Master's Thesis

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Abstract

Fill it.

1 Introduction

2 Motivation

3 Method

The idea of asymptotic homogenization. In a repeating cell Y,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{1}$$

where $C_{ijkl}(\underset{\sim}{x} + \underset{\sim}{u}\underset{\sim}{Y}) = C_{ijkl}(\underset{\sim}{x})$

$$\Rightarrow C_{ijkl}(x_1 + n_1Y_1\, x_2 + n_2Y_2\, x_3 + n_3Y_3) = C_{ijkl}(x_1, x_2, x_3) \tag{2}$$

$C_{ijkl}(\underset{\sim}{x})$ is Y-periodic

$$\underset{\sim}{y} = \tfrac{\underset{\sim}{x}}{\epsilon} \tag{3}$$

$$\Rightarrow g = g(\underset{\sim}{x}, \tfrac{\underset{\sim}{x}}{\epsilon}) = g(\underset{\sim}{x}\,\underset{\sim}{y}) \tag{4}$$

$\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ defines the domain of the composite Ω. The domain is composed of base cells of dimensions, $\varepsilon Y_1, \varepsilon Y_2, \varepsilon Y_3$ where $\underset{\sim}{y} = \frac{\underset{\sim}{x}}{\varepsilon}$

3.1 1D Elasticity

$$\sigma^\varepsilon = E^\varepsilon \tfrac{\partial u^\varepsilon}{\partial x} \tag{5}$$

$$\tfrac{\partial \sigma^\varepsilon}{\partial x} + \gamma^\varepsilon = 0 \quad E^\varepsilon\, \gamma^\varepsilon \to macroscopicallyuniform \tag{6}$$

Inside each cell,

$$E^\varepsilon(x, \frac{x}{\varepsilon}) \quad = E(y) \tag{7}$$

$$\gamma^\varepsilon(x, \frac{x}{\varepsilon}) \quad = \gamma(y) \tag{8}$$

Let

$$u^{\varepsilon}(x) = u^0 x, y + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + ... \tag{9}$$
$$\sigma^{\varepsilon}(x) = \sigma^0 x, y + \varepsilon \sigma^1(x,y) + \varepsilon^2 \sigma^2(x,y) + ... \tag{10}$$

3.2 Optimal Design of Elastic structures

$\mathbf{b} \rightarrow$ body forces
$\mathbf{t} \rightarrow$ surface tractions

Optimal choice of $\mathbb{C}_{ijkl} \in U_{ad} \leftarrow$ admissible set of elasticity
$\mathbb{C}_{ijkl}(\mathbf{x}) \forall \mathbf{x} \in \Omega$ has 21 independent components
$a_E(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbb{C}_{ijkl}\, \varepsilon_{kl}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{v}) d\mathbf{v} \rightarrow$ energy bilinear form
$L(\mathbf{v}) = \int_\Omega \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} ds \rightarrow$ load linear form.

Minimum compliance problem:

$$\begin{aligned} &minimize && L(\mathbf{v}), && (11)\\ &subject\, to && \mathbb{C}_{ijkl} \in \mathbb{U}_{ad} && (12)\\ & && a_E(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{U} && (13) \end{aligned}$$

where $\mathbb{U} \rightarrow$ kinematically admissible displacements.
For optimal shape design:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = \chi(\mathbf{x}) \overline{\mathbb{C}}_{ijkl}, \quad \text{where } \overline{\mathbb{C}}_{ijkl} \rightarrow \text{stiffness matrix of the material} \tag{14}$$

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m, \\ 0 & \text{if } \mathbf{x} \in \Omega \backslash \Omega^m \end{cases} \tag{15}$$

where $\Omega^m \rightarrow$ part of the domain occupied by the material.
For sizing problem:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = h(\mathbf{x}) \overline{\mathbb{C}}_{ijkl} \tag{16}$$
$$\int_\Omega \chi(\mathbf{x}) d\mathbf{x} = V_f \tag{17}$$
$$\& \quad \int_\Omega h(\mathbf{x}) d\mathbf{x} = V_f. \tag{18}$$

where $h(x)$ is a sizing function.
Traditionally shape design problems are initiated in the following manner:

$$Ref\ doamin : \Omega_0 \in \mathbb{R}^3 \tag{19}$$
$$\underline{\phi} : \Omega_0 \rightarrow \phi(\Omega_0) \text{is a diffeomorphism.} \tag{20}$$
$$L(\mathbf{v}) = \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| ds \tag{21}$$

$$\begin{aligned} a_E &= \int_\Omega \mathbb{C}_{ijkl}(\mathbf{x} \varepsilon_{kl}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl} \varepsilon_{kl}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \tag{22}$$

