Root Locus Method

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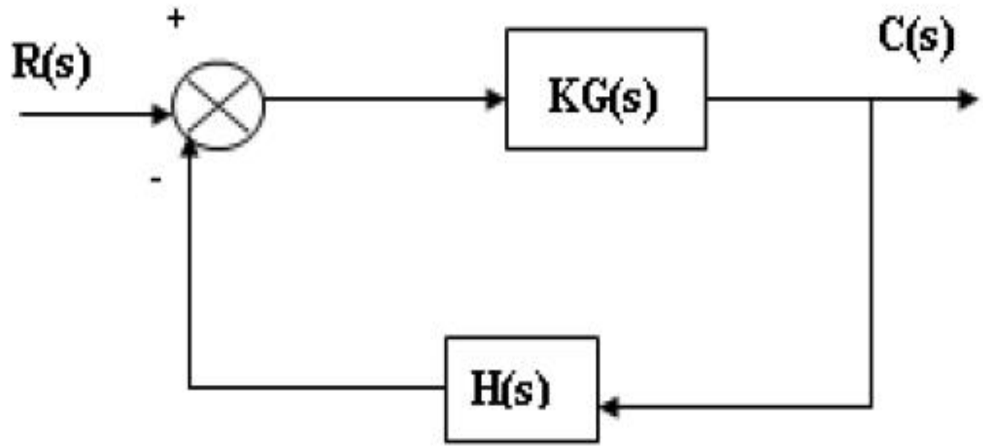
This Lecture Contains

- Open and Closed Loop Transfer Function
- Geometric Interpretation of a Transfer Function
- Evaluation of a Transfer Function
- Rules of Root Locus

Recall Open and Closed Loop Transfer Function



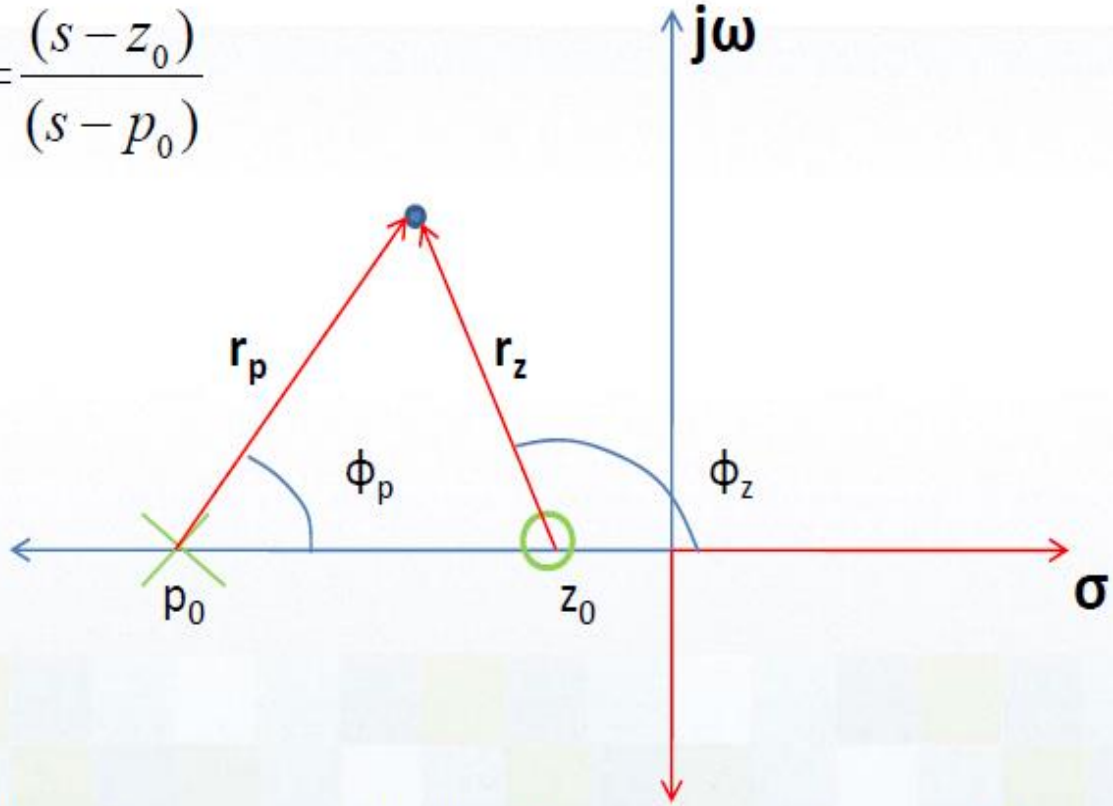
- ❑ Open Loop Transfer Function: $KG(s)H(s)$
- ❑ Close Loop Transfer Function: $\dfrac{KG(s)}{1+KG(s)H(s)}$

