

Master's Thesis

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Abstract

Fill it.

1 Introduction

2 Motivation

3 Method

The idea of asymptotic homogenization. In a repeating cell Y ,

$$\sigma_{ij} = C_{ijkl} \epsilon_{kl} \quad (1)$$

where $C_{ijkl}(\underline{x} + \underline{u}Y) = C_{ijkl}(\underline{x})$

$$\Rightarrow C_{ijkl}(x_1 + n_1 Y_1 \ x_2 + n_2 Y_2 \ x_3 + n_3 Y_3) = C_{ijkl}(x_1, x_2, x_3) \quad (2)$$

$C_{ijkl}(\underline{x})$ is Y -periodic

$$\underline{y} = \frac{\underline{x}}{\epsilon} \quad (3)$$

$$\Rightarrow g = g(x, \frac{x}{\epsilon}) = g(x \underline{y}) \quad (4)$$

$\underline{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ defines the domain of the composite Ω . The domain is composed of base cells of dimensions, $\epsilon Y_1, \epsilon Y_2, \epsilon Y_3$ where $\underline{y} = \frac{\underline{x}}{\epsilon}$

3.1 1D Elasticity

$$\sigma^\epsilon = E^\epsilon \frac{\partial u^\epsilon}{\partial x} \quad (5)$$

$$\frac{\partial \sigma^\epsilon}{\partial x} + \gamma^\epsilon = 0 \quad E^\epsilon \gamma^\epsilon \rightarrow \text{macroscopically uniform} \quad (6)$$

Inside each cell,

$$E^\epsilon(x, \frac{x}{\epsilon}) = E(y) \quad (7)$$

$$\gamma^\epsilon(x, \frac{x}{\epsilon}) = \gamma(y) \quad (8)$$

Let

$$u^\varepsilon(x) = u^0 x, y + \varepsilon u^1(x, y) + \varepsilon^2 u^2(x, y) + \dots \quad (9)$$

$$\sigma^\varepsilon(x) = \sigma^0 x, y + \varepsilon \sigma^1(x, y) + \varepsilon^2 \sigma^2(x, y) + \dots \quad (10)$$

3.2 Optimal Design of Elastic structures

$\mathbf{b} \rightarrow$ body forces

$\mathbf{t} \rightarrow$ surface tractions

Optimal choice of $\mathbb{C}_{ijkl} \in U_{ad} \leftarrow$ admissible set of elasticity

$\mathbb{C}_{ijkl}(\mathbf{x}) \forall \mathbf{x} \in \Omega$ has 21 independent components

$a_E(\mathbf{u}, \mathbf{v}) = \int_{\Omega} \mathbb{C}_{ijkl} \varepsilon_{kl}(\mathbf{u}) \varepsilon_{kl}(\mathbf{v}) d\mathbf{x} \rightarrow$ energy bilinear form

$L(\mathbf{v}) = \int_{\Omega} \mathbf{v} d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} ds \rightarrow$ load linear form.

Minimum compliance problem:

$$\text{minimize} \quad L(\mathbf{v}), \quad (11)$$

$$\text{subject to} \quad \mathbb{C}_{ijkl} \in \mathbb{U}_{ad} \quad (12)$$

$$a_E(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{U} \quad (13)$$

where $\mathbb{U} \rightarrow$ kinematically admissible displacements.

For optimal shape design:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = \chi(\mathbf{x}) \bar{\mathbb{C}}_{ijkl}, \quad \text{where } \bar{\mathbb{C}}_{ijkl} \rightarrow \text{stiffness matrix of the material} \quad (14)$$

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m, \\ 0 & \text{if } \mathbf{x} \in \Omega \setminus \Omega^m \end{cases} \quad (15)$$

where $\Omega^m \rightarrow$ part of the domain occupied by the material.

For sizing problem:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = h(\mathbf{x}) \bar{\mathbb{C}}_{ijkl} \quad (16)$$

$$\int_{\Omega} \chi(\mathbf{x}) d\mathbf{x} = V_f \quad (17)$$

$$\& \int_{\Omega} h(\mathbf{x}) d\mathbf{x} = V_f. \quad (18)$$

where $h(x)$ is a sizing function.

