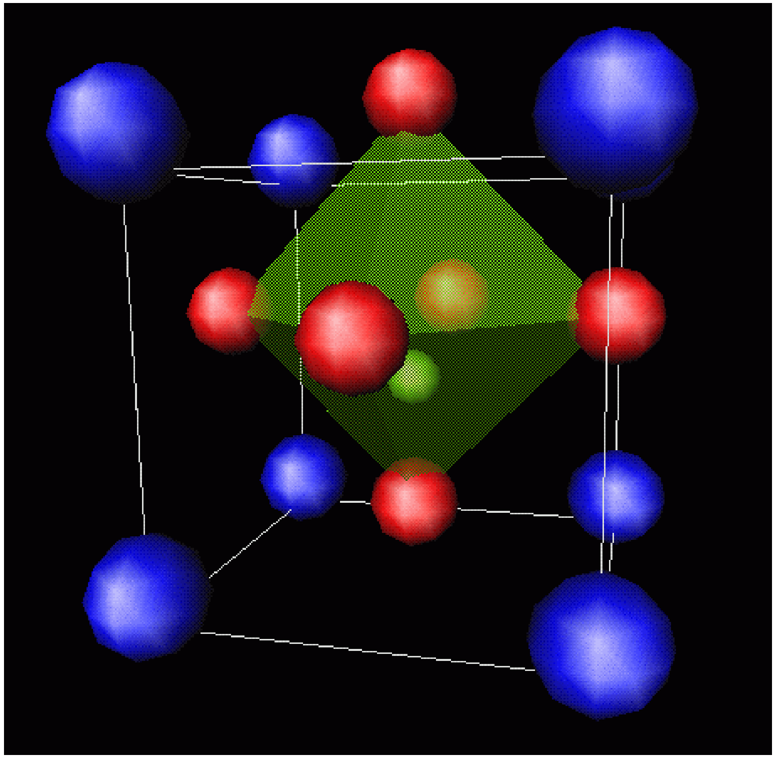
Modelling of Smart Material based Active Strain Generation



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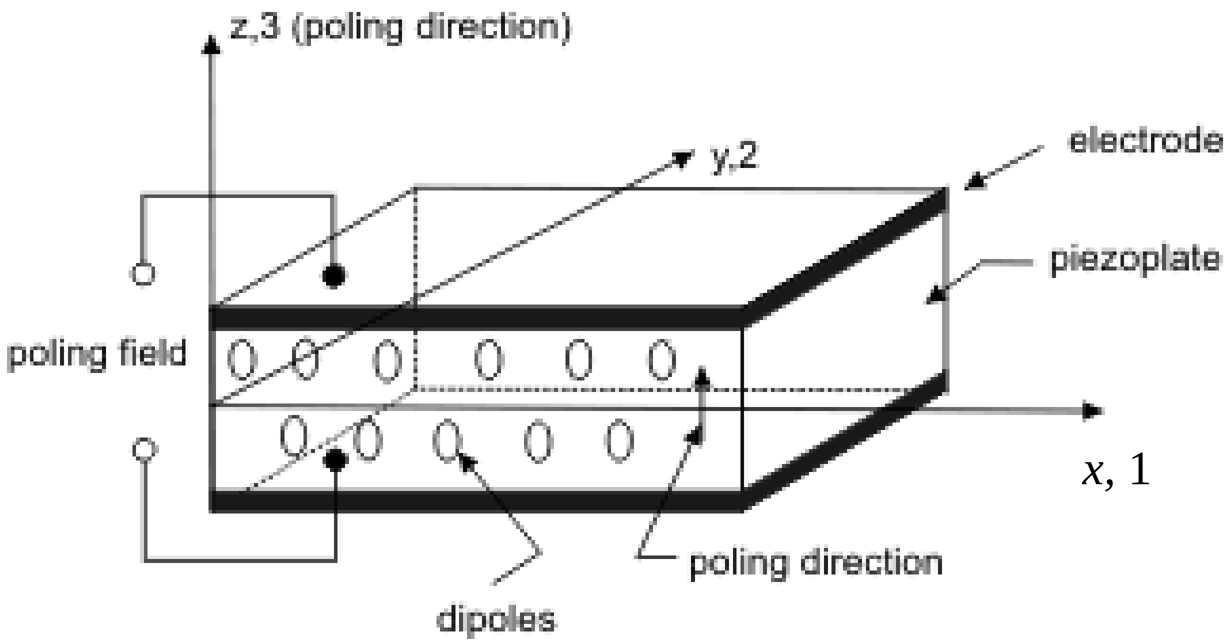
Fundamental equations of piezoelectricity

$$\sigma_{ij} = C^{E}_{ijkl} S_{kl} - d_{kij} E_k$$

$$D_i = d_{ikl} S_{kl} + \varepsilon^{S}_{ij} E_j$$

where, the subscripts ***i, j, k, l*** = 1, 2, 3 denotes **tensorial indices**.
The **stress tensor** is represented by **σ**,
S is the **strain tensor**,
E is the **electric field intensity**,
D is the electric **displacement field**.
Elastic stiffness matrix is denoted by the symbol $\boldsymbol{C^E}$, where the superscript *E* denotes that the elastic constant is measured under constant electric field;
e is the **piezoelectric stress-charge matrix**,
ϵ the **permittivity matrix**,
similar to *C*, is measured under constant strain-condition.