✓ G(s) given, H(s) Given, Select the Gain K and vary until you receive the desired response.

Geometric Interpretation of Evaluating a Transfer Function

Consider a Transfer Function to be:

$$T(s) = \frac{(s - z_0)}{(s - p_0)}$$



The pole and zero are plotted in the s-plane as shown. The magnitude of the transfer function is the ratio of the zero and the pole length (**r_z / r_p**) and the phase of the transfer function is $\phi_Z - \phi_p$.

Evaluation of a Transfer Function

Consider a Transfer Function to be

$$F(s)=\frac{(s+1)}{s(s+2)}$$

- **Evaluate the function at a test point s = -3+j4, i.e. find out the magnitude and phase of the transfer function at the test point.**
- **Following the geometric technique or using simple complex algebra you may find out that :**
- **Zero length - $\sqrt{20}$, $\varphi_z = 116.6^0$**
- **Pole at Origin: Pole length 5, $\varphi_{p1} = 126.9^0$**
- **Pole at -2: Pole length $\sqrt{17}$, $\varphi_{p2} = 104.0^0$**
- **Magnitude of the transfer function:** $$F(s)=\frac{\sqrt{20}}{5\sqrt{17}}=0.217$$
- **Total Phase = 116.6-(126.9+104) = -124.6^0**

Objective of Root Locus

- Without Factorizing 1+KG(s)H(s) Every Time Can We Find Out The Location Of Closed Loop Poles and Comment On Stability?
- Answer: Yes, Get The Root Locus

Characteristic Equation of the closed loop system

$$1+KG(s)H(s)=0, \quad Or \quad \begin{array}{l} 1+KF(s)=0 \\ F(s)=G(s)H(s) \end{array}$$

$$Or \quad F(s)=-\frac{1}{K}$$

$$Also \quad F(s)=\left|F(s)\right|e^{j\varphi_{F(s)}}, \left|F(s)\right|=\frac{1}{K}$$

The phase condition for F(s) is stated in the equation below which is further used for the root locus plot.

