time spans, fundamental change can occur in time series. They may not remain Random Walks – or they may change into different kinds of RW (say, with a different distribution of changes.)

End Sidebar.

Implications of Securities Prices being Random Walks

The autocorrelation of a RW is usually very high.[5] This indicates that past prices affect current and future prices strongly. One can do a good job of forecasting future *levels* of Dell or other securities prices that are RWs.

However, investors make money by buying low and selling higher, or by selling high and buying back lower. That is, investors make money from the *changes* in stock prices, rather than from their *levels*. So investors need to forecast the *changes* in Dell's price or in other securities, rather than their *levels*.

But it is very hard to do a good job of forecasting the *changes* in Dell or other securities prices. The explanation for this is based on a subtle but important observation: The uncertainty about the price *change* is relatively much greater than the uncertainty about the price *level*. The reason for this is that the mean change in the RW will be close to zero, unless there is a strong

[5] For example, the autocorrelation of Dell's stock prices from May 23, 2012-Aug 22, 2012 is 0.70. Contrast that with the autocorrelation of the *changes* in Dell's stock price for the same time span, which is -0.1490.

trend in the prices. Therefore, the variability in changes will be quite large relative to the average change; but the variation in the stock price will be much smaller relative to the level of the stock. In the Dell example, this is quite evident: The mean change in Dell's price is only -$0.0129; but the standard deviation of Dell's price changes is $0.2154.

To expand on this a bit, consider the *best* forecast of the next price level, which is the current level plus the mean change. Since the mean change will be close to zero (absent a strong price trend), the forecast will be close to the current level. So the current level, in itself, will be a good predictor of future levels. However, it is hard for an investor to make money by forecasting that tomorrow's price for Dell will be about the same as today's price. An investor must forecast the change in price, including the *sign* of the change. With a mean change close to zero and a relatively large standard deviation, the distribution of changes will have about as many positive changes as negative changes. So it will be almost equally likely for the *actual* change to be positive as negative. And since the autocorrelation of changes is close to zero, the current and past changes contain no clue as to whether the next change will be positive or negative.

The preceding discussion says that our method of forecasting a RW by adding the mean change to the current value will not be a reliable way for investors to make money, *although this method is the best there is*. For, the implication of securities prices being RWs extends beyond *our* method of forecasting and denies that there is *any* legal method to beat the mean change by using past price data. Here is the reason: Suppose that a financial price is a RW. So its daily changes are a Random Sample. Therefore, the daily changes are independent of each other. So future changes are independent of past changes. But past changes add up to past prices. *So the future changes are independent of past prices.* A consequence of this is that study of stock charts and other "technical" analysis of past securities prices cannot provide a reliable strategy for forecasting the change in securities prices, except for the mean change. This is the (weak form of the) Efficient Market Hypothesis. The strong form goes further and says that "fundamental" analysis of *any publicly* available information about the company's markets, management, factories, labor relations, etc. also is not useful for outperforming other investors. One more reason why economics (finance) is called the dismal science. Sorry.

A Regression Approach to Random Walks

Suppose that $Y_1, Y_2, ..., Y_t, Y_{t+1}, ...$ is a time series and we regress each next value (generically, Y_{t+1}) as response variable on each current value (generically, Y_t) as predictor variable:

$$Y_{t+1} = \alpha + \beta Y_t + \varepsilon_{t+1}$$

Suppose that the L,H,I [6] specifications of this regression model are satisfied. Finally, suppose we have sufficient evidence that $\beta = 1$. For example, we may test and accept the hypothesis $H_0 : \beta = 1$. Then the regression model can be restated as

$$Y_{t+1} = \alpha + 1Y_t + \varepsilon_{t+1} \text{ or } Y_{t+1} - Y_t = \alpha + \varepsilon_{t+1}$$

The latter says that the changes ($Y_{t+1} - Y_t$) equal a constant (α) plus a Random Sample (ε_{t+1}, which is a RS by hypothesis, since we are assuming that the regression model is valid, which

[6] The N (normal) specification is desirable, but not essential.

means that its residuals are L,H,I). But a constant plus a Random Sample ($\alpha + \varepsilon_{t+1}$) is a (different) Random Sample. So the restated regression model says that the changes are a Random Sample. Therefore, by the definition, $Y_1, Y_2, ..., Y_t, Y_{t+1}, ...$ is a Random Walk.

