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A review of homogenization and topology optimization I—homogenization theory for media with periodic structure

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Abstract

This is the first part of a three-paper review of homogenization and topology optimization, viewed from an engineering standpoint and with the ultimate aim of clarifying the ideas so that interested researchers can easily implement the concepts described. In the first paper we focus on the theory of the homogenization method where we are concerned with the main concepts and derivation of the equations for computation of effective constitutive parameters of complex materials with a periodic micro structure. Such materials are described by the base cell, which is the smallest repetitive unit of material, and the evaluation of the effective constitutive parameters may be carried out by analysing the base cell alone. For simple microstructures this may be achieved analytically, whereas for more complicated systems numerical methods such as the finite element method must be employed. In the second paper, we consider numerical and analytical solutions of the homogenization equations. Topology optimization of structures is a rapidly growing research area, and as opposed to shape optimization allows the introduction of holes in structures, with consequent savings in weight and improved structural characteristics. The homogenization approach, with an emphasis on the optimality criteria method, will be the topic of the third paper in this review.

1. Introduction

Advances in technology in recent years have been paralleled by the increased use of composite materials in industry. Since materials have different properties, it seems sensible to make use of the good properties of each single ingredient by using them in a proper combination. For example, a simple mixture of clay, sand and straw produced a composite building material which was used by the oldest known civilizations. The further development of non-metallic materials and composites has attracted the attention of scientists and engineers in various fields, for example, aerospace, transportation, and other branches of civil and mechanical engineering. Apart from the considerably low ratio of weight to strength, some composites benefit from other desirable properties, such as corrosion and thermal resistance, toughness and lower cost. Usually, composite materials comprise of a *matrix* which could be metal, polymeric (like plastics) or ceramic, and a *reinforcement* or *inclusion*, which could be particles or fibres of steel, aluminum, silicon etc.

Composite materials may be defined as a man-made material with different dissimilar constituents, which occupy different regions with distinct interfaces between them [1]. The properties of a composite are different from its individual constituents. A cellular body can be considered as a simple case of a composite, comprising solids and voids. This is the case which is used in the structural topology optimization.

In this study, composites with a regular or nearly regular structure are considered. Having sufficiently regular heterogenities enables us to assume a periodic

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structure for the composite. It should be emphasized that in comparison with the dimensions of the body the size of these non-homogeneities should be very small. Owing to this, these types of material are sometimes called *composites with periodic microstructures*.

Even with the help of high-speed modern computers, the analysis of the boundary value problems consisting of such media with a large number of heterogenities, is extremely difficult. A natural way to overcome this difficulty is to replace the composite with a kind of equivalent material model. This procedure is usually called *homogenization*. One way of finding the properties of such composites is by carrying out experimental tests. It is quite evident that because of the volume and cost of the required tests for all possible reinforcement types, experimental measurements are often impracticable.

The mathematical theory of homogenization, which has developed since the 1970 s is used as an alternative approach to find the effective properties of the equivalent homogenized material [2–4]. This theory can be applied in many areas of physics and engineering having finely heterogeneous continuous media, like heat transfer or fluid flow in porous media or, for example, electromagnetism in composites. In fact, the basic assumption of continuous media in mechanics and physics can be thought of as sort of homogenization, as the materials are composed of atoms or molecules.

From a mathematical point of view, the theory of homogenization is a limit theory which uses the asymptotic expansion and the assumption of periodicity to substitute the differential equations with rapidly oscillating coefficients, with differential equations whose coefficients are constant or slowly varying in such a way that the solutions are close to the initial equations [5].

This method makes it possible to predict both the overall and local properties of processes in composites. In the first step, the appropriate local problem on the unit cell of the material is solved and the effective material properties are obtained. In the second step, the boundary value problem for a homogenized material is solved.

2. Periodicity and Asymptotic Expansion

A heterogeneous medium is said to have a regular periodicity if the functions denoting some physical quantity of the the medium—either geometrical or some other characteristics—have the following property:

$$\mathcal{F}(\mathbf{x} + \mathbf{NY}) = \mathcal{F}(\mathbf{x}). \tag{1}$$

$\mathbf{x} = (x_1, x_2, x_3)$ is the position vector of the point, $\mathbf{N}$ is a 3×3 diagonal matrix:

$$\mathbf{N} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix},$$

where n_1, n_2 and n_3 are arbitrary integer numbers, and $\mathbf{Y} = \langle Y_1 \quad Y_2 \quad Y_3 \rangle^{\mathrm{T}}$ is a constant vector which determines the period of the structure; $\mathcal{F}$ can be a scalar or vectorial or even tensorial function of the position vector $\mathbf{x}$. For example, in a composite tissued by a periodically repeating cell $\mathbf{Y}$, the mechanical behaviour is described by the constitutional relations of the form:

$$\sigma_{ij} = c_{ijkl} e_{kl},$$

and the tensor c_{ijkl} is a periodic function of the spatial coordinate $\mathbf{x}$, so that

$$c_{ijkl}(\mathbf{x} + \mathbf{NY}) = c_{ijkl}(x) \tag{2}$$

or

$$c_{ijkl}(x_1 + n_1 Y_1, x_2 + n_2 Y_2, x_3 + n_3 Y_3) = c_{ijkl}(x_1, x_2, x_3).$$

$c_{ijkl}(\mathbf{x})$ is called the $\mathbf{Y}$-periodic (see Fig. 1). Note that σ_{ij} and e_{kl} are, respectively, the stress and strain tensors.

In the theory of homogenization the period $\mathbf{Y}$ compared with the dimensions of the overall domain is assumed to be very small. Hence, the characteristic functions of these highly heterogeneous media will rapidly vary within a very small neighbourhood of a point $\mathbf{x}$. This fact inspires the consideration of two different scales of dependencies for all quantities: one on the *macroscopic* or *global* level $\mathbf{x}$, which indicates *slow* variations, and the other on the *microscopic* or *local* level $\mathbf{y}$, which describes *rapid* oscillations.

The ratio of the real length of a unit vector in the microscopic coordinates to the real length of a unit vector in the macroscopic coordinates, is a small parameter ϵ. so $\epsilon\mathbf{y} = \mathbf{x}$ or $\mathbf{y} = \mathbf{x}/\epsilon$. Consequently, if g is a general function then we can say $g = g(\mathbf{x}, \mathbf{x}/\epsilon) = g(\mathbf{x}, \mathbf{y})$. To illustrate the technique let us assume that $\Phi(x)$ is a physical quantity of a strongly heterogeneous med-

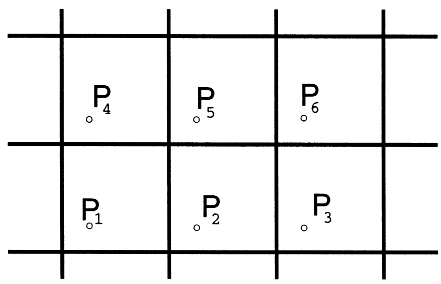


Fig. 1. Periodicity requires that the functions have equal values at points P_1, P_2,..., P_6.

