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A review of homogenization and topology optimization

Ihomogenization theory for media with periodic

structure

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Abstract

This is the ®rst part of a three-paper review of homogenization and topology optimization, viewed from an

engineering standpoint and with the ultimate aim of clarifying the ideas so that interested researchers can easily

implement the concepts described. In the ®rst paper we focus on the theory of the homogenization method where

we are concerned with the main concepts and derivation of the equations for computation of eective constitutive

parameters of complex materials with a periodic micro structure. Such materials are described by the base cell,

which is the smallest repetitive unit of material, and the evaluation of the eective constitutive parameters may be

carried out by analysing the base cell alone. For simple microstructures this may be achieved analytically, whereas

for more complicated systems numerical methods such as the ®nite element method must be employed. In the second paper, we consider numerical and analytical solutions of the homogenization equations. Topology

optimization of structures is a rapidly growing research area, and as opposed to shape optimization allows the

introduction of holes in structures, with consequent savings in weight and improved structural characteristics. The

homogenization approach, with an emphasis on the optimality criteria method, will be the topic of the third paper

in this review.

1. Introduction

Advances in technology in recent years have been

paralleled by the increased use of composite materials

in industry. Since materials have dierent properties, it

seems sensible to make use of the good properties of

each single ingredient by using them in a proper combination. For example, a simple mixture of clay, sand

and straw produced a composite building material

which was used by the oldest known civilizations. The

further development of non-metallic materials and

composites has attracted the attention of scientists and

engineers in various ®elds, for example, aerospace,

transportation, and other branches of civil and mechanical engineering. Apart from the considerably low

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ratio of weight to strength, some composites bene®t from other desirable properties, such as corrosion and

thermal resistance, toughness and lower cost. Usually,

composite materials comprise of a matrix which could

be metal, polymeric (like plastics) or ceramic, and a reinforcement or inclusion, which could be particles or

®bres of steel, aluminum, silicon etc.

Composite materials may be de®ned as a man-made

material with dierent dissimilar constituents, which

occupy dierent regions with distinct interfaces

between them [1]. The properties of a composite are

dierent from its individual constituents. A cellular

body can be considered as a simple case of a composite, comprising solids and voids. This is the case

which is used in the structural topology optimization.

In this study, composites with a regular or nearly

regular structure are considered. Having suciently

regular heterogenities enables us to assume a periodic

PII: S 0 0 4 5 - 7 9 4 9 (9 8) 0 0 1 3 1 - X

structure for the composite. It should be emphasized

that in comparison with the dimensions of the body

the size of these non-homogeneities should be very

small. Owing to this, these types of material are sometimes called composites with periodic microstructures.

Even with the help of high-speed modern computers,

the analysis of the boundary value problems consisting

of such media with a large number of heterogenities, is

extremely dicult. A natural way to overcome this dif®culty is to replace the composite with a kind of equivalent material model. This procedure is usually called

homogenization. One way of ®nding the properties of

such composites is by carrying out experimental tests.

It is quite evident that because of the volume and cost

of the required tests for all possible reinforcement

types, experimental measurements are often impracticable.

The mathematical theory of homogenization, which

has developed since the 1970 s is used as an alternative

approach to ®nd the eective properties of the equivalent homogenized material [2±4]. This theory can be

applied in many areas of physics and engineering having ®nely heterogeneous continuous media, like heat

transfer or ¯uid ¯ow in porous media or, for example, electromagnetism in composites. In fact, the basic

assumption of continuous media in mechanics and

physics can be thought of as sort of homogenization,

as the materials are composed of atoms or molecules.

From a mathematical point of view, the theory of

homogenization is a limit theory which uses the

asymptotic expansion and the assumption of periodicity to substitute the dierential equations with

rapidly oscillating coecients, with dierential

equations whose coecients are constant or slowly

varying in such a way that the solutions are close to

the initial equations [5].

This method makes it possible to predict both the

overall and local properties of processes in composites.

