PART II

HOMOGENIZATION METHOD IN THE PHYSICS OF COMPOSITE MATERIALS

Homogenization deals with the partial differential equations of physics in heterogeneous materials with a periodic structure when $\varepsilon \searrow 0$. (ε is the characteristic length of the period). Heuristically, the method is based on the consideration of two length scales associated with the microscopic and macroscopic phenomena.

Chapter 5 has an introductory character. Both formal expansions and proofs of the convergence are given. In chapter 6 to 8 we study some problems in mechanics and electromagnetism. Some results are only formal and the problem of the convergence is open.

CHAPTER 5

HOMOGENIZATION OF SECOND ORDER EQUATIONS

1.- Formal expansion - Elliptic equations of the divergence type often appear in physics. They are of the form

$$(1.1) \qquad -\frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right) = f(x) \qquad : \qquad a_{ij} = a_{ji}$$

or equivalently

$$(1.2) \qquad -\frac{\partial p_i}{\partial x_i} = f \qquad ; \qquad p_i = a_{ij}\,\frac{\partial u}{\partial x_j}$$

Equation (1.1) is, for instance the equation of electrostatics, magnetostatics or time-independent heat transfer. The function u is the electric potential, magnetic potential or temperature respectively, and $\underline{p}$ is the electric displacement, magnetic induction or heat flux, resp. the matrix $a_{ij}(x)$ is the dielectric constant, magnetic permeability or thermal conductivity, resp. and f is a given source term.

The symmetric matrix $a_{ij}(x)$ is a physical property of the material. For a homogeneous material, a_{ij} does not depend on x. On the other hand, if the material is not homogeneous, a_{ij} effectively depends on x. For materials with a <u>periodic structure</u>, such as superposition of sheets of different materials, or homogeneous materials with holes filled by another material, $a_{ij}(x)$ is a periodic function of the space variables. In certain cases, the length of the period is very small with respect to the other lengths appearing in the problem. In these cases, one may think that the solution u of (1.1) is approximately the same as the corresponding solution for a "homogenized" material with constant matrix a^h_{ij}. The homogenization method studies this problem and shows the existence and some properties of the matrix a^h_{ij}.

We now study the simplest problem in this connection in a formal way. A proof of the convergence will be given in sect. 4, and some extensions to more complicated problems will be given in the sequel.

Let Ω be a bounded domain of the space R^N of coordinates x_i, fig. 1. We shall consider it as a piece of a heterogeneous material defined as follows.

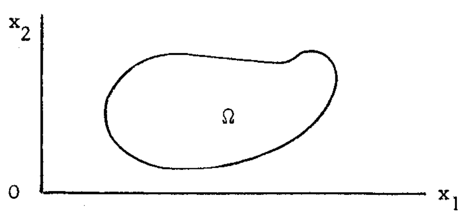


fig. 1

We consider, in the space R^N of coordinates y_i, a fixed parallelepiped Y (fig. 2) of edges y_i^o, as well as the parallelepipeds obtained by translation of length $n_i y_i^o$ (n_i integer) in the direction of the axis.

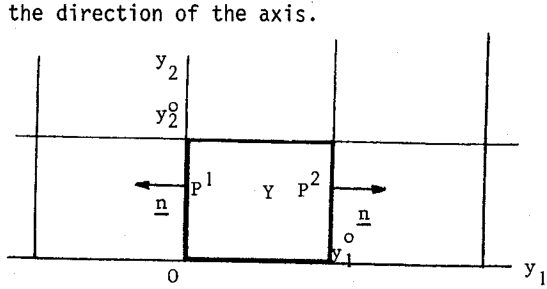


fig. 2

We now consider the Y-periodic, smooth real functions

(1.3) $$a_{ij}(y) = a_{ji}(y)$$

such that there exists $\gamma > 0$ with

(1.4) $$a_{ij}(y)\, \xi_i\, \xi_j \geqslant \gamma\ \xi_i\, \xi_i \qquad \forall\, y \in Y\ ,\ \underline{\xi} \in R^N$$

We then define the functions

(1.5) $$a^{\varepsilon}_{ij}(x) = a_{ij}(\tfrac{x}{\varepsilon})$$

where ε is a real positive parameter, say, $\varepsilon \in]\ 0,\ \varepsilon_0]$. Note that the functions $a^{\varepsilon}_{ij}(x)$ are εY-periodic in x; where the period εY is the parallelepiped of edges $\varepsilon\ y^0_i$.

Then, if f(x) is a given smooth function defined on Ω, we consider the boundary value problem

(1.6) $$-\frac{\partial}{\partial x_i}(a^{\varepsilon}_{ij}(x)\ \frac{\partial u^{\varepsilon}}{\partial x_j}) = f(x) \qquad \text{in } \Omega$$

(1.7) $$u^{\varepsilon}\Big|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega$$

<u>Remark 1.1</u> - For fixed $\varepsilon > 0$, u^{ε} exists and is unique. This is easily seen, because (1.4) is an ellipticity condition. The existence and uniqueness of u^{ε} may be studied as in chap. 3 sect. 1. In fact, <u>the abstract formulation of (1.6), (1.7) is : find $u^{\varepsilon} \in H^1_0(\Omega)$ such that</u>

(1.8) $$\int_{\Omega} a^{\varepsilon}_{ij}\ \frac{\partial u^{\varepsilon}}{\partial x_i}\,\frac{\partial v}{\partial x_j}\, dx = \int_{\Omega} fv\ dx \qquad \forall\ v \in H^1_0(\Omega).$$

The coerciveness of the form in the left hand side of (1.8) is evident by (1.4), for in fact,

$$\int_{\Omega} \frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_i}\, dx$$

may be taken as norm^2 of $H^1_0(\Omega)$ (see chap. 3, prop. 1.1). Moreover, if f, a_{ij} and $\partial\Omega$ are smooth, standard regularity theory for elliptic equations shows that u^{ε} is a smooth function. ∎

As in (1.2), we shall define $\underline{p}^{\varepsilon}$ as the vector with components

(1.9) $$p^{\varepsilon}_i(x) \equiv a^{\varepsilon}_{ij}(x)\ \frac{\partial u^{\varepsilon}}{\partial x_j}(x) \quad .$$

Now, we search for an asymptotic expansion of $u^{\varepsilon}(x)$ as a function of ε for $\varepsilon \searrow 0$. The heuristic device is to suppose that u^{ε} has a two-scale expansion of the form

(1.10) $\quad u^{\varepsilon}(x) = u^{o}(x) + \varepsilon u^{1}(x , y) + \varepsilon^{2} u^{2}(x , y) + \dots \quad ; \quad y = x/\varepsilon$

where the functions $u^{i}(x , y)$ are Y-periodic in the variable y. This means that we postulate the existence of smooth functions $u^{i}(x , y)$ defined for $x \in \Omega$, $y \in R^{N}$, independent of ε , Y-periodic in the variable y such that for $y = x/\varepsilon$, the right hand side of (1.10) is an asymptotic expansion of $u^{\varepsilon}(x)$ (as well as its derivatives).

Remark 1.2 - The function $u^{\varepsilon}(x)$ is defined on $\Omega \times]0, \varepsilon_0]$. In fact (1.10) means that there exist $u^{i}(z , y)$ defined on $\Omega \times R^{N}$, Y-periodic in y such that by taking

(1.11)
$$\begin{cases} z = x \quad , \quad y = x/\varepsilon \\ \dfrac{\partial}{\partial x_i} = \dfrac{\partial}{\partial z_i} + \dfrac{1}{\varepsilon}\dfrac{\partial}{\partial y_i} \end{cases}$$

the right hand side of (1.10) becomes a uniform expansion of $u^{\varepsilon}(x)$ and its derivatives. In practice, the use of the variable z is a little cumbersome, and we shall use only x and y, with

(1.12)
$$\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}$$

where it is understood that the total dependence on x is obtained directly and through the variable y. ■

Now, we explain the physical meaning of the expansion (1.10). Let us consider a term

(1.13) $\quad u^{i}(x , y) \quad , \quad$ Y-periodic in y , $\quad y = \dfrac{x}{\varepsilon}$.

Let us also consider $u^{i}(x , x/\varepsilon)$ with small ε . It is clear that the dependence with respect to the variable x/ε is periodic with period εY. (see fig. 3). Let us compare the values of $u^{i}(x , x/\varepsilon)$ at two points P^1 and P^2 homologous by periodicity corresponding to two adjacent periods. By periodicity, the dependence in x/ε is the same, and the dependence on x is almost the same because the distance P^1P^2 is small and u^{i} is a smooth function. On the other hand, let P^3 be a point homologous to P^1 by periodicity, but located far from P^1. The dependence of u^{i} on y is the same, but the dependence on x is very different because P^1 and P^2 are not near each other. Finally, we compare the values of u^{i} at two different points P^1 and P^4 of the same period. The dependence on x is almost the same because P^1 and P^2 are near from one another, but the dependence on y is very different because P^1 and P^4 are not homologous by periodicity (in fact, the

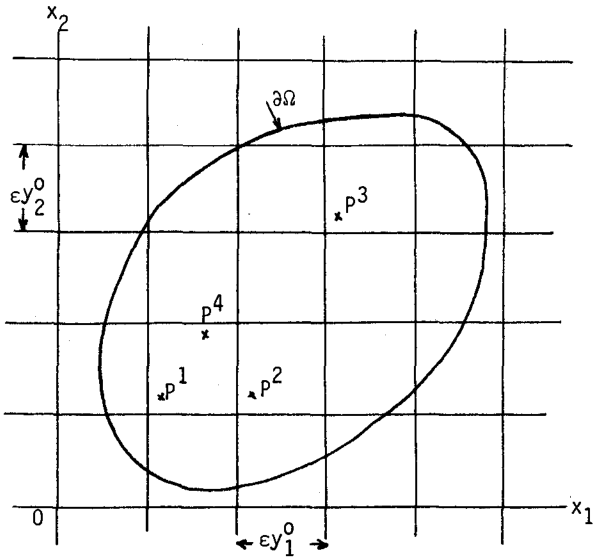


fig. 3

distance P^1P^4 is "large" when measured with the variable y !). Consequently, (1.13) means that the u^i takes values that are almost the same in neighbouring periods, but very different in distant periods. Such functions will be called "locally periodic".

Remark 1.3 - The preceeding considerations show that it is natural to search for u^ε in the form of the expansion (1.10). In fact, u^ε depends on the (periodic) coefficients a_{ij}^ε and on $\partial\Omega$ and f. It is natural to search for u^ε depending on x in two different forms. Firstly, periodically of period εY, secondly in an aperiodic fashion. In any case, the expansion (1.10) is heuristic and it fails near the boundary $\partial\Omega$, where aperiodic phenomena are preponderant and a boundary layer arises.∎

Remark 1.4 - In (1.10) we postulate that u^o depends only on x. This is a particular case of $u^o(x , y)$ (constant with respect to y). If we start with $u^o(x , y)$, the subsequent treatment of the problem shows that u^o is constant with respect to to y. In fact (1.10) means that u^ε is the smooth function $u^o(x)$ plus a little highly oscillating term (fig. 4). It is not difficult to see that u^ε must have this form (by considering the one-dimensional case, for instance)∎

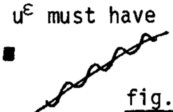


fig. 4

The following step of our study is to expand grad u^ε and p^ε according to

(1.10) and (1.12). We have

$$\frac{du^\varepsilon}{dx_i} = \left(\frac{\partial u^o}{\partial x_i} + \frac{\partial u^1}{\partial y_i}\right) + \varepsilon\left(\frac{\partial u^1}{\partial x_i} + \frac{\partial u^2}{\partial y_i}\right) + \varepsilon^2 \ldots. \tag{1.14}$$

$$p_i^\varepsilon(x) = p_i^o(x\ ,\ y) + \varepsilon p_i^1(x\ ,\ y) + \varepsilon^2 \ldots \equiv a_{ij}(y)\frac{du^\varepsilon}{dx_j} \tag{1.15}$$

$$\left\{\begin{aligned} p_i^o(x\ ,\ y) &= a_{ij}(y)\left(\frac{\partial u^o}{\partial x_j} + \frac{\partial u^1}{\partial y_j}\right) \\ p_i^1(x\ ,\ y) &= a_{ij}(y)\left(\frac{\partial u^1}{\partial x_j} + \frac{\partial u^2}{\partial y_j}\right) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right. \tag{1.16}$$

and all the terms are Y-periodic in y and the expansions hold for $y = x/\varepsilon$.

