MODULE 6
GRAPH THEORY

Varying Applications (examples)

- Computer networks
- Distinguish between two chemical compounds with the same molecular formula but different structures
- Solve shortest path problems between cities
- Scheduling exams and assign channels to television stations

Topics Covered

Graphs and Subgraphs **05** **CO4**

6.1 Definitions, Paths and circuits,
Types of Graphs,
Eulerian and Hamiltonian

6.2 Planer graphs

6.3 Isomorphism of graphs

6.4 Subgraph

Definitions - Graph

A **graph** G consists of a finite set V of objects called **vertices,** a finite set E of objects called **edges** and a function γ that assigns to each edge a subset {V, W} where v and w are vertices (and may be the same).

We will write G = (V, E, γ)

Graph

Let V={1, 2, 3, 4}

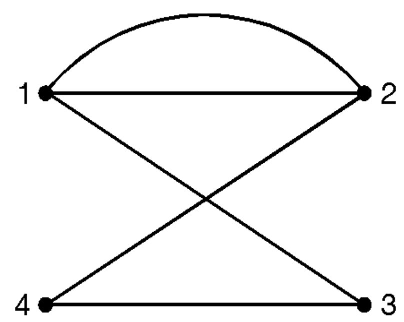
and E={ e_1, e_2, e_3, e_4, e_5}.

Let g be defined by

$\gamma(e_1) = \gamma(e_5) = \{1, 2\}$,

$\gamma(e_2) = \{4, 3\}$, $\gamma(e_3) = \{1, 3\}$, $\gamma(e_4) = \{2, 4\}$.

Then G={V, E, γ) is a graph as shown in

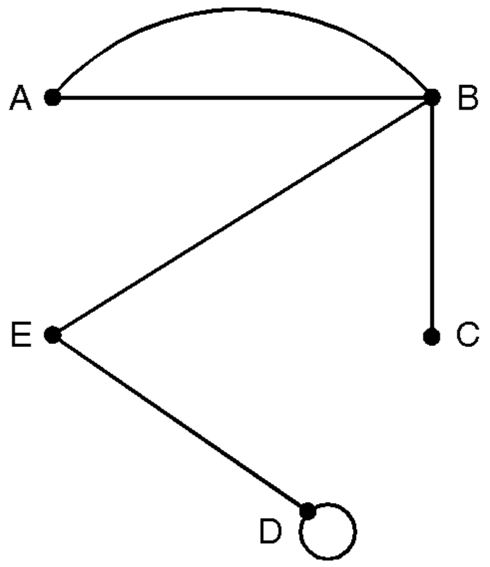


Definitions – Edge Type

Directed: Ordered pair of vertices. Represented as (u, v) directed from vertex u to v.

Undirected: Unordered pair of vertices. Represented as {u, v}. Disregards any sense of direction and treats both end vertices interchangeably.

Definitions



Degree :

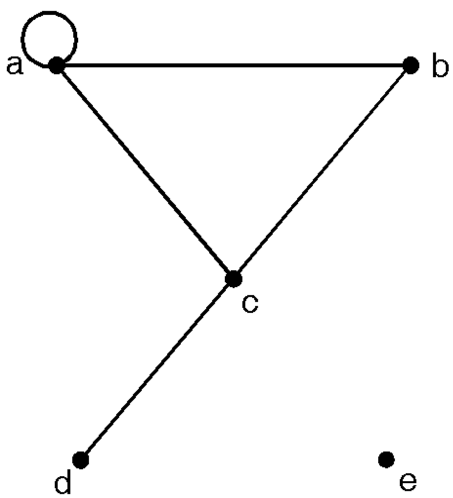
The **degree** of a vertex is the number of edges having that vertex as an end point.

Loop :

A graph may contain an edge from a vertex to itself, such an edge is referred to as a **loop**. A loop contributes 2 to the degree of a vertex. Since that vertex serves as both end points of the loop.

Isolated Vertex :

A vertex with degree 0 will be called an **isolated vertex**.



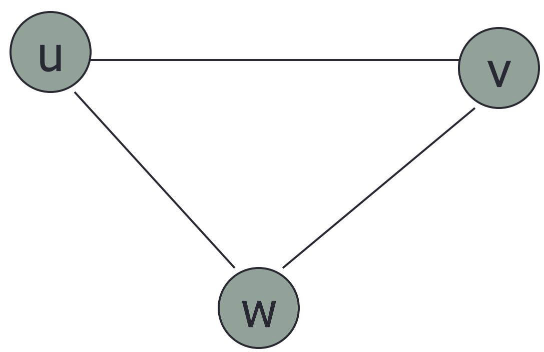
Adjacent Vertices :

A pair of vertices that determine an edge are **adjacent vertices**.

Simple undirected graph

Simple (Undirected) Graph: consists of V, a nonempty set of vertices, and E, a set of unordered pairs of distinct elements of V called edges (undirected)

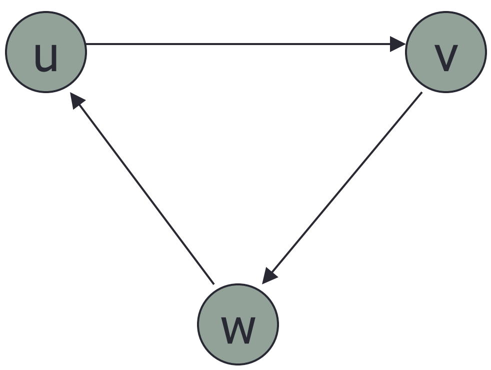
Representation Example: G(V, E), V = {u, v, w}, E = {{u, v}, {v, w}, {u, w}}



Directed graph

Directed Graph: G(V, E), set of vertices V, and set of Edges E, that are ordered pair of elements of V (directed edges)

Representation Example: G(V, E), V = {u, v, w}, E = {(u, v), (v, w), (w, u)}



Terminology – Directed graphs

In-degree (u): number of in coming edges

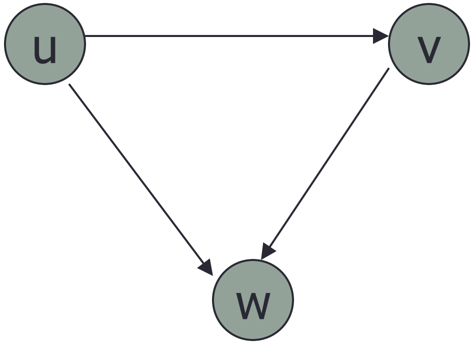
Out-degree (u): number of outgoing edges

Representation Example: For V = {u, v, w} , E = { (u, w), (v, w), (u, v) },

indeg(u) = 0, outdeg (u) = 2,

indeg(v) = 1,outdeg(v)=1

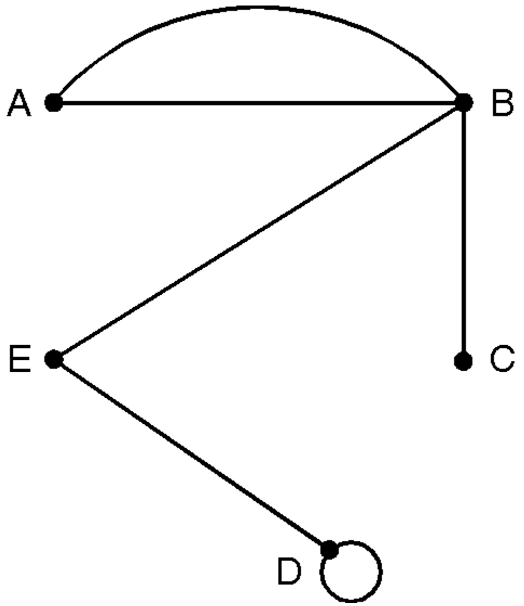
indeg(w) = 2, outdeg (w) = 0

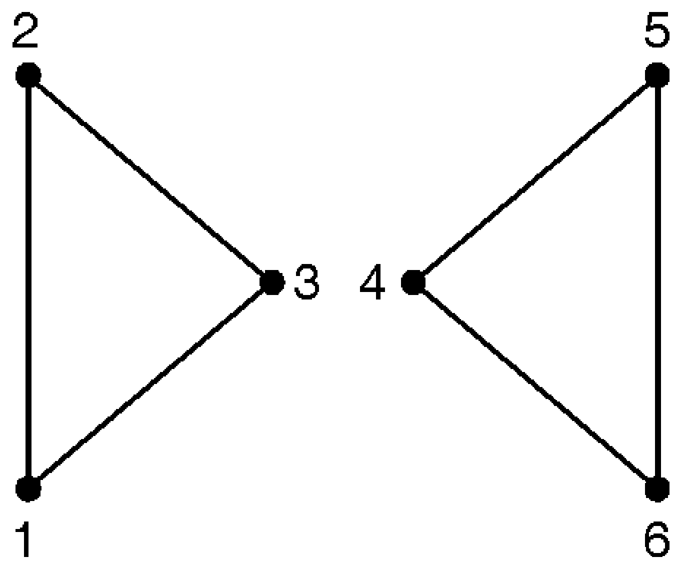


Connected graph

A graph is called **connected** if there is a path from any vertex to any other vertex in the graph.