In the ®rst step, the appropriate local problem on the

unit cell of the material is solved and the eective material properties are obtained. In the second step, the

boundary value problem for a homogenized material is

solved.

a 3 ? 3 diagonal matrix:

2
3

n1 0 0

N 4 0 n2 0 5;

0 0 n3

where n1, n2 and n3 are arbitrary integer numbers, and

Y = hY1 Y2 Y3iT is a constant vector which determines the period of the structure; F can be a scalar or

vectorial or even tensorial function of the position vector x. For example, in a composite tissued by a periodically repeating cell Y, the mechanical behaviour is

described by the constitutional relations of the form:

sij cijkl ekl ;

and the tensor cijkl is a periodic function of the spatial

coordinate x, so that

cijkl x NY cijkl x

2

or

cijkl x1 n1 Y1 ; x2 n2 Y2 ; x3 n3 Y3   cijkl x1 ; x2 ; x3 :

cijkl(x) is called the Y-periodic (see Fig. 1). Note that

sij and ekl are, respectively, the stress and strain tensors.

In the theory of homogenization the period Y compared with the dimensions of the overall domain is

assumed to be very small. Hence, the characteristic

functions of these highly heterogeneous media will

rapidly vary within a very small neighbourhood of a

point x. This fact inspires the consideration of two

dierent scales of dependencies for all quantities: one on the macroscopic or global level x, which indicates

slow variations, and the other on the microscopic or

local level y, which describes rapid oscillations.

The ratio of the real length of a unit vector in the

microscopic coordinates to the real length of a unit

vector in the macroscopic coordinates, is a small parameter E. so Ey = x or y = x/E. Consequently, if g is a

general function then we can say g = g(x, x/E) = g(x,

y). To illustrate the technique let us assume that F(x)

is a physical quantity of a strongly heterogeneous med-

2. Periodicity and Asymptotic Expansion

A heterogeneous medium is said to have a regular

periodicity if the functions denoting some physical

quantity of the the mediumeither geometrical or

some other characteristicshave the following property:

F x NY F x:

1

x = (x1, x2, x3) is the position vector of the point, N is

Fig. 1. Periodicity requires that the functions have equal

values at points P1, P2,. . ., P6.

Fig. 2. A highly oscillating function.

ium. Thus F(x) will have oscillations, see Fig. 2. To

study these oscillations using this double-scale expansion, the space can be enlarged as indicated in Fig. 3.

The small parameter E also provides an indication of

the proportion between the dimensions of the base

cells of a composite and the whole domain, known as

the characteristic inhomogeneity dimension. As a

hypothetical example, E for the skin cells of the human

body is larger than E for the atoms of which it is comprised. The quantity 1/E can be thought of as a magni-

®cation factor which enlarges the dimensions of a base

cell to be comparable with the dimensions of the

material [6±8], see Fig. 4.

In the double-scale technique, the partial dierential

equations of the problem have coecients of the form

a(x/E) or a(y), where a(y) is a periodic function of its

arguments. The corresponding boundary value problem may be treated by asymptotically expanding the

solution in powers of the small parameter E. This technique has already proved to be useful in the analysis

of slightly perturbed periodic processes in the theory of vibrations. The same principle is extendible to processes occurring in composite materials with a regular

structure.

If we assign a coordinate system x = (x1, x2, x3) in

R3 space to de®ne the domain of the composite material problem O, then assuming periodicity, the

domain can be regarded as a collection of parallelpiped cells of identical dimensions EY1, EY2, EY3, where

Y1, Y2 and Y3 are the sides of the base cell in a local

(microscopic) coordinate system y = (y1, y2, y3) = x/E.

So for a ®xed x in the macroscopic level, any dependency on y can be considered Y-periodic. Moreover, it

is assumed that the form and composition of the base

Fig. 3. One of the oscillations in the expanded scale.

Fig. 4. Characteristic dimension of inhomogeneity and scale enlargement.

cell varies in a smooth way with the macroscopic variable x. This means that for dierent points the structure of the composite may vary, but if one looks

through a microscope at a point at x, a periodic pattern can be found.

Functions determining the behaviour of the composite can be expanded as:

FE x F0 x; y EF1 x; y E2 F2 x; y ? ? ? ;

0

3. One-dimensional Elasticity Problem

To clarify the homogenization method, the simple

case of calculation of deformation of an inhomogeneous bar in the longitudinal direction is considered.

Here, we attempt to derive the modulus of elasticity

without recourse to advanced mathematics.

According to the assumptions of the theory, the

medium has a periodic composite microstructure

(Fig. 5).

