Testing For Optimality

Once the feasible solution is obtained, the next step is to check whether it is optimum or not. There are two methods used for testing the optimality:

- Stepping-stone Method
- Modified Distribution Method (MODI)

1. Stepping Stone Method:

This method content following steps:

Step (1):

Determine an initial basic feasible solution using any of the three methods discussed earlier.

Step (2):

Make sure that the number of occupied cells is exactly equal to m + n-1, where m is no. of rows and n is no. of columns.

Step (3):

Evaluate the cost – effectiveness of shipping goods via transportation routes not currently in solution.

This testing of each unoccupied cell is conducted by the following five steps as follows:

(a) Select an unoccupied cell, where a shipment should be made.

(b) Beginning at this cell, trace a closed path using the most direct route through at least three occupied cells used in the solution and then back to the original occupied cell and moving with only horizontal and vertical moves. Further, since only the cell at the turning points are considered to be on the closed path, both unoccupied and occupied boxes may be skipped over. The cell at the turning points are called "stepping stones" on the path.

(c) Assigning plus (+) and minus (-) signs alternatively on each corner cell of the closed path just traced, starting with a plus sign at the unoccupied cell to be evaluated.

(d) Compute the 'net change in the cost' along the closed path by adding together the unit cost in each square containing the minus sign.

(e) Repeat sub-step.

- Through sub-step.

- Until 'net change' in cost has been calculated for all unoccupied cells of the transportation table.

Step (4):

Check the sign of each of the net changes. If all net changes computed are greater than or equal to zero, an optimal solution has been reached. If not, it is possible to improve the current solution and decrease total shipping costs.

Step (5):

Select the unoccupied cells having the highest negative net cost change and determine the minimum number of units that can be assigned to a cell marked with a minus sign on the closed path corresponding to this cell. Add this number to the unoccupied cell and to all other cells on the path marked with a plus-sign. Subtract this number from cells on the closed path marked with a minus sign.

Step (6):

Go to step (2) and repeat the procedure until we get an optimal solution.

Example:

To understand clearly we discuss the initial solution given by least cost method as shown in table below:

Factory	Warehouse				Capacity
	W_1	W_2	W_3	W_4	
F_1	21	16	25	11 13	11
F_2	1 17	18	12 14	23	13
F_3	5 32	10 27	18	4 41	19
Requirement	6	10	12	15	43

Step 1:

The initial solution has 4+3-1 = 6 occupied cells and involves transpiration cost of Rs. 922.

Step 2:

Let us evaluate unoccupied cell (F_1, W_1). The shipment of one unit to this cell will incur an additional cost of Rs 21. This requires in turn that one unit be decreased from cell (F_1, W_4) which decreases cost by Rs 13. But to keep the balance between capacity and requirement we have to add one unit to cell (F_3, W_4) which increases cost by Rs. 41 and finally one unit is decreased from cell (F_3, W_1) which decreases cost by Rs. 32.

To determine the net cost change. Let us list down the changes as shown below:

Cell	*Change in allocation*	*Cost change (Rs)*
(F_1, W_1)	+ 1	+21
(F_1, W_4)	− 1	−13
(F_3, W_4)	+1	+41
(F_3, W_1)	−1	−32
net cost change		+17

	W_1	W_4
F_1	21 +	11 | 13 −
F_3	5 | 32 −	4 | 41 +

This indicates that if the occupied cell (F_1, W_1) is made occupied then the total transportation cost will be increased by Rs 17 per unit supplied. This transfer of shipment of one unit is also shown in the right hand side table by making a close path. Similarly other unoccupied cell can also be evaluated proceeding in the same manner.

Unoccupied Cell	*Closed Path*	*Net cost change (Rs)*	*Remarks*
(F_1, W_2)	$(F_1, W_2) \rightarrow (F_1, W_4) \rightarrow (F_3, W_4) \rightarrow (F_3, W_2)$	$16 - 13 + 41 - 27 = + 17$	Cost increases
(F_1, W_3)	$(F_1, W_3) \rightarrow (F_1, W_4) \rightarrow (F_3, W_4)$ $(F_3, W_1) \rightarrow (F_2, W_1) \rightarrow (F_2, W_3)$	$25 - 13 + 41 - 32 + 17 - 14 = + 24$	Cost increases
(F_2, W_2)	$(F_2, W_2) \rightarrow (F_3, W_2) \rightarrow (F_3, W_1) \rightarrow (F_2, W_1)$	$18 - 27 + 32 - 17 = + 6$	Cost increases
(F_2, W_4)	$(F_2, W_4) \rightarrow (F_3, W_4) \rightarrow (F_3, W_1) \rightarrow (F_2, W_1)$	$23 - 41 + 32 - 17 = - 3$	Cost increases
(F_3, W_3)	$(F_3, W_3) \rightarrow (F_3, W_1) \rightarrow (F_2, W_1) \rightarrow (F_2, W_3)$	$18 - 32 + 17 - 14 = - 11$	Cost increases

