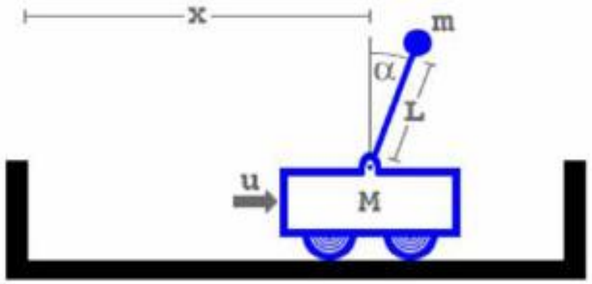
Stability Analysis and State Space Modelling

Introduction

- The response of a Dynamic System may become unbounded while subjected to a bounded input. Such systems are referred as unstable systems. One common example is an inverted pendulum on a rolling cart as shown below:



- A Control system could be designed such that by controlling the velocity of the rolling cart one can control the unstable response of the inverted pendulum.

- However, we need to first carry out a stability analysis of the system.

How to test the stability of a system

- A simple method to test the stability of a system is by checking the poles of the system transfer function.
- Consider a system which is represented by a generalized transfer function as follows:

$$T(s) = \frac{N(s)}{D(s)} = \frac{c_0 s^m + c_1 s^{m-1} + ... + c_{m-1} s + c_m}{a_0 s^n + a_1 s^{n-1} + ... + a_{n-1} s + a_n}$$

- Now, equating the denominator polynomial D(s) to zero, one can obtain the characteristic equation for the system. The roots of this characteristic equations are the poles of the system.
- If you obtain one or more poles with positive real part then the system could be predicted to be an unstable system.
- However, it is often tedious to obtain the poles of a complex system before predicting stability condition of the system.

Routh's Test for Stability

For a characteristic polynomial D(s), the number of poles in the right-half plane may be determined without actually finding the roots by using the Routh Test.

The Routh array for the polynomial D(s) could be constructed as follows:

s^n	a_0	a_2	a_4	a_6 ...
s^{n-1}	a_1	a_3	a_5	a_7 ...
s^{n-2}	b_1	b_2	b_3	...
:	:	:	:	
s^0				

Routh's Theorem

You may have observed that the first row of the Routh table consists of odd coefficients of D(s) starting from the first coefficient related to s^n . Again, the second row consists of the even coefficients starting from the second coefficient related to s^{n-1} .

The coefficients b_1 etc. For the third row could be computed as follows. The same pattern could be used for the subsequent rows.

$$b_1 = \frac{-\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1}, \quad b_2 = \frac{-\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1}, \quad b_3 = \frac{-\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}}{a_1}.$$

Routh's Theorem: The number of roots of the characteristic polynomial D(s) in the right-half plane equals the number of sign changes in the first column of the Routh Table.

A Test Case

- Consider a transfer function as:

$20/[s^8 + s^7 + 12\ s^6 + 22\ s^5 + 39\ s^4 + 59\ s^3 + 48\ s^2 + 38\ s + 20]$

S^8:	1.	12.	39.	48.	20.	
S^7 :	1.	22.	59.	38.	0.	
S^6:	-10	-20	10	20	0	
S^6:	-1	-2	1	2	0	
$S^{5:}$	1	3	2	0	0	(/20)
S^4:	1	3	2	0	0	
S^3:	2	3	0			
S^2:	3	4				
S:	1/3.	0				
S^0:	4.	0				

Unusual Case: Left Column Zero

- Consider $D(s) = 3s^4 + 6s^3 + 2s^2 + 4s + 5$
- Note appearance of zero in the first column
- Rename the row as Row A
- Create row B from Row A by sliding the A row to left until you get a non-zero pivot
- The sign of the row is changed by $(-1)^n$ where n is the number of times this row is slided
- The new non-zero row is formed by adding A and B
- [Reference Benedir and Picinbond, IEEE Trans on Automatic Control, 1990]

Other approaches for zero left column

Alternate approaches:

Put a parameter, say ε instead of zero in the pivot, assuming it to be a very small positive number

Continue and find sign changes

OR

Write the polynomial in reverse order such that the roots of the reverse polynomial will be the reciprocal of the roots of the original polynomial and follow the same procedure

It is discussed earlier that a Plant or a dynamical system could be modeled in terms of system parameters like ‘spring’, ‘mass’ and ‘damper’ for a mechanical system and ‘inductance’, ‘resistance’ and ‘capacitance’ for an electrical system.

