21-08-23

Module-2

① Fourier Series ✓ Zoe

② Fourier Transform

Fourier Series

Let $f(x)$ be a periodic function of period $2l$, defined in $(c, c+2l)$ satisfying Dirichelt's conditions

(i) $f(x)$ and its integrals are finite and single valued

(ii) $f(x)$ has discontinuities finite in number

(iii) $f(x)$ has finite no. of maxima/minima

then $f(x)$ can be expanded as an infinite trigonometric Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{l}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right)$$

which is called Fourier Series, ~~which~~ when a_0, a_n, b_n are called fourier coefficients or constants

Euler's Formulae

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x) \cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x) \sin\left(\frac{n\pi}{l}\right)x\,dx$$

Generalised Rule of Integration by parts

$\int uv = uv_1 - u'v_2 + u''v_3 - - -$

$u', u'', u''' \rightarrow$ successive derivatives of u
$v', v'', v''' \rightarrow$ successive derivatives of v

__Type-I:__ interval $(0, 2\pi) \rightarrow (c, c+2l)$

then fourier series of $f(x)$ is $(0, 2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\,dx$$

Q find fourier series for $f(x) = x$ in $(0, 2\pi)$

Solⁿ: ~~$f(x) = \frac{a_0}{2} + \sum f(x)$~~ $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$$= \frac{1}{\pi}\int_0^{2\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = \frac{1}{2\pi}\left[4\pi^2 - 0\right]$$

$$\cos n\pi = (-1)^n$$

$$\boxed{a_0 = 2\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx \, dx$$

$$= \frac{1}{\pi}\left[x\underbrace{\cancel{\left(\frac{\sin nx}{n}\right)}}_{\to 0} - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \cancel{\frac{1}{\pi}\left[x - \cos\right.} \qquad = \frac{1}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right] = 0$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin x \, dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\underbrace{\cancel{\left(-\frac{\sin nx}{n^2}\right)}}_{\to 0}\right]_0^{2\pi}$$

$$= -\frac{1}{n\pi}\left[2\pi\cos 2n\pi - 0\right]$$

$$\boxed{b_n = -\frac{2}{n}}$$

$$\boxed{\therefore f(x) = \pi + \sum_{n=1}^{\infty}\left(-\frac{2}{n}\right)\sin nx}$$

Q $f(x) = -e^{x}$ in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx \quad \text{——} \quad (1)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\,dx = \frac{1}{\pi}\left[\frac{e^{-x}}{-1}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-2\pi} - e^{0}}{-1}\right] = \frac{1}{\pi}\left[1 - e^{-2\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$$

$$= \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2+n^2}\left[-1\cos nx + n\overbrace{\sin nx}^{0}\right]\right\}_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\cos 2n\pi) - e^{0}(-\cos 0)\right]$$

$$\boxed{a_n = \frac{1 - e^{-2\pi}}{(1+n^2)\pi}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

$$b_n = \frac{1}{\pi}\left\{\frac{e^{-x}}{(-1)^2+n^2}\left[-\sin nx - n\cos nx\right]\right\}_0^{2\pi}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-n\cos 2n\pi) - e^0(-n\cos 0)\right]$$

$$b_n = \frac{n(1-e^{-2\pi})}{\pi(1+n^2)}$$

$\therefore$ Fourier series of e^{-x} in $(0, 2\pi)$ -

$$f(x) = e^{-x} = \frac{1}{2\pi}(1-e^{-2\pi}) + \sum_{n=1}^{\infty}\frac{(1-e^{-2\pi})}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{n(1-e^{-2\pi})}{\pi(1+n^2)}\sin nx$$

$$e^{-x} = \frac{(1-e^{-2\pi})}{\pi}\left[\frac{1}{2} + \sum\frac{\cos nx + n\sin nx}{1+n^2}\right]$$

Q $f(x) = \begin{cases} -\pi & 0<x<\pi \\ x-\pi & \pi<x<2\pi \end{cases}$

State the value of the series at $x=\pi$ and hence saw that $\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}$

