Root Locus Techniques

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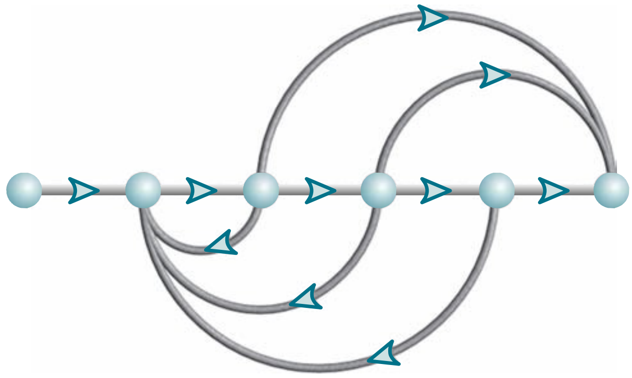

Chapter Learning Outcomes

After completing this chapter the student will be able to:

- Define a root locus (Sections 8.1–8.2)
- State the properties of a root locus (Section 8.3)
- Sketch a root locus (Section 8.4)
- Find the coordinates of points on the root locus and their associated gains (Sections 8.5–8.6)
- Use the root locus to design a parameter value to meet a transient response specification for systems of order 2 and higher (Sections 8.7–8.8)
- Sketch the root locus for positive-feedback systems (Section 8.9)
- Find the root sensitivity for points along the root locus (Section 8.10)

Case Study Learning Outcomes

You will be able to demonstrate your knowledge of the chapter objectives with case studies as follows:

- Given the antenna azimuth position control system shown on the front endpapers, you will be able to find the preamplifier gain to meet a transient response specification.
- Given the pitch or heading control system for the Unmanned Free-Swimming Submersible vehicle shown on the back endpapers, you will be able to plot the root locus and design the gain to meet a transient response specification. You will then be able to evaluate other performance characteristics.

8.1 Introduction

Root locus, a graphical presentation of the closed-loop poles as a system parameter is varied, is a powerful method of analysis and design for stability and transient response (*Evans, 1948; 1950*). Feedback control systems are difficult to comprehend from a qualitative point of view, and hence they rely heavily upon mathematics. The root locus covered in this chapter is a graphical technique that gives us the qualitative description of a control system's performance that we are looking for and also serves as a powerful quantitative tool that yields more information than the methods already discussed.

Up to this point, gains and other system parameters were designed to yield a desired transient response for only first- and second-order systems. Even though the root locus can be used to solve the same kind of problem, its real power lies in its ability to provide solutions for systems of order higher than 2. For example, under the right conditions, a fourth-order system's parameters can be designed to yield a given percent overshoot and settling time using the concepts learned in Chapter 4.

The root locus can be used to describe qualitatively the performance of a system as various parameters are changed. For example, the effect of varying gain upon percent overshoot, settling time, and peak time can be vividly displayed. The qualitative description can then be verified with quantitative analysis.

Besides transient response, the root locus also gives a graphical representation of a system's stability. We can clearly see ranges of stability, ranges of instability, and the conditions that cause a system to break into oscillation.

Before presenting root locus, let us review two concepts that we need for the ensuing discussion: (1) the control system problem and (2) complex numbers and their representation as vectors.

The Control System Problem

We have previously encountered the control system problem in Chapter 6: Whereas the poles of the open-loop transfer function are easily found (typically, they are known by inspection and do not change with changes in system gain), the poles of the closed-loop transfer function are more difficult to find (typically, they cannot be found without factoring the closed-loop system's characteristic polynomial, the denominator of the closed-loop transfer function), and further, the closed-loop poles change with changes in system gain.

A typical closed-loop feedback control system is shown in Figure 8.1(a). The open-loop transfer function was defined in Chapter 5 as $KG(s)H(s)$. Ordinarily, we

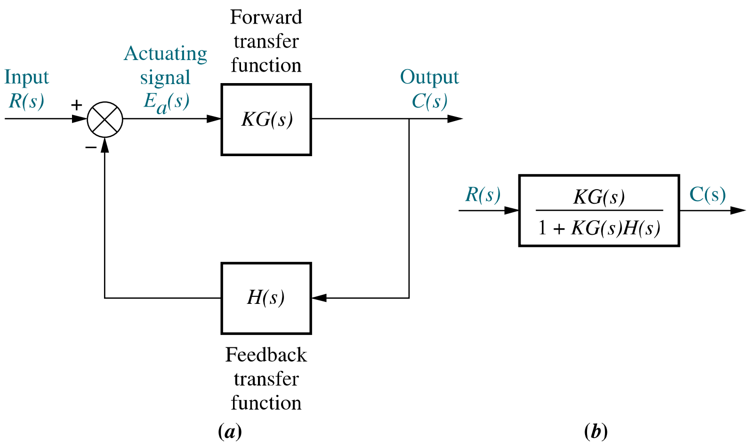

FIGURE 8.1 **a.** Closed-loop system; **b.** equivalent transfer function

can determine the poles of $KG(s)H(s)$, since these poles arise from simple cascaded first- or second-order subsystems. Further, variations in K do not affect the location of any pole of this function. On the other hand, we cannot determine the poles of $T(s) = KG(s)/[1 + KG(s)H(s)]$ unless we factor the denominator. Also, the poles of $T(s)$ change with K.

Let us demonstrate. Letting

$$G(s) = \frac{N_G(s)}{D_G(s)} \tag{8.1}$$

and

$$H(s) = \frac{N_H(s)}{D_H(s)} \tag{8.2}$$

then

$$\boxed{T(s) = \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + KN_G(s)N_H(s)}} \tag{8.3}$$

where N and D are factored polynomials and signify numerator and denominator terms, respectively. We observe the following: Typically, we know the factors of the numerators and denominators of $G(s)$ and $H(s)$. Also, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known and in fact can change with K. For example, if $G(s) = (s+1)/[s(s+2)]$ and $H(s) = (s+3)/(s+4)$, the poles of $KG(s)H(s)$ are $0, -2,$ and -4. The zeros of $KG(s)H(s)$ are -1 and -3. Now, $T(s) = K(s+1)(s+4)/[s^3 + (6+K)s^2 + (8+4K)s + 3K]$. Thus, the zeros of $T(s)$ consist of the zeros of $G(s)$ and the poles of $H(s)$. The poles of $T(s)$ are not immediately known without factoring the denominator, and they are a function of K. Since the system's transient response and stability are dependent upon the poles of $T(s)$, we have no knowledge of the system's performance unless we factor the denominator for specific values of K. The root locus will be used to give us a vivid picture of the poles of $T(s)$ as K varies.

Vector Representation of Complex Numbers

Any *complex number*, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector, as shown in Figure 8.2(a). The complex number also can be described in polar form with magnitude M and angle θ, as $M\angle\theta$. If the complex number is substituted into a complex function, $F(s)$, another complex number will result. For example, if $F(s) = (s+a)$, then substituting the complex number $s = \sigma + j\omega$ yields $F(s) = (\sigma + a) + j\omega$, another complex number. This number is shown in Figure 8.2(b). Notice that $F(s)$ has a zero at $-a$. If we translate the vector a units to the left, as in Figure 8.2(c), we have an alternate representation of the complex number that originates at the zero of $F(s)$ and terminates on the point $s = \sigma + j\omega$.

We conclude that $(s+a)$ *is a complex number and can be represented by a vector drawn from the zero of the function to the point s.* For example, $(s+7)|_{s\to 5+j2}$ is a complex number drawn from the zero of the function, -7, to the point s, which is $5 + j2$, as shown in Figure 8.2(d).

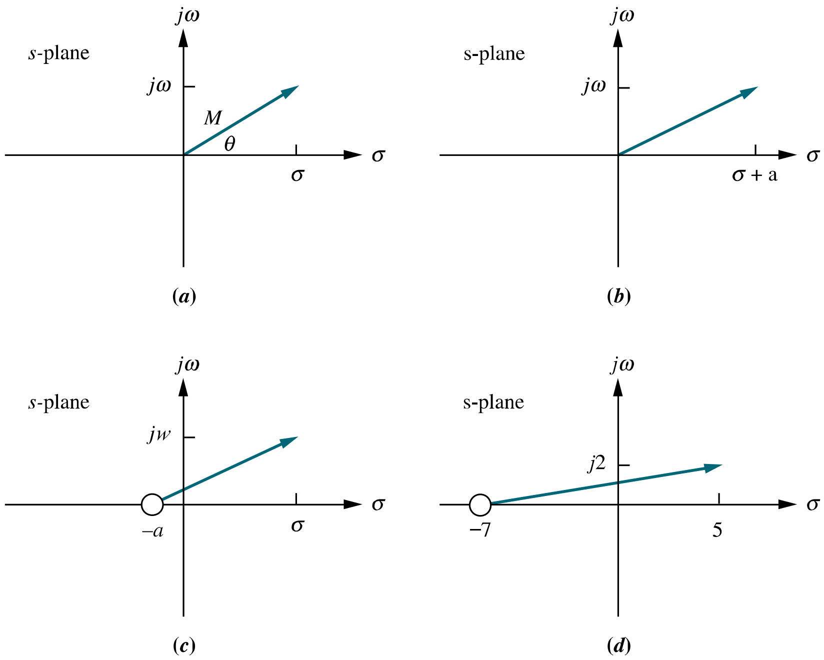

FIGURE 8.2 Vector representation of complex numbers: **a.** $s = \sigma + j\omega$; **b.** $(s + a)$; **c.** alternate representation of $(s + a)$; **d.** $(s + 7)|_{s \to 5 + j2}$

Now let us apply the concepts to a complicated function. Assume a function

$$F(s) = \frac{\prod_{i=1}^{m}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = \frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}} \tag{8.4}$$

where the symbol $\prod$ means "product," m = number of zeros, and n = number of poles. Each factor in the numerator and each factor in the denominator is a complex number that can be represented as a vector. The function defines the complex arithmetic to be performed in order to evaluate $F(s)$ at any point, s. Since each complex factor can be thought of as a vector, the magnitude, M, of $F(s)$ at any point, s, is

$$M = \frac{\prod \text{zero lengths}}{\prod \text{pole lengths}} = \frac{\prod_{i=1}^{m}|(s + z_i)|}{\prod_{j=1}^{n}|(s + p_j)|} \tag{8.5}$$

where a zero length, $|(s + z_i)|$, is the magnitude of the vector drawn from the zero of $F(s)$ at $-z_i$ to the point s, and a pole length, $|(s + p_j)|$, is the magnitude of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s. The angle, θ, of $F(s)$ at any point, s, is

$$\begin{aligned}\theta &= \sum \text{zero angles} - \sum \text{pole angles} \\ &= \sum_{i=1}^{m} \angle(s + z_i) - \sum_{j=1}^{n} \angle(s + p_j)\end{aligned} \tag{8.6}$$

where a zero angle is the angle, measured from the positive extension of the real axis, of a vector drawn from the zero of $F(s)$ at $-z_i$ to the point s, and a pole angle is the angle, measured from the positive extension of the real axis, of the vector drawn from the pole of $F(s)$ at $-p_j$ to the point s.

As a demonstration of the above concept, consider the following example.

Example 8.1

Evaluation of a Complex Function via Vectors

PROBLEM: Given

$$F(s) = \frac{(s+1)}{s(s+2)} \tag{8.7}$$

find $F(s)$ at the point $s = -3 + j4$.

SOLUTION: The problem is graphically depicted in Figure 8.3, where each vector, $(s + \alpha)$, of the function is shown terminating on the selected point $s = -3 + j4$. The vector originating at the zero at -1 is

$$\sqrt{20}\angle 116.6^\circ \tag{8.8}$$

The vector originating at the pole at the origin is

$$5\angle 126.9^\circ \tag{8.9}$$

The vector originating at the pole at -2 is

$$\sqrt{17}\angle 104.0^\circ \tag{8.10}$$

Substituting Eqs. (8.8) through (8.10) into Eqs. (8.5) and (8.6) yields

$$M\angle\theta = \frac{\sqrt{20}}{5\sqrt{17}}\angle 116.6^\circ - 126.9^\circ - 104.0^\circ = 0.217\angle - 114.3^\circ \tag{8.11}$$

as the result for evaluating $F(s)$ at the point $-3 + j4$.

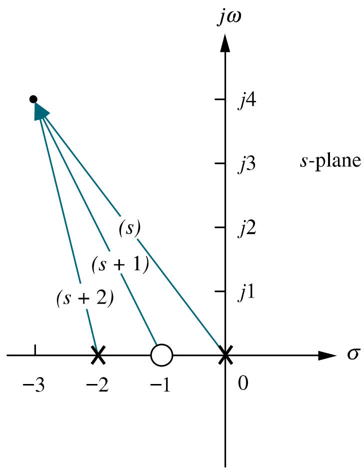

FIGURE 8.3 Vector representation of Eq. (8.7)

Skill-Assessment Exercise 8.1

PROBLEM: Given

$$F(s) = \frac{(s+2)(s+4)}{s(s+3)(s+6)}$$

find $F(s)$ at the point $s = -7 + j9$ the following ways:

a. Directly substituting the point into $F(s)$

b. Calculating the result using vectors

ANSWER:

$$-0.0339 - j0.0899 = 0.096\angle - 110.7^\circ$$

The complete solution is at www.wiley.com/college/nise.

TryIt 8.1

Use the following MATLAB statements to solve the problem given in Skill-Assessment Exercise 8.1.

```
s=-7+9j;
G=(s+2)*(s+4)/...
 (s*(s+3)*(s+6));
Theta=(180/pi)*...
 angle(G)
M=abs(G)
```

We are now ready to begin our discussion of the root locus.

8.2 Defining the Root Locus

A security camera system similar to that shown in Figure 8.4(a) can automatically follow a subject. The tracking system monitors pixel changes and positions the camera to center the changes.

The root locus technique can be used to analyze and design the effect of loop gain upon the system's transient response and stability. Assume the block diagram representation of a tracking system as shown in Figure 8.4(b), where the closed-loop poles of the system change location as the gain, K, is varied. Table 8.1, which was formed by applying the quadratic formula to the denominator of the transfer function in Figure 8.4(c), shows the variation of pole location for different values of gain, K. The data of Table 8.1 is graphically displayed in Figure 8.5(a), which shows each pole and its gain.

As the gain, K, increases in Table 8.1 and Figure 8.5(a), the closed-loop pole, which is at -10 for $K = 0$, moves toward the right, and the closed-loop pole, which is at 0 for $K = 0$, moves toward the left. They meet at -5, break away from the real axis, and move into the complex plane. One closed-loop pole moves upward while the other moves downward. We cannot tell which pole moves up or which moves down. In Figure 8.5(b), the individual closed-loop pole locations are removed and their paths are represented with solid lines. It is this *representation of the paths of the*

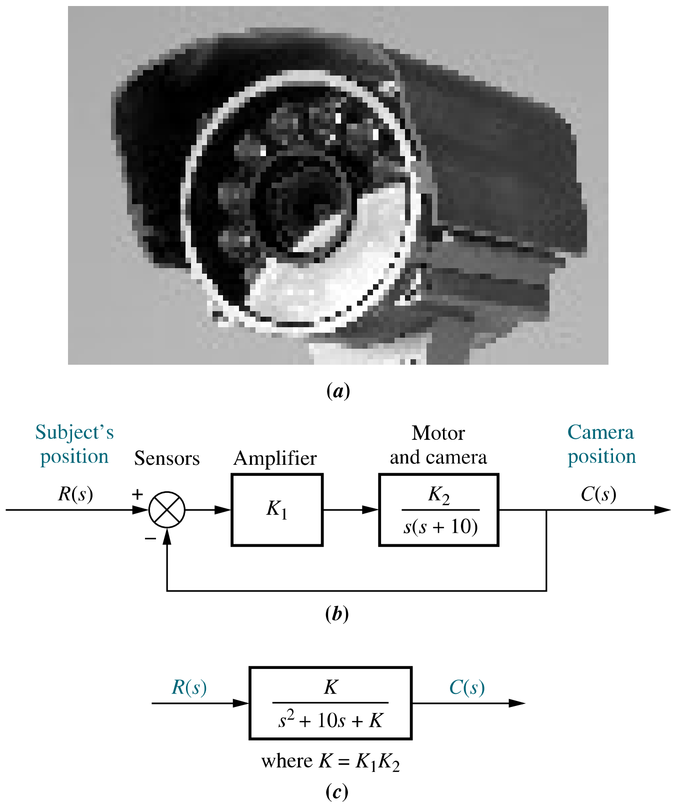

FIGURE 8.4 **a.** Security cameras with auto tracking can be used to follow moving objects automatically; **b.** block diagram; **c.** closed-loop transfer function

TABLE 8.1 Pole location as function of gain for the system of Figure 8.4

K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5+j2.24$	$-5-j2.24$
35	$-5+j3.16$	$-5-j3.16$
40	$-5+j3.87$	$-5-j3.87$
45	$-5+j4.47$	$-5-j4.47$
50	$-5+j5$	$-5-j5$

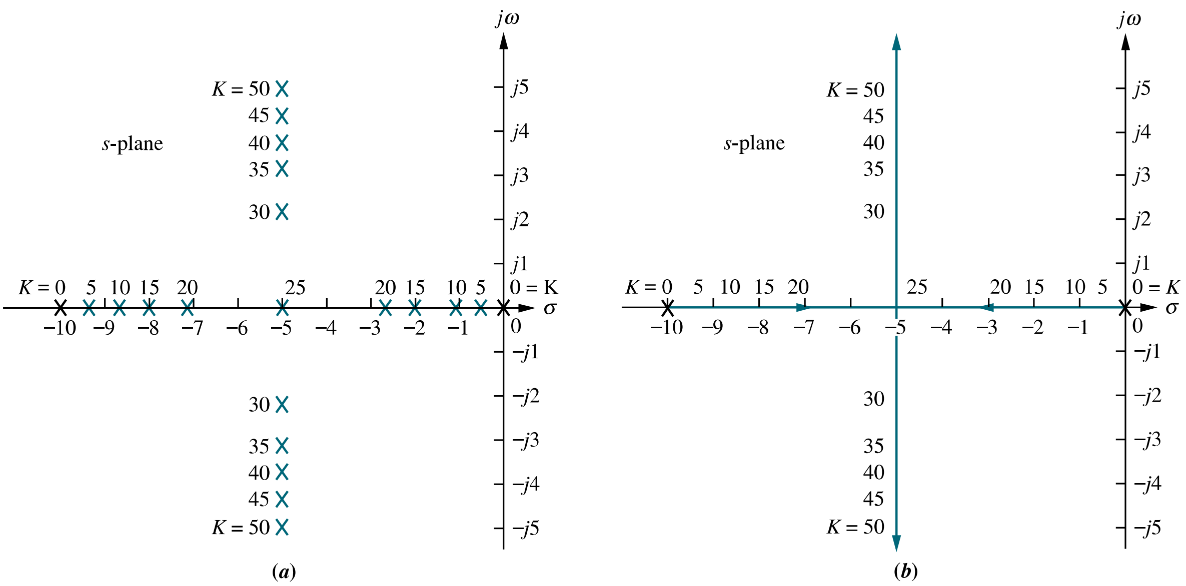

FIGURE 8.5 **a.** Pole plot from Table 8.1; **b.** root locus

closed-loop poles as the gain is varied that we call a *root locus*. For most of our work, the discussion will be limited to positive gain, or $K \geq 0$.

The root locus shows the changes in the transient response as the gain, K, varies. First of all, the poles are real for gains less than 25. Thus, the system is overdamped. At a gain of 25, the poles are real and multiple and hence critically damped. For gains above 25, the system is underdamped. Even though these preceding conclusions were available through the analytical techniques covered in Chapter 4, the following conclusions are graphically demonstrated by the root locus.

Directing our attention to the underdamped portion of the root locus, we see that regardless of the value of gain, the real parts of the complex poles are always the same.

Since the settling time is inversely proportional to the real part of the complex poles for this second-order system, the conclusion is that regardless of the value of gain, the settling time for the system remains the same under all conditions of underdamped responses.

Also, as we increase the gain, the damping ratio diminishes, and the percent overshoot increases. The damped frequency of oscillation, which is equal to the imaginary part of the pole, also increases with an increase in gain, resulting in a reduction of the peak time. Finally, since the root locus never crosses over into the right half-plane, the system is always stable, regardless of the value of gain, and can never break into a sinusoidal oscillation.

These conclusions for such a simple system may appear to be trivial. What we are about to see is that the analysis is applicable to systems of order higher than 2. For these systems, it is difficult to tie transient response characteristics to the pole location. The root locus will allow us to make that association and will become an important technique in the analysis and design of higher-order systems.

8.3 Properties of the Root Locus

In Section 8.2, we arrived at the root locus by factoring the second-order polynomial in the denominator of the transfer function. Consider what would happen if that polynomial were of fifth or tenth order. Without a computer, factoring the polynomial would be quite a problem for numerous values of gain.

We are about to examine the properties of the root locus. From these properties we will be able to make a rapid *sketch* of the root locus for higher-order systems without having to factor the denominator of the closed-loop transfer function.

The properties of the root locus can be derived from the general control system of Figure 8.1(a). The closed-loop transfer function for the system is

$$\boxed{T(s) = \frac{KG(s)}{1 + KG(s)H(s)}} \tag{8.12}$$

From Eq. (8.12), a pole, s, exists when the characteristic polynomial in the denominator becomes zero, or

$$\boxed{KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots} \tag{8.13}$$

where -1 is represented in polar form as $1\angle(2k+1)180^\circ$. Alternately, a value of s is a closed-loop pole if

$$\boxed{|KG(s)H(s)| = 1} \tag{8.14}$$

and

$$\boxed{\angle KG(s)H(s) = (2k+1)180^\circ} \tag{8.15}$$

Equation (8.13) implies that if a value of s is substituted into the function $KG(s)H(s)$, a complex number results. If the angle of the complex number is an odd multiple of 180°, that value of s is a system pole for some particular value of K. What value of K? Since the angle criterion of Eq. (8.15) is satisfied, all that remains is to satisfy the magnitude criterion, Eq. (8.14). Thus,

$$\boxed{K = \frac{1}{|G(s)||H(s)|}} \tag{8.16}$$

We have just found that a pole of the closed-loop system causes the angle of $KG(s)H(s)$, or simply $G(s)H(s)$ since K is a scalar, to be an odd multiple of 180°. Furthermore, the magnitude of $KG(s)H(s)$ must be unity, implying that the value of K is the reciprocal of the magnitude of $G(s)H(s)$ when the pole value is substituted for s.

Let us demonstrate this relationship for the second-order system of Figure 8.4. The fact that closed-loop poles exist at -9.47 and -0.53 when the gain is 5 has already been established in Table 8.1. For this system,

$$KG(s)H(s) = \frac{K}{s(s+10)} \tag{8.17}$$

Substituting the pole at -9.47 for s and 5 for K yields $KG(s)H(s) = -1$. The student can repeat the exercise for other points in Table 8.1 and show that each case yields $KG(s)H(s) = -1$.

It is helpful to visualize graphically the meaning of Eq. (8.15). Let us apply the complex number concepts reviewed in Section 8.1 to the root locus of the system shown in Figure 8.6. For this system the open-loop transfer function is

$$KG(s)H(s) = \frac{K(s+3)(s+4)}{(s+1)(s+2)} \tag{8.18}$$

The closed-loop transfer function, $T(s)$, is

$$T(s) = \frac{K(s+3)(s+4)}{(1+K)s^2 + (3+7K)s + (2+12K)} \tag{8.19}$$

If point s is a closed-loop system pole for some value of gain, K, then s must satisfy Eqs. (8.14) and (8.15).

