Master's Thesis

Rohit Gupta, Sumit Basu

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Abstract

Fill it.

1 Introduction

2 Motivation

3 Method

The idea of asymptotic homogenization. In a repeating cell Y,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1}$$

where $C_{ijkl}(\underset{\sim}{x} + \underset{\sim}{u}\underset{\sim}{Y}) = C_{ijkl}(\underset{\sim}{x})$

$$\Rightarrow C_{ijkl}(x_1 + n_1Y_1\, x_2 + n_2Y_2\, x_3 + n_3Y_3) = C_{ijkl}(x_1, x_2, x_3) \tag{2}$$

$C_{ijkl}(\underset{\sim}{x})$ is Y-periodic

$$\underset{\sim}{y} = \tfrac{x}{\epsilon} \tag{3}$$

$$\Rightarrow g = g(\underset{\sim}{x}, \tfrac{x}{\epsilon}) = g(\underset{\sim}{x}\,\underset{\sim}{y}) \tag{4}$$

$\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ defines the domain of the composite Ω. The domain is composed of base cells of dimensions, $\varepsilon Y_1, \varepsilon Y_2, \varepsilon Y_3$ where $\underset{\sim}{y} = \frac{x}{\varepsilon}$

3.1 1D Elasticity

$$\sigma^\varepsilon = E^\varepsilon \tfrac{\partial u^\varepsilon}{\partial x} \tag{5}$$

$$\tfrac{\partial \sigma^\varepsilon}{\partial x} + \gamma^\varepsilon = 0 \quad E^\varepsilon\, \gamma^\varepsilon \to macroscopicallyuniform \tag{6}$$

Inside each cell,

$$E^\varepsilon(x, \frac{x}{\varepsilon}) \quad = E(y) \tag{7}$$

$$\gamma^\varepsilon(x, \frac{x}{\varepsilon}) \quad = \gamma(y) \tag{8}$$

Let

$$u^{\varepsilon}(x) = u^0 x, y + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + ... \tag{9}$$
$$\sigma^{\varepsilon}(x) = \sigma^0 x, y + \varepsilon \sigma^1(x,y) + \varepsilon^2 \sigma^2(x,y) + ... \tag{10}$$

3.2 Optimal Design of Elastic structures

$$\mathbf{b} \rightarrow \text{body forces}$$
$$\mathbf{t} \rightarrow \text{surface tractions}$$

Optimal choice of $\mathbb{C}_{ijkl} \in U_{ad} \leftarrow$ admissible set of elasticity ??
$\mathbb{C}_{ijkl}(\mathbf{x}) \forall \mathbf{x} \in \Omega$ has 21 independent components
$a_E(\mathbf{u},\mathbf{v}) = \int_{\Omega} \mathbb{C}_{ijkl}\, \varepsilon_{kl}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{v}) d\mathbf{v} \rightarrow$ energy bilinear form
$L(\mathbf{v}) = \int_{\Omega} \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} ds \rightarrow$ load linear form.

Minimum compliance problem:

$$\textit{minimize} \qquad L(\mathbf{v}), \tag{11}$$
$$\textit{subject to} \qquad \mathbb{C}_{ijkl} \in \mathbb{U}_{ad} \tag{12}$$
$$a_E(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{U} \tag{13}$$

where $\mathbb{U} \rightarrow$ kinematically admissible displacements.
For optimal shape design:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = \chi(\mathbf{x})\overline{\mathbb{C}}_{ijkl}, \quad \text{where } \overline{\mathbb{C}}_{ijkl} \rightarrow \text{stiffness matrix of the material} \tag{14}$$
$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m, \\ 0 & \text{if } \mathbf{x} \in \Omega \backslash \Omega^m \end{cases} \tag{15}$$

where $\Omega^m \rightarrow$ part of the domain occupied by the material.
For sizing problem:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = h(\mathbf{x})\overline{\mathbb{C}}_{ijkl} \tag{16}$$
$$\textstyle\int_{\Omega} \chi(\mathbf{x}) d\mathbf{x} = V_f \tag{17}$$
$$\& \quad \textstyle\int_{\Omega} h(\mathbf{x}) d\mathbf{x} = V_f. \tag{18}$$

where $h(x)$ is a sizing function.

