Work and strain energy

The work done by surface tractions, body forces and point loads applied on a body is given by

$$W_E = \int_V \rho \boldsymbol{b} \cdot \boldsymbol{u} dV + \int_{\partial V} \boldsymbol{t} \cdot \boldsymbol{u} dS + \sum_i \boldsymbol{F}_i \cdot \boldsymbol{u}_i.$$

Work done *on the body* is considered negative.

As discussed earlier, *strain energy per unit volume* stored in an elastic body due to the stresses and strains generated is

$$U = \int_0^{\epsilon_{ij}} \sigma_{ij} d\epsilon_{ij}.$$

The *internal work* is thus given as

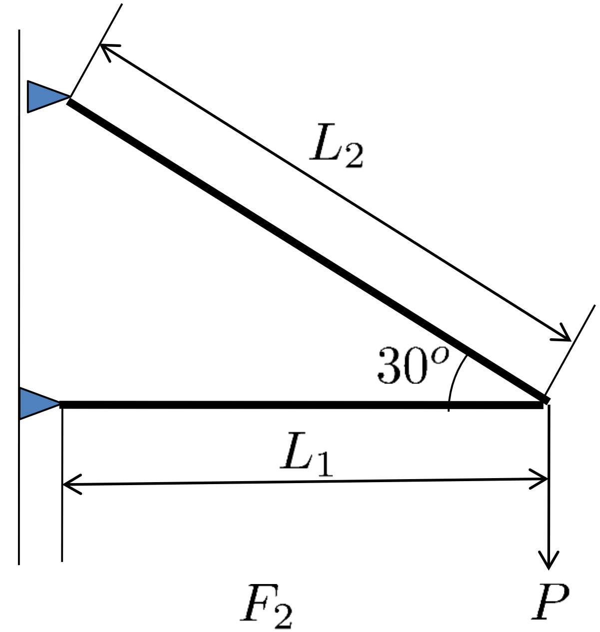
$$W_I = \int_V U dV.$$

The *potential energy* is defined as

$$\Pi = W_I - W_E.$$

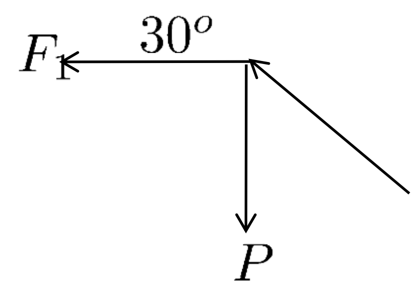
The uniaxial stress strain behaviour of the bars is given by

$$\sigma = \begin{cases} E\sqrt{\epsilon} & \epsilon \geq 0 \\ -E\sqrt{-\epsilon} & \epsilon \leq 0 \end{cases}$$



From the free body diagram of the joint, it is easy to show that

$$F_1 = -\sqrt{3}P \text{ and } F_2 = 2P.$$



The stresses and strains are consequently,

$$\sigma_1 = -\frac{\sqrt{3}P}{A_1}, \sigma_2 = \frac{2P}{A_2},$$

and

$$\epsilon_1 = -\frac{3P^2}{A_1^2E^2}, \epsilon_2 = \frac{4P^2}{A_2^2E^2}.$$

The strain energy of the structure is

$$U = \int_0^{\epsilon_1} -E\sqrt{-\epsilon_1}d\epsilon_1 + \int_0^{\epsilon_2} E\sqrt{\epsilon_2}d\epsilon_2 = \frac{2\sqrt{3}P^3}{A_1^3E^2} + \frac{2}{3}\left(\frac{8P^3}{A_2^3E^2}\right).$$

