Linear Programming Problems

Linear programming is a simple technique where we **depict** complex relationships through linear functions and then find the optimum points. The important word in the previous sentence is depicted. The real relationships might be much more complex – but we can simplify them to linear relationships.

Applications of linear programming is everywhere around you. You use linear programming at personal and professional fronts. You are using linear programming when you are driving from home to work and want to take the shortest route. Or when you have a project delivery you make strategies to make your team work efficiently for on-time delivery.

Common terminologies used in Linear Programming

Let us define some terminologies used in Linear Programming using the above example.

- **Decision Variables:** The decision variables are the variables that will decide my output. They represent my ultimate solution. To solve any problem, we first need to identify the decision variables. For the above example, the total number of units for A and B denoted by X & Y respectively are my decision variables.
- **Objective Function:** It is defined as the objective of making decisions. In the above example, the company wishes to increase the total profit represented by Z. So, profit is my objective function.
- **Constraints:** The constraints are the restrictions or limitations on the decision variables. They usually limit the value of the decision variables. In the above example, the limit on the availability of resources Milk and Choco are my constraints.
- **Non-negativity restriction:** For all linear programs, the decision variables should always take non-negative values. This means the values for decision variables should be greater than or equal to 0.

Formulation of Linear Programming

Maximization Case

Let's understand the maximization case with the help of a problem. Suppose a firm produces two products A and B. For producing the each unit of product A, 4 Kg of Raw material and 6 labor hours are required. While, for the production of each unit of product B, 4 kg of raw material and 5 labor hours is required. The total availability of raw material and labor hours is 60 Kg and 90 Hours respectively (per week). The unit price of Product A is Rs 35 and of product, B is Rs 40.

This problem can be converted into linear programming problem to determine how many units of each product should be produced per week to have the maximum profit. Firstly, the objective function is to be formulated. Suppose x1 and x2 are units produced per week of product A and B respectively. The sale of product A and product B yields Rs 35 and Rs 40 respectively. The total profit will be equal to

Z = 35×1+ 40×2 (objective function)

Since the raw material and labor is in limited supply the mathematical relationship that explains this limitation is called **inequality.** Therefore, the inequality equations will be as follows:

Product A requires 4 kg of raw material and product B also requires 4 Kg of Raw material; thus, total consumption is 4×1+4×2, which cannot exceed the total availability of 60 kg. Thus, this constraint can be expressed as:

4×1 + 4×2 ≤ 60

Similarly, the second constraint equation will be:

6×1 + 5×2 ≤ 90

Where 6 hours and 5hours of labor is required for the production of each unit of product A and B respectively, but cannot exceed the total availability of 90 hours.

Thus, the linear programming problem will be:

Maximize **Z = 35×1+ 40×2 (profit)**

Subject to:

4×1 + 4×2 ≤ 60 (raw material constraint)

6×1 + 5×2 ≤ 90 (labor hours constraint)

x1, x2 ≥ 0 (Non-negativity restriction)

Note: It is to be noted that "≤" (less than equal to) sign is used as the profit maximizing output may not fully utilize all the resources, and some may be left unused. And the non-negativity condition is used since the x1 and x2 are a number of units produced and cannot have negative values.

Minimization Case

The minimization case can be well understood through a problem. Let's say; the agricultural research institute recommended a farmer to spread out at least 5000 kg of phosphate fertilizer and not less than 7000 kg of nitrogen fertilizer to raise the productivity of his crops on the farm. There are two mixtures A and B, weighs 100 kg each, from which these fertilizers can be obtained.

The cost of each Mixture A and B is Rs 40 and 25 respectively. Mixture A contains 40 kg of phosphate and 60 kg of nitrogen while the Mixture B contains 60 kg of phosphate and 40 kg of nitrogen. This problem can be represented as a linear programming problem to find out how many bags of each type a farmer should buy to get the desired amount of fertilizers at the minimum cost.