Factory	Warehouse				Capacity
	W_1	W_2	W_3	W_4	
F_1	21 + 21	16 + 17	25 + 24	(11) 13	11
F_2	(1) 17 +	18 + 6	(12) 14 −	23 − 3	13
F_3	(5) 32 −	(10) 27	• 18 − 11 +	(4) 41	19
Requirement	6	10	12	15	43

(3) Then we observe that only (F_3, W_3) for which the largest reduction in cost change being – 11 is -ve.

∴ The unoccupied cell (F_3, W_3) will be considered for further reduction in the cell. The new solution thus obtained is shown below table

Factory	Warehouse				Capacity
	W_1	W_2	W_3	W_4	
F_1	21	16	25	(11) 13	11
F_2	(6) 17	18	(7) 14	23	13
F_3	32	(10) 27	(5) 18	(4) 41	19
Requirement	6	10	12	15	43

The total transportation cost of the improved solution is (11×13) + (6×17) + (7×14) + (10×27) + (5×18) + (4×41) = Rs. 867

Unoccupied Cell	Closed Path	Net cost change (Rs)	Remarks
(F_1, W_1)	$(F_1, W_1) \rightarrow (F_1, W_4) \rightarrow (F_3, W_4) \rightarrow (F_3, W_3) \rightarrow (F_2, W_3) \rightarrow (F_2, W_1)$	$+21-13+41-18+14-17=+28$	Cost increase
(F_1, W_2)	$(F_1, W_2) \rightarrow (F_1, W_4) \rightarrow (F_3, W_4) \rightarrow (F_3, W_2)$	$+16-13+41-27=+17$	Cost increase
(F_1, W_3)	$(F_1, W_3) \rightarrow (F_1, W_4) \rightarrow (F_3, W_4) \rightarrow (F_3, W_3)$	$+25-13+41-18=35$	Cost increase
(F_2, W_2)	$(F_2, W_2) \rightarrow (F_2, W_3) \rightarrow (F_3, W_3) \rightarrow (F_3, W_2)$	$+18-14+18-27=-5$	Cost decrease
(F_2, W_4)	$(F_2, W_4) \rightarrow (F_3, W_4) \rightarrow (F_3, W_3) \rightarrow (F_2, W_3)$	$+23-41+18-14=-14$	Cost decrease
(F_3, W_1)	$(F_3, W_1) \rightarrow (F_3, W_3) \rightarrow (F_2, W_3) \rightarrow (F_2, W_1)$	$+32-18+14-17=+11$	Cost increase

Factory	Warehouse				Capacity
	W_1	W_2	W_3	W_4	
F_1	21 + 28	16 + 17	25 + 35	11 (13)	11
F_2	6 (17)	18 – 5	7 (14) –	23 – 15	13
F_3	32 + 9	10 (27)	5 (18) +	4 (41) –	19
Requirement	6	10	12	15	43

(4) Thus we observe that only (F_2, W_4) for which the largest reduction in cost change being-15 is -ve.

∴ The unoccupied cell (F_2, W_4) will be considered for further reduction in cost.

∴The second feasible shipment plan is shown below in table

Factory	Warehouse				Capacity
	W_1	W_2	W_3	W_4	
F_1	21	16	25	(11) 13	11
F_2	(6) 17	18	(3) 14	(4) 23	13
F_3	32	(10) 27	(9) 18	41	19
Requirement	6	10	12	15	43

The total transportation cost associated with the improved solution is

(11 x 13)+ (6 x 17)+ (3 x 14)+ (4 x 23)+ (10 x 27)+ (9 x 18) = Rs. 811

(4) We now return to step 2 and examine it transportation costs can be reduced further by replacing any of the unoccupied cells with the one actually used in the second solution.