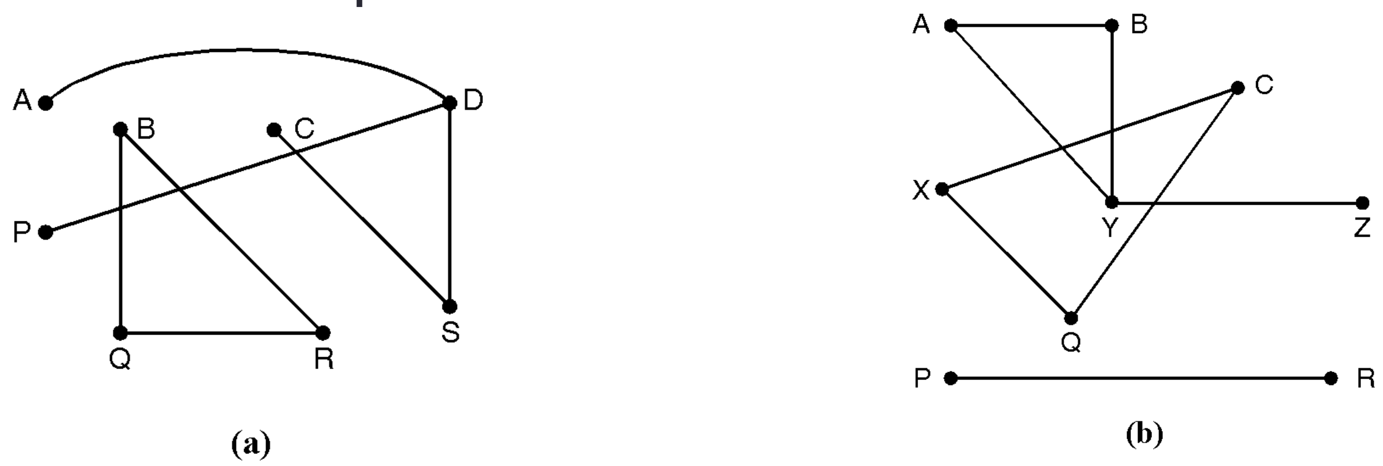
Otherwise, the graph is **disconnected**. If the graph is disconnected, the various connected pieces are called the **components** of the graph.





Problem

Determine whether the graph (shown in Fig. 6.5) is connected or disconnected. If disconnected find its connected component.



(a)Graph shown in Fig. (a) is not connected its connected components are {A, D, P, S, C} and {B, Q, R}

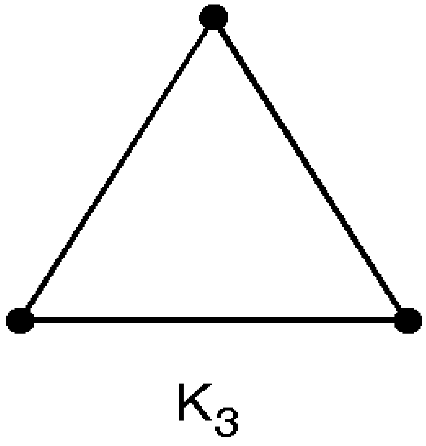
(b)Graph shown in Fig. (b) is not connected its connected components are {A, B, Y, Z}, {C, X, Q}, {P, R}

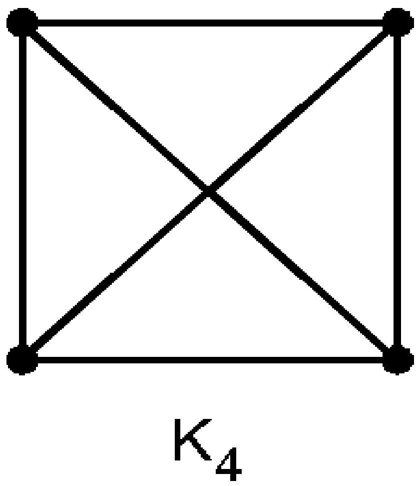
Complete Graph

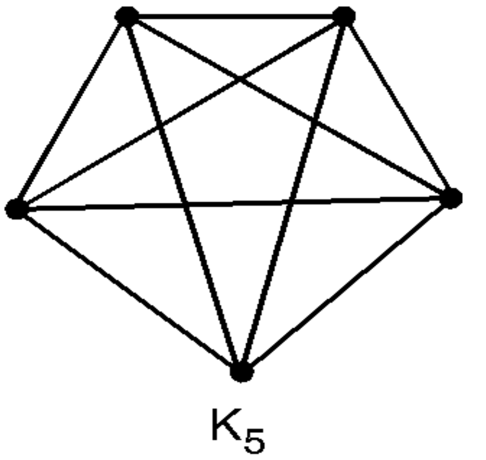
For each integer $n \geq 1$, let K_n denote the graph with vertices $\{v_1, v_2, \ldots v_n\}$ and with an edge $\{v_i, v_j\}$ for every i and j.

In other words, every vertex in K_n is connected to every other vertex.

Every pair of vertices are connected to each other







More Graphs

Linear graph: Straight lines

L_2

L_4

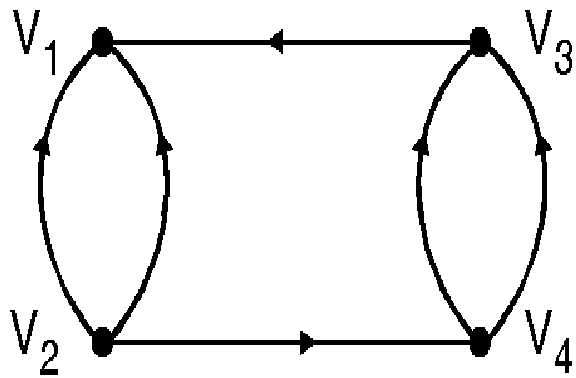
Discrete Graph: No edges between vertices

D_2

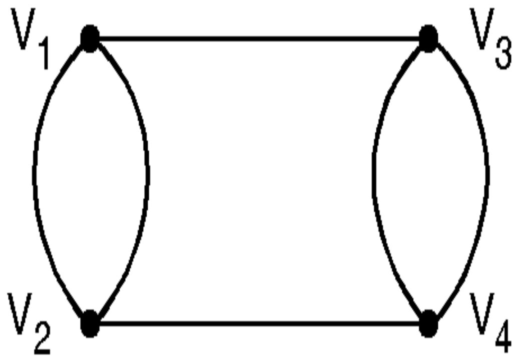
D_5

Multigraph

Directed graph having multiple edges between two vertices is called as **multigraph**.



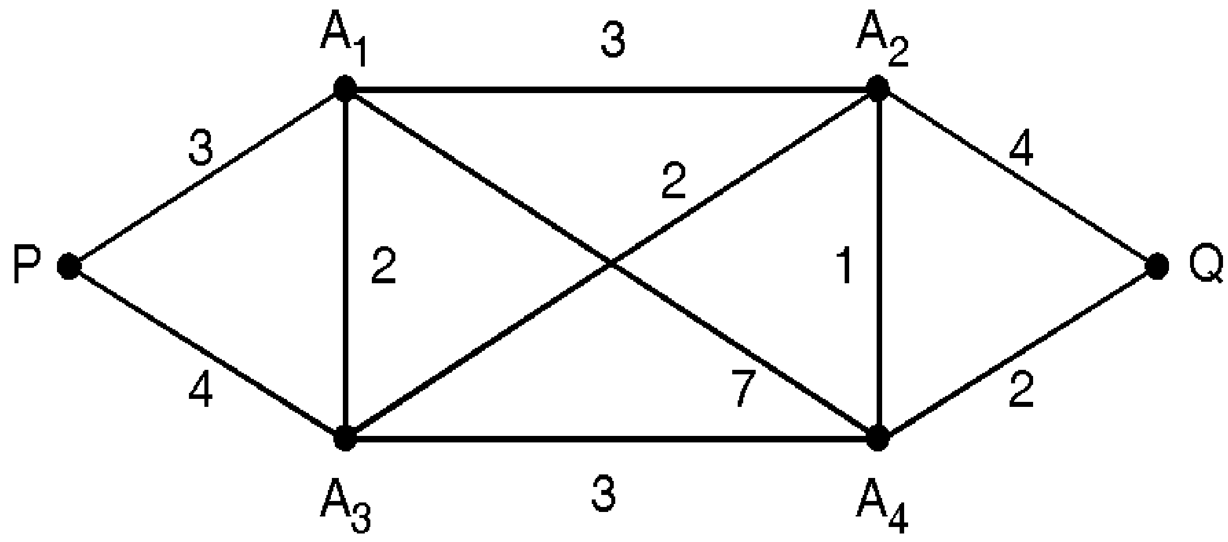
Undirected graph having more than one edge between two vertices is also called as **Multigraph**.



Labelled and weighted graph

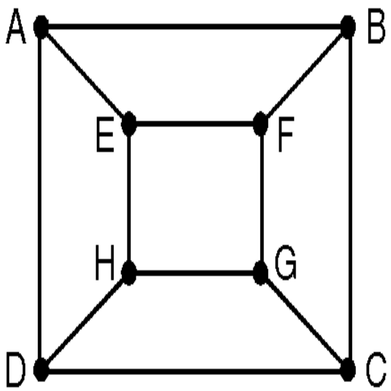
A graph G is called a **labelled graph** it its edges and /or vertices are assigned data of one kind or another.