However, these parameters seldom remain constant in nature. This may happen due to the following reasons:

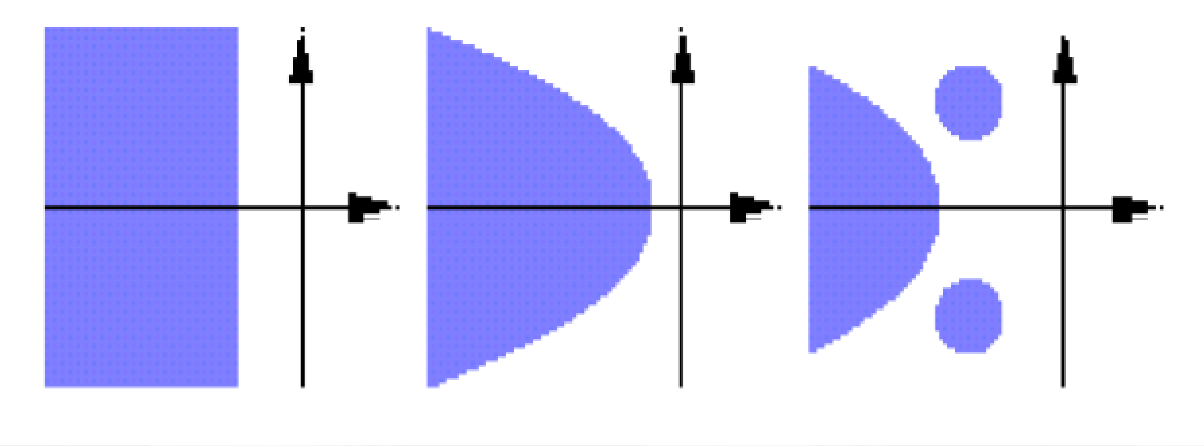
(i) Aging of a system such as ‘wear and tear’ and fatigue
(ii) Failure of components of a system
(iii) Change of environment and it’s interaction with the system
(iv) Inherent change of a system with time

The last case happens for systems like ‘Rocket’ that constantly looses mass. Considering all these possibilities of variation in system parameters, in this lecture we will explore the possibility of characterizing the stability of these systems.

Robustness of a Polynomial

A system is said to be robust if it remains stable (Bounded input corresponding to a Bounded output) while subjected to the variation of the system parameters within a specified interval.

A characteristic polynomial D(s) is stable if all its roots lie within a given region at the left half of the complex plane. The stability region varies with the nature of the system. Usually a part of the left half plane for a continuous system and unit disk for a discrete-time system. The figure below shows such regions.



A Simple Example

Consider a simple first order system given by the following equation:

$$T(s,q)=\frac{1}{s-q},\left|q\right|\leq q_0$$

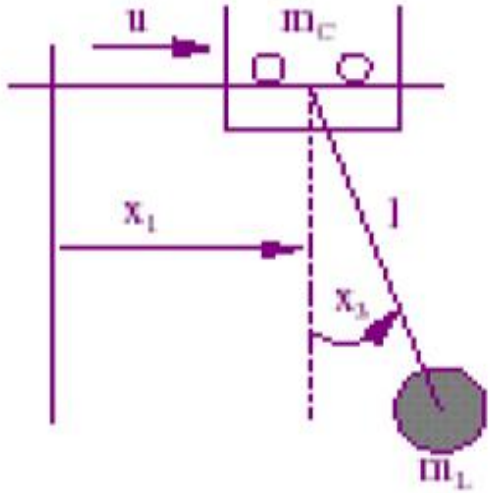
In this equation, the transfer function is shown to be varying with s and q, where q denotes the system parameter, which may vary following the given constraint.