Different Axes



Outcome of Symmetry

The crystal structure of common piezoelectric materials shows 4mm or 6mm symmetry. Following material symmetry conditions could be applied to the constitutive relationship

$$C_{ijkl} = C_{jikl} = C_{klij}$$

$$d_{kij} = d_{kji}$$

$$\varepsilon_{ij} = \varepsilon_{ji}$$

$$
\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \\ D_1 \\ D_2 \\ D_3 \end{Bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 & 0 & 0 & -e_{31} \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 & 0 & 0 & -e_{33} \\
0 & 0 & 0 & C_{44} & 0 & 0 & 0 & -e_{15} & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 & -e_{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e_{15} & 0 & \varepsilon_1 & 0 & 0 \\
0 & 0 & 0 & e_{15} & 0 & 0 & 0 & \varepsilon_2 & 0 \\
e_{31} & e_{31} & e_{33} & 0 & 0 & 0 & 0 & 0 & \varepsilon_3
\end{bmatrix}
\begin{Bmatrix} S_x \\ S_y \\ S_z \\ S_{yz} \\ S_{xz} \\ S_{xy} \\ E_1 \\ E_2 \\ E_3 \end{Bmatrix}
$$

The electro-mechanical coupling is shown inside the bordered boxes. Axes 1, 2 and 3 used for the electrical system are identical with *x*, *y* and *z*, corresponding to the mechanical system.

Simplified Equation for Piezo-patch

- Ignoring the normal stress σ_z and the shear stresses σ_{xz} and σ_{yz} for plane stress assumption:

$$\begin{Bmatrix} S_x \\ S_y \\ S_{xy} \\ D_3 \end{Bmatrix} = \begin{bmatrix} 1/E_p & -\upsilon/E_p & 0 & -d_{31} \\ -\upsilon/E_p & 1/E_p & 0 & -d_{32} \\ 0 & 0 & 2(1+\upsilon)/E_p & 0 \\ d_{31} & d_{32} & 0 & \varepsilon_{33} \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ E_3 \end{Bmatrix}$$

E_p is the **modulus of elasticity** of the piezoelectric material,
ν is the **Poisson's ratio**, and
d_{ij} are the **piezoelectric strain-charge constants**

Active Strain Expression

If a piezoelectric thin slab is subjected to mechanical load, the total strain S developed in an active layer, would consist of two parts – the **structural or elastic strain S_s** and the **piezoelectric strain S_a** such that

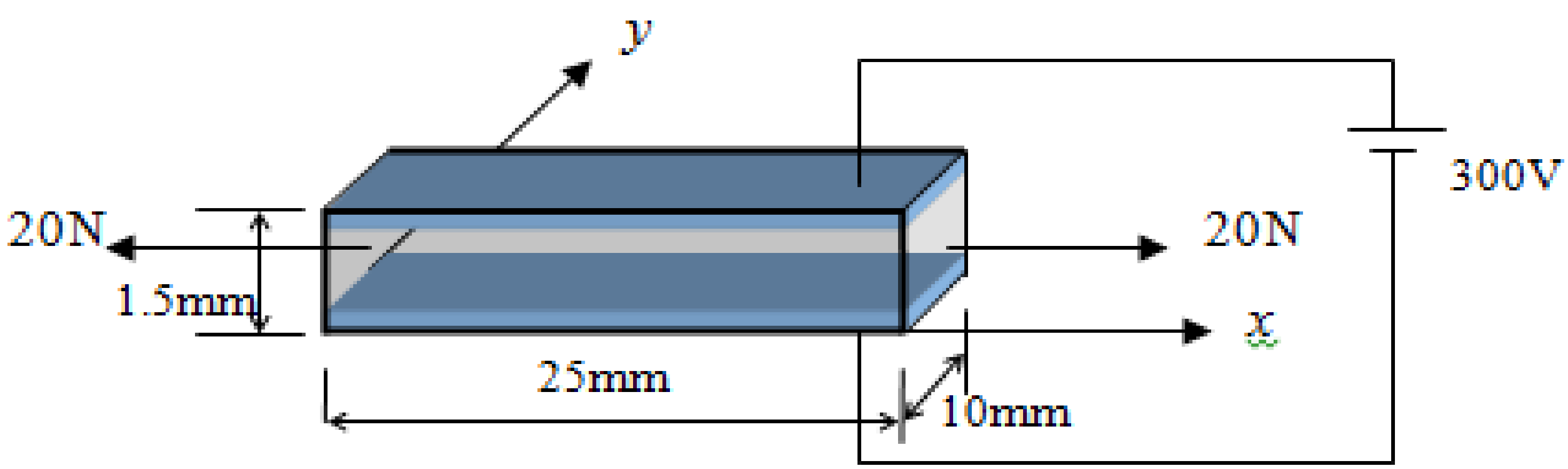
$$\boldsymbol{S} = \boldsymbol{S_s} + \boldsymbol{S_a}$$

where, $S_a = [-d_{31}E_3,\ -d_{32}E_3,\ 0]^T$.

To generate strains along the direction of the thickness of the specimen, ceramics with different crystal-cuts are used which are commonly known as Piezo-stacks. The electro elastic coupling components in the 3-3 directions, like d_{33} or e_{33}, become important in such cases.

Example 1:

A thin piezoelectric plate of size 25 mm x 10 mm and thickness 1.5 mm has electroding on top and bottom as shown in Fig (with an exaggerated view of the thickness). Assume the piezoelectric material to have elastic modulus of 65 GPa, Poisson's ratio 0.3 and electro-mechanical coupling coefficients, d_{31} and d_{33} to be -50×10^{-12} m/V and 200×10^{-12} m/V, respectively. Find out the strains in the plate when it is subjected to a voltage of 300V and a force of 20 N.



Example 1:

Solution:

- Recall, total strain developed in an active layer is given by

$$S = S_s + S_a$$

where, $S_a = [-d_{31}E_3, \ -d_{32}E_3, 0]^T.$

- In this example, we have

$$S_{a_x} = -\frac{d_{31}V}{t} = \frac{50\times10^{-12}(m/V)\times300(V)}{1.5\times10^{-3}(m)} = 1\times10^{-5}$$

$$S_{a_x} = 10\,\mu - strain$$

- since, $d_{31} = d_{32}$, $S_{a_y} = S_{a_x} = 10\,\mu$ - strain

Example 1:

Solution:

- Also, since a tensile force of 20 N is acting on the plate along the *x*-direction, the structural strain along *x*-direction is

$$S_{s_x} = \frac{20(N)}{15\times10^{-6}(m^2)\times 65e^{12}(N/m^2)} = .02\,\mu - strain$$

- and the structural strain along *y*-direction is

$$S_{s_y} = -0.3\times S_{s_x} = -0.006\ \mu - strain$$

- Hence, the **total strain along *x*-direction is 10.02 *μ-strain*** and the **total strain along *y*-direction is 9.99 *μ-strain***.

