Master's Thesis

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Abstract

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1 Introduction

2 Motivation

3 1D simulation

Bamboo is a natural composite which is composed of fibers embedded in a matrix of parenchyma cells. Bamboo fibers are mainly composed of cellulose, hemicellulose and lignin [L.Y.Mwaikambo et al.]. Fibers are spread out across the cross-section in a graded manner with higher density towards the periphery. Also, the size of parenchyma cells decreases along the radially outward direction as the air content reduces. This results in axisymmetric areal density variation in the radial direction. [Plot showing the distribution of fibers.]

This can be modeled as the distribution of two materials, and air (capture in parenchyma cells). First material being the denser and stiffer fibers and second material being the parenchyma cellular material excluding air. The properties of fibers and parenchyma are taken from [Mannan et al., L.Y.Mwaikambo et al.].

Bamboo, like any other living organism, is a result of an evolutionary process. Survival of the fittest means that bamboo species is nature's best solution for some natural condition lead to the existence of bamboo. Bamboo grows tall up to 20m to rise above the other competing plantation. With such a slender structure, bending load due to high-speed tropical winds are a significant constraint which the evolution had overcome. Bamboo has a very high specific strength, which is also the desired attribute in industrial applications. It is desired to develop composites with high stiffness and lower weight.

3.1 Formulation

Therefore, we frame a constrained problem, optimizing the radial distribution of two material, with properties corresponding to that of fibers and parenchyma, in an annular cross-section composites. The objective function to optimize is specific strength subjected to the constraints of maximum stress and maximum bending moment. The dimensions of the composites are kept the same as in [Mannan et al.]. Also, the limits of max bending moment and stress are taken from [Mannan et al.]. The general problem formulation is written as

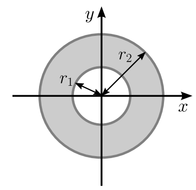
$$\begin{aligned} \underset{\chi}{\text{maximize}} \quad & strength(\chi) \\ \text{subject to} \quad & \sigma(r) \le \sigma_{max} \qquad \forall r \in [r_i, r_o] \\ & M \le M_{max} \end{aligned}$$

where, χ is the distribution of the two material in the domain whose inner radius is r_i and outer radius is r_o. σ is the stress in the longitudinal direction and M is the bending moment.

3.2 Implementation

Now, flexural rigidity is used as measure of strength which can be written as

$$
\begin{aligned}
EI(r) &= \iint\limits_R E(r) y^2 \, dA = \int_{r_1}^{r_2} \int_0^{2\pi} E(r)(r \sin\theta)^2 (r dr d\theta) \\
&= \pi \int_{r_1}^{r_2} E(r) r^3 dr \qquad \text{Integrating over } \theta
\end{aligned} \tag{1}
$$



Therefore, specific flexural rigidity will be written as

$$
\begin{aligned}
strength(\chi) &= \frac{EI}{\rho} \\
&= \frac{\pi \sum_{r_i}^{r_o} E(r) r^3 \Delta r}{2\pi r \sum_{r_i}^{r_o} \rho(r) \Delta r} \\
&= \frac{\sum_{r_i}^{r_o} E(r) r^3}{2r \sum_{r_i}^{r_o} \rho(r)}
\end{aligned} \tag{2}
$$

where

$$
\begin{aligned}
E(r) &= \chi_1(r) E_1 + \chi_2(r) E_2 \\
\rho(r) &= \chi_1(r) \rho_1 + \chi_2(r) \rho_2
\end{aligned} \tag{3}
$$

Here $\{E_1, \rho_1\}$ and $\{E_2, \rho_2\}$ are the material properties of the fibers and parenchyma respectively. χ_1 and χ_2 are the proportion of first and second material corresponding to fiber and parenchyma respectively.

