

# RELATIONS, DIGRAPHS

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# RELATIONS, DIGRAPHS (07)

- 3.1 Relations, Paths and Digraphs
- 3.2 Properties and types of binary relations
- 3.3 Manipulation of relations, Closures, Warshall's algorithm
- 3.4 Equivalence relations



# INTRODUCTION

**Definition:** A binary relation from a set  $A$  to a set  $B$  is a subset  $R \subseteq A \times B = \{ (a,b) \mid a \in A, b \in B \}$

- Let  $A$  and  $B$  be nonempty sets. A relation  $R$  from  $A$  'to'  $B$  is a subset of  $A \times B$ .
- If  $R \subseteq A \times B$  and  $(a, b) \in R$ , we say that  $a$  'is related to'  $b$  by  $R$ , and we also write  $a R b$ .
- If  $a$  is not related to  $b$  by  $R$ , we write  $a \not R b$ .
- Frequently,  $A$  and  $B$  are equal. In this case, we often say that  $R \subseteq A \times A$  'is a relation on'  $A$ , instead of a relation from  $A$  to  $A$ .



# EXAMPLES

## Ex. 1 :

Let  $A = \{ 1, 2, 3 \}$  and  $B = \{ r, s \}$

Then  $R = \{(1, r), (2, s), (3, r)\}$  is a relation from A to B.

## Ex. 2 :

Let  $A = \{ 1, 2, 3, 4, 5 \}$ .

Define the following relation R (less than) on A :

$a R b$  if and only if  $a < b$ .

Then  $R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$



# DEFINITIONS

- Let  $\{ A_1, A_2, \dots, A_n \}$  be a finite collection of sets. A subset  $R$  of  $A_1 \times A_2 \times \dots \times A_n$  is called an **n-ary relation** on  $A_1, A_2, \dots, A_n$ .
- If  $R = \phi$ , then  $R$  is called **void** or **empty relation**.
- If  $R = A_1 \times A_2 \times \dots \times A_n$ , then  $R$  is called the **universal relation**.
- If  $A_i = A$  for all  $i$ , then  $R$  is called an ' $n$  - ary relation on  $A$ '.
- If  $n=1, 2$  or  $3$ , then  $R$  is called a **unary**, **binary** or **ternary** relation respectively.
- Among the relations, binary relations are the most important being widely used in various applications.



# SET ARISING FROM RELATIONS

## Domain of Relation $R$ :

Let  $R \subseteq A \times B$  be a relation from  $A$  to  $B$ . The **domain** of  $R$ , denoted by **Dom ( $R$ )**, is the set of elements in  $A$  that are related to some element in  $B$ . In other words,  $\text{Dom } (R)$ , a subset of  $A$ , is the set of all first elements in the pairs that make up  $R$ .

## Range of relation $R$ :

Similarly, we define the **range** of  $R$ , denoted by **Ran ( $R$ )**, to be the set of elements in  $B$  that are second elements of pairs in  $R$ , that is, all elements in  $B$  that are related to some element in  $A$ .



# EXAMPLES

**Ex. 1 :**

Let  $A = \{ 1, 2, 3 \}$ ,  $B = \{ r, s \}$

and  $R = \{(1, r), (2, s), (3, r)\}$

$\text{Dom } (R) = \{ 1, 2, 3 \} = A$

$\text{Ran } (R) = \{ r, s \} = B$

**Ex. 2 :**

Let  $A = \{ 1, 2, 3, 4, 5 \}$ ,  $B = \{ 1, 2, 3, 4, 5 \}$

$a R b$ , if and only if  $a < b$

$R = \{(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

$\text{Dom } (R) = \{ 1, 2, 3, 4 \}$

$\text{Ran } (R) = \{ 2, 3, 4, 5 \}$



# REPRESENTATION OF RELATION

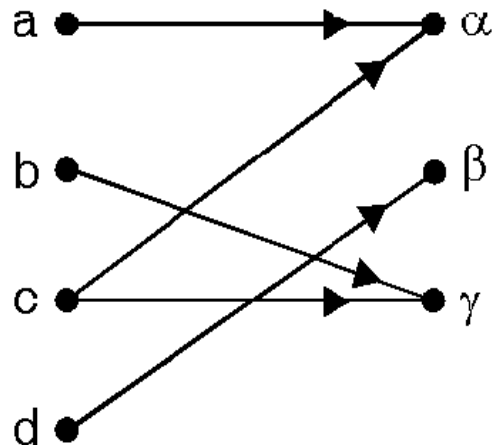
**Graphical, Tabular and Matrix forms :**

**For Example :**

Let  $A = \{a, b, c, d\}$ ,  $B = \{\alpha, \beta, \gamma\}$   
and  $R$  is a relation from  $A$  to  $B$ .

$$R = \{(a, \alpha), (b, \gamma), (c, \alpha), (c, \gamma), (d, \beta)\}$$

	$\alpha$	$\beta$	$\gamma$
a	$\checkmark$		
b			$\checkmark$
c	$\checkmark$		$\checkmark$
d		$\checkmark$	



$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$





# REPRESENTATION OF RELATION



# DIAGRAPH

If  $A$  is a finite set and  $R$  is a relation on  $A$ , we can also represent  $R$  pictorially as follows :

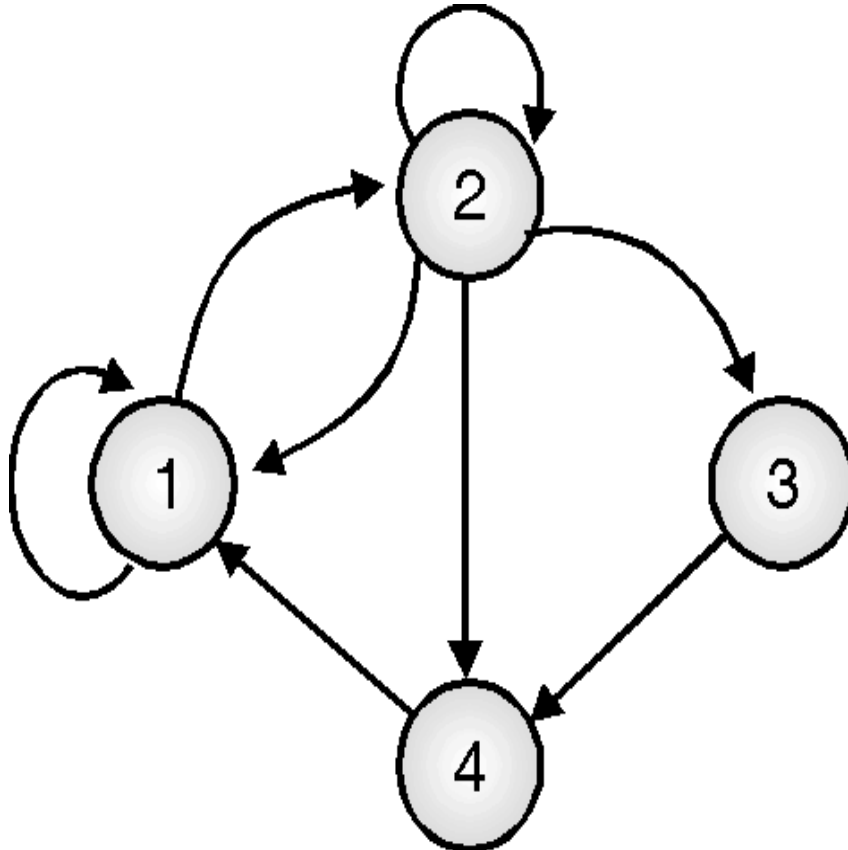
- (i) Draw a small circle for each element of  $A$  and label the circle with the corresponding element of  $A$ . These circles are called **vertices**.
- (ii) Draw an arrow, called an **edge**, from vertex  $a_i$  to vertex  $a_j$  if and only if  $a_i R a_j$ .