Now,

$$\begin{aligned}\mathbb{C}_{ijkl}\varepsilon_{kl} &= \mathbb{C}_{ijkl}\frac{1}{2}(u_{k,l}+u_{l,k})\\ &= \frac{1}{2}\mathbb{C}_{ijkl}u_{k,l}+\frac{1}{2}\mathbb{C}_{ijlk}u_{l,k}\\ &= \mathbb{C}_{ijkl}u_{k,l}\end{aligned} \tag{23}$$

$$\begin{aligned}a_E &= \int_{\Omega_0}\mathbb{C}_{ijkl}u_{k,l}(\mathbf{u})u_{i,j}(\mathbf{v})|det(D\underline{\phi}^{-1}|d\mathbf{x}\\ &= \int_{\Omega_0}\mathbb{C}_{ijkl}\frac{\partial u_k}{\partial \mathbf{x}_m}(D\underline{\phi}^{-1})_{ml}\frac{\partial u_i}{\partial \mathbf{x}_p}(D\phi^{-1})_{pj}|det(D\underline{\phi}^{-1})|d\mathbf{x}\end{aligned} \tag{24}$$

$$\Rightarrow \mathbb{C}_{ijkl}(D\underline{\phi}^{-1})_{ml}(D\underline{\phi}^{-1})_{pj}|det(D\underline{\phi}^{-1})| = \bar{\mathbb{C}}_{ipkm} \tag{25}$$

$$\bar{\mathbb{C}}_{ijkl} = \mathbb{C}_{ipkm}(D\underline{\phi}^{-1})_{lm}(D\underline{\phi}^{-1})_{jp}|det(D\underline{\phi}^{-1})| \tag{26}$$

Treating $\underline{\phi}$ as a design variable is tidious.

3.3 Homogenization method

$$E^{\varepsilon}_{ijkl}(\mathbf{x}) = E_{ijkl}(\mathbf{x},\mathbf{y}), \qquad \mathbf{y}=\frac{\mathbf{x}}{\varepsilon} \tag{27}$$

The tensor E^{ε}_{ijkl} is a material constant which satisfies the symmetry condition and is assumed to satisfy strong ellipticity condition for every $\mathbf{x}$.

$$\Rightarrow E^{\varepsilon}_{ijkl} = E^{\varepsilon}_{jikl} = E^{\varepsilon}_{ijlk} = E^{\varepsilon}_{klij} \tag{28}$$

$$E^{\varepsilon}_{ijkl}(\mathbf{x})\mathbf{X}_{ij}\mathbf{X}_{kl} \geq m\mathbf{X}_{ij}\mathbf{X}_{ij} \qquad \text{for some } m>0 \ \& \ \forall \quad \mathbf{X}_{ij}=\mathbf{X}_{ji} \tag{29}$$

Let the domainΩ has a boundary Γ. Let $\mathbf{f}$ be the body force acting on Ω and $\mathbf{t}$ be the traction acting on Γ_t part of the boundary Γ. Also, let Γ_D be the part of boundary on which displacement is defined. Then the displacement $\mathbf{u}^{\varepsilon}$ can be obtained as the solution to the following minimization problem

$$\min_{\mathbf{v}^{\varepsilon}\in U} \quad F^{\varepsilon}(\mathbf{v}^{\varepsilon}), \tag{30}$$

where F^{ε} is total potential energy given as

$$F^{\varepsilon}(\mathbf{v}^{\varepsilon}) = \frac{1}{2}\int_{\Omega}E^{\varepsilon}_{ijkl}\varepsilon_{kl}(\mathbf{v}^{\varepsilon})\varepsilon_{ij}(\mathbf{v}^{\varepsilon})dx - \int_{\Omega}\mathbf{f}\cdot\mathbf{v}^{\varepsilon}dx - \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}^{\varepsilon}ds \tag{31}$$