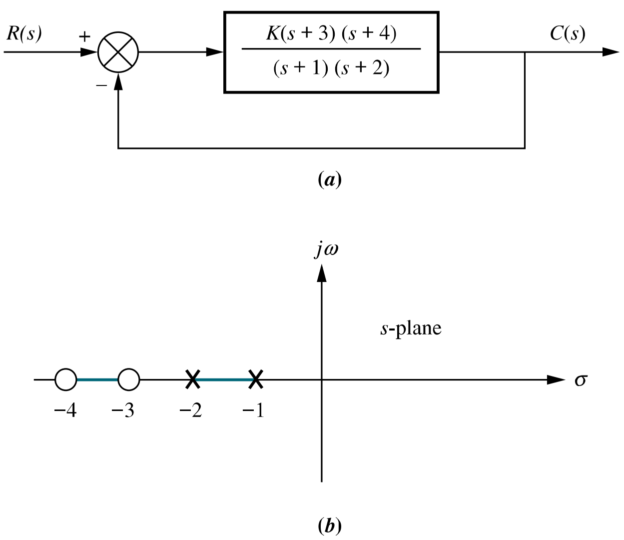

FIGURE 8.6 **a.** Example system; **b.** pole-zero plot of $G(s)$

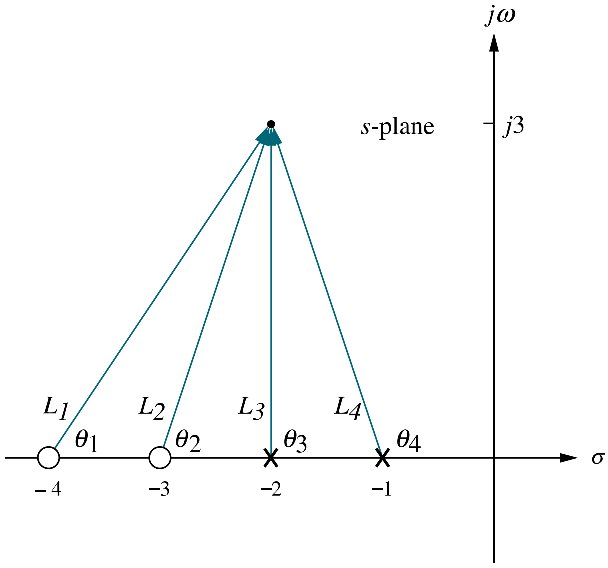

FIGURE 8.7 Vector representation of $G(s)$ from Figure 8.6(a) at $-2 + j3$

Consider the point $-2 + j3$. If this point is a closed-loop pole for some value of gain, then the angles of the zeros minus the angles of the poles must equal an odd multiple of 180°. From Figure 8.7,

$$\theta_1 + \theta_2 - \theta_3 - \theta_4 = 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ = -70.55^\circ \tag{8.20}$$

Therefore, $-2 + j3$ is not a point on the root locus, or alternatively, $-2 + j3$ is not a closed-loop pole for any gain.

If these calculations are repeated for the point $-2 + j(\sqrt{2}/2)$, the angles do add up to 180°. That is, $-2 + j(\sqrt{2}/2)$ is a point on the root locus for some value of gain. We now proceed to evaluate that value of gain.

From Eqs. (8.5) and (8.16),

$$\boxed{K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{pole lengths}}{\prod \text{zero lengths}}} \tag{8.21}$$

Looking at Figure 8.7 with the point $-2 + j3$ replaced by $-2 + j(\sqrt{2}/2)$, the gain, K, is calculated as

$$K = \frac{L_3 L_4}{L_1 L_2} = \frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)(1.22)} = 0.33 \tag{8.22}$$

Thus, the point $-2 + j(\sqrt{2}/2)$ is a point on the root locus for a gain of 0.33.

We summarize what we have found as follows: Given the poles and zeros of the open-loop transfer function, $KG(s)H(s)$, a point in the s-plane is on the root locus for a particular value of gain, K, if the angles of the zeros minus the angles of the poles, all drawn to the selected point on the s-plane, add up to $(2k + 1)180^\circ$. Furthermore, gain K at that point for which the angles add up to $(2k + 1)180^\circ$ is found by dividing the product of the pole lengths by the product of the zero lengths.

Skill-Assessment Exercise 8.2

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Control Solutions

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2+4s+13)}$$

do the following:

a. Calculate the angle of $G(s)$ at the point $(-3+j0)$ by finding the algebraic sum of angles of the vectors drawn from the zeros and poles of $G(s)$ to the given point.

b. Determine if the point specified in **a** is on the root locus.

c. If the point specified in **a** is on the root locus, find the gain, K, using the lengths of the vectors.

ANSWERS:

a. Sum of angles $= 180°$

b. Point is on the root locus

c. $K = 10$

The complete solution is at www.wiley.com/college/nise.

TryIt 8.2

Use MATLAB and the following statements to solve Skill-Assessment Exercise 8.2.

```
s=-3+0j;
G=(s+2)/(s^2+4*s+13);
Theta=(180/pi)*...
 angle(G)
M=abs(G);
K=1/M
```

8.4 Sketching the Root Locus

It appears from our previous discussion that the root locus can be obtained by sweeping through every point in the s-plane to locate those points for which the angles, as previously described, add up to an odd multiple of 180°. Although this task is tedious without the aid of a computer, the concept can be used to develop rules that can be used to *sketch* the root locus without the effort required to *plot* the locus. Once a sketch is obtained, it is possible to accurately plot just those points that are of interest to us for a particular problem.

The following five rules allow us to sketch the root locus using minimal calculations. The rules yield a sketch that gives intuitive insight into the behavior of a control system. In the next section, we refine the sketch by finding actual points or angles on the root locus. These refinements, however, require some calculations or the use of computer programs, such as MATLAB.

1. **Number of branches.** Each closed-loop pole moves as the gain is varied. If we define a *branch* as the path that one pole traverses, then there will be one branch for each closed-loop pole. Our first rule, then, defines the number of branches of the root locus:

   The number of branches of the root locus equals the number of closed-loop poles.

   As an example, look at Figure 8.5(b), where the two branches are shown. One originates at the origin, the other at -10.

2. **Symmetry.** If complex closed-loop poles do not exist in conjugate pairs, the resulting polynomial, formed by multiplying the factors containing the closed-loop poles,

would have complex coefficients. Physically realizable systems cannot have complex coefficients in their transfer functions. Thus, we conclude:

The root locus is symmetrical about the real axis.

An example of symmetry about the real axis is shown in Figure 8.5(*b*).

3. **Real-axis segments.** Let us make use of the angle property, Eq. (8.15), of the points on the root locus to determine where the real-axis segments of the root locus exist. Figure 8.8 shows the poles and zeros of a general open-loop system. If an attempt is made to calculate the angular contribution of the poles and zeros at each point, P_1, P_2, P_3, and P_4, along the real axis, we observe the following: (1) At each point the angular contribution of a pair of open-loop complex poles or zeros is zero, and (2) the contribution of the open-loop poles and open-loop zeros to the left of the respective point is zero. The conclusion is that the only contribution to the angle at any of the points comes from the open-loop, real-axis poles and zeros that exist to the right of the respective point. If we calculate the angle at each point using only the open-loop, real-axis poles and zeros to the right of each point, we note the following: (1) The angles on the real axis alternate between 0° and 180°, and (2) the angle is 180° for regions of the real axis that exist to the left of an odd number of poles and/or zeros. The following rule summarizes the findings:

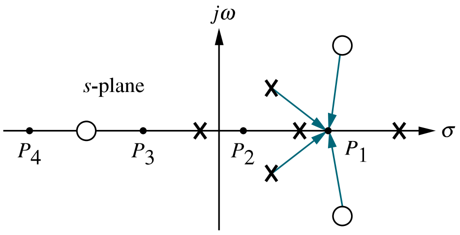


FIGURE 8.8 Poles and zeros of a general open-loop system with test points, P_i, on the real axis

On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

Examine Figure 8.6(*b*). According to the rule just developed, the real-axis segments of the root locus are between -1 and -2 and between -3 and -4 as shown in Figure 8.9.

4. **Starting and ending points.** Where does the root locus begin (zero gain) and end (infinite gain)? The answer to this question will enable us to expand the sketch of the root locus beyond the real-axis segments. Consider the closed-loop transfer function, $T(s)$, described by Eq. (8.3). $T(s)$ can now be evaluated for both large and small gains, K. As K approaches zero (small gain),

$$T(s) \approx \frac{KN_G(s)D_H(s)}{D_G(s)D_H(s) + \epsilon} \tag{8.23}$$

From Eq. (8.23) we see that the closed-loop system poles at small gains approach the combined poles of $G(s)$ and $H(s)$. We conclude that the root locus begins at the poles of $G(s)H(s)$, the open-loop transfer function.

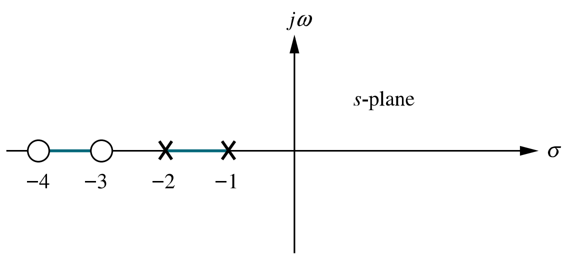


FIGURE 8.9 Real-axis segments of the root locus for the system of Figure 8.6

At high gains, where K is approaching infinity,

$$T(s) \approx \frac{KN_G(s)D_H(s)}{\epsilon + KN_G(s)N_H(s)} \tag{8.24}$$

From Eq. (8.24) we see that the closed-loop system poles at large gains approach the combined zeros of $G(s)$ and $H(s)$. Now we conclude that the root locus ends at the zeros of $G(s)H(s)$, the open-loop transfer function.

Summarizing what we have found:

The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Remember that these poles and zeros are the open-loop poles and zeros.

In order to demonstrate this rule, look at the system in Figure 8.6(*a*), whose real-axis segments have been sketched in Figure 8.9. Using the rule just derived, we find that the root locus begins at the poles at -1 and -2 and ends at the zeros at -3 and -4 (see Figure 8.10). Thus, the poles start out at -1 and -2 and move through the real-axis space between the two poles. They meet somewhere between the two poles and break out into the complex plane, moving as complex conjugates. The poles return to the real axis somewhere between the zeros at -3 and -4, where their path is completed as they move away from each other, and end up, respectively, at the two zeros of the open-loop system at -3 and -4.

5. **Behavior at infinity.** Consider applying Rule 4 to the following open-loop transfer function:

$$KG(s)H(s) = \frac{K}{s(s+1)(s+2)} \tag{8.25}$$

There are three finite poles, at $s = 0, -1,$ and -2, and no finite zeros.

A function can also have *infinite* poles and zeros. If the function approaches infinity as s approaches infinity, then the function has a pole at infinity. If the function approaches zero as s approaches infinity, then the function has a zero at infinity. For example, the function $G(s) = s$ has a pole at infinity, since $G(s)$ approaches infinity as s approaches infinity. On the other hand, $G(s) = 1/s$ has a zero at infinity, since $G(s)$ approaches zero as s approaches infinity.

Every function of s has an equal number of poles and zeros if we include the infinite poles and zeros as well as the finite poles and zeros. In this example,

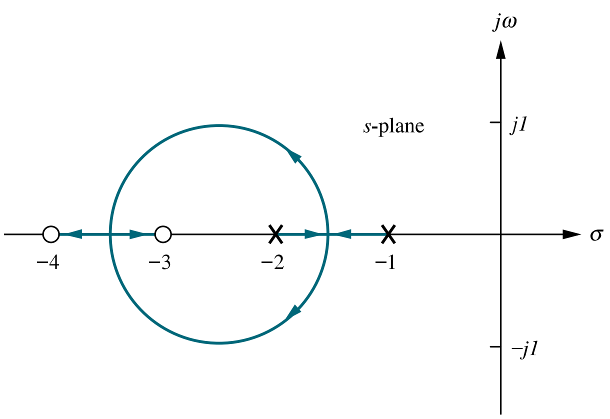


FIGURE 8.10 Complete root locus for the system of Figure 8.6

Eq. (8.25) contains three finite poles and three infinite zeros. To illustrate, let s approach infinity. The open-loop transfer function becomes

$$KG(s)H(s) \approx \frac{K}{s^3} = \frac{K}{s \cdot s \cdot s} \tag{8.26}$$

Each s in the denominator causes the open-loop function, $KG(s)H(s)$, to become zero as that s approaches infinity. Hence, Eq. (8.26) has three zeros at infinity.

Thus, for Eq. (8.25), the root locus begins at the finite poles of $KG(s)H(s)$ and ends at the infinite zeros. The question remains: Where are the infinite zeros? We must know where these zeros are in order to show the locus moving from the three finite poles to the three infinite zeros. Rule 5 helps us locate these zeros at infinity. Rule 5 also helps us locate poles at infinity for functions containing more finite zeros than finite poles.[1]

We now state Rule 5, which will tell us what the root locus looks like as it approaches the zeros at infinity or as it moves from the poles at infinity. The derivation can be found in Appendix M.1 at www.wiley.com/college/nise.

The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a as follows:

$$\boxed{\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}} \tag{8.27}$$

$$\boxed{\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}} \tag{8.28}$$

where $k = 0, \pm1, \pm2, \pm3$ and the angle is given in radians with respect to the positive extension of the real axis.

Notice that the running index, k, in Eq. (8.28) yields a multiplicity of lines that account for the many branches of a root locus that approach infinity. Let us demonstrate the concepts with an example.

Example 8.2

Sketching a Root Locus with Asymptotes

PROBLEM: Sketch the root locus for the system shown in Figure 8.11.

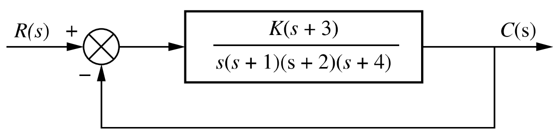


FIGURE 8.11 System for Example 8.2.

[1] Physical systems, however, have more finite poles than finite zeros, since the implied differentiation yields infinite output for discontinuous input functions, such as step inputs.

SOLUTION: Let us begin by calculating the asymptotes. Using Eq. (8.27), the real-axis intercept is evaluated as

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3} \tag{8.29}$$

The angles of the lines that intersect at $-4/3$, given by Eq. (8.28), are

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \tag{8.30a}$$

$$= \pi/3 \qquad \text{for } k = 0 \tag{8.30b}$$

$$= \pi \qquad \text{for } k = 1 \tag{8.30c}$$

$$= 5\pi/3 \qquad \text{for } k = 2 \tag{8.30d}$$

If the value for k continued to increase, the angles would begin to repeat. The number of lines obtained equals the difference between the number of finite poles and the number of finite zeros.

Rule 4 states that the locus begins at the open-loop poles and ends at the open-loop zeros. For the example there are more open-loop poles than open-loop zeros. Thus, there must be zeros at infinity. The asymptotes tell us how we get to these zeros at infinity.

Figure 8.12 shows the complete root locus as well as the asymptotes that were just calculated. Notice that we have made use of all the rules learned so far. The real-axis segments lie to the left of an odd number of poles and/or zeros. The locus starts at the open-loop poles and ends at the open-loop zeros. For the example there is only one open-loop finite zero and three infinite zeros. Rule 5, then, tells us that the three zeros at infinity are at the ends of the asymptotes.

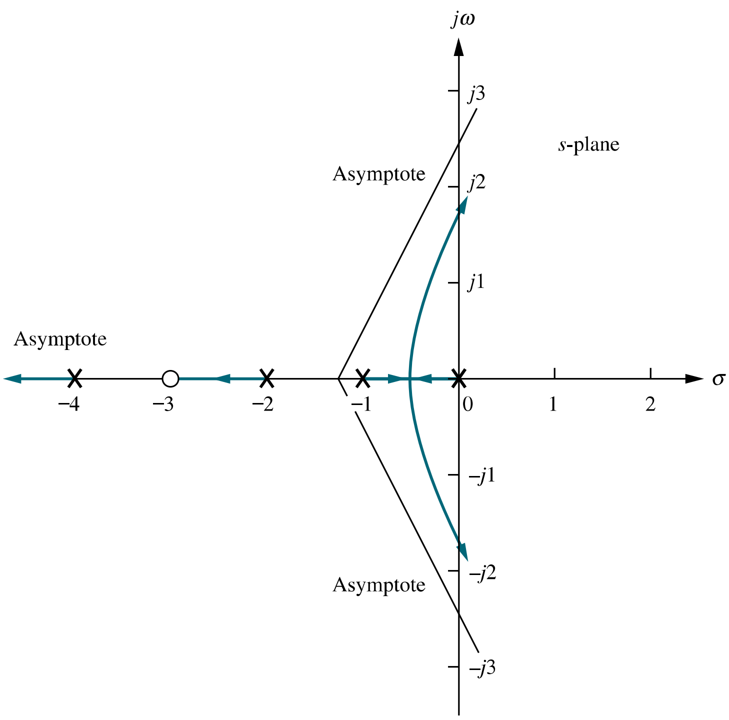


FIGURE 8.12 Root locus and asymptotes for the system of Figure 8.11

Skill-Assessment Exercise 8.3

PROBLEM: Sketch the root locus and its asymptotes for a unity feedback system that has the forward transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

8.5 Refining the Sketch

The rules covered in the previous section permit us to sketch a root locus rapidly. If we want more detail, we must be able to accurately find important points on the root locus along with their associated gain. Points on the real axis where the root locus enters or leaves the complex plane—real-axis breakaway and break-in points—and the $j\omega$-axis crossings are candidates. We can also derive a better picture of the root locus by finding the angles of departure and arrival from complex poles and zeros, respectively.

In this section, we discuss the calculations required to obtain specific points on the root locus. Some of these calculations can be made using the basic root locus relationship that the sum of the zero angles minus the sum of the pole angles equals an odd multiple of 180°, and the gain at a point on the root locus is found as the ratio of (1) the product of pole lengths drawn to that point to (2) the product of zero lengths drawn to that point. We have yet to address how to implement this task. In the past, an inexpensive tool called a Spirule™ added the angles together rapidly and then quickly multiplied and divided the lengths to obtain the gain. Today we can rely on hand-held or programmable calculators as well as personal computers.

Students pursuing MATLAB will learn how to apply it to the root locus at the end of Section 8.6. Other alternatives are discussed in Appendix H.2 at www.wiley.com/college/nise. The discussion can be adapted to programmable hand-held calculators. All readers are encouraged to select a computational aid at this point. Root locus calculations can be labor intensive if hand calculations are used.

We now discuss how to refine our root locus sketch by calculating real-axis breakaway and break-in points, $j\omega$-axis crossings, angles of departure from complex poles, and angles of arrival to complex zeros. We conclude by showing how to find accurately any point on the root locus and calculate the gain.

Real-Axis Breakaway and Break-In Points

Numerous root loci appear to break away from the real axis as the system poles move from the real axis to the complex plane. At other times the loci appear to return to the real axis as a pair of complex poles becomes real. We illustrate this in Figure 8.13. This locus is sketched using the first four rules: (1) number of branches, (2) symmetry, (3) real-axis segments, and (4) starting and ending points. The figure shows a root locus leaving the real axis between -1 and -2 and returning to the real axis between $+3$ and $+5$. The point where the locus leaves the real axis, $-\sigma_1$, is called the *breakaway point*, and the point where the locus returns to the real axis, σ_2, is called the *break-in point*.

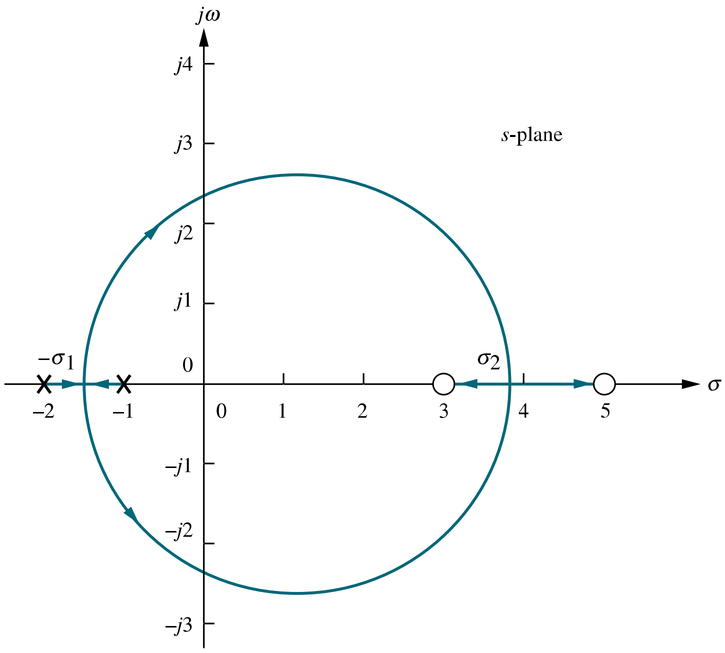


FIGURE 8.13 Root locus example showing real-axis breakaway ($-\sigma_1$) and break-in points (σ_2)

At the breakaway or break-in point, the branches of the root locus form an angle of $180°/n$ with the real axis, where n is the number of closed-loop poles arriving at or departing from the single breakaway or break-in point on the real axis (*Kuo, 1991*). Thus, for the two poles shown in Figure 8.13, the branches at the breakaway point form 90° angles with the real axis.

We now show how to find the breakaway and break-in points. As the two closed-loop poles, which are at -1 and -2 when $K = 0$, move toward each other, the gain increases from a value of zero. We conclude that the gain must be maximum along the real axis at the point where the breakaway occurs, somewhere between -1 and -2. Naturally, the gain increases above this value as the poles move into the complex plane. We conclude that the breakaway point occurs at a point of maximum gain on the real axis between the open-loop poles.

Now let us turn our attention to the break-in point somewhere between $+3$ and $+5$ on the real axis. When the closed-loop complex pair returns to the real axis, the gain will continue to increase to infinity as the closed-loop poles move toward the open-loop zeros. It must be true, then, that the gain at the break-in point is the minimum gain found along the real axis between the two zeros.

The sketch in Figure 8.14 shows the variation of real-axis gain. The breakaway point is found at the maximum gain between -1 and -2, and the break-in point is found at the minimum gain between $+3$ and $+5$.

There are three methods for finding the points at which the root locus breaks away from and breaks into the real axis. The first method is to maximize and minimize the gain, K, using differential calculus. For all points on the root locus, Eq. (8.13) yields

$$K = -\frac{1}{G(s)H(s)} \tag{8.31}$$

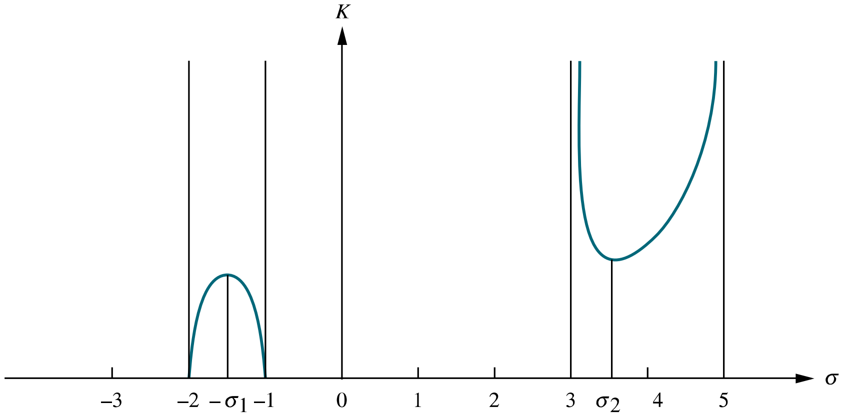


FIGURE 8.14 Variation of gain along the real axis for the root locus of Figure 8.13

For points along the real-axis segment of the root locus where breakaway and break-in points could exist, $s = \sigma$. Hence, along the real axis Eq. (8.31) becomes

$$K = -\frac{1}{G(\sigma)H(\sigma)} \tag{8.32}$$

This equation then represents a curve of K versus σ similar to that shown in Figure 8.14. Hence, if we differentiate Eq. (8.32) with respect to σ and set the derivative equal to zero, we can find the points of maximum and minimum gain and hence the breakaway and break-in points. Let us demonstrate.

Example 8.3

Breakaway and Break-in Points via Differentiation

PROBLEM: Find the breakaway and break-in points for the root locus of Figure 8.13, using differential calculus.