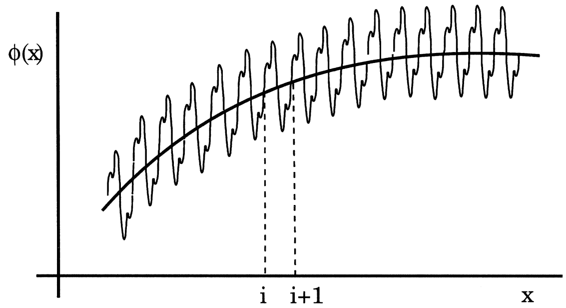


Fig. 2. A highly oscillating function.

ium. Thus $\Phi(x)$ will have oscillations, see Fig. 2. To study these oscillations using this *double-scale* expansion, the space can be enlarged as indicated in Fig. 3.

The small parameter ϵ also provides an indication of the proportion between the dimensions of the base cells of a composite and the whole domain, known as the *characteristic* inhomogeneity dimension. As a hypothetical example, ϵ for the skin cells of the human body is larger than ϵ for the atoms of which it is comprised. The quantity $1/\epsilon$ can be thought of as a magnification factor which enlarges the dimensions of a base cell to be comparable with the dimensions of the material [6–8], see Fig. 4.

In the double-scale technique, the partial differential equations of the problem have coefficients of the form $a(\mathbf{x}/\epsilon)$ or $a(\mathbf{y})$, where $a(\mathbf{y})$ is a periodic function of its arguments. The corresponding boundary value problem may be treated by asymptotically expanding the solution in powers of the small parameter ϵ. This technique has already proved to be useful in the analysis of slightly perturbed periodic processes in the theory of vibrations. The same principle is extendible to processes occurring in composite materials with a regular structure.

If we assign a coordinate system $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ space to define the domain of the composite material problem Ω, then assuming periodicity, the domain can be regarded as a collection of parallelpiped cells of identical dimensions ϵY_1, ϵY_2, ϵY_3, where Y_1, Y_2 and Y_3 are the sides of the base cell in a local (microscopic) coordinate system $\mathbf{y} = (y_1, y_2, y_3) = \mathbf{x}/\epsilon$. So for a fixed $\mathbf{x}$ in the macroscopic level, any dependency on $\mathbf{y}$ can be considered Y-periodic. Moreover, it is assumed that the form and composition of the base

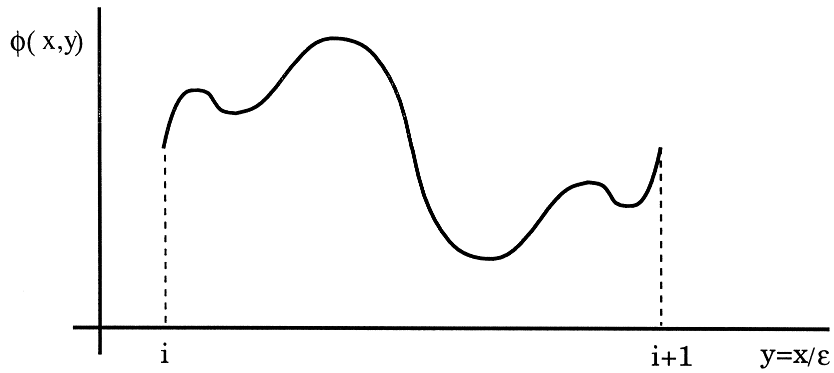


Fig. 3. One of the oscillations in the expanded scale.

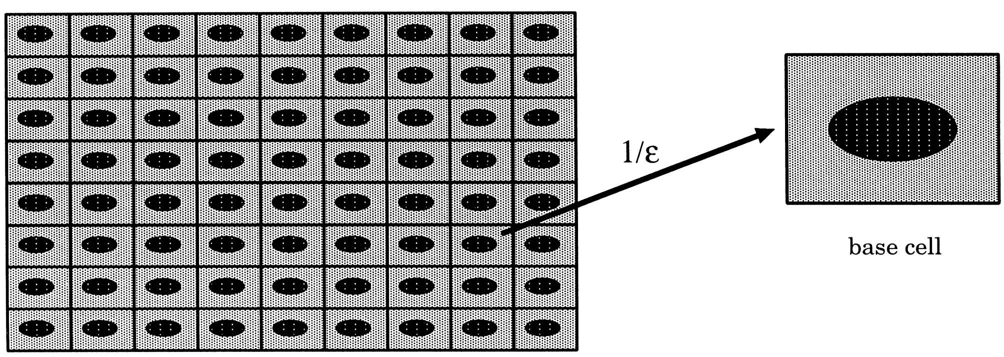


Fig. 4. Characteristic dimension of inhomogeneity and scale enlargement.

cell varies in a smooth way with the macroscopic variable **x**. This means that for different points the structure of the composite may vary, but if one looks through a microscope at a point at **x**, a periodic pattern can be found.

Functions determining the behaviour of the composite can be expanded as:

$$\Phi^{\epsilon}(\mathbf{x}) = \Phi^0(\mathbf{x},\mathbf{y}) + \epsilon\Phi^1(\mathbf{x},\mathbf{y}) + \epsilon^2\Phi^2(\mathbf{x},\mathbf{y}) + \cdots,$$

where $\epsilon \to 0$ and functions Φ^0 (**x**, **y**), Φ^1 (**x**, **y**),... are smooth with respect to **x** and Y-periodic in **y**, which means that they take equal values on the opposite sides of the parallel-piped base cell.

3. One-dimensional Elasticity Problem

To clarify the homogenization method, the simple case of calculation of deformation of an inhomogeneous bar in the longitudinal direction is considered. Here, we attempt to derive the modulus of elasticity without recourse to advanced mathematics.

According to the assumptions of the theory, the medium has a periodic composite microstructure (Fig. 5).

The governing equations, in the form of Hooke's law of linear elasticity and the Cauchy's first law of motion (equilibrium equation), are:

$$\sigma^{\epsilon} = E^{\epsilon}\frac{\partial u^{\epsilon}}{\partial x}, \tag{3}$$

$$\frac{\partial \sigma^{\epsilon}}{\partial x} + \gamma^{\epsilon} = 0. \tag{4}$$

The dependency of the quantities to the size of the unit cell of inhomogeneity is indicated by the superscript "ϵ". σ^{ϵ} is the stress, u^{ϵ} is the displacement, $E^{\epsilon}(x)$ is the Young's modulus and γ^{ϵ} is the weight per unit volume of material. It is assumed that E^{ϵ} and γ^{ϵ} are macroscopically uniform along the domain and only vary inside each cell:

$$E^{\epsilon}(x, x/\epsilon) = E^{\epsilon}(x/\epsilon) = E(y) \tag{5}$$

and

$$\gamma^{\epsilon}(x, x/\epsilon) = \gamma^{\epsilon}(x/\epsilon) = \gamma(y). \tag{6}$$

Using the double-scale asymptotic expansion:

$$u^{\epsilon}(x) = u^0(x,y) + \epsilon u^1(x,y) + \epsilon^2 u^2(x,y) + \cdots \tag{7}$$

and

$$\sigma^{\epsilon}(x) = \sigma^0(x,y) + \epsilon\sigma^1(x,y) + \epsilon^2 2(x,y) + \cdots, \tag{8}$$

where $u^i(x, y)$ and $\sigma^i(x, y)$, (i = 1, 2,...) are periodic on y and the length of period is Y. In due course the following facts will be referred to:

Fact (1). The derivative of a periodic function is also periodic with the same period.

