Reduced Order Observer Design

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Introduction to Reduced Order Observer Design

- In the last lecture, we have introduced the concept of controlling a dynamic system where none of the states are available for direct measurement and hence the states are to be estimated through the design of an observer.

- Another trivial way of solving this problem could be by the use of system output, provided the number of outputs available equal to the order of the system. Following the output eqn.:

$$Y = C\,X$$

$$X = C^{-1}\,Y$$

- However, there are many cases, where, number of outputs available are less (say 'r' numbers which is less than 'n' order of the system). In such a case you need a reduced order observer to estimate the 'n-r' number of states. This is also known as Luenberger Observer.

Governing Equation

Let us consider, the governing equation for the dynamic system (Plant) in state-space form as:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{Bu}, \quad \mathbf{Y} = \mathbf{C}_1\mathbf{X}_1$$

Consider, Y to be of size 'r', and partition the state vector into two parts such that $\mathbf{X} = [\mathbf{X}_1 \mid \mathbf{X}_2]^T$, where $\mathbf{X}_1$ is of size 'r' and $\mathbf{X}_2$ is of size 'n-r'. The governing equation could be subdivided similarly such that:

$$\dot{X}_1 = A_{11} X_1 + A_{12} X_2 + B_1 U$$

$$\dot{X}_2 = A_{21} X_1 + A_{22} X_2 + B_2 U$$

Estimator Equation

The states X_1 could be estimated based on directly the measured output Y such that:

$$X_1 = \mathbf{C}_1^{-1} Y$$

The states $\mathbf{X_2}$, however, has to be estimated following a similar strategy as had been done for the full order observer.

$$\dot{\hat{\mathbf{X}}}_2 = \mathbf{A}_{21}\ C_1^{-1} Y + A_{22} \hat{X}_2 + \mathbf{B}_2\, \mathbf{u}$$

Consider, the above equation in terms of a new state vector Z, such that

$$Z = \hat{X}_2 - LY,\ \ \hat{X}_2 = Z + LY$$

Observer Design based on Reduced States

The reduced order system may be expressed in terms of states Z as:

$$\dot{z} = Q\,z + R\,y + S\,u$$

Now, defining the error for the reduced order system, we can obtain the error dynamics as:

$$e_2 = x_2 - \hat{x}_2$$

$$\dot{e}_2 = A_{21}\,x_1 + A_{22}\,x_2 + B_2 u - LC_1 M - Q\,z - R\,y - S\,u$$

$$M = A_{11} x_1 + A_{12}\,x_2 + B_1 u$$

Observer design contd.

The error dynamics could be further simplified as:

$$\dot{e}_2 = Q e_2 + (A_{21} - L C_1 A_{11} - R C_1 + QLC_1)x_1 + (A_{22} - L C_1 A_{12} - Q)x_2 + (B_2 - L C_1 B_1 - S)u$$

To obtain an error dynamics which will be independent of x_2, x_1, and u, the following conditions must be satisfied:

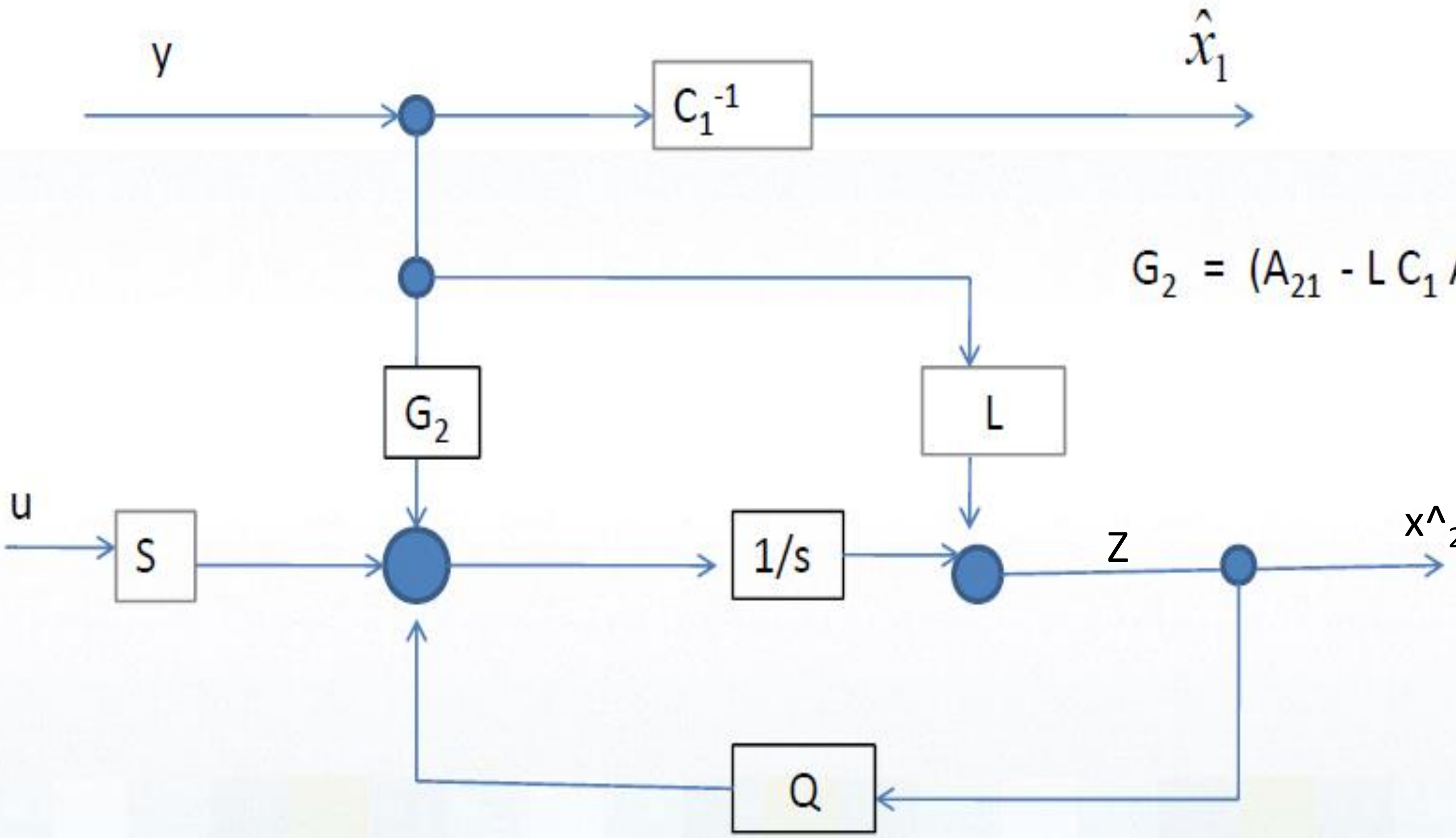
$$Q = A_{22} - LC_1 A_{12}$$

$$R = A_{21} C_1^{-1} - LC_1 A_{11} C_1^{-1} + QL$$

$$S = B_2 - LC_1 B_1$$

By selecting the observer gain *L*, one can obtain *Q*, *R* and *S*.

Reduced Order Observer in a block-diagram



$G_2 = (A_{21} - L\,C_1\,A_{11})C_1^{-1} + Q\,L$

Assignment

- Consider a networked first order hydraulic system with the following state equation

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} u$$

- Design a reduced order observer when only the first state x_1 is directly measured.

Optimal Controller Design

Introduction

- So far, we have discussed about different techniques of obtaining the control-gains to achieve desired closed-loop characteristics irrespective of the magnitude of the gains.

- It is to be understood though that higher gain implies larger power amplification which may not be possible to realize in practice.

- Hence, there is a requirement to obtain reasonable closed-loop performance using optimal control effort. A quadratic performance index may be developed in this direction, minimization of which will lead to optimal control-gain.

- The process is elaborated farther in the following discussion.

Optimal Control contd..

