State Space Control

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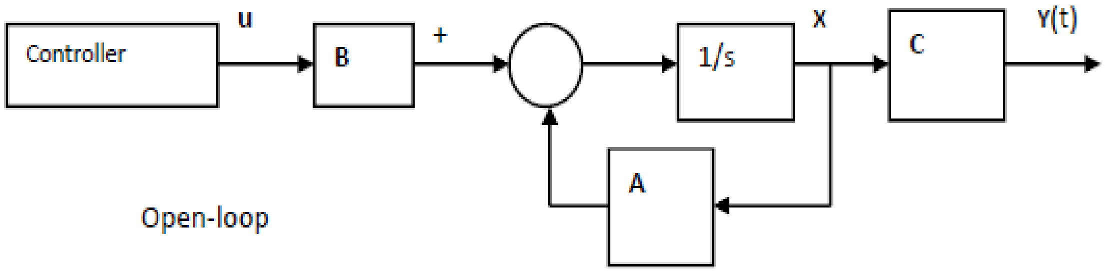
Full State Feedback Control

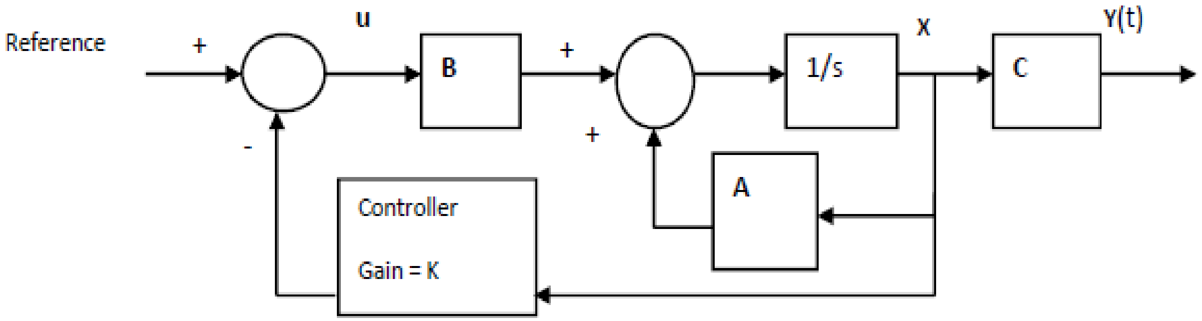
Introduction to Full-state feedback control

- Using the transfer function based technique; compensators are designed to predominantly control the response of second-order systems in frequency domain. By adjusting, the control gain, poles and zeroes of the compensator, the adverse effect of the system is compensated.
- The effect of higher-order poles are either neglected or compensated separately using notch filters.
- In case of full-state feed-back control, on the other hand, controllers could be designed to regulate the behavior of all the poles of the system.
- Although, such design is based on idealistic assumption of sensing all the states of the system, in reality, only some of the states are measured while the rest are estimated using numerical simulation.

Graphical Representation



Open-loop



Closed-loop

A System in Control Canonical Form

Let us consider the following system in control canonical form:

$$\left|s\mathbf{I}-\mathbf{A}\right| = s^{n} + a_{n-1}s^{n-1} + \ldots\ldots + a_1 s + a_0 = 0;$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$

$$\mathbf{C} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}$$

Controller Design

Let us define the control-law as

$$u = -K\,X$$

where, the control-gains ***K*** are represented in a matrix-form. For a single input system, ***u*** becomes scalar and consequently ***K*** will have a vector-form as

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$

The new state space equation could be written as

$$\dot{\mathbf{X}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{X}$$

$$\mathbf{Y} = \mathbf{C}\mathbf{X}$$

Controller Design – contd..