and $\mathcal{U}$ is the set of admissible displacements defined such that

$$\mathcal{U} = \{\mathbf{v}=v_i\mathbf{e}_i : v_i\in H^1(\Omega) \text{ and } \mathbf{v}\in\mathcal{G} \text{ on } \Gamma_D\} \tag{32}$$

where $\mathcal{G}$ is set of displacement defined along the boundary Γ_D.
Let

$$\mathbf{v}^{\varepsilon}(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \varepsilon\mathbf{v}_1(\mathbf{x},\mathbf{y}), \qquad \mathbf{y}=\frac{\mathbf{x}}{\varepsilon}. \tag{33}$$

Using chain rule for functions in two variables

$$\begin{aligned}\frac{\partial f(\mathbf{x}, \mathbf{y}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{y}} \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \\ &= \frac{\partial f}{\partial \mathbf{x}} + \frac{1}{\varepsilon} \frac{\partial f}{\partial \mathbf{y}}\end{aligned} \tag{34}$$

Using above two equations, we can write the linerized strain as

$$\begin{aligned}\epsilon_{ij}(\mathbf{v}^\varepsilon(\mathbf{x})) &= \frac{\partial(v_{0i}(\mathbf{x}) + \varepsilon v_{1i}(\mathbf{x}, \mathbf{y}))}{\partial x_j} \\ &= \frac{\partial v_{0i}}{\partial x_j} + \varepsilon \left\{ \frac{\partial v_{1i}}{\partial x_j} + \frac{1}{\varepsilon} \frac{\partial v_{1i}}{\partial y_j} \right\} \\ &\approx \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \qquad \leftarrow \{\varepsilon << 1\}\end{aligned} \tag{35}$$

Therefore, equation (31) can be written as

$$\begin{aligned}F^\varepsilon(\mathbf{v}^\varepsilon) = \frac{1}{2} \int_\Omega E^\varepsilon_{ijkl} \left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \right) dx \\ - \int_\Omega \mathbf{f} \cdot \mathbf{v}_0 dx - \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}_0 ds + \varepsilon R^\varepsilon(\mathbf{v}_0, \mathbf{v}_1)\end{aligned} \tag{36}$$

Here, R^ε is the contribution of $\varepsilon \mathbf{v}_1$ in the calculation of energy from body force and traction. Using

$$\lim_{\varepsilon \to 0} \int_\Omega \Phi(x, x/\varepsilon) dx = \frac{1}{|Y|} \int_\Omega \int_Y \Phi(x, y) dy dx, \tag{37}$$

we get,

$$\begin{aligned}\lim_{\varepsilon \to 0} F^\varepsilon(\mathbf{v}^\varepsilon) &= F(\mathbf{v}_0, \mathbf{v}_1) \\ &= \frac{1}{2|Y|} \int_\Omega \int_Y E_{ijkl}(x, y) \left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \right) dy\, dx \\ &\quad - \int_\Omega \mathbf{f} \cdot \mathbf{v}_0 dx - \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}_0 ds\end{aligned} \tag{38}$$

A minimizer $\{\mathbf{u}_0, \mathbf{u}_1\}$ of the functional F, follow the following equations:

$$\begin{aligned}&\frac{1}{|Y|} \int_\Omega \int_Y E_{ijkl}(x, y) \left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{0i}}{\partial x_j} \right) dy\, dx \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{v}_0 dx + \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0\end{aligned} \tag{39}$$

$$\frac{1}{|Y|} \int_\Omega \int_Y E_{ijkl}(x, y) \left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l} \right) \left(\frac{\partial v_i}{\partial x_j} \right) dy\, dx = 0, \qquad \text{for every } \mathbf{v}_1 \tag{40}$$