Now, if there is a unit feed-back, the characteristic equation becomes:

$$D(s)=s+1-q$$

Clearly this closed loop system is not robustly stable as for values of $q \geq 1$ the roots will be in the right half plane.

Anti sway system for a crane



Consider a more realistic problem, that of a sway of a crane on a Gantry. If the mass to be transferred by the crane is uncertain, then the robustness issue is related to find out whether the polynomial D(s) remains stable for all admissible values of the unknown payload – $m_{L.}$

Kharitonov's theorem

Assuming the uncertainties in the coefficients of a characteristic polynomial to be independent of each other, (this is also known as interval polynomial); the Russian Mathematician Vladimir L Kharitonov has proved that a continuous-time interval polynomial is robustly stable if and only if it's four characteristic polynomials are stable.

Denoting the interval polynomial as *p(s,q)*:

The Kharitonov's polynomials are defined as:

$$p(s,q)=\sum_{i=0}^{n}[q^-{}_i, q^+{}_i]s^i$$

$$\begin{aligned}
p^{--}(s) &= q^-{}_0 + q^-{}_1 s + q^+{}_2 s^2 + q^+{}_3 s^3 + q^-{}_4 s^4 + q^-{}_5 s^5 + \cdots \\
p^{-+}(s) &= q^-{}_0 + q^+{}_1 s + q^+{}_2 s^2 + q^-{}_3 s^3 + q^-{}_4 s^4 + q^+{}_5 s^5 + \cdots \\
p^{+-}(s) &= q^+{}_0 + q^-{}_1 s + q^-{}_2 s^2 + q^+{}_3 s^3 + q^+{}_4 s^4 + q^-{}_5 s^5 + \cdots \\
p^{++}(s) &= q^+{}_0 + q^+{}_1 s + q^-{}_2 s^2 + q^-{}_3 s^3 + q^+{}_4 s^4 + q^+{}_5 s^5 + \cdots
\end{aligned}$$

A Numerical Example

Consider an interval polynomial to be:

$$p(s,q) = [1,10] + [2,11]s + [3,12]s^2 + [4,8]s^3$$

The system will be stable in the given intervals provided the following four Kharitonov's polynomials are stable. These are:

$$p^{--}(s) = 1 + 2s + 12s^2 + 8s^3$$

$$p^{-+}(s) = 1 + 11s + 12s^2 + 4s^3$$

$$p^{+-}(s) = 10 + 2s + 3s^2 + 8s^3$$

$$p^{++}(s) = 10 + 11s + 3s^2 + 4s^3$$

Assignment

Consider a first order system with the transfer function as follows:

$$T(s) = \frac{s+1}{s+6}$$

The system is controlled using a negative feedback controller with transfer function as:

$$T_c(s) = \frac{1}{s+10}$$

If the open-loop pole varies between 1 and 10, find out the stability of the closed loop system using Kharitonov's polynomials.

Special References for this lecture

- *Analyzing the stability robustness of interval polynomials*, G. Beale, George Mason University

- J.M. Maciejowski, *Multivariable Feedback Design, Addison-Wesley, Reading.*

- M.J. Grimble, Robust Industrial Control, Prentice Hall, New York.

State Space Design

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This Lecture Contains

- Introduction to State Space Control
- A second order system
- State Space Representation of a SDOF system
- Solution of State Space Equation
- Invariance of Eigen-values
- Stability of a system

Introduction to State Space Control

State – The states of a system refers to the property-set of the system that relates inputs to outputs such that the knowledge of the property set at any point of time along with inputs can completely define the output/response of the system in subsequent times.

Example: for a simple rotating pendulum the choice of states could be the position and angular velocity of the pendulum.

Note that for any system, the choice of states is not unique. However, the number of states needed to represent the system is unique.

Introduction (contd..)

State-space Control – Unlike frequency transform, in this technique, we preserve the time domain governing differential equations of a system. However, the differential equations describing the dynamics of the system are transformed into a set of first order differential equations in terms of the vectored representation of the states.

