Master's Thesis

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Abstract

Fill it.

1 Introduction

2 Motivation

3 1D simulation

The distribution of material in a plant stem follows axisymmetric distribution.
⟨Discuss the objective function⟩
⟨Derive constraints, Euler Bernoulli equation⟩

3.1 Problem Formulation

$$\begin{aligned} max \quad & \frac{\sum_{r_i}^{r_o} E(r)I(r)}{\sum_{r_i}^{r_o} \rho(r)} \\ s.t. \quad & \frac{rE(r)}{R} \leq \sigma_{max} \quad \forall r \in [r_i, r_o] \\ & \frac{EI}{R} \leq M_{max} \end{aligned} \tag{1}$$

4 Method

The idea of asymptotic homogenization. In a repeating cell Y,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{2}$$

where $C_{ijkl}(\underset{\sim}{x} + \underset{\sim}{u}\underset{\sim}{Y}) = C_{ijkl}(\underset{\sim}{x})$

$$\Rightarrow C_{ijkl}(x_1 + n_1Y_1\, x_2 + n_2Y_2\, x_3 + n_3Y_3) = C_{ijkl}(x_1, x_2, x_3) \tag{3}$$

$C_{ijkl}(\underset{\sim}{x})$ is Y-periodic

$$\underset{\sim}{y} = \tfrac{\underset{\sim}{x}}{\epsilon} \tag{4}$$

$$\Rightarrow g = g(\underset{\sim}{x}, \tfrac{\underset{\sim}{x}}{\epsilon}) = g(\underset{\sim}{x}\, \underset{\sim}{y}) \tag{5}$$

$\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ defines the domain of the composite Ω. The domain is composed of base cells of dimensions, $\varepsilon Y_1, \varepsilon Y_2, \varepsilon Y_3$ where $\underset{\sim}{y} = \frac{x}{\varepsilon}$

4.1 1D Elasticity

$$\sigma^\varepsilon = E^\varepsilon \frac{\partial u^\varepsilon}{\partial x} \tag{6}$$

$$\frac{\partial \sigma^\varepsilon}{\partial x} + \gamma^\varepsilon = 0 \quad E^\varepsilon\, \gamma^\varepsilon \to macroscopicallyuniform \tag{7}$$

Inside each cell,

$$E^\varepsilon(x, \frac{x}{\varepsilon}) \quad = E(y) \tag{8}$$

$$\gamma^\varepsilon(x, \frac{x}{\varepsilon}) \quad = \gamma(y) \tag{9}$$

Let

$$u^\varepsilon(x) = u^0 x, y + \varepsilon u^1(x, y) + \varepsilon^2 u^2(x, y) + ... \tag{10}$$

$$\sigma^\varepsilon(x) = \sigma^0 x, y + \varepsilon \sigma^1(x, y) + \varepsilon^2 \sigma^2(x, y) + ... \tag{11}$$

4.2 Optimal Design of Elastic structures

$\mathbf{b} \to$ body forces
$\mathbf{t} \to$ surface tractions

Optimal choice of $\mathbb{C}_{ijkl} \in U_{ad} \leftarrow$ admissible set of elasticity
$\mathbb{C}_{ijkl}(\mathbf{x}) \forall \mathbf{x} \in \Omega$ has 21 independent components
$a_E(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbb{C}_{ijkl}\, \varepsilon_{kl}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{v}) d\mathbf{v} \to$ energy bilinear form
$L(\mathbf{v}) = \int_\Omega \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} ds \to$ load linear form.

Minimum compliance problem:

$$minimize \qquad L(\mathbf{v}), \tag{12}$$

$$subject\,to \qquad \mathbb{C}_{ijkl} \in \mathbb{U}_{ad} \tag{13}$$

$$a_E(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{U} \tag{14}$$

where $\mathbb{U} \to$ kinematically admissible displacements.
For optimal shape design:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = \chi(\mathbf{x}) \overline{\mathbb{C}}_{ijkl}, \quad \text{where } \overline{\mathbb{C}}_{ijkl} \to \text{stiffness matrix of the material} \tag{15}$$