The characteristic equation corresponding to the closed-loop plant may be expanded as:

$$|sI-(A-BK)| = s^n + (a_{n-1}+k_n)s^{n-1} + (a_{n-2}+k_{n-1})s^{n-2} + \cdots + (a_0+k_1) = 0$$

When the desired roots of the closed-loop system

$$\Lambda_c = \begin{bmatrix} \lambda_{c_1} & \cdots & \lambda_{c_i} & \cdots & \lambda_{c_n} \end{bmatrix}$$

are known, the desired characteristic equation may be obtained as:

$$\prod_{i=1}^{n}(s-\lambda_{c_i}) = 0$$

$$\text{or, } s^n + d_{n-1}s^{n-1} + \cdots + d_0 = 0$$

By comparing the coefficients of the polynomials of desired and initial characteristic polynomial one can get the elements of control gain vector **K** as

$$k_i = d_{i-1} - a_{i-1}, \quad \text{for } i = 1 \cdots n$$

Example: Controller Design for a Canonical System

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -2 & -3 & -4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Characteristic Equation: $s^4 + 4s^3 + 3s^2 + 2s + 1 = 0$

The Controller Gain Structure

Let us consider the desired roots of the new system to be [-1,-2,-5,-10]. Then, the desired characteristic polynomial may be written as:

$$s^4 + 18s^3 + 97s^2 + 180s + 100 = 0$$

The initial characteristic equation was:

$$s^4 + 4s^3 + 3s^2 + 2s + 1 = 0$$

Hence, the controller gain may be obtained as:

$$K = [99, 178, 94, 14]$$

Assignment

Consider a third order system with the following governing equation:

$$\frac{d^3y}{dt^3}+7\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=u$$

Obtain, the state space representation of the system. Design a controller such that damping factor will be 0.6 and the settling time less than 1 second.

Full state feedback control for system in non-canonical form

- If the system is not in control canonical form, you have to find out the proper transformation matrix T to convert the system into canonical form.
- If x is the state vector corresponding to non-canonical form along with the corresponding state-space parameters A, B and C and z is the state vector in canonical form along with system parameters given by A_c, B_c and C_c, then, considering T to be the transformation matrix between the two linear systems such that :

$$\mathbf{x} = \mathbf{T}\,\mathbf{z}, \quad \text{then the state space equation in non} - \text{canonical form}$$

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,\mathbf{u}, \text{gets transformed to Canonical form as}$$

$$\dot{\mathbf{z}} = \mathbf{T}^{-1}\,\mathbf{A}\,\mathbf{T}\,\mathbf{z} + \mathbf{T}^{-1}\,\mathbf{B}\,\mathbf{u} = \mathbf{A}_c\,\mathbf{z} + \mathbf{B}_c\,\mathbf{u}$$

Full state feedback control in non-canonical form contd..

- The first task here is to find out the controllability matrix corresponding to the canonical form.
- How do we find it without knowing the transformation matrix?
- Well, we can find out the roots of the characteristic equation by evaluating the determinant of $[sI-A]^{-1}$
- Once we know the roots, we can write the new plant matrix in canonical form (see the standard form discussed before)
- In order to obtain the controllability matrix you also need to know the B matrix, for a single input system it is simply

$$B=\begin{bmatrix}0 & 0 & . & . & 1\end{bmatrix}^T$$

A System not in Control Canonical Form

After evaluating the controllability matrix related to the canonical form, you can find the controllability matrix corresponding to the non-canonical form as

$$\hat{\mathbf{C}} = \begin{bmatrix} \mathbf{B} & \mathbf{A}\,\mathbf{B} & \cdots & \mathbf{A}^{n-1}\mathbf{B} \end{bmatrix} = \mathbf{T}\hat{\mathbf{C}}_c$$

This controllability matrix can be used along with the controllability matrix corresponding to canonical form to obtain the transformation matrix between the two systems as:

$$\mathbf{T} = \hat{\mathbf{C}}\,\hat{\mathbf{C}}_c^{-1}$$

Now, you can represent the system to canonical form and obtain the corresponding gain as $\mathbf{K_c}$.
Then, the gain for non-canonical form **K** could be written as

$$\mathbf{K} = \mathbf{K}_C\,\mathbf{T}^{-1}$$

Controller Design using Ackermann's algorithm

For a single input system, one can use a direct relationship to find the controller gain **K** by using Ackermann's formulation as follows:

$$\mathbf{K} = \mathbf{R}\,\hat{\mathbf{C}}^{-1}\,\mathbf{\Psi}(\mathbf{A})$$

$$\text{with } \mathbf{R} = \begin{bmatrix} 0 & \cdots & \cdots & 1 \end{bmatrix},$$

$$\hat{\mathbf{C}} = [\mathbf{B}\ \ \mathbf{AB}\ ....\ \mathbf{A}^{\mathbf{n}-1}\mathbf{B}]$$

$$\text{and } \mathbf{\Psi}(\mathbf{A}) = \mathbf{A}^{n} + d_{n-1}\mathbf{A}^{n-1} + d_{n-2}\mathbf{A}^{n-2} + ... + d_0\mathbf{I}$$

where, d_i are the coefficients of the desired characteristic polynomial.