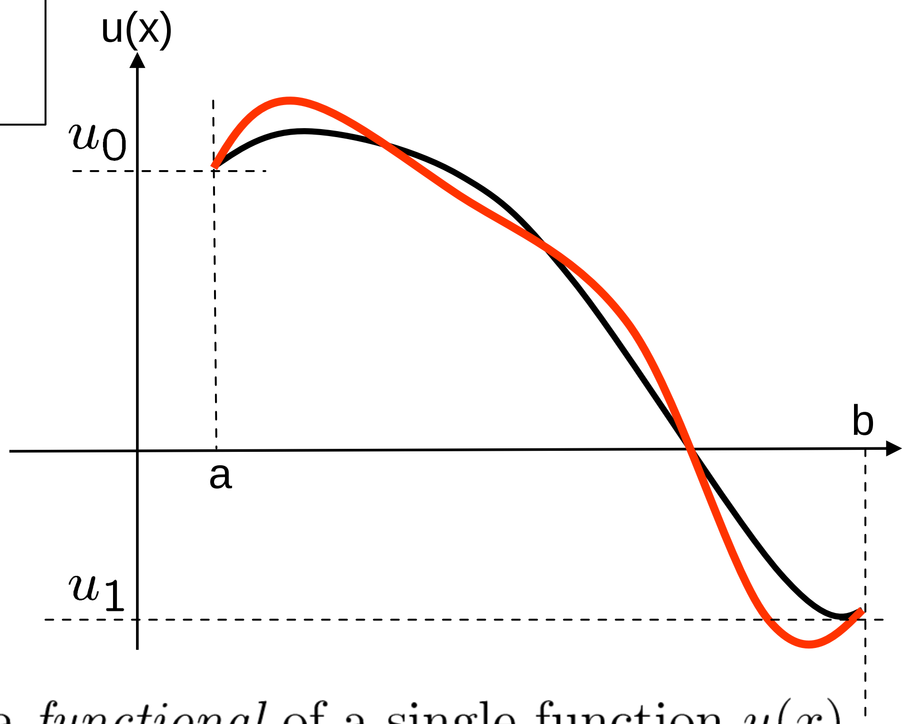
The total stored internal energy is

$$W_I = \int UdV = \frac{2\sqrt{3}P^3}{A_1^3E^2}A_1L_1 + \frac{2}{3}\left(\frac{8P^3}{A_2^3E^2}\right)A_2L_2,$$

so that

$$W_I = \frac{2}{3}\left[\left(\frac{3\sqrt{3}P^3L_1}{A_1^2E^2}\right) + \left(\frac{8P^3L_2}{A_2^2E^2}\right)\right].$$

Introduction to variational methods



Consider a function of a function or a *functional* of a single function $u(x)$

$$J[u] = \int_a^b F(x, u, u')dx$$

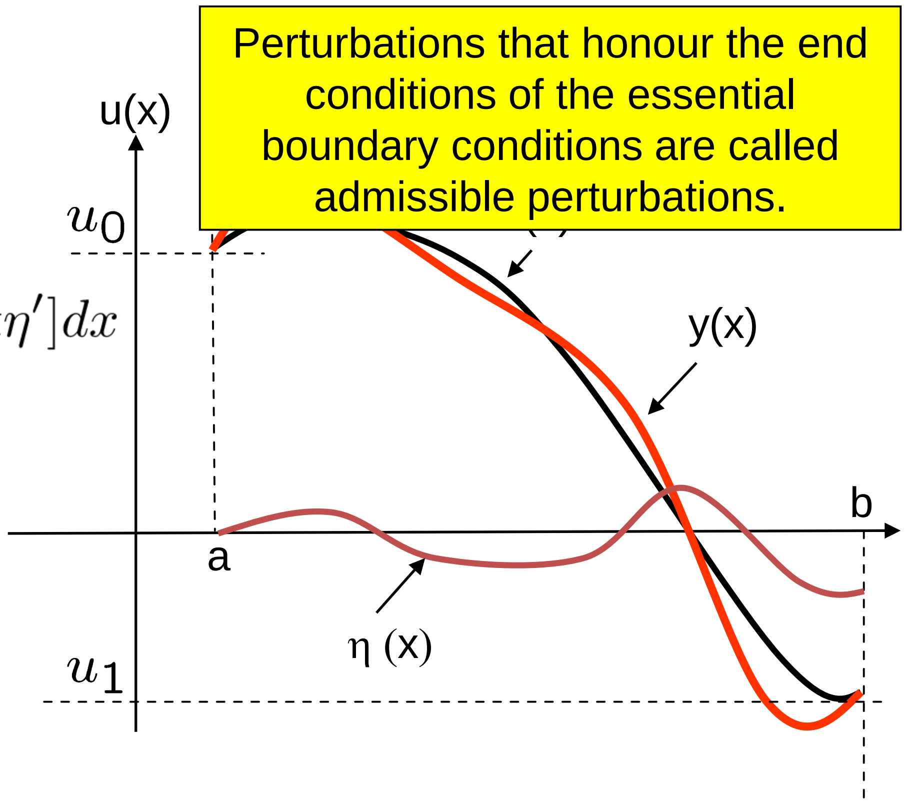
Additionally, essential boundary conditions are given as:

$$u(a) = u_0, u(b) = u_1$$

$$u(x) = y(x) + \epsilon\eta(x)$$

$$\eta(a) = \eta(b) = 0$$

$$J[y+\epsilon\eta] = \int_a^b F[x, y+\epsilon\eta(x), y' + \epsilon\eta']dx$$



Let $J[y+\epsilon\eta] = \phi(\epsilon)$
Since $\phi(\epsilon)$ is minimum at $\epsilon = 0$

$$\frac{d}{d\epsilon}[J(u+\epsilon\eta)]\Big|_{\epsilon=0} = 0$$

$$\begin{aligned}\phi'(\epsilon) \quad &= \\ &\int_a^b \left[F_u(x, y+\epsilon\eta, y'+\epsilon\eta') - \frac{d}{dx}F_{u'}(x, y+\epsilon\eta, y'+\epsilon\eta')\right]\eta(x)dx \\ &+F_{u'}(x, y+\epsilon\eta, y'+\epsilon\eta')\eta(x)|_a^b\end{aligned}$$

$$0 = \phi'(0) = \int_a^b \left[F_y(x,y,y') - \frac{d}{dx} F_{y'}(x,y,y') \right] \eta(x) dx$$

$\Rightarrow$

$$F_y(x,y,y') - \frac{d}{dx} F_{y'}(x,y,y') = 0$$ Euler equation

as $\eta(x)$ is arbitrary.

Examples: $J[u] = \int_a^b (1 + u'^2) dx = \min, u(a) = 0, u(b) = 1$

$\Rightarrow$

$$2u'' = 0$$

Again,

$$I[u] = \int_0^{\pi/2} [u'^2 - u^2] dx, u(0) = 0, u(\pi/2) = 1$$

is minimised by the curve $u = \sin x$.

Let us call

$$\delta J = \frac{d}{d\epsilon}[J(u+\epsilon\eta)]\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}[J(u+\epsilon\delta u)]\bigg|_{\epsilon=0}$$

the *first variation* of $J[u]$, identifying δu with η. Evidently,

$$\frac{d}{d\epsilon}[J(u+\epsilon\delta u)]\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}[J(u+\epsilon\delta u, u'+\epsilon\delta u')]\bigg|_{\epsilon=0},$$

which gives

$$\delta J = \frac{\partial J}{\partial u}\delta u + \frac{\partial J}{\partial u'}\delta u' = \frac{\partial J}{\partial \boldsymbol{u}}\cdot\delta\boldsymbol{u}.$$

where $\delta\boldsymbol{u}$ represents the vector of independent functions and shows that the variation of J is the *directional derivative* in the direction of $\delta\boldsymbol{u}$. Then evidently, the Euler's equations follow from

$$\delta J = 0.$$

Extending this definition, the second variation may be defined as

$$\delta^2 J = \frac{d}{d\epsilon}[\delta J(u+\epsilon\delta u)]\bigg|_{\epsilon=0}$$

The δ operator has properties similar to the differential operator, i.e.

$$\begin{aligned}
\delta(J_1 \pm J_2) &= \delta J_1 \pm \delta J_2 \\
\delta(J_1 J_2) &= \delta J_1 J_2 + J_1 \delta J_2 \\
\delta\left(\frac{J_1}{J_2}\right) &= \frac{\delta J_1 J_2 - J_1 \delta J_2}{J_2^2} \\
\delta J^n &= n J^{n-1} \delta J
\end{aligned}$$