The governing equations, in the form of Hooke's

law of linear elasticity and the Cauchy's ®rst law of

motion (equilibrium equation), are:

3

E

@s

gE 0:

@x

EE x; x=E EE x=E E y

4

The dependency of the quantities to the size of the

unit cell of inhomogeneity is indicated by the superscript "E". sE is the stress, uE is the displacement, EE(x)

is the Young's modulus and gE is the weight per unit

volume of material. It is assumed that EE and gE are

macroscopically uniform along the domain and only

5

and

gE x; x=E gE x=E g y:

6

Using the asymptotic expansion:

uE x u0 x; y Eu1 x; y E2 u2 x; y ? ? ?

1

where E 4 0 and functions F (x, y), F (x, y), . . . are

smooth with respect to x and Y-periodic in y, which

means that they take equal values on the opposite

sides of the parallel-piped base cell.

@uE

sE EE

;

@x

vary inside each cell:

7

and

sE x s0 x; y Es1 x; y E2 2 x; y ? ? ? ;

8

where ui(x, y) and si(x, y), (i = 1, 2, . . .) are periodic

on y and the length of period is Y. In due course the

following facts will be referred to:

Fact (1). The derivative of a periodic function is

also periodic with the same period.

Fact (2). The integral of the derivative of a function

over the period is zero. (These facts can easily be veri®ed by the de®nition of derivative and periodicity.)

Fact (3). If F = F(x, y) and y depends on x, then:

dF @F @F @y

:

dx @x @y @x

In this case, as y = x/E, so

dF @F 1 @F

:

dx @x E @y

Using the latter fact and substituting the series in

Eqs. (7) and (8) into Eqs. (3) and (4), we obtain:

s0 Es1 E2 s2 ? ? ?

? 0
?

@u

1 @u0

@u1 @u1

@u1

@u2

E y

E

E2

E

??? ;

@x E @y

@x

@y

@x

@y

9

and

@s0 1 @s0

@s1 @s1

E

? ? ? g y 0:

@x E @y

@x

@y

10

By equating the terms with the same power of E,

Eq. (9) yields:

? 0?

@u

0 E y

;
11

@y

? 0
?

@u

@u1

s E y

;

@x

@y

12
? 1
?

@y

@u2

s1 E y

;

@x

@y

13
0

Fig. 5. A composite bar.

and similarly from Eq. (10):

@s0

0;

@y

14

@s0 @s1

g y 0:

@x

@y

15

Dividing by E(y) and integrating both sides of Eq. (16)

over the period Y, and using fact (2), yields:

?
?

dy du0 x

0

s x Y=

:
17

dx

Y E y

Now, by substituting the value of s0(x) into Eq. (16),

we obtain:

? ?
?
? 0

@u1 x; y

dy

du x

Y= E y

ÿ1

;

@y

dx

Y E y

and by integrating this equation, we conclude that u1

has the following form:

u1 x; y w y

du0 x

x x;

dx

where Z is the dummy variable of integration and b is

a constant. Now, using the boundary condition

w(0) = w(Y) yields:

Y

0

From Eqs. (11) and (14) it is concluded that the functions u0 and s0 only depend on x [i.e. u0(x) and s0(x)].

Bearing in mind that the relationship between s0(x)

and u0(x) is sought (because they are independent of

the microscopic scale), Eq. (12) can be written as:

? 0
?

du x @u1 x; y

s0 x E y

:
16

dx

@y

18

where w(y) is the initial function of the terms inside the

square brackets and x(x) is the constant of integration

due to y. From Eqs. (16) and (18) it follows that

?
?

dw y du0 x

s0 x E y 1

:
19

dy

dx

Dierentiating Eq. (19) with respect to y, one concludes that

?
?
??

d

dw y

E y 1

0; on Y;

20

dy

dy

711

a

dZ ÿ Y 0;

E Z

24

or

?

Y

1

a 1=

Y

0
?

dZ

:

E Z

25

Note that comparig Eqs. (19) and (21) one can see

that

s0 x a

du0 x

;

dx

26

and substituting for a from Eq. (25) yields

?
1

s x 1=

Y

0

Y

0
?

dZ du0 x

:

E Z

dx

27

By integrating Eq. (15) over the length of the period

(0, Y) and using fact (2) mentioned earlier, results in:

ds0 x

g?   0;

dx

28

where g=1/Y fY g(y)dy is the volumetric average of g

inside the base cell.