The resulting pictorial representation of  $R$  is called a **directed graph** or **digraph** of  $R$ .



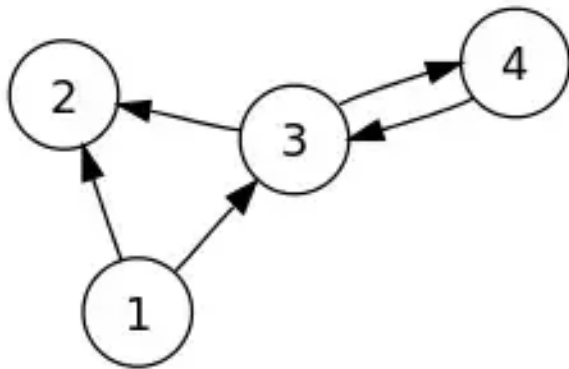
# DIAGRAPH

**Ex. 1 :** Let  $A = \{1, 2, 3, 4\}$ , Let  $R$  is a relation from  $A$  to  $A$ .  
 $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (2, 4), (3, 4), (4, 1)\}$



# DEGREE OF VERTEX IN A DIRECTED GRAPH

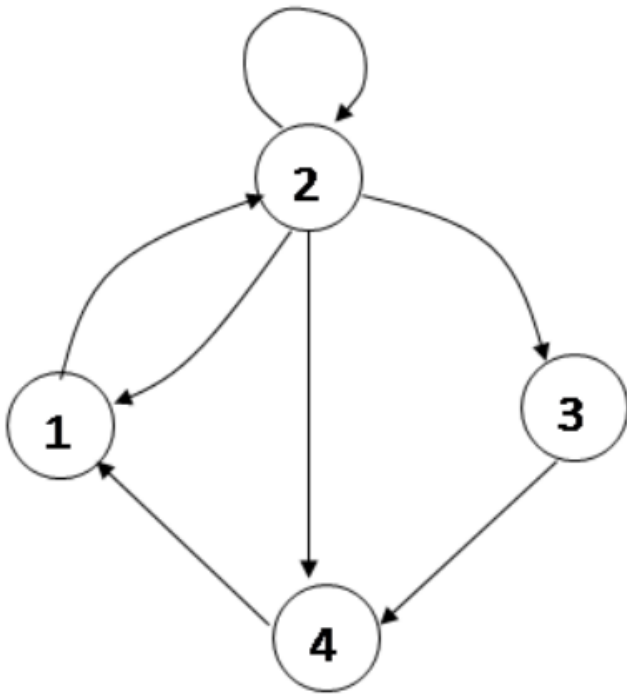
- A directed graph, each vertex has an in-degree and an out-degree.
- In-degree of a Graph-Number of edges which are coming into the vertex  $V$ .
- Out-degree of a Graph-Number of edges which are going out from the vertex  $V$



VERTEX	1	2	3	4
In Degree	0	2	2	1
Out-degree	2	0	2	1



# FIND OUT IN DEGREE AND OUT DEGREE



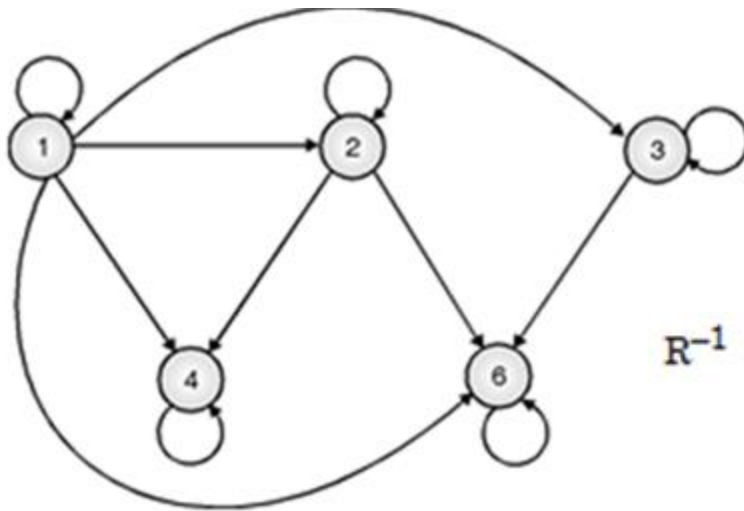
VERT EX	1	2	3	4
In Degree	2	2	1	2
Out- degree	1	4	1	1



# EXAMPLE

Let  $A = \{1, 2, 3, 4, 6\}$  and let  $R$  be the relation on  $A$  defined by 'x divides y'. Find  $R$  and draw the digraph of  $R$ . Find Matrix of  $R$ . Find inverse relation of  $R$ .

Soln.:  $R = \{(1,1), (1,2), (1,3), (1,4), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (6,6)\} \cup \{(4,4)\}$



$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$R^{-1} = \{(1,1), (2,1), (3,1), (4,1), (6,1), (2,2), (4,2), (6,2), (3,3), (6,3), (6,6)\} \cup \{(4,4)\}$$

Obtain  $M_{R^{-1}}$

# EXAMPLE

Let  $A = \{1, 2, 3, 4, 6\} = B$ ,  $a R b$  if and only if  $a$  is a multiple of  $b$ . Find  $R$  and draw the digraph of  $R$ . Find Matrix of  $R$ . Find each of the following :

- (i)  $R(3)$       (ii)  $R(6)$       (iii)  $R(\{2, 4, 6\})$

Solution:

$$R = \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (6, 1), (6, 2), (6, 3), (6, 6)\}$$

$$\text{Dom } (R) = \{1, 2, 3, 4, 6\}$$

$$\text{Ran } (R) = \{1, 2, 3, 4, 6\}$$

$$M_R = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 6 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$(i) R(3) = \{1, 3, 6\},$$

Since  $(3, 1) \in R$ ,  $(3, 3) \in R$  and  $(6, 3) \in R$

$$(ii) R(6) = \{1, 2, 3, 6\},$$

Since  $(6, 1) \in R$ ,  $(6, 2) \in R$ ,  $(6, 3) \in R$  and  $(6, 6) \in R$ .

$$(iii) R(\{2, 4, 6\}) = \{1, 2, 4, 3, 6\}$$

Since  $(2, 1) \in R$ ,  $(4, 2) \in R$ ,  $(6, 1) \in R$ ,  $(6, 2) \in R$ ,  
 $(6, 3) \in R$ ,  $(6, 6) \in R$ ,  $(4, 4) \in R$



# PATHS IN RELATIONS AND DIGRAPHS

Suppose that  $R$  is a relation on a set  $A$ . A **path of length  $n$**  in  $R$  from  $a$  to  $b$  is a finite sequence  $\pi : a, x_1, x_2, \dots, x_{n-1}, b$ , beginning with  $a$  and ending with  $b$ , such that

$$a R x_1, x_1 R x_2, \dots, x_{n-1} R b$$

Note that a path of length  $n$  involves  $n + 1$  elements of  $A$ , although they are not necessarily distinct.

The **length** of a path is the number of edges in the path, where the vertices need not all be distinct.

A path that begins and ends at the same vertex is called a **cycle**.