Solⁿ: $f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left(\int_0^{\pi} -\pi\,dx + \int_{\pi}^{2\pi}(x-\pi)\,dx\right)$$

$$= \frac{1}{\pi}\left[-\pi\left[x\right]_0^{\pi} + \left[\frac{x^2}{2} - \pi x\right]_{\pi}^{2\pi}\right]$$

$$a_0 = \frac{1}{\pi}\left[-\pi^2 + \left(2\pi^2 - 2\pi^2 - \frac{\pi^2}{2} + \pi^2\right)\right]$$

$$\boxed{a_0 = \frac{-\pi}{2}}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} -\pi \cos nx\, dx + \int_{\pi}^{2\pi} (x-\pi)\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{-\pi\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left[(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}\left(\cos 2n\pi - \cos n\pi\right)\right] = \frac{1}{\pi n^2}\left(1-(-1)^n\right) = \begin{cases} 0 & n \text{ is even} \\ \frac{2}{\pi n^2} & n \text{ is odd} \end{cases}$$

$b_n = ?$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} -\pi \sin nx\, dx + \int_{\pi}^{2\pi} (x-\pi)\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{-\pi\left(\frac{-\cos nx}{n}\right)_0^{\pi} + \left[(x-\pi)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\left(\frac{-\cos n\pi}{n} + \frac{\cos 0}{n}\right) + \left[\pi\left(\frac{-\cos 2n\pi}{n}\right) - 0\right]\right\}$$

$$= \frac{1}{\pi} \cdot \frac{\pi}{n} \{\cos n\pi - \cos 0 - \cos 2n\pi\}$$

$$\cos n\pi = (-1)^n, \cos 0 = \cos 2n\pi = 1$$

$$\boxed{b_n = \frac{1}{n}\{(-1)^n - 2\}}$$

fourier series f(x) is

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}[1-(-1)^n]\cos nx + \sum_{n=1}^{\infty} \frac{1}{n}[(-1)^n - 2] \sin nx$$

<u>At the point of discontinuity</u>

$f(x)$ is discontinuous at $x = c$
then the fourier series gives the value

$$f(c) = \frac{1}{2}\left\{\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right\}$$

Now, $f(x)$ discontinuous at $x = \pi$

$$f(\pi) = \frac{1}{2}\left\{\lim_{x\to \pi^-} f(x) + \lim_{x\to \pi^+} f(x)\right\}$$

$$= \frac{1}{2}\{-\pi + 0\} = -\frac{\pi}{2}$$

$$\frac{-\pi}{2} = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}[1-(-1)^n](-1)^n$$

$$\frac{-\pi}{2} + \frac{\pi}{4} = \frac{-2}{\pi 1^2} - \frac{2}{\pi 2^2} - \frac{2}{\pi 3^2} - \dots$$

$$-\frac{\pi}{4} = -\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$$

Q Find F.S. for $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $(0, 2\pi)$ Hence deduce

(i) $\frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$

(ii) $\frac{\pi^2}{12} = \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}-\cdots$

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$

(iv) $\frac{\pi^4}{90} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots$

Sol^n: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — ①

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\,dx = \frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}$$

$$= -\frac{1}{12\pi}\left[-\pi^3 - \pi^3\right] = \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\cos nx = \frac{1}{4\pi}\left[(\pi-x^2)\left(\frac{\sin nx}{n}\right) - (-2(\pi-x))\right.$$

$$= \frac{1}{4\pi}\left\{(\pi-x)^2\cancel{\left(\frac{\sin nx}{n}\right)}^{0} - (-2(\pi-x))\left(\frac{-\cos nx}{n^2}\right) + (2)\cancel{\left(\frac{-\sin nx}{n^3}\right)}^{0}\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-2(\pi-x)\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi}\left[-\pi\frac{\cos 2n\pi}{n^2} - \pi\frac{\cos 0}{n^2}\right]$$

$$\boxed{a_n = \frac{1}{n^2}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\sin nx\,dx$$

$$\cancel{= \frac{1}{\pi}\left[\frac{\pi^2}{4}\int_0^{\pi} -\pi\sin nx + \int_0^{2\pi}(\pi-x)(x-\pi)\sin nx\right]}$$