SOLUTION: Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.13 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)} \tag{8.33}$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$. Hence,

$$\frac{K(\sigma^2-8\sigma+15)}{(\sigma^2+3\sigma+2)} = -1 \tag{8.34}$$

Solving for K, we find

$$K = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)} \tag{8.35}$$

Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2-26\sigma-61)}{(\sigma^2-8\sigma+15)^2} = 0 \tag{8.36}$$

Solving for σ, we find $\sigma = -1.45$ and 3.82, which are the breakaway and break-in points.

The second method is a variation on the differential calculus method. Called the *transition method*, it eliminates the step of differentiation (*Franklin, 1991*). This method, derived in Appendix M.2 at www.wiley.com/college/nise, is now stated:

Breakaway and break-in points satisfy the relationship

$$\boxed{\sum_{1}^{m}\frac{1}{\sigma+z_i}=\sum_{1}^{n}\frac{1}{\sigma+p_i}} \tag{8.37}$$

where z_i and p_i are the negative of the zero and pole values, respectively, of $G(s)H(s)$.

Solving Eq. (8.37) for σ, the real-axis values that minimize or maximize K, yields the breakaway and break-in points without differentiating. Let us look at an example.

Example 8.4

Breakaway and Break-in Points Without Differentiation

PROBLEM: Repeat Example 8.3 without differentiating.

SOLUTION: Using Eq. (8.37),

$$\frac{1}{\sigma-3}+\frac{1}{\sigma-5}=\frac{1}{\sigma+1}+\frac{1}{\sigma+2} \tag{8.38}$$

Simplifying,

$$11\sigma^2-26\sigma-61=0 \tag{8.39}$$

Hence, $\sigma=-1.45$ and 3.82, which agrees with Example 8.3.

For the third method, the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise can be used to find the breakaway and break-in points. Simply use the program to search for the point of maximum gain between -1 and -2 and to search for the point of minimum gain between $+3$ and $+5$. Table 8.2 shows the results of the search. The locus leaves the axis at -1.45, the point of maximum gain between -1 and -2, and reenters the real axis at $+3.8$, the point of minimum gain between $+3$ and $+5$. These results are the same as those obtained using the first two methods. MATLAB also has the capability of finding breakaway and break-in points.

The $j\omega$-Axis Crossings

We now further refine the root locus by finding the imaginary-axis crossings. The importance of the $j\omega$-axis crossings should be readily apparent. Looking at Figure 8.12, we see that the system's poles are in the left half-plane up to a particular value of gain. Above this value of gain, two of the closed-loop system's poles move into the right half-plane, signifying that the system is unstable. The $j\omega$-axis crossing is a point on the root locus that separates the stable operation of the system from the unstable operation. The value of ω at the axis crossing yields the frequency of oscillation, while the gain at the $j\omega$-axis crossing yields, for this example, the maximum positive gain for system stability. We should note here that other examples

TABLE 8.2 Data for breakaway and break-in points for the root locus of Figure 8.13

Real-axis value	Gain		Comment
−1.41	0.008557		
−1.42	0.008585		
−1.43	0.008605		
−1.44	0.008617		
−1.45	0.008623	←	Max. gain: breakaway
−1.46	0.008622		
3.3	44.686		
3.4	37.125		
3.5	33.000		
3.6	30.667		
3.7	29.440		
3.8	29.000	←	Min. gain: break-in
3.9	29.202		

illustrate instability at small values of gain and stability at large values of gain. These systems have a root locus starting in the right–half-plane (unstable at small values of gain) and ending in the left–half-plane (stable for high values of gain).

To find the $j\omega$-axis crossing, we can use the Routh-Hurwitz criterion, covered in Chapter 6, as follows: Forcing a row of zeros in the Routh table will yield the gain; going back one row to the even polynomial equation and solving for the roots yields the frequency at the imaginary-axis crossing.

Example 8.5

Frequency and Gain at Imaginary-Axis Crossing

PROBLEM: For the system of Figure 8.11, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?

SOLUTION: The closed-loop transfer function for the system of Figure 8.11 is

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K} \tag{8.40}$$

Using the denominator and simplifying some of the entries by multiplying any row by a constant, we obtain the Routh array shown in Table 8.3.

A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros. Thus,

$$-K^2 - 65K + 720 = 0 \tag{8.41}$$

From this equation K is evaluated as

$$K = 9.65 \tag{8.42}$$

TABLE 8.3 Routh table for Eq. (8.40)

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\dfrac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

Forming the even polynomial by using the s^2 row with $K=9.65$, we obtain

$$(90-K)s^2+21K=80.35s^2+202.7=0 \tag{8.43}$$

and s is found to be equal to $\pm j1.59$. Thus the root locus crosses the $j\omega$-axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \le K < 9.65$.

Another method for finding the $j\omega$-axis crossing (or any point on the root locus, for that matter) uses the fact that at the $j\omega$-axis crossing, the sum of angles from the finite open-loop poles and zeros must add to $(2k+1)180^\circ$. Thus, we can search $j\omega$-axis until we find the point that meets this angle condition. A computer program, such as the root locus program discussed in Appendix H.2 at www.wiley .com/college/nise or MATLAB, can be used for this purpose. Subsequent examples in this chapter use this method to determine the $j\omega$-axis crossing.

Angles of Departure and Arrival

In this subsection, we further refine our sketch of the root locus by finding angles of departure and arrival from complex poles and zeros. Consider Figure 8.15, which shows the open-loop poles and zeros, some of which are complex. The root locus starts at the open-loop poles and ends at the open-loop zeros. In order to sketch the root locus more accurately, we want to calculate the root locus departure angle from the complex poles and the arrival angle to the complex zeros.

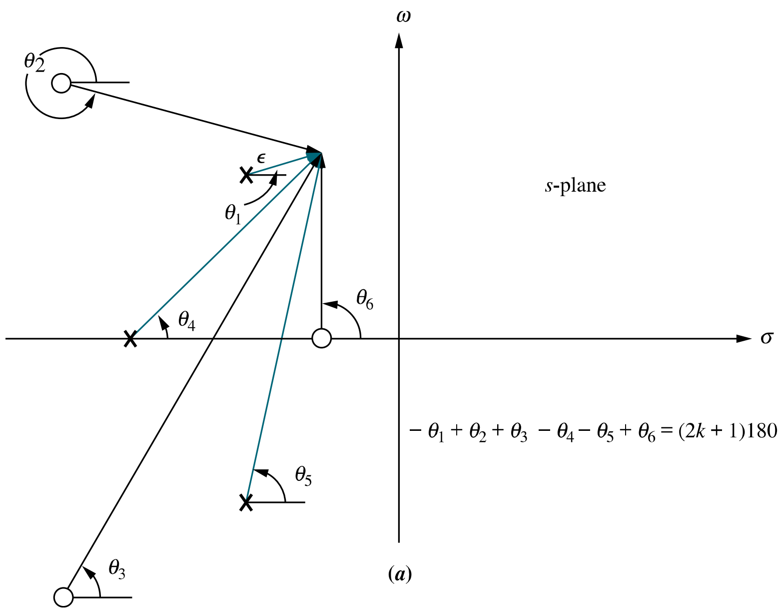
If we assume a point on the root locus ϵ close to a complex pole, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180°. Except for the pole that is ϵ close to the point, we assume all angles drawn from all other poles and zeros are drawn directly to the pole that is near the point. Thus, the only unknown angle in the sum is the angle drawn from the pole that is ϵ close. We can solve for this unknown angle, which is also the angle of departure from this complex pole. Hence, from Figure 8.15(a),

$$-\theta_1+\theta_2+\theta_3-\theta_4-\theta_5+\theta_6=(2k+1)180^\circ \tag{8.44a}$$

or

$$\theta_1=\theta_2+\theta_3-\theta_4-\theta_5+\theta_6-(2k+1)180^\circ \tag{8.44b}$$

If we assume a point on the root locus ϵ close to a complex zero, the sum of angles drawn from all finite poles and zeros to this point is an odd multiple of 180°. Except for the zero that is ϵ close to the point, we can assume all angles drawn from all other poles and zeros are drawn directly to the zero that is near the point. Thus,



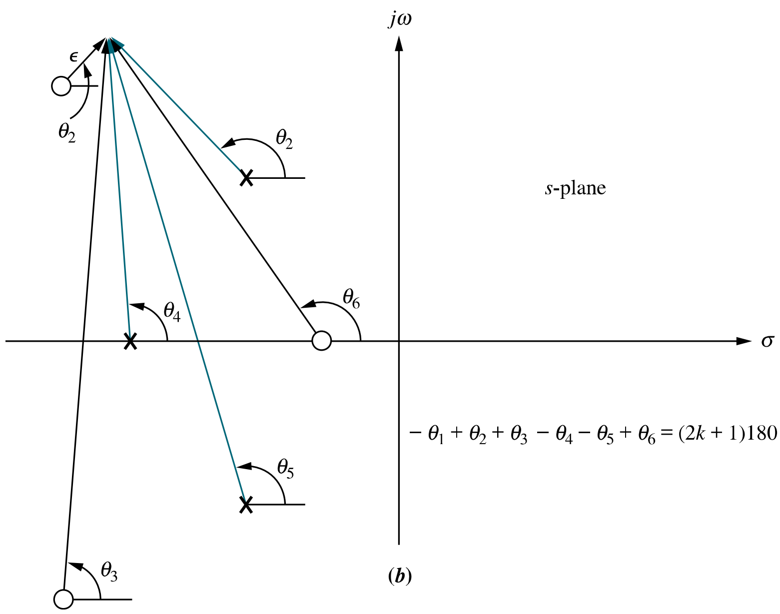


FIGURE 8.15 Open-loop poles and zeros and calculation of **a.** angle of departure; **b.** angle of arrival

the only unknown angle in the sum is the angle drawn from the zero that is ϵ close. We can solve for this unknown angle, which is also the angle of arrival to this complex zero. Hence, from Figure 8.15(*b*),

$$-\theta_1 + \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 = (2k+1)180^\circ \quad (8.45a)$$

or

$$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k+1)180^\circ \quad (8.45b)$$

Let us look at an example.

Example 8.6

Angle of Departure from a Complex Pole

PROBLEM: Given the unity feedback system of Figure 8.16, find the angle of departure from the complex poles and sketch the root locus.

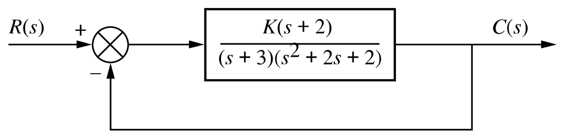


FIGURE 8.16 Unity feedback system with complex poles

SOLUTION: Using the poles and zeros of $G(s) = (s+2)/[(s+3)(s^2+2s+2)]$ as plotted in Figure 8.17, we calculate the sum of angles drawn to a point ϵ close to the complex pole, $-1+j1$, in the second quadrant. Thus,

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -\theta_1 - 90^\circ + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ \tag{8.46}$$

from which $\theta = -251.6^\circ = 108.4^\circ$. A sketch of the root locus is shown in Figure 8.17. Notice how the departure angle from the complex poles helps us to refine the shape.

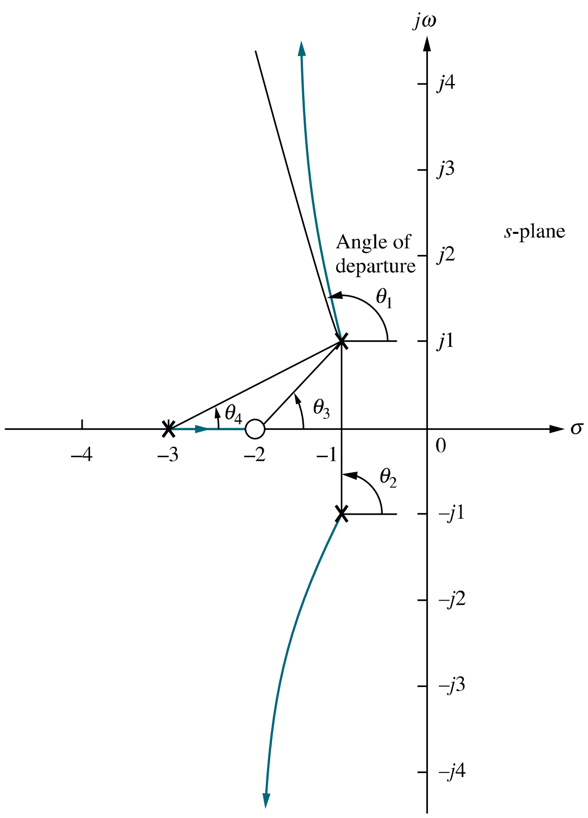


FIGURE 8.17 Root locus for system of Figure 8.16 showing angle of departure

Plotting and Calibrating the Root Locus

Once we sketch the root locus using the rules from Section 8.4, we may want to accurately locate points on the root locus as well as find their associated gain. For example, we might want to know the exact coordinates of the root locus as it crosses the radial line representing 20% overshoot. Further, we also may want the value of gain at that point.

Consider the root locus shown in Figure 8.12. Let us assume we want to find the exact point at which the locus crosses the 0.45 damping ratio line and the gain at that point. Figure 8.18 shows the system's open-loop poles and zeros along with the $\zeta = 0.45$ line. If a few test points along the $\zeta = 0.45$ line are selected, we can evaluate their angular sum and locate that point where the angles add up to an odd multiple of $180°$. It is at this point that the root locus exists. Equation (8.20) can then be used to evaluate the gain, K, at that point.

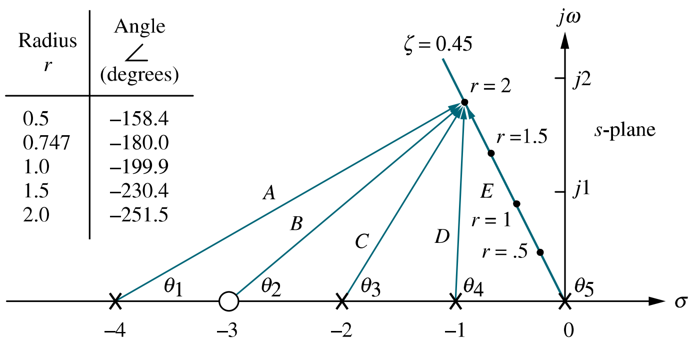
Selecting the point at radius 2 ($r = 2$) on the $\zeta = 0.45$ line, we add the angles of the zeros and subtract the angles of the poles, obtaining

$$\theta_2 - \theta_1 - \theta_3 - \theta_4 - \theta_5 = -251.5° \tag{8.47}$$

Since the sum is not equal to an odd multiple of $180°$, the point at radius $= 2$ is not on the root locus. Proceeding similarly for the points at radius $= 1.5$, 1, 0.747, and 0.5, we obtain the table shown in Figure 8.18. This table lists the points, giving their radius, r, and the sum of angles indicated by the symbol $\angle$. From the table we see that the point at radius 0.747 is on the root locus, since the angles add up to $-180°$. Using Eq. (8.21), the gain, K, at this point is

$$K = \frac{|A||C||D||E|}{|B|} = 1.71 \tag{8.48}$$

In summary, *we search a given line for the point yielding a summation of angles (zero angles–pole angles) equal to an odd multiple* of $180°$. We conclude that the point is on the root locus. The gain at that point is then found by *multiplying the pole lengths drawn to that point and dividing by the product of the zero lengths drawn to that point*. A computer program, such as that discussed in Appendix H.2 at www.wiley.com/college/nise or MATLAB, can be used.



Radius r	Angle $\angle$ (degrees)
0.5	−158.4
0.747	−180.0
1.0	−199.9
1.5	−230.4
2.0	−251.5

FIGURE 8.18 Finding and calibrating exact points on the root locus of Figure 8.12

Skill-Assessment Exercise 8.4

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2 - 4s + 13)}$$

do the following:

a. Sketch the root locus.
b. Find the imaginary-axis crossing.
c. Find the gain, K, at the $j\omega$-axis crossing.
d. Find the break-in point.
e. Find the angle of departure from the complex poles.

ANSWERS:

a. See solution at www.wiley.com/college/nise.
b. $s = \pm j\sqrt{21}$
c. $K = 4$
d. Break-in point $= -7$
e. Angle of departure $= -233.1^\circ$

The complete solution is at www.wiley.com/college/nise.

8.6 An Example

We now review the rules for sketching and finding points on the root locus, as well as present an example. The root locus is the path of the closed-loop poles of a system as a parameter of the system is varied. Each point on the root locus satisfies the angle condition, $\angle G(s)H(s) = (2k+1)180^\circ$. Using this relationship, rules for sketching and finding points on the root locus were developed and are now summarized:

Basic Rules for Sketching the Root Locus

Number of branches The number of branches of the root locus equals the number of closed-loop poles.

Symmetry The root locus is symmetrical about the real axis.

Real-axis segments On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.

Starting and ending points The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

Behavior at infinity The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes are given by the real-axis intercept and angle in radians as follows:

$$\boxed{\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}} \tag{8.49}$$

$$\boxed{\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}} \tag{8.50}$$

where $k = 0, \pm 1, \ \pm 2, \ \pm 3, \ldots$.

Additional Rules for Refining the Sketch

Real-axis breakaway and break-in points The root locus breaks away from the real axis at a point where the gain is maximum and breaks into the real axis at a point where the gain is minimum.

Calculation of $j\omega$-axis crossings The root locus crosses the $j\omega$-axis at the point where $\angle G(s)H(s) = (2k+1)180°$. Routh-Hurwitz or a search of the $j\omega$-axis for $(2k+1)180°$ can be used to find the $j\omega$-axis crossing.

Angles of departure and arrival The root locus departs from complex, open-loop poles and arrives at complex, open-loop zeros at angles that can be calculated as follows. Assume a point ϵ close to the complex pole or zero. Add all angles drawn from all open-loop poles and zeros to this point. The sum equals $(2k+1)180°$. The only unknown angle is that drawn from the ϵ close pole or zero, since the vectors drawn from all other poles and zeros can be considered drawn to the complex pole or zero that is ϵ close to the point. Solving for the unknown angle yields the angle of departure or arrival.

Plotting and calibrating the root locus All points on the root locus satisfy the relationship $\angle G(s)H(s) = (2k+1)180°$. The gain, K, at any point on the root locus is given by

$$\boxed{K = \frac{1}{|G(s)H(s)|} = \frac{1}{M} = \frac{\prod \text{finite pole lengths}}{\prod \text{finite zero lengths}}} \tag{8.51}$$

Let us now look at a summary example.

Example 8.7

Sketching a Root Locus and Finding Critical Points

PROBLEM: Sketch the root locus for the system shown in Figure 8.19(a) and find the following:

a. The exact point and gain where the locus crosses the 0.45 damping ratio line
b. The exact point and gain where the locus crosses the $j\omega$-axis
c. The breakaway point on the real axis
d. The range of K within which the system is stable

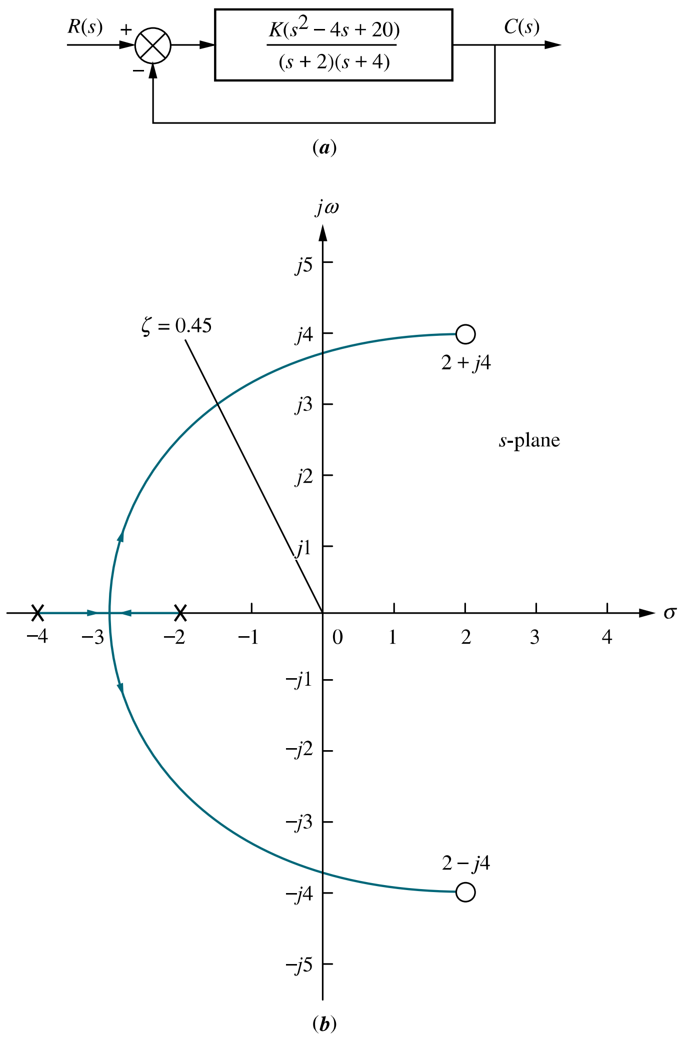


FIGURE 8.19 **a.** System for Example 8.7; **b.** root locus sketch.

SOLUTION: The problem solution is shown, in part, in Figure 8.19(*b*). First sketch the root locus. Using Rule 3, the real-axis segment is found to be between −2 and −4. Rule 4 tells us that the root locus starts at the open-loop poles and ends at the open-loop zeros. These two rules alone give us the general shape of the root locus.

a. To find the exact point where the locus crosses the $\zeta = 0.45$ line, we can use the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise to search along the line

$$\theta = 180^\circ - \cos^{-1} 0.45 = 116.7^\circ \tag{8.52}$$

for the point where the angles add up to an odd multiple of 180°. Searching in polar coordinates, we find that the root locus crosses the $\zeta = 0.45$ line at $3.4\angle 116.7^\circ$ with a gain, K, of 0.417.

b. To find the exact point where the locus crosses the $j\omega$-axis, use the root locus program to search along the line

$$\theta = 90^\circ \tag{8.53}$$

for the point where the angles add up to an odd multiple of 180°. Searching in polar coordinates, we find that the root locus crosses the $j\omega$-axis at $\pm j3.9$ with a gain of $K = 1.5$.

c. To find the breakaway point, use the root locus program to search the real axis between -2 and -4 for the point that yields maximum gain. Naturally, all points will have the sum of their angles equal to an odd multiple of 180°. A maximum gain of 0.0248 is found at the point -2.88. Therefore, the breakaway point is between the open-loop poles on the real axis at -2.88.

d. From the answer to **b**, the system is stable for K between 0 and 1.5.

MATLAB
ML

Students who are using MATLAB should now run ch8p1 in Appendix B. You will learn how to use MATLAB to plot and title a root locus, overlay constant ζ and ω_n curves, zoom into and zoom out from a root locus, and interact with the root locus to find critical points as well as gains at those points. This exercise solves Example 8.7 using MATLAB.

Skill-Assessment Exercise 8.5

WileyPLUS
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Control Solutions

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s-2)(s-4)}{(s^2+6s+25)}$$

do the following:

a. Sketch the root locus.
b. Find the imaginary-axis crossing.
c. Find the gain, K, at the $j\omega$-axis crossing.
d. Find the break-in point.
e. Find the point where the locus crosses the 0.5 damping ratio line.
f. Find the gain at the point where the locus crosses the 0.5 damping ratio line.
g. Find the range of gain, K, for which the system is stable.

TryIt 8.3

Use MATLAB, the Control System Toolbox, and the following statements to plot the root locus for Skill-Assessment Exercise 8.5. Solve the remaining parts of the problem by clicking on the appropriate points on the plotted root locus.

```
numg=poly([2 4]);
deng=[1 6 25];
G=tf(numg,deng)
rlocus(G)
z=0.5
sgrid(z,0)
```

ANSWERS:

a. See solution at www.wiley.com/college/nise.
b. $s = \pm j4.06$
c. $K = 1$
d. Break-in point $= +2.89$
e. $s = -2.42 + j4.18$
f. $K = 0.108$
g. $K < 1$

The complete solution is at www.wiley.com/college/nise.