Fact (2). The integral of the derivative of a function over the period is zero. (These facts can easily be verified by the definition of derivative and periodicity.)

Fact (3). If $\Phi = \Phi(x, y)$ and y depends on x, then:

$$\frac{d\Phi}{dx} = \frac{\partial\Phi}{\partial x} + \frac{\partial\Phi}{\partial y}\frac{\partial y}{\partial x}.$$

In this case, as $y = x/\epsilon$, so

$$\frac{d\Phi}{dx} = \frac{\partial\Phi}{\partial x} + \frac{1}{\epsilon}\frac{\partial\Phi}{\partial y}.$$

Using the latter fact and substituting the series in Eqs. (7) and (8) into Eqs. (3) and (4), we obtain:

$$\sigma^0 + \epsilon\sigma^1 + \epsilon^2\sigma^2 + \cdots = E(y)\left[\frac{\partial u^0}{\partial x} + \frac{1}{\epsilon}\frac{\partial u^0}{\partial y} + \epsilon\frac{\partial u^1}{\partial x} + \frac{\partial u^1}{\partial y} + \epsilon^2\frac{\partial u^1}{\partial x} + \epsilon\frac{\partial u^2}{\partial y} + \cdots\right], \tag{9}$$

and

$$\frac{\partial\sigma^0}{\partial x} + \frac{1}{\epsilon}\frac{\partial\sigma^0}{\partial y} + \epsilon\frac{\partial\sigma^1}{\partial x} + \frac{\partial\sigma^1}{\partial y} + \cdots + \gamma(y) = 0. \tag{10}$$

By equating the terms with the same power of ϵ, Eq. (9) yields:

$$0 = E(y)\left(\frac{\partial u^0}{\partial y}\right), \tag{11}$$

$$\sigma^0 = E(y)\left(\frac{\partial u^0}{\partial x} + \frac{\partial u^1}{\partial y}\right), \tag{12}$$

$$\sigma^1 = E(y)\left(\frac{\partial y^1}{\partial x} + \frac{\partial u^2}{\partial y}\right), \tag{13}$$

and similarly from Eq. (10):

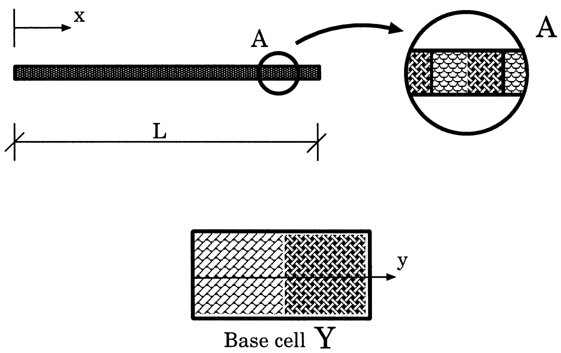


Fig. 5. A composite bar.

$$\frac{\partial\sigma^0}{\partial y} = 0, \tag{14}$$

$$\frac{\partial\sigma^0}{\partial x} + \frac{\partial\sigma^1}{\partial y} + \gamma(y) = 0. \tag{15}$$

From Eqs. (11) and (14) it is concluded that the functions u^0 and σ^0 only depend on x [i.e. $u^0(x)$ and $\sigma^0(x)$]. Bearing in mind that the relationship between $\sigma^0(x)$ and $u^0(x)$ is sought (because they are independent of the microscopic scale), Eq. (12) can be written as:

$$\sigma^0(x) = E(y)\left[\frac{\mathrm{d}u^0(x)}{\mathrm{d}x} + \frac{\partial u^1(x,y)}{\partial y}\right]. \tag{16}$$

Dividing by $E(y)$ and integrating both sides of Eq. (16) over the period Y, and using fact (2), yields:

$$\sigma^0(x) = \left(Y/\int_Y \frac{\mathrm{d}y}{E(y)}\right)\frac{\mathrm{d}u^0(x)}{\mathrm{d}x}. \tag{17}$$

Now, by substituting the value of $\sigma^0(x)$ into Eq. (16), we obtain:

$$\frac{\partial u^1(x,y)}{\partial y} = \left[Y/\left(E(y)\int_Y \frac{\mathrm{d}y}{E(y)}\right) - 1\right]\frac{\mathrm{d}u^0(x)}{\mathrm{d}x},$$

and by integrating this equation, we conclude that u^1 has the following form:

$$u^1(x,y) = \chi(y)\frac{\mathrm{d}u^0(x)}{\mathrm{d}x} + \xi(x), \tag{18}$$

where $\chi(y)$ is the initial function of the terms inside the square brackets and $\xi(x)$ is the constant of integration due to y. From Eqs. (16) and (18) it follows that

$$\sigma^0(x) = E(y)\left(1 + \frac{\mathrm{d}\chi(y)}{\mathrm{d}y}\right)\frac{\mathrm{d}u^0(x)}{\mathrm{d}x}. \tag{19}$$

Differentiating Eq. (19) with respect to y, one concludes that

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[E(y)\left(1 + \frac{\mathrm{d}\chi(y)}{\mathrm{d}y}\right)\right] = 0, \quad \text{on } Y, \tag{20}$$

and $\chi(y)$ takes equal values on the opposite faces of Y [i.e. $\chi(0) = \chi(Y)$]. Integrating Eq. (20) yields

$$E(y)\left(1 + \frac{\mathrm{d}\chi(y)}{\mathrm{d}y}\right) = a \quad (a \text{ is a constant}), \tag{21}$$

or

$$\frac{\mathrm{d}\chi(y)}{\mathrm{d}y} = \frac{a}{E(y)} - 1. \tag{22}$$

Integrating Eq. (22) it follows that

$$\chi(y) = \int_0^y \left(\frac{a}{E(\eta)} - 1\right)\mathrm{d}\eta + b, \tag{23}$$

where η is the dummy variable of integration and b is a constant. Now, using the boundary condition $\chi(0) = \chi(Y)$ yields:

$$\int_0^Y \frac{a}{E(\eta)}\mathrm{d}\eta - Y = 0, \tag{24}$$

or

$$a = 1/\left(\frac{1}{Y}\int_0^Y \frac{\mathrm{d}\eta}{E(\eta)}\right). \tag{25}$$

Note that comparig Eqs. (19) and (21) one can see that

$$\sigma^0(x) = a\frac{\mathrm{d}u0(x)}{\mathrm{d}x}, \tag{26}$$

and substituting for a from Eq. (25) yields

$$\sigma^0(x) = 1/\left(\frac{1}{Y}\int_0^Y \frac{\mathrm{d}\eta}{E(\eta)}\right)\frac{\mathrm{d}u^0(x)}{\mathrm{d}x}. \tag{27}$$

By integrating Eq. (15) over the length of the period (0, Y) and using fact (2) mentioned earlier, results in:

$$\frac{\mathrm{d}\sigma^0(x)}{\mathrm{d}x} + \bar{\gamma} = 0, \tag{28}$$

where $\gamma = 1/Y \int_Y \gamma(y)\mathrm{d}y$ is the volumetric average of γ inside the base cell.