The net changes in costs for each of the unoccupied cells are as follows:

Unoccupied Cell	*Close Path*	*Net cost change (Rs)*
(F_1, W_1)	$(F_1, W_1) \rightarrow (F_1, W_4) \rightarrow (F_2, W_4) \rightarrow (F_2, W_1)$	$21 - 13 + 23 - 17 = 14$
(F_1, W_2)	$(F_1, W_2) \rightarrow (F_1, W_4) \rightarrow (F_2, W_4) \rightarrow (F_2, W_3) \rightarrow (F_3, W_3) \rightarrow (F_3, W_2)$	$+16 - 13 + 23 - 14 + 18 - 27 = 3$
(F_1, W_3)	$(F_1, W_3) \rightarrow (F_1, W_4) \rightarrow (F_2, W_4) \rightarrow (F_2, W_3)$	$+25 - 13 + 23 - 14 = 21$
(F_2, W_2)	$(F_2, W_2) \rightarrow (F_2, W_3) \rightarrow (F_3, W_3) \rightarrow (F_3, W_2)$	$+18 - 14 + 18 - 27 = -5$
(F_3, W_1)	$(F_3, W_1) \rightarrow (F_3, W_3) \rightarrow (F_2, W_3) \rightarrow (F_2, W_3)$	$+32 - 18 + 14 - 17 = 11$
(F_3, W_4)	$(F_3, W_4) \rightarrow (F_2, W_4) \rightarrow (F_2, W_3) \rightarrow (F_3, W_3)$	$+41 - 25 + 14 - 18 = 14$

Thus the cell (F_2, W_2) with the negative value will be included in the new solution, because shipping one unit from factory F_2 to warehouse W_2 reduce the transportation cost by Rs. 8. Further since min (3, 10). 3 units can be shipped in cell (F_2, W_2) the third feasible solution is given below in table

Factory	Warehouse				Capacity
	W_1	W_2	W_3	W_4	
F_1	21	16	25	(11) 13	11
F_2	(6) 17	(3) 18	14	(4) 23	13
F_3	32	(7) 27	(12) 18	41	19
Requirement	6	10	12	15	43

The corresponding shipping cost is

(11 x3) + (6x 17) + (3x 18) + (4×23) + (7×27) + (12x 18) = Rs. 796

(5) The next step is to evaluate again all the unoccupied cells of the improved solution and see whether the total cost can be further reduced. The unoccupied cells of this improved solution are evaluated as shown below.

Unoccupied Cell	*Closed Path*	*Net cost change (Rs)*
(F_1, W_1)	$(F_1, W_1) \to (F_1, W_4) \to (F_2, W_4) \to (F_2, W_1)$	$+21-13+23-17=14$
(F_1, W_2)	$(F_1, W_2) \to (F_1, W_4) \to (F_2, W_4) \to (F_2, W_2)$	$+16-13+23-18=8$
(F_1, W_3)	$(F_1, W_3) \to (F_1, W_4) \to (F_2, W_4) \to (F_2, W_2) \to (F_3, W_2) \to (F_3, W_3)$	$+25-13+23-18+27-18=26$
(F_2, W_3)	$(F_2, W_3) \to (F_3, W_3) \to (F_3, W_2) \to (F_2, W_2)$	$+14-18+27-18=5$
(F_3, W_1)	$(F_3, W_1) \to (F_3, W_2) \to (F_2, W_2) \to (F_2, W_1)$	$+32-27+18-17=6$
(F_3, W_4)	$(F_3, W_4) \to (F_2, W_4) \to (F_2, W_2) \to (F_3, W_2)$	$41-23+18-27=+9$

Since all unoccupied ceils have positive values for the net cost change.

Hence we have reached the optimal solution.

The transportation schedule is shown in table below and the total transportation cost of the optimal solution is as given below.

From factory	*Transported to warehouse*	*Quantity*	*Unit cost*	*Total cost*
F_1	W_4	11	13	143
F_2	W_1	6	17	102
F_2	W_2	3	18	54
F_2	W_4	4	23	92
F_3	W_2	7	27	189
F_3	W_3	12	18	216
Total Transportation cost				Rs. 796

2. Modified Distribution Method (MODI):

In the modified distribution method all evaluations of all the unoccupied cells are calculated simultaneously thus only one closed path with most negative cell evaluation is traced. Therefore it provides considerable time saving as compared to the stepping stone method.