$$\cancel{= \frac{1}{\pi}\left[\frac{-\cos x}{n}\right] \frac{1}{\pi}\left[\frac{\pi-x^2}{4}\int_0^{\pi}\left(\frac{-\cos x}{n}\right)_0^{\pi} + \left(\frac{-\cos x}{n}\right)_0^{2\pi}\right]}$$

$$\cancel{= \frac{1}{\pi}\left(\frac{\pi-x^2}{4}\right)\Big[}$$

$$= \frac{1}{4\pi}\left\{(\pi-x)^2\left(\frac{-\cos nx}{n}\right) - (-2(\pi-x))\cancel{\left(\frac{-\sin x}{n^2}\right)}^{0} + (2)\left(\frac{\cos nx}{n^3}\right)\right\}_0^{2\pi}$$

$$= \frac{1}{4\pi}\left\{\left[\pi^2\left(\frac{-\cos 2n\pi}{n}\right) + \frac{2\cos 2n\pi}{n^3}\right] - \left[\pi^2\left(\frac{-\cos 0}{n}\right) + 2\left(\frac{\cos 0}{n^3}\right)\right]\right\}$$

$$= \left\{\frac{-\pi^2}{n} + \frac{2}{n^3} - \left(\frac{-\pi^2}{n} + \frac{2}{n^3}\right)\right\} \Rightarrow b_n = 0$$

fourier series is:- $\boxed{f(x) = \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx}$

(i) put $x = 0$ in F.S.

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum \frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} - - - -$$

(ii) put $x = \pi$ in F.S.

$$0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$\frac{-\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - - - -$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - - -$$

(iii) add (i) & (ii)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + - - -\right)$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - - -$$

(iv) Parseval's Identity $(c, c+2l)$:

$$\frac{1}{l}\int_{c}^{c+2l} (f(x))^2\,dx = \frac{a_0^2}{2} + \sum (a_n^2 + b_n^2)$$

$$\frac{1}{\pi}\int_0^{2\pi} [f(x)]^2\,dx = \frac{1}{\pi}\int_0^{2\pi} \left(\frac{\pi - x}{2}\right)^4 dx$$

$$= \frac{1}{16\pi}\left[\frac{(\pi - x)^5}{-5}\right]_0^{2\pi}$$

$$= -\frac{1}{80\pi}\left[-\pi^5 - \pi^5\right] = \frac{\pi^4}{40}$$

sub. in $(*)$

$$\frac{\pi^4}{40} = \frac{\pi^4}{72} + \sum \frac{1}{n^4}$$

$$\frac{\pi^4}{40} - \frac{\pi^4}{72} = \sum \frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots$$

classmate

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Type 2 :- Interval $(0, 2l)$

$$f(x) = \frac{a_0}{2} + \Sigma a_n \cos\left(\frac{n\pi}{l}\right)x\,dx + \Sigma b_n \sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx \quad \Big| \quad a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

Q $f(x) = 4 - x^2$ in $(0, 2)$

Sol^n: $(0, 2l) \xrightarrow[l=1]{} (0, 2)$

$$f(x) = \frac{a_0}{2} + \Sigma a_n \cos(n\pi x) + \Sigma b_n \sin(n\pi x) \text{ ——— (1)}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \int_0^{2} \cancel{\frac{a_0}{2} + \Sigma a_n \cos}\,(4 - x^2)\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 = \left(8 - \frac{8}{3}\right) = \frac{16}{3}$$

$$\boxed{a_0 = \frac{16}{3}}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx = \int_0^{2}(4 - x^2)\cos(n\pi x)\,dx$$

$$= \cancel{\left[\left(4x - \frac{x^3}{3}\right)\right.} \left[(4 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

a

$$= \left[-\frac{2x \cos n\pi x}{n^2\pi^2} \right]_0^2$$

$$a_n = \frac{-4\cos 2n\pi}{n^2\pi^2} - 0$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \int_0^2 (4-x^2)\sin(n\pi x)\,dx$$

