Master's Thesis

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Abstract

Fill it.

1 Introduction

2 Motivation

3 Method

The idea of asymptotic homogenization. In a repeating cell Y,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1}$$

where $C_{ijkl}(\underset{\sim}{x} + \underset{\sim}{u}\underset{\sim}{Y}) = C_{ijkl}(\underset{\sim}{x})$

$$\Rightarrow C_{ijkl}(x_1 + n_1Y_1\, x_2 + n_2Y_2\, x_3 + n_3Y_3) = C_{ijkl}(x_1, x_2, x_3) \tag{2}$$

$C_{ijkl}(\underset{\sim}{x})$ is Y-periodic

$$\underset{\sim}{y} = \tfrac{\underset{\sim}{x}}{\epsilon} \tag{3}$$

$$\Rightarrow g = g(\underset{\sim}{x}, \tfrac{\underset{\sim}{x}}{\epsilon}) = g(\underset{\sim}{x}\, \underset{\sim}{y}) \tag{4}$$

$\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ defines the domain of the composite Ω. The domain is composed of base cells of dimensions, $\varepsilon Y_1, \varepsilon Y_2, \varepsilon Y_3$ where $\underset{\sim}{y} = \frac{x}{\varepsilon}$

3.1 1D Elasticity

$$\sigma^\varepsilon = E^\varepsilon \tfrac{\partial u^\varepsilon}{\partial x} \tag{5}$$

$$\tfrac{\partial \sigma^\varepsilon}{\partial x} + \gamma^\varepsilon = 0 \quad E^\varepsilon\, \gamma^\varepsilon \rightarrow macroscopicallyuniform \tag{6}$$

Inside each cell,

$$E^\varepsilon(x, \frac{x}{\varepsilon}) \quad = E(y) \tag{7}$$

$$\gamma^\varepsilon(x, \frac{x}{\varepsilon}) \quad = \gamma(y) \tag{8}$$

Let

$$u^{\varepsilon}(x) = u^{0}x, y + \varepsilon u^{1}(x, y) + \varepsilon^{2}u^{2}(x, y) + ... \tag{9}$$

$$\sigma^{\varepsilon}(x) = \sigma^{0}x, y + \varepsilon \sigma^{1}(x, y) + \varepsilon^{2}\sigma^{2}(x, y) + ... \tag{10}$$

3.2 Optimal Design of Elastic structures

$\mathbf{b} \rightarrow$ body forces
$\mathbf{t} \rightarrow$ surface tractions

Optimal choice of $\mathbb{C}_{ijkl} \in U_{ad} \leftarrow$ admissible set of elasticity
$\mathbb{C}_{ijkl}(\mathbf{x}) \forall \mathbf{x} \in \Omega$ has 21 independent components
$a_E(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbb{C}_{ijkl}\, \varepsilon_{kl}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{v}) d\mathbf{v} \rightarrow$ energy bilinear form
$L(\mathbf{v}) = \int_{\Omega} \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} ds \rightarrow$ load linear form.

Minimum compliance problem:

$$minimize \qquad L(\mathbf{v}), \tag{11}$$

$$subject\, to \qquad \mathbb{C}_{ijkl} \in \mathbb{U}_{ad} \tag{12}$$

$$a_E(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{U} \tag{13}$$

where $\mathbb{U} \rightarrow$ kinematically admissible displacements.
For optimal shape design:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = \chi(\mathbf{x})\overline{\mathbb{C}}_{ijkl}, \quad \text{where } \overline{\mathbb{C}}_{ijkl} \rightarrow \text{stiffness matrix of the material} \tag{14}$$

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m, \\ 0 & \text{if } \mathbf{x} \in \Omega \backslash \Omega^m \end{cases} \tag{15}$$

where $\Omega^m \rightarrow$ part of the domain occupied by the material.
For sizing problem:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = h(\mathbf{x})\overline{\mathbb{C}}_{ijkl} \tag{16}$$

$$\textstyle\int_{\Omega} \chi(\mathbf{x}) d\mathbf{x} = V_f \tag{17}$$

$$\& \quad \textstyle\int_{\Omega} h(\mathbf{x}) d\mathbf{x} = V_f. \tag{18}$$

where $h(x)$ is a sizing function.
Traditionally shape design problems are initiated in the following manner:

$$Ref\ doamin : \Omega_0 \in \mathbb{R}^3 \tag{19}$$

$$\underline{\phi} : \Omega_0 \rightarrow \phi(\Omega_0) \text{is a diffeomorphism.} \tag{20}$$

$$L(\mathbf{v}) = \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| ds \tag{21}$$

$$\begin{aligned} a_E &= \int_{\Omega} \mathbb{C}_{ijkl}(\mathbf{x}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \tag{22}$$