The dynamics of a structure is represented in state space form as:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\,\mathbf{u}\,;\mathbf{x} \in \mathrm{Re}^{n}\ ,\mathbf{u} \in \mathrm{Re}^{m}$$

Where, **A** is the system matrix derived for the passive system along with the embedded actuators and sensors, and **B** represents the influence matrix corresponding to the distributed control effort **u** for the ***m*** finite number of patches.

The derivation of **A** and **B** are discussed already in the earlier section. The conventional technique of minimizing a quadratic performance index for the design of feedback controller is as follows.

The output or measurement equation can be written as

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

Controller Design contd..

Now, assume a static output feedback of the form

$$\mathbf{u} = -\mathbf{G}\mathbf{y}$$

where, **G** is the gain matrix.
The objective is to design a controller by choosing a proper controller gain **G**, which is also optimal in the sense that it minimizes a performance index:

$$J(G) = E_{x0}\left\{\int_0^\infty (\mathbf{x}^\mathbf{T}\mathbf{Q}\mathbf{x} + \mathbf{u}^\mathbf{T}\,\mathbf{R}\mathbf{u})dt\right\}$$

Where, denotes expectation with respect to the initial state, **Q** and **R** are weighting matrices. Here,

$$E_{x0} = X_0$$

Matrix Riccati Equation:

Use of X_0 eliminates the dependency of the feedback gain on the non-zero initial condition x_0 in the optimal output feedback control. The optimization results in a set of non-linear coupled matrix equations which is given by:

$$\mathbf{K M} + \mathbf{M}' \mathbf{K} + \mathbf{Q} + \mathbf{C}' \mathbf{G}' \mathbf{R G C} = \mathbf{0}$$

$$\mathbf{L M}' + \mathbf{M L} + \mathbf{X}_0 = \mathbf{0}$$

$$\mathbf{G C L C}' + \mathbf{R}^{-1} \mathbf{B}' \mathbf{K L C}' = \mathbf{0}$$

Where, **M** denotes the closed loop system as: **M = A + BGC, L** is the Liapunov Matrix.

The optimal gain **G** may be obtained by iteratively solving the eqn. set.

Algebraic Riccati Equation

- However, this method has certain difficulties in implementation. The iterative algorithm suggested in this method requires initial stabilizing gains, which may not be always available.

- The available iterative schemes in the literature are computationally intensive and the convergence is not always ensured.

- Hence, an alternate method is adopted for the controller design. In this method, a nontrivial solution of the gain G is given by:

$$\mathbf{G} = \mathbf{R}^{-1}\mathbf{B}'\mathbf{K}\mathbf{L}\mathbf{C}'(\mathbf{C}\mathbf{L}\mathbf{C}')^{-1}$$

Optimal Controller

Here, K and L are the positive definite solutions of the following equations:

$$KA + A'K + Q - KBR^{-1}B'K = 0$$

$$(A - BR^{-1}B'K)L + L(A - BR^{-1}B'K)' + X_0 = 0$$

The first equation is known as the standard Riccati equation The second one
Is known as Lyapunov equation.

By tuning the weighting matrices Q and R, a sub-optimal controller gain G can be achieved using this method. This method gives acceptable solution in a single step (no iteration is required).

The Lyapunov equation

the *Lyapunov equation* is

$$A^T P + PA + Q = 0$$

where A, P, $Q \in \mathbf{R}^{n \times n}$, and P, Q are symmetric

interpretation: for linear system $\dot{x} = Ax$, if $V(z) = z^T P z$, then

$$\dot{V}(z) = (Az)^T P z + z^T P (Az) = -z^T Q z$$

i.e., if $z^T P z$ is the (generalized)*energy*, then $z^T Q z$ is the associated (generalized) *dissipation*

linear-quadratic Lyapunov theory: *linear* dynamics, *quadratic* Lyapunov function

Stability condition

if $P > 0$, $Q > 0$ then the system $\dot{x} = Ax$ is (globally asymptotically) stable, *i.e.*, $\Re\lambda_i < 0$

to see this, note that

$$\dot{V}(z) = -z^T Q z \leq -\lambda_{\min}(Q) z^T z \leq -\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)} z^T P z = -\alpha V(z)$$

where $\alpha = \lambda_{\min}(Q)/\lambda_{\max}(P) > 0$

Comments on Optimal Control:

The principal drawback of this scheme is that it is not possible to judge as to how far the sub-optimal solution is away from the optimal (local and / or global) solution.