8.7 Transient Response Design via Gain Adjustment

Now that we know how to sketch a root locus, we show how to use it for the design of transient response. In the last section we found that the root locus crossed the 0.45 damping ratio line with a gain of 0.417. Does this mean that the system will respond with 20.5% overshoot, the equivalent to a damping ratio of 0.45? It must be emphasized that the formulas describing percent overshoot, settling time, and peak time were derived only for a system with two closed-loop complex poles and no closed-loop zeros. The effect of additional poles and zeros and the conditions for justifying an approximation of a two-pole system were discussed in Sections 4.7 and 4.8 and apply here to closed-loop systems and their root loci. The conditions justifying a second-order approximation are restated here:

1. Higher-order poles are much farther into the left half of the s-plane than the dominant second-order pair of poles. The response that results from a higher-order pole does not appreciably change the transient response expected from the dominant second-order poles.
2. Closed-loop zeros near the closed-loop second-order pole pair are nearly canceled by the close proximity of higher-order closed-loop poles.
3. Closed-loop zeros not canceled by the close proximity of higher-order closed-loop poles are far removed from the closed-loop second-order pole pair.

The first condition as it applies to the root locus is shown graphically in Figure 8.20(*a*) and (*b*). Figure 8.20(*b*) would yield a much better second-order approximation than Figure 8.20(*a*), since closed-loop pole p_3 is farther from the dominant, closed-loop second-order pair, p_1 and p_2.

The second condition is shown graphically in Figure 8.20(*c*) and (*d*). Figure 8.20(*d*) would yield a much better second-order approximation than Figure 8.20(*c*), since closed-loop pole p_3 is closer to canceling the closed-loop zero.

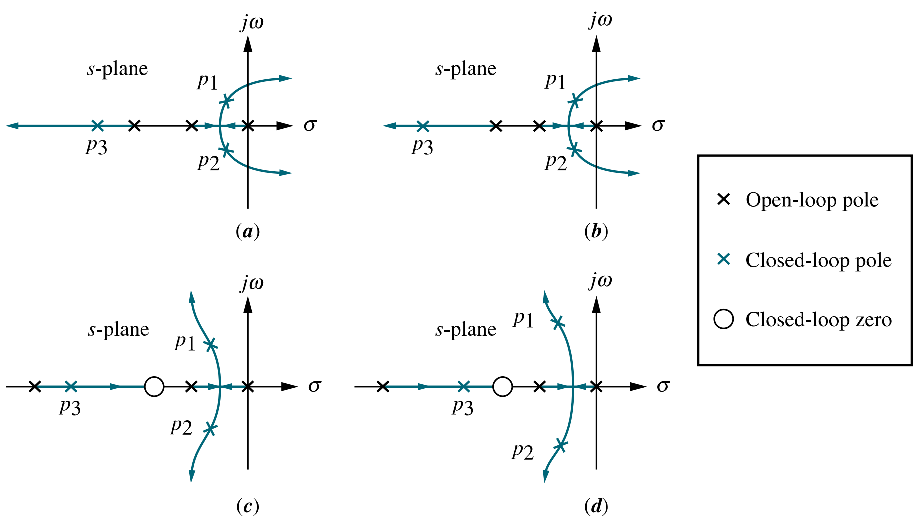

FIGURE 8.20 Making second-order approximations

Summarizing the design procedure for higher-order systems, we arrive at the following:

1. Sketch the root locus for the given system.
2. Assume the system is a second-order system without any zeros and then find the gain to meet the transient response specification.
3. Justify your second-order assumption by finding the location of all higher-order poles and evaluating the fact that they are much farther from the $j\omega$-axis than the dominant second-order pair. As a rule of thumb, this textbook assumes a factor of five times farther. Also, verify that closed-loop zeros are approximately canceled by higher-order poles. If closed-loop zeros are not canceled by higher-order closed-loop poles, be sure that the zero is far removed from the dominant second-order pole pair to yield approximately the same response obtained without the finite zero.
4. If the assumptions cannot be justified, your solution will have to be simulated in order to be sure it meets the transient response specification. It is a good idea to simulate all solutions, anyway.

We now look at a design example to show how to make a second-order approximation and then verify whether or not the approximation is valid.

Example 8.8

Third-Order System Gain Design

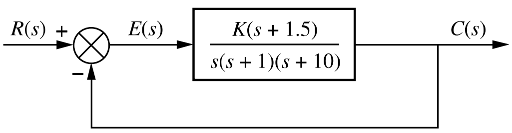

FIGURE 8.21 System for Example 8.8

PROBLEM: Consider the system shown in Figure 8.21. Design the value of gain, K, to yield 1.52% overshoot. Also estimate the settling time, peak time, and steady-state error.

SOLUTION: The root locus is shown in Figure 8.22. Notice that this is a third-order system with one zero. Breakaway points on the real

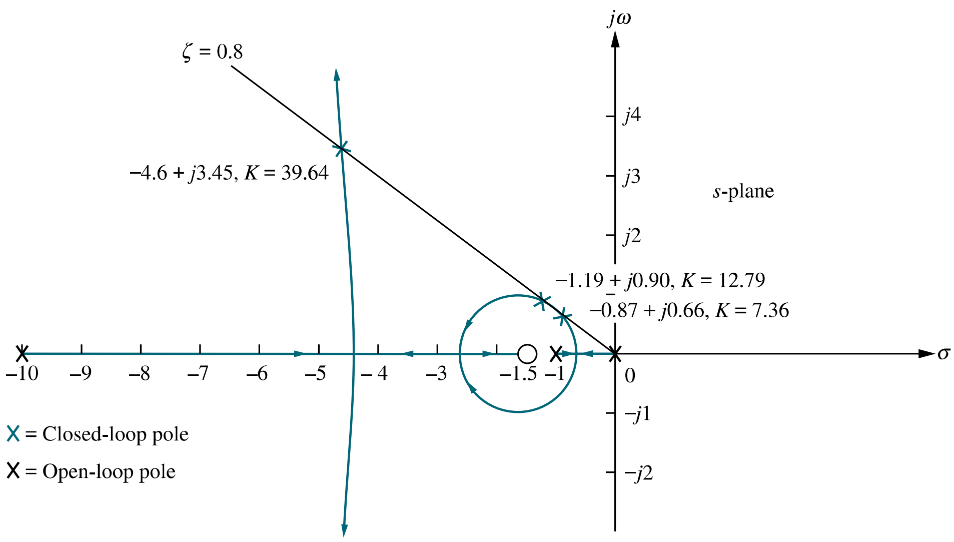

FIGURE 8.22 Root locus for Example 8.8

axis can occur between 0 and -1 and between -1.5 and -10, where the gain reaches a peak. Using the root locus program and searching in these regions for the peaks in gain, breakaway points are found at -0.62 with a gain of 2.511 and at -4.4 with a gain of 28.89. A break-in point on the real axis can occur between -1.5 and -10, where the gain reaches a local minimum. Using the root locus program and searching in these regions for the local minimum gain, a break-in point is found at -2.8 with a gain of 27.91.

Next assume that the system can be approximated by a second-order, underdamped system without any zeros. A 1.52% overshoot corresponds to a damping ratio of 0.8. Sketch this damping ratio line on the root locus, as shown in Figure 8.22.

Use the root locus program to search along the 0.8 damping ratio line for the point where the angles from the open-loop poles and zeros add up to an odd multiple of 180°. This is the point where the root locus crosses the 0.8 damping ratio or 1.52 percent overshoot line. Three points satisfy this criterion: $-0.87 \pm j0.66$, $-1.19 \pm j0.90$, and $-4.6 \pm j3.45$ with respective gains of 7.36, 12.79, and 39.64. For each point the settling time and peak time are evaluated using

$$T_s = \frac{4}{\zeta\omega_n} \tag{8.54}$$

where $\zeta\omega_n$ is the real part of the closed-loop pole, and also using

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{8.55}$$

where $\omega_n\sqrt{1-\zeta^2}$ is the imaginary part of the closed-loop pole.

To test our assumption of a second-order system, we must calculate the location of the third pole. Using the root locus program, search along the negative extension of the real axis between the zero at -1.5 and the pole at -10 for points that match the value of gain found at the second-order dominant poles. For each of the three crossings of the 0.8 damping ratio line, the third closed-loop pole is at -9.25, -8.6, and -1.8, respectively. The results are summarized in Table 8.4.

Finally, let us examine the steady-state error produced in each case. Note that we have little control over the steady-state error at this point. When the gain is set to meet the transient response, we have also designed the steady-state error. For the example, the steady-state error specification is given by K_v and is calculated as

$$K_v = \lim_{s\to 0} sG(s) = \frac{K(1.5)}{(1)(10)} \tag{8.56}$$

The results for each case are shown in Table 8.4.

How valid are the second-order assumptions? From Table 8.4, Cases 1 and 2 yield third closed-loop poles that are relatively far from the closed-loop zero. For these two cases there is no pole-zero cancellation, and a second-order system

TABLE 8.4 Characteristics of the system of Example 8.8

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time	K_v
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	-9.25	4.60	4.76	1.1
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	-8.61	3.36	3.49	1.9
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	-1.80	0.87	0.91	5.9

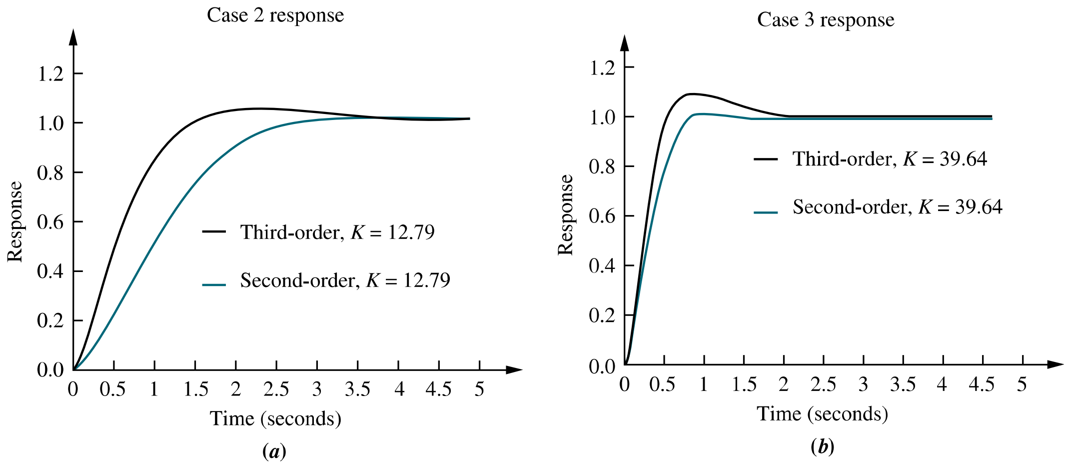

FIGURE 8.23 Second- and third-order responses for Example 8.8: **a.** Case 2; **b.** Case 3

approximation is not valid. In Case 3, the third closed-loop pole and the closed-loop zero are relatively close to each other, and a second-order system approximation can be considered valid. In order to show this, let us make a partial-fraction expansion of the closed-loop step response of Case 3 and see that the amplitude of the exponential decay is much less than the amplitude of the underdamped sinusoid. The closed-loop step response, $C_3(s)$, formed from the closed-loop poles and zeros of Case 3 is

$$\begin{aligned} C_3(s) &= \frac{39.64(s+1.5)}{s(s+1.8)(s+4.6+j3.45)(s+4.6-j3.45)} \\ &= \frac{39.64(s+1.5)}{s(s+1.8)(s^2+9.2s+33.06)} \\ &= \frac{1}{s}+\frac{0.3}{s(s+18)}-\frac{1.3(s+4.6)+1.6(3.45)}{(s+4.6)^2+3.45^2} \end{aligned} \tag{8.57}$$

Thus, the amplitude of the exponential decay from the third pole is 0.3, and the amplitude of the underdamped response from the dominant poles is $\sqrt{1.3^2+1.6^2}=2.06$. Hence, the dominant pole response is 6.9 times as large as the nondominant exponential response, and we assume that a second-order approximation is valid.

Using a simulation program, we obtain Figure 8.23, which shows comparisons of step responses for the problem we have just solved. Cases 2 and 3 are plotted for both the third-order response and a second-order response, assuming just the dominant pair of poles calculated in the design problem. Again, the second-order approximation was justified for Case 3, where there is a small difference in percent overshoot. The second-order approximation is not valid for Case 2. Other than the excess overshoot, Case 3 responses are similar.

MATLAB
ML

`Students who are using MATLAB should now run ch8p2 in Appendix B. You will learn how to use MATLAB to enter a value of percent overshoot from the keyboard. MATLAB will then draw the root locus and overlay the percent overshoot line requested. You will then interact with MATLAB and select the point of intersection of the` root locus with the requested percent overshoot line. MATLAB will respond with the value of gain, all closed-loop poles at that gain, and a closed-loop step response plot corresponding to the selected point. This exercise solves Example 8.8 using MATLAB.

Students who are using MATLAB may want to explore the SISO Design Tool described in Appendix E at www.wiley.com/college/nise. The SISO Design Tool is a convenient and intuitive way to obtain, view, and interact with a system's root locus. Section D.7 describes the advantages of using the tool, while Section D.8 describes how to use it. For practice, you may want to apply the SISO Design Tool to some of the problems at the end of this chapter.

Gui Tool
GUIT

Skill-Assessment Exercise 8.6

PROBLEM: Given a unity feedback system that has the forward-path transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

do the following:

a. Sketch the root locus.

b. Using a second-order approximation, design the value of K to yield 10% overshoot for a unit-step input.

c. Estimate the settling time, peak time, rise time, and steady-state error for the value of K designed in (b).

d. Determine the validity of your second-order approximation.

ANSWERS:

a. See solution located at www.wiley.com/college/nise.

b. $K = 45.55$

c. $T_s = 1.97$ s, $T_p = 1.13$ s, $T_r = 0.53$s, and $e_{\text{step}}(\infty) = 0.51$

d. Second-order approximation is not valid.

The complete solution is located at www.wiley.com/college/nise.

8.8 Generalized Root Locus

Up to this point we have always drawn the root locus as a function of the forward-path gain, K. The control system designer must often know how the closed-loop poles change as a function of another parameter. For example, in Figure 8.24, the parameter of interest is the open-loop pole at $-p_1$. How can we obtain a root locus for variations of the value of p_1?

If the function $KG(s)H(s)$ is formed as

$$KG(s)H(s) = \frac{10}{(s+2)(s+p_1)} \tag{8.58}$$

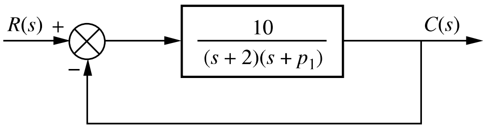

FIGURE 8.24 System requiring a root locus calibrated with p_1 as a parameter

the problem is that p_1 is not a multiplying factor of the function, as the gain, K, was in all of the previous problems. The solution to this dilemma is to create an equivalent system where p_1 appears as the forward-path gain. Since the closed-loop transfer function's denominator is $1 + KG(s)H(s)$, we effectively want to create an equivalent system whose denominator is $1 + p_1 G(s)H(s)$.

For the system of Figure 8.24, the closed-loop transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{10}{s^2 + (p_1 + 2)s + 2p_1 + 10} \tag{8.59}$$

Isolating p_1, we have

$$T(s) = \frac{10}{s^2 + 2s + 10 + p_1(s+2)} \tag{8.60}$$

Converting the denominator to the form $[1 + p_1 G(s)H(s)]$ by dividing numerator and denominator by the term not included with p_1, $s^2 + 2s + 10$, we obtain

$$T(s) = \frac{\dfrac{10}{s^2 + 2s + 10}}{1 + \dfrac{p_1(s+2)}{s^2 + 2s + 10}} \tag{8.61}$$

Conceptually, Eq. (8.61) implies that we have a system for which

$$KG(s)H(s) = \frac{p_1(s+2)}{s^2 + 2s + 10} \tag{8.62}$$

The root locus can now be sketched as a function of p_1, assuming the open-loop system of Eq. (8.62). The final result is shown in Figure 8.25.

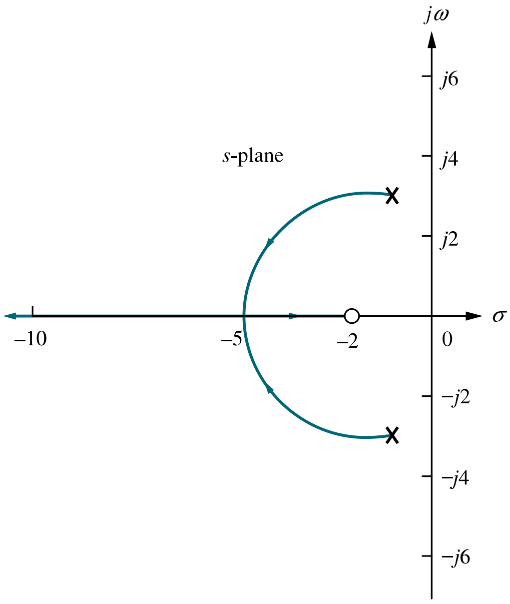

FIGURE 8.25 Root locus for the system of Figure 8.24, with p_1 as a parameter

Skill-Assessment Exercise 8.7

WileyPLUS
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Control Solutions

PROBLEM: Sketch the root locus for variations in the value of p_1, for a unity feedback system that has the following forward transfer function:

$$G(s) = \frac{100}{s(s+p_1)}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

In this section, we learned to plot the root locus as a function of any system parameter. In the next section we will learn how to plot root loci for positive-feedback systems.

8.9 Root Locus for Positive-Feedback Systems

The properties of the root locus were derived from the system of Figure 8.1. This is a negative-feedback system because of the negative summing of the feedback signal to the input signal. The properties of the root locus change dramatically if the feedback signal is added to the input rather than subtracted. A positive-feedback system can be thought of as a negative-feedback system with a negative value of $H(s)$. Using this concept, we find that the transfer function for the positive-feedback system shown in Figure 8.26 is

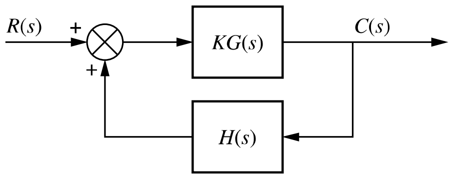


FIGURE 8.26 Positive-feedback system

$$\boxed{T(s) = \frac{KG(s)}{1 - KG(s)H(s)}} \tag{8.63}$$

We now retrace the development of the root locus for the denominator of Eq. (8.63). Obviously, a pole, s, exists when

$$\boxed{KG(s)H(s) = 1 = 1\angle k360^\circ \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots} \tag{8.64}$$

Therefore, the root locus for positive-feedback systems consists of all points on the s-plane where the angle of $KG(s)H(s) = k360^\circ$. How does this relationship change the rules for sketching the root locus presented in Section 8.4?

1. **Number of branches.** The same arguments as for negative feedback apply to this rule. There is no change.
2. **Symmetry.** The same arguments as for negative feedback apply to this rule. There is no change.
3. **Real-axis segments.** The development in Section 8.4 for the real-axis segments led to the fact that the angles of $G(s)H(s)$ along the real axis added up to either an odd multiple of 180° or a multiple of 360°. Thus, for positive-feedback systems the

root locus exists on the real axis along sections where the locus for negative-feedback systems does not exist. The rule follows:

Real-axis segments: On the real axis, the root locus for positive-feedback systems exists to the left of an even number of real-axis, finite open-loop poles and/or finite open-loop zeros.

The change in the rule is the word *even;* for negative-feedback systems the locus existed to the left of an *odd* number of real-axis, finite open-loop poles and/or zeros.

4. **Starting and ending points.** You will find no change in the development in Section 8.4 if Eq. (8.63) is used instead of Eq. (8.12). Therefore, we have the following rule.

   Starting and ending points: The root locus for positive-feedback systems begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

5. **Behavior at infinity.** The changes in the development of the asymptotes begin at Eq. (M.4) in Appendix M at www.wiley.com/college/nise since positive-feedback systems follow the relationship in Eq. (8.64). That change yields a different slope for the asymptotes. The value of the real-axis intercept for the asymptotes remains unchanged. The student is encouraged to go through the development in detail and show that the behavior at infinity for positive-feedback systems is given by the following rule:

   The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes for positive-feedback systems are given by the real-axis intercept, σ_a, and angle, θ_a, as follows:

$$\boxed{\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\,\text{finite poles} - \#\,\text{finite zeros}}} \tag{8.65}$$

$$\boxed{\theta_a = \frac{k2\pi}{\#\,\text{finite poles} - \#\,\text{finite zeros}}} \tag{8.66}$$

   where $k = 0, \pm 1, \pm 2, \pm 3, \ldots$, and the angle is given in radians with respect to the positive extension of the real axis.

   The change we see is that the numerator of Eq. (8.66) is $k2\pi$ instead of $(2k+1)\pi$.

What about other calculations? The imaginary-axis crossing can be found using the root locus program. In a search of the $j\omega$-axis, you are looking for the point where the angles add up to a multiple of 360° instead of an odd multiple of 180°. The breakaway points are found by looking for the maximum value of K. The break-in points are found by looking for the minimum value of K.

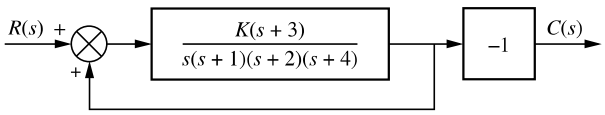
When we were discussing *negative*-feedback systems, we always made the root locus plot for positive values of gain. Since *positive*-feedback systems can also be thought of as *negative*-feedback systems with negative gain, the rules developed in this section apply equally to *negative*-feedback systems with negative gain. Let us look at an example.

Example 8.9

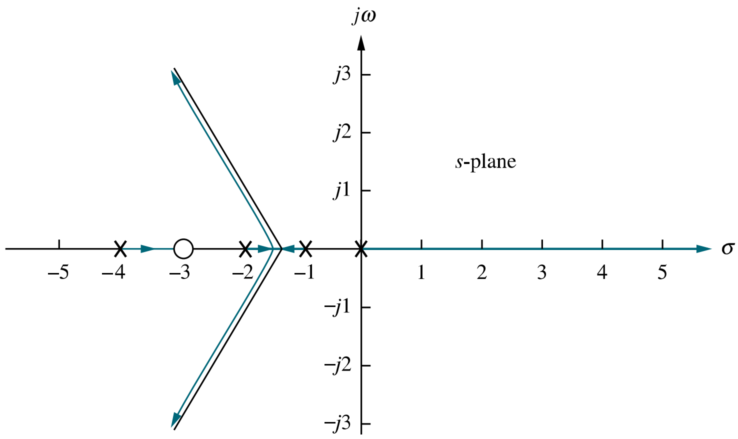
Root Locus for a Positive-Feedback System

PROBLEM: Sketch the root locus as a function of negative gain, K, for the system shown in Figure 8.11.

SOLUTION: The equivalent positive-feedback system found by pushing -1, associated with K, to the right past the pickoff point is shown in Figure 8.27(a).



(*a*)



(*b*)

FIGURE 8.27 **a.** Equivalent positive-feedback system for Example 8.9; **b.** root locus

Therefore, as the gain of the equivalent system goes through positive values of K, the root locus will be equivalent to that generated by the gain, K, of the original system in Figure 8.11 as it goes through negative values.

The root locus exists on the real axis to the left of an even number of real, finite open-loop poles and/or zeros. Therefore, the locus exists on the entire positive extension of the real axis, between -1 and -2 and between -3 and -4. Using Eq. (8.27), the σ_a intercept is found to be

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3} \tag{8.67}$$

The angles of the lines that intersect at $-4/3$ are given by

$$\theta_a = \frac{k2\pi}{\#\text{ finite poles} - \#\text{ finite zeros}} \tag{8.68a}$$

$$= 0 \qquad \text{for } k = 0 \tag{8.68b}$$

$$= 2\pi/3 \qquad \text{for } k = 1 \tag{8.68c}$$

$$= 4\pi/3 \qquad \text{for } k = 2 \tag{8.68d}$$

The final root locus sketch is shown in Figure 8.27(b)

Skill-Assessment Exercise 8.8

PROBLEM: Sketch the root locus for the positive-feedback system whose forward transfer function is

$$G(s) = \frac{K(s+4)}{(s+1)(s+2)(s+3)}$$

The system has unity feedback.

