3DCV Exercise 1

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Question 1

1. Properties of Rotation Matrices

Answer: a. Lets consider a frame with two axis, named, X1 and Y1 and rotate this frame by an angle φ. The new frame will be X2 and Y2

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \cos\varphi - y_2 \sin\varphi \\ x_2 \sin\varphi + y_2 \cos\varphi \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\psi \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

So the Rotation Matrix will be:

$$R_1^2\, R_1^2 = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

For this rotational matrix to be orthogonal $RR^T = I$

$$RR^T = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}$$

$$\begin{bmatrix} \cos^2\varphi + \sin^2\varphi & \cos\varphi\sin\varphi - \sin\varphi\cos\varphi \\ \sin\varphi\cdot\cos\varphi - \cos\varphi\sin\varphi & \cos^2\varphi + \sin^2\varphi \end{bmatrix}$$

$$W.K.T \sin^2\psi + \cos^2\psi = 1$$

So, $RR^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

The Euclidean Norm of a Rotational Matrix is 1

$$R = [x_1, y_1] = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

The Euclidean Norm of R is the largest norm of its column vector
$\|R\| = \max[\|x\|, \|x\|]=\max\left[\sqrt{\cos^2\varphi \mid +\sin^2\varphi}), \sqrt{\sin^2\varphi + \cos^2\varphi}\right]$

$\|R\| = 1$ **Hence proved that R is Orthonormal**

Answer: b.
Through lecture slides we know the $Rx(\varphi)$, $Ry(\theta)$, $Rz(\phi)$ rotational matrices.
Now considering $Rx(\varphi)$

$$Rx(\varphi)\ \mathrm{Rx}(\varphi)^T = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}\Bigg)$$

$$Rx(\varphi)\ \mathrm{Rx}(\varphi)^T = \begin{bmatrix} \cos^2\varphi + \sin^2\varphi & \cos\varphi\sin\varphi - \cos\varphi\sin\varphi & 1 \\ \cos\varphi\sin\varphi - \cos\varphi\sin\varphi & \sin^2\varphi + \cos^2\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$Rx(\varphi)\ \mathrm{Rx}(\varphi)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now considering $Ry(\theta)$

$$Ry(\theta)\ \mathrm{Ry}(\theta)^T = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$Ry(\theta)\ \mathrm{Ry}(\theta)^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$Ry(\theta)\ \mathrm{Ry}(\theta)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_y^T R_y = I \Longleftrightarrow R_y^T = R_y^- 1$$

Now considering $Rz(\phi)$

$$Rz(\phi)\ \mathrm{Rz}(\phi)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$Rz(\phi)\ \mathrm{Rz}(\phi)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos^2\phi + \sin^2\phi & 0 \\ 0 & 0 & \cos^2\phi + \sin^2\phi \end{bmatrix}$$

$$Rz(\phi)\ \mathrm{Rz}(\phi)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_z^T R_z = I \Longleftrightarrow R_z^T = R_z^- 1$$

calculating the $\det(R_z) =$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{vmatrix}$$

$\because \cos^2\phi + \sin^2\phi = 1$

Rx, Ry, Rz are rotational matrix

Now, R = $Rx(\varphi)Ry(\theta)Rz(\phi)$ is a rotation matrix

$$R=\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\varphi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}$$

$$R=\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta\sin\phi & -\sin\theta\cos\phi \\ 0 & \cos\phi & -\sin\phi \\ \sin\theta & \sin\phi\cos\theta & \cos\theta\sin\phi \end{bmatrix}$$

$$R=\begin{bmatrix} \cos\varphi\cos\theta & -\cos\varphi\sin\theta\sin\phi - \sin\psi\cos\phi & -\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi \\ \sin\varphi\cos\theta & -\sin\psi\sin\theta\sin\phi + \cos\phi\cos\psi & -\sin\theta\cos\phi\sin\psi - \sin\phi\cos\varphi \\ \sin\theta & \sin\phi\cos\theta & \cos\theta\cos\phi \end{bmatrix}$$

$$R^T=\begin{bmatrix} \cos\psi\cos\theta & \sin\psi\cos\theta & \sin\theta \\ -\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & -\sin\phi\sin\theta\sin\phi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ -\sin\theta\cos\phi\cos\psi + \sin\phi\sin\psi & -\sin\theta\cos\phi\sin\psi - \sin\phi\cos\psi & \cos\theta\cos\phi \end{bmatrix}$$

By matrix multiplication, we got $RR^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer: c. The factor by which linear transformation changes any area/Volume is called determinant of that matrix. The negative value of determinant represent the reversal of orientation.
A 3x3 Matrix geometrically represents a Parallelopipet and determinant of this matrix represents the factor by which the volume of Parallelopipet changes after the transformation. Matrix Determinant represents the factor by which area changes after the transformation. Rotation and translation belongs to a class of "Rigid Transformation'. Rigid Transformation do not change the distance between two points. thus they do not squeeze or stretch any object in anyway.