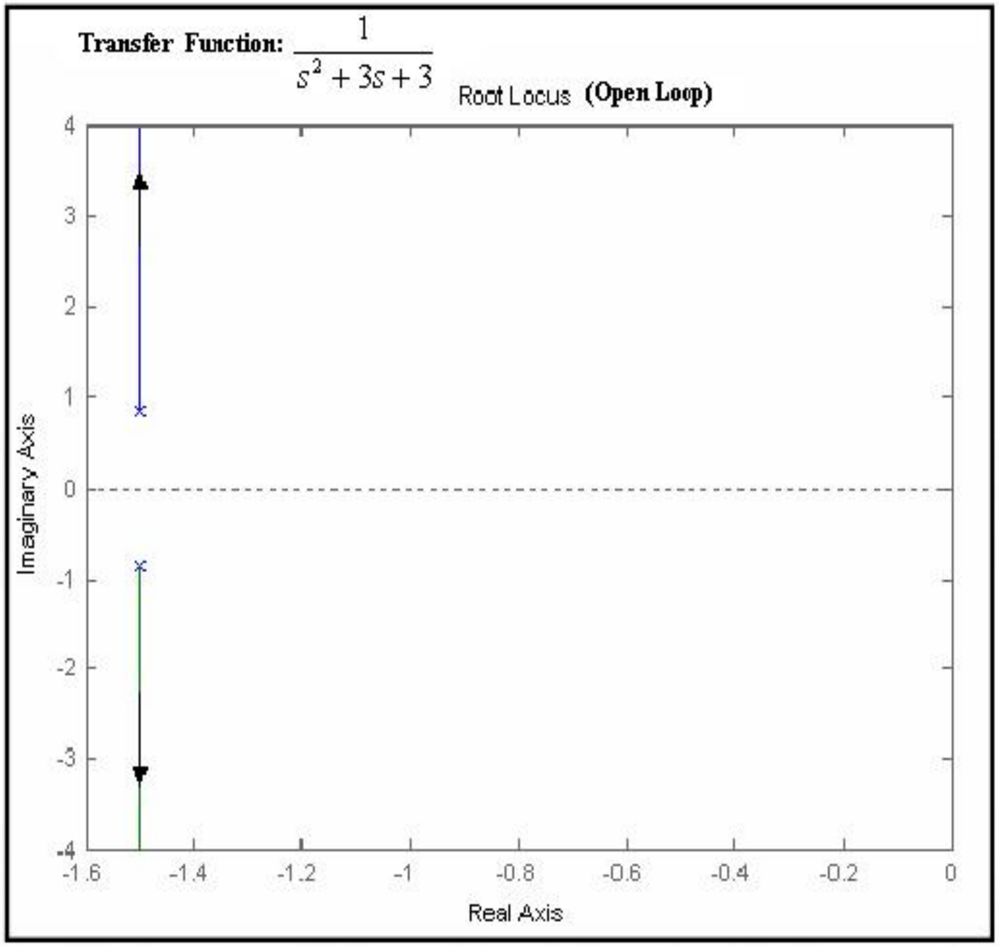
$$\boxed{\angle F(s) = (2k+1)180^0}$$

$$\text{If} \quad F(s) = \frac{\prod_{i=1}^{n}(s - zi)}{\prod_{i=1}^{n}(s - p_i)}$$

For any Values of F(s)

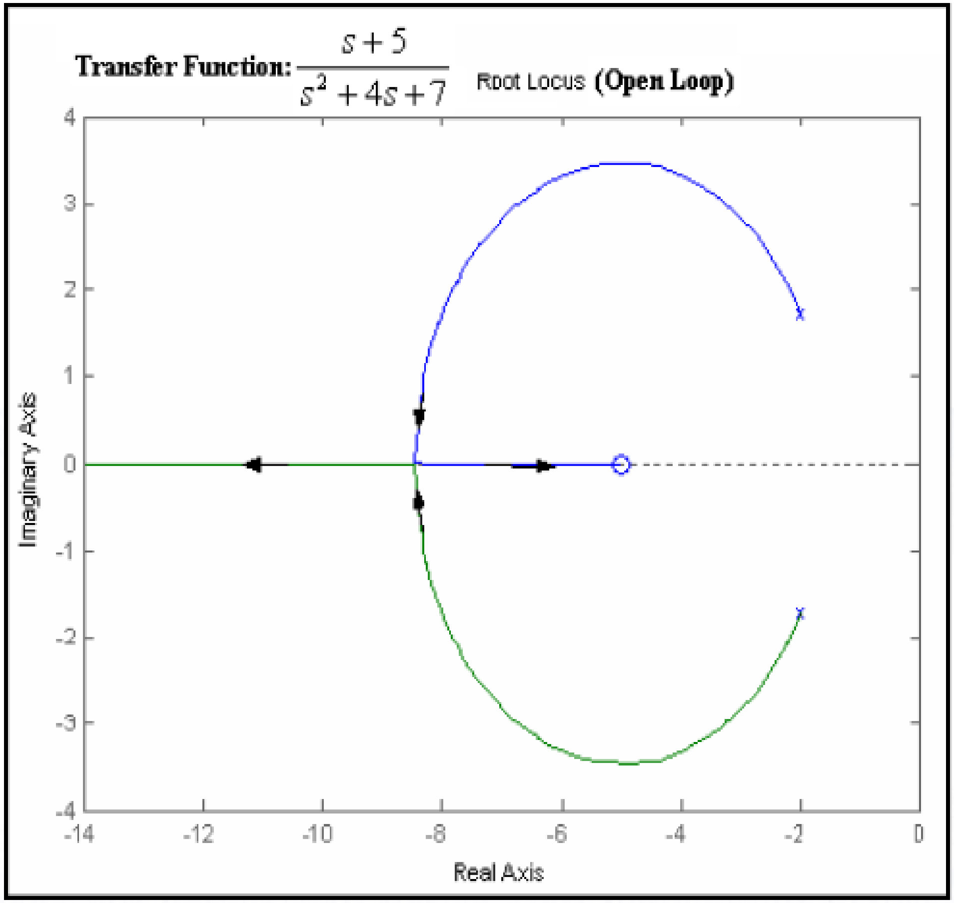
$$|F(s)| = \frac{\prod Zero \quad Lengths}{\prod Pole \quad Lenghts}$$