This is based on the fact that a matrix satisfies it's own characteristic equation, which is also known as Cayley-Hamilton's theorem

Where to place the Closed-loop poles?

- The placement of the pole often becomes one of the important prerogatives of the controller design. Given a freedom, you should design a system such that it is predominantly second order in nature. This implies that the higher order poles should be placed at least five times away from the real part of the second order poles.
- However, from the energy point of view, you should not place the closed loop poles quite far away from the open loop poles as the gain requirement would increase proportionately.
- The choice of B matrix also places an important role as the lesser controllable systems require higher gains.

Butterworth pole configurations

Following an optimization procedure, it is shown that the closed loop poles could be placed such that the characteristic equation is

$$\left(\frac{s}{\omega}\right)^{2k} = (-1)^{k+1}$$

Where, k is the number of poles required.

It can be shown that for k=1, you need to place a single pole on the –ve real axis at a distance ω from the origin. For, k=2, the radial distance remains unchanged, however, the poles will be complex and at angle 45^0 from the imaginary axis. These Configurations are known as Butterworth pole configuration.

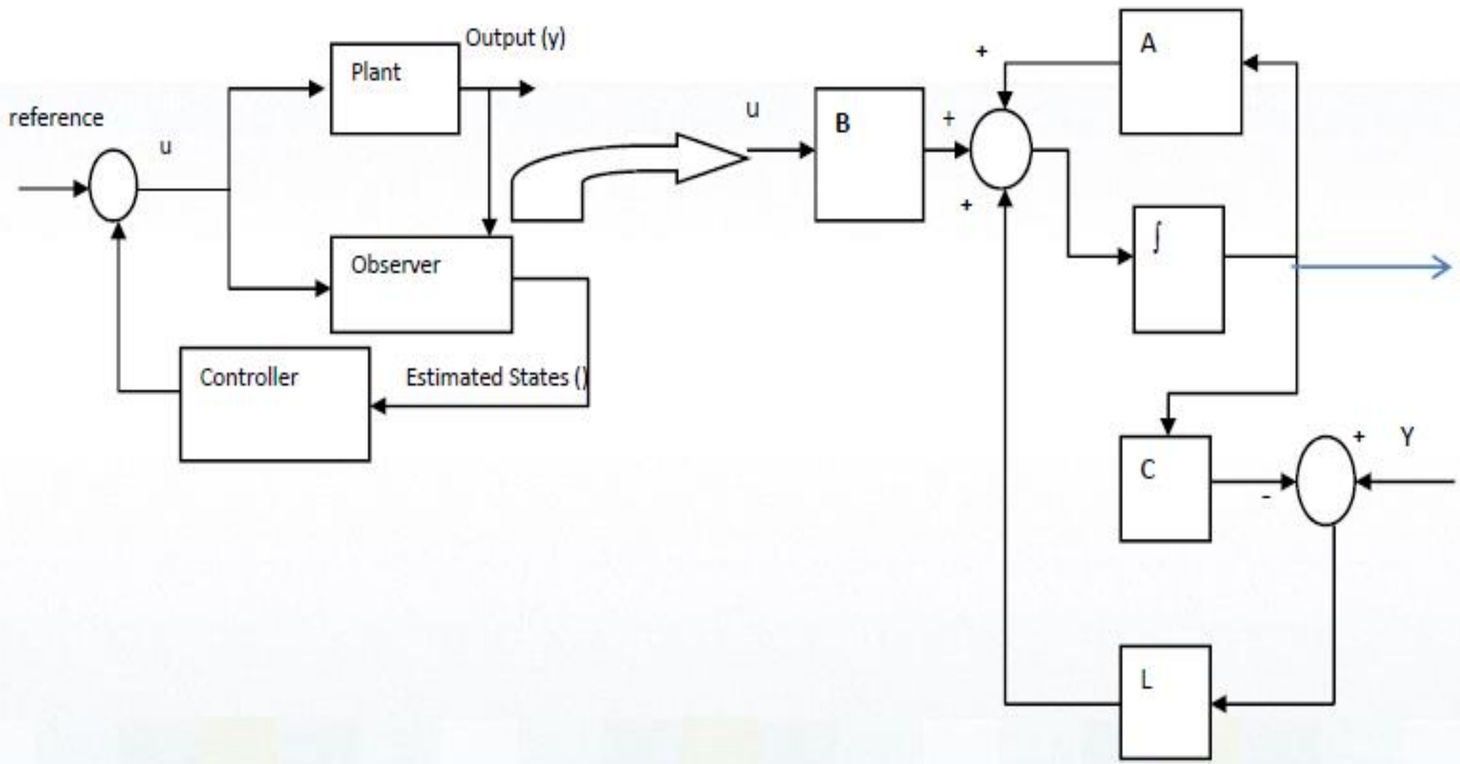
Assignment:

A SDOF system has the following mass, stiffness and damping constant in appropriate units, m=0.1, c=0.01 and k=0.8; design a full-state feedback control, with an actuator influence matrix B^T = [0 1] and a forcing function 0.1u(t) (u(t) – unit step function), such that the desired eigen-values are at -1± 2j, respectively.

Introduction to Observer Design

- The design of full-state feedback control assumes the accessibility (possibility of sensing) of the complete state vector.
- However, in reality one may have only a subset of them available for direct sensing while the other states are to be estimated via simulation.
- Accepting that there will be finite error in this process, the focus is whether the error could be driven to zero at a faster rate than the plant-dynamics.
- Obviously, such a strategy is feasible only if the states are observable.

Observer in a block-diagram



Design of an Observer

The governing equation for a dynamic system (Plant) in state-space representation may be written as:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{Bu}, \quad \mathbf{Y} = \mathbf{CX}$$

The governing equation for the Observer based on the block diagram is shown below. The superscript '^' refers to estimation.

$$\dot{\hat{\mathbf{X}}} = \mathbf{A}\hat{\mathbf{X}} + \mathbf{Bu} + \mathbf{L}(\mathbf{Y} - \hat{\mathbf{Y}})$$

$$\hat{\mathbf{Y}} = \mathbf{C}\hat{\mathbf{X}}$$

Define the error in estimation of state vector as

$$\mathbf{e}_{\mathbf{X}} = (\mathbf{X} - \hat{\mathbf{X}})$$

Observer Design based on Error Dynamics

The error dynamics could be derived now from the observer governing equation and state space equations for the system as:

$$\dot{\mathbf{e}}_X = (\mathbf{A} - \mathbf{LC})\mathbf{e}_X$$

$$\mathbf{Y} - \hat{\mathbf{Y}} = \mathbf{C}\,\mathbf{e}_X .$$

The corresponding characteristic equation may be written as:

$$\left| s\mathbf{I} - (\mathbf{A} - \mathbf{LC}) \right| = 0$$

You need to design the observer gains such that the desired error dynamics is obtained.

Case A: Observer design for canonical system

Suppose, the system [A, C] is available in observer canonical form:

$$\hat{\mathbf{A}} = \begin{bmatrix} -a_{n-1} & 1 & 0 & \cdots & 0 \\ \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_1 & \cdots & \cdots & \cdots & 1 \\ -a_0 & 0 & \cdots & \cdots & 0 \end{bmatrix}, \hat{\mathbf{L}} = \begin{bmatrix} l_1 \\ l_2 \\ \cdots \\ l_n \end{bmatrix}, \hat{\mathbf{C}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

Observer Design for Case A

- The first thing you need to check is whether the system is fully observable or not. This can be done by checking whether the rank of the observability matrix equals the order of the system as stated earlier. Once the answer is affirmative you may proceed for the observer design.
- Whenever, the desired eigen-values related to the error-dynamics are specified, one can construct the desired characteristic equation identical to controller design.
- The observer gain matrix for such cases may be obtained from the simple relationship

$$l_i = d_{n-i} - a_{n-i} \qquad i = 1 \cdots n$$

- Here 'd' and 'a' refer to the vector coefficients of the desired and the open-loop characteristic polynomial.