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m, \\ 0 & \text{if } \mathbf{x} \in \Omega \backslash \Omega^m \end{cases} \tag{16}$$

where $\Omega^m \to$ part of the domain occupied by the material.
For sizing problem:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = h(\mathbf{x}) \overline{\mathbb{C}}_{ijkl} \tag{17}$$

$$\int_\Omega \chi(\mathbf{x}) d\mathbf{x} = V_f \tag{18}$$

$$\& \quad \int_\Omega h(\mathbf{x}) d\mathbf{x} = V_f. \tag{19}$$

where $h(x)$ is a sizing function.
Traditionally shape design problems are initiated in the following manner:

$$Ref\ doamin : \Omega_0 \in \mathbb{R}^3 \tag{20}$$

$$\underline{\phi} : \Omega_0 \to \phi(\Omega_0) \text{is a diffeomorphism.} \tag{21}$$

$$L(\mathbf{v}) = \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| ds \tag{22}$$

$$\begin{aligned} a_E &= \int_{\Omega} \mathbb{C}_{ijkl}(\mathbf{x}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \tag{23}$$

Now,

$$\begin{aligned} \mathbb{C}_{ijkl}\varepsilon_{kl} &= \mathbb{C}_{ijkl}\frac{1}{2}(u_{k,l} + u_{l,k}) \\ &= \frac{1}{2}\mathbb{C}_{ijkl}u_{k,l} + \frac{1}{2}\mathbb{C}_{ijlk}u_{l,k} \\ &= \mathbb{C}_{ijkl}u_{k,l} \end{aligned} \tag{24}$$

$$\begin{aligned} a_E &= \int_{\Omega_0} \mathbb{C}_{ijkl}u_{k,l}(\mathbf{u})u_{i,j}(\mathbf{v}) |det(D\underline{\phi}^{-1}| d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl}\frac{\partial u_k}{\partial \mathbf{x}_m}(D\underline{\phi}^{-1})_{ml}\frac{\partial u_i}{\partial \mathbf{x}_p}(D\phi^{-1})_{pj} |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \tag{25}$$

$$\Rightarrow \mathbb{C}_{ijkl}(D\underline{\phi}^{-1})_{ml}(D\underline{\phi}^{-1})_{pj} |det(D\underline{\phi}^{-1})| = \bar{\mathbb{C}}_{ipkm} \tag{26}$$

$$\bar{\mathbb{C}}_{ijkl} = \mathbb{C}_{ipkm}(D\underline{\phi}^{-1})_{lm}(D\underline{\phi}^{-1})_{jp} |det(D\underline{\phi}^{-1})| \tag{27}$$

Treating $\underline{\phi}$ as a design variable is tidious.

4.3 Homogenization method

$$E^{\varepsilon}_{ijkl}(\mathbf{x}) = E_{ijkl}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \tag{28}$$

The tensor E^{ε}_{ijkl} is a material constant which satisfies the symmetry condition and is assumed to satisfy strong ellipticity condition for every $\mathbf{x}$.

$$\Rightarrow E^{\varepsilon}_{ijkl} = E^{\varepsilon}_{jikl} = E^{\varepsilon}_{ijlk} = E^{\varepsilon}_{klij} \tag{29}$$

$$E^{\varepsilon}_{ijkl}(\mathbf{x})\mathbf{X}_{ij}\mathbf{X}_{kl} \geq m\mathbf{X}_{ij}\mathbf{X}_{ij} \qquad \text{for some } m > 0 \ \& \ \forall \quad \mathbf{X}_{ij} = \mathbf{X}_{ji} \tag{30}$$