Moreover, the *commutative* property is also easy to prove:

$$\frac{d}{dx}\delta u = \delta \frac{du}{dx},$$

and

$$\delta\left(\int_0^a u dx\right) = \int_0^a \delta u dx.$$

The concept can be extended to a functional of any number of functions of any number of independent variables. For example, consider

$$J[u, v] = \int_V F(x, y, u, v, u_x, v_x, u_y, v_y) dx dx.$$

Then, the vanishing of the first variation of J implies

$$\delta J = \delta_u J + \delta_v J = 0,$$

implying that

$$\delta J = \int_V \left\{ \frac{\partial F}{\partial u} \delta u + \frac{\partial F}{\partial u_x} \delta u_x + \frac{\partial F}{\partial u_y} \delta u_y + \frac{\partial F}{\partial v} \delta u + \frac{\partial F}{\partial v_x} \delta v_x + \frac{\partial F}{\partial v_y} \delta v_y \right\} dx dy.$$

Using the *divergence theorem* on say, the second term, we get

$$\int_V \frac{\partial F}{\partial u_x} \frac{\partial \delta u}{\partial x} dx dy = \int_{\partial V} \frac{\partial F}{\partial u_x} \delta u n_x dS - \int_V \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) \delta u dx dy.$$

In the above n_x is the component of the outward unit normal to the boundary ∂V.

Setting

$$\delta J = 0,$$

yields, after collectiong terms containing δu and δv,

$$\int_V \left\{ \left[\frac{\partial F}{\partial u} \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial u_y} \right) \right] \delta u + \right.$$
$$\left. \left[\frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \left(\frac{\partial F}{\partial v_x} \right) - \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial v_y} \right) \right] \delta v \right\} dxdy$$
$$+ \int_{\partial V} \left[\left(\frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y \right) \delta u + \left(\frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y \right) \delta v \right] ds = 0$$

Since u, v are specified on ∂V, $\delta u = \delta v = 0$ on ∂V. Applying the divergence theorem to all relevant terms and assuming at δu and δv are arbitrary functions, the Euler equations become:

$$\begin{aligned} \frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) &= 0 \\ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial v_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial v_y}\right) &= 0 \end{aligned}$$

Two kinds of boundary conditions on ∂V can be derived. Firstly *essential boundary conditions*

$$\delta u = \delta v = 0,$$

on δV, and *natural boundary conditions*

$$\begin{aligned} \frac{\partial F}{\partial u_x} n_x + \frac{\partial F}{\partial u_y} n_y &= 0 \\ \frac{\partial F}{\partial v_x} n_x + \frac{\partial F}{\partial v_y} n_y &= 0 \end{aligned}$$

A simple example of functionals involving $u(x,y)$ is

$$J[u] = \int_V (u_x^2 + u_y^2) dV$$

and $\bar{u} = f(x,y)$ on ∂V.

Application of the Euler equation yields

$$u_{xx} + u_{yy} = 0$$

which is the *Laplace equation.* Thus minimising the above funtional with the given essential boundary conditions is completely equivalent to solving the Laplace equation.

As an example, consider a bar of length L, fixed at the left end ($u(0) = 0$) and subjected to a distributed axial load $f(x)$ and a point load P at $x = L$. The potential energy of the system is easily shown to be

$$\Pi[u] = \int_0^L \left[\frac{EA}{2} \left(\frac{du}{dx} \right)^2 - fu \right] dx - Pu(L).$$

Let us see what is meant by $\delta\Pi = 0$.

$$\delta\Pi = \int_0^L \left(EA \frac{du}{dx} \frac{d\delta u}{dx} - f\delta u \right) dx - P\delta u(L),$$

which, after integration by parts becomes

$$\begin{aligned} \delta\Pi[u] \quad &= \quad \int_0^L \delta u \left[-\frac{d}{dx} \left(EA \frac{du}{dx} \right) - f \right] dx + \delta u(L) \left[\left(EA \frac{du}{dx} \right)_{x=L} - P \right] \\ &\quad -\delta u(0) \left(EA \frac{du}{dx} \right)_{x=0}. \end{aligned}$$

Thus, $\delta\Pi = 0$ implies

$$-\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f = 0 \text{ on } 0 < x < L,$$

and the natural boundary condition

$$EA\frac{du}{dx} - P = 0 \text{ at } x = L,$$

alongwith the essential boundary consition $u(0) = 0$.