Assuming small deformation, from Euler Bernoulli Beam Theory, we get strain ε_{xx} as
Put Schematics for Beam Theory

$$
\begin{aligned}
\varepsilon_{xx} &= \frac{\Delta x' - \Delta x}{\Delta x} \\
&= \frac{(R+y)\Delta\theta - R\Delta\theta}{R\Delta\theta} \\
&= \frac{y}{R}
\end{aligned} \tag{4}
$$

Therefore, using constitutive relation, we get σ_{xx} as

$$
\sigma_{xx} = E\varepsilon_{xx} = \frac{E}{R} y \tag{5}
$$

$$
\sigma = \frac{E}{R} r \sin\theta \quad (\because y = r \sin\theta) \tag{6}
$$

Now, moment can be written as

$$
M(x) = \int \int y \cdot \sigma(x, y) \cdot dy dz \tag{7}
$$

For a particular x, moment for an annular structure like bamboo can be written as

$$
\begin{aligned}
M &= \int_{r_i}^{r_o} \int_0^{2\pi} \sigma(r) r^2 sin\theta \, dr \, d\theta \\
&= \int_{r_i}^{r_o} \int_0^{2\pi} \frac{E(r) r sin\theta}{R} r^2 sin\theta \, dr \, d\theta \qquad \text{(From eqn (6))} \\
&= \pi \int_{r_i}^{r_o} \frac{r^3 E(r)}{R} \, dr
\end{aligned} \tag{8}
$$

3.3 Optimization Problem

Using (3), (6), (8), the optimization problem becomes

$$
\begin{aligned}
\underset{\chi}{max} \quad & \frac{\sum_{r_i}^{r_o} E(r) r^3}{2r \sum_{r_i}^{r_o} \rho(r)} \\
s.t. \quad & \frac{rE(r)}{R} \leq \sigma_{max} \quad \forall r \in [r_i, r_o] \\
& \sum_{r_i}^{r_o} r^3 E(r) \leq \frac{M_{max} R}{\pi \Delta r}
\end{aligned} \tag{9}
$$

3.4 Results

4 Method

The idea of asymptotic homogenization. In a repeating cell Y,

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \tag{10}$$

where $C_{ijkl}(\underset{\sim}{x} + \underset{\sim}{u}\underset{\sim}{Y}) = C_{ijkl}(\underset{\sim}{x})$

$$\Rightarrow C_{ijkl}(x_1 + n_1Y_1\, x_2 + n_2Y_2\, x_3 + n_3Y_3) = C_{ijkl}(x_1, x_2, x_3) \tag{11}$$

$C_{ijkl}(\underset{\sim}{x})$ is Y-periodic

$$\underset{\sim}{y} = \tfrac{x}{\epsilon} \tag{12}$$

$$\Rightarrow g = g(\underset{\sim}{x}, \tfrac{x}{\epsilon}) = g(\underset{\sim}{x}\, \underset{\sim}{y}) \tag{13}$$

$\underset{\sim}{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ defines the domain of the composite Ω. The domain is composed of base cells of dimensions, $\varepsilon Y_1, \varepsilon Y_2, \varepsilon Y_3$ where $\underset{\sim}{y} = \frac{x}{\varepsilon}$

4.1 1D Elasticity

$$\sigma^\varepsilon = E^\varepsilon \tfrac{\partial u^\varepsilon}{\partial x} \tag{14}$$

$$\tfrac{\partial \sigma^\varepsilon}{\partial x} + \gamma^\varepsilon = 0 \quad E^\varepsilon\, \gamma^\varepsilon \rightarrow macroscopically uniform \tag{15}$$

Inside each cell,

$$E^\varepsilon(x, \frac{x}{\varepsilon}) \quad = E(y) \tag{16}$$

$$\gamma^\varepsilon(x, \frac{x}{\varepsilon}) \quad = \gamma(y) \tag{17}$$

Let

$$u^\varepsilon(x) = u^0 x, y + \varepsilon u^1(x, y) + \varepsilon^2 u^2(x, y) + ... \tag{18}$$

$$\sigma^\varepsilon(x) = \sigma^0 x, y + \varepsilon \sigma^1(x, y) + \varepsilon^2 \sigma^2(x, y) + ... \tag{19}$$