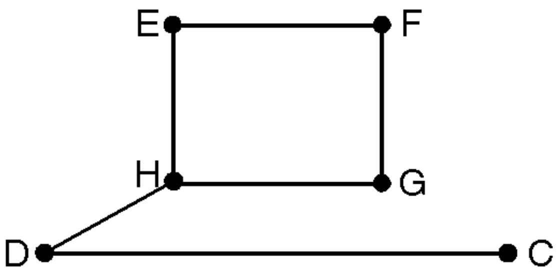
In particular, G is called a **weighted graph** if each edge 'e' of G is assigned a non–negative number called the weight or length of V.

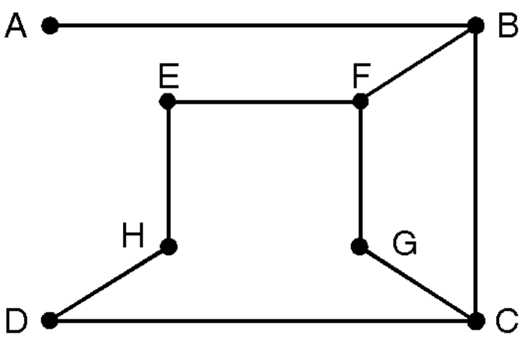


Subgraph

Let G = (V, E, γ) is a graph. Choose a subset E_1 of the edges in E and a subset V_1 of the vertices in V. So that V_1 contains all the end points of edges in E_1. Then H = (V_1, E_1, γ_1) is also a graph, where γ_1 is γrestricted to edges in E1. Such a graph H is called a **subgraph** of G.

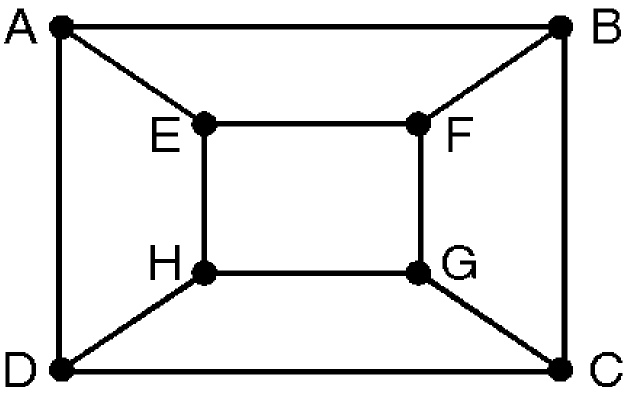


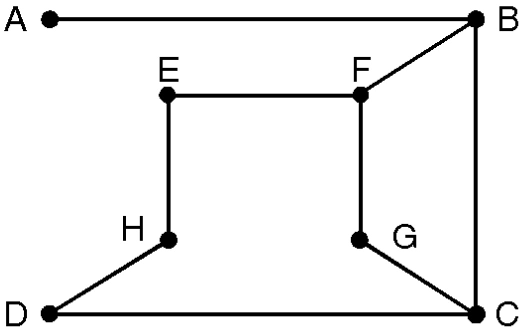




Spanning Subgraph

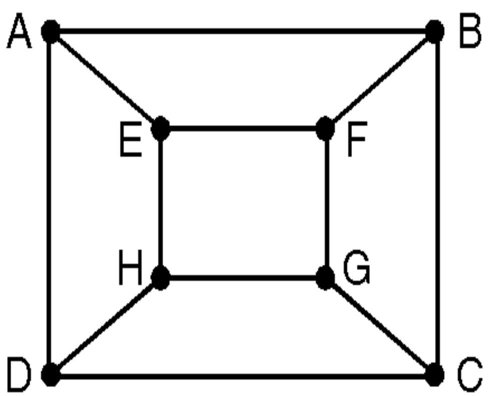
A subgraph is said to be **spanning subgraph** if it contains all the vertices of G.

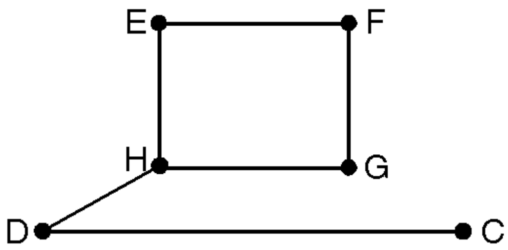


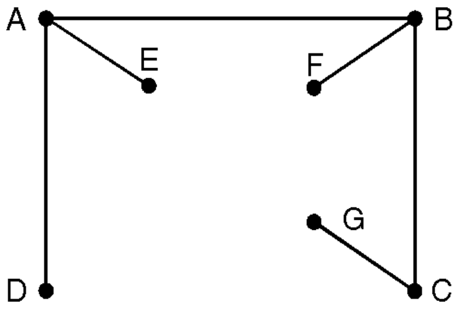


Complement of Subgraph

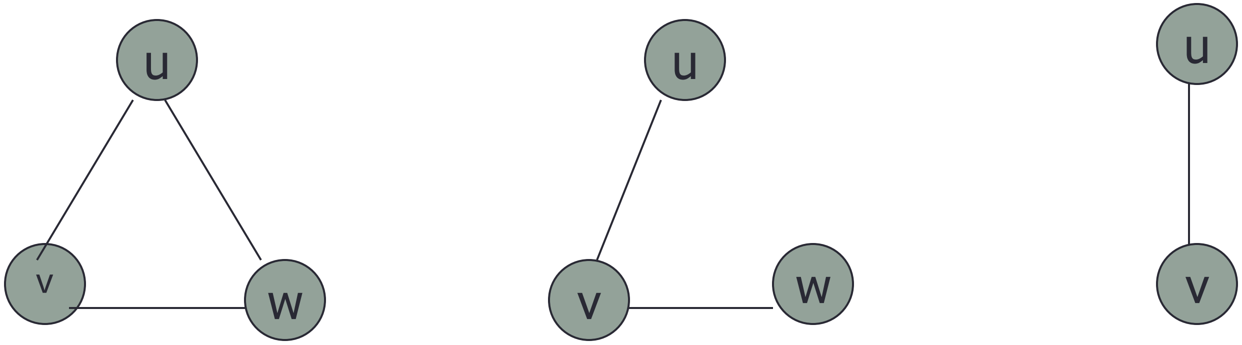
The complement of a subgraph G' = (V', E') with respect to the graph G = (V, E) is another subgraph
G" = (V", E") such that E" is equal to E – E' and V" contains only the vertices with which the edges in E" are incident.







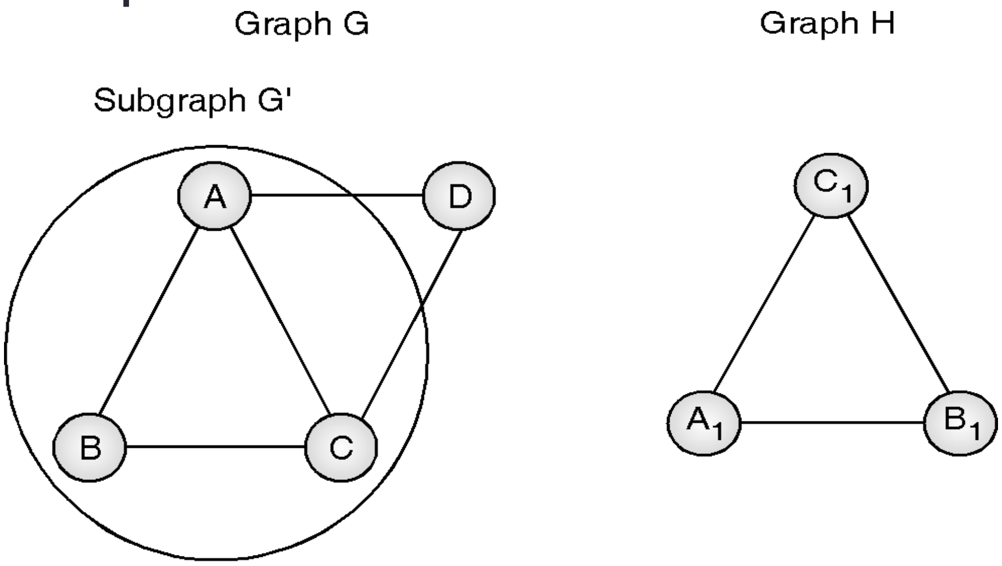
Complement of Subgraph



G H$_1$ H$_2$

Subgraph Isomorphism

The ***subgraph isomorphism*** is a computational task in which two graphs G and H are given as input, and one must determine whether G contains a subgraph that is isomorphic to H.



$f : \{(A, C_1), (B, A_1), (C, B_1)\}$

Handshaking Lemma

Consider a graph G with e number of edges and n number of vertices. Since each edge contributes two degrees, the sum of the degrees of all vertices in G is twice the number of edges in G i.e.

$$\sum \quad d(v_i) = 2e$$

Problem

How many nodes are necessary to construct a graph with exactly 6 edges in which each node is of degree 2.

Soln. : Suppose there are n vertices in the graph with 6 edges. Also, given the degree of each vertex is 2. Therefore by handshaking lemma,

$$\Sigma d\,(v_i) = 2e = 2 \times 6$$

$$\Rightarrow \quad d\,(v_1) + d\,(v_2) + \ldots + d\,(v_n) = 12.$$

$$\Rightarrow \quad = \quad 12$$

$$\Rightarrow \quad 2n \quad = \quad 12$$

$$\Rightarrow \quad n \quad = \quad 6$$

Hence, 6 nodes are required to construct a graph with 6 edges in which each node is of degree 2.