The solution of these first order ODEs yields a trajectory in space which Is spanned by the state vector. This space is known as State Space representation of the system.

Introduction (contd..)

The number of initial conditions required to solve the ODEs are unique and is equal to the number of states or the size of the state vector. This is also the same as the order of the system.

Consider the equation of motion of a single degree of freedom system described earlier:

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$

This is a second order ODE, you need two initial conditions to solve the system, hence the order of the system is also two and you need to specify any two states to define the motion of the system. Let us Choose *x* and *dx/dt*.

State-space representation of a SDOF system

Thus, we can formally write the state vector X as:

$$X = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$$

The second order ODE could be now represented as a set of two first order ODEs such that

$$\dot{X} = AX + BU$$

$$A = \begin{bmatrix} 0 & 1 \\ -K/M & -C/M \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ f/M \end{bmatrix}$$

A Standard State-Space Form

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \text{State-space equations}$$

$$\left.\begin{aligned} x &= n \times 1 \\ A &= n \times n \\ B &= n \times m \\ u &= m \times 1 \\ C &= r \times n \\ D &= r \times m \end{aligned}\right\} \text{Order of the matrices with standard notations}$$

Controller Canonical Form

State space representation is not unique in nature. Some of the commonly used forms are mentioned here:

$$\frac{d^n y}{dt^n} + \alpha_1 \frac{dy^{n-1}}{dt^{n-1}} + .. + \alpha_n y = \beta_0 \frac{d^n u}{dt^n} + \beta_1 \frac{du^{n-1}}{dt^{n-1}} + .. + \beta_n u$$

$$A = \left[\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & .. & 1 \\ \hline -\alpha_n & -\alpha_{n-1} & -\alpha_{n-2} & & -\alpha_1 \end{array}\right] \qquad B = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}$$

$$C = [\beta_n - \alpha_n \beta_0 \quad 0 \quad 0 \quad .. \quad \beta_1 - \alpha_1 \beta_0] \qquad D = \beta_0$$

Observer Canonical Form

$$\frac{d^n y}{dt^n} + \alpha_1 \frac{dy^{n-1}}{dt^{n-1}} + \ldots + \alpha_n y = \beta_0 \frac{d^n u}{dt^n} + \ldots + \beta_{n-1} \frac{du}{dt} + \beta_n u$$

$$A = \left[\begin{array}{c|cccc} -\alpha_1 & 1 & 0 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 & 0 \\ .. & .. & .. & .. & .. \\ -\alpha_{n-1} & 0 & 0 & 0 & 1 \\ -\alpha_n & 0 & 0 & 0 & 0 \end{array} \right] \qquad B = \begin{bmatrix} \beta_1 - \alpha_1 \beta_0 \\ \beta_2 - \alpha_2 \beta_0 \\ \\ \beta_{n-1} - \alpha_{n-1} \beta_0 \\ \beta_n - \alpha_n \beta_0 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0 \quad .. \quad 0] \qquad D = \beta_0$$

Jordan Canonical Form
(for non-repeated eigenvalues)

$$\frac{d^n y}{dt^n} + \alpha_1 \frac{dy^{n-1}}{dt^{n-1}} + .. + \alpha_n y = \beta_0 \frac{d^n u}{dt^n} + \beta_1 \frac{du^{n-1}}{dt^{n-1}} + .. + \beta_n u$$

$$Y(s) = [\beta_0 + \frac{P_1}{(s - \lambda_1)} + .. + + \frac{P_n}{(s - \lambda_n)}] U(s)$$

$\lambda_1 \ldots \lambda_n$ roots of $s^n + \alpha_1 s^{n-1} + .. + \alpha_n = 0$