Next, we write equation (1.6) in the form

$$-\frac{d}{dx_i} p_i^\varepsilon(x) = f \Leftrightarrow \left(-\frac{\partial}{\partial x_i} - \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)(p_i^o + \varepsilon\ p_i^1 + \ldots) = f(x) \tag{1.17}$$

and we expand it in powers of ε . We must have an identity for any small ε, and in in consequence, the coefficients of the successive powers of ε must be zero, i.e.

$$\varepsilon^{-1} \Rightarrow \frac{\partial p_i^o}{\partial y_i} = 0 \Leftrightarrow \frac{\partial}{\partial y_i}\left[a_{ij}(y)\left(\frac{\partial u^o}{\partial x_j}(x) + \frac{\partial u^1}{\partial y_j}(x\ ,\ y)\right)\right] = 0 \tag{1.18}$$

$$\varepsilon^o \Leftrightarrow -\frac{\partial p_i^o}{\partial x_i} - \frac{\partial p_i^1}{\partial y_i} = f \tag{1.19}$$

and so on. As we shall see soon, (1.18) and (1.19) are equations for the microscopic (or local) and macroscopic (or homogenized) behaviour of u^ε respectively. It is worthwhile in this connection to recall that, according to (1.10), x and y are two independent variables.

Let us begin by studying (1.19). We consider the operator "average" or "mean" defined on any Y-periodic function $\phi(y)$ by

$$\tilde{\phi} = \frac{1}{|Y|}\int_Y \phi(y)\ dy \tag{1.20}$$

where $|Y|$ is the measure of Y. It is clear that $\tilde{\phi}$ does not depend on y ; moreover, if ϕ depends also on the variable x, the mean operator commutes with differenciation with respect to x.

By applying the operator $\sim$ to (1.19), we have

$$-\frac{\partial \tilde{p}_i^o}{\partial x_i} - \left(\frac{\partial p_i^1}{\partial y_i}\right)^{\sim} = \tilde{f} = f \tag{1.21}$$

(Note that f(x) is a function of x only, it may be considered as function of x and y, constant with respect to y, and we have the second equality in (1.21).

On the other hand, by using the divergence theorem we have

$$\left(\frac{\partial p_i^1}{\partial y_i}(x\ ,\ y)\right)^{\sim} = \frac{1}{|Y|}\int_Y \frac{\partial p^1}{\partial y_i}\,dy = \frac{1}{|Y|}\int_{\partial Y} n_i\ p_i(x\ ,\ y)\ dS \tag{1.22}$$

where n is the outer unit normal to the boundary of Y, ∂Y. But $p_i(x\ ,\ y)$ is Y-periodic in y, in consequence, the integral on two opposite faces of ∂Y takes opposite values because $\underline{p}(x\ ,\ y)$ (resp. $\underline{n}(y)$) takes the same (resp. opposite) values in homologous points such as P^1, P^2 (see fig. 2). We then have

$$\left(\frac{\partial p_i^1}{\partial y_i}\right)^{\sim} = 0 \tag{1.22}$$

and (1.21) becomes

$$-\frac{\partial \tilde{p}_i^o}{\partial x_i}(x) = f(x) \tag{1.23}$$

Note that p_i^o is a function of x and y but its mean value depends only on x. In consequence, <u>(1.23) is a macroscopic equation in x</u>. <u>Moreover, (1.23) shows that $\tilde{\underline{p}}^o$ (mean value of the first term of the expansion (1.15)) satisfies an equation analogous to (1.2)</u>.

The next step is to search for a relation between $\tilde{\underline{p}}^o$ and u^o in order to obtain an equation of the type (1.1). This will be made by using the local equation (1.18).

2.- <u>Study of the local problem</u> - We write the local equation (1.18) in the form

$$-\frac{\partial}{\partial y_i}\left(a_{ij}(y)\ \frac{\partial u^1}{\partial y_j}\right) = \frac{\partial u^o}{\partial x_j}\ \frac{\partial a_{ij}}{\partial y_i}(y) \qquad \text{Y-periodic} \tag{2.1}$$

and we consider it as an equation in the unknown $u^1(y)$; u^o is considered for the time being as known. Of course, u^o and u^1 depend on the <u>parameter x.</u> The right hand side of (2. 1) is then known, and (2.1) is in fact an elliptic equation in $u^1(y)$; moreover, $u^1(y)$ must be Y-periodic, and this condition plays the role of boundary conditions. We shall see that $u^1(y)$ exists and is unique (up to an additive constant).

In order to study this problem, we introduce the two spaces of Y-periodic functions:

$$H_Y = \{ u \in L^2_{loc}(R^N) \ ; \ \ u \text{ is } Y\text{-periodic} \}$$

$$V_Y = \{ u \in H^1_{loc}(R^N) \ ; \ \ u \text{ is } Y\text{-periodic} \}$$

which are Hilbert spaces for the scalar products.:

$$(2.2) \qquad \begin{aligned} (u \ , v)_{H_Y} &= \int_Y u \, v \, dy \\ (u \ , v)_{V_Y} &= \int_Y \left(\frac{\partial u}{\partial y_i} \frac{\partial v}{\partial y_i} + u \, v\right) dy \end{aligned}$$

Remark 2.1 - A Y-periodic function defined on R^N does not belong to $L^2(R^N)$ because its "norm" is infinite. Then, H_Y, V_Y are spaces of functions whose restrictions to any bounded domain belong to L^2 or H^1 of this domain. This is the meaning of L^2_{loc} , H^1_{loc}. It is immediate to see that H_Y, V_Y are complete (and thus Hilbert spaces) for the norms associated to (2.2). In fact, Y-periodic functions may be considered as functions defined on a period Y only. In this case H_Y may be identified with $L^2(Y)$ and V_Y may be identified with the space of the functions of $H^1(Y)$ such that the traces on the opposite faces of ∂Y are the same.∎

As a consequence of the Rellich theorem, and chap. 2, (6.12), we have :

Proposition 2.1 - If H_Y is identified with its dual,

$$V_Y \subset H_Y \subset V_Y'$$

with dense and compact embeddings.

The variational formulation of (2.1) is

Find $u^1 \in V_Y$ such that

$$(2.3) \qquad \int_Y a_{ij}(y) \frac{\partial u^1}{\partial y_j} \frac{\partial v}{\partial y_i} \, dy = \frac{\partial u^o}{\partial x_j} \int_Y \frac{\partial a_{ij}}{\partial y_i}(y) \, v \, dy \qquad \forall \, v \in V_Y$$

The equivalence of (2.3) and (2.1) is easily proved. If we multiply (2.1) by a test function $v \in V_Y$ and we integrate in Y (note that this is the scalar product in H_Y) we obtain (2.3) by using the following formula of integration by parts:

$$(2.4) \qquad \int_Y \frac{\partial}{\partial y_i}\left(a_{ij} \frac{\partial u^1}{\partial y_j}\right) v \, dy + \int_Y a_{ij} \frac{\partial u^1}{\partial y_j} \frac{\partial v}{\partial y_i} \, dy = \int_{\partial Y} n_i \, a_{ij} \frac{\partial u^1}{\partial y_j} \, v \, dS = 0$$

where the surface integral in the right hand side vanishes by periodicity, as in (1.22). Conversely, if u^1 satisfies (2.3), we may use (2.4) again (note that by interior regularity theory for elliptic equations $u^1 \in H^2_{loc}$) to obtain

$$\int_Y \left[\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial u^1}{\partial y_j}\right) + \frac{\partial u^0}{\partial x_j}\frac{\partial a_{ij}}{\partial y_i}\right] v \, dy = 0 \qquad \forall\, v \in V_Y$$

which implies (2.1).

To study (2.3), we note that the form in the left hand side is not coercive on V_Y, but the form

$$(2.5) \qquad b(u^1 , v) \equiv \int_Y a_{ij}\frac{\partial u^1}{\partial y_j}\frac{\partial v}{\partial y_i}\, dy + \int_Y u^1 \, v \, dy$$

is coercive on V_Y by virtue of the ellipticity condition (1.4). This problem recalls the Neumann problem (2.6) of chap. 3 and will be solved in an analogous manner. If B is the operator associated with the form b according to the first representation theorem, we evidently have

$$(2.6) \qquad B = A + I \qquad ; \qquad A = -\frac{\partial}{\partial y_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)$$

on H_Y (the periodicity conditions are included in H_Y and they are not written explicitly). Moreover, by writing (2.1) in the form

$$(2.7) \qquad (B - I)\, u^1 = \frac{\partial u^0}{\partial x_j}\frac{\partial a_{ij}}{\partial y_i}$$

we are in the framework of chap. 2, (6.13). The necessary and sufficent condition for (2.7) to be solvable is that the right hand side is orthogonal in H_Y to the solutions of

$$(2.8) \qquad (B - I)w = 0$$

But if w is a solution of (2.8), by multiplying (2.8) by w we have

$$0 = b(w , w) - (w , w)_{H_Y} = \int_Y a_{ij}(y)\frac{\partial w}{\partial y_i}\frac{\partial w}{\partial y_j}\, dy$$

which implies w = cost. by (1.4). Consequently, the compatibility condition for equation (2.7) is

$$\frac{\partial u^0}{\partial x_j}\int_Y \frac{\partial a_{ij}}{\partial y_i}(y)\, dy = 0$$

which is in fact satisfied by the periodicity of a_{ij}:

$$\int_Y \frac{\partial a_{ij}}{\partial y_i}\, dy = \int_{\partial Y} n_i \, a_{ij}\, dS = 0$$

Consequently, $u^1(y)$ (x is always a parameter) is determined up to an additive constant. Of course, u^1 is uniquely determined if we impose that its mean value must be zero. Moreover, let w^k be the unique solution of

$$(2.9)\qquad \begin{cases} \text{Find } w^k \in V_y \quad ; \quad \text{with } \tilde{w}^k = 0 \\ \int_Y a_{ij} \frac{\partial w^k}{\partial y_j} \frac{\partial v}{\partial y_i}\, dy = \int_Y \frac{\partial a_{ik}}{\partial y_i} v\, dy \qquad \forall\, v \in V_k \end{cases}$$

Then, by virtue of the linearity of problem (2.3), we have :

$$(2.10)\qquad u^1(x\ ,\ y) = \frac{\partial u^o(x)}{\partial x_k} w^k(y) + C(x)$$

where the term C(x) is the additive constant (function of the parameter x). As a result, we have

Proposition 2.1 - If $u^o(x)$ is known, $u^1(x\ ,\ y)$ is determined up to an additive function of x. It has the form (2.10), where w^k is uniquely defined by (2.9).

3.- Formulae for the homogenized coefficients and their properties - We go back to equation (1.23). Our aim is to find a relation between $\tilde{p}^o(x)$ and $u^o(x)$. This is easily obtained from (1.16) and proposition 2.1 :

$$(3.1)\qquad p_i^o(x\ ,\ y) = a_{ij}(y) \left(\frac{\partial u^o(x)}{\partial x_j} + \frac{\partial u^1(x\ ,\ y)}{\partial y_j}\right) =$$

$$= a_{ij}(y) \left[\frac{\partial u^o(x)}{\partial x_j} + \frac{\partial u^o(x)}{\partial x_k} \frac{\partial w^k(y)}{\partial y_j}\right] = \left[a_{ik}(y) + a_{ij}(y) \frac{\partial w^k(y)}{\partial y_j}\right] \frac{\partial u^o(x)}{\partial x_k}$$

and by applying the average operator defined by (1.20)

$$(3.2)\qquad \tilde{p}_i^o(x) = \left[a_{ik}(y) + a_{ij}(y) \frac{\partial w^k(y)}{\partial y_j}\right]^{\sim} \frac{\partial u^o(x)}{\partial x_k} \quad .$$

We then have :

Proposition 3.1 - $\tilde{p}^o(x)$ is related to $u^o(x)$ by

$$(3.3)\qquad \tilde{p}_i^o(x) = a_{ik}^h \frac{\partial u^o(x)}{\partial x_k}$$

where the constant coefficients a_{ij}^h (the "homogenized "coefficients) are

$$(3.4)\qquad a_{ik}^h = \left[a_{ik}(y) + a_{ij}(y) \frac{\partial w^k(y)}{\partial y_j}\right]^{\sim} \equiv \left[a_{ij}(y) \left(\delta_{jk} + \frac{\partial w^k(y)}{\partial y_j}\right)\right]^{\sim}$$

where $\sim$ is the mean, defined by (1.20). They are well determined constants which only depend on the local coefficients $a_{ij}(y)$.

Equation (1.23) then becomes an equation in $u^o(x)$:

$$(3.5) \qquad -\frac{\partial}{\partial x_i}\left(a_{ik}^h \frac{\partial u^o(x)}{\partial x_k}\right) = f \quad .$$

We now search for symmetry and positivity relations for the homogenized coefficients.