The steps are as follow:

Step (1):

Find an initial basic feasible solution consisting of m + n -1 allocations in independent position using any of the three methods.

Step (2):

Calculate a set of numbers for each row and each column. To compute u_i (i = 1, 2 . . m) for each row and u (j = 1, 2 … n) for each column.

Set $C_{ij} = u_i + v_j$ for each of the m+n-1 occupied cells used in the initial solution.

Step (3):

For unoccupied cells calculate opportunity cost by using the relationship

$\Delta_{ij} = c_{ij} – (u_i + v_i)$; I = 1, 2. . m j = 1, 2. . n

Step (4):

(i) If all Δ_{ij} > 0. then current basic feasible solution is optimal.

(ii) If Δ_{ij} =0. then current basic feasible solution will remain unaffected but an alternative solution, exists.

(iii) If any of Δ_{ij} < 0, then an improved solution can be obtained by entering unoccupied cell (i, j) in the basis. An unoccupied cell with the largest negative value of A is chosen for entering into the basic solution.

Step (5):

Construct a closed path or loop for the unoccupied cell with the largest negative opportunity cost, start the loop with unoccupied cell and mark a plus (+) sign in this cell, trace the loop along the rows (or column) to an occupied cell, mark the corner with minus (-1) sign and continue down the column (or row) to an occupied cell and mark the corner with (+) sign and (-) sign alternatively. Close the loop back to the selected unoccupied cell.

Step (6):

Select the smallest quantity amongst the cell on the corner of closed loop marked with (-) sign. Allocate this value to the selected unoccupied cell and add it to other occupied cells marked with (+) sign and subtract it from the occupied cells marked with (-) sign, so that the solution remains feasible.

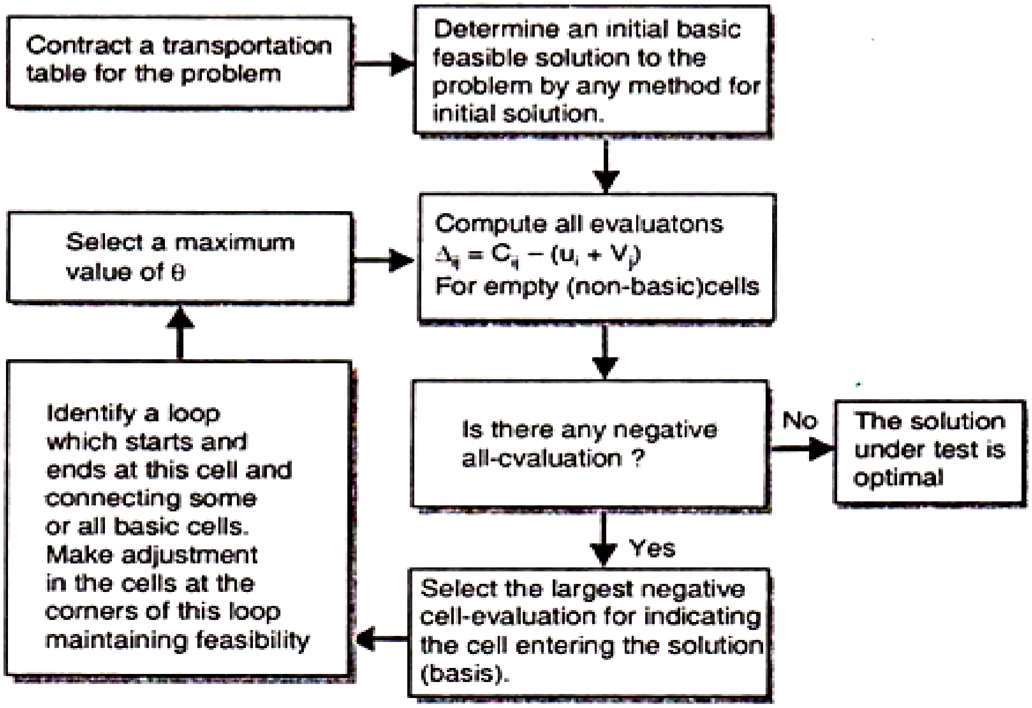
Step (7):

By allocating units to the unoccupied cell as per step (6) a new improved solution is obtained. Calculate the total transportation cost for then improved solution.

Step (8):

Test the improved solution for optimality. The procedure terminates when all $d_{ij} \geq 0$ for unoccupied cells.