# PATHS IN RELATIONS AND DIGRAPHS

$R = \{ (1, 2), (2, 3), (2, 4), (3, 3) \}$  is a relation  
on  $A = \{1, 2, 3, 4\}$

$$R^1 = R = \{(1, 2), (2, 3), (2, 4), (3, 3)\}$$

$$R^2 = \{(1, 3), (1, 4), (2, 3), (3, 3)\}$$

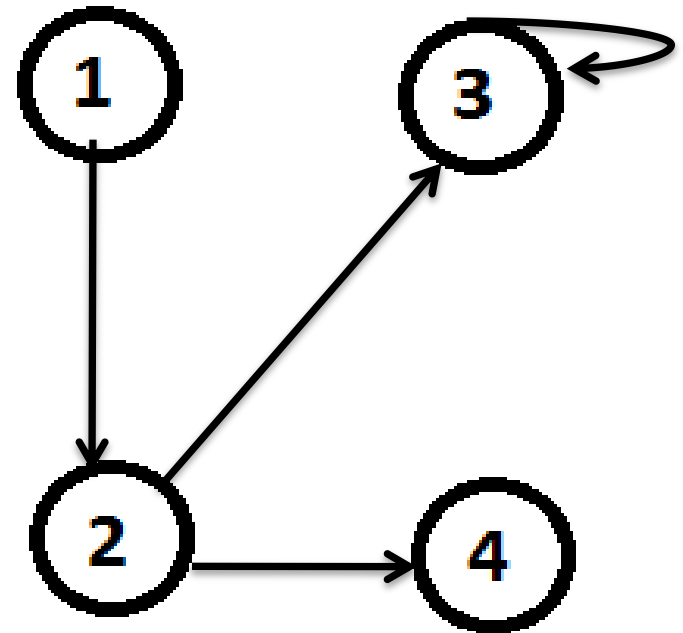
$1 R^2 3$  Since:  $1 R 2$  and  $2 R 3$

$1 R^2 4$  Since:  $1 R 2$  and  $2 R 4$

$$R^3 = \{(1, 3), (2, 3), (3, 3)\}$$

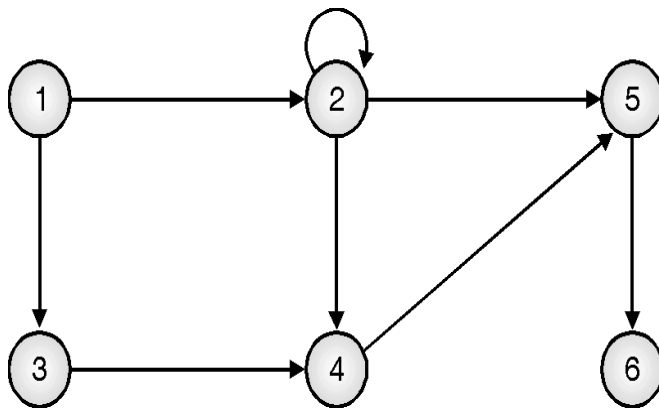
$$R^4 = \{(1, 3), (2, 3), (3, 3)\}$$

Obtain  $R^\infty$

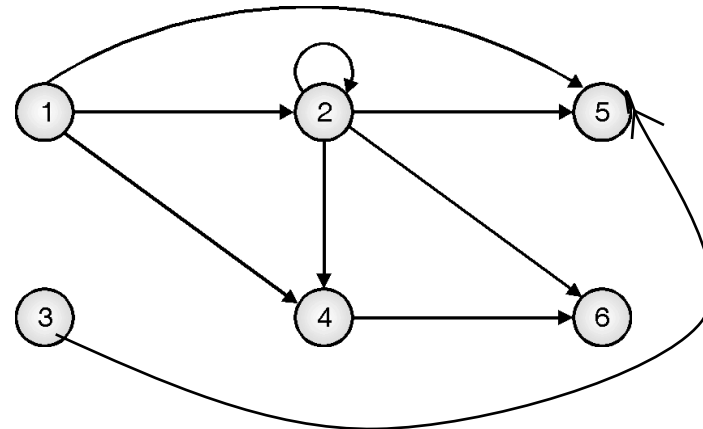


# PATHS IN RELATIONS AND DIGRAPHS

Let  $A = \{1, 2, 3, 4, 5, 6\}$ . Let  $R$  be the relation whose digraph is shown in Fig. Find  $R^2$  and draw digraph of the relation  $R^2$ .



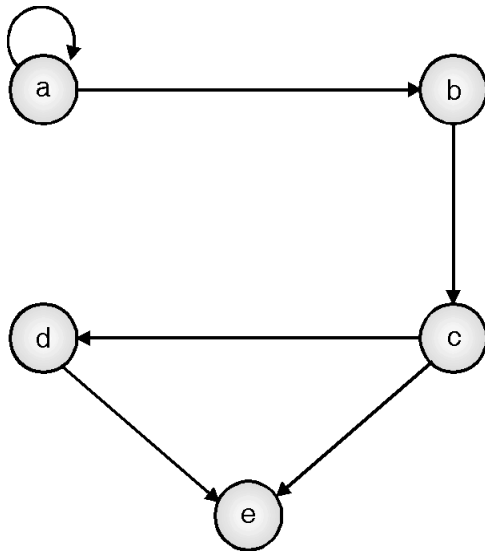
$1 R^2 2$	Since	$1 R 2$	and	$2 R 2$
$1 R^2 4$	Since	$1 R 2$	and	$2 R 4$
$1 R^2 5$	Since	$1 R 2$	and	$2 R 5$
$2 R^2 2$	Since	$2 R 2$	and	$2 R 2$
$2 R^2 4$	Since	$2 R 2$	and	$2 R 4$
$2 R^2 5$	Since	$2 R 2$	and	$2 R 5$
$2 R^2 6$	Since	$2 R 5$	and	$5 R 6$
$3 R^2 5$	Since	$3 R 4$	and	$4 R 5$
$4 R^2 6$	Since	$4 R 5$	and	$5 R 6$



# PATHS IN RELATIONS AND DIGRAPHS

Let  $A = \{a, b, c, d, e\}$   
and  $R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$

Compute (i)  $R^2$  (ii)  $R^\infty$



$a R^2 a$  Since  $a R a$  and  $a R a$

$a R^2 b$  Since  $a R a$  and  $a R b$

$a R^2 c$  Since  $a R b$  and  $b R c$

$b R^2 e$  Since  $b R c$  and  $c R e$

$b R^2 d$  Since  $b R c$  and  $c R d$

$c R^2 e$  Since  $c R d$  and  $d R e$

$R^2 = \{(a, a), (a, b), (a, c), (b, e), (b, d), (c, e)\}$

$R^\infty = \{(a, a), (a, b), (a, c), (a, d), (a, e), (b, c), (b, d), (b, e), (c, d), (c, e), (d, e)\}$ .

# BOOLEAN PRODUCT

The 'Boolean product' of A and B, denoted  $A \odot B$  is the  $m \times n$  Boolean matrix.

$C = [C_{ij}]$  defined by

$$C_{ij} = \begin{cases} 1 & \text{if } a_{ik} = 1 \text{ and } b_{kj} = 1 \text{ for some } k, 1 \leq k \leq P \\ 0 & \text{otherwise} \end{cases}$$

$$M_R^2 = M_R \odot M_R$$

$$M_R^n = M_R \odot M_R \odot \dots \odot M_R \text{ (n factors)}$$



# PATHS IN RELATIONS AND DIGRAPHS

Let  $A = \{a, b, c, d, e\}$  and

$R = \{(a, a), (a, b), (b, c), (c, e), (c, d), (d, e)\}$

$$M_R^2 = M_R \odot M_R$$

$$M_R = \begin{matrix} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$M_R$	$M_R$	$M_R^2 = M_R \odot M_R$
$(i,k)$	$(k,j)$	$\odot M_R$
(a,a)	(a,a)	(a,a)
(a,b)	(b,c)	(a,c)
(b,c)	(c,d)(c,e)	(b,d)(b,e)
(c,d)	(d,e)	(c,e)
(c,e)		-
(d,e)		-
(a,a)	(a,b)	(a,b)

# PROPERTIES/TYPES OF RELATIONS

- Reflexive
- Symmetric
- Transitive
- Antisymmetric
- Asymmetric



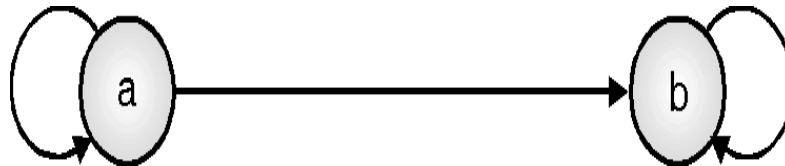
# PROPERTIES: REFLEXIVITY

A relation  $R$  on a set  $A$  is **reflexive** if for 'every' element  $a \in A$ ,  $a R a$ , i.e.  $(a, a) \in R$ .