If in (2.9) we integrate by parts the right hand side, we see that it is equal to

$$\int_{\partial Y} n_i\, a_{ik}\, v\, dS - \int_Y a_{ik} \frac{\partial v}{\partial y_i}\, dy$$

and the integral over ∂Y vanishes by periodicity, and we have, $\forall\, v \in V_Y$:

$$0 = \int_Y a_{ij}\left(\frac{\partial w^k}{\partial y_j} + \delta_{jk}\right)\frac{\partial v}{\partial y_i}\, dy = \int_Y a_{mj}\, \frac{\partial(w^k + y^k)}{\partial y_j}\, \frac{\partial\, v}{\partial y_m}\, dy$$

and by taking $v = w^i$ and adding and substracting the same quantity, we have

$$\int_Y a_{mj} \frac{\partial(w^k + y^k)}{\partial y_j}\, \frac{\partial(w^i + y^i)}{\partial y_m}\, dy = \int_Y a_{mj}\, \frac{\partial(w^k + y^k)}{\partial y_j} \delta_{im}\, dy =$$

$$= \int_Y a_{ij}\left(\frac{\partial w^k}{\partial y_j} + \delta_{kj}\right) dy$$

and by comparing this with (3.4) we have :

$$(3.6) \qquad a_{ik}^h = \frac{1}{|Y|} \int_Y a_{mj}(y)\, \frac{\partial(w^k(y) + y^k)}{\partial y_j}\, \frac{\partial(w^i(y) + y^i)}{\partial y_m}\, dy.$$

It is then evident by virtue of (1.3) that

$$(3.7) \qquad a_{ik}^h = a_{ki}^h$$

Now we derive another expression for the homogenized coefficients which will be generalized to some nonlinear problems in the next chapter. If x is a parameter we may define the following bilinear form of $\underline{\text{grad}}_x u^o$:

$$(3.8) \qquad \boxed{W\left(\frac{\partial u^o}{\partial x_i}\right) \equiv \frac{1}{2}\, a_{ik}^h\, \frac{\partial u^o}{\partial x_i}\, \frac{\partial u^o}{\partial x_k}} = \text{(by (3.6) and (2.10))} =$$

$$\boxed{= \frac{1}{2\,|Y|} \int_Y a_{mj}(y)\left(\frac{\partial u^1}{\partial y_j} + \frac{\partial u^o}{\partial x_j}\right)\left(\frac{\partial u^1}{\partial y_m} + \frac{\partial u^o}{\partial x_m}\right) dy}$$

On the other hand, (3.3) becomes

$$(3.9) \qquad \tilde{p}_i^o(x) = \frac{\partial W}{\partial\left(\frac{\partial u^o}{\partial x_i}\right)}$$

We then have a relation between $\underline{\text{grad}}\; u^o$ and $\tilde{\underline{p}}^o$ given by (3.8), (3.9) ; it contains the homogenized coefficients only in an implicit form, and it may be generalized to nonlinear problems.

Remark 3.1 - It is clear that x is only a parameter in (3.8). The function W is a function of the N components q_i of the vector $\underline{\text{grad}}_x u^0$:

$$q_i \equiv \frac{\partial u^0}{\partial x_i} \qquad \blacksquare$$

Moreover, the coefficients a^h_{ik} satisfy an ellipticity condition analogous to (1.4).

(3.10) $$a^h_{ik}\, \xi_i\, \xi_k \geqslant \delta\, \xi_i\, \xi_i \quad , \quad \delta > 0 \quad , \quad \forall \underline{\xi} \in R^N$$

(in fact relation (3.10) holds with $\delta = \gamma$, the same constant as in (1.4), but this will not be proved here (see Bensoussan, Lions, Papanicolaou [2] , chap. 1, sect. 3.4). To obtain (3.10) it suffices to prove that its left hand side is positive for any $\xi \neq 0$; this amounts to saying that $\underline{\text{grad}}\, u^0 \neq 0 \Rightarrow W \neq 0$, and this is true, because, by (1.4)

$$W = 0 \Rightarrow 0 = \int_Y \sum_s \left(\frac{\partial u^1}{\partial y_s} + \frac{\partial u^0}{\partial x_s}\right)^2 dy$$

i.e., for s = 1 ... N we have

$$\frac{\partial u^1}{\partial y_s} + \frac{\partial u^0}{\partial x_s} = 0 \Rightarrow 0 = \int_Y \frac{\partial u^1}{\partial y_s}\, dy = - \frac{\partial u^0}{\partial x_s}\, |Y| \Rightarrow \underline{\text{grad}}_x u^0 = 0 \quad .$$

Finally to obtain a well posed problem for u^0, we only need a boundary condition for u^0. From (1.7) and (1.10) we obtain

(3.11) $$u^0(x)\Big|_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega$$

Note that this relation is formal, (see Remark 1.3) but it will be rigorously proved in next section.

We summarize the results of the preceeding considerations :

Proposition 3.2 - If we postulate an expansion of the form (1.10), the first term $u^0(x)$ is determined as a solution of the elliptic equation (3.5) with the boundary condition (3.11). The coefficients a^h_{ik} are given by any of the formulae (3.4), (3.6) or (3.8) and (3.9). They satisfy the symmetry and positivity relations (3.7) and (3.10).

The only formal point in the preceeding considerations is the form of the expansion (1.10). In fact, it is possible to prove that u^ε converges to the function u^0 given by proposition 3.2. An elementary proof, based on the maximum principle is given in Bensoussan–Lions–Papanicolaou [2], chap. 1, sect 2.4. Unfortunately such a proof does not hold for other problems (such as Neumann conditions, elliptic systems ...). In the next section we shall give a proof due to Tartar which holds with minor modifications for many other problems.

4.- Proof of the convergence - This section is devoted to the proof of the following theorem of De Giorgi and Spagnolo (Tartar's proof) :

Theorem 4.1 - Under the hypothesis of the preceeding sections, if u^ε (resp. u^0) is the solution of (1.6), (1.7) (resp. (3.5), (3.11)), we have

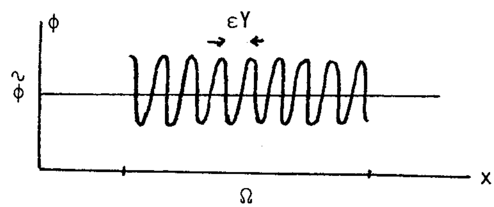
(4.1) $$u^\varepsilon \to u^0 \quad \text{in } H^1_0(\Omega) \text{ weakly}$$

We begin by a lemma which helps us to understand the sense of convergence in $L^2(\Omega)$ weakly.

Lemma 4.1 - Let $\phi \in L^2(Y)$. If we extend it periodically to R^N, we have

(4.2) $$\phi(\tfrac{x}{\varepsilon}) \to \tilde{\phi} \qquad \text{in } L^2(\Omega) \text{ weakly}$$

[(4.2) means that $\phi(x/\varepsilon)$ tends to the function defined on Ω which is equal to the constant $\tilde{\phi}$ given by (1.20)]



Proof - It is clear that $\phi(x/\varepsilon)$ is bounded in the $L^2(\Omega)$ norm as $\varepsilon \to 0$. Because $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$ it suffices to prove (see chap. 2, prop. 1.2) that

(4.3) $$\int_\Omega \phi(x/\varepsilon)\ \theta(x)\ dx \longrightarrow \tilde{\phi} \int_\Omega \theta(x)\ dx \qquad \forall\ \theta \in \mathcal{D}(\Omega)$$

But this is immediate. Let θ^*_ε be the function which takes in each εY-period a constant value equal to θ at the center of the period. Because θ is smooth, we clearly have

$$\int_\Omega \phi(\tfrac{x}{\varepsilon})(\theta - \theta^*_\varepsilon)\ dx \underset{\varepsilon \to 0}{\longrightarrow} 0$$

$$\int_\Omega \phi(\tfrac{x}{\varepsilon})\ \theta^*_\varepsilon(x)\ dx = \tilde{\phi} \int_\Omega \theta^*_\varepsilon(x)\ dx \underset{\varepsilon \to 0}{\longrightarrow} \tilde{\phi} \int_\Omega \theta(x)\ dx \quad . \blacksquare$$

Now, we prove theorem 4.1. The function u^ε (resp. u^0) is the element of $H^1_0(\Omega)$ which satisfy (1.8) (resp. an analogous relation with a^h_{ij}, u^0 instead of

a^{ε}_{ij} , u^{ε}).

By taking $v = u^{\varepsilon}$ in (1.8) and taking into account (1. 4) we have

$$\gamma \int_{\Omega} \frac{\partial u^{\varepsilon}}{\partial x_i} \frac{\partial \bar{u}^{\varepsilon}}{\partial x_i} dx \leqslant \int_{\Omega} a^{\varepsilon}_{ij} \frac{\partial u^{\varepsilon}}{\partial x_i} \frac{\partial u^{\varepsilon}}{\partial x_j} dx \leqslant C' \|u^{\varepsilon}\|_{L^2} \leqslant C'' \|u^{\varepsilon}\|_{H^1_0}$$

But the left hand side may be taken as the norm^2 in H^1_0 (see chap. 3, prop. 1.1) and thus

(4.4) $$\|u^{\varepsilon}\|_{H^1_0} \leqslant C$$

where C is some constant independent of ε . This means that u^{ε} remains in a bounded set of H^1_0, i.e., in a precompact set for the weak topology of H^1_0 (see chap. 2, prop. 1.6). Consequently, we can extract a sequence $\varepsilon \to 0$ such that

(4.5) $$u^{\varepsilon} \to u^{*} \qquad (u^{*} \in H^1_0) \quad \text{in } H^1_0(\Omega) \quad \text{weakly}$$

<u>the theorem will be proved if we show that for any sequence as (4.5), we have $u^{*} = u^{0}$</u>. From (4.5), the partial derivatives of u^{ε} are bounded in $L^2(\Omega)$; by multiplying them by $a_{ij}(x/\varepsilon)$, which are smooth bounded functions, we see that

(4.6) $$\|p^{\varepsilon}_i(x)\|_{L^2} \equiv \|a^{\varepsilon}_{ij}(x) \frac{\partial u^{\varepsilon}(x)}{\partial x_j}\|_{L^2} \leqslant C$$

and by extracting a sequence $\varepsilon \to 0$ from the preceeding one, we have

(4.7) $$p^{\varepsilon}_i \to p^{*}_i \qquad (p^{*}_i \in L^2(\Omega)) \quad \text{in } L^2(\Omega) \text{ weakly} \quad .$$

Now, by writing (1.8) in the form

$$\int_{\Omega} p^{\varepsilon}_i \frac{\partial v}{\partial x_i} dx = \int_{\Omega} f\ v\ dx \qquad \forall\ v \in H^1_0$$

we can pass to the limit for any fixed $v \in H^1_0$ and get

(4.8) $$\int_{\Omega} p^{*}_i \frac{\partial v}{\partial x_i} dv = \int_{\Omega} f\ v\ dx \qquad \forall\ v \in H^1_0$$

Let us suppose that

(4.9) $$p^{*}_i(x) = a^{h}_{ij} \frac{\partial u^{*}}{\partial x_j}(x) \qquad \text{in } \Omega$$

Then, (4.8) shows that $u^{*} \in H^1_0$ satisfies the variational formulation of the problem u^0 ; by uniqueness, $u^{*} = u^{0}$. Consequently, <u>we only have to prove (4.9)</u>. To show this, we shall take test functions of a special suitable form.

If $w^k(y)$ is the function defined by (2.9), we write

(4.10) $$w_{\varepsilon}(x) \equiv x_k + \varepsilon\, w^k(x/\varepsilon)$$

(note that this function is in fact the sum of the two first terms of the expansion, $u^0(x) + \varepsilon u^1(x/\varepsilon)$ for $u^0 = x_k$). It is clear that

$$w_\varepsilon(x) \to x_k \quad \text{in } L^2(\Omega) \text{ strongly.} \tag{4.11}$$

Moreover, w_ε satisfies the equation

$$-\frac{\partial}{\partial x_j}\left(a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial w_\varepsilon(x)}{\partial x_i}\right) = 0 \qquad \text{in } R^N \tag{4.12}$$

for, (2.9) gives, in the sense of distributions :

$$-\frac{\partial}{\partial y_j}\left(a_{ij}\frac{\partial w^k}{\partial y_i}\right) = \frac{\partial a_{ik}}{\partial y_i}$$

which is equivalent to (4.12). Then, by multiplying (4.12) by any $v \in H_0^1$ we have :

$$\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial w_\varepsilon}{\partial x_j}\frac{\partial v}{\partial x_i}\, dx = 0 \qquad \forall v \in H_0^1(\Omega) \tag{4.13}$$

Now, to avoid difficulties with the boundary condition, we take a function $\phi \in \mathcal{D}(\Omega)$ and we write (1.8) with $v = \phi\, w_\varepsilon$ and (4.13) with $v = \phi\, u^\varepsilon$. By substracting and taking into account that $a_{ij} = a_{ji}$ we have

$$\int_\Omega a_{ij}\left(\frac{x}{\varepsilon}\right)\left[\frac{\partial u^\varepsilon}{\partial x_j}\frac{\partial \phi}{\partial x_i} w_\varepsilon - \frac{\partial w_\varepsilon}{\partial x_i}\frac{\partial \phi}{\partial x_j} u^\varepsilon\right] dx = \int_\Omega f\,\phi\, w_\varepsilon\, dx \tag{4.14}$$

Now we can pass to the limit $\varepsilon \to 0$ in (4.14) because each term is the scalar product in L^2 of an element which converges weakly and another which converges strongly (chap. 2 , prop. 1.3). Indeed :

$p_i^\varepsilon(x) \equiv a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u^\varepsilon}{\partial x_j}$ converges in $L^2(\Omega)$ weakly by (4.7).