ANSWER: The complete solution is at www.wiley.com/college/nise.

8.10 Pole Sensitivity

The root locus is a plot of the closed-loop poles as a system parameter is varied. Typically, that system parameter is gain. Any change in the parameter changes the closed-loop poles and, subsequently, the performance of the system. Many times the parameter changes against our wishes, due to heat or other environmental conditions. We would like to find out the extent to which changes in parameter values affect the performance of our system.

The root locus exhibits a nonlinear relationship between gain and pole location. Along some sections of the root locus, (1) very small changes in gain yield very large changes in pole location and hence performance; along other sections of the root locus, (2) very large changes in gain yield very small changes in pole location. In the first case we say that the system has a high sensitivity to changes in gain. In the second case, the system has a low sensitivity to changes in gain. We prefer systems with low sensitivity to changes in gain.

In Section 7.7, we defined sensitivity as the ratio of the fractional change in a function to the fractional change in a parameter as the change in the parameter approaches zero. Applying the same definition to the closed-loop poles of a system that vary with a parameter, we define *root sensitivity* as the ratio of the fractional change in a closed-loop pole to the fractional change in a system parameter, such as gain. Using Eq. (7.75), we calculate the sensitivity of a closed-loop pole, s, to gain, K:

$$\boxed{S_{s:K} = \frac{K}{s}\frac{\delta s}{\delta K}} \tag{8.69}$$

where s is the current pole location, and K is the current gain. Using Eq. (8.69) and converting the partials to finite increments, the actual change in the closed-loop poles can be approximated as

$$\boxed{\Delta s = s(S_{s:K})\frac{\Delta K}{K}} \tag{8.70}$$

where Δs is the change in pole location, and $\Delta K/K$ is the fractional change in the gain, K. Let us demonstrate with an example. We begin with the characteristic equation from which $\delta s/\delta K$ can be found. Then, using Eq. (8.69) with the current closed-loop pole, s, and its associated gain, K, we can find the sensitivity.

Example 8.10

Root Sensitivity of a Closed-Loop System to Gain Variations

PROBLEM: Find the root sensitivity of the system in Figure 8.4 at $s = -9.47$ and $-5 + j5$. Also calculate the change in the pole location for a 10% change in K.

SOLUTION: The system's characteristic equation, found from the closed-loop transfer function denominator, is $s^2 + 10s + K = 0$. Differentiating with respect to K, we have

$$2s\frac{\delta s}{\delta K} + 10\frac{\delta s}{\delta K} + 1 = 0 \tag{8.71}$$

from which

$$\frac{\delta s}{\delta K} = \frac{-1}{2s + 10} \tag{8.72}$$

Substituting Eq. (8.72) into Eq. (8.69), the sensitivity is found to be

$$S_{s:K} = \frac{K}{s}\frac{-1}{2s + 10} \tag{8.73}$$

For $s = -9.47$, Table 8.1 shows $K = 5$. Substituting these values into Eq. (8.73) yields $S_{s:K} = -0.059$. The change in the pole location for a 10% change in K can be found using Eq. (8.70), with $s = -9.47$, $\Delta K/K = 0.1$, and $S_{s:K} = -0.059$. Hence, $\Delta s = 0.056$, or the pole will move to the right by 0.056 units for a 10% change in K.

For $s = -5 + j5$, Table 8.1 shows $K = 50$. Substituting these values into Eq. (8.73) yields $S_{s:K} = 1/(1 + j1) = (1/\sqrt{2})\angle - 45°$. The change in the pole location for a 10% change in K can be found using Eq. (8.70), with $s = -5 + j5$, $\Delta K/K = 0.1$, and $S_{s:K} = (1/\sqrt{2})\angle - 45°$. Hence, $\Delta s = -j5$, or the pole will move vertically by 0.5 unit for a 10% change in K.

In summary, then, at $K = 5$, $S_{s:K} = -0.059$. At $K = 50$, $S_{s:K} = (1/\sqrt{2})\angle - 45°$. Comparing magnitudes, we conclude that the root locus is less sensitive to changes in gain at the lower value of K. Notice that root sensitivity is a complex quantity possessing both the magnitude and direction information from which the change in poles can be calculated.

Skill-Assessment Exercise 8.9

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PROBLEM: A negative unity feedback system has the forward transfer function

$$G(s) = \frac{K(s + 1)}{s(s + 2)}$$

If K is set to 20, find the changes in closed-loop pole location for a 5% change in K.

ANSWER: For the closed-loop pole at -21.05, $\Delta s = -0.9975$; for the closed-loop pole at -0.95, $\Delta s = -0.0025$.

The complete solution is at www.wiley.com/college/nise.

Case Studies

Antenna Control: Transient Design via Gain

Design
D

The main thrust of this chapter is to demonstrate design of higher-order systems (higher than two) through gain adjustment. Specifically, we are interested in determining the value of gain required to meet transient response requirements, such as percent overshoot, settling time, and peak time. The following case study emphasizes this design procedure, using the root locus.

PROBLEM: Given the antenna azimuth position control system shown on the front endpapers, Configuration 1, find the preamplifier gain required for 25% overshoot.

SOLUTION: The block diagram for the system was derived in the Case Studies section in Chapter 5 and is shown in Figure 5.34(c), where $G(s) = 6.63K/[s(s+1.71)(s+100)]$.

First a sketch of the root locus is made to orient the designer. The real-axis segments are between the origin and -1.71 and from -100 to infinity. The locus begins at the open-loop poles, which are all on the real axis at the origin, -1.71, and -100. The locus then moves toward the zeros at infinity by following asymptotes that, from Eqs. (8.27) and (8.28), intersect the real axis at -33.9 at angles of $60°$, $180°$, and $-60°$. A portion of the root locus is shown in Figure 8.28.

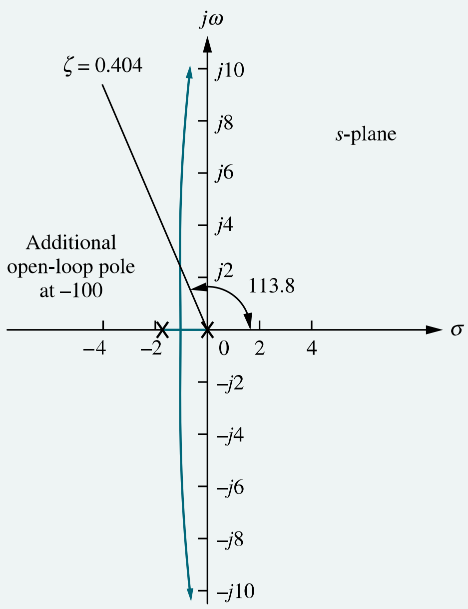

FIGURE 8.28 Portion of the root locus for the antenna control system

From Eq. (4.39), 25% overshoot corresponds to a damping ratio of 0.404. Now draw a radial line from the origin at an angle of $\cos^{-1}\zeta = 113.8$. The intersection of this line with the root locus locates the system's dominant, second-order closed-loop poles. Using the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise to search the radial line for $180°$ yields the closed-loop dominant poles as $2.063\angle 113.8° = -0.833 \pm j1.888$. The gain value yields $6.63K = 425.7$, from which $K = 64.21$.

Checking our second-order assumption, the third pole must be to the left of the open-loop pole at -100 and is thus greater than five times the real part of the dominant pole pair, which is -0.833. The second-order approximation is thus valid.

The computer simulation of the closed-loop system's step response in Figure 8.29 shows that the design requirement of 25% overshoot is met.

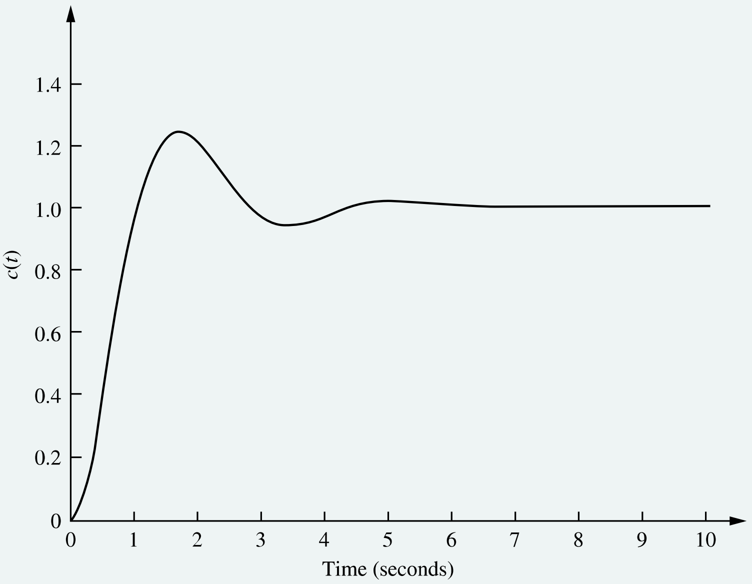

FIGURE 8.29 Step response of the gain-adjusted antenna control system

CHALLENGE: You are now given a problem to test your knowledge of this chapter's objectives. Referring to the antenna azimuth position control system shown on the front endpapers, Configuration 2, do the following:

a. Find the preamplifier gain, K, required for an 8-second settling time.

b. `Repeat, using MATLAB.`

MATLAB
ML

UFSS Vehicle: Transient Design via Gain

Design
D

In this case study, we apply the root locus to the UFSS vehicle pitch control loop. The pitch control loop is shown with both rate and position feedback on the back endpapers. In the example that follows, we plot the root locus without the rate feedback and then with the rate feedback. We will see the stabilizing effect that rate feedback has upon the system.

PROBLEM: Consider the block diagram of the pitch control loop for the UFSS vehicle shown on the back endpapers (*Johnson*, *1980*).

a. If $K_2 = 0$ (no rate feedback), plot the root locus for the system as a function of pitch gain, K_1, and estimate the settling time and peak time of the closed-loop response with 20% overshoot.

b. Let $K_2 = K_1$ (add rate feedback) and repeat **a**.

SOLUTION:

a. Letting $K_2 = 0$, the open-loop transfer function is

$$G(s)H(s) = \frac{0.25K_1(s+0.435)}{(s+1.23)(s+2)(s^2+0.226s+0.0169)} \tag{8.74}$$

from which the root locus is plotted in Figure 8.30. Searching along the 20% overshoot line evaluated from Eq. (4.39), we find the dominant second-order poles to be $-0.202 \pm j0.394$ with a gain of $K = 0.25K_1 = 0.706$, or $K_1 = 2.824$.

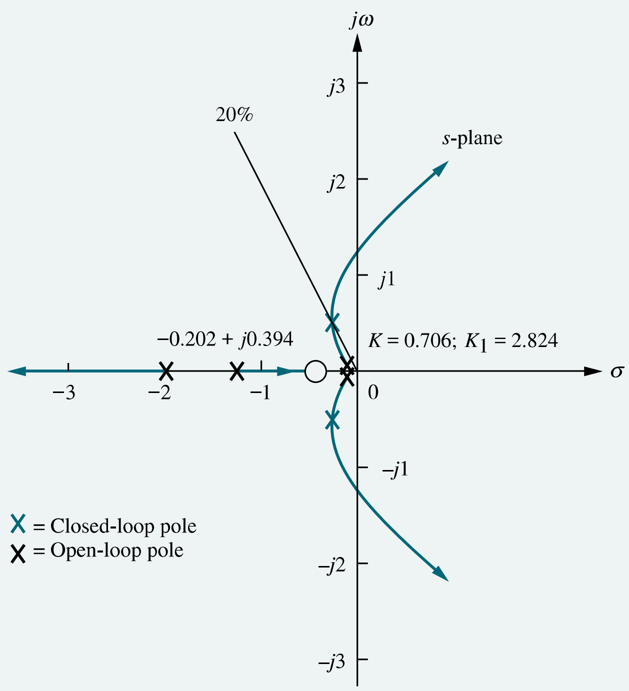

FIGURE 8.30 Root locus of pitch control loop without rate feedback, UFSS vehicle

From the real part of the dominant pole, the settling time is estimated to be $T_s = 4/0.202 = 19.8$ seconds. From the imaginary part of the dominant pole, the peak time is estimated to be $T_p = \pi/0.394 = 7.97$ seconds. Since our estimates are based upon a second-order assumption, we now test our assumption by finding the third closed-loop pole location between -0.435 and -1.23 and the fourth closed-loop pole location between -2 and infinity. Searching each of these regions for a gain of $K = 0.706$, we find the third and fourth poles at -0.784 and -2.27, respectively. The third pole, at -0.784, may not be close enough to the zero at -0.435, and thus the system should be simulated. The fourth pole, at -2.27, is 11 times as far from the imaginary axis as the dominant poles and thus meets the requirement of at least five times the real part of the dominant poles.

A computer simulation of the step response for the system, which is shown in Figure 8.31, shows a 29% overshoot above a final value of 0.88, approximately 20-second settling time, and a peak time of approximately 7.5 seconds.

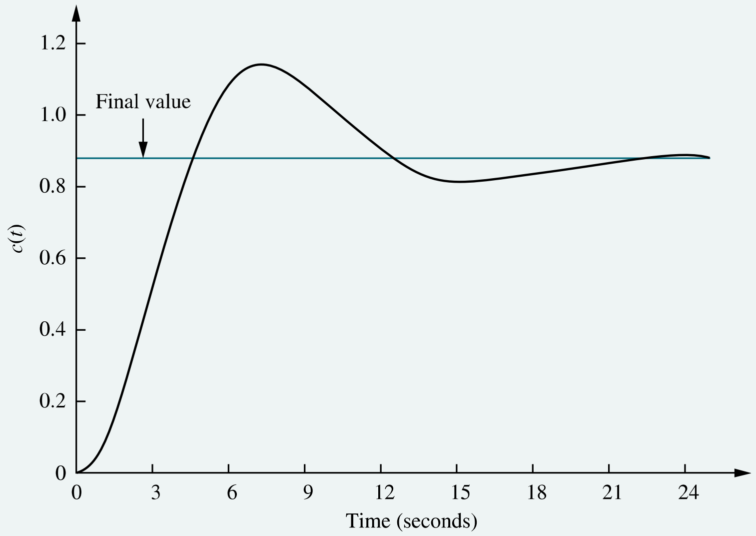

FIGURE 8.31 Computer simulation of step response of pitch control loop without rate feedback, UFSS vehicle

b. Adding rate feedback by letting $K_2 = K_1$ in the pitch control system shown on the back endpapers, we proceed to find the new open-loop transfer function. Pushing $-K_1$ to the right past the summing junction, dividing the pitch rate sensor by $-K_1$, and combining the two resulting feedback paths obtaining $(s+1)$ give us the following open-loop transfer function:

$$G(s)H(s) = \frac{0.25K_1(s+0.435)(s+1)}{(s+1.23)(s+2)(s^2+0.226s+0.0169)} \tag{8.75}$$

Notice that the addition of rate feedback adds a zero to the open-loop transfer function. The resulting root locus is shown in Figure 8.32. Notice that this root locus, unlike the root locus in **a**, is stable for all values of gain, since the locus does not enter the right half of the s-plane for any value of positive gain,

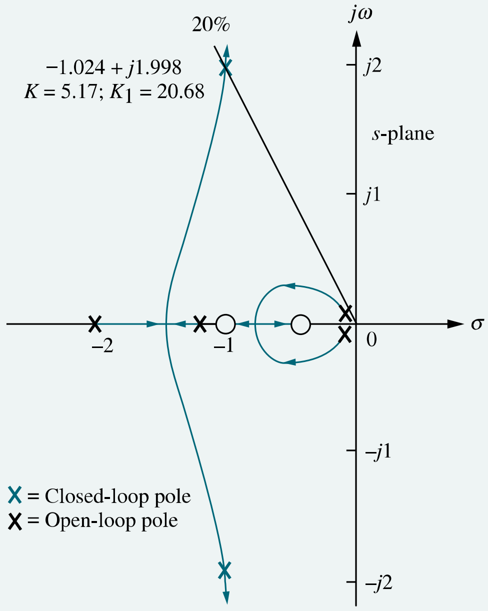

FIGURE 8.32 Root locus of pitch control loop with rate feedback, UFSS vehicle

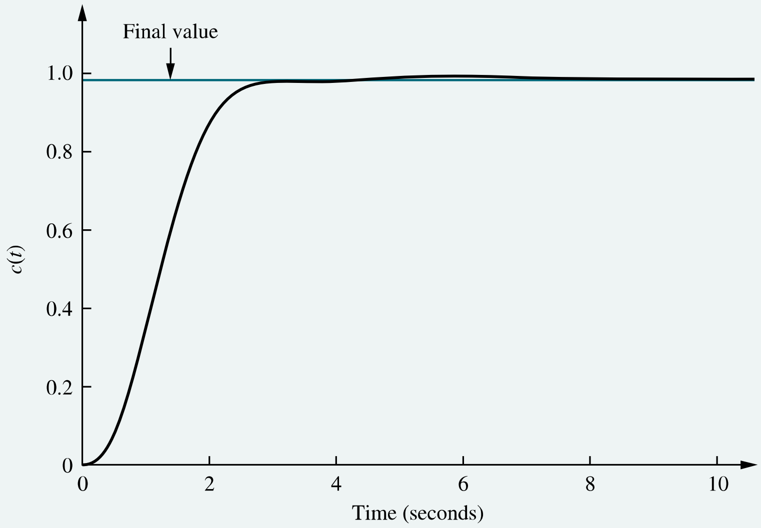

FIGURE 8.33 Computer simulation of step response of pitch control loop with rate feedback, UFSS vehicle

$K = 0.25K_1$. Also notice that the intersection with the 20% overshoot line is much farther from the imaginary axis than is the case without rate feedback, resulting in a faster response time for the system.

The root locus intersects the 20% overshoot line at $-1.024 \pm j1.998$ with a gain of $K = 0.25K_1 = 5.17$, or $K_1 = 20.68$. Using the real and imaginary parts of the dominant pole location, the settling time is predicted to be $T_s = 4/1.024 = 3.9$ seconds, and the peak time is estimated to be $T_p = \pi/1.998 = 1.57$ seconds. The new estimates show considerable improvement in the transient response as compared to the system without the rate feedback.

Now we test our second-order approximation by finding the location of the third and fourth poles between -0.435 and -1. Searching this region for a gain of $K = 5.17$, we locate the third and fourth poles at approximately -0.5 and -0.91. Since the zero at -1 is a zero of $H(s)$, the student can verify that this zero is not a zero of the closed-loop transfer function. Thus, although there may be pole-zero cancellation between the closed-loop pole at -0.5 and the closed-loop zero at -0.435, there is no *closed-loop* zero to cancel the closed-loop pole at -0.91.[2] Our second-order approximation is not valid.

A computer simulation of the system with rate feedback is shown in Figure 8.33. Although the response shows that our second-order approximation is invalid, it still represents a considerable improvement in performance over the system without rate feedback; the percent overshoot is small, and the settling time is about 6 seconds instead of about 20 seconds.

CHALLENGE: You are now given a problem to test your knowledge of this chapter's objectives. For the UFSS vehicle (*Johnson, 1980*) heading control system shown on the back endpapers, and introduced in the case study challenge in Chapter 5, do the following:

a. Let $K_2 = K_1$ and find the value of K_1 that yields 10% overshoot.

MATLAB
ML

b. `Repeat, using MATLAB.`

[2] The zero at -1 shown on the root locus plot of Figure 8.32 is an open-loop zero since it comes from the numerator of $H(s)$.

We have concluded the chapter with two case studies showing the use and application of the root locus. We have seen how to plot a root locus and estimate the transient response by making a second-order approximation. We saw that the second-order approximation held when rate feedback was not used for the UFSS. When rate feedback was used, an open-loop zero from $H(s)$ was introduced. Since it was not a closed-loop zero, there was no pole-zero cancellation, and a second-order approximation could not be justified. In this case, however, the transient response with rate feedback did represent an improvement in transient response over the system without rate feedback. In subsequent chapters we will see why rate feedback yields an improvement. We will also see other methods of improving the transient response.

Summary

In this chapter, we examined the *root locus*, a powerful tool for the analysis and design of control systems. The root locus empowers us with qualitative and quantitative information about the stability and transient response of feedback control systems. The root locus allows us to find the poles of the closed-loop system by starting from the open-loop system's poles and zeros. It is basically a graphical root-solving technique.

We looked at ways to sketch the root locus rapidly, even for higher-order systems. The sketch gave us qualitative information about changes in the transient response as parameters were varied. From the locus we were able to determine whether a system was unstable for any range of gain.

Next we developed the criterion for determining whether a point in the s-plane was on the root locus: The angles from the open-loop zeros, minus the angles from the open-loop poles drawn to the point in the s-plane, add up to an odd multiple of 180°.

The computer program discussed in Appendix G.2 at www.wiley.com/college/nise helps us to search rapidly for points on the root locus. This program allows us to find points and gains to meet certain transient response specifications as long as we are able to justify a second-order assumption for higher-order systems. Other computer programs, such as MATLAB, plot the root locus and allow the user to interact with the display to determine transient response specifications and system parameters.

Our method of design in this chapter is gain adjustment. We are limited to transient responses governed by the poles on the root locus. Transient responses represented by pole locations outside of the root locus cannot be obtained by a simple gain adjustment. Further, once the transient response has been established, the gain is set, and so is the steady-state error performance. In other words, by a simple gain adjustment, we have to trade off between a specified transient response and a specified steady-state error. Transient response and steady-state error cannot be designed independently with a simple gain adjustment.

We also learned how to plot the root locus against system parameters other than gain. In order to make this root locus plot, we must first convert the closed-loop transfer function into an equivalent transfer function that has the desired system parameter in the same position as the gain. The chapter discussion concluded with positive-feedback systems and how to plot the root loci for these systems.

The next chapter extends the concept of the root locus to the design of compensation networks. These networks have as an advantage the separate design of transient performance and steady-state error performance.

Review Questions

1. What is a root locus?
2. Describe two ways of obtaining the root locus.
3. If $KG(s)H(s) = 5\angle 180°$, for what value of gain is s a point on the root locus?
4. Do the zeros of a system change with a change in gain?
5. Where are the zeros of the closed-loop transfer function?
6. What are two ways to find where the root locus crosses the imaginary axis?
7. How can you tell from the root locus if a system is unstable?
8. How can you tell from the root locus if the settling time does not change over a region of gain?
9. How can you tell from the root locus that the natural frequency does not change over a region of gain?
10. How would you determine whether or not a root locus plot crossed the real axis?
11. Describe the conditions that must exist for all closed-loop poles and zeros in order to make a second-order approximation.
12. What rules for plotting the root locus are the same whether the system is a positive- or a negative-feedback system?
13. Briefly describe how the zeros of the open-loop system affect the root locus and the transient response.