$$= \left[(4-x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-2x)\cancel{\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)}^{0} + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right) \right]_0^2$$

$$= \left[\left(0 - \frac{2\cos 2n\pi}{n^3\pi^3}\right) - \left[4\left(\frac{-\cos 0}{n\pi}\right) - 2\frac{\cos 0}{n^3\pi^3}\right]\right]$$

$$\boxed{b_n = \frac{4}{n\pi}}$$

$\therefore$ Fourier series for $f(x)$ is $f(x) = 4 - x^2 = \frac{8}{3} + \sum \frac{-4}{n^2\pi^2}\cos n\pi x$

$$+ \sum \frac{4}{n\pi}\sin n\pi x$$

Q $f(x) = \begin{cases} x & 0 < x < \frac{1}{2} \\ 1-x & \frac{1}{2} < x < 1 \end{cases}$

Solⁿ: $(0, 2l) \rightarrow (0, l)$

$l = \frac{1}{2}$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos(2n\pi x) + \sum b_n \sin(2n\pi x) \quad \text{--- (1)}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= 2\left[\int_0^{1/2} x\,dx + \int_{1/2}^{1} (1-x)\,dx\right]$$

$$= 2\left[\left[\frac{x^2}{2}\right]_0^{1/2} + \left[x - \frac{x^2}{2}\right]_{1/2}^{1}\right]$$

$$\cancel{a_0} = 2\left[\frac{1}{8} + \left(\left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{8}\right)\right)\right] = \boxed{\frac{1}{2} = a_0}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= 2\left[\int_0^{1/2} x\cos(2n\pi x)\,dx + \int_{1/2}^{1} (1-x)\cos(2n\pi x)\,dx\right]$$

$$= 2\left\{\left[x\left(\frac{\sin(2n\pi x)}{2n\pi}\right) - 1\left(\frac{-\cos(2n\pi x)}{4n^2\pi^2}\right)\right]_0^{1/2} + \left[(1-x)\left(\frac{\sin n\pi x}{2n\pi}\right) - (-1)\left(\frac{-\cos(2n\pi x)}{4n^2\pi^2}\right)\right]\right.$$

$$= 2\left\{\left(\frac{\cos n\pi}{4n^2\pi^2} - \frac{\cos 0}{4n^2\pi^2}\right) + \left(-\frac{\cos 2n\pi}{4n^2\pi^2} + \frac{\cos n\pi}{4n^2\pi^2}\right)\right\}$$

$$= 2\left\{\frac{2(-1)^n}{4n^2\pi^2} - \frac{2}{4n^2\pi^2}\right\}$$

$$a_n = \left\{\frac{(-1)^n - 1}{n^2\pi^2}\right\} = \begin{cases} 0 & n \text{ is even} \\ \frac{-2}{n^2} & n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$= 2\left\{\int_0^{1/2} x\sin(2n\pi x)\,dx + \int_{1/2}^{1}(1-x)\sin(2n\pi x)\,dx\right\}$$

$$= 2\left\{\left[x\left(\frac{-\cos 2n\pi x}{2n\pi}\right) - (1)\left(\cancel{\frac{-\sin 2n\pi x}{4n^2\pi^2}}\right)\right]_0^{1/2}\right.$$

$$\left. + \left[(1-x)\left(\frac{-\cos 2n\pi x}{2n\pi}\right) - (-1)\left(\frac{-\sin(2n\pi x)}{4n^2\pi^2}\right)\right]_{1/2}^{1}\right\}$$

$$= 2\left\{\left[\frac{1}{2}\left(\frac{-\cos n\pi}{2n\pi} - 0\right)\right] + \left[0 - \frac{1}{2}\left(\frac{-\cos n\pi}{2n\pi}\right)\right]\right\}$$

$$\boxed{b_n = 0}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2\pi^2}\cos(2n\pi x)$$

Type-3 : $(-l, l) \,/\, (-\pi, \pi)$

$$\int_{-l}^{l} \text{even } f^n = 2\int_{0}^{l} \text{even } f^n \quad \int_{-l}^{l} \text{odd } f^n = 0$$