Therefore, the regression approach provides an alternative way to determine whether a time series is a RW. The procedure is:

1. Run the regression model $Y_{t+1} = \alpha + \beta Y_t + \varepsilon_{t+1}$
2. Test whether the regression model is valid. This means testing whether the L,H,I specifications are met for the residuals.
3. If the L,H,I specifications are met, then test $H_0 : \beta = 1$. If this hypothesis is accepted, then $Y_1, Y_2, ..., Y_t, Y_{t+1}, ...$ is a RW.

The regression approach also provides an alternative way to forecast the future value Y_{t+1}: Plug the current value Y_t into the estimated regression equation: $\hat{Y}_{t+1} = \hat{\alpha} + \hat{\beta} Y_t$ (one might also use $\hat{Y}_{t+1} = \hat{\alpha} + Y_t$). The estimated margin of error for the forecast is the RMSE of the regression.[7]

The regression approach can also be used to forecast more than one time period in the future. For forecasting two time periods into the future, run the regression model $Y_{t+2} = \alpha + \beta Y_t + \varepsilon_{t+2}$, test whether the regression specifications are met; test $H_0 : \beta = 1$. If accepted, then the changes between alternate dates are a RW. Then the forecast for two time periods ahead can be calculated by plug-in: $\hat{Y}_{t+2} = \hat{\alpha} + \hat{\beta} Y_t$, with the margin of error estimated by the RMSE of the regression. This approach can be extended to forecasting more than two time periods into the future.

Finally, the regression approach can also illuminate the relationship between the RW time series and the RS time series. Suppose that we start out as in the regression approach to RW by regressing each next value (generically, Y_{t+1}) as response variable on each current value (generically, Y_t) as predictor variable:

$$Y_{t+1} = \alpha + \beta Y_t + \varepsilon_{t+1}$$

Suppose that the L,H,I specifications of this regression model are satisfied. Finally, suppose we have sufficient evidence that $\beta = 0$. For example, we may test and accept the hypothesis $H_0 : \beta = 0$. Then the regression model can be restated as

$$Y_{t+1} = \alpha + 0 Y_t + \varepsilon_{t+1} \text{ or } Y_{t+1} = \alpha + \varepsilon_{t+1}$$

The latter says that the original time series (Y_{t+1}) equals a constant (α) plus a Random Sample (ε_{t+1}). But a constant plus a Random Sample ($\alpha + \varepsilon_{t+1}$) is a Random Sample. So the restated regression model says that the original time series (Y_{t+1}) is a Random Sample. That is, $Y_1, Y_2, ..., Y_t, Y_{t+1}, ...$ is a Random Sample.

An interpretation is that the RS and RW are at opposite ends of a spectrum of ways in which the future can depend upon the past. The RW depends completely upon the past ($\beta = 1$), whereas the RS depends not at all upon the past ($\beta = 0$). If $0 < \beta < 1$, then the future depends

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[7] RMSE is a little low. A more accurate margin of error is RMSE $\sqrt{1 + 1/n + (y_t - \bar{y})^2 / \sum_{i=1}^{n} (y_i - \bar{y})^2}$.

partly upon the past. The RS and RW are special cases of the general first-order autoregression model $Y_{t+1} = \alpha + \beta Y_t + \varepsilon_{t+1}$, which will be covered in a future Topic Note.

A More Sophisticated Random Walk Model

The RW model that has been presented above in these notes is often a satisfactory model for securities prices in the short run. In the long run, the model may not fit very well. The problem is that variability often changes in proportion to the level of prices, rather than remaining fixed. When a stock is priced at $10 per share, a daily fluctuation in price of $0.25 may be reasonable. However, if the stock price advances to $100 per share, it may not be reasonable to expect the daily price fluctuation to remain at $0.25. If the variation is proportional to price, then a daily price fluctuation of $2.50 might be expected if the price advances to $100. That is, heteroscedasticity may be more reasonable than homoscedasticity. If variation is proportional to price, the price level may not change much in the short run, and so **H** may be a reasonable approximation. In the long run, prices change and so **H** may be less tenable.