By studying Eqs. (27) and (28), we realize that they are very similar to the equations of one-dimensional (1D) elasticity in homogeneous material, and σ^0 and u^0 are independent of the microscopic scale y. The only difference is the elasticity coefficient, which should be replaced by the homogenized one. Hence, the problem can be summarized as:

$$\begin{cases} \sigma^0(x) = E^{\mathrm{H}}\mathrm{d}u^0(x)/\mathrm{d}x \\ \mathrm{d}\sigma^0(x)/\mathrm{d}x + \bar{\gamma} = 0, \end{cases} \tag{29}$$

where

$$E^{\mathrm{H}} = 1/\left(\frac{1}{Y}\int_0^Y \frac{\mathrm{d}\eta}{E(\eta)}\right), \tag{30}$$

is the *homogenized modulus of elasticity*.

To find displacements, following the same as for the homogeneous material, the bar problem is now straightforward. Combining the two parts of Eq. (29), we obtain:

$$\frac{\partial^2 u^0(x)}{\partial x^2} = -\frac{\gamma}{E^{\mathrm{H}}}.$$

By two times integration and using the boundary conditions ($x = 0$; $u = 0$) and ($x = L$; $\mathrm{d}u/\mathrm{d}x = 0$) it results in:

$$u(x) = -\frac{\bar{\gamma}}{E^{\mathrm{H}}}\frac{x^2}{2} + \frac{\bar{\gamma}}{E^{\mathrm{H}}}Lx.$$

4. Problem of Heat Conduction

The 1D heat conduction is very similar to the 1D elasticity problem. The governing equations, Fourier's law of heat conduction and the equation of heat balance, are:

$$\begin{cases} q^{\epsilon}(x) = K^{\epsilon}\mathrm{d}T^{\epsilon}(x)/\mathrm{d}x \\ \partial q^{\epsilon}/\partial x + f = 0. \end{cases} \tag{31}$$

q^{ϵ} is the heat flux, T^{ϵ} is the temperature, and $K^{\epsilon}(x)$ is the conductivity coefficient. Following a very similar procedure to the 1D elasticity problem, the homogenized coefficient of heat conduction can be obtained as:

$$K^{\mathrm{H}} = 1/\left(\frac{1}{Y}\int_0^Y \frac{\mathrm{d}\eta}{K(\eta)}\right),$$

which as is expected, is the same as Eq. (30).

Similarly, starting from the equations of heat conduction in the general 3D case, and following the same procedure as for 1D problem, the following results will be obtained [6]:

$$\begin{cases} \bar{q}_i\mathbf{x} = K^{\mathrm{H}}_{ij}\partial\bar{T}(\mathbf{x})/\partial x_j \\ \partial\bar{q}_i/\partial x_i + f = 0, \end{cases} \tag{32}$$

where

$$K^{\mathrm{H}}_{ij} = \frac{1}{|\mathbf{Y}|}\left[\int_Y K(\mathbf{y})\left(\delta_{ij} + \frac{\partial\chi^j}{\partial y_i}\right)\mathrm{d}\mathbf{y}\right], \tag{33}$$

and $\chi^j(\mathbf{y})$ is the solution of the partial differential equation:

$$\frac{\partial}{\partial y_i}\left[K(\mathbf{y})\left(\delta_{ij} + \frac{\partial\chi^j}{\partial y_i}\right)\right] = 0 \quad \text{on } \mathbf{Y}. \tag{34}$$

δ_{ij} is the Kronecker symbol and the boundary conditions are concluded from the periodicity, i.e. χ^j takes equal values on the opposite sides of the base cell. In Eqs. (31) and (32), q and $\partial q_i/\partial x_i$ are the volumetric average value of $q_i^0(x)$ and $\partial q_i^0/\partial x_i$ over $\mathbf{Y}$. The volumetric average of a quantity $a(\mathbf{x}, \mathbf{y})$ over $\mathbf{Y}$ is defined by:

$$\bar{a}(\mathbf{x}) = \frac{1}{|\mathbf{Y}|}\int_Y a(\mathbf{x}, \mathbf{y})\mathrm{d}\mathbf{y}. \tag{35}$$

5. General Boundary Value Problem

Many physical systems which do not change with time—sometimes called steady-state problems—can be modelled by elliptic equations. As a general problem, the divergent elliptic equation in a non-homogeneous medium with regular structure is now explained.

Let $\Omega\subset\mathbb{R}^3$ be an unbounded medium tissued by parallelepiped unit cells $\mathbf{Y}$, whose material properties are determined by a symmetric matrix $a_{ij}(\mathbf{x}, \mathbf{y}) = a_{ij}(\mathbf{y})$, where $\mathbf{y} = \mathbf{x}/\epsilon$ and $\mathbf{x} = (x_1, x_2, x_3)$ and the functions a_{ij} are periodic in the spatial variables $\mathbf{y} = (y_1, y_2, y_3)$. The boundary value problem to be dealt with is:

$$\mathcal{A}^{\epsilon}u^{\epsilon} = f \quad \text{in } \Omega, \tag{36}$$

$$u^{\epsilon} = 0 \quad \text{on } \partial\Omega, \tag{37}$$

where the function f is defined in Ω and

$$\mathcal{A}^{\epsilon} = \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial x_j}\right) \tag{38}$$

is the elliptical operator. The superscript "ϵ" is used to show the dependency of the operator and the solution to the characteristic inhomogeneity dimension.

Using a double-scale asymptotic expansion, the solution to Eqs. (36) and (37) can be written as:

$$u^{\epsilon}(\mathbf{x}) = u^0(\mathbf{x}, \mathbf{y}) + \epsilon^1u^1(\mathbf{x}, \mathbf{y}) + \epsilon^2u^2(\mathbf{x}, \mathbf{y}) + \cdots, \tag{39}$$

where functions $u^j(\mathbf{x}, \mathbf{y})$ are $\mathbf{Y}$-periodic in $\mathbf{y}$. Recalling the rule of indirect differentiation (fact 3) yields

$$\mathcal{A}^{\epsilon} = \frac{1}{\epsilon^2}\mathcal{A}^1 + \frac{1}{\epsilon}\mathcal{A}^2 + \mathcal{A}, \tag{40}$$

where

$$\mathcal{A}^1 = \frac{\partial}{\partial y_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right); \quad \mathcal{A}^3 = \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial x_j}\right)$$

and

$$\mathcal{A}^2 = \frac{\partial}{\partial y_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial x_j}\right) + \frac{\partial}{\partial x_i}\left(a_{ij}(\mathbf{y})\frac{\partial}{\partial y_j}\right).$$

Applying Eqs. (39) and (40) into Eq. (36) yields

$$(\epsilon^{-2}\mathcal{A}^1 + \epsilon^{-1}\mathcal{A}^2 + \mathcal{A}^3)(u^0 + \epsilon u^1 + \epsilon^2u^2 + \cdots) = f, \tag{41}$$

and by equating terms with the same power of ϵ, we obtain:

$$\mathcal{A}^1u^0 = 0, \tag{42}$$

$$\mathcal{A}^1u^1 + \mathcal{A}^2u^0 = 0, \tag{43}$$

$$\mathcal{A}^1u^2 + \mathcal{A}^2u^1 + \mathcal{A}^3u^0 = f; \cdots \tag{44}$$

If $\mathbf{x}$ and $\mathbf{y}$ are considered as independent variables, these equations form a recurrent system of differential equations with the functions u^0, u^1 and u^2 parameterized by $\mathbf{x}$. Before proceeding to the analysis of this system, it is useful to notice the following fact:

Fact (4). The equation

$$\mathcal{A}^1 u = F \quad \text{in } \mathbf{Y} \tag{45}$$

for a $\mathbf{Y}$-periodic function u has a unique solution if:

$$\bar{F} = \frac{1}{|Y|}\int_Y F\mathrm{d}\mathbf{y} = 0, \tag{46}$$

where $|Y|$ denotes the volume of the base cell.