2. Transformation Chain
At first we will consider a 3D point from world coordinate system and convert it into Homogeneous coordinates which makes it a 4x1 X matrix.

$$X=\begin{pmatrix} X_s \\ Y_s \\ Z_s \\ 1 \end{pmatrix}$$

Then we consider the extrinsic parameters like, Rotational matrix and translation matrix. These two matrices will be added and transformed to homogeneous coordinate system. This matrix will be 4x4 Rt matrix.

$$\text{Rt=}\begin{pmatrix} \boldsymbol{I.i} & \boldsymbol{J.i} & \boldsymbol{K}\cdot\boldsymbol{i} & t_x \\ \boldsymbol{I}\cdot\boldsymbol{j} & \boldsymbol{J}\cdot\boldsymbol{j} & \boldsymbol{K}\cdot\boldsymbol{j} & t_y \\ \boldsymbol{I}\cdot\boldsymbol{k} & \boldsymbol{J}\cdot\boldsymbol{k} & \boldsymbol{K}\cdot\boldsymbol{k} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now, we will take dot product of Rt and X to get camera coordinates, which will be 4x1 CC matrix.

$$\begin{pmatrix} x_s \\ y_s \\ z_s \\ 1 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I.i} & \boldsymbol{J}\cdot\boldsymbol{i} & \boldsymbol{K}\cdot\boldsymbol{i} & t_x \\ \boldsymbol{I}\cdot\boldsymbol{j} & \boldsymbol{J}\cdot\boldsymbol{j} & \boldsymbol{K}\cdot\boldsymbol{j} & t_y \\ \boldsymbol{I}\cdot\boldsymbol{k} & \boldsymbol{J}\cdot\boldsymbol{k} & \boldsymbol{K}\cdot\boldsymbol{k} & t_z \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} X_s \\ Y_s \\ Z_s \\ 1 \end{pmatrix}$$

Now, we will consider the camera intrinsic parameters, the upper triangular matrix K and transform this to homogeneous matrix of size 3x4 KI matrix.

$$\text{KI=}\begin{pmatrix} \alpha_x & s & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Now to get the image plane coordinates we have to take dot product of KI and CC matrix, This will give us image plane coordinates 3x1 matrix.

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}$$

Now by dividing the first two elements from this matrix with the third element we will get the pixel coordinates.

$$x_{pix} = \frac{u'}{w'}$$
$$y_{pix} = \frac{v'}{w'}$$

Homogeneous Coordinates : It makes the transformation chain a sequence of matrix multiplication and it avoids division by z, which makes the transformation chain non linear. Points at infinity can be represented using Homogeneous Coordinates.

Implementation

1. Please refer to the script main.py for implementation.

2. Please find the images below:

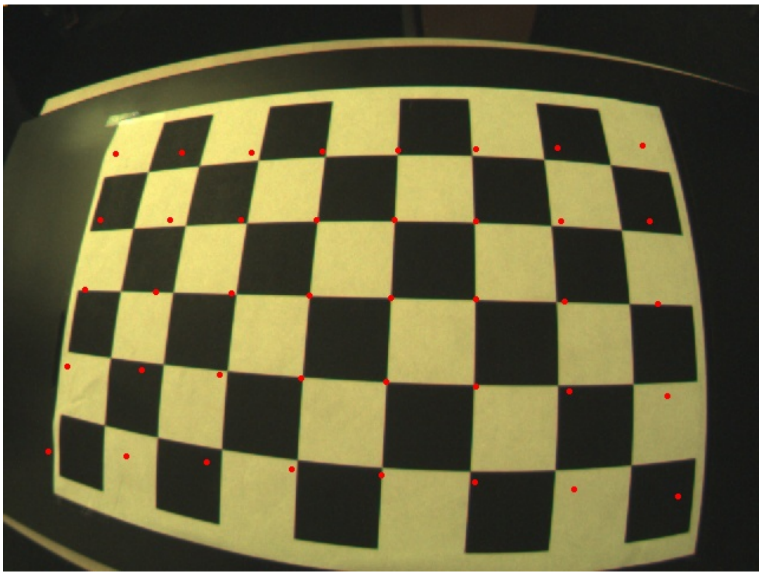

Figure 1: Projected points without distortion correction

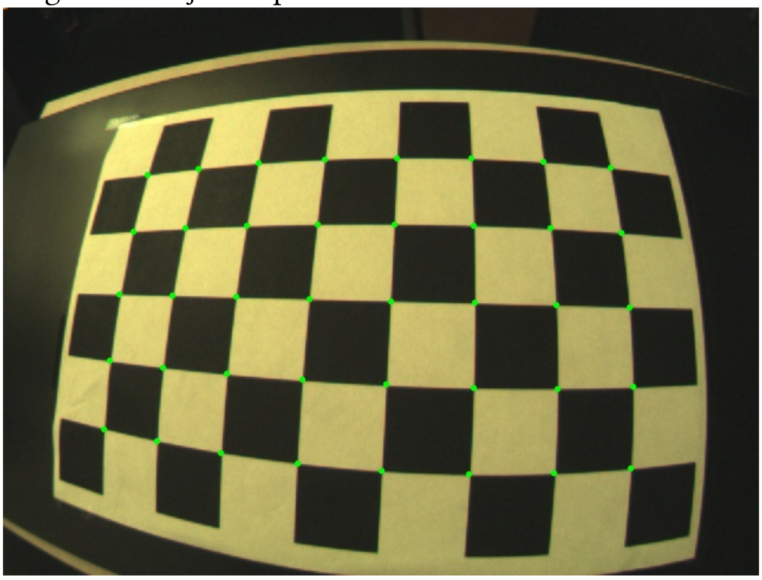

Figure 2: Projected points with distortion correction