Now,

$$\begin{aligned}
\mathbb{C}_{ijkl}\varepsilon_{kl} &= \mathbb{C}_{ijkl}\frac{1}{2}(u_{k,l}+u_{l,k}) \\
&= \frac{1}{2}\mathbb{C}_{ijkl}u_{k,l}+\frac{1}{2}\mathbb{C}_{ijlk}u_{l,k} \\
&= \mathbb{C}_{ijkl}u_{k,l}
\end{aligned} \tag{23}$$

$$\begin{aligned}
a_E &= \int_{\Omega_0} \mathbb{C}_{ijkl}u_{k,l}(\mathbf{u})u_{i,j}(\mathbf{v})|det(D\underline{\phi}^{-1}|d\mathbf{x} \\
&= \int_{\Omega_0} \mathbb{C}_{ijkl}\frac{\partial u_k}{\partial \mathbf{x}_m}(D\underline{\phi}^{-1})_{ml}\frac{\partial u_i}{\partial \mathbf{x}_p}(D\phi^{-1})_{pj}|det(D\underline{\phi}^{-1})|d\mathbf{x}
\end{aligned} \tag{24}$$

$$\Rightarrow \mathbb{C}_{ijkl}(D\underline{\phi}^{-1})_{ml}(D\underline{\phi}^{-1})_{pj}|det(D\underline{\phi}^{-1})| = \bar{\mathbb{C}}_{ipkm} \tag{25}$$

$$\bar{\mathbb{C}}_{ijkl} = \mathbb{C}_{ipkm}(D\underline{\phi}^{-1})_{lm}(D\underline{\phi}^{-1})_{jp}|det(D\underline{\phi}^{-1})| \tag{26}$$

Treating $\underline{\phi}$ as a design variable is tidious.

3.3 Homogenization method

$$E^{\varepsilon}_{ijkl}(\mathbf{x}) = E_{ijkl}(\mathbf{x},\mathbf{y}), \qquad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \tag{27}$$

The tensor E^{ε}_{ijkl} is a material constant which satisfies the symmetry condition and is assumed to satisfy strong ellipticity condition for every $\mathbf{x}$.

$$\Rightarrow E^{\varepsilon}_{ijkl} = E^{\varepsilon}_{jikl} = E^{\varepsilon}_{ijlk} = E^{\varepsilon}_{klij} \tag{28}$$

$$E^{\varepsilon}_{ijkl}(\mathbf{x})\mathbf{X}_{ij}\mathbf{X}_{kl} \geq m\mathbf{X}_{ij}\mathbf{X}_{ij} \qquad \text{for some } m>0 \;\&\; \forall \quad \mathbf{X}_{ij}=\mathbf{X}_{ji} \tag{29}$$

Let the domainΩ has a boundary Γ. Let $\mathbf{f}$ be the body force acting on Ω and $\mathbf{t}$ be the traction acting on Γ_t part of the boundary Γ. Also, let Γ_D be the part of boundary on which displacement is defined. Then the displacement $\mathbf{u}^{\varepsilon}$ can be obtained as the solution to the following minimization problem

$$\min_{\mathbf{v}^{\varepsilon}\in U} \quad F^{\varepsilon}(\mathbf{v}^{\varepsilon}), \tag{30}$$

where F^{ε} is total potential energy given as

$$F^{\varepsilon}(\mathbf{v}^{\varepsilon}) = \frac{1}{2}\int_{\Omega} E^{\varepsilon}_{ijkl}\varepsilon_{kl}(\mathbf{v}^{\varepsilon})\varepsilon_{ij}(\mathbf{v}^{\varepsilon})dx - \int_{\Omega}\mathbf{f}\cdot\mathbf{v}^{\varepsilon}dx - \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}^{\varepsilon}ds \tag{31}$$

and $\mathcal{U}$ is the set of admissible displacements defined such that

$$\mathcal{U} = \{\mathbf{v}=v_i\mathbf{e}_i : v_i \in H^1(\Omega) \text{ and } \mathbf{v}\in\mathcal{G} \text{ on } \Gamma_D\} \tag{32}$$

where $\mathcal{G}$ is set of displacement defined along the boundary Γ_D.
Let

$$\mathbf{v}^{\varepsilon}(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \varepsilon\mathbf{v}_1(\mathbf{x},\mathbf{y}), \qquad \mathbf{y}=\frac{\mathbf{x}}{\varepsilon}. \tag{33}$$