PIEZO-ELECTRIC COEFFICIENTS

- Four constants are frequently used for the comparison of the performances of different piezoelectric materials for sensing and actuation.
- These are:

i. Piezoelectric Charge Constant (d),

ii. Piezoelectric Voltage Constant (g),

iii. Electro-mechanical coupling factor (k)

iv. Frequency constant (N_p)

- The piezoelectric charge constant *d*, expressed in ‘m/V’ or ‘pC/N’ (1 Pico-Coulomb (pC) = 10^{-12} Coulomb), is defined by the following simple relationship:

$$d_{31} = \frac{\Delta l / l}{V_3 / t} = \frac{q}{F_1}, \qquad d_{32} = \frac{\Delta w / w}{V_3 / t} = \frac{q}{F_2}, \qquad d_{33} = \frac{\Delta t / t}{V_3 / t} = \frac{q}{F_3}$$

q denote **charge collected** in the electrode surfaces,
$\boldsymbol{F_i}$, *i=1..3*, denote the **forces** along the respective directions,
$\boldsymbol{V_3}$ denotes the **voltage applied** along the *z* direction.

- The piezoelectric voltage constant, *g* expressed in V-m/N, is similarly defined as:

$$g_{31} = \frac{V_3}{F_1 / w}, \quad g_{32} = \frac{V_3}{F_2 / l}, \quad g_{33} = \frac{V_3 / t}{F_3 /(w \times l)}$$

- V_3 is the voltage sensed along the *z*-direction due to the application of pressure.
- Like d_{31}, g_{31} is also usually negative signifying the generation of positive voltage upon the application of tensile force to the system.

- The electro-mechanical coupling factor, ***k*** measures the electro-mechanical energy conversion efficiency. It is expressed by the simple relationship

$$k^2 = \frac{{d_{33}}^2}{S^E \varepsilon^T}$$

Superscripts, *E* and *T* signify the values of *S* and ε under constant electric field and constant stress condition, respectively.

- When an unconstrained piezoelectric material is subjected to an alternating current, the material shows resonating behaviour at certain frequencies.
- For a disc element of diameter D_p, and thickness t, the frequency constants corresponding to the radial and thickness modes (f_r and f_t, respectively) are

$$N_d = f_r D_p, \quad N_t = f_t t$$

Property	**PZT** (Hard)	**PZT** (soft)	**PZT-PVDF**	**PMN-PT**	**$LiNbO_3$**	**PVDF**
d_{33} (pC/N)	190	425	120	1240	6	30
d_{31} (pC/N)	-55	-170	-	-	-0.85	-16
g_{33} (mV-m/N)	54	27	300	43		150
g_{31} (mV-m/N)	-16	-11	-	-		-150
k_{33}	0.67	0.70	0.80	0.92	0.17	0.11
E_p (GPa)	63	45	~30	100	20	2.7
Density (ρ) (Kg/m^3)	7500	7500	3300	8120	4600	1760
Λ	1500	1980	400	3100	1210	700

- Composite of PZT-PVDF has high electro-mechanical coupling with a moderate density which is in between those of PZT and PVDF.
- The elastic modulus is also seen to be quite high in comparison with that of PVDF.
- Thus, such composites present a good trade-off between excellent actuation potential of PZT and sensing capability of PVDF.

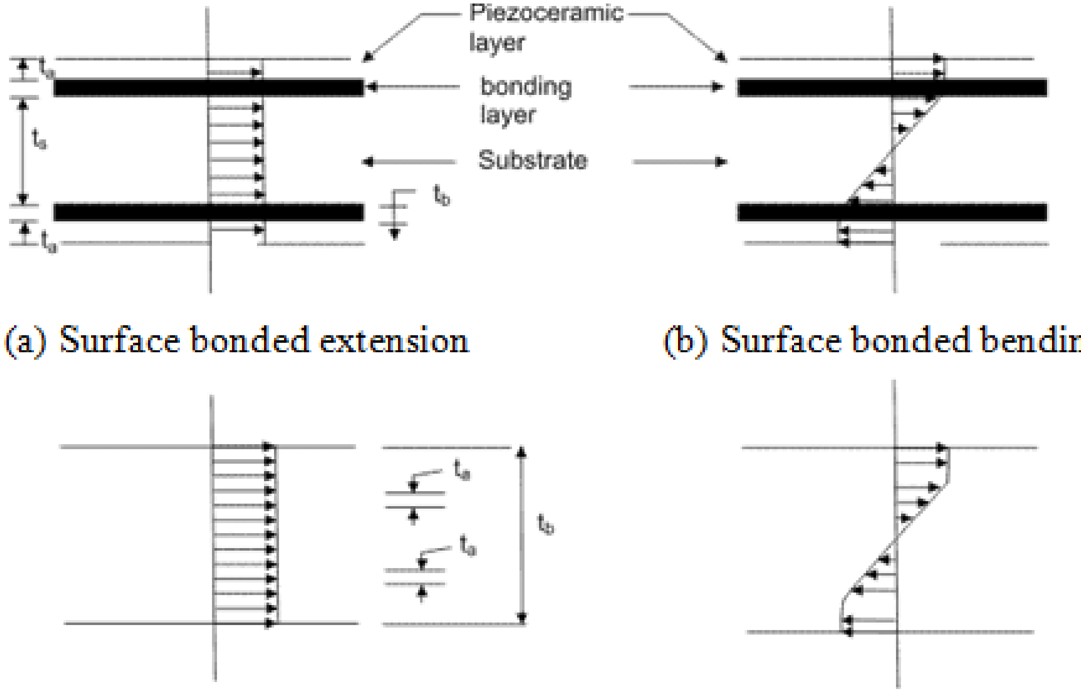
- Often, the high-actuation strain generating capability of SMA is exploited by doping elements of SMA into piezo-ceramic PLZST.
- The product is known as **shape memory ceramic active material**. As high as 6000 μ-*strain* along with memory effect is achieved through this material.
- However, such materials are still in the developmental stage.

