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Generalized plane strain elasticity problems

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Abstract

The generalized plane strain problem in elasticity theory is characterized by a geometric body and external load that does not change along the longitudinal direction. A two-dimensional solution becomes possible. The theoretical basis for general boundary value problems and the BEM formulation are presented.

1 Introduction

For an elastic solid whose geometric form does not change along a longitudinal axis, it is natural to solve such a problem as a two-dimensional one. Figure 1 showing cylindrical and prismatic surfaces depicts examples of such geometry. The elastic domain can be the interior or exterior of a cylindrical (left diagram) or a prismatic (right diagram) surface, or the annular region formed by two such surfaces (middle diagram). However, geometry alone is not sufficient to have a dimensionally reduced problem. The material properties and the boundary conditions must not vary along the longitudinal direction. When these conditions are satisfied, every plane that is perpendicular to the longitudinal axis is a symmetry plane. The stresses and strains hence are invariant along the longitudinal direction. These type of problems are defined as *generalized plane strain* problems.

The definition of generalized plane strain is not unified. The broadest definition so far appears to be that by Wu and Li [1]. That definition how-

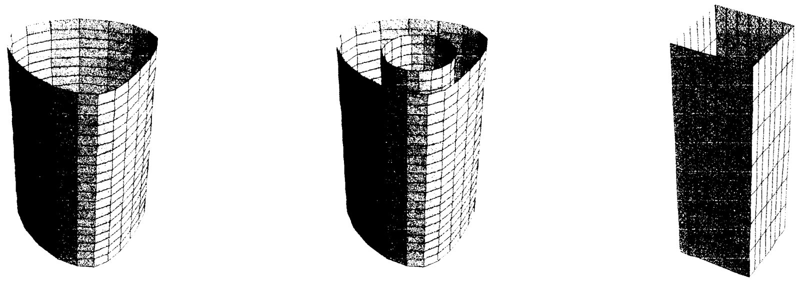

Figure 1: Geometry for generalized plane strain problems.

ever is somewhat narrower than the present one. Other definitions (sometimes referred to as *complete plane strain*) found in various sources [2, 3, 4, 5] are further reduced cases. Finally, the classical *plane strain* problem is a degenerated case of all those above.

In the present work, the generalized plane strain problem is defined based on the above reasoning. We then demonstrate that the three-dimensional problem can be decomposed and reduced to several lower-dimensional problems. The boundary value problems are presented and the boundary element formulations are stated.

2 Elasticity Equations

Elasticity equations in three-dimensional Cartesian geometry are presented as follows:

- Kinematic definitions:

$$
\begin{aligned}
&e_x = \frac{\partial u_x}{\partial x}, \quad e_y = \frac{\partial u_y}{\partial y}, \quad e_z = \frac{\partial u_z}{\partial x}, \\
&\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}, \\
&\gamma_{zx} = \frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}
\end{aligned} \tag{1}
$$

- Equilibrium equations:

$$
\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0
\end{aligned}
$$

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$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{2}$$

- Linear, isotropic constitutive relations:

$$\begin{aligned} &\sigma_x = 2\mu e_x + \lambda e, \quad \sigma_y = 2\mu e_y + \lambda e, \quad \sigma_z = 2\mu e_z + \lambda e, \\ &\tau_{xy} = \mu\gamma_{xy}, \quad \tau_{yz} = \mu\gamma_{yz}, \quad \tau_{zx} = \mu\gamma_{zx} \end{aligned} \tag{3}$$

In the above, u_x, u_y, u_z are displacements, e_x, e_y, e_z are extensional strains, $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are shearing strains, $\sigma_x, \sigma_y, \sigma_z$ are normal stresses, $\tau_{xy}, \tau_{yz}, \tau_{zx}$ are shear stresses, λ, μ are Lamé constants, and

$$e = e_x + e_y + e_z \tag{4}$$

is the dilatation.

3 Generalized Plane Strain

We choose the longitudinal axis, *i.e.* the axis parallel to the cylindrical or prismatic surfaces, as the z-axis. The generalized plane strain condition can be stated as "*all strains are independent of the z coordinate*":

$$\begin{aligned} &e_x = e_x(x, y), \quad e_y = e_y(x, y), \quad e_z = e_z(x, y), \\ &\gamma_{xy} = \gamma_{xy}(x, y), \quad \gamma_{yz} = \gamma_{yz}(x, y), \quad \gamma_{zx} = \gamma_{zx}(x, y) \end{aligned} \tag{5}$$

By the virtue of stress-strain relations (3), all stresses are functions of x and y only.