$\frac{\partial \phi}{\partial x_i} w_\varepsilon$ converges to $\frac{\partial \phi}{\partial x_i} x_k$ in L^2 strongly by (4.11) (note that ϕ is smooth and fixed).

$a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial w_\varepsilon}{\partial x_i}$ is εY-periodic and tends in L^2 weakly to its mean value :

$$\left[a_{ij}(y)\left(\delta_{ik} + \frac{\partial w^k(y)}{\partial y_i}\right)\right]^\sim = \text{by (3.4)} = a_{jk}^h \quad .$$

Finally $\frac{\partial \phi}{\partial x_j} u^\varepsilon$ converges to $\frac{\partial \phi}{\partial x_j} u^*$ in L^2 strongly by (4.5) and the Rellich theorem.

We obtain

$$\int_\Omega \left(p_j^* x_k - a_{jk}^h u^*\right) \frac{\partial \phi}{\partial x_j}\, dx = \int_\Omega f\,\phi\, x_k\, dx \tag{4.15}$$

Moreover, by (4.8) with $v = \phi\, x_k$, the right hand side of (4.15) is

$$= \int_\Omega p_j^* \frac{\partial(\phi x_k)}{\partial x_j}\, dx \tag{4.16}$$

The relation (4.15) (with the right hand side (4.16)) holds for any $\phi \in \mathcal{D}(\Omega)$; this means that, in the sense of distributions on Ω, we have :

$$- \frac{\partial}{\partial x_j} (p_j^* x_k - a_{jk}^h u^*) = - \frac{\partial p_j^*}{\partial x_j} x_k \Leftrightarrow p_k^* = a_{jk}^h \frac{\partial u^*}{\partial x_j}$$

which is the desired relation (4.9). ∎

5.- Generalization to other elliptic problems and convergence of the resolvents -

The considerations of the sections 1 to 4 apply with minor modifications to a great variety of problems.

Let us consider, instead of (1.6), (1.7) a Neumann problem of the type :

$$- \frac{\partial}{\partial x_i} \left(a_{ij}^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_j}\right) + a_0^\varepsilon(x)\, u^\varepsilon = f^\varepsilon(x) \qquad \text{in } \Omega \tag{5.1}$$

$$a_{ij}^\varepsilon(x) \frac{\partial u^\varepsilon}{\partial x_j} n_i = 0 \qquad \text{on } \partial\Omega \tag{5.2}$$

where n is the outer unit normal to $\partial\Omega$ and $a_0^\varepsilon(x) = a_0(\frac{x}{\varepsilon})$ and $a_0(y)$ is a Y-periodic smooth function with

$$a_0(y) \geqslant \gamma \qquad y \in R^N \tag{5.3}$$

and $f^\varepsilon(x)$ is a sequence of functions such that

$$f^\varepsilon(x) \to f^* \qquad \text{in } L^2(\Omega) \text{ weakly.} \tag{5.4}$$

Then we have :

Theorem 5.1 - $u^\varepsilon \to u^o$ in $H^1(\Omega)$ weakly where u^o is the (unique) solution of

$$- \frac{\partial}{\partial x_i} \left(a_{ij}^h \frac{\partial u^o}{\partial x_j}\right) + \tilde{a}_o\, u^o = f^* \qquad \text{in } \Omega \tag{5.5}$$

$$a_{ij}^h \frac{\partial u^o}{\partial x_j} n_i = 0 \qquad \text{on } \partial\Omega \quad . \tag{5.6}$$

Remark 5.1 - Note that the homogenized coefficients a_{ij}^h are given by the same formulas as in the Dirichlet problem, (3.4). The new term a_0^ε gives in the homogenized equation its mean value : $a_0^h \equiv \tilde{a}_0$ (This fact also holds for Dirichlet conditions !). On the other hand, the Neumann condition (5.6) is associated with the homogenized coefficients ; while (5.2) is associated with the given coefficients (which depend on ε). ∎

The proof of Th. 5.1 is almost the same as that of Th. 4.1. The variational formulation of (5.1), (5.2) is : (as in chap. 3, (2.1), (2.2)) : Find $u^\varepsilon \in H^1(\Omega)$ such that

$$(5.7) \qquad \int_\Omega (a^\varepsilon_{ij} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j} + a^\varepsilon_o \; u^\varepsilon \; v) dx = \int_\Omega f^\varepsilon v \; dx \qquad \forall v \in H^1(\Omega)$$

The convergence in (4.5) now holds in H^1 weakly and by the Rellich theorem, in $L^2(\Omega)$ strongly. Moreover, by lemma 4.1 :

$$a^\varepsilon_o \to \tilde{a} \qquad \text{in } L^2(\Omega) \text{ weakly}$$

and then, we obtain instead of (4.8) :

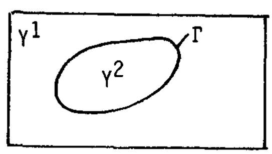
$$(5.8) \qquad \int_\Omega (p^*_i \frac{\partial v}{\partial x_i} + \tilde{a}_o \; u^* v) \; dx = \int_\Omega f^* v \; dx \qquad \forall v \in H^1$$

and this shows that p^* and u^* satisfy a certain equation and <u>an associated boundary condition as in the classical Neumann problems</u>(chap. 3, sect. 2). Indeed, by integrating (5.8) by parts we have

$$\int_\Omega \left(- \frac{\partial p^*_i}{\partial x_i} + \tilde{a}_o \; u^* - f^*\right) v \; dx + \int_{\partial\Omega} (n_i \; p^*_i) v \; dx = 0 \quad \forall v \in H^1$$

which implies that the two functions in parenthesis are zero. By proving (4.9) we have (5.5) and (5.6), as desired. The proof of (4.9) is of course the same as in th. 4.1 because it is a local property independent of the boundary conditions.

It is also possible to consider transmission problems with coefficients $a_{ij}(y)$ which are not smooth. To fix ideas, we take a_{ij} to be piecewise constant, (or piecewise smooth). We may consider the period Y divided in two regions Y^1, Y^2 separated by the smooth surface Γ , and $a_{ij}(y)$ takes the constant values a^1_{ij} and



<u>fig. 5</u>

a^2_{ij} in Y^1 and Y^2 respectively. Then, equation (1.6) (or (5.1), because the term a_o is irrelevant) must be interpretated in the sense of distributions ;

$$(5.9) \qquad [u^\varepsilon] = 0 \qquad \text{on } \Gamma$$

$$(5.10) \qquad [a_{ij}(y) \frac{\partial u^\varepsilon}{\partial x_i} n_j] = 0 \qquad \text{on } \Gamma$$

where the brakets mean, as usual, the discontinuity across Γ . In the usual weak sense, the variational formulation of the problem for u^ε is (1.8) (or (5.7)) ;

this problem is analogous to that of chap. 3, sect. 3 ; condition (5.9) is contained in $u^{\varepsilon} \in H_0^1(\Omega)$, and (5.10) is the natural condition associated with the variational formulation.

Theorems 4.1 or 5.1 holds in the present case (the proofs are exactely the same but the formal asymptotic expansion and the expressions for the homogenized coefficients are slightly modified, in the following way :

In the formal expansion, we note that $a_{ij}(y)$ is piecewise constant, but we must include the transmission conditions on Γ .

Relation (1.18) becomes (5.11), (5.12) :

$$\text{(5.11)} \qquad \frac{\partial}{\partial y_i}\left(a_{ij}(y)\,\frac{\partial u^1(x\,,\,y)}{\partial y_j}\right) = 0 \qquad \text{in } Y^1 \text{ and } Y^2$$

$$\text{(5.12)} \qquad \left[a_{ij}(y)\left(\frac{\partial u^o}{\partial x_j} + \frac{\partial u^1}{\partial y_j}\right)n_j\right] = 0 \qquad \text{on } \Gamma$$

where (5.12) is the first term of the expansion of (5.10) with (1.10). To find u^1 we also must include the Y-periodicity condition.

It is easily seen that (5.11) and (5.12) may be written as

$$\frac{\partial}{\partial y_i}\left(a_{ij}(y)\left(\frac{\partial u^o}{\partial x_j} + \frac{\partial u^1}{\partial y_j}\right)\right) = 0 \qquad \text{on } Y$$

in the sense of distributions. Then, the variational formulation of the local problem is (instead of (2.3)) :

$$\text{(5.13)} \qquad \left|\begin{array}{l} \text{Find } u^1 \in V_Y \text{ such that} \\ \displaystyle\int_Y a_{ij}(y)\left(\frac{\partial u^o}{\partial x_j} + \frac{\partial u^1}{\partial y_j}\right)\frac{\partial v}{\partial y_i}\,dy = 0 \qquad \forall\, v \in V_Y \end{array}\right.$$

with an analogous modification for (2.9). Note that (2.3) in sect. 2 may also be written in the form (5.13) because, by integrating by parts the right hand side of (2.3) we have, by virtue of the periodicity :

$$\int_Y \frac{\partial a_{ij}}{\partial y_i}\,v\,dy = -\int_Y a_{ij}\,\frac{\partial v}{\partial y_i}\,dy$$

Consequently, <u>(5.13) is more general than (2.3), because (5.13) holds for continuous as well as discontinuous coefficients.</u>

Of course, it is also possible to consider the case where a_{ij} is dicontinuous on Γ and smoothly variable in Y^1 and Y^2. Formula (5.13) also holds in this case.

It is also possible to consider elliptic equations with complex coefficients. In this case, we consider (to use the preceeding calculations) only the symmetric case (1.3). It is clear that the operator (1.6) is no longer selfadjoint, and in fact, the homogenized operator is not symmetric in general.

To fix ideas, we consider the problem of sections 1 to 4 with the following modifications. Instead of (1.4), the ellipticity condition is ($\bar{}$ is for the complex conjugate)

(5.14) $$| a_{ij}(y) \xi_i \bar{\xi}_j | \geq \gamma |\xi|^2 \qquad \forall \xi \quad C^N$$

and (1.8) becomes

(5.15) $$\int_\Omega a_{ij}(y) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial \bar{v}}{\partial x_j} dx = \int_\Omega f \bar{v} \, dx \qquad \forall v \in H_0^1(\Omega)$$

Existence and uniqueness follow from the Lax Milgram theorem (see chap. 2, formula (5.5))

In (2.3) and (2.9), we must write $\bar{v}$ instead of v. The expression (3.4) for the homogenized coefficients holds in this case. Moreover, instead of (3.6), we obtain (we take $v = \bar{w}^i$ instead of $v = w^i$ in the proof) :

$$a_{ik}^h = \frac{1}{|Y|} \int_Y a_{mj}(y) \frac{\partial(w^k + y^k)}{\partial y_j} \frac{\partial(\bar{w}^i + y^i)}{\partial y_m} dy$$

which is not in general symmetric. But we have an ellipticity property of the homogenized problem

(5.16) $$| a_{ij}^h \xi_i \bar{\xi}_j | \geq \gamma |\xi|^2 \qquad \forall \xi \in C^N .$$

This can be proved as in sect. 3, because (5.16) amounts to proving that grad $u^0 \neq 0$ implies that the right hand side of (3.8) (with the complex conjugates in the second factors) is not zero.

Moreover, theorem 4.1 holds in the case of complex coefficients. The proof is the same as in sect. 4, but in (4.8) and (4.13), we have $\bar{v}$ instead of v. Moreover, to obtain (4.14) we must write $v = \bar{\phi}\, \bar{w}_\varepsilon$ in (1.8) and $v = \bar{\phi}\, \bar{u}^\varepsilon$ in (1.13) ; this gives (4.14) without modification, and the proof follows.

To finish this section, we give a result about the convergence of the resolvents at the point zero, which is useful to obtain, in a simple way, results about the convergence of eigenvectors and eigenvalues (without hypothesis of seladjointness, in particular in the framework of complex coefficients) (see chap.11, sect 3).

Let A_ε and A_h be the elliptic operators in (1.6), and (3.5), with the Dirichlet boundary conditions, or the operators in (5.1), (5.5) with the Neumann boundary conditions. According to the first representation theorem, we consider A_ε, A_h as unbounded maximal accretive operators in $L^2(\Omega)$ (Ω is bounded). Then, we have the following theorem which applies to spectral properties of the homogenization process (see chap. 11, sect. 3) :

Theorem 5.2 - Under the preceeding conditions

(5.17) $$\| A_\varepsilon^{-1} - A_h^{-1} \|_{\mathcal{L}(L^2,L^2)} \to 0 \qquad \text{as } \varepsilon \to 0$$

The proof is exactly analogous to that of chapter 9, formula (6.7), and will not be given here. It is based on the consideration of A_ε, A_h as bounded operators from V into V' where

$$V \subset H \subset V'$$

with dense and compact embedding. Here, we shall take $H = L^2(\Omega)$, $V = H_0^1(\Omega)$ for the Dirichlet problem and $V = H^1(\Omega)$ for the Neumann problem.