Let the domainΩ has a boundary Γ. Let $\mathbf{f}$ be the body force acting on Ω and $\mathbf{t}$ be the traction acting on Γ_t part of the boundary Γ. Also, let Γ_D be the part of boundary on which displacement is defined. Then the displacement $\mathbf{u}^\varepsilon$ can be obtained as the solution to the following minimization problem

$$\min_{\mathbf{v}^\varepsilon \in U} \quad F^\varepsilon(\mathbf{v}^\varepsilon), \tag{31}$$

where F^ε is total potential energy given as

$$F^\varepsilon(\mathbf{v}^\varepsilon) = \frac{1}{2}\int_\Omega E^\varepsilon_{ijkl}\varepsilon_{kl}(\mathbf{v}^\varepsilon)\varepsilon_{ij}(\mathbf{v}^\varepsilon)dx - \int_\Omega \mathbf{f}\cdot\mathbf{v}^\varepsilon dx - \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}^\varepsilon ds \tag{32}$$

and $\mathcal{U}$ is the set of admissible displacements defined such that

$$\mathcal{U} = \{\mathbf{v} = v_i\mathbf{e}_i : \, v_i \in H^1(\Omega) \text{ and } \mathbf{v} \in \mathcal{G} \text{ on } \Gamma_D\} \tag{33}$$

where $\mathcal{G}$ is set of displacement defined along the boundary Γ_D.
Let

$$\mathbf{v}^\varepsilon(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \varepsilon\mathbf{v}_1(\mathbf{x},\mathbf{y}), \qquad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \tag{34}$$

Using chain rule for functions in two variables

$$\begin{aligned} \frac{\partial f(\mathbf{x},\mathbf{y}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \\ &= \frac{\partial f}{\partial \mathbf{x}} + \frac{1}{\varepsilon}\frac{\partial f}{\partial \mathbf{y}} \end{aligned} \tag{35}$$

Using above two equations, we can write the linerized strain as

$$\begin{aligned} \epsilon_{ij}(\mathbf{v}^\varepsilon(\mathbf{x})) &= \frac{\partial(v_{0i}(\mathbf{x}) + \varepsilon v_{1i}(\mathbf{x},\mathbf{y}))}{\partial x_j} \\ &= \frac{\partial v_{0i}}{\partial x_j} + \varepsilon\left\{\frac{\partial v_{1i}}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial v_{1i}}{\partial y_j}\right\} \\ &\approx \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \qquad \leftarrow \{\varepsilon << 1\} \end{aligned} \tag{36}$$

Therefore, equation (32) can be written as

$$\begin{aligned} F^\varepsilon(\mathbf{v}^\varepsilon) = \frac{1}{2}\int_\Omega E^\varepsilon_{ijkl}\left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j}\right)dx \\ - \int_\Omega \mathbf{f}\cdot\mathbf{v}_0 dx - \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds + \varepsilon R^\varepsilon(\mathbf{v}_0,\mathbf{v}_1) \end{aligned} \tag{37}$$

Here, R^ε is the contribution of $\varepsilon\mathbf{v}_1$ in the calculation of energy from body force and traction. Using

$$\lim_{\varepsilon\to 0}\int_\Omega \Phi(x, x/\varepsilon)dx = \frac{1}{|Y|}\int_\Omega\int_Y \Phi(x,y)dydx, \tag{38}$$

we get,

$$\begin{aligned}\lim_{\varepsilon\to 0} F^{\varepsilon}(\mathbf{v}^{\varepsilon}) &= F(\mathbf{v}_0,\mathbf{v}_1)\\ &= \frac{1}{2|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial v_{0k}}{\partial x_l}+\frac{\partial v_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}+\frac{\partial v_{1i}}{\partial y_j}\right)dy\,dx\\ &\quad -\int_{\Omega}\mathbf{f}\cdot\mathbf{v}_0 dx-\int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds\end{aligned} \tag{39}$$

A minimizer $\{\mathbf{u}_0,\mathbf{u}_1\}$ of the functional F, follow the following equations:

$$\begin{aligned}&\frac{1}{|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l}+\frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}\right)dy\,dx\\ &=\int_{\Omega}\mathbf{f}\cdot\mathbf{v}_0 dx+\int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0\end{aligned} \tag{40}$$

$$\frac{1}{|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l}+\frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_i}{\partial x_j}\right)dy\,dx=0, \qquad \text{for every } \mathbf{v}_1 \tag{41}$$