Problems

1. For each of the root loci shown in Figure P8.1, tell whether or not the sketch can be a root locus. If the sketch cannot be a root locus, explain why. Give *all* reasons. [Section: 8.4]

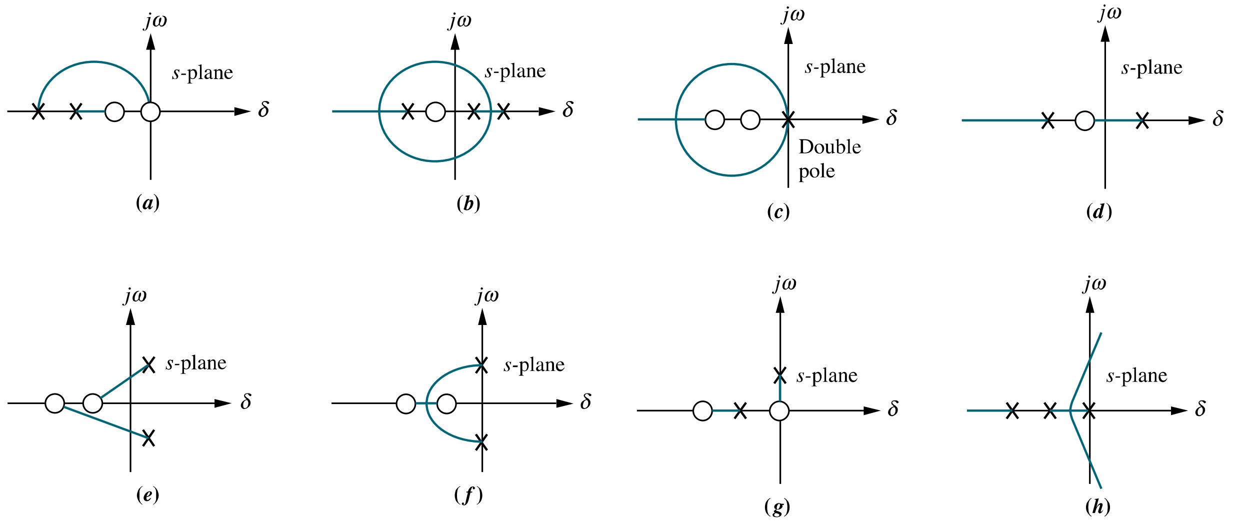

FIGURE P8.1

2. Sketch the general shape of the root locus for each of the open-loop pole-zero plots shown in Figure P8.2. [Section: 8.4]

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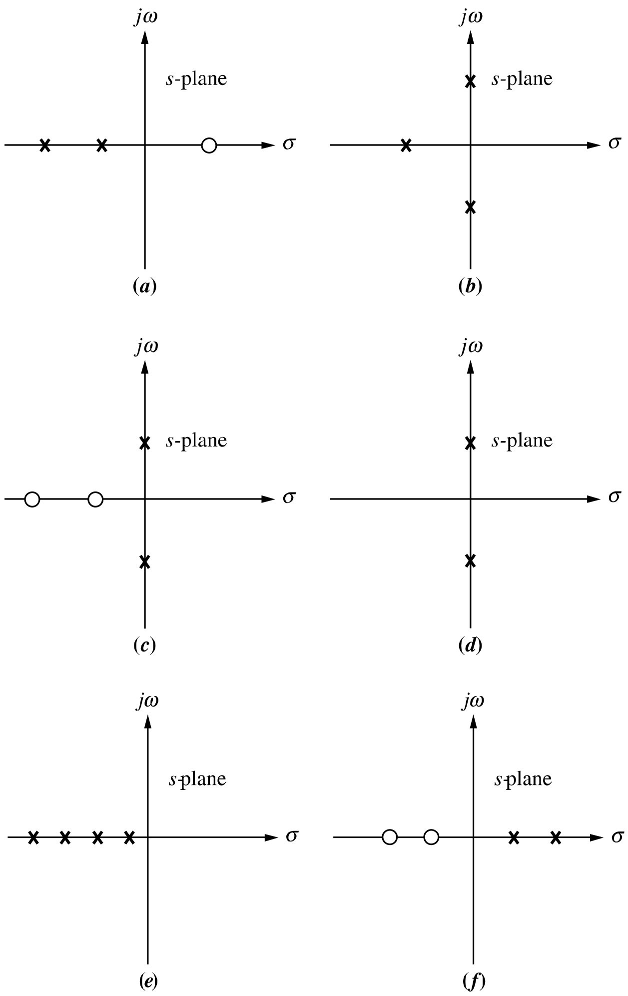

FIGURE P8.2

3. Sketch the root locus for the unity feedback system shown in Figure P8.3 for the following transfer functions: [Section: 8.4]

a. $G(s) = \dfrac{K(s+2)(s+6)}{s^2+8s+25}$

b. $G(s) = \dfrac{K(s^2+4)}{(s^2+1)}$

c. $G(s) = \dfrac{K(s^2+1)}{s^2}$

d. $G(s) = \dfrac{K}{(s+1)^3(s+4)}$

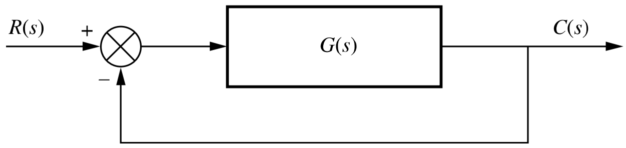

FIGURE P8.3

4. Let

$$G(s) = \frac{K\left(s+\frac{2}{3}\right)}{s^2(s+6)}$$

in Figure P8.3. [Section: 8.5]

a. Plot the root locus.

b. Write an expression for the closed-loop transfer function at the point where the three closed-loop poles meet.

5. Let

$$G(s) = \frac{-K(s+1)^2}{s^2+2s+2}$$

with $K > 0$ in Figure P8.3. [Sections: 8.5, 8.9]

a. Find the range of K for closed-loop stability.

b. Sketch the system's root locus.

c. Find the position of the closed-loop poles when $K = 1$ and $K = 2$.

6. For the open-loop pole-zero plot shown in Figure P8.4, sketch the root locus and find the break-in point. [Section: 8.5]

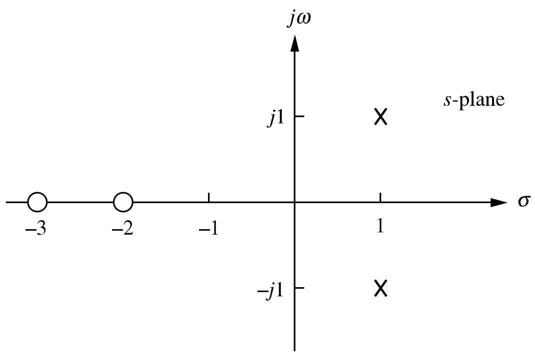

FIGURE P8.4

7. Sketch the root locus of the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s+3)(s+5)}{(s+1)(s-7)}$$

and find the break-in and breakaway points. [Section: 8.5]

8. The characteristic polynomial of a feedback control system, which is the denominator of the closed-loop transfer function, is given by $s^3 + 2s^2 + (20K + 7)s + 100K$. Sketch the root locus for this system. [Section: 8.8]

9. Figure P8.5 shows open-loop poles and zeros. There are two possibilities for the sketch of the root locus. Sketch each of the two possibilities. Be aware that only one can be the *real* locus for specific open-loop pole and zero values. [Section: 8.4]

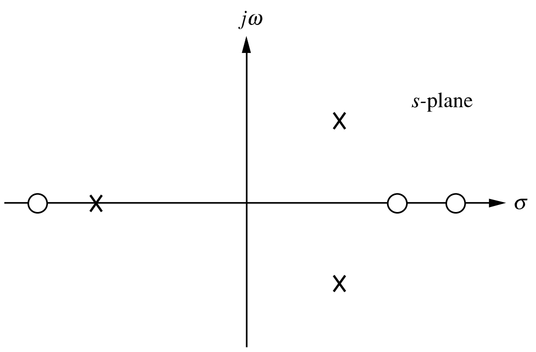

FIGURE P8.5

10. Plot the root locus for the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s+2)(s^2+4)}{(s+5)(s-3)}$$

For what range of K will the poles be in the right half-plane? [Section: 8.5]

11. For the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s^2-9)}{(s^2+4)}$$

sketch the root locus and tell for what values of K the system is stable and unstable. [Section: 8.5]

12. Sketch the root locus for the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s^2+2)}{(s+3)(s+4)}$$

Give the values for all critical points of interest. Is the system ever unstable? If so, for what range of K? [Section: 8.5]

13. For each system shown in Figure P8.6, make an accurate plot of the root locus and find the following: [Section: 8.5]

a. The breakaway and break-in points

b. The range of K to keep the system stable

c. The value of K that yields a stable system with critically damped second-order poles

d. The value of K that yields a stable system with a pair of second-order poles that have a damping ratio of 0.707

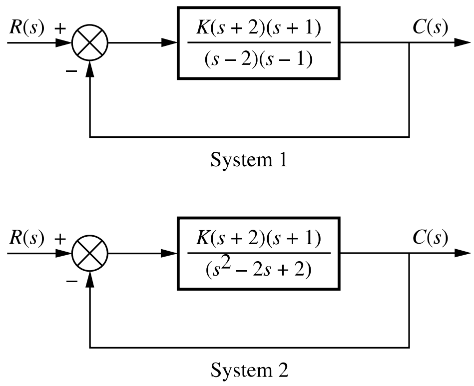

FIGURE P8.6

14. Sketch the root locus and find the range of K for stability for the unity feedback system shown in Figure P8.3 for the following conditions: [Section: 8.5]

a. $G(s) = \dfrac{K(s^2+1)}{(s-1)(s+2)(s+3)}$

b. $G(s) = \dfrac{K(s^2-2s+2)}{s(s+1)(s+2)}$

15. For the unity feedback system of Figure P8.3, where

$$G(s) = \frac{K(s+3)}{(s^2+2)(s-2)(s+5)}$$

sketch the root locus and find the range of K such that there will be only two right–half-plane poles for the closed-loop system. [Section: 8.5]

16. For the unity feedback system of Figure P8.3, where

$$G(s) = \frac{K}{s(s+6)(s+9)}$$

plot the root locus and calibrate your plot for gain. Find all the critical points, such as breakaways, asymptotes, $j\omega$-axis crossing, and so forth. [Section: 8.5]

17. Given the unity feedback system of Figure P8.3, make an accurate plot of the root locus for the following:

a. $G(s) = \dfrac{K(s^2 - 2s + 2)}{(s+1)(s+2)}$

b. $G(s) = \dfrac{K(s-1)(s-2)}{(s+1)(s+2)}$

Calibrate the gain for at least four points for each case. Also find the breakaway points, the $j\omega$-axis crossing, and the range of gain for stability for each case. Find the angles of arrival for Part **a**. [Section: 8.5]

18. Given the root locus shown in Figure P8.7, [Section: 8.5]

a. Find the value of gain that will make the system marginally stable.

b. Find the value of gain for which the closed-loop transfer function will have a pole on the real axis at -5.

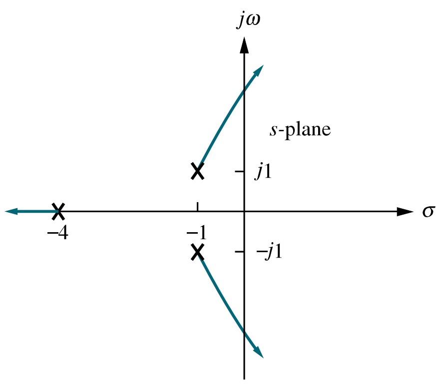

FIGURE P8.7

19. Given the unity feedback system of Figure P8.3, where

$$G(s) = \frac{K(s+1)}{s(s+2)(s+3)(s+4)}$$

do the following: [Section: 8.5]

a. Sketch the root locus.

b. Find the asymptotes.

c. Find the value of gain that will make the system marginally stable.

d. Find the value of gain for which the closed-loop transfer function will have a pole on the real axis at -0.5.

20. For the unity feedback system of Figure P8.3, where

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$$G(s) = \frac{K(s+\alpha)}{s(s+3)(s+6)}$$

find the values of α and K that will yield a second-order closed-loop pair of poles at $-1 \pm j100$. [Section: 8.5]

21. For the unity feedback system of Figure P8.3, where

$$G(s) = \frac{K(s-1)(s-2)}{s(s+1)}$$

sketch the root locus and find the following: [Section: 8.5]

a. The breakaway and break-in points

b. The $j\omega$-axis crossing

c. The range of gain to keep the system stable

d. The value of K to yield a stable system with second-order complex poles, with a damping ratio of 0.5

22. For the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s+10)(s+20)}{(s+30)(s^2 - 20s + 200)}$$

do the following: [Section: 8.7]

a. Sketch the root locus.

b. Find the range of gain, K, that makes the system stable.

c. Find the value of K that yields a damping ratio of 0.707 for the system's closed-loop dominant poles.

d. Find the value of K that yields closed-loop critically damped dominant poles.

23. For the system of Figure P8.8(a), sketch the root locus and find the following: [Section: 8.7]

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a. Asymptotes

b. Breakaway points

c. The range of K for stability

d. The value of K to yield a 0.7 damping ratio for the dominant second-order pair

To improve stability, we desire the root locus to cross the $j\omega$-axis at $j5.5$. To accomplish this, the open-loop function is cascaded with a zero, as shown in Figure P8.8(b).

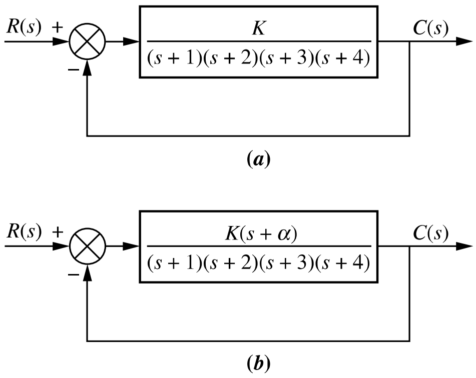

FIGURE P8.8

e. Find the value of α and sketch the new root locus.

f. Repeat Part **c** for the new locus.

g. Compare the results of Part **c** and Part **f**. What improvement in transient response do you notice?

24. Sketch the root locus for the positive-feedback system shown in Figure P8.9. [Section: 8.9]

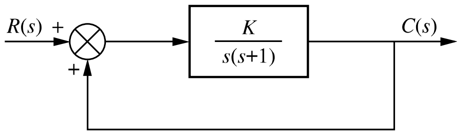

FIGURE P8.9

25. Root loci are usually plotted for variations in the gain. Sometimes we are interested in the variation of the closed-loop poles as other parameters are changed. For the system shown in Figure P8.10, sketch the root locus as α is varied. [Section: 8.8]

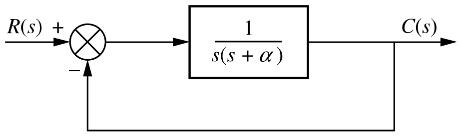

FIGURE P8.10

26. Given the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K}{(s+1)(s+2)(s+3)}$$

do the following problem parts by first making a second-order approximation. After you are finished with all of the parts, justify your second-order approximation. [Section: 8.7]

a. Sketch the root locus.

b. Find K for 20% overshoot.

c. For K found in Part **b**, what is the settling time, and what is the peak time?

d. Find the locations of higher-order poles for K found in Part **b**.

e. Find the range of K for stability.

27. For the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s^2 - 2s + 2)}{(s+2)(s+4)(s+5)(s+6)}$$

do the following: [Section: 8.7]

a. Sketch the root locus.

b. Find the asymptotes.

c. Find the range of gain, K, that makes the system stable.

d. Find the breakaway points.

e. Find the value of K that yields a closed-loop step response with 25% overshoot.

f. Find the location of higher-order closed-loop poles when the system is operating with 25% overshoot.

g. Discuss the validity of your second-order approximation.

h. `Use MATLAB to obtain the closed-loop step response to validate or refute your second-order approximation.` MATLAB **ML**

28. The unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s+2)(s+3)}{s(s+1)}$$

is to be designed for minimum damping ratio. Find the following: [Section: 8.7]

a. The value of K that will yield minimum damping ratio

b. The estimated percent overshoot for that case

c. The estimated settling time and peak time for that case

d. The justification of a second-order approximation (discuss)

e. The expected steady-state error for a unit ramp input for the case of minimum damping ratio

29. For the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s+2)}{s(s+6)(s+10)}$$

find K to yield closed-loop complex poles with a damping ratio of 0.55. Does your solution require a justification of a second-order approximation? Explain. [Section: 8.7]

30. For the unity feedback system shown in Figure P8.3, where

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$$G(s) = \frac{K(s+\alpha)}{s(s+1)(s+10)}$$

find the value of α so that the system will have a settling time of 4 seconds for large values of K. Sketch the resulting root locus. [Section: 8.8]

31. For the unity feedback system shown in Figure 8.3, where

$$G(s) = \frac{K(s+6)}{(s^2+10s+26)(s+1)^2(s+\alpha)}$$

design K and α so that the dominant complex poles of the closed-loop function have a damping ratio of 0.45 and a natural frequency of 9/8 rad/s.

32. For the unity feedback system shown in Figure 8.3, where

$$G(s) = \frac{K}{s(s+3)(s+4)(s+8)}$$

do the following: [Section: 8.7]

a. Sketch the root locus.

b. Find the value of K that will yield a 10% overshoot.

c. Locate all nondominant poles. What can you say about the second-order approximation that led to your answer in Part **b**?

d. Find the range of K that yields a stable system.

33. `Repeat Problem 32 using MATLAB. Use one program to do the following:`

MATLAB ML

a. `Display a root locus and pause.`

b. `Draw a close-up of the root locus where the axes go from` -2 `to 0 on the real axis and` -2 `to 2 on the imaginary axis.`

c. `Overlay the 10% overshoot line on the close-up root locus.`

d. `Select interactively the point where the root locus crosses the 10% overshoot line, and respond with the gain at that point as well as all of the closed-loop poles at that gain.`

e. `Generate the step response at the gain for 10% overshoot.`

34. For the unity feedback system shown in Figure 8.3, where

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$$G(s) = \frac{K(s^2+4s+5)}{(s^2+2s+5)(s+3)(s+4)}$$

do the following: [Section: 8.7]

a. Find the gain, K, to yield a 1-second peak time if one assumes a second-order approximation.

b. `Check the accuracy of the second-order approximation using MATLAB to simulate the system.`

MATLAB ML

35. For the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K(s+2)(s+3)}{(s^2+2s+2)(s+4)(s+5)(s+6)}$$

do the following: [Section: 8.7]

a. Sketch the root locus.

b. Find the $j\omega$-axis crossing and the gain, K, at the crossing.

c. Find all breakaway and break-in points.

d. Find angles of departure from the complex poles.

e. Find the gain, K, to yield a damping ratio of 0.3 for the closed-loop dominant poles.

36. Repeat Parts **a** through **c** and **e** of Problem 35 for [Section: 8.7]

$$G(s) = \frac{K(s+8)}{s(s+2)(s+4)(s+6)}$$

37. For the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K}{(s+3)(s^2+4s+5)}$$

do the following: [Section: 8.7]

a. Find the location of the closed-loop dominant poles if the system is operating with 15% overshoot.

b. Find the gain for Part **a.**

c. Find all other closed-loop poles.

d. Evaluate the accuracy of your second-order approximation.

38. For the system shown in Figure P8.11, do the following: [Section: 8.7]

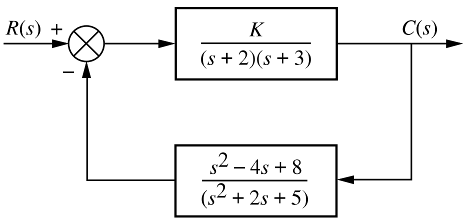

FIGURE P8.11

a. Sketch the root locus.

b. Find the $j\omega$-axis crossing and the gain, K, at the crossing.

c. Find the real-axis breakaway to two-decimal-place accuracy.

d. Find angles of arrival to the complex zeros.

e. Find the closed-loop zeros.

f. Find the gain, K, for a closed-loop step response with 30% overshoot.

g. Discuss the validity of your second-order approximation.

39. Sketch the root locus for the system of Figure P8.12 and find the following: [Section: 8.7]

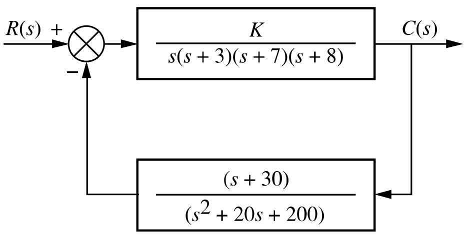

FIGURE P8.12

a. The range of gain to yield stability

b. The value of gain that will yield a damping ratio of 0.707 for the system's dominant poles

c. The value of gain that will yield closed-loop poles that are critically damped

40. Repeat Problem 39 using MATLAB. The program will do the following in one program:

MATLAB ML

a. Display a root locus and pause.

b. Display a close-up of the root locus where the axes go from −2 to 2 on the real axis and −2 to 2 on the imaginary axis.

c. Overlay the 0.707 damping ratio line on the close-up root locus.

d. Allow you to select interactively the point where the root locus crosses the 0.707 damping ratio line, and respond by displaying the gain at that point as well as all of the closed-loop poles at that gain. The program will then allow you to select interactively the imaginary-axis crossing and respond with a display of the gain at that point as well as all of the closed-loop poles at that gain. Finally, the program will repeat the evaluation for critically damped dominant closed-loop poles.

e. Generate the step response at the gain for 0.707 damping ratio.

41. Given the unity feedback system shown in Figure P8.3, where

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$$G(s) = \frac{K(s+z)}{s^2(s+20)}$$

do the following: [Section: 8.7]

a. If $z = 6$, find K so that the damped frequency of oscillation of the transient response is 10 rad/s.

b. For the system of Part **a**, what static error constant (finite) can be specified? What is its value?

c. The system is to be redesigned by changing the values of z and K. If the new specifications are $\%OS = 4.32\%$ and $T_s = 0.4$ s, find the new values of z and K.

42. Given the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K}{(s+1)(s+3)(s+6)^2}$$

find the following: [Section: 8.7]

a. The value of gain, K, that will yield a settling time of 4 seconds

b. The value of gain, K, that will yield a critically damped system

43. Let

$$G(s) = \frac{K(s-1)}{(s+2)(s+3)}$$

in Figure P8.3. [Section: 8.7].

a. Find the range of K for closed-loop stability.

b. Plot the root locus for $K > 0$.

c. Plot the root locus for $K < 0$.

d. Assuming a step input, what value of K will result in the smallest attainable settling time?

e. Calculate the system's e_{ss} for a unit step input assuming the value of K obtained in Part **d**.

f. Make an approximate hand sketch of the unit step response of the system if K has the value obtained in Part **d**.

44. Given the unity feedback system shown in Figure P8.3, where

$$G(s) = \frac{K}{s(s+1)(s+5)}$$

evaluate the pole sensitivity of the closed-loop system if the second-order, underdamped closed-loop poles are set for [Section: 8.10]

a. $\zeta = 0.591$

b. $\zeta = 0.456$

c. Which of the two previous cases has more desirable sensitivity?

45. Figure P8.13(a) shows a robot equipped to perform arc welding. A similar device can be configured as a six-degrees-of-freedom industrial robot that can transfer objects according to a desired program. Assume the block diagram of the swing motion system shown in Figure P8.13(b). If $K = 64{,}510$, make a second-order approximation and estimate the following (*Hardy, 1967*):

a. Damping ratio

b. Percent overshoot

c. Natural frequency

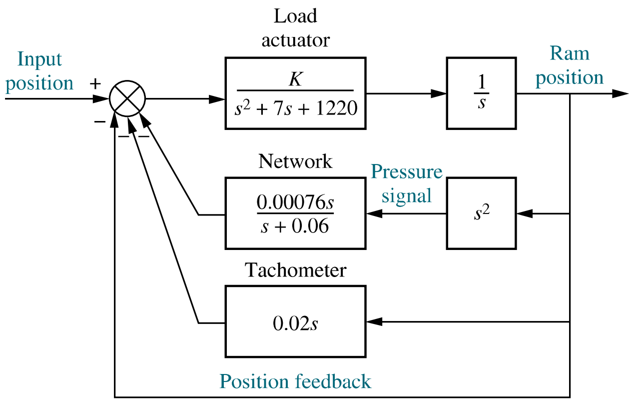
d. Settling time

e. Peak time

What can you say about your original second-order approximation?

46. During ascent, the automatic steering program aboard the space shuttle provides the interface between the low-rate processing of guidance (commands) and the high-rate processing of flight control (steering in response to the commands). The function performed is basically that of smoothing. A simplified representation of a maneuver smoother linearized for coplanar maneuvers is shown in Figure P8.14. Here $\theta_{CB}(s)$ is the commanded body angle as calculated by guidance, and $\theta_{CB}(s)$ is the desired body angle sent to flight control after smoothing.[3] Using the methods of Section 8.8, do the following:

(a)



(b)

FIGURE P8.13 **a.** Robot equipped to perform arc welding; **b.** block diagram for swing motion system

[3] Source: Rockwell International.