As a good engineering practice, Q is tuned such that accepted closed loop response is obtained. Normally, Q is taken as the Modal matrix of the system, while R is taken as an identity matrix.

The other drawback of this system is that the closed loop system is not robust. This means that a slight variation of system parameters may drastically affect the system performance.

Direct Output Feedback Control

Introduction

- Often we come across dynamic systems which are controlled by distributed actuators and sensors. One common idea is to use distributed piezoelectric actuators and sensors for controlling the noise of a vibrating system. This is also known as “Smart Structural Control”

- A crucial issue for such systems is to place the sensors and actuators in a collocated manner, which guarantees stability of the system. In fact, one can design self sensing actuators using piezoelectric transducers.

- Such systems clearly does not need to contain any state estimator or observer. This systems can be controlled directly by generating control inputs based on the sensor outputs. The strategy is commonly known as ‘Output feedback control’.

Governing equation for a dynamic system

Consider the following governing equation of motion of a system as:

$$\mathbf{M}\ddot{\mathbf{q}}+\mathbf{C}\dot{\mathbf{q}}+\mathbf{K}\mathbf{q}=\mathbf{f}(t)$$

where, **q** represents the DOF of a flexible body, **M**, **C** and **K** represent the mass, damping and stiffness matrices of the system while **f** represent the control force acting on it.

Now consider the input f(t) of the following form

$$\mathbf{f}(t)=-C_d\,q(t)-C_v\,\dot{q}(t)$$

where, C_d and C_v represents the control gain matrices corresponding to displacement and velocity.

Governing equation for the closed loop system

Based on the sensed states, the governing equations for the closed loop system may be written as

$$\mathbf{M}\,\ddot{\mathbf{q}} + (\mathbf{C} + C_v)\,\dot{\mathbf{q}} + (\mathbf{K} + C_d)\,\mathbf{q} = 0$$

Note that the displacement feedback is particularly necessary for positive semi definite systems for which **K** could have zero diagonal terms. The governing equation could also be represented in energy form by pre-multiplying the above eqn. with $\dot{q}$

$$\frac{d}{dt}\left[\frac{1}{2}\left\{\dot{q}^{T} M \dot{q} + q^{T}(K + C_d)\,q\right\}\right] = -\dot{q}^{T}(C + C_v)\,\dot{q}$$

Decoupled Governing Equation:

The last equation suggests that C_d and C_v should be positive definite such that even if the stiffness matrix (K) and/or the damping matrix C are positive semi-definite, the system could be stabilized by the control gains.

Expressing the governing equation in state-space form we obtain

$$X = \begin{Bmatrix} q \\ \dot{q} \end{Bmatrix}$$

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -M^{-1}(K + C_d) & -M^{-1}(C + C_v) \end{bmatrix} X$$