$R$  is not a reflexive relation if for 'some' element  $a \in A$ ,  $(a, a) \notin R$

**Ex. 1 :** Let  $A = \{a, b\}$  and let  $R = \{(a, a), (a, b), (b, b)\}$ .

Then  $R$  is reflexive.



**Ex. 2 :** Let  $A = \{1, 2\}$  and let  $R = \{(1, 1), (1, 2)\}$ .

$R$  is not reflexive since  $(2, 2) \notin R$ .



# PROPERTIES: SYMMETRY

A relation  $R$  on a set  $A$  is **symmetric** if whenever  $a R b$ , then  $b R a$ . It then follows that  $R$  is not symmetric if we have some  $a$  and  $b \in A$  with  $a R b$ , but  $b \not R a$ .

Ex. 1 :  $A = \{ 1, 2, 3 \}$  , Is  $R$  symmetric ?

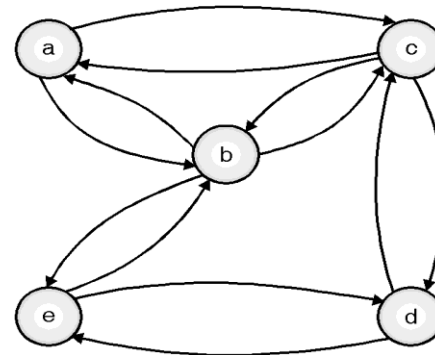
$R = \{ (1,2), (2,1), (2,3), (3,2), (1,1) \}$

Ex. 2 :  $A = \{ 1, 2, 3, 4 \}$  , Is  $R$  symmetric ?

$R = \{ (1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3) \}$

Ex. 3 :  $A = \{a, b, c, d, e\}$

$R = \{(a, b), (b, a), (a, c), (c, a), (b, c), (c, b), (b, e), (e, b), (e, d), (d, e), (c, d), (d, c)\}$





# PROPERTIES: ASYMMETRIC RELATION

A relation  $R$  on a set  $A$  is **asymmetric** if whenever  $(a,b) \in R$ , then  $(b, a) \notin R$ . It then follows that  $R$  is not asymmetric if we have some  $a$  and  $b \in A$  with both  $(a,b) \in R$  and  $(b, a) \in R$

## Examples :

1. Let  $A = \mathbb{R}$ , the set of real numbers and let  $R$  be the relation ' $<$ '. If  $a < b$ , then  $b \not< a$  ( $b$  is not less than  $a$ ), so ' $<$ ' is asymmetric.

2. Let  $A = \{1, 2, 3, 4\}$  and let

$R_1 = \{(1,1), (2,2), (3,3), (4,4)\}$  Reflexive - Yes, Symmetric- Yes, Asymmetric - No

$R_2 = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$

Then,  $R_2$  is not asymmetric, since  $(2, 2) \in R$ .

3. Let  $A = \mathbb{Z}^+$ , the set of positive integers, and let

$R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$ .

If  $a = b = 3$ , say then  $a R b$  and  $b R a$ , so  $R$  is not asymmetric.

# ANTISYMMETRIC RELATIONS

A relation  $R$  on a set  $A$  is **antisymmetric** if whenever  $a R b$  and  $b R a$ , then  $a = b$ .

- ✓ If  $a = b$  then,  $a R b$  and  $b R a$  is **Antisymmetric**
- ✓ If  $a \neq b$  then  $(a,b) \notin R$  or  $(b,a) \notin R$  is **Antisymmetric**
- ✓ (If  $a \neq b$  then both  $a R b$  and  $b R a$  is **NOT Antisymmetric**)

The contrapositive of this definition is that  $R$  is antisymmetric if whenever  $a \neq b$ , then  $(a,b) \notin R$  or  $(b,a) \notin R$ .

It follows that  $R$  is not antisymmetric if we have  $a$  and  $b$  in  $A$ ,  $a \neq b$ , and both  $a R b$  and  $b R a$ .



# SYMMETRY VERSUS ANTISYMMETRY

- In a symmetric relation  $aRb \Leftrightarrow bRa$
- In an antisymmetric relation, if we have  $aRb$  and  $bRa$  hold only when  $a = b$

- Example:  $A = \{1, 2, 3\}$

$R_1 = \{(1, 2), (2, 2), (2, 1)\}$

——> Symmetric- Yes

——> Antisymmetric- No

$R_2 = \{(1, 2), (2, 2), (1, 3)\}$

——> Symmetric- No

——> Antisymmetric- Yes



# SYMMETRY VERSUS ANTISYMMETRY

- An antisymmetric relation is not necessarily a reflexive relation

$$A = \{1, 2, 3\}$$

$$R_1 = \{(1, 1), (2, 2), (3, 3)\}$$

$$R_2 = \{(1, 1), (2, 2)\}$$

Type of Relation	R <sub>1</sub>	R <sub>2</sub>
Antisymmetric	Yes	Yes
Reflexive	Yes	No
Symmetric	Yes	Yes



# SYMMETRY VERSUS ANTISYMMETRY

- A relation that is not symmetric is not necessarily asymmetric

$$A=\{1,2,3\}$$

$$R=\{(1,2),(2,2)\}$$

Type of Relation	R
Symmetric	No
Asymmetric	No



## EXAMPLES

**Ex. :** Let  $A = \mathbb{Z}$ , the set of integers, and let  $R = \{(a, b) \in A \times A \mid a < b\}$  Is  $R$  symmetric, asymmetric, or antisymmetric ?

**Soln.:**

Symmetry : If  $a < b$ , then it is not true that  $b < a$ , so  $R$  is **not symmetric**.

Asymmetry : If  $a < b$ , then  $b \not< a$  ( $b$  is not less than  $a$ ), so  $R$  is **asymmetric**.

Antisymmetry : If  $a \neq b$ , then either  $a < b$  or  $b < a$ , so that  $R$  is **antisymmetric**.



# EXAMPLES

**Ex. :** Let  $A = \{1, 2, 3\}$  and let  $R = \{(1, 2), (2, 1), (2, 3)\}$ . Is  $R$  symmetric, asymmetric, or antisymmetric?

**Soln.:**

Symmetry :  $R$  is not symmetric either since  $(2, 3) \in R$   
but  $(3, 2) \notin R$

Asymmetry :  $R$  is also not asymmetric since both  $(1, 2)$  and  $(2, 1) \in R$ .

Antisymmetry :  $R$  is not antisymmetric since  $(1, 2)$  and  $(2, 1) \in R$ .



# EXAMPLES

**Ex. :** Let  $A = \{1, 2, 3, 4\}$  and let  $R = \{(1, 2), (2, 2), (3, 4), (4, 1)\}$ . Is  $R$  symmetric, asymmetric, or antisymmetric?

**Soln.:**

Symmetry :  $R$  is not symmetric, since  $(1, 2) \in R$ , but  $(2, 1) \notin R$ .