From this fact, and using Eq. (42), it immediately follows that

$$u^0 = u(\mathbf{x}), \tag{47}$$

and by substituting into Eq. (43) we find:

$$\mathcal{A}^1 u^1 = -\mathcal{A}^2_u 0 = -\frac{\partial a_{ij}(\mathbf{y})}{\partial y_i}\frac{\partial u(\mathbf{x})}{\partial x_j}. \tag{48}$$

As in the right-hand side of Eq. (48) the variables are separated, the solution of this equation may be represented in the form

$$u^1(\mathbf{x},\mathbf{y}) = \chi^i(\mathbf{y})\frac{\partial u(\mathbf{x})}{\partial x_j} + \xi(\mathbf{x}), \tag{49}$$

where $\chi^j(\mathbf{y})$ is the $\mathbf{Y}$-periodic solution of the local equation

$$\mathcal{A}^1\chi^j(\mathbf{y}) = \frac{\partial a_{ij}(\mathbf{y})}{\partial y_i} \quad \text{in } \mathbf{Y}. \tag{50}$$

Now, turning to Eq. (44) for u^2 and taking $\mathbf{x}$ as a parameter, it follows from fact (4) that Eq. (44) will have a unique solution if

$$-\frac{1}{|Y|}\int_Y (\mathcal{A}^2 u^1 + \mathcal{A}^3 u^0)\mathrm{d}\mathbf{y} + f = 0, \tag{51}$$

which when combined with Eq. (49) results in the following homogenized (macroscopic) equation for $u(\mathbf{x})$:

$$a^{\mathrm{H}}_{ij}\frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j} = f, \tag{52}$$

where the quantities

$$a^{\mathrm{H}}_{ij} = \frac{1}{|Y|}\int_Y\left(a_{ij}(\mathbf{y}) + a_{ik}(\mathbf{y})\frac{\partial \chi^j}{\partial y_k}\right)\mathrm{d}\mathbf{y} \tag{53}$$

are the effective coefficients of the homogenized operator:

$$\mathcal{A}^{\mathrm{H}} = a^{\mathrm{H}}_{ij}\frac{\partial^2}{\partial x_i \partial x_j}.$$

Thus, it is demonstrated that the initial equation has been split into two different problems:

1. Determine $\chi^j(\mathbf{y})$ from Eq. (50) which is solved on the base cell.
2. Solve Eq. (52) on Ω with $u = 0$ on $\partial\Omega$. The homogenized coefficients a^{H}_{ij} are obtained from Eq. (53).

6. General Elasticity Problem

So far, the application of the homogenization theory in 1D elasticity, heat conduction, and as a more general problem in elliptic partial differential equations, has been discussed. For the sake of completeness the homogenization method for cellular media in weak form, which is suitable for the derivation of the finite element formulation, using the procedure and notation used by Guedes and Kikuchi in Ref. [9], is briefly explained. This is the case applied in topological structural optimization by Bendsøe and Kikuchi [10–14].

Let us consider the elasticity problem constructed from a material with a porous body with a periodic cellular microstructure. Body forces $\mathbf{f}$ and tractions $\mathbf{t}$ are applied. See Fig. 6 Ω is assumed to be an open subset of $\mathbb{R}^3$ with a smooth boundary on Γ comprising Γ_{d} (where displacements are prescribed) and Γ_{t} (the traction boundary). The base cell[1] of the cellular body $\mathbf{Y}$ is illustrated in Fig. 7. $\mathbf{Y}$ is assumed to be an open rectangular parallel-piped in $\mathbb{R}^3$ defined by

$$\mathbf{Y} =]0, Y_1[\times]0, Y_2[\times]0, Y_3[,$$

with a hole v in it. The boundary of v is defined by s ($\partial v = s$) and is assumed to be sufficiently smooth, and as a more general case the tractions $\mathbf{p}$ can also exist inside the holes. The solid part of the cell is denoted by ¥, therefore, the solid part of the domain can be defined as

$$\Omega^{\epsilon} = \{\mathbf{x} \in \Omega | (\mathbf{y} = \mathbf{x}/\epsilon) \in ¥\}.$$

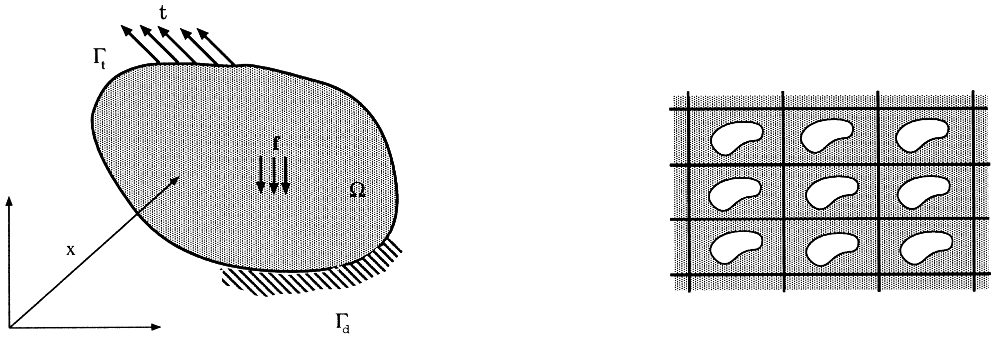
Also, we define

$$S^{\epsilon} = \bigcup_{i=1}^{\text{all cells}} s_i.$$

It is also assumed that none of the holes v_i intersect the boundary Γ. (i.e. $\Gamma \cap S^{\epsilon} = \emptyset$).

Now, considering the stress–strain and strain–displacement relations:

[1] Having periodic microstructure does not mean that the form and composition of the base cell cannot vary,but the variations in the macroscopic scale are assumed to be smooth enough.



General elasticity problem

Cellular structure

Fig. 6. Elasticity problem in a cellular body.

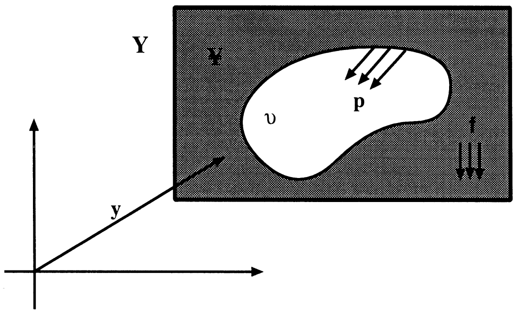


Fig. 7. Base cell of the cellular body.

$$\sigma^{\epsilon}_{ij} = E^{\epsilon}_{ijkl} e^{\epsilon}_{kl}, \tag{54}$$

$$e^{\epsilon}_{kl} = \frac{1}{2}\left(\frac{\partial u^{\epsilon}_k}{\partial x_l} + \frac{\partial u^{\epsilon}_l}{\partial x_k}\right), \tag{55}$$

the virtual displacement equation can be constructed as:

Find $\mathbf{u}^{\epsilon} \in \mathbf{V}^{\epsilon}$, such that

$$\int_{\Omega_\epsilon} E_{ijkl} \frac{\partial u^{\epsilon}_k}{\partial x_l}\frac{\partial v_i}{\partial x_j} \mathrm{d}\Omega = \int_{\Omega_\epsilon} f^{\epsilon}_i v_i \mathrm{d}\Omega + \int_{\Gamma_t} t_i v_i \mathrm{d}\Gamma + \int_{S^\epsilon} p^{\epsilon}_i v_i \mathrm{d}S, \quad \forall \mathbf{v} \in \mathbf{V}^{\epsilon}, \tag{56}$$

where

$$\mathbf{V} = \{\mathbf{v} \in [H^1(\Omega^{\epsilon})]^3 \quad \text{and} \quad \mathbf{v}|_{\Gamma_d} = 0\},$$

and H^1 is the Sobolev space[2]. The elastic constants of the solid are assumed to have symmetry and coercivity properties:

$$E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij},$$
$$\exists \alpha > 0 : E^{\epsilon}_{ijkl} e_{kl} = \alpha e_{ij} e_{ij}, \quad \forall e_{ij} = e_{ij}.$$

Now, using the double-scale asymptotic expansion and fact (3), Eq. (56) becomes

$$\begin{aligned} &\int_{\Omega_\epsilon} E_{ijkl}\left\{\frac{1}{\epsilon^2}\frac{\partial u^0_k}{\partial g_l}\frac{\partial u_i}{\partial y_j} + \frac{1}{\epsilon}\left[\left(\frac{\partial u^0_k}{\partial x_l} + \frac{\partial u^1_k}{\partial y_l}\right)\frac{\partial v_i}{\partial_j} + \frac{\partial u^0_k}{\partial y_l}\frac{\partial v_i}{\partial x_j}\right]\right. \\ &\quad \left. + \left[\left(\frac{\partial u^0_k}{\partial x_l} + \frac{\partial u^1_k}{\partial y_l}\right)\frac{\partial v_i}{\partial x_j} + \left(\frac{\partial u^1_k}{\partial x_l} + \frac{\partial u^2}{\partial y_l}\right)\frac{\partial v_i}{\partial y_j}\right] + \epsilon(\cdots)\right\}\mathrm{d}\Omega \\ &\quad = \int_{\Omega_\epsilon} f_i v_i \mathrm{d}\Omega + \int_{\Gamma_t} t_i v_i \mathrm{d}\Gamma + \int_{S^\epsilon} p^{\epsilon}_i v_i \mathrm{d}s, \quad \forall \mathbf{v} \in \mathbf{V}_{\mathbf{\Omega}\times\text{¥}}, \end{aligned} \tag{57}$$

where

$$\mathbf{V}_{\mathbf{\Omega}\times\text{¥}} = \{\mathbf{v}(\mathbf{x}, \mathbf{y}); (\mathbf{x}, \mathbf{y}) \in \Omega \times \text{¥} | \mathbf{v}(., \mathbf{y})\mathbf{Y} - \text{periodic}; \; \mathbf{v} \text{ smooth enough and } \mathbf{v}|_{\Gamma_d} = 0\}.$$

Similarly, we define $\mathbf{V}_{\Omega}$ and $\mathbf{V}_{\text{¥}}$ as:

$$\mathbf{V}_{\Omega} = \{\mathbf{v}(\mathbf{x}) \text{ defined in } \Omega | \mathbf{v} \text{ smooth enough and } \mathbf{v}|_{\Gamma_d} = 0\}.$$
$$\mathbf{V}_{\text{¥}} = \{\mathbf{v}(\mathbf{y}) \text{ defined in } \text{¥} | \; \mathbf{v}(\mathbf{y}), \; \mathbf{Y} - \text{periodic and smooth enough}\}.$$

Introducing the following facts:

Fact (5). For a Y-periodic function $\Psi(\mathbf{y})$ when $\epsilon \to 0$ we have

$$\int_{\Omega_\epsilon} \Psi\left(\frac{\mathbf{x}}{\epsilon}\right)\mathrm{d}\Omega = \frac{1}{|Y|}\int_{\Omega}\int_{\text{¥}} \Psi(\mathbf{y})\mathrm{d}Y \,\mathrm{d}\Omega, \tag{58}$$

$$\int_{S^\epsilon} \Psi\left(\frac{\mathbf{x}}{\epsilon}\right)\mathrm{d}\Omega = \frac{1}{\epsilon|Y|}\int_{\Omega}\int_{s} \Psi(\mathbf{y})\mathrm{d}s\mathrm{d}\Omega, \tag{59}$$

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[2] $H^1(\Omega^{\epsilon})$ is defined as:

$$H^1(\Omega^{\epsilon}) = \{w(\mathbf{x}) | w(\mathbf{x}) \in L_2(\Omega^{\epsilon}) \quad \text{and} \quad \frac{\partial w(\mathbf{x})}{\partial x_i} \in L_2(\Omega^{\epsilon})\},$$

where

$$L_2(\Omega^{\epsilon}) = \{w(\mathbf{x}) | \int_{\Omega^\epsilon} [w(\mathbf{x})]^2 < \infty \quad \text{and} \quad \mathbf{x} \in \Omega^{\epsilon}\},$$

which assures the integrability of the functions and their derivatives.

and assuming that the functions are all smooth so that when $\epsilon \to 0$ all integrals exist, and by equating the terms with the same power of ϵ we obtain:

$$\frac{1}{|Y|}\int_\Omega \int_{¥} E_{ijkl}\frac{\partial u_k^0}{\partial y_l}\frac{\partial v_i}{\partial y_i}\mathrm{d}Y\,\mathrm{d}\Gamma = 0, \quad \forall \mathbf{v} \in \mathbf{V}_{\mathbf{\Omega}\times ¥}, \tag{60}$$

$$\int_\Omega \left\{\frac{1}{|Y|}\int_{¥} E_{ijkl}\left[\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)\frac{\partial v_i}{\partial y_j}+\frac{\partial u_k^0}{\partial y_l}\frac{\partial v_i}{\partial x_j}\right]\mathrm{d}Y\right\}\mathrm{d}\Omega = \int_\Omega \left(\frac{1}{|Y|}\int_s p_i v_i \mathrm{d}S\right)\mathrm{d}\Omega, \quad \forall \mathbf{v} \in \mathbf{V}_{\mathbf{\Omega}\times ¥}, \tag{61}$$

$$\int_\Omega \left\{\frac{1}{|Y|}\int_{¥} E_{ijkl}\left[\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)\frac{\partial v_i}{\partial x_j}+\left(\frac{\partial u_k^1}{\partial y_l}+\frac{\partial u_k^2}{\partial y_l}\right)\frac{\partial v_i}{\partial y_j}\right]\mathrm{d}Y\right\}\mathrm{d}\Omega = \int_\Omega \left(\frac{1}{|Y|}\int_{¥} f_i v_i \mathrm{d}Y\right)\mathrm{d}\Omega + \int_{\Gamma_t} t_i v_i \mathrm{d}\Gamma, \quad \forall \mathbf{v} \in \mathbf{V}_{\mathbf{\Omega}\times ¥}. \tag{62}$$