By studying Eqs. (27) and (28), we realize that they

are very similar to the equations of one-dimensional

(1D) elasticity in homogeneous material, and s0 and u0

are independent of the microscopic scale y. The only

dierence is the elasticity coecient, which should be

replaced by the homogenized one. Hence, the problem

can be summarized as:

?

s0 x EH du0 x=dx

ds0 x=dx g?   0;

29

and w(y) takes equal values on the opposite faces of Y

[i.e. w(0) = w(Y)]. Integrating Eq. (20) yields

?
?

dw y

E y 1

a a is a constant;

21

dy

where

or

22

is the homogenized modulus of elasticity.

To ®nd displacements, following the same as for the

homogeneous material, the bar problem is now

straightforward. Combining the two parts of Eq. (29),

we obtain:

23

@2 u0 x

g

ÿ H:

@x2

E

dw y

a

ÿ 1:

dy

E y

Integrating Eq. (22) it follows that

?

y?

a

w y

ÿ 1 dZ b;

0 E Z

EH 1=

?
1

Y

Y

0
?

dZ

;

E Z

30

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By two times integration and using the boundary conditions (x = 0; u = 0) and (x = L; du/dx = 0) it

results in:

5. General Boundary Value Problem

The 1D heat conduction is very similar to the 1D

elasticity problem. The governing equations, Fourier's

law of heat conduction and the equation of heat balance, are:

Many physical systems which do not change with

timesometimes called steady-state problemscan be

modelled by elliptic equations. As a general problem,

the divergent elliptic equation in a non-homogeneous

medium with regular structure is now explained.

Let OWR3 be an unbounded medium tissued by parallelepiped unit cells Y, whose material properties are

determined by a symmetric matrix aij(x, y) = aij(y),

where y = x/E and x = (x1, x2, x3) and the functions

aij are periodic in the spatial variables y = (y1, y2, y3).

The boundary value problem to be dealt with is:

?

AE uE f

u x ÿ

2

g? x

g?

Lx:

EH 2 EH

4. Problem of Heat Conduction

qE x KE dTE x=dx

@qE =@x f 0:

E

31

E

E

q is the heat ¯ux, T is the temperature, and K (x) is

the conductivity coecient. Following a very similar

procedure to the 1D elasticity problem, the homogenized coecient of heat conduction can be obtained

as:

? Y

?
1

dZ

KH 1=

;

Y 0 K Z

which as is expected, is the same as Eq. (30).

Similarly, starting from the equations of heat conduction in the general 3D case, and following the same

procedure as for 1D problem, the following results will

be obtained [6]:

?
?

q?i x KH

ij @T x=@xj

@?

qi =@xi f 0;

32

where

KH

ij

36

uE 0 on @O;

37

where the function f is de®ned in O and

?
?
@
@

E

A

aij y

@xi

@xj

?
?
? ?

@wi

K y dij

dy ;

@yi

Y

33

and wj(y) is the solution of the partial dierential

equation:

?
?
??
@

@wj

K y dij

0

@yi

@yi

on Y:

34

dij is the Kronecker symbol and the boundary conditions are concluded from the periodicity, i.e. wj takes

equal values on the opposite sides of the base cell. In

Eqs. (31) and (32), q and @qi/@xi are the volumetric

average value of q0i (x) and @q0i /@xi over Y. The volumetric average of a quantity a(x, y) over Y is de®ned

by:

1

jYj

Y

a x; ydy:

35
38

is the elliptical operator. The superscript "E" is used to

show the dependency of the operator and the solution

to the characteristic inhomogeneity dimension.

Using a double-scale asymptotic expansion, the solution to Eqs. (36) and (37) can be written as:

uE x u0 x; y E1 u1 x; y E2 u2 x; y ? ? ? ;

39

j

where functions u (x, y) are Y-periodic in y. Recalling

the rule of indirect dierentiation (fact 3) yields

AE

1 1 1 2

A A A;

E2

E

40

where

1

jYj

a? x

in O;

A1

and

A2

?
?
@
@

aij y

;

@yi

@yj

A3

?
?
@
@

aij y

@xi

@xj

?
?
?
?
@
@
@
@

aij y

aij y

:

@yi

@xj

@xi

@yj

Applying Eqs. (39) and (40) into Eq. (36) yields

Eÿ2 A1 Eÿ1 A2 A3 u0 Eu1 E2 u2 ? ? ? f;

41

and by equating terms with the same power of E, we

obtain:

A1 u0 0;

2 0
43
2 1
3 0
44

A u A u 0;

1 2
42
1 1

A u A u A u f; ? ? ?