Case-① : $f(x)$ is even

a_0, a_n, b_n

Then $b_n = 0$

Case-② : $f(x)$ is odd

$a_0 = a_n = 0$

Case-③ : neither odd ~~nor~~ even

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Ex: $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & 0 < x < \pi \end{cases}$

Sol^n: $f(-x) = \begin{cases} 1 - \frac{2x}{\pi} & -\pi < -x < 0 \\ 1 + \frac{2x}{\pi} & 0 < -x < \pi \end{cases}$

$= \begin{cases} 1 - \frac{2x}{\pi} & \pi > x > 0 \\ 1 + \frac{2x}{\pi} & 0 > x > -\pi \end{cases}$

$= \begin{cases} 1 + \frac{2x}{\pi} & -\pi < x < 0 \\ 1 - \frac{2x}{\pi} & 0 < x < \pi \end{cases} = f(x)$

$\therefore$ $f(x)$ is an even f^n $\therefore b_n = 0$

$\therefore$ Fourier series: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi-\frac{\pi^2}{\pi}-0\right]$$

$$= 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\underbrace{\left(\frac{\sin nx}{n}\right)}_{0} - \left(-\frac{2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{-4}{n^2\pi^2}\left[\cos n\pi - \cos 0\right] = \frac{-4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

when n = even

0

when n = odd

$$\frac{8}{n^2\pi^2}$$

$\therefore$ fourier series $\Rightarrow f(x) = \cancel{\frac{8}{n^2\pi^2}} \sum_{n=1}^{\infty} \frac{-4((-1)^n - 1)}{n^2\pi^2} \cos nx$

$$= \frac{8}{1^2\pi^2}\cos x + \frac{8}{3^2\pi^2}\cos 3x + \cdots$$

Q fourier series for $x \cos x$ in $(-\pi, \pi)$

Sol^n: $f(-x) = (-x)\cancel{(\cos x)}(\cos(-x))$

$= -x\cos x$

$= -f(x)$

$\therefore f(x)$ odd f^n

$\therefore a_0 = a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} x\cos x \sin nx \, dx$$

$$\sin A \cos B = \frac{\sin(A+B) + \sin(A-B)}{2}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\left[\frac{\sin(n+1)x + \sin(n-1)x}{2}\right] dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(n+1)x \, dx + \int_0^{\pi} x\sin(n-1)x \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos(n+1)x}{n+1}\right) - (1)\cancel{\left(\frac{-\sin(n+1)x}{(n+1)^2}\right)}^{0}\right]_0^{\pi}\right.$$

$$\left. + \left[x\left(\frac{-\cos(n-1)x}{n-1}\right) - (1)\left(\frac{-\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi \cos(n+1)\pi}{n+1} - \frac{\pi \cos(n-1)\pi}{n-1}\right] \quad n \neq 1$$

$$b_n = -\left[\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right]$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right]$$

$$\cos n\pi = (-1)^n$$

$$= -(-1)^{n-1}\left[\frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= (-1)^n \left[\frac{n-1+n+1}{n^2-1}\right]$$

$$b_n = \frac{2n(-1)^n}{n^2-1} \quad n \neq 1$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin x \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos x \sin x \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \left(\frac{\sin 2x}{2}\right) dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi(-\cos 2\pi)}{2}\right]$$

$$b_1 = \frac{-1}{2}$$

$$= b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

$$f(x) = x\cos x = \sum b_n \sin nx = \frac{-1}{2}\sin x + \sum \frac{2n(-1)^n}{n^2-1} \sin nx$$

Q $f(x) = \begin{cases} x - \pi & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$

$$\cancel{f(-x) = \begin{cases} \pi - x \;\; (-x-\pi) & -\pi < -x < 0 \\ x - \pi & 0 < -x < \pi \end{cases}}$$

$$\cancel{= \begin{cases} \pi - x & \pi > x > 0 \\ x - \pi & 0 > x > -\pi \end{cases}}$$

$$\cancel{= \begin{cases} \pi - x \\ x - \pi \end{cases} = \begin{cases} x - \pi & 0 > x > -\pi \\ \pi - x & \pi > x > 0 \end{cases} \neq f(x)}$$

$$f(-x) = \begin{cases} -x - \pi \;\; (-x-\pi) & -\pi < -x < 0 \;\; (\pi > x > 0) \\ \pi - (-x) & 0 < -x < \pi \\ \pi + x & 0 > x > -\pi \end{cases}$$