6.- Homogenization of evolution equations - Now we consider parabolic and hyperbolic equations analogous to (1.6), (1.7) and the generalizations of sect. 5. To fix ideas, we consider real smooth and symmetric coefficients $a_{ij}(y)$ as in sect. 1. Moreover, let $\rho(y)$ be a real smooth Y-periodic function with

(6.1) $$\rho(y) \geqslant \gamma \qquad \gamma > 0 \qquad \forall\, y \in Y$$

(It is of course possible to take the same γ as in (1.4). We also consider the corresponding homogenized coefficients a_{ij}^h of (3.4). Let A_ε , A_h be the operators

(6.2) $$A_\varepsilon = -\frac{\partial}{\partial x_i}\left(a_{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial}{\partial x_j}\right)$$
$$A_h = -\frac{\partial}{\partial x_i}\left(a_{ij}^h \frac{\partial}{\partial x_j}\right)$$

with Dirichlet boundary condition (1.7) on the boundary $\partial\Omega$ of a bounded domain Ω of R^N.

Definition 6.1 - Let u^ε be the solution of the parabolic problem

(6.3) $$\rho\left(\frac{x}{\varepsilon}\right)\frac{\partial u^\varepsilon}{\partial t} + A_\varepsilon u^\varepsilon = 0 \qquad ; \qquad (u^\varepsilon|_{\partial\Omega} = 0)$$

(6.4) $$u^\varepsilon(0) = u_0 \quad L^2(\Omega)$$

for $x \in \Omega$, $t \in [\,0,\infty[$. Let also u^h be the solution of

(6.5) $$\tilde{\rho}\,\frac{\partial u^h}{\partial t} + A_h u^h = 0 \qquad (u^h|_{\partial\Omega} = 0)$$

(6.6) $$u^h(0) = u_0$$

for $x \in \Omega$, $t \in [0,\infty[$.

Remark 6.1 - (6.4) and (6.6) are initial conditions. Equations (6.3) and (6.5) are written for functions with values in $L^2(\Omega)$; then the boundary conditions on $\partial\Omega$ are included in A_ε and A_h . This is the reason why these boundary conditions are written in parentheses. On the other hand, the initial value u_0 is the same in (6.4) and (6.6). The right hand sides of (6.3) and (6.5) are zero, but it is also possible to consider non homogeneous problems (see remark 6.3 later). Note also that the homogenized form of $\rho(x/\varepsilon)$ is its mean value $\tilde{\rho}$.■

Theorem 6.1 - If $\rho(y)$ is independent of y, we have

(6.7) $$u^\varepsilon(t) \to u^h(t) \qquad \text{in } L^2(\Omega) \text{ strongly}$$

for any fixed t. (see def. 6.1).

Proof - This is an immediate consequence of the Trotter-Kato theorem (see chap. 4 Th. 3.3 and chap. 10). Indeed, if ρ is a constant we may take it equal to 1. We consider the semigroups associated with (6.3) and (6.5) in the framework of chapter 4, sect. 4 and by the Trotter-Kato theorem, (6.7) is equivalent to

(6.8) $$(I + A_\varepsilon)^{-1} v \to (I + A_\varepsilon)^{-1} v \quad \text{in } L^2(\Omega) \text{ strongly}$$

for any test function $v \in L^2(\Omega)$. But this amounts to proving that if w^ε, w^h are defined by

(6.9) $$w^\varepsilon - \frac{\partial}{\partial x_i}(a_{ij}(\frac{x}{\varepsilon}) \frac{\partial w^\varepsilon}{\partial x_j}) = v \qquad ; \qquad w^\varepsilon\big|_{\partial\Omega} = 0$$

and an analogous equation with w^h and a^h_{ij}, we have

(6.10) $$w^\varepsilon \to w^h \text{ in } L^2(\Omega) \text{ strongly.}$$

But from theorem 4.1 (or rather theorem 5.1 with Dirichlet boundary condition, due to the term ρ) w^ε converges to w^h in H^1_0 weakly and by the Rellich theorem we have (6.10).

Let us now consider the case where ρ is variable.

Remark 6.2 - Let us consider (6.3), (6.4) with fixed ε . It is possible to obtain the existence and uniqueness of u^ε from semigroup theory. If we take the L^2-scalar product of (6.3) with $v \in H^1_0$, we see that (6.3) is equivalent to

(6.11) $$b^\varepsilon(\frac{\partial u^\varepsilon}{\partial t} , v) + a^\varepsilon(u^\varepsilon , v) = 0 \qquad \forall v \in H^1_0$$

where

(6.12) $$b^{\varepsilon}(u\ ,\ v) \equiv \int_{\Omega} \rho(\frac{x}{\varepsilon})\ u\ v\ dx$$

(6.13) $$a^{\varepsilon}(u\ ,\ v) \equiv \int_{\Omega} a_{ij}(\frac{x}{\varepsilon})\ \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\ dx$$

Now, from (6.1), we see that b^{ε} is a scalar product on L^2 with norm equivalent to the standard one. Let H^{ε} be the space L^2 equipped with the scalar product b^{ε} . On the other hand, a^{ε} is a symmetric bounded and coercive form on H^1_o, which is a Hilbert space densely embedded into H^{ε} as well as in L^2. Then, (6.11) is equivalent to

(6.14) $$\frac{\partial u^{\varepsilon}}{\partial t} + A^*_{\varepsilon}\ u^{\varepsilon} = 0$$

where A^*_{ε} is the operator associated with the form a^{ε} in the framework of the first representation theorem <u>in H^{ε}</u> (in the same way as A_{ε} is associated with a^{ε} <u>in L^2</u>). The operator A^*_{ε} is not to be confused with $\rho^{-1}A_{\varepsilon}$. Semigroup theory then applies to (6.14) (see chap. 4, sect. 4). It is then immediate to come back to (6.3).

<u>Theorem 6.2</u> - In the framework of def. 6.1, we have

$$u^{\varepsilon} \to u^h \qquad \text{as } \varepsilon \searrow 0$$

in $L^{\infty}(0\ ,\infty\ ;\ L^2(\Omega))$ weakly $*$ and in $L^2(0\ ,\infty\ ;\ H^1_o(\Omega))$ weakly.

<u>Proof</u> - The classical a priori estimate (see remark 6.2 and chap. 4, (4.2)) give

$$\begin{cases} \|u^{\varepsilon}(t)\|_{H^{\varepsilon}} \leqslant \|u_o\|_{H^{\varepsilon}} \leqslant C & \forall\ t \geqslant 0 \\ \int_0^{\infty} \|u^{\varepsilon}(t)\|^2_{H^1_o}\ dt \leqslant \|u_o\|_{H^{\varepsilon}} \leqslant C & \end{cases}$$

and by extracting subsequences, we have

(6.15) $$u^{\varepsilon} \underset{\varepsilon \to 0}{\longrightarrow} u^* \qquad \begin{cases} \text{in } L^{\infty}(0\ ,\infty\ ;\ L^2) \text{ weakly } * \\ \text{in } L^2(0\ ,\infty\ ;\ H^1_o) \text{ weakly} \end{cases}$$

The theorem will be proved if we show that $u^* = u^h$ for any subsequence.

By taking the Laplace transform of (6.15),

(6.16) $$\hat{u}^{\varepsilon} \to \hat{u}^* \quad \begin{cases} \text{in } L^2 \text{ weakly} \\ \text{in } H^1_o \text{ weakly} \end{cases}$$

for any p with Re$p > 0$. (This is obtained by taking $e^{-pt}v$ as test function in (6.15)).

On the other hand, $\hat{u}^\varepsilon$ can be written as (see remark 6.2 and chap. 4, prop. 6.3) :

$$\hat{u}^\varepsilon = (p + A_\varepsilon)^{-1} u_0 \quad \text{in} \quad H^\varepsilon, \text{ for } \operatorname{Re} p > 0 \ .$$

and this amounts to saying that

$$p\rho(\tfrac{x}{\varepsilon})\, \hat{u}^\varepsilon + A_\varepsilon\, \hat{u}^\varepsilon = u_0 \quad \text{in } L^2 \ , \ \operatorname{Re} p > 0 \tag{6.17}$$

and in the same way

$$p\, \tilde{\rho}\ \hat{u}^h + A_h\, \hat{u}^h = u_0 \quad \text{in} \quad L^2 \ , \operatorname{Re} p > 0 \ . \tag{6.18}$$

Now we consider (6.17) and (6.18) with real p. By theorem 5.1 (with Dirichlet boundary condition !), $\hat{u}^\varepsilon$ converges to $\hat{u}^h$ in H_0^1 weakly, and by (6.16), $\hat{u}^h = \hat{u}^*$ for real p. Moreover, the Laplace transforms are holomorphic functions in the half plane Rep > 0 ; thus $\hat{u}^h = \hat{u}^*$ for any p, and this implies $u^h = u^*$.■

Now we study <u>hyperbolic problems</u>.

<u>Definition 6.2</u> - u^ε is the solution of the hyperbolic problem

$$\rho(\tfrac{x}{\varepsilon})\, \frac{\partial^2 u^\varepsilon}{\partial t^2} + A_\varepsilon\, u^\varepsilon = 0 \qquad (u^\varepsilon\big|_{\partial\Omega} = 0) \tag{6.19}$$

$$u^\varepsilon(0) = u_0 \in H_0^1(\Omega) \quad ; \quad u^{\varepsilon\prime}(0) = u_1 \in L^2(\Omega) \tag{6.20}$$

for $x \in \Omega$, $t \in [0 , \infty[$. Let also u^h be the solution of

$$\tilde{\rho}\, \frac{\partial^2 u^h}{\partial t^2} + A_h\, u^h = 0 \tag{6.21}$$

$$u^h(0) = u_0 \quad ; \quad u^{h\prime}(0) = u_1 \tag{6.22}$$

for $x \in \Omega$, $t \in [0,\infty[$.

<u>Theorem 6.3</u> - In the framework of def. 6.2, we have

$$u^\varepsilon \rightharpoonup u^h \qquad \text{in } L^\infty(0 , \infty ; H_0^1) \text{ weakly } *$$

$$u^{\varepsilon\prime} \rightharpoonup u^{h\prime} \qquad \text{in } L^\infty(0 , \infty ; L^2) \text{ weakly } * \ .$$

<u>Proof</u> - The proof is analogous to that of th. 6.2. By remark 6.2, (6.19) is equivalent to (we define V^ε as H_0^1 with the scalar product $(A^\varepsilon u , v)$)

$$\frac{\partial^2 u^\varepsilon}{\partial t^2} + A_\varepsilon^* u = 0 \qquad \text{in} \quad H^\varepsilon$$

and for fixed ε, the problem is in the framework of chap. 4, remark 5.1 in $V^\varepsilon \times H^\varepsilon$. In this space, u^ε, $u^{\varepsilon\prime}$ are associated with a unitary group and thus

(6.23) $$\| u^\varepsilon \|^2_{V^\varepsilon} + \| u^{\varepsilon\prime} \|^2_{H^\varepsilon} = \| u_0 \|^2_{V^\varepsilon} + \| u_1 \|^2_{H^\varepsilon} \leq C$$

for any t. By extraction of subsequences,

$$u^\varepsilon \to u^* \quad \text{in } L^\infty(0\,,\infty\,;H^1_0) \text{ weakly } *$$

$$u^{\varepsilon\prime} \to u^{*\prime} \quad \text{in } L^\infty(0\,,\infty,\ L^2) \text{ weakly } *$$

and by taking the Laplace transform :

(6.24) $$\hat{u}^\varepsilon \to \hat{u}^* \quad \text{in} \quad H^1_0 \text{ weakly} \quad .$$

By taking the Laplace transform of the semigroup we have

$$\left.\begin{aligned} p\ \hat{u}^\varepsilon - (u^{\varepsilon\prime})^\wedge &= u_0 \\ p(u^{\varepsilon\prime})^\wedge + A^*_\varepsilon\ \hat{u}^\varepsilon &= u_1 \end{aligned}\right\} \Rightarrow p^2\ \hat{u}^\varepsilon + A^*_\varepsilon\ \hat{u}_\varepsilon = u_1 + pu_0 \quad .$$

and this is equivalent to

$$p^2\ b(\tfrac{x}{\varepsilon})\ \hat{u}^\varepsilon + A_\varepsilon\ \hat{u}_\varepsilon = b(\tfrac{x}{\varepsilon})\ (u_1 + p\ u_0)$$

for any p with Rep > 0. We then finish the proof as in th. 6.2.

Remark 6.3 - It is not difficult to obtain theorems analogous to th. 6.1 to 6.3 if the right hand side of the equations is a fixed function f instead of zero. If f has a Laplace transform, the proofs are analogous to the preceeding ones. If f has not a Laplace transform (for example $f = e^{t^2}$),one study $t \in [0\ ,\ T]$ for any fixed T it is then possible to take f = 0 for $t > T$, and estimates (6.15) and (6.23) hold.