Using chain rule for functions in two variables

$$\begin{aligned}\frac{\partial f(\mathbf{x},\mathbf{y}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \\ &= \frac{\partial f}{\partial \mathbf{x}} + \frac{1}{\varepsilon}\frac{\partial f}{\partial \mathbf{y}}\end{aligned} \tag{34}$$

Using above two equations, we can write the linerized strain as

$$\begin{aligned}\epsilon_{ij}(\mathbf{v}^{\varepsilon}(\mathbf{x})) &= \frac{\partial(v_{0i}(\mathbf{x}) + \varepsilon v_{1i}(\mathbf{x},\mathbf{y}))}{\partial x_j} \\ &= \frac{\partial v_{0i}}{\partial x_j} + \varepsilon\left\{\frac{\partial v_{1i}}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial v_{1i}}{\partial y_j}\right\} \\ &\approx \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \qquad \leftarrow \{\varepsilon << 1\}\end{aligned} \tag{35}$$

Therefore, equation (31) can be written as

$$\begin{aligned}F^{\varepsilon}(\mathbf{v}^{\varepsilon}) = \frac{1}{2}\int_{\Omega} E^{\varepsilon}_{ijkl}\left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j}\right)dx \\ - \int_{\Omega}\mathbf{f}\cdot\mathbf{v}_0 dx - \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds + \varepsilon R^{\varepsilon}(\mathbf{v}_0,\mathbf{v}_1)\end{aligned} \tag{36}$$

Here, R^{ε} is the contribution of $\varepsilon\mathbf{v}_1$ in the calculation of energy from body force and traction. Using

$$\lim_{\varepsilon\to 0}\int_{\Omega}\Phi(x, x/\varepsilon)dx = \frac{1}{|Y|}\int_{\Omega}\int_{Y}\Phi(x,y)dydx, \tag{37}$$

we get,

$$\begin{aligned}\lim_{\varepsilon\to 0} F^{\varepsilon}(\mathbf{v}^{\varepsilon}) &= F(\mathbf{v}_0,\mathbf{v}_1) \\ &= \frac{1}{2|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j}\right)dy\,dx \\ &\quad - \int_{\Omega}\mathbf{f}\cdot\mathbf{v}_0 dx - \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds\end{aligned} \tag{38}$$

A minimizer $\{\mathbf{u}_0,\mathbf{u}_1\}$ of the functional F, follow the following equations:

$$\begin{aligned}&\frac{1}{|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}\right)dy\,dx \\ &= \int_{\Omega}\mathbf{f}\cdot\mathbf{v}_0 dx + \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0\end{aligned} \tag{39}$$

$$\frac{1}{|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_i}{\partial x_j}\right)dy\,dx = 0, \qquad \text{for every } \mathbf{v}_1 \tag{40}$$

Now, from localizing u_{1k}

$$u_{1k}(x,y) = -\chi_k^{pq}(y)\frac{\partial u_{0p}}{\partial x_q}(x), \tag{41}$$

$$\begin{aligned}
&\Rightarrow \int_\Omega \int_Y E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} - \frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0\\
&\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijkl}\frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0\\
&\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0\\
&\int_\Omega \int_Y \frac{\partial u_{0k}}{\partial x_l}\left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0\\
&\int_\Omega \frac{\partial u_{0k}}{\partial x_l} dx \cdot \int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy \ = 0
\end{aligned}$$

$$\Rightarrow \int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy = 0 \qquad \text{for k, l = 1 and 2,} \tag{42}$$

Similarly, substituting equation (41) in (39) gives the homogenized equation.

$$\begin{aligned}
\text{LHS} &= \frac{1}{|Y|}\int_\Omega \int_Y E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}\right) dy\, dx\\
&= \frac{1}{|Y|}\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_{0i}}{\partial x_j} dy\, dx\\
&= \frac{1}{|Y|}\int_\Omega \left\{\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy\right\}\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j} dx\\
&= \int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx
\end{aligned}$$

Homogenized equation

$$\int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx = \int_\Omega \mathbf{f}\cdot \mathbf{v}_0 dx + \int_{\Gamma_t} \mathbf{t}\cdot \mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0 \tag{43}$$

where $E_{ijkl}^H(x)$ is

$$\boxed{E_{ijkl}^H = \frac{1}{|Y|}\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy} \tag{44}$$

Now, Define