Problem

Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw two such graphs.

Soln: Suppose the graph with 6 vertices has e number of edges. Therefore, by handshaking lemma.

$$\Sigma d(v_i) = 2e$$

$$\Rightarrow \quad d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 2e$$

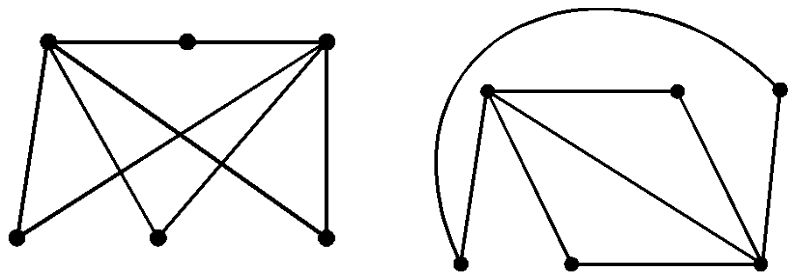
Now, given 2 vertices are of degree 4 and 4 vertices are of degree 2.

Hence from the above equation

$$\Rightarrow \quad (4 + 4) + (2 + 2 + 2 + 2) = 2e$$

$$\Rightarrow \quad 16 = 2e$$

$$\Rightarrow \quad e = 8$$



Path & Circuit

Path : A path is a sequence of vertices where no edge is chosen more than once

A path is called simple if no vertex repeats more than once

Length of Path : Number of edges in a path is called as length of path

Circuit: A circuit is a path that begins and ends with the same vertex

Euler path and Euler circuit

EULER PATH

- A path in a graph G is called an Euler path if it includes every edge exactly once

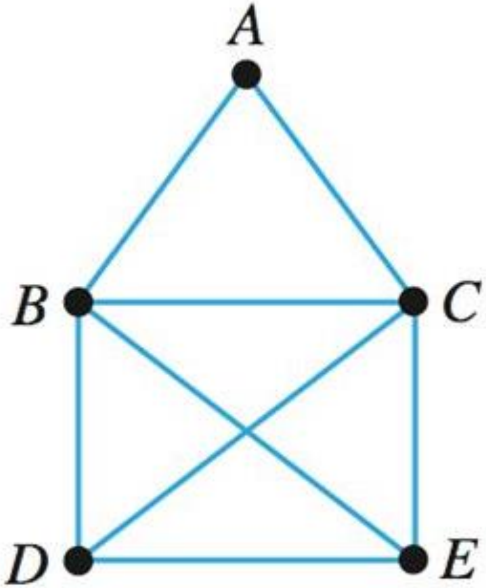
EULER CIRCUIT

- A Euler path that is a circuit

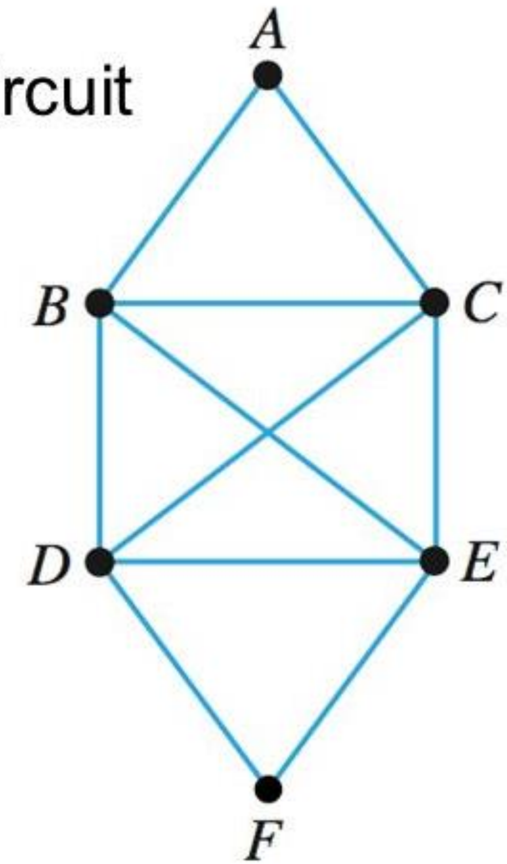
Example

- Euler path

D, E, B, C, A, B, D, C, E

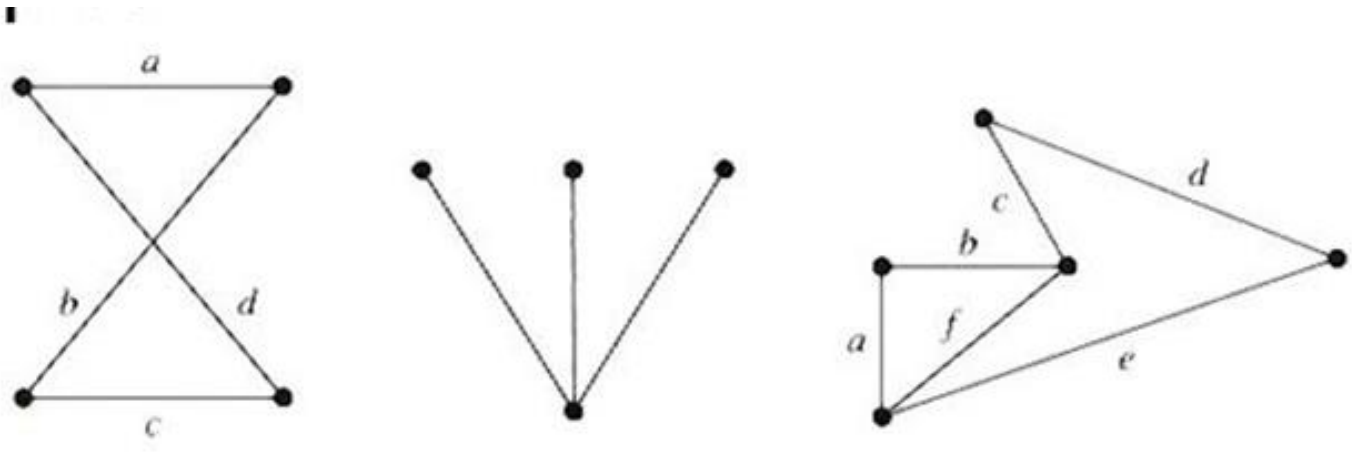


- Euler circuit



D, E, B, C, A, B, D, C, E, F, D

Identify Euler path and circuit



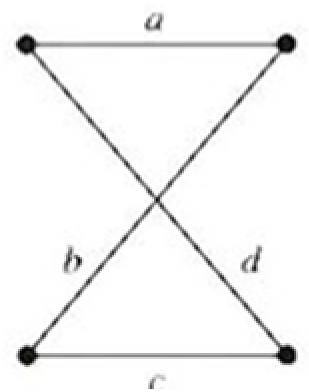
(a) (b) (c)

- The path a, b, c, d in (a) is an Euler circuit since all edges are included exactly once.
- The graph (b) has neither an Euler path nor circuit.
- The graph (c) has an Euler path a, b, c, d, e, f but not an Euler circuit.

Theorem: EULER CIRCUIT

A) If graph G has a vertex of odd degree , then there can be <u>no Euler circuit in G</u>

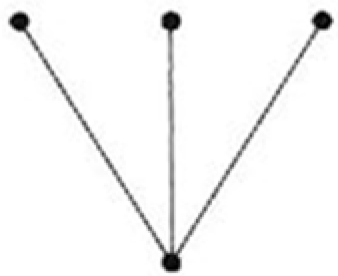
B) If G is a connected graph and every vertex has an even degree then <u>there is a Euler circuit in G</u>



All vertices have even degree

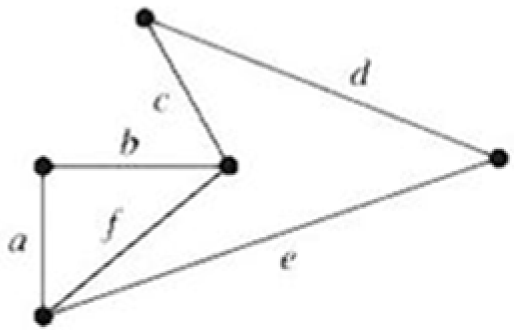
Theorem: EULER PATH

A) If a graph G has more than two vertices of odd degree then there can be <u>no Euler path in G</u>



All vertices have odd degree

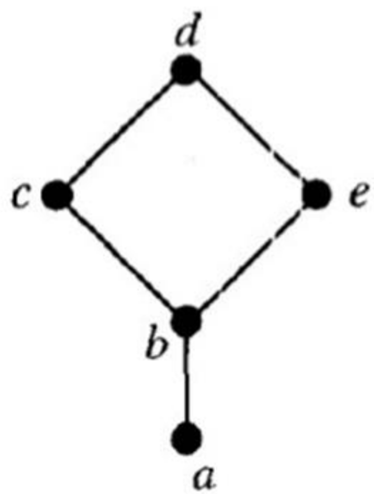
B) If G is connected and has exactly two vertices of odd degree then <u>there is a Euler path in G</u>



