Master's Thesis

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Abstract

Fill it.

1 Introduction

2 Motivation

3 Method

The idea of asymptotic homogenization. In a repeating cell Y,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1}$$

where $C_{ijkl}(\underset{\sim}{x} + \underset{\sim}{u}\underset{\sim}{Y}) = C_{ijkl}(\underset{\sim}{x})$

$$\Rightarrow C_{ijkl}(x_1 + n_1Y_1\, x_2 + n_2Y_2\, x_3 + n_3Y_3) = C_{ijkl}(x_1, x_2, x_3) \tag{2}$$

$C_{ijkl}(\underset{\sim}{x})$ is Y-periodic

$$\underset{\sim}{y} = \tfrac{x}{\epsilon} \tag{3}$$

$$\Rightarrow g = g(\underset{\sim}{x}, \tfrac{x}{\epsilon}) = g(\underset{\sim}{x}\,\underset{\sim}{y}) \tag{4}$$

$\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ defines the domain of the composite Ω. The domain is composed of base cells of dimensions, $\varepsilon Y_1, \varepsilon Y_2, \varepsilon Y_3$ where $\underset{\sim}{y} = \frac{x}{\varepsilon}$

3.1 1D Elasticity

$$\sigma^\varepsilon = E^\varepsilon \tfrac{\partial u^\varepsilon}{\partial x} \tag{5}$$

$$\tfrac{\partial \sigma^\varepsilon}{\partial x} + \gamma^\varepsilon = 0 \quad E^\varepsilon\, \gamma^\varepsilon \rightarrow macroscopically uniform \tag{6}$$

Inside each cell,

$$E^\varepsilon(x, \frac{x}{\varepsilon}) \quad = E(y) \tag{7}$$

$$\gamma^\varepsilon(x, \frac{x}{\varepsilon}) \quad = \gamma(y) \tag{8}$$

Let

$$u^{\varepsilon}(x) = u^0 x, y + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + ... \tag{9}$$

$$\sigma^{\varepsilon}(x) = \sigma^0 x, y + \varepsilon \sigma^1(x,y) + \varepsilon^2 \sigma^2(x,y) + ... \tag{10}$$

3.2 Optimal Design of Elastic structures

$$\mathbf{b} \rightarrow \text{body forces}$$
$$\mathbf{t} \rightarrow \text{surface tractions}$$

Optimal choice of $\mathbb{C}_{ijkl} \in U_{ad} \leftarrow$ admissible set of elasticity ??
$\mathbb{C}_{ijkl}(\mathbf{x}) \forall \mathbf{x} \in \Omega$ has 21 independent components
$a_E(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbb{C}_{ijkl}\, \varepsilon_{kl}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{v}) d\mathbf{v} \rightarrow$ energy bilinear form
$L(\mathbf{v}) = \int_\Omega \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} ds \rightarrow$ load linear form.

Minimum compliance problem:

$$minimize \qquad L(\mathbf{v}), \tag{11}$$

$$subject\, to \qquad \mathbb{C}_{ijkl} \in \mathbb{U}_{ad} \tag{12}$$

$$a_E(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{U} \tag{13}$$

where $\mathbb{U} \rightarrow$ kinematically admissible displacements.
For optimal shape design:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = \chi(\mathbf{x}) \overline{\mathbb{C}}_{ijkl}, \quad \text{where } \overline{\mathbb{C}}_{ijkl} \rightarrow \text{stiffness matrix of the material} \tag{14}$$

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m, \\ 0 & \text{if } \mathbf{x} \in \Omega \backslash \Omega^m \end{cases} \tag{15}$$

where $\Omega^m \rightarrow$ part of the domain occupied by the material.
For sizing problem:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = h(\mathbf{x}) \overline{\mathbb{C}}_{ijkl} \tag{16}$$

$$\int_\Omega \chi(\mathbf{x}) d\mathbf{x} = V_f \tag{17}$$

$$\& \quad \int_\Omega h(\mathbf{x}) d\mathbf{x} = V_f. \tag{18}$$

where $h(x)$ is a sizing function.

Traditionally shape design problems are initiated in the following manner:

$$Ref\, doamin : \qquad \Omega_0 \in \mathbb{R}^3 \tag{19}$$

$$\underline{\phi} : \qquad \Omega_0 \rightarrow \phi(\Omega_0) \text{is a diffeomorphism.} \tag{20}$$

$$L(\mathbf{v}) \quad = \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| ds \tag{21}$$

$$\begin{aligned} a_E &= \int_\Omega \mathbb{C}_{ijkl}(\mathbf{x} \varepsilon_{kl}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl} \varepsilon_{kl}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \tag{22}$$

Now,

$$\begin{aligned}
\mathbb{C}_{ijkl}\varepsilon_{kl} &= \mathbb{C}_{ijkl}\frac{1}{2}(u_{k,l} + u_{l,k}) \\
&= \frac{1}{2}\mathbb{C}_{ijkl}u_{k,l} + \frac{1}{2}\mathbb{C}_{ijlk}u_{l,k} \\
&= \mathbb{C}_{ijkl}u_{k,l}
\end{aligned} \tag{23}$$

$$\begin{aligned}
a_E &= \int_{\Omega_0} \mathbb{C}_{ijkl}u_{k,l}(\mathbf{u})u_{i,j}(\mathbf{v})|det(D\underline{\phi}^{-1}|d\mathbf{x} \\
&= \int_{\Omega_0} \mathbb{C}_{ijkl}\frac{\partial u_k}{\partial \mathbf{x}_m}
\end{aligned} \tag{24}$$