Now, from localizing u_{1k}

$$u_{1k}(x,y)=-\chi_k^{pq}(y)\frac{\partial u_{0p}}{\partial x_q}(x), \tag{42}$$

$$\begin{aligned}&\Rightarrow \int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l}-\frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j}dy\,dx=0\\ &\int_{\Omega}\int_{Y}\left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l}-E_{ijkl}\frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j}dy\,dx=0\\ &\int_{\Omega}\int_{Y}\left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l}-E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_i}{\partial x_j}dy\,dx=0\\ &\int_{\Omega}\int_{Y}\frac{\partial u_{0k}}{\partial x_l}\left(E_{ijkl}-E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j}dy\,dx=0\\ &\int_{\Omega}\frac{\partial u_{0k}}{\partial x_l}dx\cdot\int_{Y}\left(E_{ijkl}-E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j}dy\ =0\end{aligned}$$

$$\Rightarrow \int_{Y}\left(E_{ijkl}-E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j}dy=0 \qquad \text{for k, l = 1 and 2,} \tag{43}$$

Similarly, substituting equation (42) in (40) gives the homogenized equation.

$$
\begin{aligned}
\text{LHS} &= \frac{1}{|Y|}\int_\Omega \int_Y E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}\right) dy\, dx \\
&= \frac{1}{|Y|}\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_{0i}}{\partial x_j} dy\, dx \\
&= \frac{1}{|Y|}\int_\Omega \left\{ \int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy \right\} \frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j} dx \\
&= \int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx
\end{aligned}
$$

Homogenized equation

$$
\int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx = \int_\Omega \mathbf{f}\cdot\mathbf{v}_0 dx + \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0 \tag{44}
$$

where $E_{ijkl}^H(x)$ is

$$
\boxed{E_{ijkl}^H = \frac{1}{|Y|}\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy} \tag{45}
$$

Now, Define

$$
a_H(\mathbf{u},\mathbf{v}) = \int_\Omega E_{ijkl}^H(\mathbf{x})\frac{\partial u_k}{\partial x_l}\frac{\partial v_i}{\partial x_j}\, dx, \tag{46}
$$

$$
a_Y(\chi^{kl},\mathbf{v}) = \int_Y E_{ijpq}(\mathbf{x},\mathbf{y})\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial v_i}{\partial y_j}\, dy, \tag{47}
$$

$$
L_Y^{kl}(\mathbf{v}) = \int_Y E_{ijkl}\frac{\partial v_i}{\partial y_j}\, dy \tag{48}
$$

At microscopic level, we have

$$
a_Y(\chi^{kl},\mathbf{v}) = L_Y^{kl}(\mathbf{v}) \qquad \forall \mathbf{v}\in\mathcal{U}_Y, \tag{49}
$$

At macroscopic level, we have

$$
a_H(\mathbf{u},\mathbf{v}) = L(\mathbf{v}) \qquad \forall \mathbf{v}\in\mathcal{U}_0 \tag{50}
$$

where $\mathcal{U}_0$ is homogeneous case of $\mathcal{U}$, i.e., $\mathbf{g}=0$.

4.4 Implementation 2D Homogenization

Basic homogenization equation,

$$
u_{1i}(\mathbf{x},\mathbf{y}) = -\chi_i^{pq}\frac{\partial u_{0p}(\mathbf{x})}{\partial x_q} \tag{51}
$$

Solve χ_p^{kl} from:

$$\int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) \frac{\partial v_{1i}}{\partial y_j} \, dy = 0 \tag{52}$$

Compute:

$$E_{ijkl}^H = \frac{1}{|Y|} \int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) dy \tag{53}$$

4.5 Examples

Consider: k=1, l=1

$$\begin{aligned} \int_Y E_{ijkl} \frac{\partial v_i}{\partial y_j} \, dy &= \int_Y E_{ij11} \frac{\partial v_i}{\partial y_j} \, dy \\ &= \int_Y \left(E_{1111} \frac{\partial v_1}{\partial y_1} + E_{2211} \frac{\partial v_2}{\partial y_2} \right) dy \end{aligned} \tag{54}$$