Uniform strain model of induced strain actuation

- In the ‘Uniform Strain’ model, it is assumed that the strain remains constant across the piezo-actuator while it varies linearly inside the substructure. The model has been used for surface bonded actuation.
- In ‘Bernoulli- Euler’ model, on the other hand, a linear variation of strain is assumed for the entire cross-section which is considered for embedded actuation.
- For each of these models, the actuators embedded/ bonded on top and bottom of the beam are excitable in the same phase to cause uniform extension or contraction.
- Otherwise, bending can also be generated through out-of-phase excitation of the piezo-ceramics.

Uniform strain model of induced strain actuation

- Figures show detailed sketches of all the strain-diagrams



(a) Surface bonded extension (b) Surface bonded bending

(c) Embedded extension (d) Embedded bending

Uniform strain model of induced strain actuation

- For surface bonded actuation, the static equilibrium corresponding to the induced strain can be expressed as

$$2\,F_a + F_s = F_i\ ,$$

where F_a is the reactive force developed in each active layer, F_s is that in the substrate, and F_i is the total force.

- From above equation, we obtain

$$2(E_p\,S_a\,A_p) + E_s\,S_s\,A_s = 2\Lambda\,E_p\,A_p$$

where ***S*** denotes the **strain**,

E the modulus of **elasticity**,

A the **area of cross-section** and

Λ the **free strain**.

The subscript ***p*** denotes the **piezoelectric material** and ***s*** denotes the **substrate**.

Uniform strain model of induced strain actuation

- Since it is assumed that near the actuator-substrate interface, strain remains unchanged, we can write

$$S_a = S_s = \frac{2\Lambda}{2+\Psi_e} \qquad \text{............ [A]}$$

where, the in-plane stiffness ratio $\psi_e = (E_s A_s) / (E_p A_p)$.

- Similarly, for bending, considering equilibrium of active and reactive moments one gets

$$E_p S_a A_p t_s + \frac{2 E_s I_s S_a}{t_s} = \Lambda E_p A_p t_s \qquad \text{............ [B]}$$

Uniform strain model of induced strain actuation

- Once again the strain compatibility at the interface leads to

$$S_a = S_s = \frac{6\Lambda}{6+\psi_b}$$

with the bending stiffness ratio $\psi_b = 12\,(EI)_s \,/\, [t_s^2\ (EA)_p]$.

- However, when the thickness of the bonding layer is finite, the presence of viscoelastic bonding material reduces the transmission of stress from actuator to substrate and the induced strain to actuation strain ratio is shown to be given by the following relationships

Uniform strain model of induced strain actuation

$$S_a = \frac{\alpha}{\alpha+\psi}\Lambda\left[1+\frac{\psi\cosh(\Gamma\bar{x})}{\alpha\cosh(\Gamma)}\right] \quad \text{............ [C]}$$

$$S_s = \frac{\alpha}{\alpha+\psi}\Lambda\left[1-\frac{\cosh(\Gamma\bar{x})}{\alpha\cosh(\Gamma)}\right] \quad \text{............ [D]}$$

With

$$\Gamma^2 = \bar{D}\,\frac{\alpha+\psi}{\psi}, \qquad \bar{D} = \frac{(G/E_a)(t_b/t_p)}{(t_b/l_p)^2} \quad \text{............ [E]}$$

REF: Crawley, E.F. and Anderson, E.L., Detailed models of piezoceramic actuation of beams, *Journal of Intelligent Material Systems and Structures*, Vol. 1 (1), 4- 25, 1991.

Uniform strain model of induced strain actuation

- $\bar{x}$ is the non-dimensional length parameter varying from –1 to +1 (edge to edge of the actuator).
- The geometric constant α is 2 for extension and 6 for bending and ψ is the stiffness ratio related to bending or extension based on the appropriate case.
- G is the shear modulus of the bonding layer and Γ is the shear lag contributed by the bonding layer of thickness t_b.
- A high value of Γ signifies a thin layer with stiff bonding.
- As $\Gamma \rightarrow \infty$, the above equations become identical to the earlier set of equations [A] and [B] indicating perfect bonding.

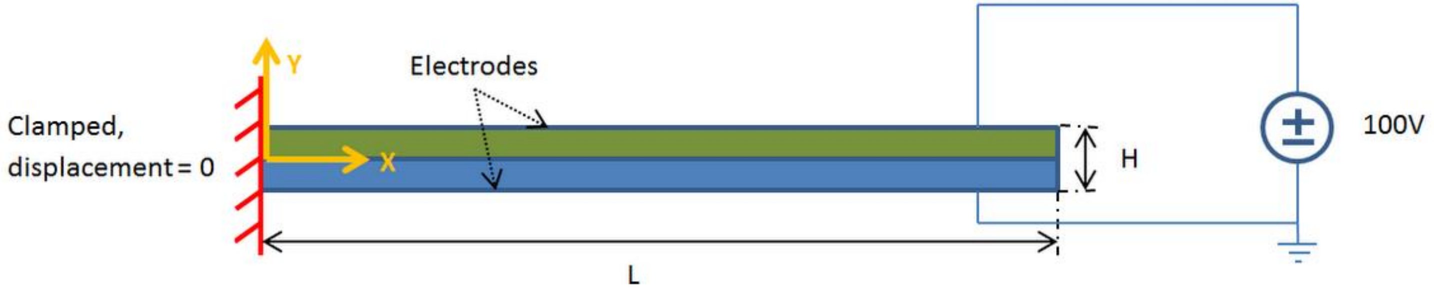
Uniform strain model of induced strain actuation