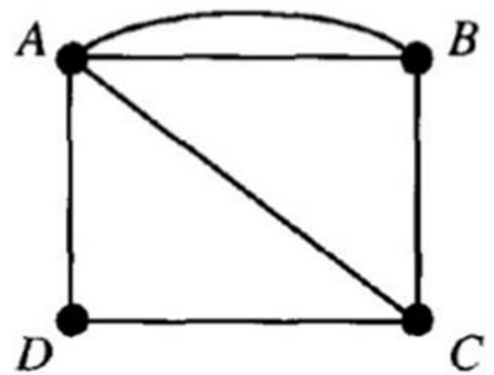
Two vertices have odd degree

Hamiltonian Path & Circuit

- A Hamiltonian path contains each vertex exactly once
- A Hamiltonian circuit is a circuit that contains each vertex exactly once except for the first vertex which is also the last

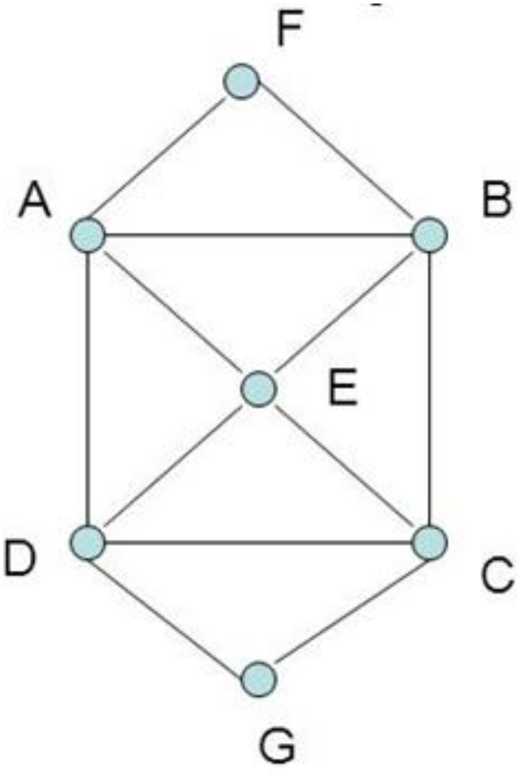


Hamiltonian path: a, b, c, d, e



Hamiltonian circuit: A, D, C, B, A

Example



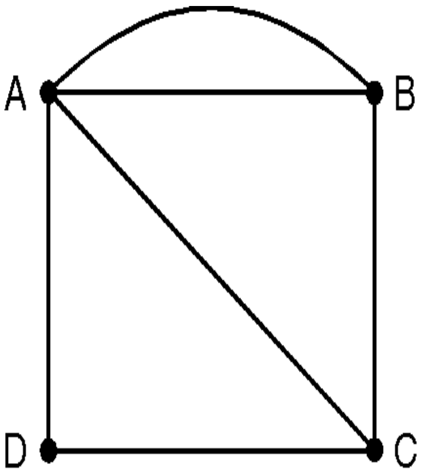
Has many **Hamilton circuits**:

1) A, F, B, E, C, G, D, A
2) A, F, B, C, G, D, E, A

Has many **Hamilton paths**:

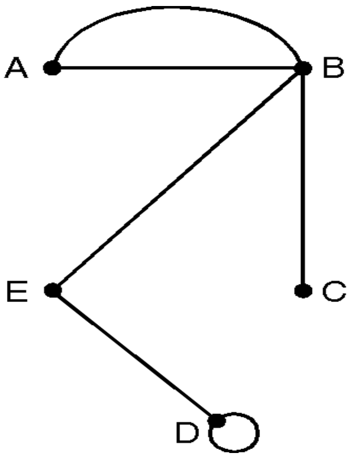
1) A, F, B, E, C, G, D
2) A, F, B, C, G, D, E

Examples



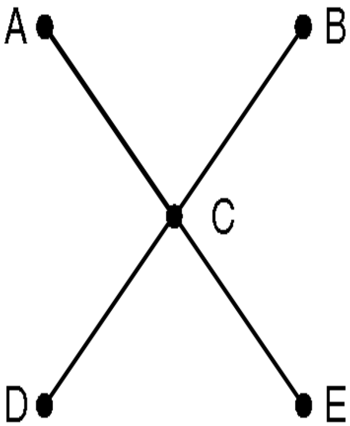
Hamiltonian Circuit:ADCBA

Euler Path: BACDABC



Hamiltonian Path: NO

Euler Path: DDEBABC

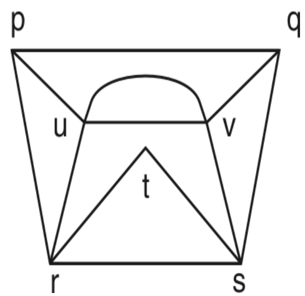


Hamiltonian Path: NO

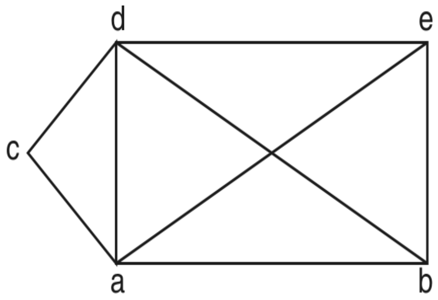
Euler Path: NO

Problem

Determine the Eulerian and Hamiltonian path, if exists, in the following graphs.

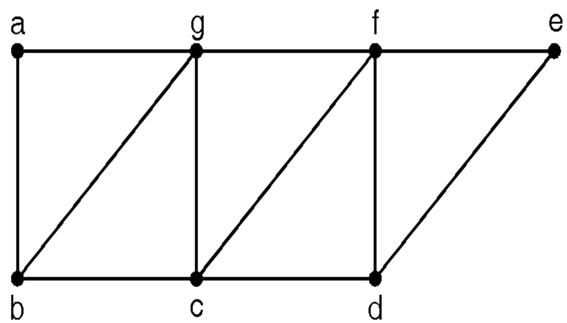


Hamiltonian path : p, u, v, q, s, t, r
Hamiltonian circuit : r, p, u, v, q, s, t, r
Eulerian path : (p, u, v, q, s, v, u, r, t, s, r, p, q)



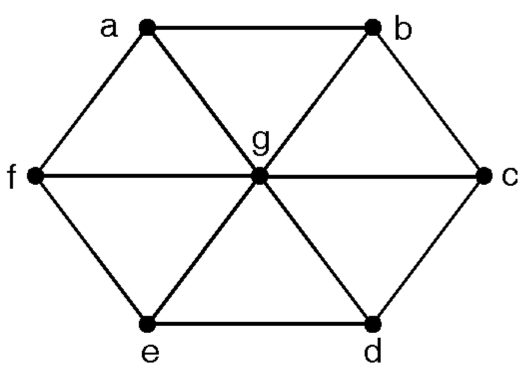
Hamiltonian path : c, d, e, b, a
Hamiltonian circuit : c, d, e, b, a, c
Eulerian path : (e, d, b, a, d, c, a, e, b)

Identify Euler path, circuit, Hamiltonian path and circuit



(a)

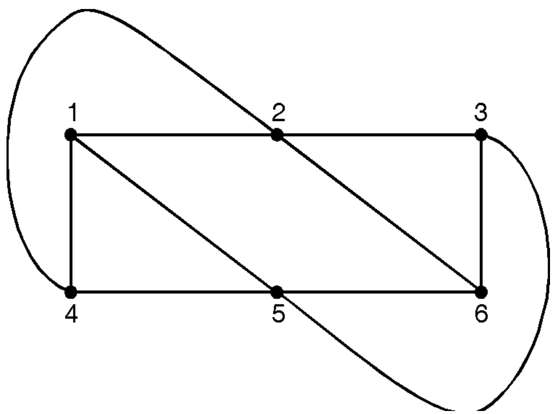
(a) two vertices b and d have odd degree.
Hence there is an Euler path.
π: b, a, g, f, e, d, c, b, g, c, f, d



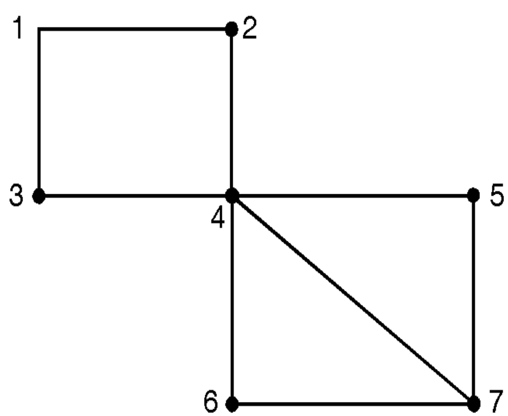
(b)

(b)6 vertices have odd degree, 3 and 1 vertex of even degree, 6.
So Euler path does not exist in this graph.

Identify Euler path, circuit, Hamiltonian path and circuit

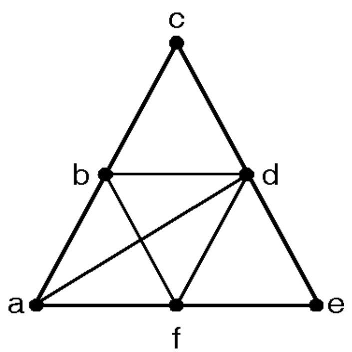


Number of vertices is 6. Each vertex has degree greater than equal to 6/2. So there is an Hamiltonian circuit.
π : 1, 4, 5, 6, 3, 2, 1

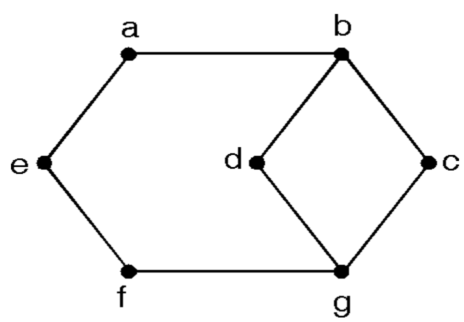


There is no Hamiltonian circuit.
But there is an Hamiltonian path
π: 3, 1, 2, 4, 6, 7, 5.