4.2 Optimal Design of Elastic structures

$\mathbf{b} \rightarrow$ body forces
$\mathbf{t} \rightarrow$ surface tractions

Optimal choice of $\mathbb{C}_{ijkl} \in U_{ad} \leftarrow$ admissible set of elasticity
$\mathbb{C}_{ijkl}(\mathbf{x}) \forall \mathbf{x} \in \Omega$ has 21 independent components
$a_E(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbb{C}_{ijkl}\, \varepsilon_{kl}(\mathbf{u})\, \varepsilon_{kl}(\mathbf{v}) d\mathbf{v} \rightarrow$ energy bilinear form
$L(\mathbf{v}) = \int_\Omega \mathbf{v}\, d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} ds \rightarrow$ load linear form.

Minimum compliance problem:

$$minimize \qquad L(\mathbf{v}), \tag{20}$$

$$subject\, to \qquad \mathbb{C}_{ijkl} \in \mathbb{U}_{ad} \tag{21}$$

$$a_E(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbb{U} \tag{22}$$

where $\mathbb{U} \rightarrow$ kinematically admissible displacements.
For optimal shape design:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = \chi(\mathbf{x})\overline{\mathbb{C}}_{ijkl}, \quad \text{where } \overline{\mathbb{C}}_{ijkl} \rightarrow \text{stiffness matrix of the material} \tag{23}$$

$$\chi(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^m, \\ 0 & \text{if } \mathbf{x} \in \Omega \backslash \Omega^m \end{cases} \tag{24}$$

where $\Omega^m \rightarrow$ part of the domain occupied by the material.
For sizing problem:

$$\mathbb{C}_{ijkl}(\mathbf{x}) = h(\mathbf{x})\overline{\mathbb{C}}_{ijkl} \tag{25}$$

$$\textstyle\int_\Omega \chi(\mathbf{x}) d\mathbf{x} = V_f \tag{26}$$

$$\& \quad \textstyle\int_\Omega h(\mathbf{x}) d\mathbf{x} = V_f. \tag{27}$$

where $h(x)$ is a sizing function.
Traditionally shape design problems are initiated in the following manner:

$$Ref\ doamin : \Omega_0 \in \mathbb{R}^3 \tag{28}$$

$$\underline{\phi} : \Omega_0 \rightarrow \phi(\Omega_0) \text{is a diffeomorphism.} \tag{29}$$

$$L(\mathbf{v}) = \int_{\Omega_0} \mathbf{f} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| d\mathbf{x} + \int_{\partial\Omega_t} \mathbf{t} \cdot \mathbf{v} |det(D\underline{\phi}^{-1})| ds \tag{30}$$

$$\begin{aligned} a_E &= \int_\Omega \mathbb{C}_{ijkl}(\mathbf{x}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl}\varepsilon_{kl}(\mathbf{v})\varepsilon_{ij}(\mathbf{v}) |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \tag{31}$$

Now,

$$\begin{aligned} \mathbb{C}_{ijkl}\varepsilon_{kl} &= \mathbb{C}_{ijkl}\frac{1}{2}(u_{k,l} + u_{l,k}) \\ &= \frac{1}{2}\mathbb{C}_{ijkl}u_{k,l} + \frac{1}{2}\mathbb{C}_{ijlk}u_{l,k} \\ &= \mathbb{C}_{ijkl}u_{k,l} \end{aligned} \tag{32}$$

$$\begin{aligned} a_E &= \int_{\Omega_0} \mathbb{C}_{ijkl}u_{k,l}(\mathbf{u})u_{i,j}(\mathbf{v}) |det(D\underline{\phi}^{-1}| d\mathbf{x} \\ &= \int_{\Omega_0} \mathbb{C}_{ijkl}\frac{\partial u_k}{\partial \mathbf{x}_m}(D\underline{\phi}^{-1})_{ml}\frac{\partial u_i}{\partial \mathbf{x}_p}(D\phi^{-1})_{pj} |det(D\underline{\phi}^{-1})| d\mathbf{x} \end{aligned} \tag{33}$$

$$\Rightarrow \mathbb{C}_{ijkl}(D\underline{\phi}^{-1})_{ml}(D\underline{\phi}^{-1})_{pj} |det(D\underline{\phi}^{-1})| = \bar{\mathbb{C}}_{ipkm} \tag{34}$$

$$\bar{\mathbb{C}}_{ijkl} = \mathbb{C}_{ipkm}(D\underline{\phi}^{-1})_{lm}(D\underline{\phi}^{-1})_{jp} |det(D\underline{\phi}^{-1})| \tag{35}$$

Treating $\underline{\phi}$ as a design variable is tedious.