In fact these strains cannot be arbitrarily defined. They must satisfy the compatibility conditions. Taking into consideration the z-independence, they reduce to the following:

$$\begin{aligned} &\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \quad \frac{\partial^2 e_z}{\partial y^2} = 0, \\ &\frac{\partial^2 \gamma_{zx}}{\partial x \partial y} - \frac{\partial^2 \gamma_{yz}}{\partial x^2} = 0, \quad \frac{\partial^2 e_z}{\partial x^2} = 0, \\ &\frac{\partial^2 \gamma_{yz}}{\partial x \partial y} - \frac{\partial^2 \gamma_{zx}}{\partial y^2} = 0, \quad \frac{\partial^2 e_z}{\partial x \partial y} = 0 \end{aligned} \tag{6}$$

The second, third and sixth equations in the above show that e_z must be restricted to

$$e_z = Cx + Dy + E \tag{7}$$

where C, D, E are constants. In terms of displacements, these conditions exist:

$$\begin{aligned} &u_x = Az + f(x, y), \quad u_y = Bz + g(x, y), \\ &u_z = (Cx + Dy + E)z + h(x, y) \end{aligned} \tag{8}$$

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where A, B are constants, and f, g, h are continuous functions of x and y. Equation (8) defines the most general displacement field under the generalized plane strain condition. In Wu and Li [1], the condition of $A = B = 0$ was assumed. In the more traditional definition [2, 3, 4], E is further set to zero. Rencis and Huang [6] considered $A = B = C = D = h = 0$. The classical plane strain condition requires $A = B = C = D = E = h = 0$.

4 Decomposition of Generalized Plane Strain

The governing equations are linear partial differential equations. This fact allows the decomposition and superposition of solution. The displacement field (8) can be decomposed into three parts:

- Classical plane strain problem

$$u_x^{(1)} = f(x, y), \quad u_y^{(1)} = g(x, y), \quad u_z^{(1)} = 0 \tag{9}$$

- Anti-plane shear problem

$$u_x^{(2)} = 0, \quad u_y^{(2)} = 0, \quad u_z^{(2)} = h(x, y) \tag{10}$$

- Out-of-plane strain problem

$$u_x^{(3)} = Az, \quad u_y^{(3)} = Bz, \quad u_z^{(3)} = (Cx + Dy + E)z \tag{11}$$

These three problems are discussed in the next three sections.

5 Out-of-Plane Strain Problem

We shall discuss the third problem first. The solution to this problem is trivial, as the displacements are already defined in (11). Following (1) we find the strains as

$$\begin{aligned} &e_x = e_y = \gamma_{xy} = 0, \quad e_z = Cx + Dy + E, \\ &\gamma_{yz} = B + Dz, \quad \gamma_{zx} = A + Cz \end{aligned} \tag{12}$$

where we note that all the in-plane components of strain vanish. From (3) the stresses are obtained as

$$\begin{aligned} &\sigma_x = \sigma_y = \lambda(Cx + Dy + E), \\ &\sigma_z = (2\mu + \lambda)(Cx + Dy + E), \\ &\tau_{xy} = 0, \quad \tau_{yz} = \mu(B + Dz), \quad \tau_{zx} = \mu(A + Cz) \end{aligned} \tag{13}$$

The boundary conditions of this problem cannot be arbitrarily specified. They must be compatible with the solution at the boundary. From some compatible boundary conditions, the five unknown constants, A, B, C, D, E, are solved.

These conditions are highly restrictive. It is difficult to find a practical problem that fits such a specification, other than a constant longitudinal strain problem

$$e_x = e_y = \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0, \quad e_z = E \tag{14}$$

The stresses for such a problem are

$$\begin{aligned} &\sigma_x = \sigma_y = \lambda E, \quad \sigma_z = (2\mu + \lambda)E, \\ &\tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \end{aligned} \tag{15}$$

6 Extended Plane Strain Problem

Since the constant longitudinal strain problem shown in (14) and (15) is more often encountered, we shall incorporate it into the classical plane strain problem in 9 to form an *extended plane strain problem.* The displacements are

$$u_x = f(x, y), \quad u_y = g(x, y), \quad u_z = Ez \tag{16}$$

which are identical to those studied by Rencis and Huang [6]. We note that the following out-of-plane shear strains and stresses vanish:

$$\gamma_{yz} = \gamma_{zx} = \tau_{yz} = \tau_{zx} = 0 \tag{17}$$

and the z-component strain and stress are:

$$e_z = E, \quad \sigma_z = (2\mu + \lambda)E \tag{18}$$

The idea here is to form an in-plane stress and strain system that can be solved as a two-dimensional problem. Defining the stress components as

$$\hat{\sigma}_x = \sigma_x - \lambda E, \quad \hat{\sigma}_y = \sigma_y - \lambda E, \quad \hat{\tau}_{xy} = \tau_{xy} \tag{19}$$

the constitutive equations (3) can be expressed as

$$\hat{\sigma}_x = 2\mu\hat{e}_x + \lambda\hat{e}, \quad \hat{\sigma}_y = 2\mu\hat{e}_y + \lambda\hat{e}, \quad \hat{\tau}_{xy} = \mu\hat{\gamma}_{xy} \tag{20}$$

In the above, definitions of strains remain unchanged

$$\hat{e}_x = e_x, \quad \hat{e}_y = e_y, \quad \hat{\gamma}_{xy} = \gamma_{xy} \tag{21}$$

while the dilatation takes the two-dimensional form

$$\hat{e} = e_x + e_y \tag{22}$$

The first two equilibrium equations in (2) reduce to

$$\begin{aligned}\frac{\partial \hat{\sigma}_x}{\partial x} + \frac{\partial \hat{\tau}_{xy}}{\partial y} &= 0 \\ \frac{\partial \hat{\tau}_{xy}}{\partial x} + \frac{\partial \hat{\sigma}_y}{\partial y} &= 0\end{aligned} \tag{23}$$

The boundary conditions are transformed to:

$$\begin{aligned}&\hat{u}_x = u_x, \quad \hat{u}_y = u_y, \\ &\hat{t}_x = t_x - \lambda E n_x, \quad \hat{t}_y = t_y - \lambda E n_y\end{aligned} \tag{24}$$

where t_x, t_y are components of boundary traction, and n_x, n_y are components of the unit outward normal $\boldsymbol{n}$. The system represented by the accented quantities, (20), (23) and (24), is fully analogous with the classical plane strain governing equations, hence can be solved as a plane strain (two-dimensional) problem.

The above results were given by Rencis and Huang [6]. Since boundary element formulation follows exactly that of the classical plane strain problem, it is not repeated here.

7 Anti-Plane Shear Problem

By the definition of displacements in (10)

$$u_x = 0, \quad u_y = 0, \quad u_z = h(x, y) \tag{25}$$

it is easily shown that

$$e_x = e_y = e_z = \gamma_{xy} = \sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0 \tag{26}$$

The only surviving components are the anti-plane shear strains and stresses, $\gamma_{yz}, \gamma_{zx}, \tau_{yz}, \tau_{zx}$. The following boundary conditions also vanish

$$u_x = u_y = t_x = t_y = 0 \tag{27}$$

such that only u_z and t_z exist. This type of anti-plane problems have been extensively studied [7].

As a consequence of (26), the third of equilibrium equations (2) becomes

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} = 0 \tag{28}$$

Using the kinematic definitions (1), the constitutive equations in 3 can be written as

$$\tau_{yz} = \mu \frac{\partial u_z}{\partial y}, \quad \tau_{zx} = \mu \frac{\partial u_z}{\partial x} \tag{29}$$

Substituting (29) into (28), we obtain the Laplace equation

$$\nabla^2 u_z = 0 \tag{30}$$

This result was given by Horgan and Miller [7]. The boundary conditions of this problem are in terms of u_z and

$$\frac{\partial u_z}{\partial n} = \frac{t_z}{\mu} \tag{31}$$

Since the solution of Laplace equation and its boundary element formulation are well-known, they are not repeated herein.

8 Conclusion

The generalized plane strain problem of elasticity theory is presented in this work following the broadest definition. It states that the stresses and strains are functions of Cartesian coordinates x and y only. Such a condition can exist for a linear, isotropic elasticity problems with cylindrical or prismatic geometry. The boundary conditions must not vary in the z-direction. The material can be inhomogeneous, yet the material variation can only be a function of x and y. Although only the isotropic case is presented here, material anisotropy is in fact allowed. However, the decomposition as utilized above is permissible only for a limited class of anisotropy conditions [3, 7]. In particular, for materials possessing elastic symmetry in the x-y plane, the decomposition is valid.

In accordance to the definition presented, it has been shown that the generalized plane strain problem can be decomposed into three sub-problems. The out-of-plane strain problem can be solved as an exact solution. With a transformation, the extended plane strain problem becomes analogous to the classical plane strain problem. Finally, the anti-plane shear problem is formulated as a Laplace problem. Boundary element implementations of the last two problems are straightforward.

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