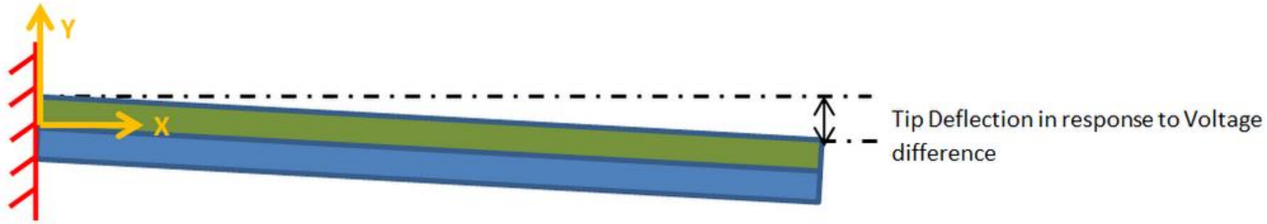
- Using eqns. (A) to (D), one can also obtain the reactive force corresponding to perfect bonding, generated by a pair of smart patches fixed at the top and bottom surface of a host beam of width *b* as

$$F_a = \frac{E_b\, t_b\, b}{\psi + \alpha} \Lambda \qquad \text{[F]}$$

Example 2:

In a bimorph configuration with smart actuators, a host specimen has two actuators placed at the top and bottom of the specimen. One can apply opposite voltages into these actuators and bend the beam either way. A fixed-free bimorph-beam suitable for MEMS (micro-electro-mechanical system) application consists of a host beam of length 2 mm, width 200 μm and thickness 100 μm.





Example 2: cont….

Two 50 μm thick piezoelectric actuators of same length and width as the host beam are fixed on the top and bottom of the beam.

The piezoelectric material has elastic modulus of 65 GPa, and the electro-mechanical coupling coefficient, d_{31} = -50x10^{-12} m/V.

The host beam is made of silicon and of elastic modulus 100 GPa. The top-actuator is actuated with a voltage of 100 V and the bottom one is reversely connected to a -100 V source.

Find out the deflection at the free-end of the bimorph beam.

Example 2:

Solution:

- The second moment of area of the host beam is

$$I_s = \frac{200\times10^{-6}\times(100\times10^{-6})^3}{12} = 16.7\times10^{-18}\ \text{m4}$$

and the flexural rigidity is $(EI)_s = 16.67\text{x}10^{-6}\ \text{Nm}^2$.

- The bending stiffness ratio of the host beam and the piezoelectric actuator is expressed as

$$\psi_b = 12\ (EI)_s\ /\ [t_s^2\ \ (EA)_p\ ] = 1.54.$$

Example 2:

Solution:

- Also, for bending, α = 6. The free-strain Λ is $-d_{31}V/t_a$ = 100 μ*-strain*.

- Using eqn. (F), the force acting on the top of the host beam at its free end is

$$F_a = \frac{100\times10^9\times100\times10^{-6}\times200\times10^{-6}}{(6+1.54)}\times100\times10^{-6} = 0.026\,\mathrm{N}$$

- Moment applied at the free-end of the cantilever beam is

$$F_a\, t_b = 2.652\text{x}10^{-6}\ \text{N-m.}$$

Example 2:

Solution:

- Hence, the deflection at the tip is

$$(ML^2/2EI) = \frac{2.652\times10^{-6}\times(2\times10^{-3})^2}{2\times100\times10^9\times16.67\times10^{-18}} = 3.18\,\mu\text{m}$$

Piezo-actuators and sensors

- PZT based actuators can normally generate a maximum strain of about 0.2%

  (about 2000 μ-*strain*).
- Single crystals of PZN and PMN may generate strains of the range of 8000 μ-*strain*, the use of such crystals as actuators are limited due to their high cost and difficulty of integrating in a structure.

Various displacement and force amplification techniques are developed for Piezoelectric actuators