$\left.\begin{matrix} \neq f(x) \\ \neq -f(x) \end{matrix}\right\}$ neither even nor odd

$$f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x-\pi)\,dx + \int_{0}^{\pi}(\pi-x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}-\pi x\right)_{-\pi}^{0} + \left(\pi x - \frac{x^2}{2}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[0 - \left(\frac{\pi^2}{2} - -(\pi^2)\right] + \left[\left(\pi^2 - \frac{\pi^2}{2}\right) - 0\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{-3\pi^2}{2} + \frac{\pi^2}{2}\right] = -\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(x-\pi)\cos nx\,dx + \int_{0}^{\pi}(\pi-x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(x-\pi)\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[(\pi-x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2}(\cos 0 - \cos n\pi) - \frac{1}{n^2}(\cos n\pi - \cos 0)\right]$$

$$= \frac{2}{\pi n^2}\left[1-(-1)^n\right] = \begin{cases} \frac{4}{\pi n^2} & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{\pi}^{0}(x-\pi)\sin nx\,dx + \int_{0}^{\pi}(\pi-x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(x-\pi)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[(\pi-x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}$$

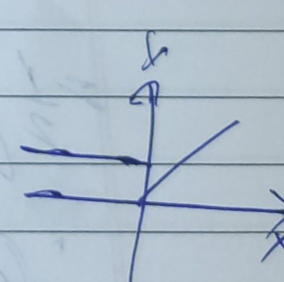
29-08-23

Q $f(x) = \begin{cases} 2 & -2<x<0 \\ x & 0<x<2 \end{cases}$

Sol^n: $(-l, l) \equiv (-2, 2)$
$l = 2$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi}{l}\right)x + \sum b_n \sin\left(\frac{n\pi}{l}\right)x$$

$$= \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi}{2}\right)x + \sum b_n \sin\left(\frac{n\pi}{l}\right)x$$



$$f(-x) = \begin{cases} 2 & -2<-x<0 & 2>x>0 \\ -x & 0<-x<2 & 0>x>-2 \end{cases}$$

$\neq f(x)$

$\neq -f(x)$ neither even nor odd

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{2}\left[\left[2x\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2}\right] = \frac{1}{2}\left[0-(-4)+\left(\frac{4}{2}-0\right)\right] = \frac{1}{2}\times 6 = 3$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\cos\left(\frac{n\pi}{2}\right)x\,dx$$

$$= \frac{1}{2}\left\{\int_{-2}^{0} 2\cos\left(\frac{n\pi}{2}\right)x\,dx + \int_{0}^{2} x\cos\left(\frac{n\pi}{2}\right)x\,dx\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{2\sin\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)}\right]_{-2}^{0} + \left[x\,\frac{\sin\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)} - (1)\left(\frac{-\cos\left(\frac{\pi}{2}\right)x}{\frac{n^2\pi^2}{4}}\right)\right]_{0}^{2}\right\}$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\left(\cos n\pi - \cos 0\right)\right]$$

$$\therefore \boxed{a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]} = \begin{cases} 0 & \text{even} \\ -\frac{4}{n^2\pi^2} & \text{odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi}{l}\right)x\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} f(x)\sin\left(\frac{n\pi}{2}\right)x\,dx$$

$$= \frac{1}{2}\left[2\left[\frac{-\cos\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)}\right]_{-2}^{0} + \left[x\left(\frac{-\cos\left(\frac{n\pi}{2}\right)x}{\left(\frac{n\pi}{2}\right)}\right) - (1)\left(\frac{-\sin\left(\frac{n\pi}{2}\right)x}{\frac{n^2\pi^2}{4}}\right)\right]_0^2 \right\}$$

$$= \frac{1}{2}\left\{\left[\frac{-2\cos 0}{\left(\frac{n\pi}{2}\right)} + \frac{2\cos n\pi}{\left(\frac{n\pi}{2}\right)}\right] + \left[\frac{-2\cos n\pi}{\left(\frac{n\pi}{2}\right)} - 0\right]\right\}$$

$$b_n = \frac{-2}{n\pi}$$

$$f(x) = \frac{3}{2} + \sum \frac{2[(-1)^n - 1]}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)x + \sum \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)x$$