Asymmetry :  $R$  is not asymmetric, since  $(2, 2) \in R$ .

Antisymmetry :  $R$  is antisymmetric, since if  $a \neq b$ , either  $(a, b) \notin R$  or  $(b, a) \notin R$ .





## PROPERTIES: TRANSITIVITY

Definition: A relation  $R$  on a set  $A$  is called transitive if whenever  $(a,b) \in R$  and  $(b,c) \in R$  then  $(a,c) \in R$  for all  $a,b,c \in A$ .

$$\forall a,b,c \in A ((aRb) \wedge (bRc)) \Rightarrow aRc$$

Example

Let  $A = \mathbb{Z}^+$ , the set of positive integers, and let  $R = \{(a, b) \in A \times A \mid a \text{ divides } b\}$  Is  $R$  transitive?

Soln.:  $a$  divides  $b$ ,  $aRb$  and  $b$  divides  $c$ ,  $bRc$   
 $a$  divides  $c$ ,  $aRc$ . Thus  $R$  is transitive.



## SPECIAL CASES

1) Let  $A = \{ 1, 2, 3, 4 \}$

$$R = \{ (1, 2), (1, 3), (4, 2) \}$$

Is  $R$  transitive?

YES

2)  $R = \{ \}$

3) A relation that is symmetric and anti-symmetric

$R = \{(1,1), (2,2)\}$  on the set  $A = \{1,2,3\}$



## EXAMPLE

Give examples of relations  $R$  on  $A = \{1, 2, 3\}$  having the stated property.

- (i)  $R$  is transitive but not symmetric.
- (ii)  $R$  is symmetric but not transitive.
- (iii)  $R$  is both symmetric and anti-symmetric.
- (iv)  $R$  is neither symmetric nor anti-symmetric.

Solution:

- i.  $R = \{(1, 2), (2, 3), (1, 3)\}$
- ii.  $R = \{(1, 2), (2, 1)\}$
- iii.  $R = \{(1, 1), (2, 2)\}$
- iv.  $R = \{(1, 2), (2, 3), (3, 2)\}$



## EXAMPLE

Define a relation on the set  $\{a, b, c, d\}$  that is

- (i) transitive, reflexive and symmetric,
- (ii) symmetric and transitive.

Solution:

- (i) Transitive, reflexive and symmetric,

$$A = \{a, b, c, d\}$$

$$R = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, a), (a, c), (c, a), (a, d), (d, a), (b, c), (c, b), (b, d), (d, b), (c, d), (d, c)\}$$

- (ii) Symmetric and transitive.

$$A = \{a, b, c, d\}$$

$$R = \{(a, b), (b, a), (c, d), (d, c), (a, a), (c, c)\}$$



# IRREFLEXIVE RELATIONS

A relation  $R$  on a set  $A$  is **irreflexive** if a not related to  $a$ , i.e.  $(a,a) \notin R$  for every  $a \in A$ . Thus  $R$  is irreflexive if no element is related to itself.

## Examples

1. Let  $A = \{1, 2\}$  and let  $R = \{(1, 2), (2, 1)\}$ .

$R$  is not reflexive  $(1,1) (2,2) \notin R$

Then  $R$  is irreflexive since  $(1, 1) (2, 2) \notin R$ .

2. Let  $A = \{1, 2\}$  and let  $R = \{(1, 2), (2, 2)\}$ .

Then  $R$  is not irreflexive since  $(2, 2) \in R$ .

Note:  $R$  is not reflexive either; since  $(1, 1) \notin R$ .



# IDENTITY RELATION

Identity relation  $I$  on set  $A$  is reflexive, transitive and symmetric.

**Example:**

$$A = \{1, 2, 3\}$$

$$R = \{(1, 1), (2, 2), (3, 3)\}$$



# VOID RELATION

It is given by  $R: A \rightarrow B$  such that  $R = \emptyset (\subseteq A \times B)$  is a null relation.

Void Relation  $R = \emptyset$  is **symmetric and transitive but not reflexive.**



# UNIVERSAL RELATION

A relation  $R: A \rightarrow B$  such that  $R = A \times B (\subseteq A \times B)$  is a universal relation.

Universal Relation from  $A \rightarrow B$  is **reflexive, symmetric and transitive.**

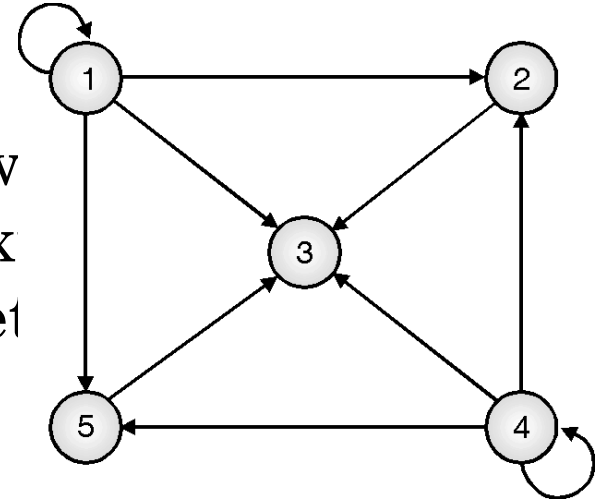




## EXAMPLE

Let  $A = \{1, 2, 3, 4, 5\}$

Determine whether the relation  $R$  w  
digraph is given is reflexive, irreflex  
symmetric, asymmetric, antisymmet  
transitive.



$R$  is not reflexive (See slide No. 23)

$R$  is not irreflexive (See slide No. 35)

$R$  is not symmetric (See slide No. 24)

$R$  is not asymmetric (See slide No. 25)

$R$  is antisymmetric (See slide No. 26)

$R$  is transitive (See slide No. 31)



## EXERCISE : PROPERTIES OF RELATIONS

State whether R satisfies property of reflexive ,  
irreflexive , symmetry, asymmetry , antisymmetry ,  
transitivity for  $A=\{1,2,3,4\}$

$$R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(4,3),(3,4),(4,4)\}$$

$$R= \{(1,3),(1,1),(3,1),(1,2),(3,3),(4,4)\}$$

$$R=\{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4)\}$$

$$R=\{(1,3),(1,4),(2,3),(2,4),(3,1),(3,4)\}$$

$$R=\{(1,1),(2,2),(3,3),(4,4)\}$$



# EQUIVALENCE RELATION

A relation is an Equivalence Relation if it is **reflexive, symmetric, and transitive.**

Let  $A = \{ a , b , c \}$  and

$R = \{ (a,a), (b,b), (b,c), (c,b), (c,c) \}$

is an equivalence relation since it is reflexive, symmetric, and transitive.



# DETERMINE WHETHER R IS AN EQUIVALENCE RELATION

Let  $A = \{a, b, c\}$  and let ,  $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Determine whether R is an equivalence relation.

**Soln.:**  $R = \{(a, a), (b, b), (b, c), (c, b), (c, c)\}$

R is reflexive since  $(a, a), (b, b), (c, c) \in R$

R is symmetric since  $(b, c) \in R \rightarrow (c, b) \in R$

R is transitive since,

$(b, b)$	and	$(b, c) \in R$	implies	$(b, c) \in R,$
$(b, c)$	and	$(c, b) \in R$	implies	$(b, b) \in R,$
$(c, c)$	and	$(c, b) \in R$	implies	$(c, b) \in R,$
$(c, b)$	and	$(b, b) \in R$	implies	$(c, b) \in R,$
$(c, b)$	and	$(b, c) \in R$	implies	$(c, c) \in R,$
$(b, c)$	and	$(c, c) \in R$	implies	$(b, c) \in R,$

Hence R is an equivalence relation.