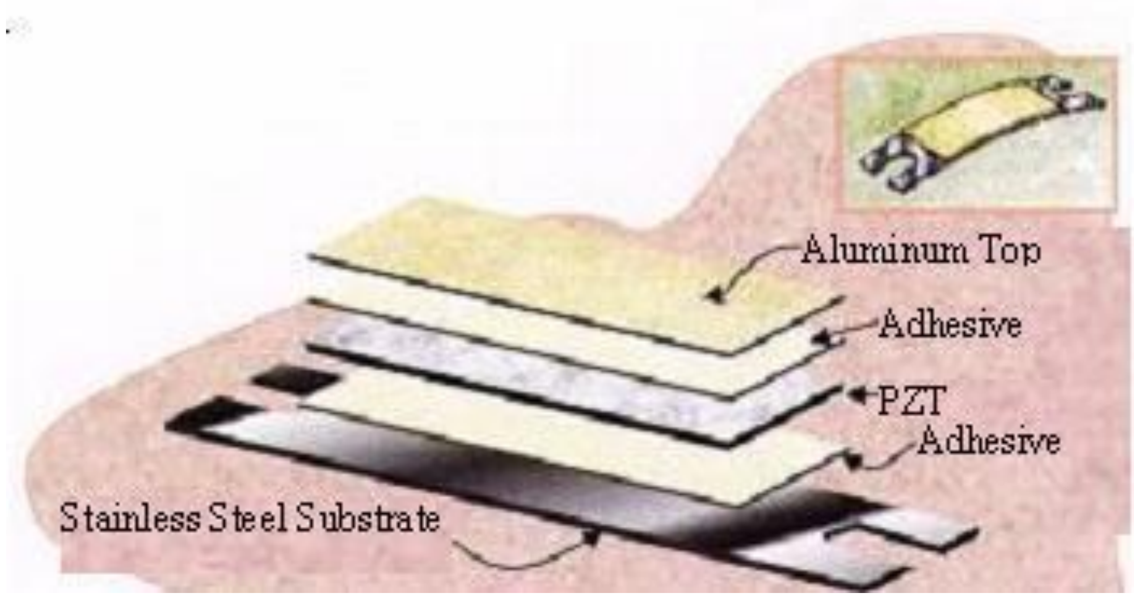
Internally Leveraged System

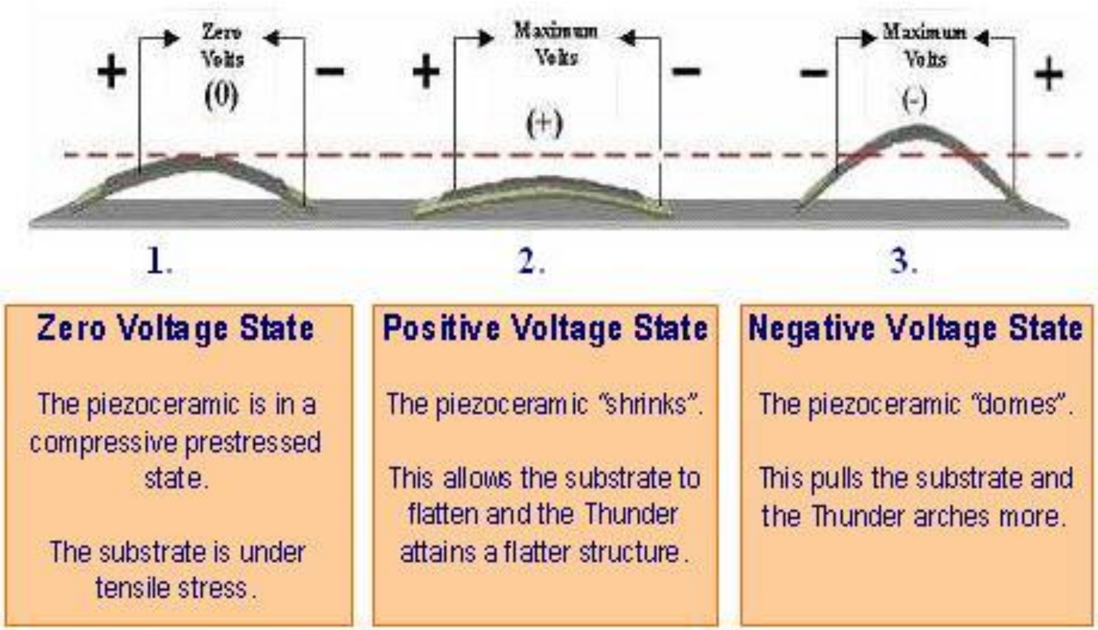
- The actuators contain multiple piezoelectric elements to get an amplified effect.
- Simplest example- **Piezo-stack** where many piezoelectric wafers are stacked in such a way that a comparatively larger deformation is obtained in the d_{33} mode by applying a smaller voltage.
- Other configurations: **Rainbow**, **C-block**, and **Crescent Forms**.

Rainbow Actuator

- **RAINBOW**s or **R**educed **A**nd **I**nternally **B**iased **Ox**ide **W**afers are piezoelectric wafers with an additional heat treatment step to increase their mechanical displacements.
- In the RAINBOW process, PZT wafers are lapped, placed on a graphite block, and heated in a furnace at 975 ^{0}C for 1 hour. The heating process causes one side of the wafer to become chemically reduced.
- This reduced layer, approximately 1/3 of the wafer thickness, causes the wafer to have internal strains that shape the once flat wafer into a dome. The internal strains cause the material to have higher displacements and higher mechanical strength than a typical PZT wafer. RAINBOWs with 3 mm of displacements and 10 kg point loads have been reported.

Thunder – Thin Layer Uniform Ferroelectric Driver

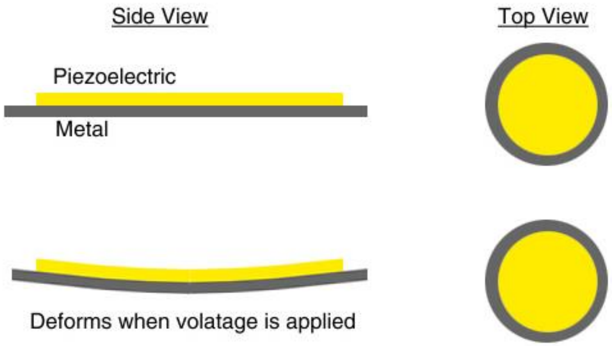




Externally Leveraged System

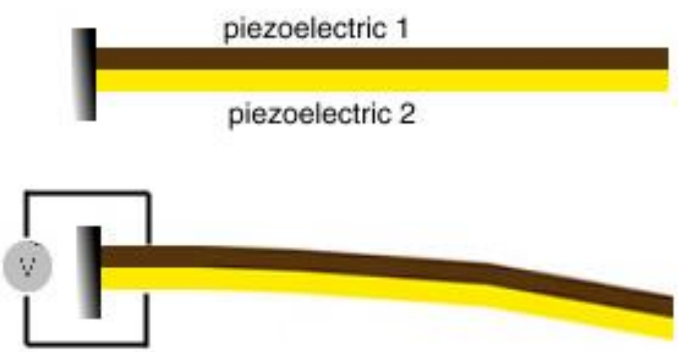
- In these actuators mechanical systems are utilized to amplify the output of a piezoelectric actuator – these include actuators like **unimorph**, **bimorph**, **flexure based actuator**, **moonie, cymbal** etc.