7.- Homogenization of a boundary in heat transfer theory. Formal expansion -

Now we study the heat equation in a bounded domain whose boundary is a waved surface of very small period, with the boundary condition

(7.1) $$\frac{\partial u}{\partial n} + \lambda\ u = 0$$

where λ is a real positive constant and n is the outer unit normal to the domain. The boundary condition (7.1) is classical in some problems of heat transfer. Physically, if k is the conductivity of the medium, $-k\ \partial u/\partial n$ is the flux of heat at the surface of the body ; (7.1) means that this flux is proportional to the temperature near the boundary. This is the case if the body is cooled by a flow of fluid at temperature 0. The more the temperature u of the body is, the more the heat flux towards the fluid is, but this one is not heated because it is in motion and the heated particles are replaced by cool particules.

In the basis of the preceeding physical considerations, it is natural to expect that the waved surface shall radiate more heat than a smooth (homogenized !)

surface. The boundary condition for the homogenized problem shall be different from (7.1). This is the reason why the radiators are waved !

We shall study the problem in the two dimensional case, but the three-dimensional case is obtained in the same way (see remark 7.2 later).

Let Ω_0 be a bounded open domain of R^2 with smooth boundary $\partial\Omega_0$ and with outer unit normal N. Let s be the curvilinear abscissa of the curve $\partial\Omega_0$. In a neighbourhood of $\partial\Omega_0$, s and N are curvilinear coordinates of the plane.

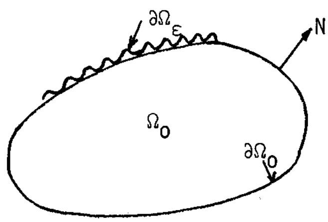


fig. 6

Moreover, in the rectangular coordinates y_1, y_2, we consider a smooth periodic function $y_2 = F(y_1)$ of period 1 (fig. 7). We then define the boundary $\partial\Omega_\varepsilon$ of the domain Ω_ε as the curve defined in the coordinates s, N by $N = \varepsilon F(s/\varepsilon)$ (see fig. 6) where ε is a real positive small parameter.

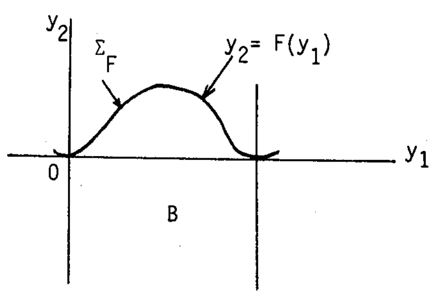


fig. 7

Note that the waves of $\partial\Omega_\varepsilon$ are "almost" homothetic to the curve of fig. 7, "almost" because there is a little distortion due to the curvilinear coordinates, but this distortion tends to zero as $\varepsilon \searrow 0$. We then consider the domain Ω_ε enclosed by $\partial\Omega_\varepsilon$.

Now we consider the heat equation in Ω_ε with fixed initial value $u_0(x)$ (which is a smooth function defined on a neighbourhood of Ω_0, and thus on any Ω_ε). $u^\varepsilon(x , t)$, $x \in \Omega_\varepsilon$, $t \in [0 , \infty [$ is defined by

(7.2) $$\frac{\partial u^\varepsilon}{\partial t} - \Delta \ u^\varepsilon = 0 \qquad \text{for } x \in \Omega_\varepsilon \ , \ t \in]0 , \infty[$$

$$(7.3) \qquad \frac{\partial u^\varepsilon}{\partial n} + \lambda\, u^\varepsilon = 0 \qquad \text{on } \partial\Omega_\varepsilon$$

$$(7.4) \qquad u^\varepsilon(x\,,\,0) = u_o(x) \qquad \text{on } \Omega_\varepsilon$$

This problem has a unique solution, as we shall see in the next section. (It is of course also possible to consider the non homogeneous problem with f instead of 0 at the right hand side of (7.2)).

We now postulate an asymptotic expansion for u^ε . The parameter ε only occurs in the boundary. It is then natural to hope that the expansion of u^ε will have a boundary layer term depending on the x and y variables, for $y_1 = s/\varepsilon$, $y_2 = N/\varepsilon$, and that this term will be periodic of period 1 in the y_1 variable. Consequently, we shall write :

$$(7.5) \qquad \begin{aligned} &u^\varepsilon(x\,,\,t) = u^o(x\,,\,t) + \varepsilon\, u^1(x\,,\,y\,,\,t) + \varepsilon^2 \ldots \\ &\text{for } y_1 = \frac{s}{\varepsilon} \;,\; y_2 = \frac{N}{\varepsilon} \;,\; u^1 \text{ is 1-periodic in } y_1 \end{aligned}$$

$$(7.6) \qquad \underline{\mathrm{grad}}_y\, u^1 \xrightarrow[y_2 \to -\infty]{} 0$$

Condition (7.6) means that the "waving" introduced by u^1 in (7.5) tends to zero far the boundary, i.e. u^1 is a boundary layer term. In fig. 7, the region

$$(7.7) \qquad B = \{y_1\,,\,y_2\;;\;\; y_1 \in \,]\,0\,,\,1[\;\;;\;\; y_2 < F(y_1)\}$$

is in fact the period of u^1 if x and t are parameters.

In order to replace (7.5) into (7.2), we remark that for small ε , the coordinates y are asymptotically orthogonal and we can handle them as orthogonal coordinates for the study of the first term u^1. Then

$$(7.8) \qquad \begin{aligned} &\frac{d}{dx_i} = \frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i} \\ &\frac{du^\varepsilon}{dx_i} = \frac{\partial u^o}{\partial x_i} + \frac{\partial u^1}{\partial y_i} + \varepsilon \ldots . \end{aligned}$$

$$(7.9) \qquad \Delta\, u^\varepsilon = \frac{d}{dx_i}\frac{du^\varepsilon}{dx_i} = \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\frac{\partial u^1}{\partial y_i} + \ldots$$

In the same way, (7.3) gives

$$\frac{du^\varepsilon}{dn} = n_i\,\frac{du^\varepsilon}{dx_i} = n_i\left(\frac{\partial u^o}{\partial x_i} + \frac{\partial u^1}{\partial y_i}\right) + \varepsilon \ldots$$

Consequently, taking the terms ε^{-1} of (7.2) and ε^{0} of (7.3) we obtain the following local problem for $u^1(y)$ (where x, t are parameters)

$$(7.10)\qquad \begin{cases} \Delta_y\, u^1 = 0 & \text{in } B \\ \dfrac{\partial u^1}{\partial n} = -\left(\dfrac{\partial u^o}{\partial n} + \lambda\, u^o\right) & \text{for } y_2 = F(y_1) \\ u^1 \text{ is B-periodic and satisfies (7.6)} & \end{cases}$$

We shall see later that this problem has a solution which is unique up to an additive constant if a compatibility condition is satisfied by u^o. In order to obtain such a condition, let us suppose that u^1 exists. Then, by integrating by parts, with the periodicity conditions, we have (Σ_F is the curve $y_2 = F(y_1)$).:

$$(7.11)\qquad 0 = \int_B \Delta u^1\, dy = \int_{\partial B} \frac{\partial u^1}{\partial n}\, d\sigma = \int_{\Sigma_F} \frac{\partial u^1}{\partial n}\, d\sigma = -\int_{\Sigma_F} \left(\frac{\partial u^o}{\partial n} + \lambda\, u^o\right) d\sigma =$$

$$= -\frac{\partial u^o}{\partial x_i} \int_{\Sigma_F} n_i\, d\sigma - \lambda\, u^o \int_{\Sigma_F} d\sigma$$

but

$$(7.12)\qquad \int_{\Sigma_F} d\sigma = |\Sigma_F| \quad ; \quad \int_{\Sigma_F} n_i\, d\sigma = \delta_{i2}$$

where Σ_F denotes the measure (length) of the arc Σ_F. δ_{i2} is the Kronecker symbol (if the period of F in the direction y_1 is not one, the second relation (7.12) must be multiplied by the period). Let us define the "waving coefficient" Γ of the boundary by

$$\Gamma = |\Sigma_F|$$

this is evidently the ratio of the lenght of $\partial\Omega_\varepsilon$ to the lenght of $\partial\Omega_o$.

Note that (7.11) is written in the basis associated with y_1, y_2 ; consequently we have $x_1 = s$, $x_2 = N$; (7.11) becomes

$$(7.13)\qquad \frac{\partial u^o}{\partial N} + \lambda\, \Gamma\, u^o = 0$$

which is the desired compatibility condition.

Consequently, the "limit problem" for $u^o(x\ ,\ t)$ is

$$(7.14)\qquad \begin{cases} \dfrac{\partial u^0}{\partial t} = \Delta\, u^0 & \text{for } x \in \Omega_0 \ , \ t \in \,]\,0\ ,\ \infty\,[\\ u^0(x\ ,\ 0) = u_0(x) & \text{for } \ x \in \Omega_0 \\ (7.13) & \text{for } x \in \partial\Omega_0 \ , \ t \in \,]\,0\ ,\ \infty\,[\ . \end{cases}$$

Note that the first equation of (7.14) is immediately obtained from (7.5) out of the boundary layer (see (7.6)).

Proposition 7.1 - If we postulate an expansion of the type (7.5), the first term $u^0(x\ ,\ t)$ (i.e., the limit of u^ε as $\varepsilon \searrow 0$) is uniquely determined by (7.14). This is a problem analogous to (7.2) - (7.4) but in Ω_0 instead of Ω_ε and with the boundary condition (7.13) instead of (7.3). Note that the coefficient λ is multiplied by the waving coefficient Γ of $\partial\Omega_\varepsilon$.

It is worthwhile to prove the existence of u^1 (i.e. of the boundary layer).

Theorem 7.1 - If u^0 is the solution of (7.14) the local problem (7.10) for $u^1(y)$ has a solution, which is unique up to an additive constant.

The remainder of this section is devoted to the proof of this theorem. The exact definition of "solution" will be clear later.

Let us construct a B-periodic function $a(y)$ satisfying

$$(7.15)\qquad \frac{\partial a}{\partial n} = - \frac{\partial u^0}{\partial x_i}\, n_i \ - \ \lambda u^0 \qquad \text{on } \ \Sigma_F$$

and identically zero for sufficiently large $-y_2$. It is clear that for the study of the local problem, x and t are parameters ; moreover $x_1 = s$, $x_2 = N$. The function evidently exists and is smooth (because F does). Moreover

$$(7.17)\qquad \int_B \Delta\ a\ dy = 0$$

For, by integrating by parts, (7.17) is equal to

$$\int_{\partial B} \frac{\partial a}{\partial n}\, d\sigma$$

this integral vanishes on Σ_F by (7.15) and (7.11) (which is equivalent to (7.13)) ; on the remainder of ∂B the integral vanishes by the periodicity and nullity conditions.

Now, we take the new unknown $v = u^1 - a$. The problem for v is

$$(7.18)\qquad \Delta\ v = -\ \Delta\ a$$

(7.19) $$\left.\frac{\partial v}{\partial n}\right|_{\Sigma_F} = 0 \qquad ; \qquad \underline{\text{grad}}\ v \xrightarrow[y_2 \to -\infty]{} 0$$

(7.20) v is B-periodic

Let us define the set V of the functions w which are B-periodic and smooth, constant for sufficiently large $-y_2$. We immediatly obtain

(7.21) $$\int_B \Delta\, v\, w\, dy = \int_{\partial B} \frac{\partial v}{\partial n}\, w\, d\sigma - \int_B \frac{\partial v}{\partial y_i}\frac{\partial w}{\partial y_i}\, dy \qquad \forall\, w \in V$$

and the term $\int_{\partial B}$ vanishes by periodicity and (7.19). We then have

(7.22) $$\int_B \frac{\partial v}{\partial y_i}\frac{\partial w}{\partial y_i}\, dy = \int_B \Delta\, a\, w\, dy \qquad \forall w \in V$$

Moreover, if v satisfies (7.20), the second relation of (7.19) and (7.22), by using (7.21) we see that (7.18) and the first relation of (7.19) are satisfied. It is then easy to obtain a variational formulation of (7.18) - (7.20) up to an additive constant.

We consider the equivalence class obtained by identifying the elements of V difference of which is a constant. We introduce the scalar product

(7.23) $$(\hat{v}\ ,\ \hat{w})_{\hat{V}} = \int_B \frac{\partial v}{\partial y_i}\frac{\partial w}{\partial y_i}\, dy$$

in the space of the equivalence class, where v, (or w) is any element of the equivalence class $\hat{v}$ (or $\hat{w}$). Note that $(\hat{v}\ ,\ \hat{v}) = 0 \Rightarrow v = \text{const.} \Rightarrow \hat{v} = 0$. We then define $\hat{V}$ as the Hilbert space obtained by completion of the equivalence class space with the norm associated with (7.23).

The variational formulation of (7.18) - (7.20) is :

(7.24) Find $\hat{v} \in \hat{V}$ such that
$$(\hat{v}\ ,\ \hat{w})_{\hat{V}} = \int_B \Delta\ a\ \hat{w}\ dy \qquad \forall\ \hat{w} \in \hat{V}$$

where the right hand side is for the right hand side of (7.22) with any $w \in \hat{w}$ (note that by virtue of (7.17), this value is independent of the particular w chosen, it only depends on the equivalence class $\hat{w}$).