Traditionally shape design problems are initiated in the following manner:

$$\text{Ref domain } : \Omega_0 \in \mathbb{R}^3 \quad (19)$$

$$\underline{\phi} : \Omega_0 \rightarrow \phi(\Omega_0) \text{ is a diffeomorphism.} \quad (20)$$

$$L(\mathbf{v}) = \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| ds \quad (21)$$

$$\begin{aligned} a_E &= \int_{\Omega} \mathbb{C}_{ijkl}(\mathbf{x}) \varepsilon_{kl}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl} \varepsilon_{kl}(\mathbf{v}) \varepsilon_{ij}(\mathbf{v}) |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \quad (22)$$

Now,

$$\begin{aligned}
\mathbb{C}_{ijkl}\varepsilon_{kl} &= \mathbb{C}_{ijkl}\frac{1}{2}(u_{k,l} + u_{l,k}) \\
&= \frac{1}{2}\mathbb{C}_{ijkl}u_{k,l} + \frac{1}{2}\mathbb{C}_{ijlk}u_{l,k} \\
&= \mathbb{C}_{ijkl}u_{k,l}
\end{aligned} \tag{23}$$

$$\begin{aligned}
a_E &= \int_{\Omega_0} \mathbb{C}_{ijkl}u_{k,l}(\mathbf{u})u_{i,j}(\mathbf{v})|det(D\underline{\phi}^{-1})|d\mathbf{x} \\
&= \int_{\Omega_0} \mathbb{C}_{ijkl}\frac{\partial u_k}{\partial \mathbf{x}_m}(D\underline{\phi}^{-1})_{ml}\frac{\partial u_i}{\partial \mathbf{x}_p}(D\underline{\phi}^{-1})_{pj}|det(D\underline{\phi}^{-1})|d\mathbf{x}
\end{aligned} \tag{24}$$

$$\Rightarrow \mathbb{C}_{ijkl}(D\underline{\phi}^{-1})_{ml}(D\underline{\phi}^{-1})_{pj}|det(D\underline{\phi}^{-1})| = \bar{\mathbb{C}}_{ipkm} \tag{25}$$

$$\bar{\mathbb{C}}_{ijkl} = \mathbb{C}_{ipkm}(D\underline{\phi}^{-1})_{lm}(D\underline{\phi}^{-1})_{jp}|det(D\underline{\phi}^{-1})| \tag{26}$$

Treating $\underline{\phi}$ as a design variable is tedious.

3.3 Homogenization method

$$E_{ijkl}^\varepsilon(\mathbf{x}) = E_{ijkl}(\mathbf{x}, \mathbf{y}), \quad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \tag{27}$$

The tensor E_{ijkl}^ε is a material constant which satisfies the symmetry condition and is assumed to satisfy strong ellipticity condition for every \mathbf{x} .

$$\Rightarrow E_{ijkl}^\varepsilon = E_{jikl}^\varepsilon = E_{ijlk}^\varepsilon = E_{klji}^\varepsilon \tag{28}$$

$$E_{ijkl}^\varepsilon(\mathbf{x})\mathbf{X}_{ij}\mathbf{X}_{kl} \geq m\mathbf{X}_{ij}\mathbf{X}_{ij} \quad \text{for some } m > 0 \text{ \& \forall } \mathbf{X}_{ij} = \mathbf{X}_{ji} \tag{29}$$

Let the domain Ω has a boundary Γ . Let \mathbf{f} be the body force acting on Ω and \mathbf{t} be the traction acting on Γ_t part of the boundary Γ . Also, let Γ_D be the part of boundary on which displacement is defined. Then the displacement \mathbf{u}^ε can be obtained as the solution to the following minimization problem

$$\min_{\mathbf{v}^\varepsilon \in U} F^\varepsilon(\mathbf{v}^\varepsilon), \tag{30}$$

where F^ε is total potential energy given as

$$F^\varepsilon(\mathbf{v}^\varepsilon) = \frac{1}{2} \int_{\Omega} E_{ijkl}^\varepsilon \varepsilon_{kl}(\mathbf{v}^\varepsilon) \varepsilon_{ij}(\mathbf{v}^\varepsilon) dx - \int_{\Omega} \mathbf{f} \cdot \mathbf{v}^\varepsilon dx - \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}^\varepsilon ds \tag{31}$$

and \mathcal{U} is the set of admissible displacements defined such that

$$\mathcal{U} = \{\mathbf{v} = v_i \mathbf{e}_i : v_i \in H^1(\Omega) \text{ and } \mathbf{v} \in \mathcal{G} \text{ on } \Gamma_D\} \tag{32}$$

where \mathcal{G} is set of displacement defined along the boundary Γ_D .