Now, as $\mathbf{v}$ is an arbitrary function we choose $\mathbf{v} = \mathbf{v}(\mathbf{y})$ (i.e. $\mathbf{v} \in \mathbf{V}_{¥}$). Then integrating by parts, applying the divergence theorem to the integral in **¥**, and using periodicity from Eq. (60), we obtain:

$$\frac{1}{|Y|}\int_\Omega \left\{\int_{¥}\left[-\frac{\partial}{\partial y_j}\left(E_{ijkl}\frac{\partial u_k^0}{\partial y_l}\right)\right]v_i \mathrm{d}Y + \int_s E_{ijkl}\frac{\partial u_k^0}{\partial y_l}n_j v_i \mathrm{d}S\right\}\mathrm{d}\Omega = 0, \quad \forall \mathbf{v}. \tag{63}$$

$\mathbf{v}$ being arbitrary results in:

$$-\frac{\partial}{\partial y_j}\left(E_{ijkl}\frac{\partial u_k^0}{\partial y_l}\right) = 0, \quad \forall \mathbf{y} \in ¥, \tag{64}$$

$$E_{ijkl}\frac{u_k^0}{\partial y_l}n_j = 0 \quad \text{on} \quad s. \tag{65}$$

Considering fact (4) and Eq. (64) it is concluded that:

$$\mathbf{u}^0(\mathbf{x}, \mathbf{y}) = \mathbf{u}^0(\mathbf{x}). \tag{66}$$

This means that the first term of the asymptotic expansion only depends on the macroscopic scale $\mathbf{x}$.

Now, as $\mathbf{v}$ is an arbitrary function, if we choose $\mathbf{v} = \mathbf{v}(\mathbf{x})$ (i.e. $\mathbf{v}$ is only a function of $\mathbf{x}$), then from Eq. (61) it is concluded that:

$$\int_\Omega \left(\frac{1}{|Y|}\int_s p_i \mathrm{d}S\right)v_i(\mathbf{x})\mathrm{d}\Omega = 0, \quad \forall \mathbf{v} \in \mathbf{V}_{\mathbf{\Omega}}, \tag{67}$$

which implies that

$$\int_s p_i(\mathbf{x}, \mathbf{y})\mathrm{d}S = 0. \tag{68}$$

This means that the applied tractions have to be self-equilibrating. So the possible applied tractions are restricted.

On the other hand, introducing Eq. (66) into Eq. (61) and choosing $\mathbf{v} = \mathbf{v}(\mathbf{y})$ yields

$$\int_{¥} E_{ijkl}\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)\frac{\partial v_i(\mathbf{y})}{\partial y_j}\mathrm{d}Y = \int_s p_i v_i \mathrm{d}S, \quad \forall \mathbf{v} \in \mathbf{V}_{¥}. \tag{69}$$

Integrating by parts, using the divergence theorem and applying the periodicity conditions on the opposite faces of Y, it follows from Eq. (69) that:

$$-\int_{¥}\frac{\partial}{\partial y_j}\left[E_{ijkl}\left(\frac{\partial u_k^0(\mathbf{x})}{\partial x_l}+\frac{\partial u_k^1}{\partial y_t}\right)\right]v_i \mathrm{d}Y + \int_s E_{ijkl}\left(\frac{\partial u_k^0(\mathbf{x})}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)v_i n_j \mathrm{d}S = \int_s p_i v_i \mathrm{d}S, \quad \forall \mathbf{v} \in \mathbf{V}_{¥}. \tag{70}$$

Since $\mathbf{v}$ is arbitrary, it is concluded that

$$-\frac{\partial}{\partial y_j}\left(E_{ijkl}\frac{\partial u_k^1}{\partial y_l}\right) = \frac{\partial}{\partial y_j}\left(E_{ijkl}\frac{\partial u_k^0(\mathbf{x})}{\partial x_l}\right) \quad \text{on } ¥, \tag{71}$$

$$E_{ijkl}\frac{\partial u_k^1}{\partial y_l} = -E_{ijkl}\frac{\partial u_k^0(\mathbf{x})}{\partial x_l}n_j + p_i \quad \text{on } s. \tag{72}$$

Now, considering Eq. (62) and choosing $\mathbf{v} = \mathbf{v}(\mathbf{x})$ results in a statement of equilibrium in the macroscopic level:

$$\int_\Omega \left[\frac{1}{|Y|}\int_{¥} E_{ijkl}\left(\frac{\partial u_k^0}{\partial x_l}+\frac{\partial u_k^1}{\partial y_l}\right)\mathrm{d}Y\right]\frac{\partial v_i(\mathbf{x})}{\partial x_j}\mathrm{d}\Omega = \int_\Omega \left(\frac{1}{|Y|}\int_{¥} f_i \mathrm{d}Y\right)v_i(\mathbf{x})\mathrm{d}\mathbf{\Omega} + \int_{\mathbf{\Gamma}_t} \mathbf{t_i v_i(x)}\mathrm{d}\mathbf{\Gamma}, \quad \forall \mathbf{v} \in \mathbf{V}_{\mathbf{\Omega}}. \tag{73}$$

If in Eq. (62) we assume that $\mathbf{v} = \mathbf{v}(\mathbf{y})$, this leads to:

$$\int_\Omega \left[\frac{1}{|Y|}\int_{¥} E_{ijkl}\left(\frac{\partial u_k^1}{\partial x_l}+\frac{\partial u_k^2}{\partial y_l}\right)\frac{\partial v_i(\mathbf{y})}{\partial y_j}\mathrm{d}Y\right]d\Omega = \int_\Omega \left(\frac{1}{|Y|}\int_{¥} f_i v_i(\mathbf{y})\mathrm{d}Y\right)\mathrm{d}\Omega, \quad \forall \mathbf{v} \in \mathbf{V}_{¥}, \tag{74}$$

or equivalently,

$$\int_{¥} E_{ijkl}\left(\frac{\partial u_k^1}{\partial x_1}+\frac{\partial u_k^2}{\partial y_l}\right)\frac{\partial v_i(\mathbf{y})}{\partial y_j}\mathrm{d}Y = \int_{¥} f_i v_i(\mathbf{y})\mathrm{d}Y, \quad \forall \mathbf{v} \in \mathbf{V}_{¥}, \tag{75}$$

which represents the equilibrium of the base cell in the microscopic level.

The procedure followed so far can be applied for higher terms of the expansion. However, in this case the first-order terms are enough. The macroscopic mechanical behaviour is represented by $\mathbf{u}^0$, and $\mathbf{u}^1$ represents the microscopic behaviour.