$$\varphi_{F(s)} = \sum Zero \quad angles - \sum Pole \quad angles$$



- ✓ Number of root loci – as many as open loop poles
- ✓Origin at poles

Root Locus of the Open Loop Transfer Function

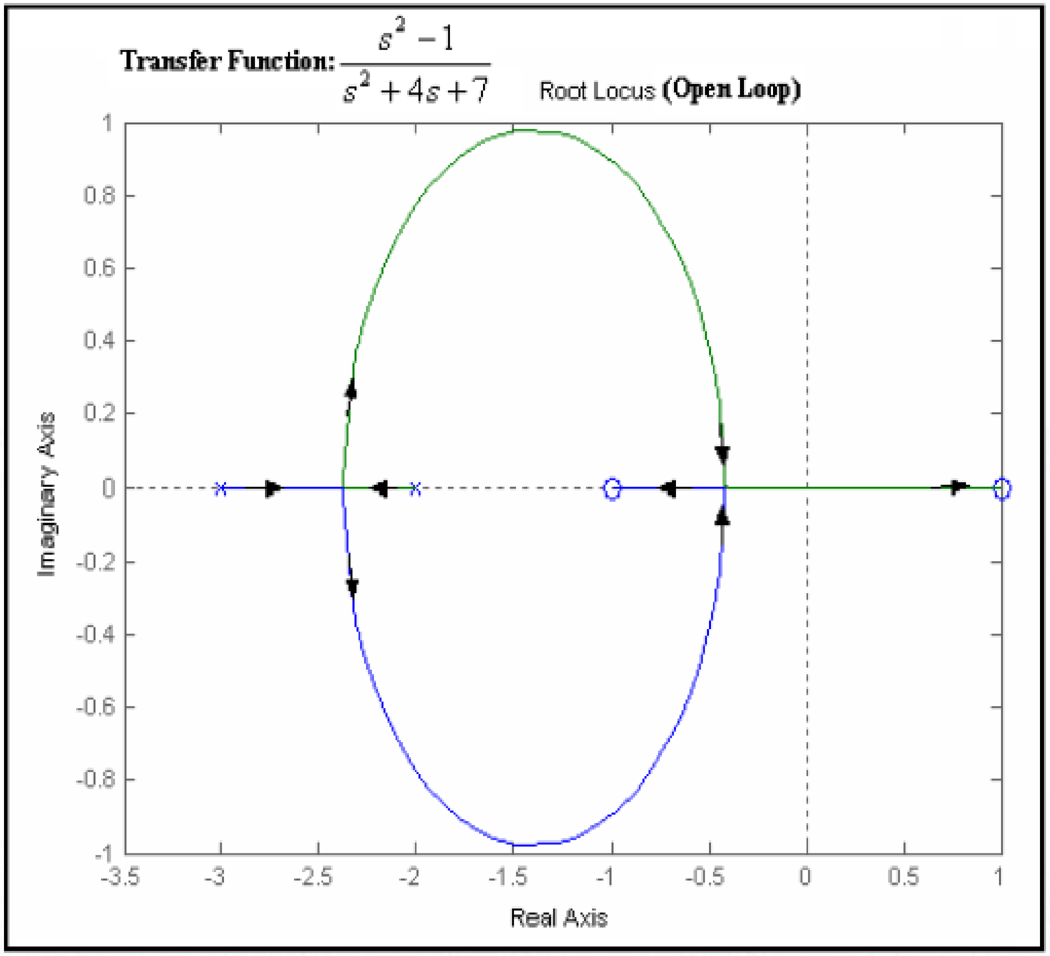


✓Symmetry always about the real axis.