The existence and uniqueness of $\hat{v}$ will be proved if we show that the right hand side of (7.24) is a bounded functional on $\hat{V}$. To this end, we note that Δa is zero for sufficiently large $-y_2$; consequently the domain of integration is in fact a bounded set, denoted by Bd, where the Poincaré's inequality (see Mikhlin [1] p. 337) :

$$(7.25) \qquad \int_{Bd} w^2\, dy \leqslant C \left[\int_{Bd} \frac{\partial w}{\partial y_i} \frac{\partial w}{\partial y_i}\, dy + \left(\int_{Bd} w\, dy \right)^2 \right]$$

holds. Moreover, for a given equivalence class $\hat{w}$, we may choose $w \in \hat{w}$ in such a way that the mean value of w on Bd is zero. We then have :

$$\left| \int_B \Delta\, a\, \hat{w}\, dy \right| \leqslant C \left(\int_{Bd} w^2\, dy \right)^{1/2} \leqslant C' \left(\int_{Bd} \frac{\partial w}{\partial y_i} \frac{\partial w}{\partial y_i}\, dy \right)^{1/2} = C' \| \hat{w} \|_{\hat{V}}$$

and the functional is bounded. The theorem is proved.

Remark 7.1 - The local problem does not hold in a bounded domain and the Rellich compactness theorem does not hold. Consequently it is not easy to see if the Fredholm's alternative holds or not. This is the reason why we introduced the space of the equivalence class.■

Remark 7.2 - If we consider the three-dimensional case, all the results hold but Γ is defined as the ratio of the areas of the surfaces $\partial\Omega_\varepsilon$ and $\partial\Omega_0$.■

8.- Proof of the convergence - This section is devoted to the proof of the convergence $u^\varepsilon \to u^0$ in the problem of the preceeding section. We begin with the associated stationary problem. For spectral properties, see chap. 11, sect. 6.

Let Ω_ε and Ω_0 be defined as in the preceeding section. If Γ is the waving coefficient and λ, μ are two real positive constants, we consider

$$(8.1) \qquad (-\Delta + \mu)\, u^\varepsilon = f \qquad \text{in } \Omega_\varepsilon$$

$$(8.2) \qquad \frac{\partial u^\varepsilon}{\partial n} + \lambda\, u^\varepsilon = 0 \qquad \text{on } \partial\Omega_\varepsilon$$

and

$$(8.3) \qquad (-\Delta + \mu)\, u^0 = f \qquad \text{in } \Omega_0$$

$$(8.4) \qquad \frac{\partial u^0}{\partial n} + \lambda\, \Gamma\, u^0 = 0 \qquad \text{on } \partial\Omega_0$$

where f is a given function of $L^2(R^2)$ (and thus of $L^2(\Omega_\varepsilon)$ for any ε).

It is easily seen that u^ε and u^0 are uniquely determined. In fact, the variational formulations of (8.1), (8.2) (proof of which is immediate) is :

Find $u^\varepsilon \in H^1(\Omega_\varepsilon)$ such that

$$(8.5) \qquad \int_{\Omega_\varepsilon} \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx + \mu \int_{\Omega_\varepsilon} u^\varepsilon v\, dx + \lambda \int_{\partial\Omega_\varepsilon} u^\varepsilon\, v\, d\sigma = \int_{\Omega_\varepsilon} f\, v\, dx \quad \forall v \in H^1(\Omega_\varepsilon)$$

and an analogous one for u^0.

Theorem 8.1 - If u^ε, u^0 are defined by (8.1) - (8.4),

(8.6) $$u^\varepsilon\big|_{\Omega_0} \xrightarrow[\varepsilon \searrow 0]{} u^0 \qquad \text{in } H^1(\Omega_0) \text{ weakly} \quad .$$

If we take $v = u^\varepsilon$ in (8.5), we immediately have

(8.7) $$\| u^\varepsilon \|_{H^1(\Omega_\varepsilon)} \leqslant C \qquad (\text{indep. of } \varepsilon)$$

Then, by extracting a sequence $\varepsilon \to 0$:

(8.8) $$u^\varepsilon\big|_{\Omega_0} \longrightarrow u^* \qquad \text{in } H^1(\Omega_0) \quad \text{weakly}$$

and it suffices to prove that $u^* = u^0$. Because Ω_0 has a smooth boundary it suffices to prove that

(8.9) $$\int_{\Omega_0} \frac{\partial u^*}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx + \mu \int_{\Omega_0} u^* \, v \, dx + \lambda \, \Gamma \int_{\partial\Omega_0} u^* \, v \, d\sigma = \int_{\Omega_0} f \, v \, dx$$

for any smooth function v. It is clear that, by (8.8), the integrals over Ω_ε in (8.5) converge to the corresponding integrals over Ω_0 in (8.9) (Note that the integrals over $\Omega_\varepsilon - \Omega_0$ are bounded above by

(8.10) $$\left| \int_{\Omega_\varepsilon - \Omega_0} u^\varepsilon \, v \, dx \right| \leqslant \| u^\varepsilon \|_{L^2(\Omega_\varepsilon - \Omega_0)} \| v \|_{L^2(\Omega_\varepsilon - \Omega_0)}$$

and an analogous one for the derivatives. The right hand side of (8.10) tends to zero by (8.7) and $|\Omega_\varepsilon - \Omega_0| \to 0$).

Consequently, theorem 8.1 will be proved if we prove the following lemma.

Lemma 8.1 -

(8.11) $$\int_{\partial\Omega_\varepsilon} u^\varepsilon \, v \, d\sigma \longrightarrow \Gamma \int_{\partial\Omega_0} u^* \, v \, d\sigma$$

for any smooth function v.

Proof - We shall write (8.11) in the curvilinear coordinates s, N.

(8.12) $$\partial\Omega_0 \text{ is } N = 0 \quad ; \quad d\sigma = ds$$

$$\partial\Omega_\varepsilon \text{ is } N = \varepsilon \, F_\varepsilon(s) \text{ where } F_\varepsilon(s) \equiv F\left(\frac{s}{\varepsilon}\right) \quad .$$

Then, if g_{ij} are the components of the metric tensor,

$$d\sigma^2 = g_{11} \, ds^2 + g_{22} \, dN^2 + 2 \, g_{12} \, ds \, dN \quad ; \quad dN = F_\varepsilon'(s) \, ds$$

But it is clear that, because the curvilinear coordinates are smooth,

$$g_{11} \to 1 \qquad ; \qquad g_{12} \to 0 \qquad ; \qquad g_{22} \to 1$$

uniformly as $N \to 0$. Thus, we have

$$d\sigma = [\, 1 + F'_\varepsilon(s)^2 \,]^{1/2} \, ds \quad (1 + \delta(\varepsilon)) \quad \text{on } \partial\Omega_\varepsilon \tag{8.13}$$

where $\delta(\varepsilon)$ tends to zero as $\varepsilon \searrow 0$ (uniformly).

Consequently, the left hand side of (8.11) is :

$$\int_{\partial\Omega_0} u^\varepsilon(s, \varepsilon F_\varepsilon(s)) v(s, \varepsilon F_\varepsilon(s)) \sqrt{1 + F'_\varepsilon(s)^2} \, (1 + \delta(\varepsilon)) \, ds \tag{8.14}$$

On the other hand, by the trace theorem, from (8.8) we have

$$u^\varepsilon \Big|_{\partial\Omega_0} \to u^* \Big|_{\partial\Omega_0} \qquad \text{in } L^2(\partial\Omega_0) \ \text{ strongly} \tag{8.15}$$

Moreover, using (8.7) and a calculation analogous to that of the trace theorem, chap. 1, sect. 3, we have, by using (8.7) :

$$|u^\varepsilon(s, \varepsilon F_\varepsilon(s)) - u^\varepsilon(s,0)|^2 = \left| \int_0^{\varepsilon F_\varepsilon} \frac{\partial u^\varepsilon}{\partial N}(s,\xi) \, d\xi \right|^2 \leqslant \varepsilon F_\varepsilon \int_0^{\varepsilon F_\varepsilon} \left| \frac{\partial u^\varepsilon}{\partial N} \right|^2 d\xi \Longrightarrow$$

$$\int_{\partial\Omega_0} \left| u^\varepsilon(s, \varepsilon F_\varepsilon) - u^\varepsilon(s,0) \right|^2 ds \leqslant C\varepsilon \int_{\Omega_\varepsilon - \Omega_0} |\mathrm{grad}\, u^\varepsilon|^2 \, dx \leqslant C\varepsilon \tag{8.16}$$

This, with (8.15) implies that :

$$u^\varepsilon(s, \varepsilon F_\varepsilon(s)) \quad \text{is bounded in } L^2(\partial\Omega_0) \tag{8.17}$$

As a consequence, we can neglect the term $\delta(\varepsilon)$ in (8.14) because it tends to zero. For the same reason, we may write $v(s,0)$ instead of $v(s,\varepsilon F_\varepsilon)$ in (8.14). Thus, it suffices to prove that

$$\int_{\partial\Omega_0} u^\varepsilon(s, \varepsilon F_\varepsilon) \, v(s,0) \sqrt{1 + F'^2_\varepsilon} \, ds \to \Gamma \int_{\partial\Omega_0} u^*(s,0) \, ds \quad . \tag{8.18}$$

But from (8.15) and (8.16), $u^\varepsilon(s,\varepsilon F_\varepsilon)$ converges to $u^*(s,0)$ in $L^2(\partial\Omega_0)$ strongly. Moreover, $\sqrt{1 + F'^2_\varepsilon}$ is a ε-periodic function, thus by lemma 4.1 it tends in $L^2(\partial\Omega_0)$ weakly to its mean value which is Γ, and (8.18) follows. Lemma 8.1 (and thus theorem 8.1) is proved.∎

Now we consider the parabolic problem of the preceeding section.

<u>Theorem 8.2</u> - If u^ε (resp. u^0) are the solutions of (7.2) - (7.4) (resp. (7.14)),

$$\left|\begin{array}{l} (8.19) \qquad u^\varepsilon\big|_{\Omega_0} \xrightarrow[\varepsilon \searrow 0]{} u^0 \\ u^\varepsilon \text{ and } \dfrac{\partial u^\varepsilon}{\partial x_k} \text{ converge in } L^\infty(0,\infty;\ L^2(\Omega^0)) \text{ weakly } * \end{array}\right.$$

<u>Proof</u> - First, taking into account that the Laplace operator with the boundary condition (8.2) is associated with the form in (8.5), we see that u^ε and u^0 are well determined in the framework of semigroup theory. The proof is then analogous to that of sect. 6. From (7.2) - (7.4) we see that u^ε is bounded in $L^\infty(0,\infty;L^2(\Omega_\varepsilon))$ and $\frac{\partial u^\varepsilon}{\partial x_k}$ in $L^2(0,\infty;L^2(\Omega_\varepsilon))$ and by extracting a subsequence

$$(8.20) \qquad u^\varepsilon\big|_{\Omega_0} \to u^*$$

in the topologies indicated in theorem 8.2. By taking the Laplace transform, (which commutes with $|_{\Omega_0}$) :

$$(8.21) \qquad \hat{u}^\varepsilon\big|_{\Omega_0} \to \hat{u}^* \qquad \text{in } H^1(\Omega_0) \text{ weakly}$$

but $\hat{u}^\varepsilon$ satisfies

$$A_\varepsilon \hat{u}^\varepsilon + p\,\hat{u}^\varepsilon = u_0 \quad \text{for Re } p > 0$$

where A_ε is the Laplace operator in Ω_ε with the boundary condition (7.3). We have an analogous relation for $\hat{u}^0$. Theorem 8.1 with (8.21) then shows that $\hat{u}^0 = \hat{u}^*$ for real p (and by analytic continuation, for Re $p > 0$). Then $u^* = u^0$ and (8.20), the desired result, follows ■

9.- <u>Asymptotic expansion of an integral identity</u> - We consider here another method of expansion to find again the results of sections 1 - 4. This method is based on the variational formulation (1.8) instead of the classical formulation (1.6), (1.7). The advantage of the new method is that it deals only with first order derivatives and the calculations are shorter than with the classical method.

Let us begin with a formal calculation. Let $F(x, y)$ be a function of x and y, Y-periodic in y in the framework of (1.10). Let Φ be the function defined by

$$(9.1) \qquad \Phi(\varepsilon) = \int_\Omega F\left(x, \frac{x}{\varepsilon}\right) dx$$

We then have

$$(9.2) \qquad \lim_{\varepsilon \to 0} \Phi(\varepsilon) = \int_\Omega \tilde{F}(x)\, dx$$

where, as usual, $\tilde{F}(x) = \dfrac{1}{|Y|} \displaystyle\int_Y F(x, y)\, dy$.

To obtain (9.2), we consider (9.1) as the sum of the integrals on the periods εY.

(9.3) $$\Phi(\varepsilon) = \sum_{\text{periods}} |\varepsilon Y| \left(\frac{1}{|\varepsilon Y|} \int_{\varepsilon Y} F(x , \frac{x}{\varepsilon})\, dx \right)$$

and in each period, we may consider x constant in the first argument because the corresponding variation of F tends to zero. Then, by writing each integral in the variable y, we have

$$\Phi(\varepsilon) \cong \sum_{\text{periods}} |\varepsilon Y| \left(\frac{1}{|Y|} \int_Y F(x , y)\, dy \right)$$

because the jacobian of the transformation is $|\varepsilon Y| / |Y|$, and (9.2) follows.