Now, from localizing u_{1k}

$$u_{1k}(x,y) = -\chi_k^{pq}(y)\frac{\partial u_{0p}}{\partial x_q}(x), \tag{41}$$

$$\begin{aligned}
&\Rightarrow \int_\Omega \int_Y E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} - \frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
&\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijkl}\frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
&\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
&\int_\Omega \int_Y \frac{\partial u_{0k}}{\partial x_l}\left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
&\int_\Omega \frac{\partial u_{0k}}{\partial x_l} dx \cdot \int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy \;= 0
\end{aligned}$$

$$\Rightarrow \int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy = 0 \qquad \text{for k, l = 1 and 2,} \tag{42}$$

Similarly, substituting equation (41) in (39) gives the homogenized equation.

$$\begin{aligned}
\text{LHS} &= \frac{1}{|Y|}\int_\Omega \int_Y E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}\right) dy\, dx \\
&= \frac{1}{|Y|}\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_{0i}}{\partial x_j} dy\, dx \\
&= \frac{1}{|Y|}\int_\Omega \left\{\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy\right\}\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j} dx \\
&= \int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx
\end{aligned}$$

Homogenized equation

$$\int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}_0 dx + \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0 \tag{43}$$

where $E_{ijkl}^H(x)$ is

$$\boxed{E_{ijkl}^H = \frac{1}{|Y|}\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy} \tag{44}$$

Now, Define

$$a_H(\mathbf{u},\mathbf{v}) = \int_\Omega E^H_{ijkl}(\mathbf{x})\frac{\partial u_k}{\partial x_l}\frac{\partial v_i}{\partial x_j}\,dx, \tag{45}$$

$$a_Y(\chi^{kl},\mathbf{v}) = \int_Y E_{ijpq}(\mathbf{x},\mathbf{y})\frac{\partial \chi^{kl}_p}{\partial y_q}\frac{\partial v_i}{\partial y_j}\,dy, \tag{46}$$

$$L^{kl}_Y(\mathbf{v}) = \int_Y E_{ijkl}\frac{\partial v_i}{\partial y_j}\,dy \tag{47}$$

At microscopic level, we have

$$a_Y(\chi^{kl},\mathbf{v}) = L^{kl}_Y(\mathbf{v}) \qquad \forall \mathbf{v}\in\mathcal{U}_Y, \tag{48}$$

At macroscopic level, we have

$$a_H(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \qquad \forall \mathbf{v}\in\mathcal{U}_0 \tag{49}$$

where $\mathcal{U}_0$ is homogeneous case of $\mathcal{U}$, i.e., $\mathbf{g}=0$.

3.4 Implementation 2D Homogenization

Basic homogenization equation,

$$u_{1i}(\mathbf{x},\mathbf{y}) = -\chi^{pq}_i\frac{\partial u_{0p}(\mathbf{x})}{\partial x_q} \tag{50}$$

Solve χ^{kl}_p from:

$$\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi^{kl}_p}{\partial y_q}\right)\frac{\partial v_{1i}}{\partial y_j}\,dy = 0 \tag{51}$$

Compute:

$$E^H_{ijkl} = \frac{1}{|Y|}\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi^{kl}_p}{\partial y_q}\right)dy \tag{52}$$

3.5 Examples

Consider: k=1, l=1

$$\begin{aligned}\int_Y E_{ijkl}\frac{\partial v_i}{\partial y_j}\,dy &= \int_Y E_{ij11}\frac{\partial v_i}{\partial y_j}\,dy \\ &= \int_Y \left(E_{1111}\frac{\partial v_1}{\partial y_1} + E_{2211}\frac{\partial v_2}{\partial y_2}\right)dy\end{aligned} \tag{53}$$