Identify Euler path, circuit, Hamiltonian path and circuit



(i) Eulerian Path : π: a, b, c, d, b, f, d, a, f, e, d
G has 2 vertices of odd degree.
Hamiltonian Circuit : a, b, c, d, e, f, a.
Hamiltonian Path : a, b, c, d, e, f



(ii) Eulerian Circuit : -
Eulerian Path : g, d, b, a, e, f, g, c, b.
Hamiltonian Path : d, b, a, e, f, g, c

Isomorphic Graph

Graphs G = (V, E) and H = (U, F) are isomorphic if we can set up a bijection $f : V \rightarrow U$ such that

x and y are adjacent in G

$\Leftrightarrow$ f(x) and f(y) are adjacent in H

Function f is called isomorphism

1. Same no. of vertices
2. Same no. of edges
3. Equal no. of vertices with a given degree
4. Adjacency of vertices

Graph - Isomorphism

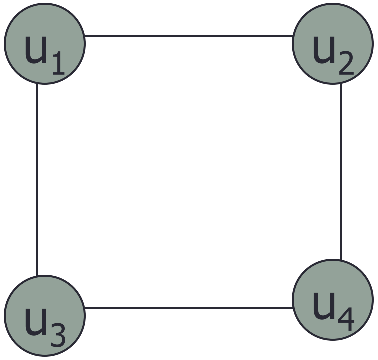
Representation example: G1 = (V1, E1) , G2 = (V2, E2)

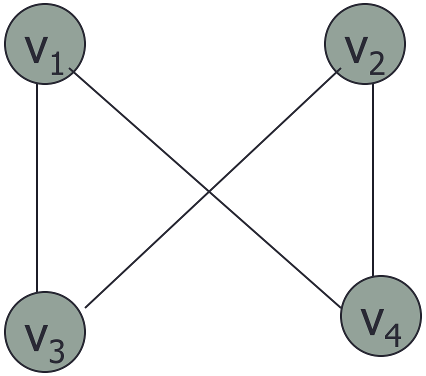
f(u1) = v1, f(u2) = v4, f(u3) = v3, f(u4) = v2

No. of vertices:4

No. of edges:4

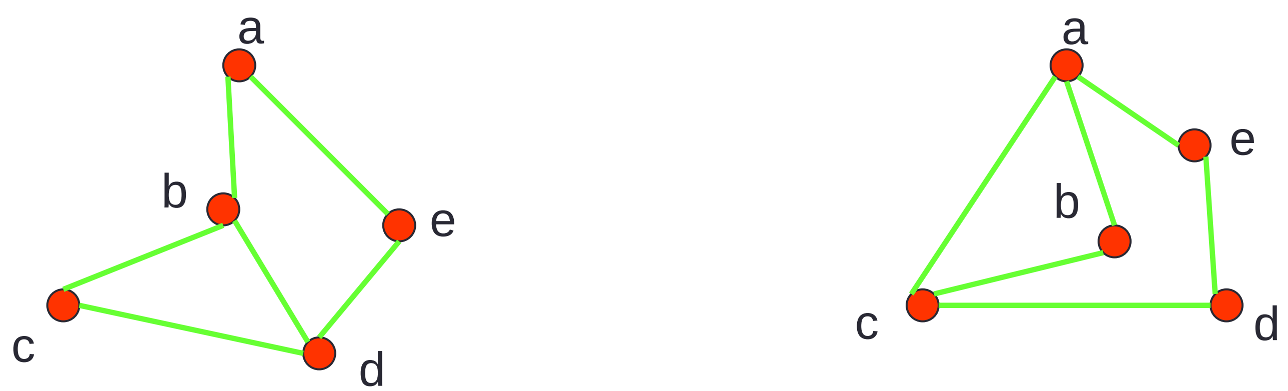
All vertices have degree 2





Isomorphism of Graphs

Example I: Are the following two graphs isomorphic?



Solution: No. of vertices: 6, No. of edges: 6

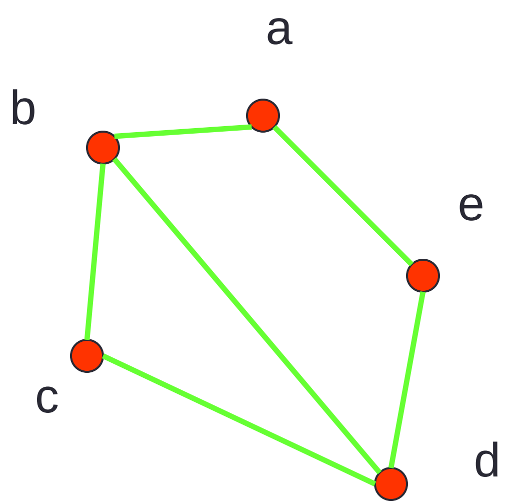
3 vertices with degree 2, 2 vertices with degree 3

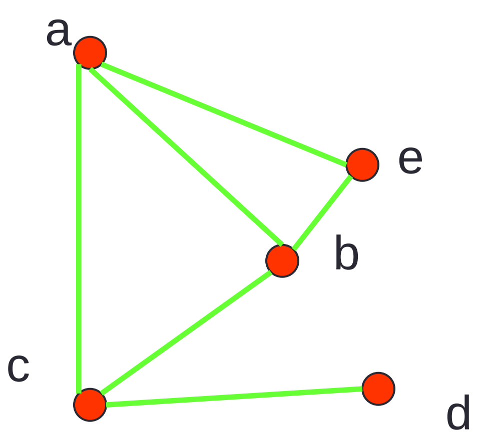
Then the isomorphism f from the left to the right graph is:

f(a)=e, f(b)=a, f(c) = b, f(d) = c, f(e) = d.

Isomorphism of Graphs

Example II: How about these two graphs?





Solution: No. of vertices: 5, No. of edges:6

No, they are not isomorphic, because they differ in the degrees of their vertices.Vertex d in right graph is of degree one, but there is no such vertex in the left graph.

Isomorphism of Graphs

Example III: Are the following two graphs isomorphic?

Solution:

Both graphs contain

8 vertices and 10 edges

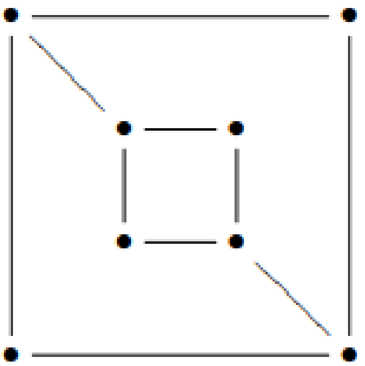
Nos of vertices of degree 2 = 4

Nos of vertices of degree 3 = 4

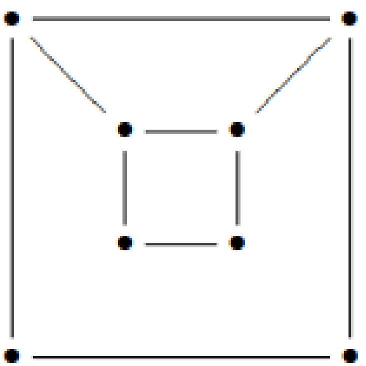
Adjacency : There exists no vertex of degree 3 whose adjacent vertices have same degree in both graphs

So its not ISOMORPHIC

G :

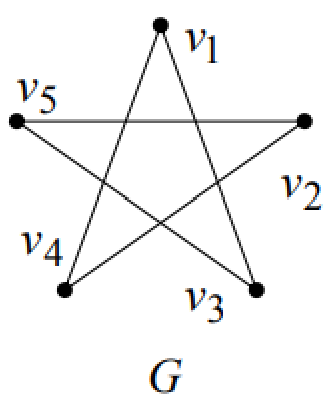


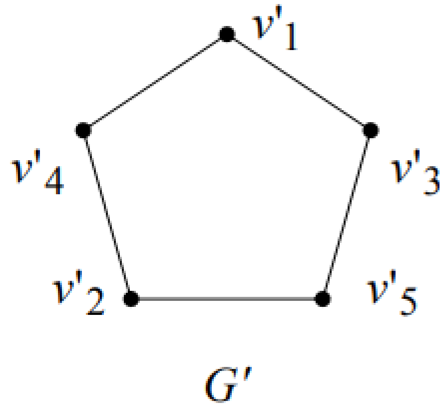
H :



Isomorphism of Graphs

Example IV: Are the following two graphs isomorphic?



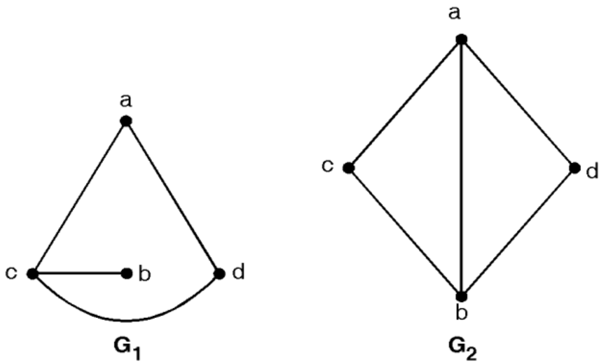


Solution: Both graphs have 5 vertices and 5 edges. All vertices have degree 2.