If x and y are considered as independent variables,

these equations form a recurrent system of dierential

equations with the functions u0, u1 and u2 parameterized by x. Before proceeding to the analysis of this system, it is useful to notice the following fact:

Fact (4). The equation

A1 u F

in Y

45

for a Y-periodic function u has a unique solution if:

1

F?

jYj

Y

Fdy 0;

46

where vYv denotes the volume of the base cell.

From this fact, and using Eq. (42), it immediately

follows that

u0 u x;

A1 u1 ÿA2u 0 ÿ

@aij y @u x

:

@yi @xj

u1 x; y wi y

@u x

x x;

@xj

49

where wj(y) is the Y-periodic solution of the local

equation

@aij y

@yi

in Y:

50

Now, turning to Eq. (44) for u2 and taking x as a parameter, it follows from fact (4) that Eq. (44) will have

a unique solution if

1

jYj

Y

A2 u1 A3 u0 dy f 0;

51

which when combined with Eq. (49) results in the following homogenized (macroscopic) equation for u(x):

aH

ij

Thus, it is demonstrated that the initial equation has

been split into two dierent problems:

1. Determine wj(y) from Eq. (50) which is solved on

the base cell.

2. Solve Eq. (52) on O with u = 0 on @O. The homogenized coecients aH

ij are obtained from Eq. (53).

6. General Elasticity Problem

48

As in the right-hand side of Eq. (48) the variables are

separated, the solution of this equation may be represented in the form

ÿ

@2
:

@xi @xj

47

and by substituting into Eq. (43) we ®nd:

A1 wj y

AH aH

ij

713
2

@ u x

f;

@xi @xj

where the quantities

?
?
1

@wj

aH

aij y aik y

dy

ij

@yk

jYj Y

52
53

are the eectivе coecients of the homogenized operator:

So far, the application of the homogenization theory

in 1D elasticity, heat conduction, and as a more general problem in elliptic partial dierential equations,

has been discussed. For the sake of completeness the

homogenization method for cellular media in weak

form, which is suitable for the derivation of the ®nite

element formulation, using the procedure and notation

used by Guedes and Kikuchi in Ref. [9], is brie¯y

explained. This is the case applied in topological structural optimization by Bendsùe and Kikuchi [10±14].

Let us consider the elasticity problem constructed

from a material with a porous body with a periodic cellular microstructure. Body forces f and tractions t

are applied. See Fig. 6 O is assumed to be an open

subset of R3 with a smooth boundary on G comprising

Gd (where displacements are prescribed) and Gt (the

traction boundary). The base cell1 of the cellular body

Y is illustrated in Fig. 7. Y is assumed to be an open

rectangular parallel-piped in R3 de®ned by

Y 0; Y1 ?0; Y2 ?0; Y3 ;

with a hole v in it. The boundary of v is de®ned by s

(@v = s) and is assumed to be suciently smooth, and

as a more general case the tractions p can also exist

inside the holes. The solid part of the cell is denoted

by

Y, therefore, the solid part of the domain can be

de®ned as

OE fx 2 Oj y x=E 2

Yg:

Also, we de®ne

SE

all[

cells

si :

i1

1

Having periodic microstructure does not mean that the

form and composition of the base cell cannot vary,but the

variations in the macroscopic scale are assumed to be smooth

enough.

It is also assumed that none of the holes vi intersect

the boundary G. (i.e. G\SE=;).

Now, considering the stress±strain and strain±displacement relations:

Fig. 6. Elasticity problem in a cellular body.

V fv 2 H1 OE 3

and

vjGd 0g;

and H1 is the Sobolev space2. The elastic constants of

the solid are assumed to have symmetry and coercivity

properties:

Eijkl Ejikl Eijlk Eklij ;

9a > 0 : EEijkl ekl aeij eij ;

Fig. 7. Base cell of the cellular body.

sEij EEijkl eEkl ;

eEkl

54
?
?

1 @uEk @uEl

;

2 @xl @xk

Find uE$VE, such that

OE

Eijkl

@uEk @vi

dO

@xl @xj

OE

fEi vi dO

Gt

ti vi dG

sE

pEi vi dS;

8v 2 VE ;

56

where

2

H1(OE) is de®ned as:

H1 OE   fw xjw x 2 L2 OE

and

@w x

2 L2 OE g;

@xi

where

L2 OE   fw xj

OE

Now, using the double-scale asymptotic expansion and

fact (3), Eq. (56) becomes

?
??
?
?