A more sophisticated RW model, capable of handling proportional price variation, is provided by the **Geometric Random Walk** (**GRW**). To provide some motivation for the GRW, let us compare two equivalent ways to write the current value Y_t of a time series in terms of its past:

A. $Y_t = Y_0 + (Y_1 - Y_0) + (Y_2 - Y_1) + \cdots + (Y_{t-1} - Y_{t-2}) + (Y_t - Y_{t-1})$

B. $Y_t = Y_0 + \dfrac{(Y_1 - Y_0)}{Y_0} Y_0 + \dfrac{(Y_2 - Y_1)}{Y_1} Y_1 + \cdots + \dfrac{(Y_{t-1} - Y_{t-2})}{Y_{t-2}} Y_{t-2} + \dfrac{(Y_t - Y_{t-1})}{Y_{t-1}} Y_{t-1}$

Both methods are mathematical identities, true for all time series (assuming non-zero values for Method B). Method A writes the current value in terms of past *changes* and leads to the RW model presented up to this point in this document. Method B writes the current value in terms of past *proportional changes* and leads to the GRW model, as presented below. In any given real problem, the RW and GRW may be competing models offered to explain the same time series. Which one is better is a matter to be decided by empirical tests. Often, there is very little difference, especially if the level of Y_t does not change a great deal over the time period under study – which is often the case over the short term. However, in the longer term, Y_t often displays considerable secular growth or decline. In that case, the GRW is often significantly better. In fact, in finance, analysis of securities prices often begins with the GRW as a matter of course, especially in its logarithmic form (see below). Over the short run, the RW form might be preferred for its easier interpretability.

In order for Y_t to qualify as a RW, the period-to-period changes must qualify as a RS. In order to qualify as a GRW, the period-to-period *rates of change* must qualify as a RS. The period-to-period rates of change are $R_1 = \dfrac{(Y_1 - Y_0)}{Y_0}, R_2 = \dfrac{(Y_2 - Y_1)}{Y_1}, \cdots, R_{t-1} = \dfrac{(Y_{t-1} - Y_{t-2})}{Y_{t-2}}, R_t = \dfrac{(Y_t - Y_{t-1})}{Y_{t-1}}$.

Recall that the meaning of $R_1, R_2, ..., R_t, ...$ being a Random Sample is that $R_1, R_2, ..., R_t, ...$ satisfies two specifications:

- $R_1, R_2, ..., R_t, ...$ are independent
- $R_1, R_2, ..., R_t, ...$ are identically distributed

In finance, $R_1, R_2, ..., R_t, ...$ are called **returns** because they equal the rate of return an investor can obtain (neglecting transaction costs) from buying an investment at time *t* and selling it at time *t* + 1. For example, a stock price may be Y_0 = 100 today. If the price tomorrow is 110, then R_1 = 0.10 (a 10% increase). If the price the day after tomorrow is 104.50, then R_2 = -0.05 (a 5% decrease). Model B can be rewritten in a "geometric" or multiplicative form that reveals the origin of the "geometric" part of the name: For the first time period,

$Y_1 = Y_0 + \frac{(Y_1 - Y_0)}{Y_0} Y_0 = Y_0 + R_1 Y_0 = Y_0(1 + R_1)$. Similarly for the second:

$Y_2 = Y_1 + \frac{(Y_2 - Y_1)}{Y_1} Y_1 = Y_1(1 + R_2)$. Substitute the first expression for Y_1 into the second to yield $Y_2 = Y_0(1 + R_1)(1 + R_2)$. Continue "growing forward" the time series in this manner to obtain $Y_t = Y_0(1 + R_1)(1 + R_2)\cdots(1 + R_t)$ as a general expression for the value at time *t*. The multiplication of rates of change suggests a similarity to a geometric series.

Here is the definition: A time series $Y_0, Y_1, Y_2, ..., Y_t, ...$ is a **Geometric Random Walk** (**GRW**) if $Y_t = Y_0(1 + R_1)(1 + R_2)\cdots(1 + R_t)$ and $R_1, R_2, ..., R_t, ...$ is a Random Sample.

I have suggested that the "geometric" part of the name comes from the resemblance to a geometric series. The "random walk" part of the name comes from the property (that I will demonstrate in just a little bit) that changes in the series (in logarithmic scale) are approximately a Random Sample.