For an elastic bar under uniaxial loading, note that

$$\frac{d}{dx}\left(EA\frac{du}{dx}\right) - f = 0$$

implies the equilibrium equation

$$\frac{d\sigma_{xx}}{dx} + f = 0$$

Principle of minimum potential energy

Define a *kinematically admissible* deformation field $\hat{u}_i(\boldsymbol{x})$ in a solid as a field that satisfies the essential boundary consition on ∂V_u and is everywhere continuous. Further assume that this field is differentiable as well so that strain field may be computed as

$$\hat{\epsilon}_{ij} = \frac{1}{2}\left(\hat{u}_{i,j} + \hat{u}_{j,i}\right).$$

Note that this is not necessarily the actual displacement field in the solid. The potential energy for any kinematically admissible field is given as

$$\Pi[\hat{\boldsymbol{u}}] = \int_V U(\hat{\boldsymbol{u}})dV - \int_V \rho b_i \hat{u}_i dV - \int_{\partial V} t_i \hat{u}_i dS.$$

The principle of *minimum potential energy* states that $\Pi[\hat{\boldsymbol{u}}]$ is a minimum for $\hat{\boldsymbol{u}} = \boldsymbol{u}$, where $\boldsymbol{u}$ is the actual displacement field.

As an example, consider a cylinder subjected to uniform pressure p at the top face and sitting on a rigid frictionless base at $x_3 = 0$. Let us assume that the kinematically admissible field

$$\hat{u}_1 = \lambda_1 x_1, \hat{u}_2 = \lambda_2 x_2, \hat{u}_3 = \lambda_3 x_3,$$

is the solution to this problem. Then,

$$\hat{\epsilon}_{11} = \lambda_1, \hat{\epsilon}_{22} = \lambda_2, \hat{\epsilon}_{33} = \lambda_3.$$

The strain energy density turns out to be (assuming isotropic elasticity)

$$U = \frac{E}{2(1+\nu)} \left\{ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \frac{\nu}{1-2\nu} (\lambda_1 + \lambda_2 + \lambda_3)^2 \right\}.$$

Further,

- On the sides $\boldsymbol{t} = 0$,
- On the bottom face, $t_1 = t_2 = 0$ and
- on the top $t_3 = -p$.

Thus, the potential energy can be expressed as

$$\Pi = \int_V U dV + \int \int_A \lambda_3 L p dA,$$

leading to,

$$\Pi = \frac{ALE}{2(1+\nu)} \left\{ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \frac{\nu}{1-2\nu}(\lambda_1 + \lambda_2 + \lambda_3)^2 \right\} + A\lambda_3 Lp.$$

For the actual fields

$$\frac{\partial \Pi}{\partial \lambda_i} = 0 \text{ for } i \in [1, 3].$$

The equations to be solved are

$$\begin{aligned}
\frac{ALE}{2(1+\nu)}\left\{2\lambda_1 + \frac{2\nu}{1-2\nu}(\lambda_1+\lambda_2+\lambda_3)\right\} &= 0 \\
\frac{ALE}{2(1+\nu)}\left\{2\lambda_2 + \frac{2\nu}{1-2\nu}(\lambda_1+\lambda_2+\lambda_3)\right\} &= 0 \\
\frac{ALE}{2(1+\nu)}\left\{2\lambda_3 + \frac{2\nu}{1-2\nu}(\lambda_1+\lambda_2+\lambda_3)\right\} + ALp &= 0
\end{aligned}$$

the solution to which is

$$\lambda_1 = \lambda_2 = \nu p/E, \lambda_3 = p/E.$$

The final displacement field looks plausible showing that our initial guess of the kinematically admissible field was good.