Observer design for system in non-canonical form

- If a system is not in observer canonical form, then one needs to transform the system matrices first into the particular canonical form.
- The transformation matrix required for such cases has been derived as

$$T = O^{-1}\hat{O}$$

o

- Here, 'O' and '$\hat{O}$' are the observability matrices related to the non-canonical and canonical forms respectively.
- After obtaining the observer gains in observer canonical form, one can transform the gain vector to the original non-canonical form as:

$$L = T\hat{L}$$

Assignment:

- The system matrices for a plant (A, B and C) are as follows:

$$\mathbf{A}=\begin{bmatrix}-4 & 1 & 0\\ 0 & -2 & 1\\ 0 & 0 & -1\end{bmatrix}, \mathbf{B}=\begin{bmatrix}0\\0\\1\end{bmatrix} \text{ and } \mathbf{C}=\begin{bmatrix}1 & 0 & 0\end{bmatrix}$$

- Design an observer for the plant, where the desired characteristic polynomial is given by:

$$s^3+12s^2+25s+50$$

Introduction to Reduced Order Observer Design

- In the last lecture, we have introduced the concept of controlling a dynamic system where none of the states are available for direct measurement and hence the states are to be estimated through the design of an observer.

- Another trivial way of solving this problem could be by the use of system output, provided the number of outputs available equal to the order of the system. Following the output eqn.:

$$Y = C\,X$$

$$X = C^{-1}\,Y$$

- However, there are many cases, where, number of outputs available are less (say 'r' numbers which is less than 'n' order of the system). In such a case you need a reduced order observer to estimate the 'n-r' number of states. This is also known as Luenberger Observer.

Governing Equation

Let us consider, the governing equation for the dynamic system (Plant) in state-space form as:

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{Bu}, \quad \mathbf{Y} = \mathbf{C}_1\mathbf{X}_1$$

Consider, Y to be of size ‘r’, and partition the state vector into two parts such that $\mathbf{X} = [\mathbf{X_1} \mid \mathbf{X_2}]^T$, where $\mathbf{X_1}$ is of size ‘r’ and $\mathbf{X_2}$ is of size ‘n-r’. The governing equation could be subdivided similarly such that:

$$\dot{X}_1 = A_{11} X_1 + A_{12} X_2 + B_1 U$$

$$\dot{X}_2 = A_{21} X_1 + A_{22} X_2 + B_2 U$$

Estimator Equation

The states X_1 could be estimated based on directly the measured output Y such that:

$$X_1 = \mathbf{C}_1^{-1} Y$$

The states $\mathbf{X_2}$, however, has to be estimated following a similar strategy as had been done for the full order observer.

$$\dot{\hat{\mathbf{X}}}_2 = \mathbf{A}_{21}\ C_1^{-1} Y + A_{22}\hat{X}_2 + \mathbf{B}_2\,\mathbf{u}$$

Consider, the above equation in terms of a new state vector Z, such that

$$Z = \hat{X}_2 - LY,\ \ \hat{X}_2 = Z + LY$$

Observer Design based on Reduced States

The reduced order system may be expressed in terms of states Z as:

$$\dot{z} = Q\,z + R\,y + S\,u$$

Now, defining the error for the reduced order system, we can obtain the error dynamics as:

$$e_2 = x_2 - \hat{x}_2$$

$$\dot{e}_2 = A_{21}\,x_1 + A_{22}\,x_2 + B_2 u - LC_1 M - Q\,z - R\,y - S\,u$$

$$M = A_{11} x_1 + A_{12}\,x_2 + B_1 u$$

Observer design contd.

The error dynamics could be further simplified as:

$$\dot{e}_2 = Q e_2 + (A_{21} - L C_1 A_{11} - R C_1 + QLC_1)x_1 + (A_{22} - L C_1 A_{12} - Q)x_2 + (B_2 - L C_1 B_1 - S)u$$

To obtain an error dynamics which will be independent of x_2, x_1, and u, the following conditions must be satisfied:

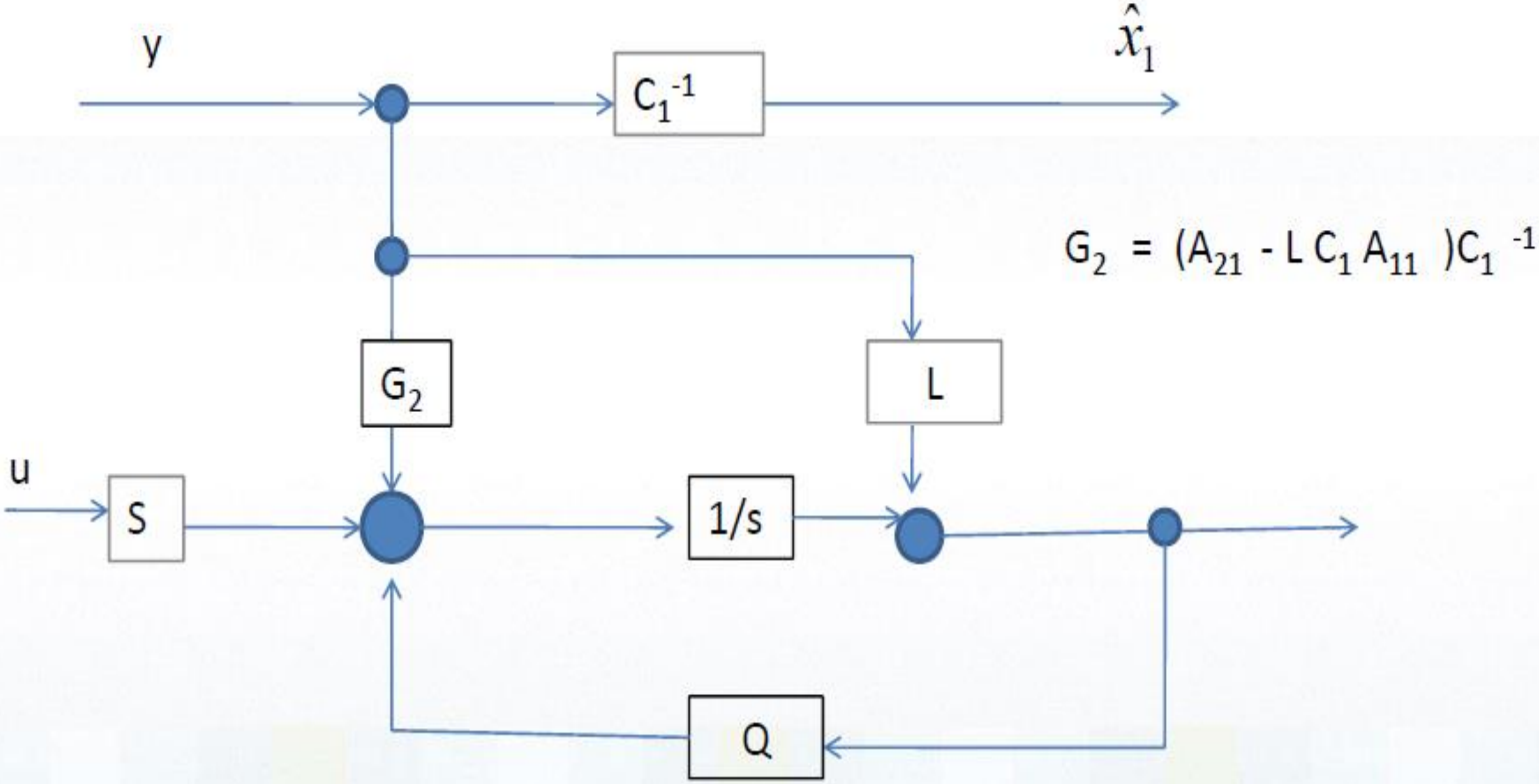
$$Q = A_{22} - LC_1 A_{12}$$

$$R = A_{21} C_1^{-1} - LC_1 A_{11} C_1^{-1} + QL$$

$$S = B_2 - LC_1 B_1$$

By selecting the observer gain *L*, one can obtain *Q*, *R* and *S*.

Reduced Order Observer in a block-diagram



Assignment

- Consider a networked first order hydraulic system with the following state equation

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 1 & -5 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix} u$$

- Design a reduced order observer when only the first state x_1 is directly measured.

Special References for this lecture

- Control System Design, Bernard Friedland, Dover
- *Control Systems Engineering* – Norman S Nise, John Wiley & Sons
- *Design of Feedback Control Systems* – Stefani, Shahian, Savant, Hostetter

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- Feedback Control of Dynamic Systems, Frankline, Powell and Emami, Pearson

- Control System Design, Bernard Friedland, Dover Publications Inc