The series of rate changes $R_1, R_2, ..., R_t, ...$ need not have a mean of zero, nor does it need to be normally distributed. If the uncertainty distribution of *R* has a positive mean, then there will be a tendency for the price to drift up over time; a negative mean indicates a tendency for the price to drift down over time.

Now recall the first-order Taylor's series approximations $e^x \approx 1 + x$ and $\log_e(1 + x) \approx x$. Then $e^R \approx 1 + R$ and $\log_e(1 + R) \approx R$. So the value of the time series at time *t* is $Y_t = Y_0(1 + R_1)(1 + R_2)\cdots(1 + R_t) \approx Y_0 e^{R_1} e^{R_2} \cdots e^{R_t} = Y_0 e^{R_1 + R_2 + \cdots + R_t}$. In log scale (always base *e*), and using common properties of logarithms, $\log\{Y_0(1 + R_1)(1 + R_2)\cdots(1 + R_t)\} =$ $\log(Y_0) + \log(1 + R_1) + \log(1 + R_2) + \cdots + \log(1 + R_t) \approx \log(Y_0) + R_1 + R_2 + \cdots + R_t$. The approximation $Y_0(1 + R_1)(1 + R_2)\cdots(1 + R_t) \approx Y_0 e^{R_1 + R_2 + \cdots + R_t}$ yields the same formula in log scale: $\log(Y_0 e^{R_1 + R_2 + \cdots + R_t}) = \log(Y_0) + R_1 + R_2 + \cdots + R_t$. From this expression, we see that the change in $\log(Y_t)$ from the previous time period is R_t, which is a Random Sample.

But that is the definition of a RW! Since the change in $\log(Y_t)$ is a RS, then $\log(Y_t)$ is a RW. These mathematical musings give us a characterization of a GRW: Y_t **is a GRW if and only if** $\log(Y_t)$ **is a RW if and only if the changes in** $\log(Y_t)$ **are a RS.** Furthermore, the change in $\log(Y_t)$ is $\log(Y_t) - \log(Y_{t-1}) = \log(Y_t / Y_{t-1}) = \log((Y_t - Y_{t-1} + Y_{t-1}) / Y_{t-1}) =$ $\log(1 + (Y_t - Y_{t-1}) / Y_{t-1}) \approx (Y_t - Y_{t-1}) / Y_{t-1} = R_t$ by the first-order Taylor's series expansion for log.

Note that I implicitly assume that the time series contemplated for the GRW model must not have negative values. This is because one cannot calculate the logarithm of a negative number. The rates of change can be negative, but not the original values. This is not a limitation for securities prices, which are never negative or zero.

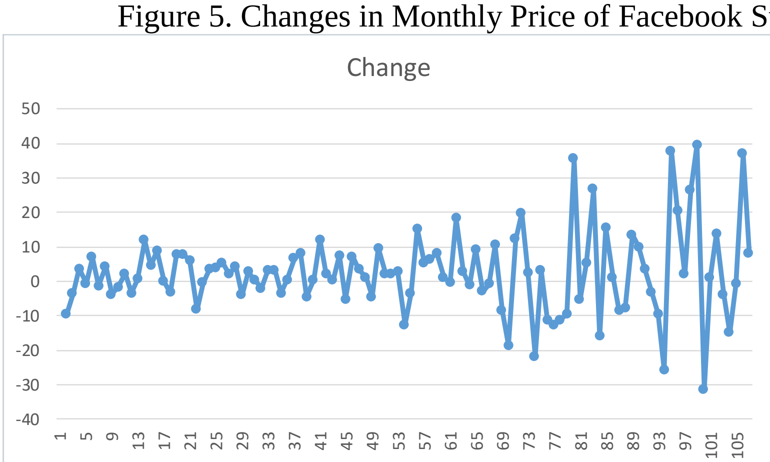
The takeaway: To apply the GRW, transform the original time series $Y_0, Y_1, Y_2, \ldots, Y_t, \ldots$ into $\log(Y_0), \log(Y_1), \log(Y_2), \ldots, \log(Y_t), \ldots$. Test the log(Y) series for being a RW: Are the changes in log(Y) a RS? If so, then analyze and forecast the log(Y) as you would any other RW. Finally, exponentiate the results to convert them back into original scale.