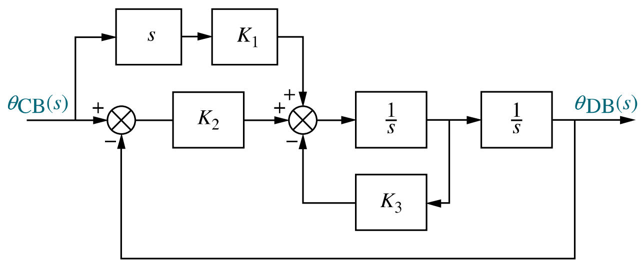


FIGURE P8.14 Block diagram of smoother

a. Sketch a root locus where the roots vary as a function of K_3.

b. Locate the closed-loop zeros.

c. Repeat Parts **a** and **b** for a root locus sketched as a function of K_2.

47. Repeat Problem 3 but sketch your root loci for negative values of K. [Section: 8.9]

48. Large structures in space, such as the space station, have to be stabilized against unwanted vibration. One method is to use an active vibration absorber to control the structure, as shown in Figure P8.15(a) (*Bruner, 1992*). Assuming that all values except the mass of the active vibration absorber are known and are equal to unity, do the following:

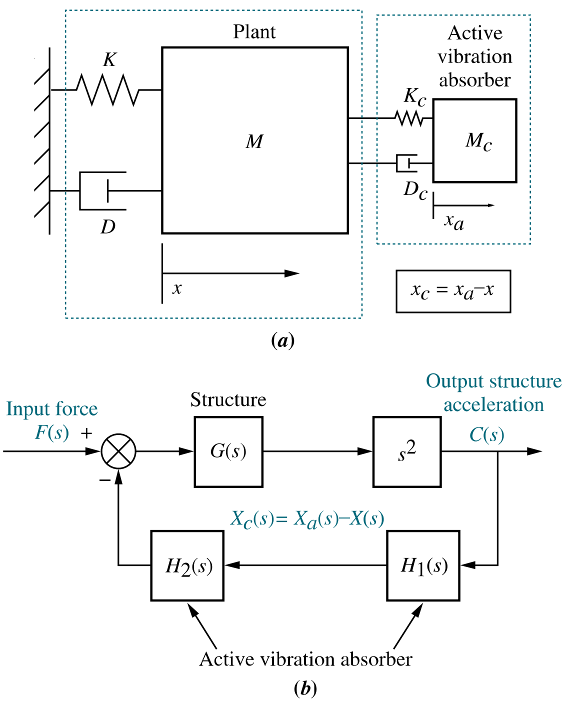


FIGURE P8.15 **a.** Active vibration absorber (© 1992 AIAA); **b.** control system block diagram

a. Obtain $G(s)$ and $H(s) = H_1(s)H_2(s)$ in the block diagram representation of the system of Figure 8.15(b), which shows that the active vibration absorber acts as a feedback element to control the structure. (Hint: Think of K_c and D_c as producing inputs to the structure.)

b. Find the steady-state position of the structure for a force disturbance input.

c. Sketch the root locus for the system as a function of active vibration absorber mass, M_c.

49. Figure P8.16 shows the block diagram of the closed-loop control of the linearized magnetic levitation system described in Chapter 2, Problem 58. (*Galvão, 2003*).

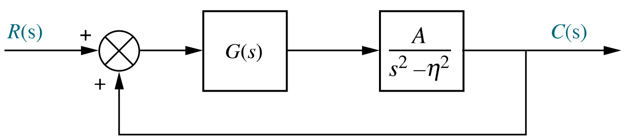


FIGURE P8.16 Linearized magnetic levitation system block diagram

Assuming $A = 1300$ and $\eta = 860$, draw the root locus and find the range of K for closed-loop stability when:

a. $G(s) = K$;

b. $G(s) = \dfrac{K(s+200)}{s+1000}$

50. The simplified transfer function model from steering angle $\delta(s)$ to tilt angle $\varphi(s)$ in a bicycle is given by

$$G(s) = \frac{\varphi(s)}{\delta(s)} = \frac{aV}{bh}\,\frac{s + \dfrac{V}{a}}{s^2 - \dfrac{g}{h}}$$

In this model, h represents the vertical distance from the center of mass to the floor, so it can be readily verified that the model is open-loop unstable. (*Åström, 2005*). Assume that for a specific bicycle, $a = 0.6$ m, $b = 1.5$ m, $h = 0.8$ m, and $g = 9.8$ m/sec. In order to stabilize the bicycle, it is assumed that the bicycle is placed in the closed-loop configuration shown in Figure P8.3 and that the only available control variable is V, the rear wheel velocity.

a. Find the range of V for closed-loop stability.

b. Explain why the methods presented in this chapter cannot be used to obtain the root locus.

c. `Use MATLAB to obtain the system's root locus.` MATLAB ML

51. A technique to control the steering of a vehicle that follows a line located in the middle of a lane is to define a look-ahead point and measure vehicle deviations with respect to the point. A linearized model for such a vehicle is

$$\begin{bmatrix} \dot{V} \\ \dot{r} \\ \dot{\psi} \\ \dot{Y}_g \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & -b_1K & \dfrac{b_1K}{d} \\ a_{21} & a_{22} & -b_2K & \dfrac{b_2K}{d} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & U & 0 \end{bmatrix} \begin{bmatrix} V \\ r \\ \psi \\ Y_g \end{bmatrix}$$

where V = vehicle's lateral velocity, r = vehicle's yaw velocity, ψ = vehicle's yaw position, and Y_g = the y-axis coordinate of the vehicle's center of gravity. K is a parameter to be varied depending upon trajectory changes. In a specific vehicle traveling at a speed of $U = -10$ m/sec, the parameters are $a_{11} = -11.6842$, $a_{12} = 6.7632$, $b_1 = -61.5789$, $a_{21} = -3.5143$, $a_{22} = 24.0257$, and $b_2 = 66.8571$. $d = 5$ m is the look-ahead distance (*Ünyelioğlu, 1997*). Assuming the vehicle will be controlled in closed loop:

a. Find the system's characteristic equation as a function of K.

b. Find the system's root locus as K is varied.

c. Using the root locus found in Part **b**, show that the system will be unstable for all values K.

52. It is known that mammals have hormonal regulation mechanisms that help maintain almost constant calcium plasma levels (0.08–0.1 g/L in dairy cows). This control is necessary to maintain healthy functions, as calcium is responsible for diverse physiological functions, such as bone formation, intracellular communications, and blood clotting. It has been postulated that the mechanism of calcium control resembles that of a PI (proportional-plus-integral) controller. PI controllers (discussed in detail in Chapter 9) are placed in cascade with the plant and used to improve steady-state error. Assume that the PI controller has the form $G_c(s) = \left[K_P + \dfrac{K_I}{S}\right]$ where K_P and K_I are constants. Also assume that the mammal's system accumulates calcium in an integrator-like fashion, namely $P(s) = \dfrac{1}{V_s}$, where V is the plasma volume. The closed-loop model is similar to that of Figure P8.3, where $G(s) = G_c(s)P(s)$ (*Khammash, 2004*).

a. Sketch the system's root locus as a function of K_P, assuming $K_I > 0$ is constant.

b. Sketch the system's root locus as a function of K_I, assuming $K_P > 0$ is constant.

53. Problem 65 in Chapter 7 introduced the model of a TCP/IP router whose packet-drop probability is controlled by using a random early detection (RED) algorithm (*Hollot, 2001*). Using Figure P8.3 as a model, a specific router queue's open-loop transfer function is

$$G(s) = \frac{7031250Le^{-0.2s}}{(s+0.667)(s+5)(s+50)}$$

The function $e^{-0.2s}$ represents delay. To apply the root locus method, the delay function must be replaced with a rational function approximation. A first-order Padé approximation can be used for this purpose. Let $e^{-sD} \approx 1 - Ds$. Using this approximation, plot the root locus of the system as a function of L.

54. For the dynamic voltage restorer (DVR) discussed in Problem 47, Chapter 7, do the following:

a. When $Z_L = \dfrac{1}{sC_L}$, a pure capacitance, the system is more inclined toward instability. Find the system's characteristic equation for this case.

b. Using the characteristic equation found in Part **a**, sketch the root locus of the system as a function of C_L. Let $L = 7.6$ mH, $C = 11$ μF, $\alpha = 26.4$, $\beta = 1$, $K_m = 25$, $K_v = 15$, $K_T = 0.09565$, and $\tau = 2$ ms (*Lam, 2004*).

55. The closed-loop vehicle response in stopping a train depends on the train's dynamics and the driver, who is an integral part of the feedback loop. In Figure P8.3, let the input be $R(s) = v_r$ the reference velocity, and the output $C(s) = v$, the actual vehicle velocity. (*Yamazaki, 2008*) shows that such dynamics can be modeled by $G(s) = G_d(s)G_t(s)$ where

$$G_d(s) = h\left(1 + \frac{K}{s}\right)\frac{s - \dfrac{L}{2}}{s + \dfrac{L}{2}}$$

represents the driver dynamics with h, K, and L parameters particular to each individual driver. We assume here that $h = 0.003$ and $L = 1$. The train dynamics are given by

$$G_t(s) = \frac{k_b f K_p}{M(1+k_e)s(\tau s + 1)}$$

where $M = 8000$ kg, the vehicle mass; $k_e = 0.1$ the inertial coefficient; $k_b = 142.5$, the brake gain; $K_p = 47.5$, the pressure gain; $\tau = 1.2$ sec, a time constant; and $f = 0.24$, the normal friction coefficient.

a. Make a root locus plot of the system as a function of the driver parameter K.

b. Discuss why this model may not be an accurate description of a real driver-train situation.

56. Voltage droop control is a technique in which loads are driven at lower voltages than those provided by the source. In general, the voltage is decreased as current demand increases in the load. The advantage of voltage droop is that it results in lower sensitivity to load current variations.

Voltage droop can be applied to the power distribution of several generators and loads linked through a dc bus. In (*Karlsson, 2003*) generators and loads are driven by 3-phase ac power, so they are interfaced to the bus through ac/dc converters. Since each one of the loads works independently, a feedback system shown in Figure P8.17 is used in each to respond equally to bus voltage variations. Given that $C_s = C_r = 8{,}000\ \mu\text{F}$, $L_{cable} = 50\ \mu\text{H}$, $R_{cable} = 0.06\ \Omega$, $Z_r = R_r = 5\ \Omega$, $\omega_{lp} = 200$ rad/s, $G_{conv}(s) = 1$, $V_{dc\text{-}ref} = 750$ V, and $P_{ref\text{-}ext} = 0$, do the following:

a. If $Z_{r_{eq}}$ is the parallel combination of R_r and C_r, and $G_{conv}(s) = 1$, find

$$G(s) = \frac{V_s(s)}{I_s(s)} = \frac{V_s(s)}{I_{s-ref}(s)}.$$

b. `Write a MATLAB M-file to plot and copy the full root locus for that system, then zoom-in the locus by setting the x-axis (real-axis) limits to −150 to 0 and the y-axis (imaginary-axis) limits to −150 to 150. Copy that plot, too, and find and record the following:` MATLAB ML

- **(1)** `The gain, K, at which the system would have complex-conjugate closed-loop dominant poles with a damping ratio` $\zeta = 0.707$
- **(2)** `The coordinates of the corresponding point selected on the root-locus`
- **(3)** `The values of all closed-loop poles at that gain`
- **(4)** `The output voltage` $v_s(t)$ `for a step input voltage` $v_{dc\text{-}ref}(t) = 750\ u(t)$ `volts`

c. `Plot that step response and use MATLAB` **`Characteristics`** `tool (in the graph window) to note on the curve the following parameters:`

- **(1)** `The actual percent overshoot and the corresponding peak time,` T_p
- **(2)** `The rise time,` T_r`, and the settling time,` T_s
- **(3)** `The final steady-state value in volts`

DESIGN PROBLEMS

57. A disk drive is a position control system in which a read/write head is positioned over a magnetic disk. The system responds to a command from a computer to position itself at a particular track on the disk. A physical representation of the system and a block diagram are shown in Figure P8.18.

a. Find K to yield a settling time of 0.1 second.

b. What is the resulting percent overshoot?

c. What is the range of K that keeps the system stable?

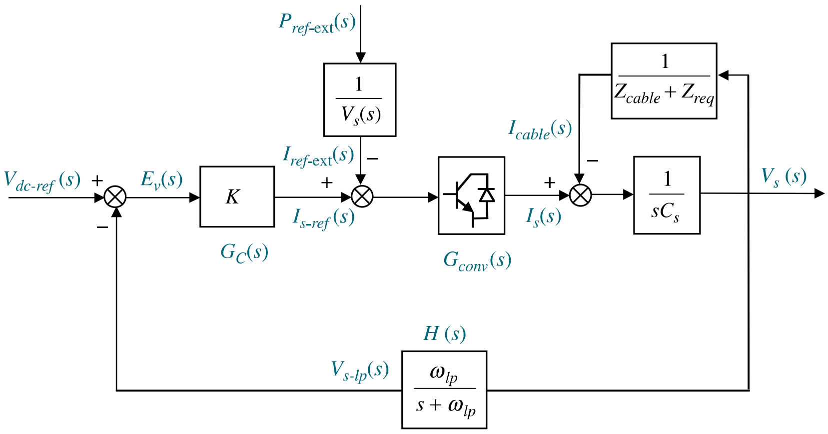


FIGURE P8.17 (© 2003 IEEE)

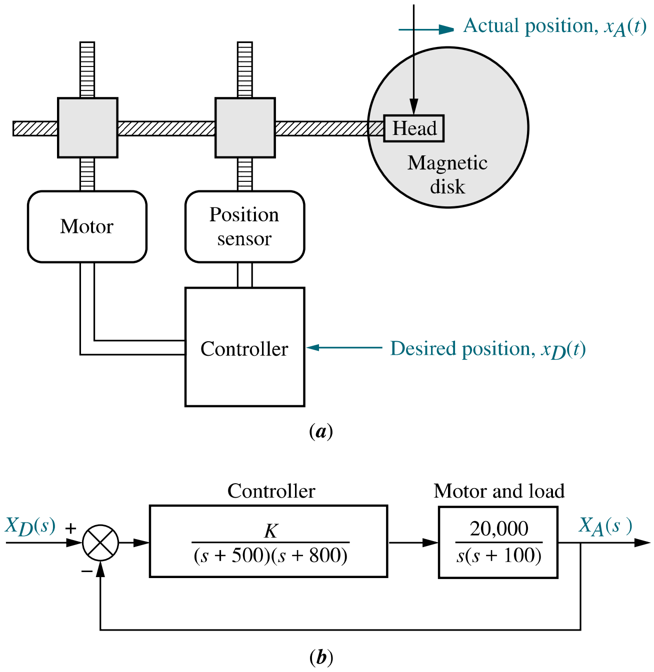


FIGURE P8.18 Disk drive: **a.** physical representation; **b.** block diagram

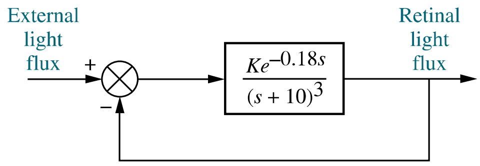


FIGURE P8.19 Simplified block diagram of pupil servomechanism

58. A simplified block diagram of a human pupil servomechanism is shown in Figure P8.19. The term $e^{-0.18s}$ represents a time delay. This function can be approximated by what is known as a Padé approximation. This approximation can take on many increasingly complicated forms, depending upon the degree of accuracy required. If we use the Padé approximation

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$$e^{-x} = \frac{1}{1 + x + \frac{x^2}{2!}}$$

then

$$e^{-0.18s} = \frac{61.73}{s^2 + 11.11s + 61.73}$$

Since the retinal light flux is a function of the opening of the iris, oscillations in the amount of retinal light flux imply oscillations of the iris (*Guy, 1976*). Find the following:

a. The value of K that will yield oscillations

b. The frequency of these oscillations

c. The settling time for the iris if K is such that the eye is operating with 20% overshoot

59. An active suspension system for AMTRAK trains has been proposed. The system uses a pneumatic actuator in parallel with the passive suspension system, as shown in Figure P8.20. The force of the actuator subtracts from the force applied by the ground, as represented by displacement, $y_g(t)$. Acceleration is sensed by an accelerometer, and signals proportional to acceleration and velocity are fed back to the force actuator. The transfer function relating acceleration to ground displacement is

$$\frac{\ddot{Y}_m(s)}{Y_g(s)} = \frac{s^2(D_s + K)}{(C_a + M)s^2 + (C_v + D)s + K}$$

Assuming that $M = 1$ and $D = K = C_v = 2$, do the following (*Cho, 1985*):

a. Sketch a root locus for this system as C_a varies from zero to infinity.

b. Find the value of C_a that would yield a damping ratio of 0.69 for the closed-loop poles.

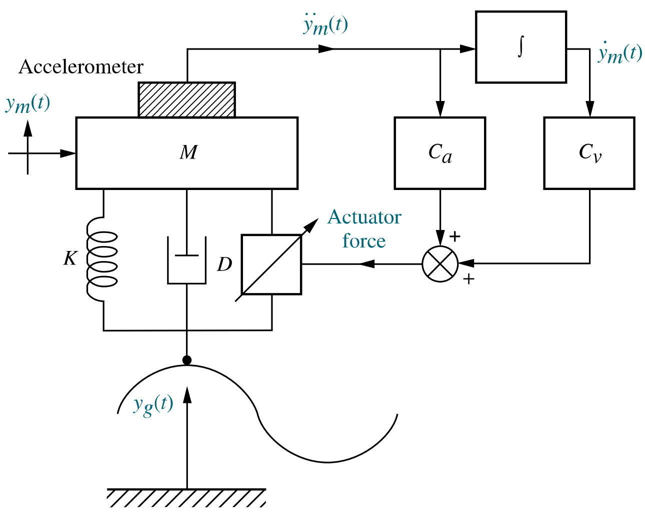

FIGURE P8.20 Active suspension system (Reprinted with permission of ASME)

60. The pitch stabilization loop for an F4-E military aircraft is shown in Figure P8.21. δ_{com} is the elevator

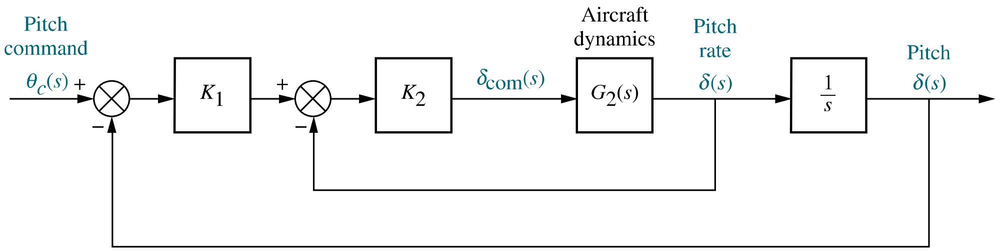

FIGURE P8.21 F4-E pitch stabilization loop

and canard input deflection command to create a pitch rate (see Problem 22, Chapter 3). If

$$G_2(s) = \frac{-508(s+1.6)}{(s+14)(s-1.8)(s+4.9)}$$

do the following (*Cavallo, 1992*):

a. Sketch the root locus of the inner loop.

b. Find the range of K_2 to keep the inner loop stable with just pitch-rate feedback.

c. Find the value of K_2 that places the inner-loop poles to yield a damping ratio of 0.5.

d. For your answer to Part **c**, find the range of K_1 that keeps the system stable.

e. Find the value of K_1 that yields closed-loop poles with a damping ratio of 0.45.

61. Accurate pointing of spacecraft is required for communication and mapping. Attitude control can be implemented by exchanging angular momentum between the body of the spacecraft and an onboard momentum wheel. The block diagram for the pitch axis attitude control is shown in Figure P8.22, where $\theta_c(s)$ is a commanded pitch angle and $\theta(s)$ is the actual pitch angle of the spacecraft. The compensator, which improves pointing accuracy, provides a commanded momentum, $H_c(s)$, to the momentum wheel assembly. The spacecraft momentum, $H_{sys}(s)$, is an additional input to the momentum wheel. This body momentum is given by

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$$h_{sys}(t) = I_2\dot{\theta}(t) + h_w(t)$$

where I_2 is the spacecraft moment of inertia about the pitch axis and $h_w(t)$ is the momentum of the wheel. The total torque output from the momentum wheel, T_w, as shown in Figure P8.22, is

$$T_w(t) = \frac{h_{sys}(t) - h_w(t) + h_c(t)}{\tau}$$

If $\tau = 23$ seconds and $I_2 = 9631$ in-lb-s^2, do the following (*Piper, 1992*):

a. Sketch the root locus for the pitch axis control system.

b. Find the value of K to yield a closed-loop step response with 25% overshoot.

c. Evaluate the accuracy of any second-order approximations that were made.

62. During combustion in such devices as gas turbines and jet engines, acoustic waves are generated. These pressure waves can lead to excessive noise as well as mechanical failure. Active control is proposed to reduce this thermoacoustic effect. Specifically, a microphone is used as a sensor to read the sound waves, while a loudspeaker is used as an actuator to set up opposing pressure waves to reduce the effect. A proposed diagram showing the microphone and loudspeaker positioned in the combustion chamber is

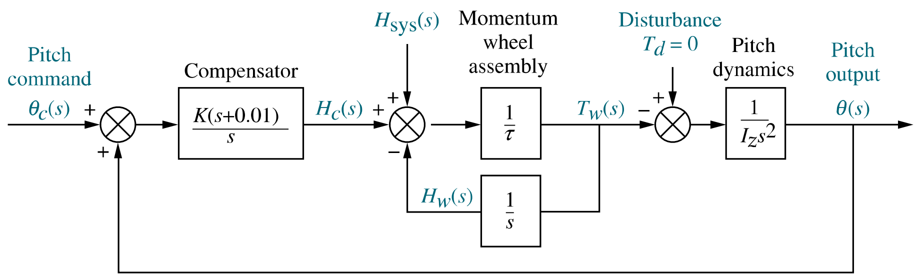

FIGURE P8.22 Pitch axis attitude control system utilizing momentum wheel

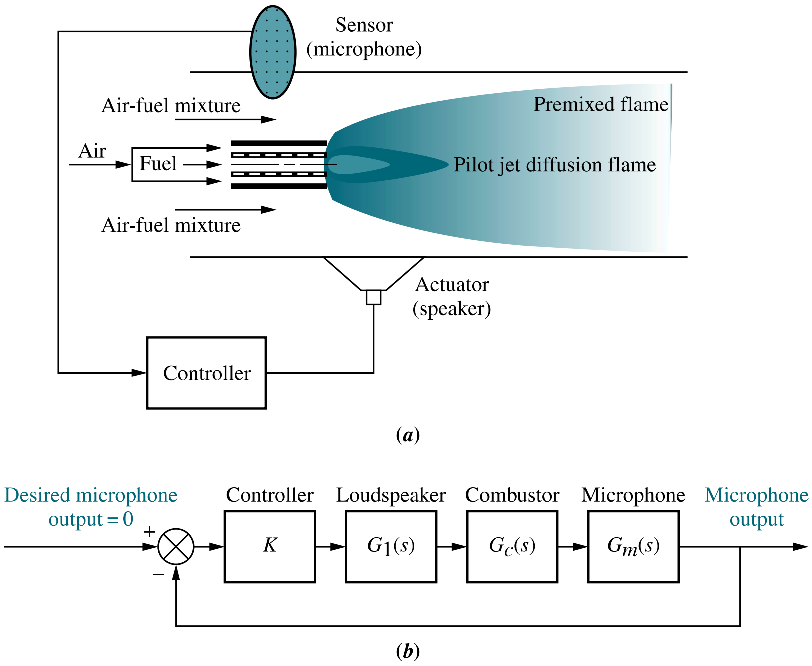

FIGURE P8.23 **a.** Combustor with microphone and loudspeaker (© 1995 IEEE); **b.** block diagram (© 1995 IEEE)

shown in Figure P8.23(*a*). A simplified block diagram of the active control system is shown in Figure P8.23(*b*). The transfer functions are dependent upon microphone and loudspeaker placement and parameters as well as flame placement and parameters. The forward-path transfer function is of the form

$$G(s) = KG_1(s)G_c(s)G_m(s)$$
$$= \frac{K\left(s+z_f\right)\left(s^2+2\zeta_2\omega_2 s+\omega_2^2\right)}{\left(s+p_f\right)\left(s^2-2\zeta_1\omega_1 s+\omega_1^2\right)\left(s^2+2\zeta_2\omega_2 s+\omega_2^2\right)}$$

where the values for three configurations (A, B, and C) are given in the following table for Part **b** (*Annaswamy, 1995*).