Let

$$\mathbf{v}^\varepsilon(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \varepsilon \mathbf{v}_1(\mathbf{x}, \mathbf{y}), \quad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \tag{33}$$

Using chain rule for functions in two variables

$$\begin{aligned}\frac{\partial f(\mathbf{x}, \mathbf{y}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{y}} \frac{\partial \mathbf{y}}{\partial \mathbf{x}} \\ &= \frac{\partial f}{\partial \mathbf{x}} + \frac{1}{\varepsilon} \frac{\partial f}{\partial \mathbf{y}}\end{aligned}\quad (34)$$

Using above two equations, we can write the linerized strain as

$$\begin{aligned}\epsilon_{ij}(\mathbf{v}^\varepsilon(\mathbf{x})) &= \frac{\partial(v_{0i}(\mathbf{x}) + \varepsilon v_{1i}(\mathbf{x}, \mathbf{y}))}{\partial x_j} \\ &= \frac{\partial v_{0i}}{\partial x_j} + \varepsilon \left\{ \frac{\partial v_{1i}}{\partial x_j} + \frac{1}{\varepsilon} \frac{\partial v_{1i}}{\partial y_j} \right\} \\ &\approx \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \quad \leftarrow \{\varepsilon \ll 1\}\end{aligned}\quad (35)$$

Therefore, equation (31) can be written as

$$\begin{aligned}F^\varepsilon(\mathbf{v}^\varepsilon) &= \frac{1}{2} \int_{\Omega} E_{ijkl}^\varepsilon \left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \right) dx \\ &\quad - \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_0 dx - \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}_0 ds + \varepsilon R^\varepsilon(\mathbf{v}_0, \mathbf{v}_1)\end{aligned}\quad (36)$$

Here, R^ε is the contribution of $\varepsilon \mathbf{v}_1$ in the calculation of energy from body force and traction. Using

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \Phi(x, x/\varepsilon) dx = \frac{1}{|Y|} \int_{\Omega} \int_Y \Phi(x, y) dy dx, \quad (37)$$

we get,

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0} F^\varepsilon(\mathbf{v}^\varepsilon) &= F(\mathbf{v}_0, \mathbf{v}_1) \\ &= \frac{1}{2|Y|} \int_{\Omega} \int_Y E_{ijkl}(x, y) \left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \right) dy dx \\ &\quad - \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_0 dx - \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}_0 ds\end{aligned}\quad (38)$$

A minimizer $\{\mathbf{u}_0, \mathbf{u}_1\}$ of the functional F , follow the following equations:

$$\begin{aligned}&\frac{1}{|Y|} \int_{\Omega} \int_Y E_{ijkl}(x, y) \left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{0i}}{\partial x_j} \right) dy dx \\ &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_0 dx + \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}_0 ds \quad \text{for every } \mathbf{v}_0\end{aligned}\quad (39)$$

$$\frac{1}{|Y|} \int_{\Omega} \int_Y E_{ijkl}(x, y) \left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{1i}}{\partial x_j} \right) dy dx = 0, \quad \text{for every } \mathbf{v}_1 \quad (40)$$

Now, from localizing u_{1k}

$$u_{1k}(x, y) = -\chi_k^{pq}(y) \frac{\partial u_{0p}}{\partial x_q}(x), \quad (41)$$