Here is an example that illustrates the advantage of GRW over RW in a lengthy time period.
<u>Example:</u> The stock price of Facebook (FB).[8]

Figure 4. Monthly Price of Facebook Stock

Over the time period shown, FB went from a price of about \$20 to about \$300 (Figure 4). The variation appears to have increased over time. This is seen in the timeplot of the month-to-month changes in prices (Figure 5):

Figure 5. Changes in Monthly Price of Facebook Stock



autocorr =	-0.0552						
mean change	2.5634			Apr-21	mean cha	Forecast	+/- MOE
std change =	11.7190		Forecast May-21:	302.82	2.5634	305.38	11.72

<u>The RW model</u>: The changes in FB prices are clearly not a RS. Although the timeplot appears level and the autocorrelation tests satisfactorily close to zero, the variation increases over time.

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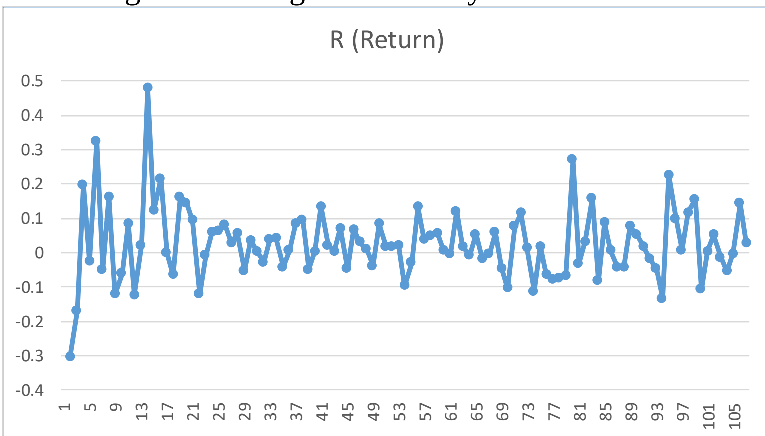
[8] See "Facebook-RW and GRW.xlsx" for the data and analysis.

The value of the forecast for May 2021 (305.38) is OK, but the margin-of-error (MOE) (11.72), appropriate for a RW, is too small, as it represents an average of variability over the whole time period, rather than the much larger variability characteristic of FB more recently.

Figure 6. Changes in Monthly Returns of Facebook Stock



autocorr=	-0.0237							
mean R =	0.0267				Apr-21	mean R	Forecast	+/- MOE
stdev R =	0.1034		Forecast May-21:		302.82	0.0267	310.91	31.32

<u>The GRW model:</u> Except for an initial 2-year period after listing, the returns (proportional returns – Figure 6) appear roughly level, homoscedastic, and have an autocorrelation satisfactorily close to zero. The forecast (310.91) is OK and the margin-of-error (10.34% = 31.32 recently) is much more believable than that of the straight RW model.

Asides.

(1) Discrete time and continuous time.

Time series may be considered a branch of a larger field known as stochastic processes, which studies indexed variables, but in which the indexing need not represent time. The indexing can be discrete or continuous. This course deals with discrete-time time series, in which the index is an integer: $Y_0, Y_1, Y_2, ..., Y_t, ...$ But there are continuous-time time series, Y_t, in which the index *t* lies in some continuous range, like $0 \le t \le 1$. This just means that there is an uncertainty distribution of *Y* for every *t* in the range.

(2) Brownian Motion, informally.

In much of mathematical finance, it is usually assumed that the distribution of the rate of change [9] (our *R* variable) is normal. So far in these notes, that assumption has not been made. The assumption of normality is a testable hypothesis, discussed in another set of Notes. With the normal assumption, the GRW is a discrete Geometric Brownian Motion (GBM). Brownian Motion [10] (BM) is a stochastic process with continuous *t* in which the changes are independent

[9] Also called the **return** or yield.

[10] Named after botanist Robert Brown (1827), who observed, but could not explain, the random motion of particles suspended in water. Albert Einstein (1905) explained the motion as resulting from random collisions with water molecules, thus providing indirect evidence for the atomic theory of matter, and derived the distribution. Curiously, the distribution had been derived independently at least twice before – most notably by Louis Bachelier (1900) in his dissertation, "The Theory of Speculation", as a model for stock prices.

normally distributed random variables with mean 0 and variance proportional to the duration of the time interval of the change.[11] BM can be obtained as the limit of a RW in which the time intervals are made smaller and smaller. Similarly, GBM can be obtained as the limit of a GRW in which the time intervals are made smaller and smaller. So you can think of BM as a continuous-time RW; likewise, you can think of a GBM as a continuous-time GRW.