Firstly, the objective function is to be formulated. Suppose, x1 and x2are the number of bags of mixture A and mixture B. The cost of both the mixture is 40×1 + 25×2 and thus, the objective function will be:

Minimize

Z = 40×1+25×2

In this problem, there are two constraints, minimum 5000 kg of phosphate and minimum 7000 kg of nitrogen is required. The Bag A contains 40 kg of phosphate while Bag B contains 60 kg of phosphate. Thus, the phosphate constraint can be expressed as:

$40 \times 1 + 60 \times 2 \geq 5000$

Similarly, the second constraint equation can be expressed as:

$60 \times 1 + 40 \times 2 \geq 7000$

Where, Bag A contains 60 kg of nitrogen and Bag B contains 40 kg of nitrogen, and The minimum requirement of nitrogen is 7000 kg.

Thus, the linear programming problem is:

Minimize **$Z = 40 \times 1 + 25 \times 2$ (cost)**

Subject to:

$40 \times 1 + 60 \times 2 \geq 5000$ (Phosphate Constraint)

$60 \times 1 + 40 \times 2 \geq 7000$ (Nitrogen Constraint)

$x1, x2 \geq 0$ (Non-negativity Restriction)

Note: It is to be noted that, "≥" (greater than equal to) sign shows the full utilization of resources at the minimum cost. The non-negativity condition is used, since x1 and x2 represent the number of bags of both the mixture and hence cannot have the negative values.

Graphical method for solving LPP

Question – 1

A calculator company produces a scientific calculator and a graphing calculator. Long-term projections indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. Because of limitations on production capacity, no more than 200 scientific and 170 graphing calculators can be made daily. To satisfy a shipping contract, a total of at least 200 calculators much be shipped each day.

If each scientific calculator sold results in a \$2 loss, but each graphing calculator produces a \$5 profit, how many of each type should be made daily to maximize net profits?

The question asks for the optimal number of calculators, so my variables will stand for that:

x: number of scientific calculators produced

y: number of graphing calculators produced

Since they can't produce negative numbers of calculators, I have the two constraints, $x \geq 0$ and $y \geq 0$. But in this case, I can ignore these constraints, because I already have that $x \geq 100$ and $y \geq 80$. The exercise also gives maximums: $x \leq 200$ and $y \leq 170$. The minimum shipping requirement gives me $x + y \geq 200$; in other words, $y \geq -x + 200$. The profit relation will be my optimization equation: $P = -2x + 5y$. So the entire system is:

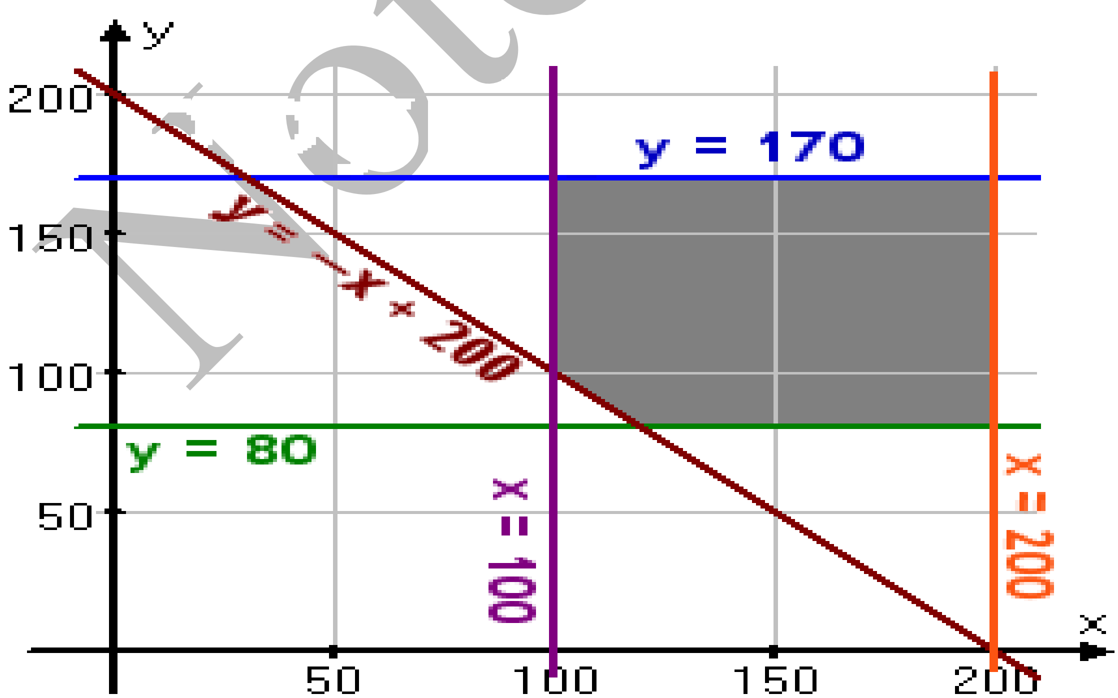
$P = -2x + 5y$, subject to:

$100 \leq x \leq 200$

$80 \leq y \leq 170$

$y \geq -x + 200$

The feasibility region graphs as:



When you test the corner points at (100, 170), (200, 170), (200, 80), (120, 80), and (100, 100), you should obtain the maximum value of $P = 650$ at $(x, y) = (100, 170)$. That is, the solution is "100 scientific calculators and 170 graphing calculators".

Question – 2

You need to buy some filing cabinets. You know that Cabinet X costs \$10 per unit, requires six square feet of floor space, and holds eight cubic feet of files. Cabinet Y costs \$20 per unit, requires eight square feet of floor space, and holds twelve cubic feet of files. You have been given \$140 for this purchase, though you don't have to spend that much. The office has room for no more than 72 square feet of cabinets. How many of which model should you buy, in order to maximize storage volume?

The question asks for the number of cabinets I need to buy, so my variables will stand for that:

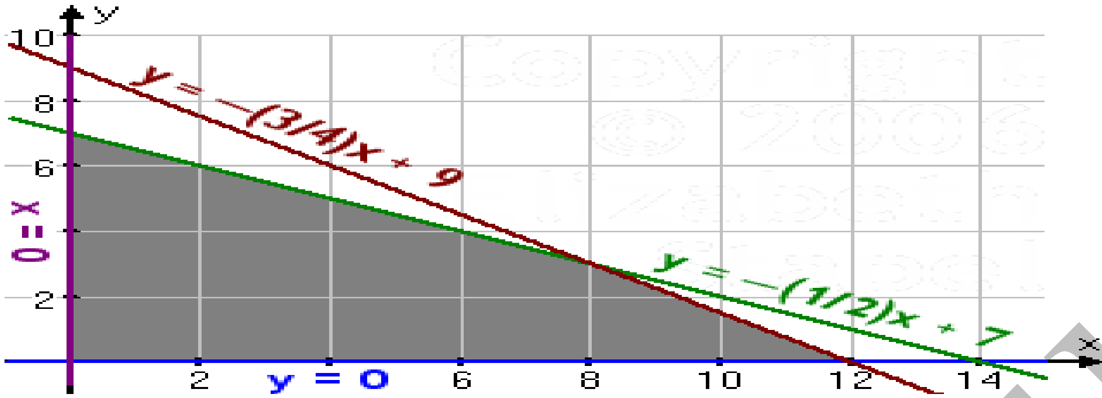
x: number of model X cabinets purchased

y: number of model Y cabinets purchased

Naturally, $x \geq 0$ and $y \geq 0$. I have to consider costs and floor space (the "footprint" of each unit), while maximizing the storage volume, so costs and floor space will be my constraints, while volume will be my optimization equation.

cost: $10x + 20y \leq 140$, or $y \leq -(\,^{1}/_{2}\,)x + 7$
space: $6x + 8y \leq 72$, or $y \leq -(\,^{3}/_{4}\,)x + 9$
volume: $V = 8x + 12y$

This system (along with the first two constraints) graphs as:



When you test the corner points at (8, 3), (0, 7), and (12, 0), you should obtain a maximal volume of 100 cubic feet by buying eight of model X and three of model Y.