1 @u0 @ui 1 @u0k @u1k @vi @u0k @vi

Eijkl 2 k

E @gl @yj E @xl @yl @j

@yl @xj

OE

?? 0
?
? 1
? ?

@uk @u1k @vi

@uk @u2 @vi

E ? ? ?gdO

@xl @yl @xj

@xl @yl @yj

55

the virtual displacement equation can be constructed

as:

w x2 < 1

and x 2 OE g;

which assures the integrability of the functions and their derivatives.

8eij eij :

OE

fEi vi dO

Gt

ti vi dG

sE

pEi vi ds;

8v 2 VO ?

Y;

57

where

VO ?

Yjv :; yY ÿ periodic;

Y fv x; y; x; y 2 O ?

v smooth enough and vjGd 0g:

Similarly, we de®ne VO and V

Y as:

VO fv x defined in Ojv smooth enough and vjGd 0g:

V

Yj v y;

Y fv y defined in

Y ÿ periodic and smooth enoughg:

Introducing the following facts:

Fact (5). For a Y-periodic function C(y) when

E 4 0 we have

?x?

1

C

C ydY dO;

58

dO

E

jYj O

OE

Y

SE

C

?x?

E

dO

1

EjYj

O

s

C ydsdO;

59

and assuming that the functions are all smooth so that

when E 4 0 all integrals exist, and by equating the

terms with the same power of E we obtain:

1

jYj

O

Y

?

@u0k @vi

dY dG 0;

@yl @yi

8v 2 VO ?

Y;

?? 0
?
? ?

@uk @u1k @vi @u0k @vi

Eijkl

dY dO

@xl @yl @yj @yl @xj

Y

?
?
1

pi vi dS dO; 8v 2 VO ?

Y;

O jYj s

60
1

jYj

O

?

O

Eijkl

61
?? 0
?
? 1
? ? ?

@uk @u1k @vi

@uk @u2k @vi

dY dO

Eijkl

@xl @yl @xj

@yl @yl @yj

Y

?
?
1

fi vi dY dO

ti vi dG; 8v 2 VO ?

Y:

O jYj

Y

Gt

1

jYj

62

Now, as v is an arbitrary function we choose v = v(y)

(i.e. v $ V

Y). Then integrating by parts, applying the

divergence theorem to the integral in

Y, and using

periodicity from Eq. (60), we obtain:

? ?
?
??

@u0

1
@

ÿ

Eijkl k vi dY

jYj O

@yj

@yl

Y

@u0

Eijkl k nj vi dSgdO 0;

@yl

s

v being arbitrary results in:

?
?

@u0

@

ÿ

;

Eijkl k 0; 8y 2 Y

@yj

@yl

Eijkl

u0k

nj 0 on

@yl

s:

8v:

63
64
65

Considering fact (4) and Eq. (64) it is concluded that:

u0 x; y u0 x:

66

This means that the ®rst term of the asymptotic

expansion only depends on the macroscopic scale x.

Now, as v is an arbitrary function, if we choose

v = v(x) (i.e. v is only a function of x), then from

Eq. (61) it is concluded that:

?
?
1

pi dS vi xdO 0; 8v 2 VO ;

67

O jYj s

which implies that

s

pi x; ydS 0:

68

This means that the applied tractions have to be selfequilibrating. So the possible applied tractions are

restricted.

715

On the other hand, introducing Eq. (66) into Eq. (61)

and choosing v = v(y) yields

? 0
?

@uk @u1k @vi y

Eijkl

dY pi vi dS; 8v 2 V

Y:

@yj

@xl @yl

Y

s

69

Integrating by parts, using the divergence theorem and

applying the periodicity conditions on the opposite

faces of Y, it follows from Eq. (69) that:

?
? 0
??

@uk x @u1k

@

ÿ

Eijkl

vi dY

@xl

@yt

Y @yj

? 0
?

@uk x @u1k

Eijkl

vi nj dS

@yl

@xl

s

s

pi vi dS;

8v 2 V

Y:

70

Since v is arbitrary, it is concluded that

?
?
?
?

@u0 x

@

@u1

@

ÿ

on

Y;

Eijkl k

Eijkl k

@yj

@yj

@xl

@yl

Eijkl

@u0 x

@u1k

ÿEijkl k nj pi

@xl

@yl

on s:

71
72

Now, considering Eq. (62) and choosing v = v(x)

results in a statement of equilibrium in the macroscopic level:

?
? 0
? ?