Decoupled form

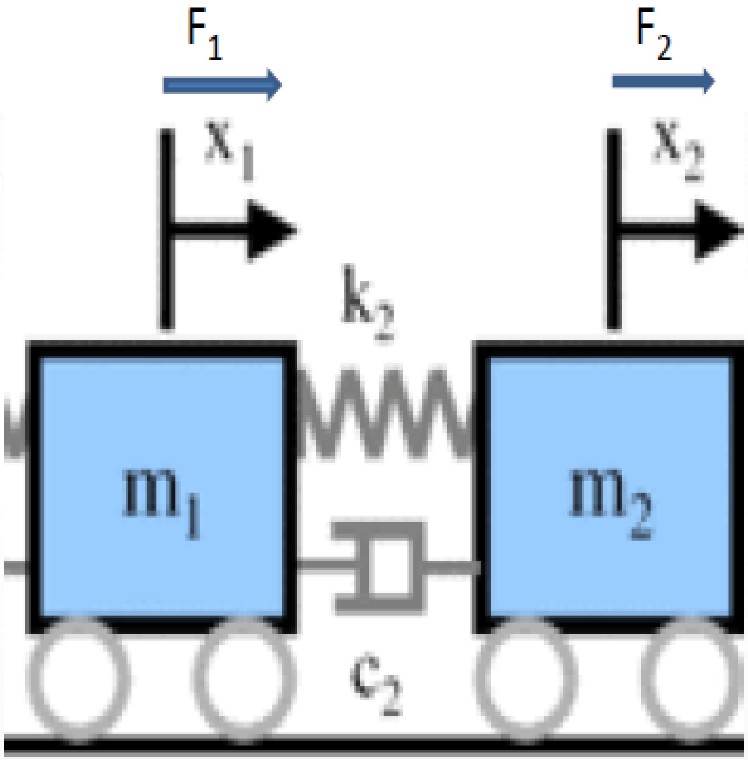
The last governing equation may also be expressed in terms of eigen values such that

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Lambda} & -C_{diag} \end{bmatrix} \mathbf{x}$$

Where, Λ denotes the eigen-value matrix and C_{diag} represents the damping Matrix in diagonal form . By choosing the control gain C_d you can choose the closed loop frequencies and then by varying C_v , you can add suitable damping to the mode.

Example: Modelling a Joined-Bus system

Example: Abstraction of the System



The Governing equation of motion of the system will be same as discussed earlier with the following matrices:

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix},\ C = \begin{bmatrix} c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix},\ K = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

$$X = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix},\ F = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

Let us choose some numerical values – $m_1 = m_2 = 1$, $c_2 = 0.5$ and $k_2 = 10$

Governing Equation

In this problem, let us assume that two sensors are used to measure the position and velocity of mass $m_{1,}$ The driving force is also proportional to the measured values of these two states such that

$$f_1 = -(G_1 x_1 + G_2 x_2),\ f_2 = 0$$

The modified K and C matrices will now be

$$K = \begin{bmatrix} 10+G_1 & -10 \\ -10 & 10 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.5+G_2 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

Independent Modal Space Control

Introduction

- State-space based representation of a flexible-body system even though is attractive from the mathematical point of view – there are two major difficulties associated with such representation.

- Firstly, the representation of any *n*-DOF system through 2n state-space equations is computationally challenging. Considering for example, dynamic analysis of a crude rocket structure would need about 1200 DOF model. Solving the eigen-value problem of 2400 states is usually avoided.

- Secondly, the solution would produce 2400 poles all of which are impossible to control. One way, to alleviate this problem is to decouple the states by representing them in terms of modal co-ordinates and then consider only the low frequency modes for control.

Governing Equation for a flexible system

The process is described farther with the following governing equation of motion

$$\mathbf{M}\,\ddot{\mathbf{q}} + \mathbf{C}\,\dot{\mathbf{q}} + \mathbf{K}\,\mathbf{q} = \mathbf{f}$$

where, **q** represents the DOF of a flexible body, **M**, **C** and **K** represent the mass, damping and stiffness matrices of the system while **f** represent the control force acting on it.

The control input force could be farther represented as

$$\mathbf{f} = \mathbf{B}\,\mathbf{u}$$

where, **B** represents the actuator influence matrix and **u** represents the control input.

Governing equation in Modal form

The output of the system may be represented in terms of displacement and velocity feedback as

$$\mathbf{y} = \mathbf{C}_d\,\mathbf{q} + \mathbf{C}_v\,\dot{\mathbf{q}}$$

Now, considering the generalized eigen-value problem associated with the system to be

$$\mathbf{K}\,\boldsymbol{\Phi} = \lambda\mathbf{M}\boldsymbol{\Phi}$$

you can represent the governing equation in terms of modal coordinates **z** as follows in the next slide.