Simplex Method for solving LPP

LINEAR PROGRAMMING PROBLEM

SIMPLEX METHOD

Q. Solve using simplex method the given problem:

Maximize $Z = x_1 + 9x_2 + x_3$

Subject to: $x_1 + 2x_2 + 3x_3 < 9$

$3x_1 + 2x_2 + 2x_3 \leq 15$

Solution →

Step I – To remove inequality we add slack variables to our equations.

$x_1 + 2x_2 + 3x_3 + S_1 = 9$

$3x_1 + 2x_2 + 2x_3 + S_2 = 15$

$Z = x_1 + 9x_2 + x_3 + 0S_1 + 0S_2$

NOTE As Z was already equal to given variables and there were no inequalities in the eq^n so value of S_1 and S_2 will be zero.

Step II – Prepare Initial Simplex Table.

C_B	C_j	1	9	1	0	0		
	Basic variables	X_1	X_2	X_3	S_1	S_2	Solution	Ratio
0	S_1	1	2	3	1	0	9	
0	S_2	3	2	2	0	1	15	
	Z_j	0	0	0	0	0		
	$C_j - Z_j$	1	9	0	0	0		

In this step, value of C_j are obtained from Z equation. The values of CB are also based on S_1 and S_2 values of Z. The values in row S_1 and S_2 are obtained from the two subject equations.

For finding Z_j formula is

$$Z_j = \Sigma\left[(C_B) \times (a_{ij})\right]$$

(as S_1 & S_2 are zero so all values will be zero).

Step III Find Key Row, key column & key element.

★ To find key column we need to know the biggest value of $C_j - Z_j$ row. Here it is 9 so that will be key column.

★ Now divide each key column value by solution value to get least Ratio and that will be our Key Row.

	C_j							
C_B	Basic Variables	X_1	X_2	X_3	S_1	S_2	Solution	Ratio
	S_1		2				9	9/2
	S_2		2				15	15/2
	Z_j		0					
	$C_j - Z_j$		9					

Key column

Divide by solution to get least Ratio

Least Ratio

So S_1 will be our Key Row.

	C_j							
C_B	Basic Variable	X_1	X_2	X_3	S_1	S_2	Solution	Ratio
	S_1	1	2	3	1	0	9	
	S_2		2	↑ Key Row			15	
			0					
			9					

← Key column

2 will be our key element as here at this point our key row & key column intersect.

Now S_1 will be our leaving variable.

and x_2 will be entering variable.

<u>Step IV</u> – Make 1st Iteration Table

	C_j	1	9	1	0	0		
C_B	Basic Variable	X_1	X_2	X_3	S_1	S_2	Solution	Ratio
9	X_2	1/2	1	3/2	1/2	0	9/2	
0	S_2	2	0	-1	-1	1	6	
	Z_j	9/2	9	27/2	9/2	0	81/2	
	$C_j - Z_j$	-7/2	0	-25/2	-9/2	-9	-	

We need to make our key element so dividing the whole key row by 2. S_2 will have all the new values.

Formula for finding values of S_2 is -

$$\text{New Value} = \text{Old value} - \left(\frac{\text{Corresponding Key Row} \times \text{Corresponding Key Column}}{\text{Key element}}\right)$$

eg. old value was 3 so

$$\text{New value} = 3 - \left(\frac{1 \times 2}{2}\right) = 3 - \frac{2}{2} = 2$$

old value was 2 so

$$\text{New value} = 2 - \left(\frac{2 \times 2}{2}\right) = 2 - 2 = 0$$

old value was 2 so

$$\text{New value} = 2 - \left(\frac{2 \times 3}{2}\right) = 2 - 3 = -1$$

Similarly find other values also.

To find z, use the formula previously discussed.

$$z_j = \Sigma\left[(C_B) \times (a_{ij})\right]$$

For Maximization

$c_j - z_j$ should be less than zero. our values are also either zero or negative.