# DETERMINE WHETHER $R$ IS AN EQUIVALENCE RELATION

Let  $A = \mathbb{Z}$ , the set of integers, and let  $R$  be defined by  $a R b$  if and only if  $a \leq b$ . Is  $R$  an equivalence relation?

- Since  $a \leq a$ ,  $R$  is reflexive.
- If  $a \leq b$ , it need not follow that  $b \leq a$ , so  $R$  is not symmetric.
- Incidentally,  $R$  is transitive, since  $a \leq b$  and  $b \leq c$  imply that  $a \leq c$ .
- We see that  $R$  is **not an equivalence relation**.



# DETERMINE WHETHER R IS AN EQUIVALENCE RELATION

Let  $A = \{1, 2, 3, 4\}$  and

Let  $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1), (3, 3), (1, 3), (4, 1), (4, 4)\}$

Determine whether the relation R on the set A is an equivalence relation.

**Soln.:**

R is reflexive since  $(1, 1), (2, 2), (3, 3), (4, 4) \in R$ . R is not symmetric since,  $(4, 1) \in R$  but  $(1, 4) \notin R$ .

R is not transitive since,

$(2, 1), (1, 3) \in R$  but  $(2, 3) \notin R$

Hence given relation R is not an equivalence relation.



Let R be a binary relation on the set of all positive integers such that,

$$R = \{(a, b) \mid a - b \text{ is an odd positive integer}\}$$

Is R reflexive ? Symmetric ? Transitive ? An equivalence relation ?

**Soln.:** R is not Reflexive since

$$a - a = 0 \neq \text{odd positive integer}$$

$$\therefore a \not R a$$

R is not symmetric also, since, as if  $a R b$  then

$$a - b = 2n + 1 \text{ where } n = \text{integer number}$$

if  $b R a$  then  $b - a = -2n - 1$  where  $n = \text{integer number}$

$$\therefore b - a \neq \text{odd positive integer}$$

R is not transitive since,

Let  $a R b$  and  $b R c$

$$\text{i.e. } a - b = 2n_1 + 1$$

$$b - c = 2n_2 + 1 \dots \text{i.e. odd positive integer}$$

$$a - c = (2n_1 + 1) + (2n_2 + 1) = 2(n_1 + n_2 + 1)$$

$$\neq \text{odd positive integer}$$

Hence R is not transitive.

Therefore R is not an equivalence relation.



# EQUIVALENCE CLASS AND PARTITIONS

Let  $A = \{ 1, 2, 3, 4 \}$  and consider the partition

$$P = \{ \{ 1, 2, 3 \}, \{ 4 \} \} \text{ of } A.$$

Find the equivalence relation  $R$  on  $A$  determined by  $P$

**Soln:** “ Each element in a block is related to every other element in the same block and only to those elements”

$$R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3), (4,4)\}$$





## PROBLEMS

Find the equivalence relation on A by P

1) Let  $A = \{a, b, c, d\}$  and  $P = \{\{a, b\}, \{c\}, \{d\}\}$

$R = \{(a,a), (a,b), (b,b), (b,a), (c,c), (d,d)\}$

2) Let  $A = \{1, 2, 3, 4, 5\}$  and  $P = \{\{1, 2\}, \{3\}, \{4, 5\}\}$

$R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (4,4), (4,5), (5,5), (5,4)\}$

3) If  $\{\{1, 3, 5\}, \{2, 4\}\}$  is a partition on the set  $A = \{1, 2, 3, 4, 5\}$ , determine the corresponding equivalence relation

$R = \{(1,1), (1,3), (1,5), (3,1), (3,3), (3,5), (5,1), (5,3), (5,5), (2,2), (2,4), (4,2), (4,4)\}$



# EQUIVALENCE CLASS

Let  $A = \{1, 2, 3, 4, 5, 6\}$  and let  $R$  be the equivalence relation on  $A$  defined by

$R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), (6, 3), (6, 6)\}$

Find the equivalence classes of  $R$  and find the partition of  $A$  induced by  $R$



$$R = \{(1,1), (1,5), (2,2), (2,3), (2,6), (3,2), (3,3), \\ (3,6), (4,4), (5,1), (5,5), (6,2), (6,3), (6,6)\}$$

Equivalence Classes:  $R(1)$  ,  $R(2)$  ,  $R(3)$  ,  $R(4)$  ,  $R(5)$  ,  $R(6)$ .

$$R(1) = \{1,5\}$$

$$R(2) = \{2,3,6\}$$

$$R(3) = \{2,3,6\}$$

$$R(4) = \{4\}$$

$$R(5) = \{1,5\}$$

$$R(6) = \{2,3,6\}$$

Therefore, the partition of  $A$  induced by  $R$  i.e

$$A | R = \{\{1,5\}, \{2,3,6\}, \{4\}\}$$

$$\text{Rank} = R = \text{Number of distinct equivalence classes} = 3$$



# PROBLEMS: FIND EQUIVALENCE CLASSES, PARTITION AND RANK

1. Let  $A=\{1,2,3\}$  and let  $R=\{(1,1),(2,2),(1,3),(3,1),(3,3)\}$ .

Find  $A | R$ .

Ans:  $R(1)=\{1,3\}$   $R(2)=\{2\}$   $R(3)=\{1,3\}$

$A | R = \{\{1,3\}, \{2\}\}$ , Rank=2

2. Let  $A = \{1,2,3,4\}$ , and let  $R=\{(1,1),(1,2),(2,1),(2,2),(3,4),(4,3),(3,3),(4,4)\}$

Determine  $A | R$ .

Ans:  $R(1)=\{1,2\}$   $R(2)=\{1,2\}$   $R(3)=\{3,4\}$   $R(4)=\{3,4\}$

$A | R = \{\{1,2\}, \{3,4\}\}$  Rank=2

3. Let  $A = \{1,2,3,4\}$ , and let

$R=\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1),(2,3),(3,2),(3,3),(4,4)\}$  Show that  $R$  is an equivalence relation and determine the equivalence classes and hence find the rank of  $R$

Ans:  $R(1)=\{1,2,3\}$   $R(2)=\{1,2,3\}$   $R(3)=\{1,2,3\}$   $R(4)=\{4\}$

$A | R = \{\{1,2,3\}, \{4\}\}$

Rank=2



# COMBINING RELATIONS

- Relations are simply... sets (of ordered pairs); subsets of the Cartesian product of two sets
- Therefore, in order to combine relations to create new relations, it makes sense to use the usual set operations
  - Compliment  $\bar{R}$
  - Intersection ( $R_1 \cap R_2$ )
  - Union ( $R_1 \cup R_2$ )
  - Set difference ( $R_1 \setminus R_2$ )
  - Inverse  $R^{-1}$