$f: V \to V'$	
V	V'
v_1	v'_1
v_2	v'_2
v_3	v'_3
v_4	v'_4
v_5	v'_5

Isomorphism of Graphs

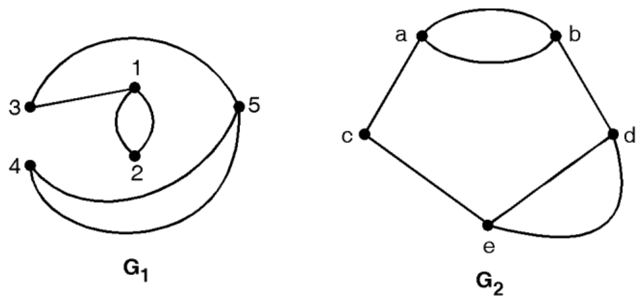
Example V: Are the following two graphs isomorphic?



Solution: Here G1 and G2 both have 4 vertices but G1 has 4 edges and G2 has 5 edges. Hence G1 is not isomorphic to G2.

Isomorphism of Graphs

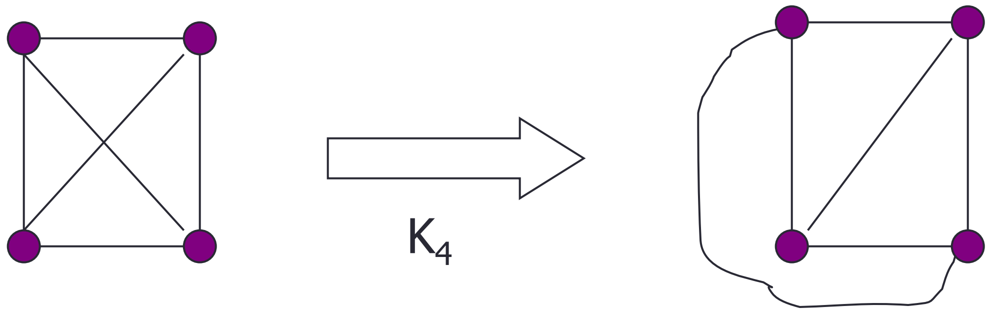
Example VI: Are the following two graphs isomorphic?



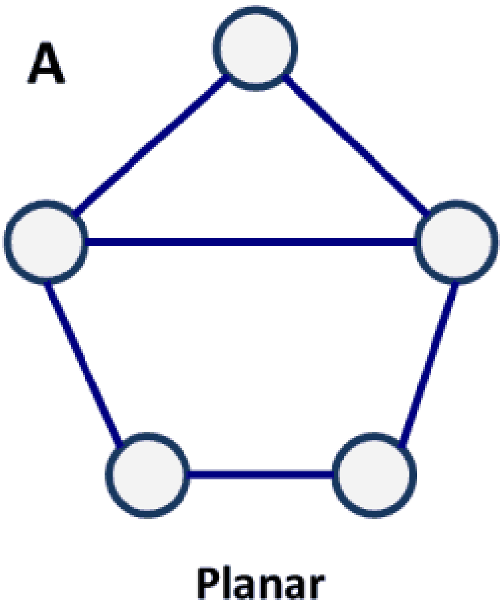
Solution: G1 and G2 both have 5 vertices but G1 has 6 edges while G2 has 7 edges. Hence G1 $\ncong$ G2. That is G1 is not isomorphic to G2.

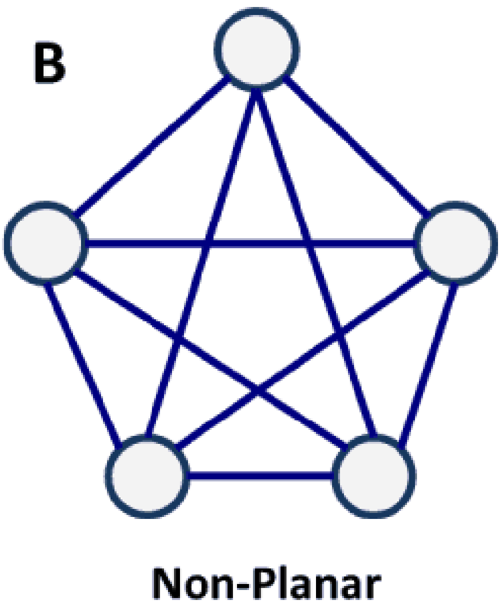
Planar Graphs

- A graph (or multigraph) G is called planar if G can be drawn in the plane with its edges intersecting only at vertices of G, such a drawing of G is called an embedding of G in the plane.
- Application Example: VLSI design (overlapping edges requires extra layers), Circuit design (cannot overlap wires on board)
- Representation examples: K_1,K_2,K_3,K_4 are planar, Kn for n>4 and if all the vertices are fully connected with each other then it will be a non-planar.

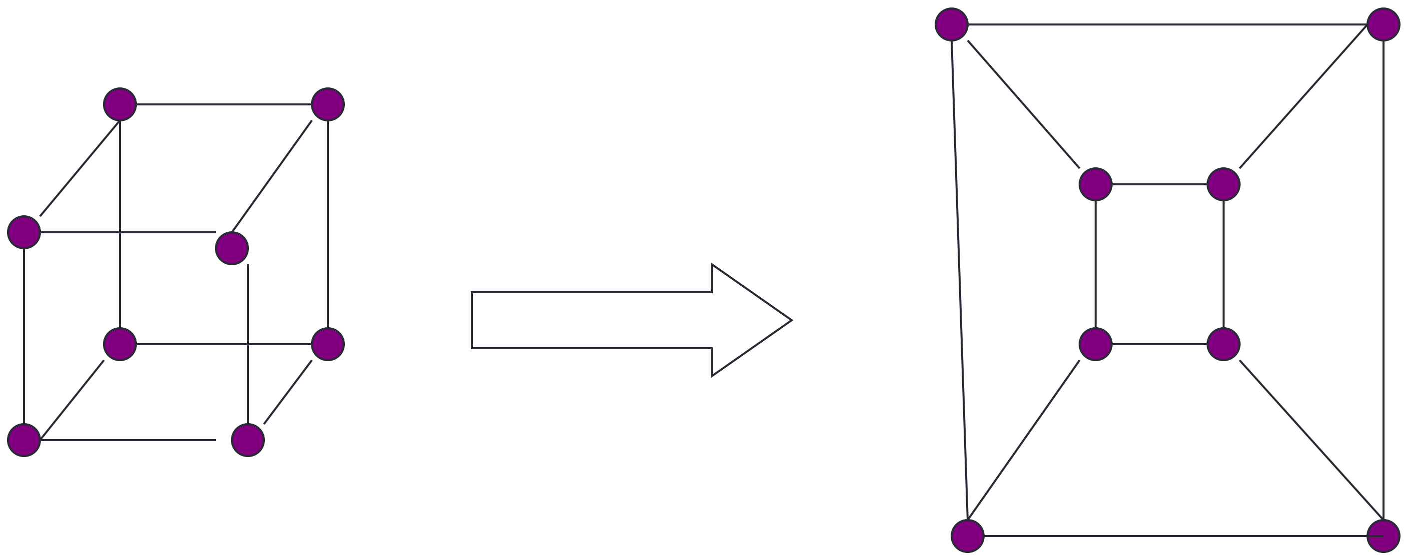


Examples





Planar Graphs Example



Planer Graph

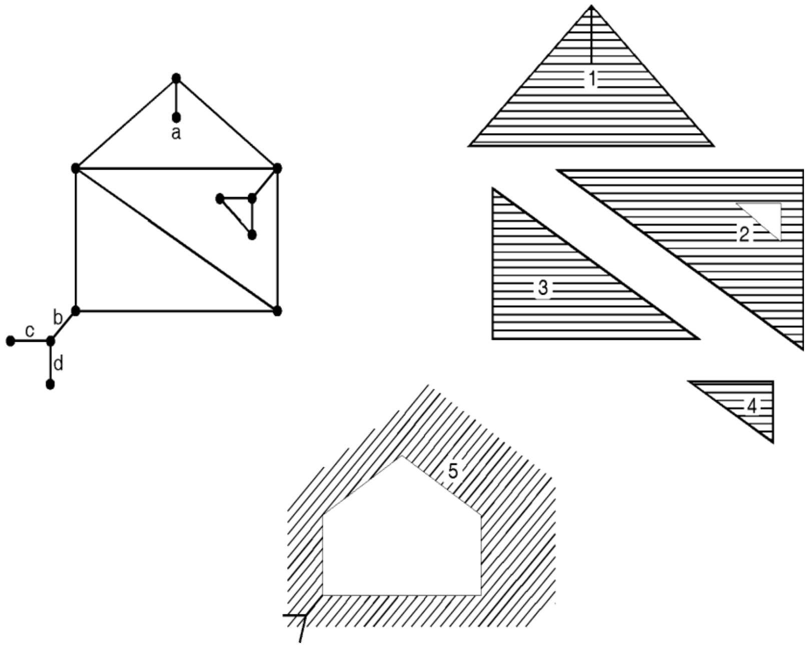
- **Theorem :** *Euler's connected planar graph theorem*