Traditionally shape design problems are initiated in the following manner:

$$\textit{Ref doamin}: \Omega_0 \in \mathbb{R}^3 \tag{19}$$
$$\underline{\phi}: \Omega_0 \rightarrow \phi(\Omega_0) \text{is a diffeomorphism.} \tag{20}$$
$$L(\mathbf{v}) = \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| ds \tag{21}$$

$$\begin{aligned} a_E &= \int_\Omega \mathbb{C}_{ijkl}(\mathbf{x}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v})|det(D\underline{\phi}^{-1})|d\mathbf{x} \end{aligned} \tag{22}$$

Now,

$$\begin{aligned} \mathbb{C}_{ijkl}\varepsilon_{kl} &= \mathbb{C}_{ijkl}\frac{1}{2}(u_{k,l} + u_{l,k}) \\ &= \frac{1}{2}\mathbb{C}_{ijkl}u_{k,l} + \frac{1}{2}\mathbb{C}_{ijlk}u_{l,k} \\ &= \mathbb{C}_{ijkl}u_{k,l} \end{aligned} \tag{23}$$

$$\begin{aligned} a_E &= \int_{\Omega_0} \mathbb{C}_{ijkl}u_{k,l}(\mathbf{u})u_{i,j}(\mathbf{v})|det(D\underline{\phi}^{-1}|d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl}\frac{\partial u_k}{\partial \mathbf{x}_m}(D\underline{\phi}^{-1})_{ml}\frac{\partial u_i}{\partial \mathbf{x}_p}(D\phi^{-1})_{pj}|det(D\underline{\phi}^{-1})|d\mathbf{x} \end{aligned} \tag{24}$$

$$\Rightarrow \mathbb{C}_{ijkl}(D\underline{\phi}^{-1})_{ml}(D\underline{\phi}^{-1})_{pj}|det(D\underline{\phi}^{-1})| = \bar{\mathbb{C}}_{ipkm} \tag{25}$$

$$\bar{\mathbb{C}}_{ijkl} = \mathbb{C}_{ipkm}(D\underline{\phi}^{-1})_{lm}(D\underline{\phi}^{-1})_{jp}|det(D\underline{\phi}^{-1})| \tag{26}$$

Treating $\underline{\phi}$ as a design variable is tidious.

3.3 Homogenization method

$$E^\varepsilon_{ijkl}(\mathbf{x}) = E_{ijkl}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \tag{27}$$

The tensor E^ε_{ijkl} is a material constant which satisfies the symmetry condition and is assumed to satisfy strong ellipticity condition for every $\mathbf{x}$.

$$\Rightarrow E^\varepsilon_{ijkl} = E^\varepsilon_{jikl} = E^\varepsilon_{ijlk} = E^\varepsilon_{klij} \tag{28}$$

$$E^\varepsilon_{ijkl}(\mathbf{x})\mathbf{X}_{ij}\mathbf{X}_{kl} \geq m\mathbf{X}_{ij}\mathbf{X}_{ij} \qquad \text{for some } m > 0 \ \& \ \forall \quad \mathbf{X}_{ij} = \mathbf{X}_{ji} \tag{29}$$

Let the domainΩ has a boundary Γ. Let $\mathbf{f}$ be the body force acting on Ω and $\mathbf{t}$ be the traction acting on Γ_t part of the boundary Γ. Also, let Γ_D be the part of boundary on which displacement is defined. Then the displacement $\mathbf{u}^\varepsilon$ can be obtained as the solution to the following minimization problem

$$\min_{\mathbf{v}^\varepsilon \in U} \quad F^\varepsilon(\mathbf{v}^\varepsilon), \tag{30}$$

$$\tag{31}$$

where F^ε is total potential energy given as

$$F^\varepsilon(\mathbf{v}^\varepsilon) = \frac{1}{2}\int_\Omega E^\varepsilon_{ijkl}\varepsilon_{kl}(\mathbf{v}^\varepsilon)\varepsilon_{ij}(\mathbf{v}^\varepsilon)dx - \int_\Omega \mathbf{f}\cdot\mathbf{v}^\varepsilon dx - \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}^\varepsilon ds \tag{32}$$

and $\mathcal{U}$ is the set of admissible displacements defined such that

$$\mathcal{U} = \{\mathbf{v} = v_i \mathbf{e}_i : \, v_i \in H^1(\Omega) \text{ and } \mathbf{v} \in \mathcal{G} \text{ on } \Gamma_D\} \tag{33}$$

where $\mathcal{G}$ is set of displacement defined along the boundary Γ_D.
Let

$$\mathbf{v}^{\varepsilon}(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \varepsilon \mathbf{v}_1(\mathbf{x}, \mathbf{y}), \qquad y = \frac{\mathbf{x}}{\varepsilon} \tag{34}$$