✓Termination as $K \rightarrow \alpha$ m open loop poles to finite zeros

Of the open loop system, n-m number of poles approach zeroes at infinity.

Root Locus of the Open Loop Transfer Function



✓Real axis Segment: - Root locus exists on the left of an odd number of real axis finite open loop poles and zeroes

Root Locus of the Open Loop Transfer Function

Angle of the Asymptotes

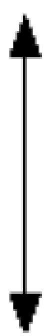
- The root loci approaches the zeros at infinity along asymptotes. The real axis intercept and angle of which are given by the following rules:

$$R_{\text{int}} = \frac{\sum Poles - \sum Zeros}{m}, \quad m = n_p - n_z$$

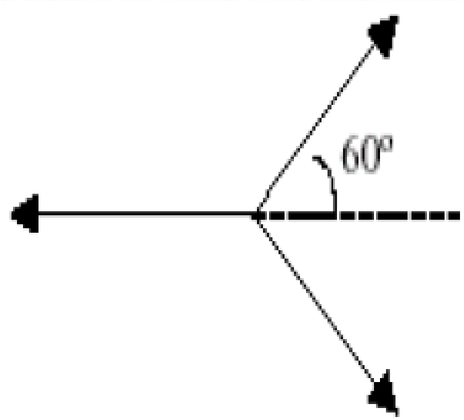
$$\Phi_{\text{int}} = \frac{(2i+1)}{m} 180^0, i = 0,1,2..(m-1)$$