$$v - e + r = 2$$

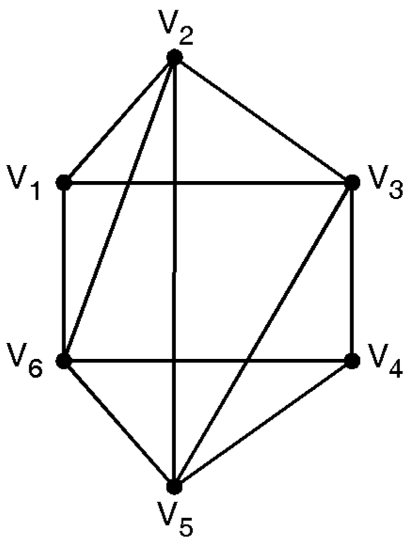
number of vertices

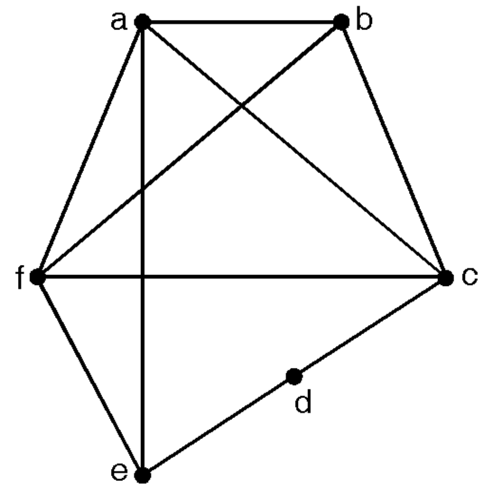
number of edges

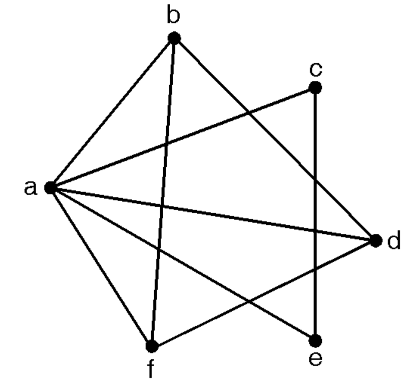
number of regions

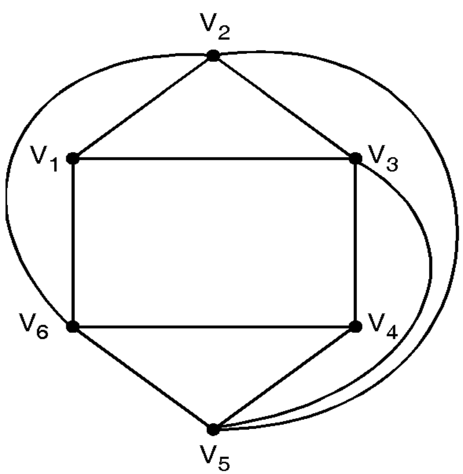


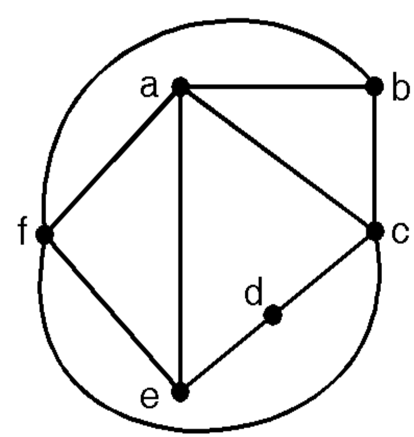
Q. 1) By drawing the graph, show that following graphs are planar graphs

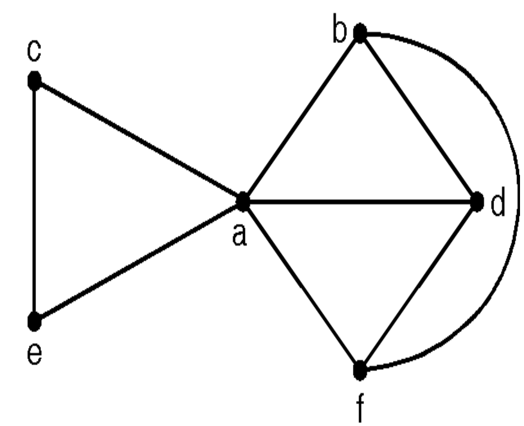












Q. 2 : How many edges must a planar graph have if it has 7 regions and 5 nodes. Draw one such graph.

Soln. :

According to Euler's formula, in a planar graph

$v - e + r = 2$

where v, e, r are the number of vertices, edges and regions in a planar graph.

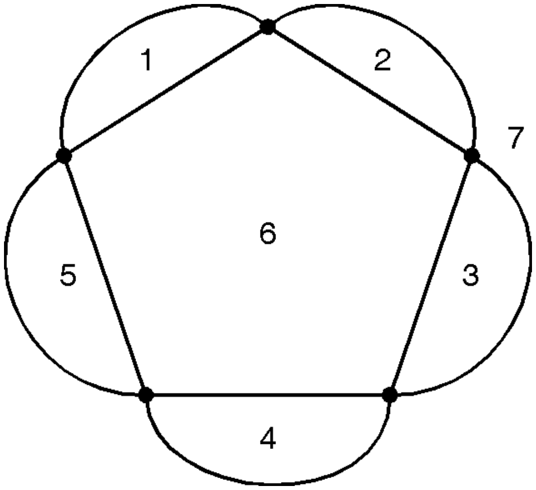
Here $v = 5, r = 7, e = ?$

$v - e + r = 2$

$5 - e + 7 = 2$

$e = 10$

Hence the given graph must have 10 edges.



Q. 3 : Determine the number of regions defined by a connected planar graph with 6 vertices and 10 edges. Draw a simple and a multi-graph.

Soln. :

Given $v = 6, e = 10$

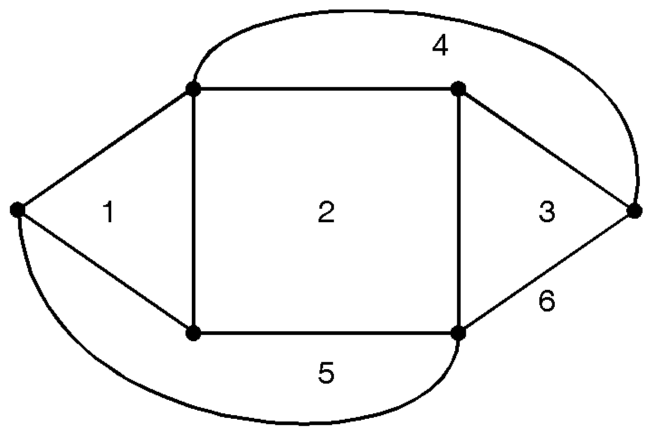
Hence by Euler's formula for a planar graph

$$v - e + r = 2$$

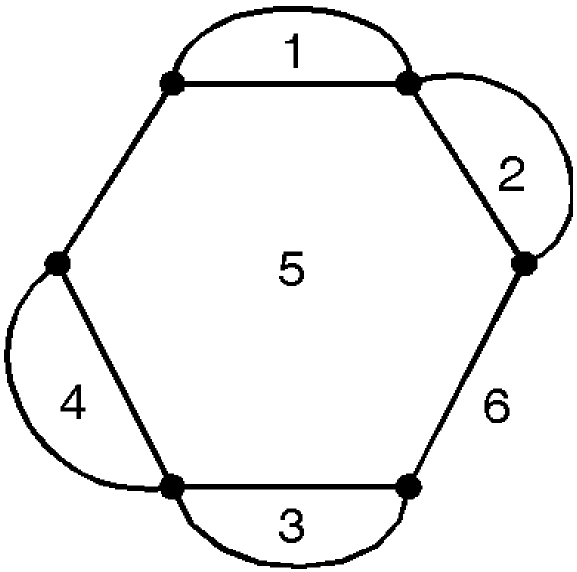
$$6 - 10 + r = 2$$

$$r = 6$$

Hence the graph should have 6 regions.



(a) Simple Graph



(a) Multi-Graph

Q. 4 : A connected planar graph has 9 vertices having degrees 2, 2, 2, 3, 3, 3, 4, 4 and 5. How many edges are there ?

Soln. :

By handshaking lemma

$$\Sigma d\ (v_i) = 2e$$

where $d\ (v_i)$ = degree of ith vertex

e = number of edges

For given graph

$$2 + 2 + 2 + 3 + 3 + 3 + 4 + 4 + 5 = 2.e$$

$$28 = 2e$$

$$e = 14$$

There are 14 edges.

Ex. 5 : Suppose that a connected planer graph has 20 vertices, each of degree 3 into how many regions does a representation of this plan graph split the plane ?

Soln. :

$$|V| = 20 \quad = \quad \text{number of vertices}$$

$$\text{degree of each vertex} \quad = \quad 3$$

By hand shaking Lemma

$$\Sigma\, d(Vi) \quad = \quad 2e$$

$$20 \times 3 \quad = \quad 2e$$

$$\Rightarrow \quad e \quad = \quad 30$$

By Euler's theorem,

$$|V| - |E| + |R| = 2$$

$$20 - 30 + |R| = 2$$

$$|R| = 12$$

Planar graph will split the plane in 12 regions.