$$\begin{aligned} & \Rightarrow \int_{\Omega} \int_Y E_{ijkl}(x, y) \left(\frac{\partial u_{0k}}{\partial x_l} - \frac{\partial \chi_k^{pq}}{\partial y_l} \frac{\partial u_{0p}}{\partial x_q} \right) \frac{\partial v_i}{\partial x_j} dy dx = 0 \\ & \int_{\Omega} \int_Y \left(E_{ijkl} \frac{\partial u_{0k}}{\partial x_l} - E_{ijkl} \frac{\partial \chi_k^{pq}}{\partial y_l} \frac{\partial u_{0p}}{\partial x_q} \right) \frac{\partial v_i}{\partial x_j} dy dx = 0 \\ & \int_{\Omega} \int_Y \left(E_{ijkl} \frac{\partial u_{0k}}{\partial x_l} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \frac{\partial u_{0k}}{\partial x_l} \right) \frac{\partial v_i}{\partial x_j} dy dx = 0 \\ & \int_{\Omega} \int_Y \frac{\partial u_{0k}}{\partial x_l} \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) \frac{\partial v_i}{\partial x_j} dy dx = 0 \\ & \int_{\Omega} \frac{\partial u_{0k}}{\partial x_l} dx \cdot \int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) \frac{\partial v_i}{\partial x_j} dy = 0 \\ & \Rightarrow \int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) \frac{\partial v_i}{\partial x_j} dy = 0 \quad \text{for } k, l = 1 \text{ and } 2, \end{aligned} \quad (42)$$

Similarly, substituting equation (41) in (39) gives the homogenized equation.

$$\begin{aligned} \text{LHS} &= \frac{1}{|Y|} \int_{\Omega} \int_Y E_{ijkl}(x, y) \left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l} \right) \left(\frac{\partial v_{0i}}{\partial x_j} \right) dy dx \\ &= \frac{1}{|Y|} \int_{\Omega} \int_Y \left(E_{ijkl} \frac{\partial u_{0k}}{\partial x_l} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \frac{\partial u_{0k}}{\partial x_l} \right) \frac{\partial v_{0i}}{\partial x_j} dy dx \\ &= \frac{1}{|Y|} \int_{\Omega} \left\{ \int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) dy \right\} \frac{\partial u_{0k}}{\partial x_l} \frac{\partial v_{0i}}{\partial x_j} dx \\ &= \int_{\Omega} E_{ijkl}^H(x) \frac{\partial u_{0k}}{\partial x_l} \frac{\partial v_{0i}}{\partial x_j} dx \end{aligned}$$

Homogenized equation

$$\int_{\Omega} E_{ijkl}^H(x) \frac{\partial u_{0k}}{\partial x_l} \frac{\partial v_{0i}}{\partial x_j} dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_0 dx + \int_{\Gamma_t} \mathbf{t} \cdot \mathbf{v}_0 ds \quad \text{for every } \mathbf{v}_0 \quad (43)$$

where $E_{ijkl}^H(x)$ is

$$\boxed{E_{ijkl}^H = \frac{1}{|Y|} \int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) dy} \quad (44)$$

Now, Define

$$a_H(\mathbf{u}, \mathbf{v}) = \int_{\Omega} E_{ijkl}^H(\mathbf{x}) \frac{\partial u_k}{\partial x_l} \frac{\partial v_i}{\partial x_j} dx, \quad (45)$$

$$a_Y(\chi^{kl}, \mathbf{v}) = \int_Y E_{ijpq}(\mathbf{x}, \mathbf{y}) \frac{\partial \chi_p^{kl}}{\partial y_q} \frac{\partial v_i}{\partial y_j} dy, \quad (46)$$

$$L_Y^{kl}(\mathbf{v}) = \int_Y E_{ijkl} \frac{\partial v_i}{\partial y_j} dy \quad (47)$$

At microscopic level, we have

$$a_Y(\chi^{kl}, \mathbf{v}) = L_Y^{kl}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{U}_Y, \quad (48)$$

At macroscopic level, we have

$$a_H(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathcal{U}_0 \quad (49)$$

where \mathcal{U}_0 is homogeneous case of \mathcal{U} , i.e., $\mathbf{g} = 0$.