m=1

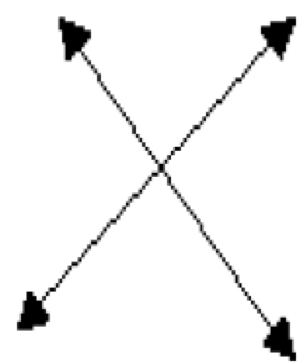
m=2



m=3



m=4



Jω intercept: Use $1+KG(j\omega)H(j\omega) = 0$

- Find ω and K and for any Complex Pole Find Angle

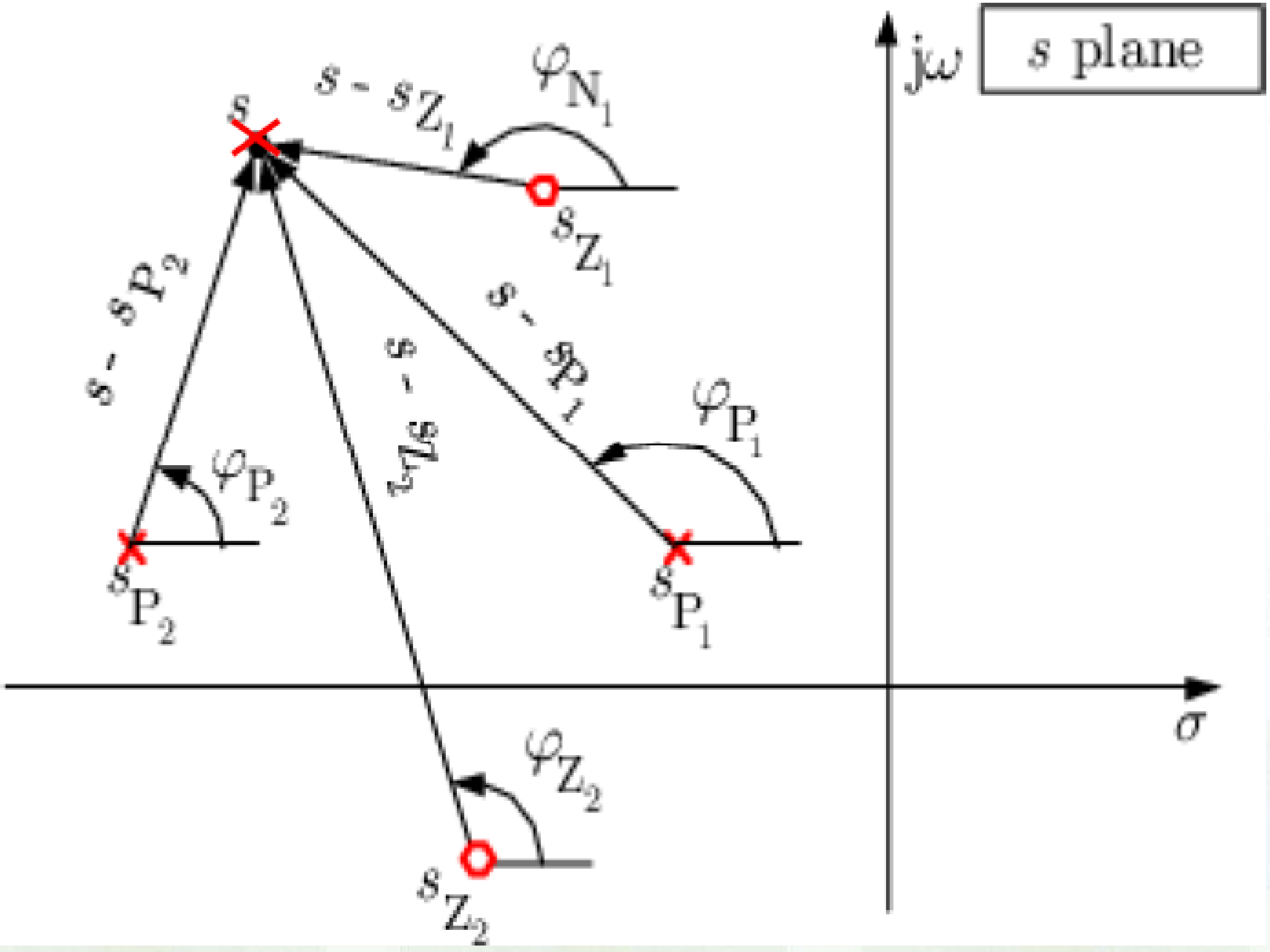
- Another way: Use Routh's Array, find stability condition for a complete zero row,

Obtain, K, go back to the upper row, obtain ω

- Take points along the imaginary axis and check the phase condition
- Try the problem $T=K(s+3)/(s(s+1)(s+2)(s+4))$ (OLTF with unity feedback)

$-K^2 - 65K + 720 = 0 \Rightarrow K = 9.65$

Angle Checking at a Test Point



Departure and Arrival Angle

- **Angle of Departure from a complex Pole with ‘r’ multiplicity:**

$$\boxed{r\varphi_{l,dep} = \sum \psi_i - \sum_{i \neq l} \varphi_i - (180^\circ + 360^\circ (l-1)), l = 1..r}$$

- **Angle of Arrival at a complex Pole with ‘r’ multiplicity:**

$$\boxed{r\psi_{l,arr} = \sum \varphi_i - \sum_{i \neq l} \psi_i + (180^\circ + 360^\circ (l-1)), l = 1..r}$$

❑ Example:

$$G(s)H(s) = \frac{K}{s(s+2)(s+3)} = KF(s)$$

$$F(s) = \frac{1}{s(s+2)(s+3)}$$

(1) 3 Loci Starting From $P_1 = 0, P_2 = -2, \quad P_3 = -3$

(2) All Roots Loci Terminate at ∞

Open Loop Zeros $= \infty$

$$\angle F(s) = -\angle s - \angle s+2 - \angle s+3 = (2i+1)180^{\circ}, i = 0,1,2..$$

$$|F(s)| = \frac{1}{K}, Or, \quad \left|\frac{K}{s(s+2)(s+3)}\right| = 1$$

(a) Root Loci on The Real Axis

(i) Positive Real Axis → $\angle s = \angle s+2 = \angle s+3 = 0^{\circ}$

(ii) $s: 0 \leftrightarrow -2, \quad \angle = 180^{\circ}, \quad \angle s = 180^{\circ}, \quad \angle s+2 = \angle s+3 = 0$