$$
\begin{aligned}
\int_Y E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \frac{\partial v_i}{\partial y_j}\, dy &= \int_Y E_{ijpq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_i}{\partial y_j}\, dy \\
&= \int_Y \left\{ E_{11pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_1}{\partial y_1} + E_{12pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_1}{\partial y_2} \right. \\
&\qquad \left. + E_{21pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_2}{\partial y_1} + E_{22pq} \frac{\partial \chi_p^{11}}{\partial y_q} \frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1112} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1121} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + \left(E_{1211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{1222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_2} \\
&\quad + \left(E_{2111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2112} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{2121} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{2122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_1} \\
&\quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{2221} \frac{\partial \chi_2^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + \cancelto{0}{E_{1112} \frac{\partial \chi_1^{11}}{\partial y_2}} + \cancelto{0}{E_{1121} \frac{\partial \chi_2^{11}}{\partial y_1}} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + \left(\cancelto{0}{E_{1211} \frac{\partial \chi_1^{11}}{\partial y_1}} + E_{1212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \frac{\partial \chi_2^{11}}{\partial y_1} + \cancelto{0}{E_{1222} \frac{\partial \chi_2^{11}}{\partial y_2}} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + \cancelto{0}{E_{2212} \frac{\partial \chi_1^{11}}{\partial y_2}} + \cancelto{0}{E_{2221} \frac{\partial \chi_2^{11}}{\partial y_1}} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + E_{1212} \left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy
\end{aligned}
\tag{55}
$$

Therefore, using equations (49) , (55) and (54) for k=1, l=1 we have:

$$\begin{aligned}
&\int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + E_{1212} \left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
&\int_Y \left(E_{1111} \frac{\partial v_1}{\partial y_1} + E_{2211} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{56}$$

From equation (53), we can write

$$E^H_{1111} = \frac{1}{|Y|} \int_Y \left(E_{1111} - E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} - E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{57}$$

$$E^H_{2211} = \frac{1}{|Y|} \int_Y \left(E_{2211} - E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} - E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{58}$$

$$E^H_{1211} = -\frac{1}{|Y|} \int_Y \left(E_{1212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \frac{\partial \chi_2^{11}}{\partial y_1} \right) dy \tag{59}$$

Let $\chi_1^{11} = \Phi_1, \chi_2^{11} = \Phi_2$ and $E_{1111} = D_{11}, E_{2222} = D_{22}, E_{1212} = D_{66}, E_{1122} = E_{2211} = D_{12}$

$$\begin{aligned}
&\int_Y \left\{ \left(D_{11} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{12} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\quad + D_{66} \left(\frac{\partial \Phi_1^{11}}{\partial y_2} + \frac{\partial \Phi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(D_{12} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{22} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
&\int_Y \left(D_{11} \frac{\partial v_1}{\partial y_1} + D_{12} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{60}$$

Also,

$$D^H_{11} = \frac{1}{|Y|} \int_Y \left(D_{11} - D11 \frac{\partial \Phi_1}{\partial y_1} - D_{12} \frac{\partial \Phi_2}{\partial y_2} \right) dy \tag{61}$$

Rearranging Eq. (60)

$$\int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D66 \end{bmatrix} \times \begin{bmatrix} \frac{\partial \Phi_1}{\partial y_1} \\ \frac{\partial \Phi_2}{\partial y_2} \\ \frac{\partial \Phi_1}{\partial y_2} + \frac{\partial \Phi_2}{\partial y_1} \end{bmatrix} dY = \int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} \\ D_{12} \\ 0 \end{bmatrix} dY \tag{62}$$

Let us define

$$\mathbf{b} = \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 \\ 0 & \frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} \end{bmatrix} \tag{63}$$

and

$$\mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix} \tag{64}$$

Then Eq (60), can be written as

$$\int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{D} \mathbf{b} \Phi \, dY = \int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{d}_1 \qquad \forall \mathbf{v} \in \mathbf{V}_Y \tag{65}$$

and eq. (61) becomes:

$$\boxed{D_{11}^H = \frac{1}{|Y|} \int_Y \left(D_{11} - \mathbf{d}_1^T \mathbf{b} \Phi \right) dy} \tag{66}$$

5 Results

[1] [2]

6 References

References

[1] John Doe. Title. *Journal*, 2017.

[2] Intel. Example website. `http://example.com`, Dec 1988. Accessed on 2012-11-11.