## EXAMPLES

Let  $A = \{1, 2, 3\}$  and  $B = \{u, v\}$  and

$R_1 = \{(1, u), (2, u), (2, v), (3, u)\}$  and

$R_2 = \{(1, v), (3, u), (3, v)\}$

$R_1 \cup R_2 = \{(1, u), (1, v), (2, u), (2, v), (3, u), (3, v)\}$

$R_1 \cap R_2 = \{(3, u)\}$

$R_1 - R_2 = \{(1, u), (2, u), (2, v)\}$

$R_2 - R_1 = \{(1, v), (3, v)\}$



## EXAMPLES

Let  $A = \{ 1, 2, 3, 4 \}$  and  $B = \{ a, b, c \}$  and let

$R = \{(1,a), (1,b), (2,b), (2,c), (3,b), (4,a)\}$  and

$S = \{(1,b), (2,c), (3,b), (4,b)\}$

Compute  $R \cap S$ ,  $R \cup S$ ,  $R^{-1}$

$R \cap S = \{(1,b), (2,c), (3,b)\}$

$R \cup S = \{(1,a), (1,b), (2,b), (2,c), (3,b), (4,a), (4,b)\}$

$R^{-1} = \{(a,1), (b,1), (b,2), (c,2), (b,3), (a,4)\}$



# COMBINING RELATIONS: EXAMPLE

Let

$$A = \{1, 2, 3, 4\}$$

$$B = \{1, 2, 3, 4\}$$

$$R_1 = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$$

$$R_2 = \{(1, 1), (1, 2), (1, 3), (2, 3)\}$$

$$R_1 \cup R_2 =$$

$$\{(1, 2), (1, 3), (1, 4), (2, 2), (3, 4), (4, 1), (4, 2), (1, 1), (2, 3)\}$$

$$R_1 \cap R_2 =$$

$$\{(1, 2), (1, 3)\}$$

$$R_1 - R_2 =$$

$$\{(1, 4), (2, 2), (3, 4), (4, 1), (4, 2)\}$$

$$R_2 - R_1 =$$

$$\{(1, 1), (2, 3)\}$$





# COMPOSITE OF RELATIONS

- **Definition:** Let  $R_1$  be a relation from the set  $A$  to  $B$  and  $R_2$  be a relation from  $B$  to  $C$ , i.e.

$$R_1 \subseteq A \times B \text{ and } R_2 \subseteq B \times C$$

the composite of  $R_1$  and  $R_2$  is the relation consisting of ordered pairs  $(a,c)$  where  $a \in A$ ,  $c \in C$  and for which there exists an element  $b \in B$  such that  $(a,b) \in R_1$  and  $(b,c) \in R_2$ . We denote the composite of  $R_1$  and  $R_2$  by

$$R_2 \circ R_1$$



# COMPOSITE OF RELATIONS

Ex: Let  $A = \{1,2,3\}$  ,  $B = \{0,1,2\}$  and  $C = \{a,b\}$

$R = \{(1,0),(1,2),(3,1),(3,2)\}$

$S = \{(0,b),(1,a),(2,b)\}$

$S \circ R = ?$

Since  $(1,0) \in R$  and  $(0,b) \in S$ ,  $\therefore (1,b) \in S \circ R$

Since  $(1,2) \in R$  and  $(2,b) \in S$ ,  $\therefore (1,b) \in S \circ R$

Since  $(3,1) \in R$  and  $(1,a) \in S$ ,  $\therefore (3,a) \in S \circ R$

Since  $(3,2) \in R$  and  $(2,b) \in S$ ,  $\therefore (3,b) \in S \circ R$

$$S \circ R = \{ (1, b), (3, a), (3, b) \}$$



# PROBLEMS

1) Let  $A=\{1,2,3\}$  and let

$R=\{(1,1),(1,3),(2,1),(2,2),(2,3),(3,2)\}$  and

$S=\{(1,1),(2,2),(2,3),(3,1),(3,3)\}$ .

Find  $\text{SoR}$  and  $M_{\text{SoR}}$

$\text{SoR}=\{(1,1),(1,3),(2,1),(2,2),(2,3),(3,2),(3,3)\}$

2) Let  $A=\{1,2,3,4\}$

$R=\{(1,1),(1,2),(2,3),(2,4),(3,4),(4,1),(4,2)\}$

$S=\{(3,1),(4,4),(2,3),(2,4),(1,1),(1,4)\}$

Compute  $\text{SoR}$ ,  $\text{RoS}$ ,  $\text{RoR}$ ,  $\text{SoS}$

$\text{SoR}=\{(1,1),(1,3),(2,1),(2,4),(3,4),(4,1),(4,4),(1,4)\}$

$\text{RoS}=\{(3,1),(3,2),(4,1),(4,2),(2,4),(2,1),(2,2),(1,1),(1,2)\}$

$\text{RoR}$

$\text{SoS}$



# CLOSURES

The 'smallest' relation  $R_1$  on  $A$  that contains  $R$  and possesses the property we desire. Sometimes  $R_1$  does not exist. If a relation such as  $R_1$  does exist, we call it the 'closure' of  $R$  with respect to the property in question.



# REFLEXIVE CLOSURE

Let  $R$  be a relation on a set  $A$ , and  $R$  is not reflexive (i.e. some pairs of the diagonal relation  $\Delta$  are not in  $R$ ).

A relation  $R_1 = R \cup \Delta$  is the reflexive closure of the relation  $R$  if  $R \cup \Delta$  is the smallest relation containing  $R$  which is reflexive.

$$R_1 = R \cup \Delta$$

where  $\Delta$  is the set of elements of the type  $(a, a)$   
where  $a \in A$ .



## EXAMPLE

$A = \{1, 2, 3\}$  and the relation  $R$  is given by

$R = \{(1, 1), (1, 2), (2, 3)\}$  then

$R_1 = R \cup \Delta$  where

$\Delta = \{(1, 1), (2, 2), (3, 3)\}$

$R \cup \Delta = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$

**Reflexive closure is,**

$$R_1 = \{(1, 1), (1, 2), (2, 2), (2, 3), (3, 3)\}$$



## SYMMETRIC CLOSURE

Suppose that  $R$  is a relation on  $A$  that is not symmetric. Then there must exist pairs  $(x, y)$  in  $R$  such that  $(y, x)$  is not in  $R$ . Of course,  $(y, x) \in R^{-1}$ , so if  $R$  is to be symmetric we must add all pairs from  $R^{-1}$ ; that is we must enlarge  $R$  to  $R \cup R^{-1}$ . Clearly  $(R \cup R^{-1})^{-1} = R \cup R^{-1}$ , So  $R \cup R^{-1}$  is the smallest symmetric relation containing  $R$ ; that is  $R \cup R^{-1}$  is the 'symmetric closure' of  $R$ .



## EXAMPLE

$A = \{a, b, c, d\}$  and

$R = \{(a, b), (b, c), (a, c), (c, d)\}$  then

$R^{-1} = \{(b, a), (c, b), (c, a), (d, c)\}$

so the symmetric closure of  $R$  is

$R \cup R^{-1} = \{(a, b), (b, a), (b, c), (c, b), (a, c), (c, a), (c, d), (d, c)\}$





# TRANSITIVE CLOSURE

Let  $R$  be a relation on a set  $A$ . Then the 'transitive closure' of a relation  $R$  is the smallest transitive relation containing  $R$ . The transitive closure of  $R$  is just the connectivity relation  $R^\infty$ .

$R^* = \text{Transitive closure of } R$

$$= R \cup \{(a, c), \text{ if and only if } (a, b), (b, c) \in R\}$$



## EXAMPLE

Find the transitive closure  $R^*$  of the relation  $R$  on  $A = \{1, 2, 3, 4\}$  defined by the directed graph shown

**Soln.:**

$R = \{(1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$

Here transitive closure of  $R$  is

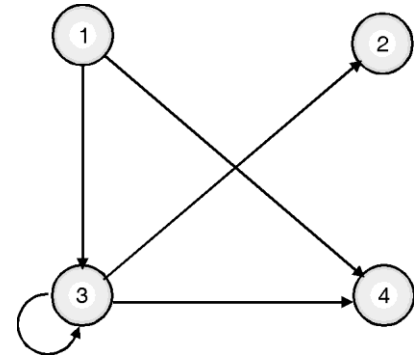
$= R \cup \{(a, c) \mid \text{if } (a, b), (b, c) \in R\}$