The steps of MODI method can also be described by the following flow chart:



Flow chart of MODI mehtod.

Example 1:

To understand clearly MODI method we have to take initial solution obtained by Vogel's method. Then check of optimality.

Factory	Warehouse				Capacity	Row no. u_i
	W_1	W_2	W_3	W_4		
F_1	21 / 14	16 / 8	25 / 26	11 13	11	41
F_2	6 17	3 18	14 / 5	4 23	13	42
F_3	32 / 6	7 27	12 18	41	19	43
Requirement	6	10	12	15	43	
Column no vj	v_1	v_2	v_3	v_4		

Step (1):

Here we have altered the transportation table by assigning an additional row and column.

Since the no. of occupied cells are m + n – 1 = 3 + 4 – 1 = 6 the initial solution is non – degenerate. Thus an optimal solution can be obtained. The total transportation cost is = (13 x 11) + (6 x 17) + (3 x 18) + (4 x 23) + (7 x 27) + (12 x 18) = Rs. 798.

Step (2):

Here u_i indicate row values & v_j indicate column values.

u_i — value for the i^{th} row (factory) i = 1,2,3

v_j = – – j^{th} column (warehouse) j= 1,2,3,4

For the occupied cell $e_{ij} = u_i + v_j$.

Here for six occupied cells which can be described as

$$c_{14} = u_1 + v_4 = 13$$
$$c_{21} = u_2 + v_1 = 17$$
$$c_{22} = u_2 + v_2 = 18$$
$$c_{24} = u_2 + v_4 = 23$$
$$c_{32} = u_3 + v_2 = 27$$
$$c_{33} = u_3 + v_3 = 18$$

To calculate this we select u_1 and assign it zero $u_1 = 0$

$$u_1 + v_4 = 13 \Rightarrow 0 + v_4 = 13 \Rightarrow v_4 = 13$$
$$u_2 + v_1 = 17 \Rightarrow 10 + v_1 = 17 \Rightarrow v_1 = 7$$
$$u_2 + v_2 = 18 \Rightarrow 10 + v_2 = 18 \Rightarrow v_2 = 8$$
$$u_2 + v_4 = 23 \Rightarrow u_2 + 13 = 23 \Rightarrow u_2 = 10$$
$$u_3 + v_2 = 27 \Rightarrow 43 + 8 = 27 \Rightarrow u_3 = 19$$
$$u_3 + v_3 = 18 \Rightarrow 19 + v_3 = 18 \Rightarrow v_3 = -1$$

Step (3):

We now proceed to calculate $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for all the unoccupied cell

Unoccupied cell	$\Delta ij = cij - cj\ (ui + vj)$	Net cost change
(F_1, W_1)	$\Delta_{11} = 21-(0+7)=14$	+14
(F_1, W_2)	$\Delta_{12} = 16-(0+8)=8$	+ 8
(F_1, W_3)	$\Delta_{13} = 25-[0+(-1)]=26$	+ 26
(F_2, W_3)	$\Delta_{23} = 14-[10-1]=+5$	+ 5
(F_3, W_1)	$\Delta_{31} = 32-[19+7]=+6$	+ 6
(F_3, W_4)	$\Delta_{34} = 41-[19+13]=+9$	+ 9

Step (4):

Here all $\Delta_{ij} > 0$ then current basic feasible solution is optimal. The following table represents the solution of the problem with corresponding row and column numbers.

Factory	Warehouse				Capacity	Row no. ui
	W_1	W_2	W_3	W_4		
F_1	21 / 14	16 / 8	25 / 26	11 / 13	11	$u_1 = 0$
F_2	6 / 17	3 / 18	14 / 5	4 / 23	13	$u_2 = 10$
F_3	32 / 6	7 / 27	12 / 18	41 / 9	19	$u_3 = 19$
Requirement	6	10	12	15	43	
Column no vj	$v_1 = 7$	$v_2 = 8$	$v_3 = 1$	$v_4 = 13$		

According to the optimality criteria for minimization transportation problem the current solution is optimal one, since the opportunity costs of the unoccupied cells are all zero or positive.

The optimal solution to the given problem is

$x_{14} = 11,$ $\quad x_{24} = 4$

$x_{21} = 6,$ $\quad x_{32} = 7$ $\quad$ and all other $x_{ij} = 0$

$x_{22} = 3,$ $\quad x_3 = 12$

The total transportation cost is Rs. 796.