As we have noticed earlier, our goal is to find the homogenized elastic constants such that the equilibrium equation (or equivalently the equation of virtual displacements) can be constructed in the macroscopic system of coordinates. These homogenized constants should be such that the corresponding equilibrium equation reflects the mechanical behaviour of the microstructure of the cellular material without explicitly using the parameter ϵ. To accomplish this we consider Eq. (69) once again. As this equation is linear with respect to $\mathbf{u}^0$ and $\mathbf{p}$, we consider the two following problems:

(i) Let $\boldsymbol{\chi}^{kl} \in V_{\text{¥}}$ be the solution of

$$\int_{\text{¥}} E_{ijpq} \frac{\partial \boldsymbol{\chi}_p^{kl}}{\partial y_q} \frac{\partial v_i(\mathbf{y})}{\partial y_j} \mathrm{d}Y = \int_{\text{¥}} E_{ijkl} \frac{\partial v_i(\mathbf{y})}{\partial y_i} \mathrm{d}Y, \quad \forall \mathbf{v} \in \mathbf{V}_{\text{¥}}; \tag{76}$$

(ii) and let $\Psi \in \mathbf{V}_{\text{¥}}$ be the solution of

$$\int_{\text{¥}} E_{ijkl} \frac{\partial \Psi_k}{\partial y_l} \frac{\partial v_i(\mathbf{y})}{\partial y_j} \mathrm{d}Y = \int_S p_i v_i(\mathbf{y}) \mathrm{d}Y, \quad \forall \mathbf{v} \in \mathbf{V}_{\text{¥}}, \tag{77}$$

where $\mathbf{x}$ plays the role of a parameter. It can be shown that the solution $\mathbf{u}^1$ will be

$$u_i^1 = -\chi_i^{kl}(\mathbf{x}, \mathbf{y}) \frac{\partial u_k^0(\mathbf{x})}{\partial x_l} - \Psi_i(\mathbf{x}, \mathbf{y}) + \tilde{u}_i^1(\mathbf{x}), \tag{78}$$

where $\tilde{u}_i^1$ are arbitrary constants of integration in $\mathbf{y}$.

Introducing Eq. (78) into Eq. (73) yields

$$\int_\Omega \left[\frac{1}{|Y|} \int_{\text{¥}} \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) \mathrm{d}Y \right] \frac{\partial u_k^0(\mathbf{x})}{\partial x_l} \frac{\partial u_i(x)}{\partial x_j} \mathrm{d}\Omega$$
$$= \int_\Omega \left(\frac{1}{|Y|} \int_{\text{¥}} E_{ijkl} \frac{\partial \Psi_k}{\partial y_l} \mathrm{d}Y \right) \frac{\partial v_i(\mathbf{x})}{\partial x_j} \mathrm{d}\Omega$$
$$+ \int_\Omega \left(\frac{1}{|Y|} \int_{\text{¥}} f_i \mathrm{d}Y \right) v_i(\mathbf{x}) \mathrm{d}\Omega$$
$$+ \int_{\Gamma_t} t_i v_i(\mathbf{x}) \mathrm{d}\Gamma, \quad \forall \mathbf{v} \in \mathbf{V}_{\boldsymbol{\Omega}}. \tag{79}$$

Now, denoting

$$E_{ijkl}^{\mathrm{H}}(\mathbf{x}) = \frac{1}{|Y|} \int_{\text{¥}} \left(E_{ijkl} - E_{ijpq} \frac{\partial \boldsymbol{\chi}_p^{kl}}{\partial y_q} \right) \mathrm{d}Y, \tag{80}$$

$$\tau_{ij}(\mathbf{x}) = \int_{\text{¥}} E_{ijkl} \frac{\partial \Psi_k}{\partial y_l} \mathrm{d}Y, \tag{81}$$

and

$$b_i(\mathbf{x}) = \frac{1}{|Y|} \int_{\text{¥}} f_i \mathrm{d}Y, \tag{82}$$

Eq. (79) can be written as:

$$\int_\Omega E_{ijkl}^{\mathrm{H}} \frac{\partial u_k^0(\mathbf{x})}{\partial x_l} \frac{\partial v_i(\mathbf{x})}{\partial x_j} \mathrm{d}\Omega = \int_\Omega \tau_{ij}(\mathbf{x}) \frac{\partial v_i(\mathbf{x})}{\partial x_j} \mathrm{d}\Omega$$
$$+ \int_\Omega b_i(\mathbf{x}) v_i(\mathbf{x}) \mathrm{d}\Omega + \int_{\Gamma_t} t_i(\mathbf{x}) \mathrm{d}\Gamma, \quad \forall \mathbf{v} \in \mathbf{V}_{\boldsymbol{\Omega}}. \tag{83}$$

This is very similar to the equation of virtual displacement, Eq. (56), and it represents the macroscopic equilibrium. E_{ijkl}^{H} defined by Eq. (80) represents the homogenized elastic constants. τ_{ij} are average "residual" stresses within the cell due to the tractions $\mathbf{p}$ inside the holes, and b_i are the average body forces.

As we notice, the microscopic and macroscopic problems are not coupled and the solution of the elasticity problem can be summarized as:

(i) Find $\boldsymbol{\chi}$ and $\boldsymbol{\Psi}$ within the base cell by solving the integral Eqs. (76) and (77) on the base cell.
(ii) Find D_{ijkl}^{H}, τ_{ij} and b_i by using Eqs. (80)–(82).
(iii) Construct Eq. (83) in macroscopic coordinates.

If the whole domain of the cellular material comprises a uniform cell structure, as well as uniform applied tractions on the boundaries of the holes of the cells, then it is only necessary to solve the microscopic Eqs. (76) and (77) once. Otherwise these equations must be solved for every point $\mathbf{x}$ of Ω.

7. Conclusion and Final Remarks

In this first part of a three paper review we have focused on the theory of the homogenization method for the computation of effective constitutive parameters of complex materials with a periodic microstructure. In the second part of this review we will consider the motives for using the homogenization theory for topological structural optimization. In particular, the finite element formulation will be explained for the material model based on a microstructure consisting of an isotropic material with rectangular voids. Some examples will also be provided.

References

[1] Kalamkarov AL. Composite and reinforced elements of construction. Wiley, New York, 1992.
[2] Sanchez-Palencia E. Non-homogenous media and vibration theory. Lect Notes Phys, 127, 1980.
[3] Benssousan A, Lions JL, Papanicoulau G. Asymptotic analysis for periodic structures. Amesterdam, North Holland, 1978.
[4] Cioranescu D, Paulin JSJ. Homogenization in open sets with holes. J Math Aanal Appl 1979;71:590–607.

[5] Oleinik OA. On homogenization problems. In: Trends and application of pure mathematics in mechanics. Springer, Berlin, 1984.
[6] Caillerie D. Homogenization of periodic media tissued composite materials, Technical report. Institute of Mechanics, Grenoble, France.
[7] Bourgat JF. Numerical experiments of the homogenization method for operators with periodic coefficients. Lect Notes Math 1977:330–56.
[8] Lene F., Duvaut G. Resultats d'isotropie pour des milieux homogeneises. C R Acad Sc Paris 1981; 7 293 (II): 477–480.
[9] Guedes JM, Kikuchi N. Pre and post processing for materials based on the homogenization method with adaptive finite element methods. Comp Meth Appl Mech Engng 1990;83:143–98.
[10] Bendsøe MP, Kikuchi N. Generating optimal topologies in structural design using homogenization method. Comp Meth Appl Mech Engng 1988;71:197–224.
[11] Bendsøe MP. Optimal shape design as a material distribution problem. Struct Optim 1989;1:193–202.
[12] Bendsøe MP, Díaz AR, Kikuchi N. Topology and generalized layout optimization of elastic structures. In: Bendsøe MP, Mota Soares CA, editors. Topology design of structures. Kluwer, Dordrecht, 1993. p. 159–205.
[13] Suzuki K, Kikuchi N. A homogenization method for shape and topology optimization. Comp Meth Appl Mech Engng 1991;93:291–318.
[14] Jog CS, Haber RB, Bendsøe MP. Topology design with optimized, self-adaptive materials, Technical Report DCAMM 457. Technical University of Denmark, 1993.