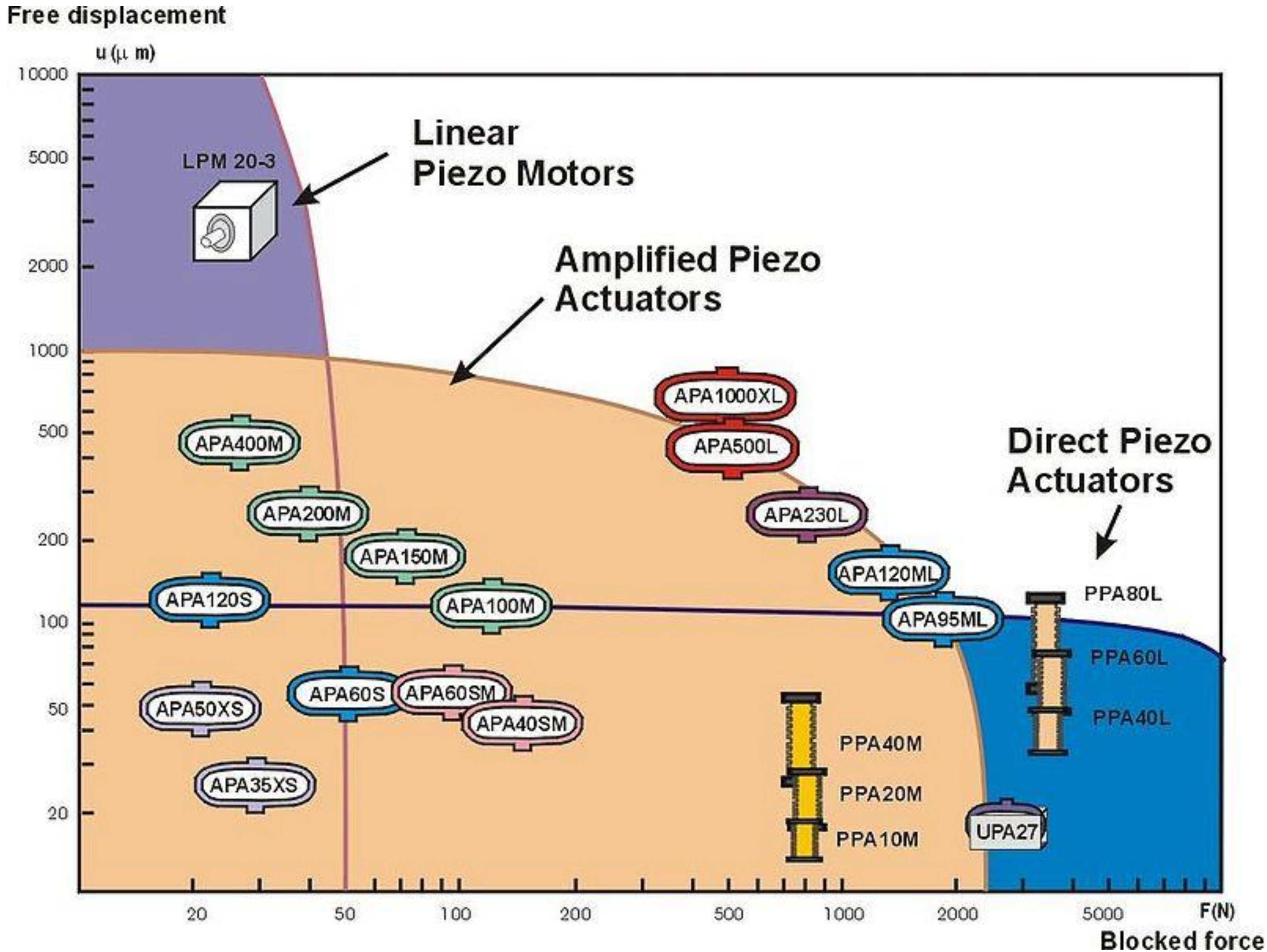
Uni and Bimorph Actuators



Unimorph

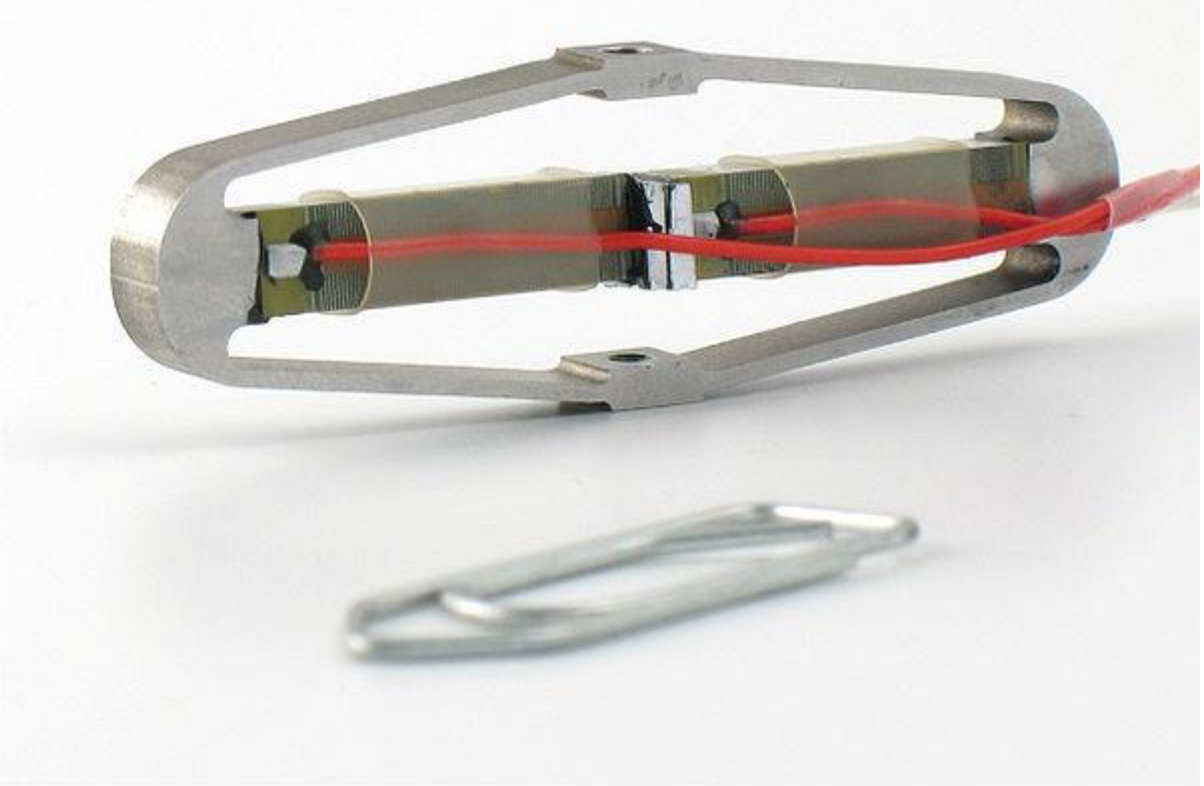
Bimorph




Free displacement
u (μ m)
10000
5000
2000
1000
500
200
100
50
20
LPM 20-3
Linear
Piezo Motors
Amplified Piezo
Actuators
APA1000XL
APA500L
APA400M
APA200M
APA150M
APA230L
APA120ML
APA120S
APA100M
APA95ML
Direct Piezo
Actuators
PPA80L
PPA60L
PPA40L
APA60S
APA60SM
APA40SM
APA50XS
APA35XS
PPA40M
PPA20M
PPA10M
UPA27
20
50
100
200
500
1000
2000
5000
F(N)
Blocked force