@uk @u1k

1

@vi x

Eijkl

dO

dY

jYj

@xj

@x

@y

l

l

O

Y

?
?
1

fi dY vi xdO

O

ti vi xdG

G; 8v 2 VO :

jYj

O

Y

Gt

73

If in Eq. (62) we assume that v = v(y), this leads to:

? 1
?
?
?
1

@uk @u2k @vi y

Eijkl

dY dO

@yj

@xl @yl

O jYj

Y

?
?
1

fi vi ydY dO; 8v 2 V

74

Y;

O jYj

Y

or equivalently,

? 1
?

@uk @u2k @vi y

Eijkl

dY

@yj

@x1 @yl

Y

Y

fi vi ydY;

8v 2 V

Y;

75

which represents the equilibrium of the base cell in the

microscopic level.

The procedure followed so far can be applied for

higher terms of the expansion. However, in this case

the ®rst-order terms are enough. The macroscopic

mechanical behaviour is represented by u0, and u1 represents the
microscopic behaviour.

As we have noticed earlier, our goal is to ®nd the

homogenized elastic constants such that the equilibrium equation (or equivalently the equation of virtual

displacements) can be constructed in the macroscopic

system of coordinates. These homogenized constants

should be such that the corresponding equilibrium

equation re¯ects the mechanical behaviour of the

microstructure of the cellular material without explicitly using the parameter E. To accomplish this we consider Eq. (69) once again. As this equation is linear

with respect to u0 and p, we consider the two following

problems:

(i) Let w kl$V

Y be the solution of

Y

Eijpq

@w

wkl

p @vi y

dY

@yq @yj

Y

Eijkl

@vi y

dY;

@yi

8v 2 V

Y;

76

(ii) and let C $ V

Y be the solution of

Y

Eijkl

@Ck @vi y

dY

@yl @yj

S

pi vi ydY;

8v 2 V

Y;

77

where x plays the role of a parameter. It can be

shown that the solution u1 will be

u1i ÿwkl

i x; y

@u0k x

ÿ Ci x; y u˜ 1i x;

@xl

Introducing Eq. (78) into Eq. (73) yields

“

! #

@wkl

@u0k x @ui x

1

p

Eijkl ÿ Eijpq

dO

dY

@yq

@xl @xj

O jYj

Y

?

?

1

@Ck

@vi x

Eijkl

dY

dO

jYj

@xj

@y

l

O

Y

?
?
1

fi dY vi xdO

jYj

O

Y

Gt

ti vi xdG;

8v 2 VO :

Now, denoting

EH

ijkl x

tij x

1

jYj

Y

Y

Eijkl

Eijkl ÿ Eijpq

@Ck

dY;

@yl

!

@w

wkl

p

dY;

@yq

79
80
81

and

bi x

1

jYj

Y

fi dY;

Eq. (79) can be written as:

@u0k x @vi x

dO

@xl @xj

O

bi xvi xdO

O

Gt

tij x

@vi x

dO

@xj

ti xdG;

8v 2 VO :

83

This is very similar to the equation of virtual displacement, Eq. (56), and it represents the macroscopic equilibrium. EH

ijkl de®ned by Eq. (80) represents the

homogenized elastic constants. tij are average "residual" stresses within the cell due to the tractions p

inside the holes, and bi are the average body forces.

As we notice, the microscopic and macroscopic problems are not coupled and the solution of the elasticity

problem can be summarized as:

(i) Find w and C within the base cell by solving the

integral Eqs. (76) and (77) on the base cell.

(ii) Find DH

ijkl, tij and bi by using Eqs. (80)±(82).

(iii) Construct Eq. (83) in macroscopic coordinates.

If the whole domain of the cellular material comprises

a uniform cell structure, as well as uniform applied

tractions on the boundaries of the holes of the cells,

then it is only necessary to solve the microscopic

Eqs. (76) and (77) once. Otherwise these equations

must be solved for every point x of O.

78

where uÄ1i are arbitrary constants of integration

in y.

O

EH

ijkl

82

7. Conclusion and Final Remarks

In this ®rst part of a three paper review we have

focused on the theory of the homogenization method

for the computation of eective constitutive parameters of complex materials with a periodic microstructure. In the second part of this review we will

consider the motives for using the homogenization theory for topological structural optimization. In particular, the ®nite element formulation will be explained for

the material model based on a microstructure consisting of an isotropic material with rectangular voids.

Some examples will also be provided.

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