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & 0 & 0 & \lambda_{n-1} & 0 \\ \hline 0 & 0 & .. & 0 & \lambda_n \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \\ . \\ 1 \\ 1 \end{bmatrix}$$

$$C = [P_1 \quad P_2 \quad .. \quad .. P_n] \qquad D = \beta_0$$

Solution of State-Space Equation

State space equations could be solved by following simple ODE solving procedure. Thus, at any time t_f the response of the state space system in standard form could be expressed as:

$$x(t_f) = e^{At_f} x(0) + \int_0^{t_f} e^{A(t_f-\tau)} B u(\tau)\, d\tau$$

The expression e^{At} is also known as state transition matrix and could be solved by procedures like Eigen-vector representation, Cayley-Hamilton Method and Resolvent Matrix Method.

$$e^{At} = I + \sum_{k=1}^{\infty} (At)^k / k! = \sum_{i=1}^{n} t_i e^{\lambda_i t} q_i$$

t_i, q_i are left & right eigen - vectors

Invariance of Eigen Values

$$\dot{x} = Ax + Bu, \qquad Use \qquad x = Tz$$

$$T\dot{z} = ATz + Bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}Bu$$

$$\left|\lambda I - T^{-1}AT\right| = \left|\lambda T^{-1}T - T^{-1}AT\right|$$

$$= \left|T^{-1}(\lambda I - A)T\right|$$

$$= \left|T^{-1}\right|\left|\lambda I - A\right|\left|T\right|$$

$$= \left|\lambda I - A\right|$$

How do we check the System Stability?

In state space form, the stability of a system depends on the Eigen Values of A which may be obtained from the characteristic equation as follows. If the real parts of the roots of this equation are strictly negative then the system is considered to be asymptotically stable

$$|\lambda I - A| = 0$$

$e.g.$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda + 6 \end{vmatrix}$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

$$\lambda = -1, -2, -3 : \quad Stable \quad System$$

Controllability & Observability of Dynamic Systems

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Controllability of a System

- A state x_1 of a system is "controllable" if all initial conditions x_0 at any previous time t can be transferred to x_1 in a finite time by some control function $u(t,x_0)$.
- If all the states are controllable then the system is completely controllable
- If controllability is restricted to depend upon t_0, then the system is *controllable at time* t_0.
- If a particular output can be obtained from any arbitrary x_0 at t_0, then the system is *output controllable*.

How to test the Controllability of a system?

- A system is state controllable at $t=t_0$ if there exists a continuous input u(t) such that it will drive the initial states $x(t_0)$ to any final state $x(t_f)$ within a finite time interval (t_f-t_0)

- The Controllability matrix for a system (A,B) is defined as:

$$C = [B \quad AB \quad A^2B \quad .. \quad A^{n-1}B]$$

- The state matrix A is of size *nxn*.

- The system is fully controllable iff Rank(*C*) = n

The concept of Stabilizability

- In general, controllability is considered to be a very strong constraint for a multi-degrees of freedom system.
- Hence, in practice, there exists an weaker definition of controllability – this is known as stabilizability.
- Let us consider the following system which is represented in modal or block diagram form:

$$\dot{x} = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

- This system has two roots one at 3 and hence unstable, the other at -2 and hence stable. However, the control effort exists only for the unstable mode at 3 and hence the system is partly controllable or stabilizable.

Observability of a System

A state $x_1(t)$ at some given time is 'observable' if knowledge of the input u(t) and output y(t) over a finite segment of time completely determines $x_1(t)$.

The Observability matrix for a system (A,C) is defined as:

$$O^T = [C \quad C\,A \quad ...... \, C\,A^{n-1}]$$

The state matrix A is of size *n x n.*

The system is fully Observable iff Rank(*O*) = n

Example 1:

Check whether the following system is controllable and observable

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Solution 1:

- The order of the Plant is 2 here. Let us obtain the Controllability and the Observability Matrices

- Following earlier definitions:

$$C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$O = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- Since both *C* and *O* are of rank 2 which equals to the order of the system – hence, this system is fully controllable and observable.

Assignment: Check Controllability and Observability of the given system

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

Special References for this lecture

- Feedback Control of Dynamic Systems, Frankline, Powell and Emami, Pearson
- *Control Systems Engineering* – Norman S Nise, John Wiley & Sons
- *Design of Feedback Control Systems* – Stefani, Shahian, Savant, Hostetter

  Oxford