3.4 Implementation 2D Homogenization

Basic homogenization equation,

$$u_{1i}(\mathbf{x}, \mathbf{y}) = -\chi_i^{pq} \frac{\partial u_{0p}(\mathbf{x})}{\partial x_q} \quad (50)$$

Solve χ_p^{kl} from:

$$\int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) \frac{\partial v_{1i}}{\partial y_j} dy = 0 \quad (51)$$

Compute:

$$E_{ijkl}^H = \frac{1}{|Y|} \int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) dy \quad (52)$$

3.5 Examples

Consider: $k=1, l=1$

$$\begin{aligned} \int_Y E_{ijkl} \frac{\partial v_i}{\partial y_j} dy &= \int_Y E_{ij11} \frac{\partial v_i}{\partial y_j} dy \\ &= \int_Y \left(E_{1111} \frac{\partial v_1}{\partial y_1} + E_{2211} \frac{\partial v_2}{\partial y_2} \right) dy \end{aligned} \quad (53)$$

$$\begin{aligned}
\int_Y E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \frac{\partial v_i}{\partial y_j} dy &= \int_Y E_{ijpq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_i}{\partial y_j} dy \\
&= \int_Y \left\{ E_{11pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_1}{\partial y_1} + E_{12pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_1}{\partial y_2} \right. \\
&\quad \left. + E_{21pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_2}{\partial y_1} + E_{22pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1112} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1121} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + \left(E_{1211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{1222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_2} \\
&\quad + \left(E_{2111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2112} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{2121} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{2122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_1} \\
&\quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{2221} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1112} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1121} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + \left(E_{1211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{1222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{2221} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \Big\} dy \\
&= \int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + E_{1212} \left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy
\end{aligned} \tag{54}$$

Therefore, using equations (48) , (54) and (53) for k=1, l=1 we have:

$$\begin{aligned}
& \int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
& \quad + E_{1212} \left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
& \quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
& \int_Y \left(E_{1111} \frac{\partial v_1}{\partial y_1} + E_{2211} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{55}$$

From equation (52), we can write

$$E_{1111}^H = \frac{1}{|Y|} \int_Y \left(E_{1111} - E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} - E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{56}$$

$$E_{2211}^H = \frac{1}{|Y|} \int_Y \left(E_{2211} - E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} - E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{57}$$

$$E_{1211}^H = -\frac{1}{|Y|} \int_Y \left(E_{1212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \frac{\partial \chi_2^{11}}{\partial y_1} \right) dy \tag{58}$$

Let $\chi_1^{11} = \Phi_1, \chi_2^{11} = \Phi_2$ and $E_{1111} = D_{11}, E_{2222} = D_{22}, E_{1212} = D_{66}, E_{1122} = E_{2211} = D_{12}$

$$\begin{aligned}
& \int_Y \left\{ \left(D_{11} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{12} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
& \quad + D_{66} \left(\frac{\partial \Phi_1^{11}}{\partial y_2} + \frac{\partial \Phi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
& \quad \left. + \left(D_{12} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{22} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
& \int_Y \left(D_{11} \frac{\partial v_1}{\partial y_1} + D_{12} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{59}$$

Also,

$$D_{11}^H = \frac{1}{|Y|} \int_Y \left(D_{11} - D_{11} \frac{\partial \Phi_1}{\partial y_1} - D_{12} \frac{\partial \Phi_2}{\partial y_2} \right) dy \tag{60}$$

Rearranging Eq. (59)

$$\begin{aligned}
& \int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \\
& \quad \times \begin{bmatrix} \frac{\partial \Phi_1}{\partial y_1} \\ \frac{\partial \Phi_2}{\partial y_2} \\ \frac{\partial \Phi_1}{\partial y_2} + \frac{\partial \Phi_2}{\partial y_1} \end{bmatrix} dY \\
& = \int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} \\ D_{12} \\ 0 \end{bmatrix} dY
\end{aligned} \tag{61}$$

Let us define

$$\mathbf{b} = \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 \\ 0 & \frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} \end{bmatrix} \tag{62}$$

and

$$\mathbf{D} = [\mathbf{d}_1 \quad \mathbf{d}_2 \quad \mathbf{d}_3] \tag{63}$$

Then Eq (59), can be written as

$$\int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{D} \mathbf{b} \Phi dY = \int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{d}_1 \quad \forall \mathbf{v} \in \mathbf{V}_Y \tag{64}$$

and eq. (60) becomes:

$$\boxed{D_{11}^H = \frac{1}{|Y|} \int_Y \left(D_{11} - \mathbf{d}_1^T \mathbf{b} \Phi \right) dy} \tag{65}$$

4 Results

[1] [2]

5 References

References

- [1] John Doe. Title. *Journal*, 2017.
- [2] Intel. Example website. <http://example.com>, Dec 1988. Accessed on 2012-11-11.