$$
\begin{aligned}
\int_Y E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial v_i}{\partial y_j}\,dy &= \int_Y E_{ijpq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_i}{\partial y_j}\,dy \\
&= \int_Y \Bigg\{ E_{11pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_1}{\partial y_1} + E_{12pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_1}{\partial y_2} \\
&\qquad + E_{21pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_2}{\partial y_1} + E_{22pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_2}{\partial y_2} \Bigg\}\,dy \\
&= \int_Y \Bigg\{ \left(E_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{1112}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{1121}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{1122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_1} \\
&\quad + \left(E_{1211}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{1212}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{1222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_2} \\
&\quad + \left(E_{2111}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{2112}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{2121}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{2122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_1} \\
&\quad + \left(E_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{2212}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{2221}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{2222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_2} \Bigg\}\,dy \\
&= \int_Y \Bigg\{ \left(E_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} + \cancelto{0}{E_{1112}\frac{\partial \chi_1^{11}}{\partial y_2}} + \cancelto{0}{E_{1121}\frac{\partial \chi_2^{11}}{\partial y_1}} + E_{1122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_1} \\
&\quad + \left(\cancelto{0}{E_{1211}\frac{\partial \chi_1^{11}}{\partial y_1}} + E_{1212}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221}\frac{\partial \chi_2^{11}}{\partial y_1} + \cancelto{0}{E_{1222}\frac{\partial \chi_2^{11}}{\partial y_2}} \right)\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\right) \\
&\quad + \left(E_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} + \cancelto{0}{E_{2212}\frac{\partial \chi_1^{11}}{\partial y_2}} + \cancelto{0}{E_{2221}\frac{\partial \chi_2^{11}}{\partial y_1}} + E_{2222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_2} \Bigg\}\,dy \\
&= \int_Y \Bigg\{ \left(E_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_1} \\
&\quad + E_{1212}\left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1}\right)\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1}\right) \\
&\quad + \left(E_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_2} \Bigg\}\,dy
\end{aligned}
\tag{54}
$$

Therefore, using equations (48) , (54) and (53) for k=1, l=1 we have:

$$\begin{aligned}
&\int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + E_{1212} \left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
&\int_Y \left(E_{1111} \frac{\partial v_1}{\partial y_1} + E_{2211} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{55}$$

From equation (52), we can write

$$E_{1111}^H = \quad \tfrac{1}{|Y|} \int_Y \left(E_{1111} - E_{1111} \tfrac{\partial \chi_1^{11}}{\partial y_1} - E_{1122} \tfrac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{56}$$

$$E_{2211}^H = \quad \tfrac{1}{|Y|} \int_Y \left(E_{2211} - E_{2211} \tfrac{\partial \chi_1^{11}}{\partial y_1} - E_{2222} \tfrac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{57}$$

$$E_{1211}^H = \quad -\tfrac{1}{|Y|} \int_Y \left(E_{1212} \tfrac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \tfrac{\partial \chi_2^{11}}{\partial y_1} \right) dy \tag{58}$$

Let $\chi_1^{11} = \Phi_1, \chi_2^{11} = \Phi_2$ and $E_{1111} = D_{11}, E_{2222} = D_{22}, E_{1212} = D_{66}, E_{1122} = E_{2211} = D_{12}$

$$\begin{aligned}
&\int_Y \left\{ \left(D_{11} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{12} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + D_{66} \left(\frac{\partial \Phi_1^{11}}{\partial y_2} + \frac{\partial \Phi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(D_{12} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{22} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
&\int_Y \left(D_{11} \frac{\partial v_1}{\partial y_1} + D_{12} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{59}$$

Also,

$$D_{11}^H = \frac{1}{|Y|} \int_Y \left(D_{11} - D11 \frac{\partial \Phi_1}{\partial y_1} - D_{12} \frac{\partial \Phi_2}{\partial y_2} \right) dy \tag{60}$$

Rearranging Eq. (59)

$$\int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D66 \end{bmatrix}$$

$$\times \begin{bmatrix} \frac{\partial \Phi_1}{\partial y_1} \\ \frac{\partial \Phi_2}{\partial y_2} \\ \frac{\partial \Phi_1}{\partial y_2} + \frac{\partial \Phi_2}{\partial y_1} \end{bmatrix} dY \tag{61}$$

$$= \int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} \\ D_{12} \\ 0 \end{bmatrix} dY$$

Let us define

$$\mathbf{b} = \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 \\ 0 & \frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} \end{bmatrix} \tag{62}$$

and

$$\mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix} \tag{63}$$

Then Eq (59), can be written as

$$\int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{D} \mathbf{b} \Phi \, dY = \int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{d}_1 \qquad \forall \mathbf{v} \in \mathbf{V}_Y \tag{64}$$

and eq. (60) becomes:

$$\boxed{D_{11}^H = \frac{1}{|Y|} \int_Y \left(D_{11} - \mathbf{d}_1^T \mathbf{b} \Phi \right) dy} \tag{65}$$