	A	B	C
z_f	1500	1500	1500
p_f	1000	1000	1000
ζ_z	0.45	0.45	−0.45
ω_z	4500	4500	4500
ζ_1	0.5	−0.5	−0.5
ω_1	995	995	995
ζ_2	0.3	0.3	0.3
ω_2	3500	3500	3500

a. Draw the root locus for each configuration.

b. For those configurations where stable regions of operation are possible, evaluate the range of gain, K, for stability.

63. Wind turbines, such as the one shown in Figure P8.24(*a*), are becoming popular as a way of generating electricity. Feedback control loops are designed to control the output power of the turbine, given an input power demand. Blade-pitch control may be used as part of the control loop for a constant-speed, pitch-controlled wind turbine, as shown in Figure P8.24(*b*). The drivetrain, consisting of the windmill rotor, gearbox, and electric generator (see Figure P8.24(*c*)), is part of the control loop. The torque created by the wind drives the rotor. The windmill rotor is connected to the generator through a gearbox.
The transfer function of the drivetrain is

$$\frac{P_o(s)}{T_R(s)} = G_{dt}(s)$$
$$= \frac{3.92K_{LSS}K_{HSS}K_G N^2 s}{\{N^2K_{HSS}(J_R s^2+K_{LSS})(J_G s^2[\tau_{el}s+1] + K_G s) + J_R s^2 K_{LSS}[(J_G s^2+K_{HSS})(\tau_{el}s+1)+K_G s]\}}$$

where $P_o(s)$ is the Laplace transform of the output power from the generator and $T_R(s)$ is the Laplace

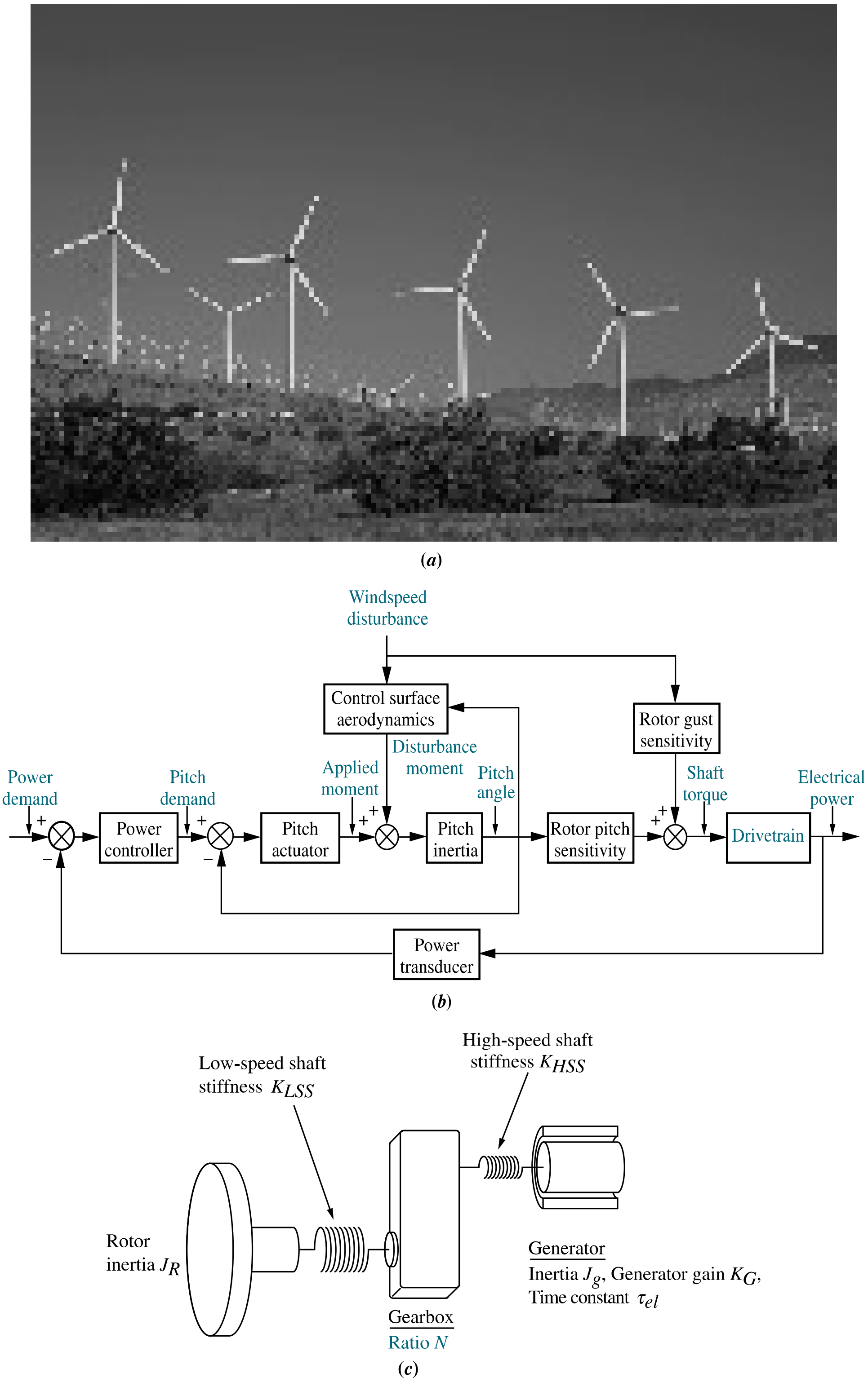

FIGURE P8.24 **a.** Wind turbines generating electricity near Palm Springs, California **b.** control loop for a constant-speed pitch-controlled wind turbine (© 1998 IEEE); **c.** drivetrain (© 1998 IEEE) transform of the input torque on the rotor. Substituting typical numerical values into the transfer function yields

$$\frac{P_o(s)}{T_R(s)} = G_{dt}(s)$$
$$= \frac{(3.92)(12.6\times10^6)(301\times10^3)(688)N^2 s}{\begin{array}{l}\{N^2(301\times10^3)(190,120s^2+12.6\times10^6)\\ \times(3.8s^2[20\times10^{-3}s+1]+668s)\\ +190,120s^2(12.6\times10^6)\\ \times[(3.8s^2+301\times10^3)\\ \times(20\times10^{-3}s+1)+668s]\}\end{array}}$$

(*Anderson*, *1998*). Do the following for the drive-train dynamics, making use of any computational aids at your disposal:

a. Sketch a root locus that shows the pole locations of $G_{dt}(s)$ for different values of gear ratio, N.

b. Find the value of N that yields a pair of complex poles of $G_{dt}(s)$ with a damping ratio of 0.5.

64. A hard disk drive (HDD) arm has an open-loop unstable transfer function,

$$P(s) = \frac{X(s)}{F(s)} = \frac{1}{I_b s^2}$$

where $X(s)$ is arm displacement and $F(s)$ is the applied force (*Yan*, *2003*). Assume the arm has an inertia of $I_b = 3\times10^{-5}$ kg-m^2 and that a lead controller, $G_c(s)$ (used to improve transient response and discussed in Chapter 9), is placed in cascade to yield

$$P(s)G_c(s) = G(s) = \frac{K}{I_b}\frac{(s+1)}{s^2(s+10)}$$

as in Figure P8.3.

a. Plot the root locus of the system as a function of K.

b. Find the value of K that will result in dominant complex conjugate poles with a $\zeta = 0.7$ damping factor.

65. A robotic manipulator together with a cascade PI controller (used to improve steady-state response and discussed in Chapter 9) has a transfer function (*Low*, *2005*)

$$G(s) = \left(K_p + \frac{K_1}{s}\right)\frac{48,500}{s^2+2.89s}$$

Assume the robot's joint will be controlled in the configuration shown in Figure P8.3.

a. Find the value of K_I that will result in $e_{ss} = 2\%$ for a parabolic input.

b. Using the value of K_I found in Part **a**, plot the root locus of the system as a function of K_P.

c. Find the value of K_P that will result in a real pole at -1. Find the location of the other two poles.

66. An active system for the elimination of floor vibrations due to human presence is presented in (*Nyawako*, *2009*). The system consists of a sensor that measures the floor's vertical acceleration and an actuator that changes the floor characteristics. The open-loop transmission of the particular setup used can be described by $G(s) = KG_a(s)F(s)G_m(s)$, where the actuator's transfer function is

Gui Tool
GUIT

$$G_a(s) = \frac{10.26}{s^2+11.31s+127.9}$$

The floor's dynamic charactristics can be modeled by

$$F(s) = \frac{6.667\times10^{-5}s^2}{s^2+0.2287s+817.3}$$

The sensor's transfer function is

$$G_m(s) = \frac{s}{s^2+5.181s+22.18}$$

and K is the gain of the controller. The system operations can be described by the unity-gain feedback loop of Figure P8.3.

a. Use MATLAB's SISO Design Tool to obtain the root locus of the system in terms of K.

b. Find the range of K for closed-loop stability.

c. Find, if possible, a value of K that will yield a closed-loop overdamped response.

67. Many implantable medical devices such as pacemakers, retinal implants, deep brain stimulators, and spinal cord stimulators are powered by an in-body battery that can be charged through a transcutaneous inductive device. Optimal battery charge can be obtained when the out-of-body charging circuit is in resonance with the implanted charging circuit (*Baker, 2007*). Under certain conditions, the coupling of both resonant circuits can be modeled by the feedback system in Figure P8.3 where

$$G(s) = \frac{Ks^4}{(s^2+2\zeta\omega_n s+\omega_n^2)^2}$$

The gain K is related to the magnetic coupling between the external and in-body circuits. K may vary due to positioning, skin conditions, and other variations. For this problem let $\zeta = 0.5$ and $\omega_n = 1$.

a. Find the range of K for closed-loop stability.

b. Draw the corresponding root locus.

68. It is important to precisely control the amount of organic fertilizer applied to a specific crop area in order to provide specific nutrient quantities and to avoid unnecessary environmental pollution. A precise delivery liquid manure machine has been developed for this purpose (*Saeys, 2008*). The system consists of a pressurized tank, a valve, and a rheological flow sensor. After simplification, the system can be modeled as a closed-loop negative-feedback system with a forward-path transfer function

MATLAB **ML**
Gui Tool **GUIT**

$$G(s) = \frac{2057.38K(s^2 - 120s + 4800)}{s(s + 13.17)(s^2 + 120s + 4800)}$$

consisting of an electrohydraulic system in cascade with the gain of the manure flow valve and a variable gain, K. The feedback path is comprised of

$$H(s) = \frac{10(s^2 - 4s + 5.333)}{(s + 10)(s^2 + 4s + 5.333)}$$

a. Use the SISO Design Tool in MATLAB to obtain the root locus of the system.

b. Use the SISO Design Tool to find the range of K for closed-loop stability.

c. Find the value of K that will result in the smallest settling time for this system.

d. Calculate the expected settling time for a step input with the value of K obtained in Part **c**.

e. Check your result through a step-response simulation.

69. Harmonic drives are very popular for use in robotic manipulators due to their low backlash, high torque transmission, and compact size (*Spong, 2006*). The problem of joint flexibility is sometimes a limiting factor in achieving good performance. Consider that the idealized model representing joint flexibility is shown in Figure P8.25. The input to the drive is from an actuator and is applied at θ_m. The output is connected to a load at θ_l. The spring represents the joint flexibility and B_m and B_l represent the viscous damping of the actuator and load, respectively. Now we insert the device into the feedback loop shown in Figure P8.26. The first block in the forward path is a PD controller, which we will study in the next chapter. The PD controller is used to improve transient response performance.

MATLAB **ML**

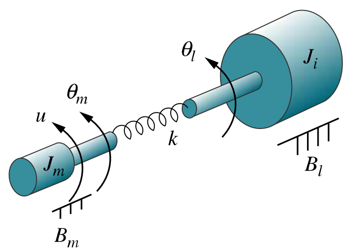

FIGURE P8.25 Idealized model representing joint flexibility (Reprinted with permission of John Wiley & Sons, Inc.)

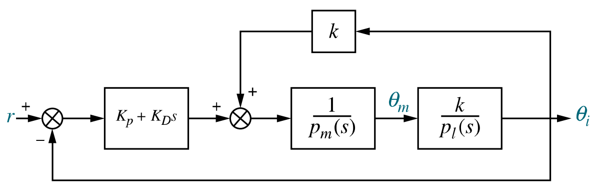

FIGURE P8.26 Joint flexibility model inserted in feedback loop. (Reprinted with permission of John Wiley & Sons, Inc.)

Use MATLAB to find the gain K_D to yield an approximate 5% overshoot in the step response given the following parameters: J_l=10; B_l=1; k=100; J_m=2; B_m=0.5; $\frac{K_P}{K_D} = 0.25$; $p_l(s) = J_l s^2 + B_l s + k$; and $p_m(s) = J_m s^2 + B_m s + k$

70. Using LabVIEW, the Control Design and Simulation Module, and the MathScript RT Module, open and customize the Interactive Root Locus VI from the Examples to implement the system of Problem 69. Select the parameter K_D to meet the requirement of Problem 69 by varying the location of the closed-loop poles on the root locus. Be sure

LabVIEW **LV**

your front panel shows the following: (1) open-loop transfer function, (2) closed-loop transfer function, (3) root locus, (4) list of closed-loop poles, and (5) step response.

71. An automatic regulator is used to control the field current of a three-phase synchronous machine with identical symmetrical armature windings (*Stapleton, 1964*). The purpose of the regulator is to maintain the system voltage constant within certain limits. The transfer function of the synchronous machine is

MATLAB ML

$$G_{sm}(s) = \frac{\Delta\delta(s)}{\Delta P_m(s)} = \frac{M(s - z_1)(s - z_2)}{(s - p_1)(s - p_2)(s - p_3)}$$

which relates the variation of rotor angle, $\Delta\delta(s)$, to the change in the synchronous machine's shaft power, $\Delta P_m(s)$. The closed-loop system is shown in Figure P8.3, where $G(s) = KG_C(s)G_{sm}(s)$ and K is a gain to be adjusted. The regulator's transfer function, $G_c(s)$, is given by:

$$G_c(s) = \frac{\mu/T_e}{s + \frac{1}{T_e}}$$

Assume the following parameter values:

$\mu = 4, M = 0.117, T_e = 0.5, z_{1,2} = -0.071 \pm j6.25,$
$p_1 = -0.047$, and $p_{2,3} = -0.262 \pm j5.1,$

and do the following:

Write a MATLAB M-file to plot the root locus for the system and to find the following:

a. The gain K at which the system becomes marginally stable

b. The closed-loop poles, p, and transfer function, $T(s)$, corresponding to a 16% overshoot

c. The coordinates of the point selected on the root-locus corresponding to 16% overshoot

d. A simulation of the unit-step response of the closed-loop system corresponding to your 16% overshoot design. Note in your simulation the following values: (1) actual percent overshoot, (2) corresponding peak time, T_p, (2) rise time, T_r, (3) settling time, T_s, and (4) final steady-state value.

PROGRESSIVE ANALYSIS AND DESIGN PROBLEMS

72. High-speed rail pantograph. Problem 21 in Chapter 1 discusses the active control of a pantograph mechanism for high-speed rail systems. In Problem 79, Chapter 5, you found the block diagram for the active pantograph control system. Use your block diagram to do the following (*O'Connor, 1997*):

a. Sketch the root locus.

b. Assume a second-order approximation and find the gain, K, to yield a closed-loop step response that has 38% overshoot.

c. Estimate settling time and peak time for the response designed in Part **b.**

d. Discuss the validity of your second-order approximation.

e. Use MATLAB to plot the closed-loop step response for the value of K found in Part **b**. Compare the plot to predicted values found in Parts **b** and **c**.

MATLAB ML

73. Control of HIV/AIDS. In the linearized model of Chapter 6, Problem 68, where virus levels are controlled by means of RTIs, the open-loop plant transfer function was shown to be

$$P(s) = \frac{Y(s)}{U_1(s)} = \frac{-520s - 10.3844}{s^3 + 2.6817s^2 + 0.11s + 0.0126}$$

The amount of RTIs delivered to the patient will automatically be calculated by embedding the patient in the control loop as $G(s)$ shown in Figure P6.20 (*Craig, 2004*).

a. In the simplest case, $G(s) = K$, with $K > 0$. Note that this effectively creates a positive-feedback loop because the negative sign in the numerator of $P(s)$ cancels out with the negative-feedback sign in the summing junction. Use positive-feedback rules to plot the root locus of the system.

b. Now assume $G(s) = -K$ with $K > 0$. The system is now a negative-feedback system. Use negative-feed-back rules to draw the root locus. Show that in this case the system will be closed-loop stable for all $K > 0$.

74. Hybrid vehicle. In chapter 7, Figure P7.34 shows the block diagram of the speed control of

MATLAB ML

an HEV rearranged as a unity feedback system (*Preitl, 2007*).

Let the transfer function of the speed controller be

$$G_{SC}(s) = K_{P_{sc}} + \frac{K_{I_{sc}}}{s} = \frac{K_{P_{sc}}\left(s + \frac{K_{I_{sc}}}{K_{P_{sc}}}\right)}{s}$$

a. Assume first that the speed controller is configured as a proportional controller ($K_{I_{SC}} = 0$ and $G_{SC}(s) = K_{P_{SC}}$). Calculate the forward-path open-loop poles. Now use MATLAB to plot the system's root locus and find the gain, $K_{P_{SC}}$ that yields a critically damped closed-loop response. Finally, plot the time-domain response, $c(t)$, for a unit-step input using MATLAB. Note on the curve the rise time, T_r, and settling time, T_s.

b. Now add an integral gain, $K_{I_{SC}}$, to the controller, such that $K_{I_{SC}}/K_{P_{SC}} = 0.4$. Use MATLAB to plot the root locus and find the proportional gain, $K_{P_{SC}}$, that could lead to a closed-loop unit-step response with 10% overshoot. Plot $c(t)$ using MATLAB and note on the curve the peak time, T_p, and settling time, T_s. Does the response obtained resemble a second-order underdamped response?

Cyber Exploration Laboratory

Experiment 8.1

Objective To verify the effect of open-loop poles and zeros upon the shape of the root locus. To verify the root locus as a tool for estimating the effect of open-loop gain upon the transient response of closed-loop systems.

Minimum Required Software Packages MATLAB and the Control System Toolbox

Prelab

1. Sketch two possibilities for the root locus of a unity negative-feedback system with the open-loop pole-zero configuration shown in Figure P8.27.

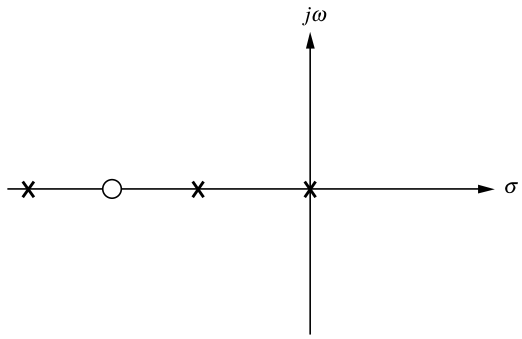

FIGURE P8.27

2. If the open-loop system of Prelab 1 is $G(s) = \dfrac{K(s+1.5)}{s(s+0.5)(s+10)}$, estimate the percent overshoot at the following values of gain, K: 20, 50, 85, 200, and 700.

Lab

1. Using Matlab's SISO Design Tool, set up a negative unity feedback system with $G(s) = \dfrac{K(s+6)}{s(s+0.5)(s+10)}$ to produce a root locus. For convenience, set up the zero

at -6 using SISO Design Tool's compensator function by simply dragging a zero to -6 on the resulting root locus. Print the root locus for the zero at -6. Move the zero to the following locations and print out a root locus at each location: -2, -1.5, -1.37, and -1.2.

2. Using Matlab's SISO Design Tool, set up a negative unity feedback system with $G(s) = \dfrac{K(s+1.5)}{s(s+0.5)(s+10)}$ to produce a root locus. Open the LTI Viewer for SISO Design Tool to show step responses. Using the values of K specified in Prelab 2, record the percent overshoot and settling time and print the root loci and step response for each value of K.

Postlab

1. Discuss your findings from Prelab 1 and Lab 1. What conclusions can you draw?
2. Make a table comparing percent overshoot and settling time from your calculations in Prelab 2 and your experimental values found in Lab 2. Discuss the reasons for any discrepancies. What conclusions can you draw?

Experiment 8.2

Objective To use MATLAB to design the gain of a controller via root locus.

Minimum Required Software Package MATLAB with the Control Systems Toolbox.

Prelab The open-loop system dynamics model for the NASA eight-axis Advanced Research Manipulator II (ARM II) electromechanical shoulder joint/link, actuated by an armature-controlled dc servomotor is shown in Figure P8.28.

The ARM II shoulder joint constant parameters are $K_a = 12$, $L = 0.006$ H, $R = 1.4\ \Omega$, $K_b = 0.00867$, $n = 200$, $K_m = 4.375$, $J = J_m + J_L/n^2$, $D = D_m + D_L/n^2$, $J_L = 1$, $D_L = 0.5$, $J_m = 0.00844$, and $D_m = 0.00013$(*Craig, 2005*), (*Nyzen, 1999*), (*Williams, 1994*).

a. Obtain the equivalent open-loop transfer function, $G(s) = \dfrac{\theta_L(s)}{V_{ref}(s)}$.

b. The loop is to be closed by cascading a controller, $G_c(s) = K_D s + K_P$, with $G(s)$ in the forward path forming an equivalent forward-transfer function, $G_e(s) = G_c(s)G(s)$. Parameters of $G_c(s)$ will be used to design a desired transient performance. The input to the closed-loop system is a voltage, $V_I(s)$, representing the desired angular displacement of the robotic joint with a ratio of 1 volt equals 1 radian. The output of the closed-loop system is the actual angular displacement of the joint, $\theta_L(s)$. An encoder in the feedback path, K_e, converts the actual joint displacement to a voltage with a ratio of 1 radian equals 1 volt. Draw the closed-loop system showing all transfer functions.

c. Find the closed-loop transfer function.

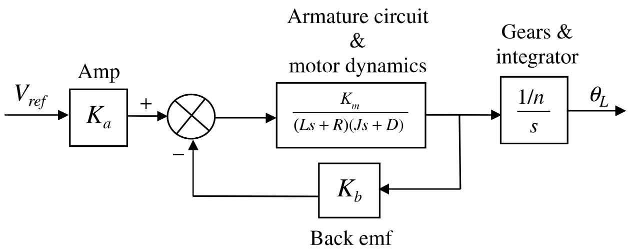

FIGURE P8.28 Open-loop model for ARM II

Lab Let $\frac{K_P}{K_D} = 4$ and use MATLAB to design the value of K_D to yield a step response with a maximum percent overshoot of 0.2%.

Postlab

1. Discuss the success of your design.
2. Is the steady-state error what you would expect? Give reasons for your answer.

Experiment 8.3

Objective To use LabVIEW to design the gain of a controller via root locus.

Minimum Required Software Package LabVIEW with the Control Design and Simulation Module, and the MathScript RT Module.

Prelab Complete the Prelab to Experiment 8.2 if you have not already done so.

Lab Let $\frac{K_P}{K_D} = 4$. Use LabVIEW to open and customize the Interactive Root Locus VI from the Examples in order to implement a design of K_D to yield a step response with a maximum percent overshoot of 0.2%. Use a hybrid graphical/MathScript approach.

Postlab

1. Discuss the success of your design.
2. Is the steady-state error what you would expect? Give reasons for your answer.

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