Q find F.S. for $f(x) = |x|$ in $(-3,3)$ Hence deduce $\sum \frac{1}{(2n-1)^x} = \frac{\pi^4}{96}$

Sol^n: $(-l, l) = (-3, 3)$

$l = 3$

$$f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi}{3}\right)x + \sum b_n \sin\left(\frac{n\pi}{3}\right)x$$

Now, $f(x) = |x|$ is an even function

$\therefore b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum a_n \cos\left(\frac{n\pi}{3}\right)x$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{3}\int_{-3}^{3} f(x)\,dx = \frac{2}{3}\int_0^3 f(x)\,dx$$

$|x| = f(x) = -x \quad x < 0$

$\qquad = x \quad x \geq 0$

$$= \frac{2}{3}\int_0^3 x\,dx = 3$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi}{l}\right)x\,dx = \frac{2}{3}\int_0^3 x\cos\left(\frac{n\pi}{3}\right)x\,dx$$

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$$\frac{1}{l}\int_{c}^{c+2l}[f(x)]^2 dx = \frac{a_0^2}{2}+\sum(a_n^2+b_n^2)$$

$$\cancel{=\frac{2}{3}\left[-3\sin\left(\frac{n\pi}{3}\right)x\,dx\right]_0^3}$$

$$=\frac{2}{3}\left\{x\left(\frac{\cancel{\sin\left(\frac{n\pi}{3}\right)x}}{\left(\frac{n\pi}{3}\right)}\right)-(1)\left(\frac{-\cos\left(\frac{n\pi}{3}x\right)}{\frac{n^2\pi^2}{9}}\right)\right\}_0^3$$

$$=\frac{2}{3}\left\{\frac{9}{n^2\pi^2}(\cos n\pi-\cos 0)\right\}$$

$$a_n=\frac{6}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$f(x)=\frac{3}{2}+\sum\frac{6}{n^2\pi^2}\left[(-1)^n-1\right]\cos\left(\frac{n\pi}{3}\right)x\,dx$$

Q. $f(x)=x|x|$ $\quad -3<x<3$

odd f^n

$$f(x)=\begin{cases}-x^2 & -3<x<0\\ x^2 & 0<x<3\end{cases}$$

$$a_0, a_n=0$$

$$b_n=\frac{1}{3}\int_{-3}^{3}f(x)\sin\left(\frac{n\pi}{3}\right)x\,dx$$

$$=\frac{2}{3}\int_0^3 x^2\sin\left(\frac{n\pi}{3}\right)x\,dx$$

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$$\Rightarrow \frac{1}{3}\int_{-3}^{3}[f(x)]^2dx = \frac{a_0^2}{2} + \sum a_n^2$$

$$LHS: \frac{1}{3}\int_{-3}^{3}[f(x)]^2dx = \frac{2}{3}\int_0^3 x^2dx = \frac{2}{3}\left(\frac{x^3}{3}\right)_0^3 = 6$$

$$6 = \frac{9}{2} + \sum \frac{6}{n^2\pi^2}\left((-1)^n - 1\right)^2$$

$$\frac{3}{2} = \frac{36}{\pi^4 n^4}\left[\left((-1)^n - 1\right)\right]^2$$

$$\Rightarrow \frac{\pi^4}{24} = \frac{4}{1^4} + \frac{4}{3^4} + \frac{4}{5^4} + \cdots$$

$$\Rightarrow \frac{\pi^4}{96} = \sum \frac{1}{(2n-1)^4}$$