Amplified Piezo-actuator

The principle is based on the deformation of an elliptic shell to amplify the ceramic strain. The ceramic stack is aligned with the great axis of the ellipse. A small deformation of the great axis creates a large displacement of the small axis. The amplification ratio can typically reach 20 times, that means such actuators can reach strokes of 1 mm.



Strain at any point of the Beam CS

- Following Euler-Bernoulli assumptions, the strain at any point at a distance **z** from the neutral axis of the structure can be expressed as

$$S_t = S_s - z\kappa \qquad \text{[G]}$$

- where S_s is the mid-plane strain and κ is the curvature of the beam.

Total Stress at any point

- Total stress at any point consists of the combination of elastic stress and induced stress and is given by

$$\sigma_t = \sigma_s + E_a \Lambda \quad \text{[H]}$$

- where the subscripts *s* and *a* stands for the passive and active substrate, respectively

Using the last two equations (G) and (H) for strains and stress and integrating over the cross-section of the smart beam, the equations of force and moment balance corresponding to perfect bending may be obtained as shown in the next slide..

Final Equations of Equilibrium

$$\begin{bmatrix} EA & EB \\ EB & EC \end{bmatrix} \begin{Bmatrix} S_s \\ \kappa \end{Bmatrix} = \begin{Bmatrix} P_s + P_\Lambda \\ M_s + M_\Lambda \end{Bmatrix}$$

where $(EA) = \int_z E(z)\, dz$, $(EB) = \int_z E(z) z\, dz$,

$$(EC) = \int_z E(z) z^2 dz ,$$

$$P_s = \int_z \sigma(z) dz, \qquad M_s = \int_z \sigma(z) z dz ,$$

$$P_\Lambda = \int_z E_a(z)\Lambda(z) dz \quad \text{and } M_\Lambda = \int_z E_a(z)\Lambda(z) z dz .$$

New Stiffness Ratio for axial deformation

parameter Ψ_e now becomes

$$\Psi_e = \frac{\left(E_S A_S - E_S A_a\right)}{E_a A_a}$$

Substitution of passive material by the active piezo-layer has been taken into account in this model

Induced Moment by a Single Piezo Patch on a Host Beam

- Moment and Force Balance Equation

$$\int_{-t_b/2}^{t_{b/2}} \sigma_b(z) z\,dz + \int_{t_b/2}^{t_{b/2}+t_p} \sigma_p(z) z\,dz = 0$$

$$\int_{-t_b/2}^{t_{b/2}} \sigma_b(z)\,dz + \int_{t_b/2}^{t_{b/2}+t_p} \sigma_p(z)\,dz = 0$$

Curvature and Mid-plane Strain

Final Relations with free-strain

$$\chi = \frac{6\,E_b E_p t_b t_p (t_b + t_p)}{E_b^2 t_b^4 + E_p E_b (4t_b^3 t_p + 6t_b^2 t_p^2 + 4t_b t_p^3) + E_p^2\, t_p^4}\,\Lambda$$

$$\varepsilon_0 = \frac{(E_b t_b^3 + E_p t^3{}_p)(E_p t_b / 2)}{E_b^2 t_b^4 + E_p E_b (4t_b^3 t_p + 6t_b^2 t_p^2 + 4t_b t_p^3) + E_p^2 t_p^4}\,\Lambda$$

Where, $\Lambda = \ d_{31}(\frac{\mathrm{V}}{\mathrm{t}})$

$$M_x = E_b I_b\, \chi$$

$$\boxed{Hence, M_x = \ \gamma_c\, V}$$

Work Done by the Active Patch

- The constant γ_c depends on the geometry, configuration and piezoelectric device constants. $V(t)$ is the time-dependent voltage
- The work done by the piezoelectric patches in moving or extracting the electrical charge is

$$W = \int_0^{L_c} M(x,t)\kappa(x)dx$$

where L_c is the active length of the piezoelectric material, which is assumed to be attached at the clamped end of the beam.

- The quantity $\kappa(x)$ is the curvature of the beam and this is approximately expressed by the second-derivative of the displacement
- Using the approximation for κ we have

$$W = \theta V$$

where the coupling coefficient

$$\theta = \gamma_c \int_0^{L_c} \frac{\partial^2 \varphi(x)}{\partial x^2} dx = \gamma_c \varphi'(L_c)$$

- Using the first mode shape, for $L_c = L$, this can be evaluated

$$\varphi'(1) = 2L \frac{\lambda\ (\cos(\lambda)\sinh(\lambda) + \cosh(\lambda)\sin(\lambda))}{\cosh(\lambda) + \cos(\lambda)}$$

Governing Equations of Motion

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) - \theta v(t) = f_b(t)$$

$$C_p\dot{v}(t) + \frac{1}{R_l}v(t) + \theta\dot{x}(t) = 0$$

References

- Gauenzi, P., Smart Structures, Wiley, 2009
- Cady, W. G., Piezoelectricity, Dover Publication, 1950
- Crawley, E. F., Intelligent Structures for Aerospace: a technology overview and assessment, AIAA, 33 (8), 1994, pp. 1689-1699

END OF LECTURE