To find transitive closure

$(1, 3) \in R$  and  $(3, 4) \in R$ , hence add  $(1, 4)$  in  $R$

$(1, 3) \in R$  and  $(3, 3) \in R$ , hence add  $(1, 3)$  in  $R$

$(1, 3) \in R$  and  $(3, 2) \in R$ , hence add  $(1, 2)$  in  $R$



**Transitive closure of  $R = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4)\}$**

# MATRIX METHOD

Let  $A = \{1, 2, 3, 4\}$  and let  $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$ . Find the transitive closure of  $R$ . The matrix of  $R$  is

$$M_R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)_0^2 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (M_R)_0^3 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(M_R)_0^4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^2 = \{(1, 1), (1, 3), (2, 2), (2, 4)\}$$

$$R^3 = \{(1, 2), (1, 4), (2, 1), (2, 3)\}$$

$$R^4 = \{(1, 1), (1, 3), (2, 2), (2, 4)\}$$

$$R^\infty = R \cup R^2 \cup R^3 \cup R^4$$

$$R^\infty = \{(1, 1), (1, 2), (1, 3), (1, 4), \\ (2, 1), (2, 2), (2, 3), \\ (2, 4), (3, 4)\}$$

$$M_R^\infty = M_R \vee (M_R)_0^2 \vee (M_R)_0^3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# WARSHALL'S ALGORITHM

**Ex. 1 :** Let  $A = \{1, 2, 3, 4\}$  and let  $R = \{(1, 2), (2, 3), (3, 4), (2, 1)\}$ .

Find transitive closure of  $R$  using Warshall's algorithm.

Solution:

$$W_0 = M_R = \begin{bmatrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$

First we find  $W_1$ , so that  $k = 1$ .  $W_0$  has 1's in location 2 of column 1 i.e. (2, 1) and location 2 of row 1 i.e. (1, 2)

$i \quad j$   
 $p_1: (2, 1)$

$i \quad j$   
 $q_1: (1, 2)$

add  $(p_i, q_j)$  i.e. (2, 2) in  $W_k$

Thus  $W_1$  is just  $W_0$  with a new 1 in position (2, 2)

$$W_1 = \begin{bmatrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{bmatrix}$$



$$W_1 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Matrix  $W_1$  has 1's at row 1 and 2 of column 2 and columns 1, 2, and 3 of row 2. i.e.

$$\begin{array}{ll} \begin{matrix} i & j \\ p_1 & : (1, 2) \end{matrix} & \begin{matrix} i & j \\ p_2 & : (2, 2) \end{matrix} \\ \begin{matrix} i & j \\ q_1 & : (2, 1) \end{matrix} & \begin{matrix} i & j \\ q_2 & : (2, 2) \end{matrix} & \begin{matrix} i & 1 \\ q_3 & : (2, 3) \end{matrix} \end{array}$$

We must put 1's in positions  $(p_i, q_j)$  i.e.  $(1, 1)$ ,  $(1, 2)$ ,  $(1, 3)$ ,  $(2, 1)$ ,  $(2, 2)$  and  $(2, 3)$  of matrix  $W_1$  (if 1's are not already there).

$$W_2 = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} i & j & & i & j \\ p_1 & : (1, 3) & & p_2 & : (2, 3) \\ & & i & j \\ q_1 & : (3, 4) & & (1, 4) \text{ and } (2, 4) \end{matrix}$$

$$W_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finally,  $W_3$  has 1's in locations 1, 2, 3 of column 4 and no 1's in row 4, so no new 1's are added and  $MR_\infty = W_4 = W_3$ .



## EXAMPLE:

Let  $A = \{1,2,3,4,5\}$   $R = \{(1,1), (1,2), (2,1), (2,2), (3,3), (3,4), (4,3), (4,4), (5,5)\}$  and  $S = \{(1,1), (2,2), (3,3), (4,4), (4,5), (5,4), (5,5)\}$ . The reader may verify that both  $R$  and  $S$  are equivalence relations. The partition  $A \mid R$  of  $A$  corresponding to  $R$  is  $\{\{1,2\}, \{3,4\}, \{5\}\}$ , and the partition  $A \mid S$  of  $A$  corresponding to  $S$  is  $\{\{1\}, \{2\}, \{3\}, \{4,5\}\}$ . Find the smallest equivalence relation containing  $R$  and  $S$ , and compute the partition of  $A$  that it produces.



$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\text{So } M_{R \cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

We now compute  $M_{(R \cup S)^\infty}$  by Warshall's algorithm. First,  $W_0 = M_{R \cup S}$ . We next compute  $W_1$  so  $k = 1$ . Since  $W_0$  has 1's in locations 1 and 2 of column 1 and in locations 1 and 2 of row 1, we find that no new 1's must be adjoined to  $W_1$ . Thus

$$W_0 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

**K=1**

$$\begin{array}{cc} i & j \\ p_1 & : (1, 1) \quad p_2 : (2, 1) \end{array}$$

$$\begin{array}{cc} i & j \\ q_1 & : (1, 1) \quad q_2 : (1, 2) \end{array}$$

To obtain  $W_1$ , we must put 1's in positions (1, 1), (1, 2), (2, 1) and (2, 2). We see that

$$W_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus  $W_1 = W_0$

We now compute  $W_2$ , so  $k = 2$ .

Since  $W_1$  has 1's in locations 1 and 2 : of column 2 and in locations 1 and 2 of row 2, we find that no new 1's must be added to  $W_1$ . That is,

$$\begin{array}{cc} \begin{array}{c} i \quad j \\ p_1 : (1, 2) \end{array} & \begin{array}{c} i \quad j \\ p_2 : (2, 2) \end{array} \\ \\ \begin{array}{c} i \quad j \\ q_1 : (2, 1) \end{array} & \begin{array}{c} i \quad j \\ q_2 : (2, 2) \end{array} \end{array}$$

To obtain  $W_2$ , we must put 1's in positions (1, 1), (1, 2), (2, 1), (2, 2). We see that

$$W_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Thus  $W_2 = W_1$

We next compute  $W_3$ , so  $k = 3$ . Since  $W_2$  has 1's in locations 3 and 4 of column 3 and in locations 3 and 4 of row 3, we find that no new 1's must be added to  $W_2$ . That is

$$\begin{array}{cc} \begin{array}{c} i \quad j \\ p_1 : (3, 3) \end{array} & \begin{array}{c} i \quad j \\ p_2 : (4, 3) \end{array} \\ \\ \begin{array}{c} i \quad j \\ q_1 : (3, 3) \end{array} & \begin{array}{c} i \quad j \\ q_2 : (3, 4) \end{array} \end{array}$$

To obtain  $W_3$ , we must put 1's in position (3, 3), (3, 4), (4, 3), (4, 4). We see that

$$W_3 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$





Thus  $W_3 = W_2$

Things change when we now compute  $W_4$ . Since  $W_3$  has 1's in locations 3, 4, and 5 of column 4 and in locations 3, 4 and 5 of column 4, and in locations 3, 4 and 5 of row 4 we must add new 1's to  $W_3$  in positions 3, 5, and 5, 3, i.e.

$$\begin{array}{ccc} \begin{array}{c} i \quad j \\ p_1 : (3, 4) \end{array} & \begin{array}{c} i \quad j \\ p_2 : (4, 4) \end{array} & \begin{array}{c} i \quad j \\ p_3 : (5, 4) \end{array} \\ \begin{array}{c} i \quad j \\ q_1 : (4, 3) \end{array} & \begin{array}{c} i \quad j \\ q_2 : (4, 4) \end{array} & \begin{array}{c} i \quad j \\ q_3 : (4, 5) \end{array} \end{array}$$

To obtain  $W_4$ , we must put 1's in positions (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5). We see that,

$$W_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

You may verify that  $W_5 = W_4$  and thus

$$(R \cup S)^\infty = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3), (3, 4), (3, 5), (4, 3), (4, 4), (4, 5), (5, 3), (5, 4), (5, 5)\}$$