Now, <u>we consider the Dirichlet problem with periodic coefficients under the form (1.8), i.e.</u> :

(9.4) $$\begin{cases} \text{Find } u^\varepsilon \in H_0^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega a_{ij}(\frac{x}{\varepsilon}) \frac{\partial u^\varepsilon}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx - \int_\Omega f\, v\, dx = 0 \qquad \forall\, v \in H_0^1(\Omega) \end{cases}$$

We then introduce the asymptotic expansion (1.10) for the solution, i.e. :

(9.5) $$u^\varepsilon(x) = u^0(x) + \varepsilon u^1(x,y) + \varepsilon^2 \ldots \quad , \qquad y = \frac{x}{\varepsilon} \quad , \quad Y\text{-periodic in } y$$

Moreover, (9.4) is satisfied for any $v \in H_0^1$. In particular, for small ε we may take test functions of the form (see also remark 9.1 later)

(9.6) $$v = v^\varepsilon = v^0(x) + \varepsilon\, v^1(x , y) + \ldots. \quad ; \quad y = \frac{x}{\varepsilon} \quad ; \quad Y\text{-periodic in } y.$$

As usual, we have

$$\frac{\partial u^\varepsilon}{\partial x_i} = \frac{\partial u^0}{\partial x_i} + \frac{\partial u^1}{\partial y_i}$$

and an analogous relation for v. The integral identity (9.4) becomes <u>at the first order</u> :

(9.7) $$\int_\Omega a_{ij} \frac{\partial u^0}{\partial x_i} \frac{\partial v^0}{\partial x_j}\, dx + \int_\Omega a_{ij} \frac{\partial u^1}{\partial y_i} \frac{\partial v^0}{\partial x_j}\, dx + \int_\Omega a_{ij} \frac{\partial u^0}{\partial x_i} \frac{\partial v^1}{\partial y_j}\, dx + \int_\Omega a_{ij} \frac{\partial u^1}{\partial y_i} \frac{\partial v^1}{\partial y_j}\, dx - \int_\Omega f\, v^0\, dx = 0$$

for any v^0, v^1 (Y-periodic in y). If in particular we take $v^1 = 0$, v^0 arbitrary, we have

(9.8) $$\int_\Omega a_{ij}(y) \left(\frac{\partial u^0}{\partial x_i} + \frac{\partial u^1}{\partial y_i}\right) \frac{\partial v^0}{\partial x_j}\, dx - \int_\Omega f\, v^0\, dx = 0$$

and by taking the limit as $\varepsilon \to 0$ according to (9.1), (9.2) :

$$(9.9) \qquad \int_\Omega \tilde{p}_j^o \frac{\partial v^o}{\partial x_j} dx - \int_\Omega f\, v^o\, dx = 0 \qquad \text{where}$$

$$(9.10) \qquad p_j^o \equiv a_{ij}\left(\frac{\partial u^o}{\partial x_i} + \frac{\partial u^1}{\partial y_i}\right)$$

and (9.10) shows that (in the sense of distributions) :

$$(9.11) \qquad - \frac{\partial \tilde{p}_j^o}{\partial x_j} = f \quad .$$

Moreover, by replacing (9.8) into (9.7), only the terms in v^1 remain ; and by taking the limit value according to (9.1), (9.2),

$$(9.12) \qquad \int_\Omega \left[a_{ij}\left(\frac{\partial u^o}{\partial x_i} + \frac{\partial u^1}{\partial y_i}\right) \frac{\partial v^1}{\partial y_j} \right]^\sim dx = 0 \qquad \forall\, v^1 \ \ Y\text{-periodic}$$

This relation may be considered as an equation to find $u^1(x , y)$ (defined on $\Omega \times Y$) if $u^o(x)$ is given. In fact, it is easier to take

$$(9.13) \qquad v^1 = \theta(x)\, w(y) \quad ; \quad \theta \in \mathcal{D}(\Omega) \quad ; \quad w \in V_Y$$

where V_Y is the space of periodic functions defined in sect. 2. We then have

$$\int_\Omega [\, a_{ij}(\frac{\partial u^o}{\partial x_i} + \frac{\partial u^1}{\partial y_i}) \frac{\partial w}{\partial y_j}\,]^\sim\, \theta\, dx = 0 \qquad \forall\, \theta\, ,\, w$$

and this implies

$$(9.14) \qquad [\, a_{ij}(\frac{\partial u^o}{\partial x_i} + \frac{\partial u^1}{\partial y_i}) \frac{\partial w}{\partial y_j}\,]^\sim = 0 \qquad \forall\, w \in V_Y$$

which is exactly the equation for the local behaviour, written under the form (5.13).

Consequently, the homogenized problem is obtained under the form (9.9) (or 9.11), (9.14). Of course, $u^o \in H_0^1(\Omega)$ gives the boundary condition on $\partial\Omega$.

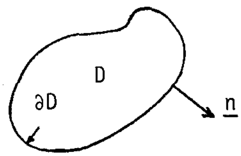
Remark 9.1 - We have only used test functions v^1 of the form (9.13). In fact, we may consider instead of (9.6) :

$$(9.15) \qquad \begin{cases} v = v^o(x) + \theta(x)\, w(y) \quad ; \\ v^o \in H_0^1(\Omega) \quad ; \quad \theta \in \mathcal{D}(\Omega) \quad ; \quad w \in V_Y \end{cases}$$

and the expansion (9.5) of the unknown may be assumed only out of a boundary layer near $\partial\Omega$ (see remark 1.3). The preceeding considerations furnish in this case the behaviour out of a neighbourhood of $\partial\Omega$. ∎

10.- Method of the conservation law - In physics it is customary to consider an equation of the type (1.1), (1.2) as equivalent to the conservation law

(10.1) $$\int_{\partial D} p_i \, n_i \, ds + \int_D f \, dx = 0 \qquad \forall \, D$$



where D is any subdomain of the domain of definition of u. If u satisfies (1.2), we obtain (10.1) by integration of (1.2) on D and utilisation of the divergence theorem. Conversely, if (10.1) is satisfied, for any D, by the divergence theorem, we have

(10.2) $$\int_D \left(\frac{\partial p_i}{\partial x_i} + f \right) dx = 0$$

and, if the integrand is continuous (which is a hypothesis of regularity for u), this shows that it is zero. (For, if it is positive or negative at a point, it is also positive or negative in a neighbourhood of this point, and (10.2) should not hold by taking D equal to that neighbourhood).

This remark is the basis of another method to obtain the macroscopic equations. This method is often useful in physics because it shows the physical meaning of the equation (conservation of mass, momentum, energy ...).

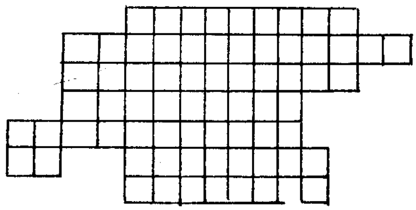
If in (1.6) we take only the first term of the expansion, we have (1.18) and consequently the study of the local behaviour and the homogenized coefficients. The macroscopic equation may be obtained in the following way (without using equation (1.19)).

We consider a domain D whose dimensions are of order O(1) (independent of ε), made of whole periods, but otherwise arbitrary. From equations (1.6) and (1.9), the remark at the begining of this section gives :

$$\int_{\partial D} p_i^{\varepsilon} \, n_i \, dS + \int_D f \, dx = 0$$

and thus, at the first order,

(10.3) $$\int_{\partial D} p_i^{o} \, n_i \, dS + \int_D f \, dx = 0$$



Let us now suppose (this will be proved later) that the integral of $p_i^0 n_i$ on each face of a period is the same as that of its mean value $\tilde{p}_i^0\, n_i$. We then have

$$(10.4) \qquad \int_{\partial D} \tilde{p}_i^0\, n_i\, dS + \int_D f\, dx = 0$$

which is a conservation law for functions of the macroscopic variable x. This implies

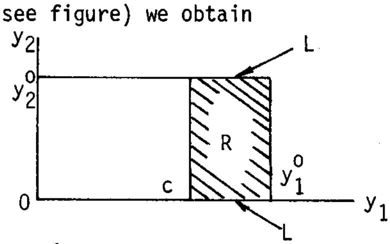
$$(10.5) \qquad \frac{\partial \tilde{p}_i^0}{\partial x_i} + f = 0$$

which is the desired relation (1.23). (Note that the integrand of (10.4) is independent of ε and the reasoning leading to (10.5) holds for small ε as for arbitrary D).

Now, we prove the assumed property on the integrals of $p_i^0\, n_i$ and $\tilde{p}_i^0\, n_i$.

Proposition 10.1 - The integral of $p_i^0(y)n_i$ on a face of a period is the same as the integral of its mean value $p_i^0 n_i$.

Proof - We consider a period (x is a parameter). By integrating (1.18) on a region R (see figure) we obtain



$$(10.6) \qquad \int_{\partial R} p_i^0\, n_i\, dS = 0$$

and by Y-periodicity, the surface integral on the lateral surface L vanishes, and we see that the integral

$$(10.7) \qquad \Phi(c) = \int_0^{y_2^0} \int_0^{y_3^0} p_1^0(c\,,\, y_2\,,\, y_3)\, dy_2\, dy_3$$

is equal to

$$\int_0^{y_2^o}\int_0^{y_3^o} p_i^o(y_1^o, y_2 , y_3)\, dy_2\, dy_3$$

i.e., Φ is independent of c. We then have

$$(10.8) \qquad \boxed{\Phi = \frac{1}{y_1^o}\int_0^{y_1^o} \Phi\, dc = \frac{y_2^o\, y_3^o}{|Y|}\int_Y p_1^o\, dy = y_2^o\, y_3^o\, \tilde{p}_1^o}$$

and the equality of the right hand sides of (10.7) and (10.8) is the desired relation.

Remark 10.1 - Proposition 10.1 may be stated by saying that "The volume and surface means coincide". It is clear that this fact is not general ; it holds for vectors of zero divergence in the variable y (see (1.18), which implies (10.6)). ∎

11.- Comments and bibliographical notes - The homogenization method for the study of composite materials is based on the study of periodic solutions of partial differential equations, and their asymptotic behaviour as the period tends to zero. It is fitted for the study of composite materials with periodic structure, and it may also be taken as a model of other composite materials. The hypothesis of periodic structure permits a rigorous treatment of the problem, which is useful in very controversed problem, such that mixtures of solids and fluids, fluid flow in porous media and so on (see the following chapters). Generally speaking, we shall not give comparaisons and references for other methods ; this is made in the papers cited in the litterature. Nevertheless, we point out Sendeckyj [1] as a general reference for other methods.

First papers on homogenization were Sanchez-Palencia [1], [2], [3], where only a formal study relevant of asymptotic methods (such as in Cole [1] or Van Dyke [1]) was given. De Giorgi and Spagnolo [1] gave the first proof of the convergence of the method, based on the G-convergence theory of partial differential equations (see Spagnolo [1], [2]). For recent developments of G-convergence, including non linear problems, see for instance Biroli [1], Boccardo et Marcellini [1], Carbone [1], Marcellini and Sbordone [1], Murat [2], Tartar [3] and the recent review paper of De Giorgy [1]. Tartar [3], [4] gave a simplified proof of convergence (sect. 4) which applies to other problems. Bensoussan, Lions and Papanicolaou [2] studied a great number of homogenization problems ; this is the general reference in homogenization theory. It includes time-varying coefficients, probabilistic methods and wave propagation for small wave length. An abridged version of some parts of this book may be seen in Lions [5].

The convergence of the resolvents in the norm (sect. 5) is suggested by Boccardo and Marcellini [1] and gives in a natural way the perturbation of eigen-values and eigenvectors (see chapter 11, sect. 3) without a hypothesis of self-adjointness (compare with Kesavan [1]). The homogenization of a boundary (sect. 7 and 8) seems new (see Brizzi et Chalot for other related problems). The method of section 9 is fitted the asymptotic study of problems in variational form, and is systematically used in the following chapters, including variational inequalities (chap. 6, sect. 6 and 7). See also, in this convection, Bensoussan, Lions et Papanicolaou [2] , remarks at the end of chapter 1, sect. 18.

For other homogenization problems, see Babuska [1] , Bakhvalov [1] , [2], Bensoussan, Lions and Papanicolaou [3] , Desgraupes [1] ; for some non linear problems, see Artola et Duvaut [2] , Damlamian [1] . The problems of homogenization of media with holes (see also the following chapters and the appendix by L. Tartar) were studied by Tartar [3] , [4] , Cioranescu and Saint Jean Paulin [1] and Vanninathan [1] , [2] . Scattering problems were studied by Codegone [1] . Numerical results and comparaison with exact solutions were given by Bourgat [1] , Bourgat et Dervieux [1] and Bourgat et Lanchon [1] .