(3) Brownian Motion, formally.

Formally, a GBM can be defined as a continuously indexed stochastic process (the continuous-time analog of a discrete-time series) S_t such that $dS_t = \mu S_t + \sigma S_t dW_t$. This is a stochastic differential equation (SDE), in which μ and σ are constants and W_t is Brownian Motion [12], and dS_t and dW_t are differentials representing small changes in S_t and in W_t. SDE's are beyond the scope of this course, but the meaning of this SDE can be made intelligible if it is first put into another form: Divide through by S_t to get $\frac{dS_t}{S_t} = \mu + \sigma\, dW_t$. The left-hand side is the *proportionate* change (i.e., the *return*) in the security price S_t that results from a small change dW_t in the Wiener Process at time *t*. There is a built-in constant change (μ), which could be positive or negative or zero and which equals the expected (mean) proportionate change: $E\left(\frac{dS_t}{S_t}\right) = E(\mu + \sigma\, dW_t) = \mu + \sigma\, E(dW_t) = \mu$ since the small change dW_t in BM has mean 0.

So the model $\frac{dS_t}{S_t} = \mu + \sigma\, dW_t$ says that the proportionate change in the GBM S_t is a constant μ plus zero-mean error $\sigma\, dW_t$, where σ governs the variance of the error term. In other words, the SDE says that the proportionate change is a "Random Sample." This is exactly what the discrete-time GRW says. To complete the parallelism, note that the model $\frac{dS_t}{S_t} = \mu + \sigma\, dW_t$ can be rewritten $d\log(S_t) = \mu + \sigma\, dW_t$. This says that small changes in $\log(S_t)$ are small changes in a "random walk" – i.e., a "random sample". Thus, in continuous time, as in discrete time, we have Y_t **is a GBM if and only if** $\log(Y_t)$ **is a BM if and only if the changes in** $\log(Y_t)$ **are a "RS."**

[11] In discrete time, the time intervals are all of unit length (0 to 1 to 2 to 3, etc.) so the variance of the intervals remains constant (H) and so the changes are a RS.

[12] Another name for Brownian Motion is a Wiener Process (named after Norbert Wiener, also considered the founder of cybernetics) – hence the notation W_t.

SUMMARY

A Random Walk (RW) is a time series for which the period-to-period changes are a Random Sample (RS).
This definition provides a simple means to test whether a time series is a RW:
Plot the changes in the putative RW and check:

- L – does the timeplot of changes appear level?
- H – does the timeplot of changes appear homoscedastic?
- I – is the timeplot of changes autocorrelated? (The autocorrelation of changes should be close to zero. This may need to be calculated. As a rule of thumb, the autocorrelation is close to zero if it is between $\pm 2/\sqrt{n}$.)

The changes do not need to be normally distributed.
The best forecast of the next value of a RW is

Forecast of Future price = Current price + Mean change in past prices

The uncertainty of this forecast is assessed by its standard deviation = st dev of changes in prices.

To the extent that stock prices are RWs, stocks challenge investors' ability to make money from them. Stock prices can be predicted well; but their changes cannot be predicted well. Stock prices correlate strongly with past stock prices; but changes in stock prices have no correlation with past changes (nor with past stock prices). Investors make money from the changes in stock prices, not from stock prices themselves. Because stock price changes are RS, past changes offer few clues to future changes. An important consequence is that study of past stock prices offer few reliable clues to the future price changes that could bring profits to investors.

The Geometric Random Walk (GRW) offers a more sophisticated model for securities prices that accommodates heteroscedastic prices. Y_t **is a GRW if and only if** $\log(Y_t)$ **is a RW if and only if the changes in** $\log(Y_t)$ **are a RS.** Variability of a GRW is proportional to the value of the GRW. The continuous-time analog of the normal GRW is the Geometric Brownian Motion (GBM). Y_t **is a GBM if and only if** $\log(Y_t)$ **is a BM if and only if the changes in** $\log(Y_t)$ **are a "RS."**