Castigliano's theorem follows from the principle of minimum potential energy. Consider that the body is subjected to point loads $\boldsymbol{F}_i$ only and the displacements at the points of application of these loads is $\boldsymbol{u}_i$.

Thus, the potential energy can be written as

$$\Pi = W_I - \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \boldsymbol{u}_i,$$

so that

$$\delta\Pi = \frac{\partial W_E}{\partial \boldsymbol{u}_i} \cdot \delta\boldsymbol{u}_i - \sum_{i=1}^{N} \boldsymbol{F}_i \cdot \delta\boldsymbol{u}_i = 0.$$

As the variations in $\boldsymbol{u}_i$ are arbitrary, the above implies that

$$\frac{\partial W_E}{\partial \boldsymbol{u}_i} = \boldsymbol{F}_i,$$

which is the Castigliano theorem.

For an elastic solid, the potential energy is given by

$$\Pi[\boldsymbol{u}] = \int_V \left(\frac{1}{2}\sigma_{ij}\epsilon_{ij} - \rho b_i u_i \right) dV - \int_{\partial V_t} \hat{t}_i u_i dS.$$

with essential boundary conditions $u_i = \hat{u}_i$ on ∂V_u. Further, for an isotropic material,

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda\delta_{ij}\epsilon_{kk}.$$

Further, using the strain displacement relation

$$\epsilon_{ij} = (1/2)(u_{i,j} + u_{j,i}),$$

yields

$$\Pi[\boldsymbol{u}] = \int_V \left[\frac{\mu}{2}(u_{i,j} + u_{j,i})^2 + \frac{\lambda}{2}u_{i,i}u_{k,k} - \rho b_i u_i \right] dV - \int_{\partial V_t} \hat{t}_i u_i dS.$$

The first variation of the above with $\delta u_i = 0$ on ∂V_u gives

$$\delta\Pi = \int_V \left[\frac{\mu}{2}(\delta u_{i,j} + \delta u_{j,i})(u_{i,j} + u_{j,i}) + \frac{\lambda}{2}\delta u_{i,i}u_{k,k} - \rho b_i \delta u_i \right] dV - \int_{\partial V_t} \hat{t}_i \delta u_i dS.$$

Use divergence theorem to show that

$$\begin{aligned}\int_V \delta u_{i,j}(u_{i,j}+u_{j,i})dV &= -\int_V \delta u_i(u_{i,j}+u_{j,i})_{,j}dV \\ &+ \int_{\partial V} \delta u_i(u_{i,j}+u_{j,i})n_j dS.\end{aligned}$$

Further,

$$\begin{aligned}0 &= \int_V \left[-\mu(u_{i,j}+u_{j,i})_{,j} - \lambda u_{k,ki} - \rho b_i\right]\delta u_i dV \\ &+ \int_{\partial V} \left[\mu(u_{i,j}+u_{j,i}) + \lambda u_{k,k}\delta_{ij}\right] n_j \delta u_i dS - \int_{\partial V_t} \delta u_i \hat{t}_i dS\end{aligned}$$

Recognising that

$$\mu(u_{i,j}+u_{j,i}) + \lambda u_{k,k}\delta_{ij} = \sigma_{ij},$$

we get

$$\int_V \left[-\mu(u_{i,j}+u_{j,i})_{,j} - \lambda u_{k,ki} - \rho b_i\right]\delta u_i dV + \int_{\partial V_t} \delta u_i(t_i - \hat{t}_i)dS = 0.$$

Arbitrariness of δu_i then leads to the *Navier's equation* of elasticity

$$\mu(u_{i,j} + u_{j,i})_{,j} + \lambda u_{k,ki} + \rho b_i = 0 \text{ in } V, \text{ and}$$

$$t_i = \hat{t}_i \text{ on } \partial V_t.$$

Recall that the Navier's equation is the equilibrium equation expressed in terms of the displacements.