So Answer is -

$$\boxed{\begin{aligned} x_2 &= 9/2 \\ S_2 &= 6 \\ z_j &= 81/2 \end{aligned}}$$

Duality in LPP

The Duality in Linear Programming states that every linear programming problem has another linear programming problem related to it and thus can be derived from it. The original linear programming problem is called **“**Primal,” while the derived linear problem is called **“**Dual**”.**

Before solving for the duality, the original linear programming problem is to be formulated in its standard form. Standard form means, all the variables in the problem should be non-negative and “≥,” ”≤” sign is used in the minimization case and the maximization case respectively.

For every Linear programming Problem, there is a corresponding unique problem involving the same data and it also describes the original problem. The original problem is called primal programme and the corresponding unique problem is called Dual programme. The two programmes are very closely related and optimal solution of dual gives complete information about optimal solution of primal and vice versa.

Different useful aspects of this property are:

(a) If primal has large number of constraints and small number of variables, computation can be considerably reduced by converting problem to Dual and then solving it.

(b) Duality in linear programming has certain far reaching consequence of economic nature. This can help managers answer questions about alternative courses of action and their relative values.

(c) Calculation of the dual checks the accuracy of the primal solution.

(d) Duality in linear programming shows that each linear programme is equivalent to a two-person zero-sum game. This indicates that fairly close relationships exist between linear programming and the theory of games.

Procedure

Step 1: Convert the objective function if maximization in the primal into minimization in the dual and vice versa. Write the equation considering the transpose of RHS of the constraints

Step 2: The number of variables in the primal will be the number of constraints in the dual and vice versa.

Step 3: The co-efficient in the objective function of the primal will be the RHS constraints in the dual and vice versa.

Step 4: In forming the constraints for the dual, consider the transpose of the body matrix of the primal problems.

Note: Constraint inequality signs are reversed

Example: Construct the dual to the primal problem

Maximize $Z = 6x_1 + 10x_2$

Subject to constraints,

$2x_1 + 8x_2 \leq 60$ ....(i)

$3x_1 + 5x_2 \leq 45$ ....(ii)

$5x_1 - 6x_2 \leq 10$ ....(iii)

$x_2 \leq 40$ ....(iv)

where $x_1, x_2 \geq 0$

Solution:

Minimize $W = 60y_1 + 45y_2 + 10y_3 + 40y_4$

Subject to constraints,

$2y_1 + 3y_2 + 5y_3 + 0y_4 \geq 6$

$8y_1 + 5y_2 + 6y_3 + y_4 \geq 10$

where $y_1, y_2, y_3, y_4 \geq 0$

Here are some uses of the dual problem.

1. **Understanding the dual problem leads to specialized algorithms for some important classes of linear programming problems.** Examples include the transportation simplex

method, the Hungarian algorithm for the assignment problem, and the network simplex method. Even column generation relies partly on duality.

2. **The dual can be helpful for sensitivity analysis.** Changing the primal's right-hand side constraint vector or adding a new constraint to it can make the original primal optimal solution infeasible. However, this only changes the objective function or adds a new variable to the dual, respectively, so the original dual optimal solution is still feasible (and is usually not far from the new dual optimal solution).

3. **Sometimes finding an initial feasible solution to the dual is much easier than finding one for the primal.** For example, if the primal is a minimization problem, the constraints are often of the form $Ax \geq bAx \geq b$, $x \geq 0x \geq 0$, for $b \geq 0b \geq 0$. The dual constraints would then likely be of the form $ATy \leq cATy \leq c$, $y \geq 0y \geq 0$, for $c \geq 0c \geq 0$. The origin is feasible for the latter problem but not for the former.

4. **The dual variables give the shadow prices for the primal constraints.** Suppose you have a profit maximization problem with a resource constraint ii. Then the value yiyi of the corresponding dual variable in the optimal solution tells you that you get an increase of yiyi in the maximum profit for each unit increase in the amount of resource ii (absent degeneracy and for small increases in resource ii).

5. **Sometimes the dual is just easier to solve.** Aseem Dua mentions this: A problem with many constraints and few variables can be converted into one with few constraints and many variables.