Decoupled Governing Equation:

The governing equation in modal coordinates may be written as:

$$\mathbf{q} = \mathbf{\Phi}\,\mathbf{z}$$

$$\mathbf{I}\ddot{\mathbf{z}} + \overline{\mathbf{C}}\,\dot{\mathbf{z}} + \mathbf{\Lambda}\,\mathbf{z} = \overline{\mathbf{B}}\mathbf{u}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} 2\xi\omega_1 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 2\xi\omega_n \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} \omega_1{}^2 & 0 & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & \omega_n{}^2 \end{bmatrix}, \ \overline{\mathbf{B}} = \mathbf{\Phi}^{-1}\mathbf{B}$$

Where, **ϕ** denotes the eigen vectors., **I** the identity matrix, the rest of the terms are standard as explained earlier.

Note that the left hand side of the new governing equation is completely decoupled, the coupling only occurs through the right hand side (forcing function).

State Space form

The governing equation may be expressed in state space form as follows:

$$\mathbf{x} = \begin{bmatrix} \mathbf{z} \\ \dot{\mathbf{z}} \end{bmatrix}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Lambda} & -\overline{\mathbf{C}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{0} \\ \overline{\mathbf{B}} \end{bmatrix} \mathbf{u}$$

$$\mathbf{y} = \begin{bmatrix} \mathbf{C}_\mathbf{d} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_\mathbf{v} \end{bmatrix} \mathbf{x}$$

Decoupling the states:

The complete set of state vector could now be divided into two parts modal vector that is retained in a model could be denoted as $\mathbf{x}_R$ and the modal vector that is not modelled could be denoted as $\mathbf{x}_N$.

Considering the location of *m* no. of actuators, are from a_{-1} to a_m the new decoupled state-space equations could be expressed as

$$\dot{\mathbf{x}}_{\mathrm{R}} = \mathbf{A}_{\mathrm{R}}\,\mathbf{x}_{\mathrm{R}} + \mathbf{B}_{\mathrm{R}}\,\mathbf{u}$$

$$\dot{\mathbf{x}}_{\mathrm{N}} = \mathbf{A}_{\mathrm{N}}\,\mathbf{x}_{\mathrm{N}} + \mathbf{B}_{\mathrm{N}}\,\mathbf{u}$$

$$\mathbf{A}_{\mathrm{R}} = \begin{bmatrix} \mathbf{0} & \mathbf{I}_{\mathrm{R}} \\ -\boldsymbol{\Lambda}_{\mathrm{R}} & -\overline{\mathbf{C}}_{\mathrm{R}} \end{bmatrix}, \mathbf{B}_{\mathrm{R}} = \begin{bmatrix} \mathbf{0} \\ \overline{\mathbf{B}}_{\mathrm{R}} \end{bmatrix}, \overline{\mathbf{B}}_{\mathrm{R}} = \begin{bmatrix} \boldsymbol{\Phi}_1(a_1) & \cdots & \boldsymbol{\Phi}_1(a_m) \\ \vdots & \vdots & \vdots \\ \boldsymbol{\Phi}_{\mathrm{R}}(a_1) & \cdots & \boldsymbol{\Phi}_{\mathrm{R}}(a_m) \end{bmatrix}$$

Design of Controller:

Let us consider the controller in the form

$$\mathbf{u} = -\mathbf{K}_R\ \mathbf{x}_R$$

where, K_R represents the gain matrix corresponding to the retained states of the system.

The closed-loop poles corresponding to the truncated model could be designed by following the same technique as discussed in earlier lecture for full state feedback control.
However, you may note that the control input **u** would also be used partly to excite the un-modelled modes x_N of the system. This may cause control-spillover due to the excitation of the ignored modes which may create instability of the system.

Special References for this lecture

- Dynamics and Control of Structures, Meirovitch, Wiley-Eastern
- *Design of Feedback Control Systems* – Stefani, Shahian, Savant, Hostetter

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