(iii) $s: -2 \leftrightarrow -3, \quad \angle = 360^{\circ}, \quad \angle s = 180^{\circ}, \quad \angle s+2 = 180^{\circ}, \quad \angle s+3 = 0$

(iv) $s: \triangleright -3, \quad \angle = 540^{\circ}, \quad \angle s = 180^{\circ}, \quad \angle s+2 = 180^{\circ}, \quad \angle s+3 = 180^{0}$

Six basic rules of Root-Locus Construction

1. ‘n’ branches of root locus starts at the open loop poles and ‘m’ of them meet the zeroes of the same
2. Loci are on the real axis to the left of odd number of poles and zeroes
3. For large s and K, n-m of the root loci are asymptotic. Get Asymptote angle and Real intercept by using the earlier equations
4. Calculate the Angle of Departure from poles and arrival at zeroes by using the rule derived
5. Calculate the ‘jω’ crossing using Routh’s stability
6. Calculate the break away and break-in points using $dK/d\sigma = 0$

Consider another system with unity feedback.

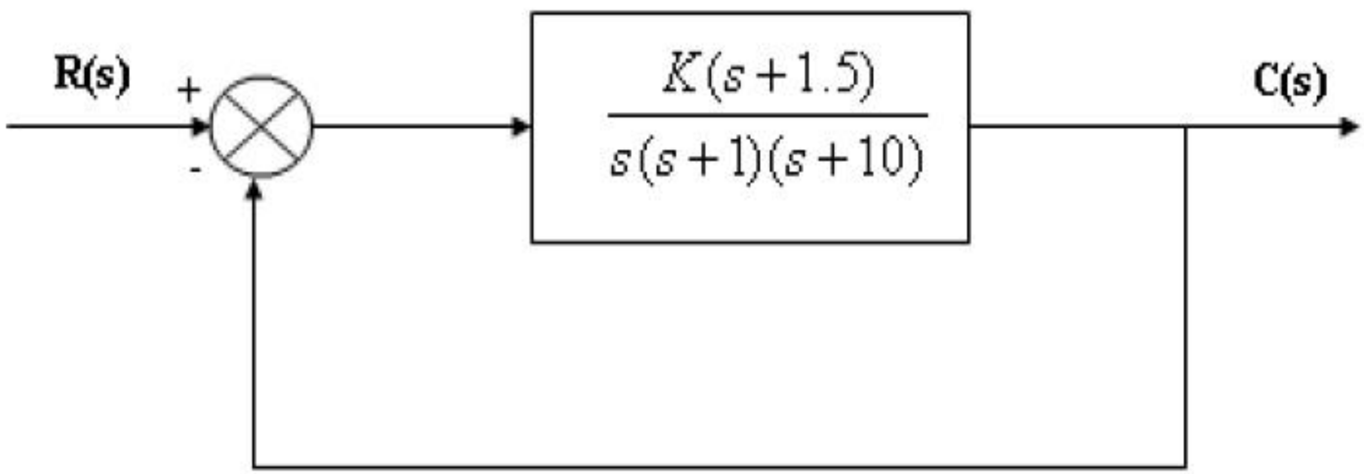
$$GH = \frac{10}{(s+2)(s+p_1)}, H = 1$$

$$\mathrm{KGH} = \frac{10}{(s+2)(s+p_1)}, \mathrm{CLT} = \frac{10}{s^2+(p_1+2)s+2p_1+10} = \frac{10}{(s^2+2s+10)+p_1(s+2)}$$

$$= \frac{10/(s^2+2s+10)}{1 + p_1(s+2)/(s^2+2s+10)}$$

Find the root locus for gain p_1 and TF: $(s+2)/(s^2+2s+10)$

Assignment: Find the position of the leftmost pole by using root locus corresponding to the Control Gains K= 7 and 40



Special References for this lecture

- Feedback Control of Dynamic Systems, Frankline, Powell and Emami, Pearson
- *Control Systems Engineering* – Norman S Nise, John Wiley & Sons
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