4.3 Homogenization method

$$E^{\varepsilon}_{ijkl}(\mathbf{x}) = E_{ijkl}(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon} \tag{36}$$

The tensor E^{ε}_{ijkl} is a material constant which satisfies the symmetry condition and is assumed to satisfy strong ellipticity condition for every $\mathbf{x}$.

$$\Rightarrow E^{\varepsilon}_{ijkl} = E^{\varepsilon}_{jikl} = E^{\varepsilon}_{ijlk} = E^{\varepsilon}_{klij} \tag{37}$$

$$E^{\varepsilon}_{ijkl}(\mathbf{x})\mathbf{X}_{ij}\mathbf{X}_{kl} \geq m\mathbf{X}_{ij}\mathbf{X}_{ij} \qquad \text{for some } m > 0 \ \&\ \forall \quad \mathbf{X}_{ij} = \mathbf{X}_{ji} \tag{38}$$

Let the domainΩ has a boundary Γ. Let $\mathbf{f}$ be the body force acting on Ω and $\mathbf{t}$ be the traction acting on Γ_t part of the boundary Γ. Also, let Γ_D be the part of boundary on which displacement is defined. Then the displacement $\mathbf{u}^{\varepsilon}$ can be obtained as the solution to the following minimization problem

$$\min_{\mathbf{v}^{\varepsilon} \in U} \quad F^{\varepsilon}(\mathbf{v}^{\varepsilon}), \tag{39}$$

where F^{ε} is total potential energy given as

$$F^{\varepsilon}(\mathbf{v}^{\varepsilon}) = \frac{1}{2}\int_{\Omega} E^{\varepsilon}_{ijkl}\varepsilon_{kl}(\mathbf{v}^{\varepsilon})\varepsilon_{ij}(\mathbf{v}^{\varepsilon})dx - \int_{\Omega} \mathbf{f}\cdot\mathbf{v}^{\varepsilon}dx - \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}^{\varepsilon}ds \tag{40}$$

and $\mathcal{U}$ is the set of admissible displacements defined such that

$$\mathcal{U} = \{\mathbf{v} = v_i\mathbf{e}_i : v_i \in H^1(\Omega) \text{ and } \mathbf{v} \in \mathcal{G} \text{ on } \Gamma_D\} \tag{41}$$

where $\mathcal{G}$ is set of displacement defined along the boundary Γ_D.
Let

$$\mathbf{v}^{\varepsilon}(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) + \varepsilon\mathbf{v}_1(\mathbf{x}, \mathbf{y}), \qquad \mathbf{y} = \frac{\mathbf{x}}{\varepsilon}. \tag{42}$$

Using chain rule for functions in two variables

$$\begin{aligned} \frac{\partial f(\mathbf{x}, \mathbf{y}(\mathbf{x}))}{\partial \mathbf{x}} &= \frac{\partial f}{\partial \mathbf{x}} + \frac{\partial f}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{x}} \\ &= \frac{\partial f}{\partial \mathbf{x}} + \frac{1}{\varepsilon}\frac{\partial f}{\partial \mathbf{y}} \end{aligned} \tag{43}$$

Using above two equations, we can write the linerized strain as

$$\begin{aligned} \epsilon_{ij}(\mathbf{v}^{\varepsilon}(\mathbf{x})) &= \frac{\partial(v_{0i}(\mathbf{x}) + \varepsilon v_{1i}(\mathbf{x}, \mathbf{y}))}{\partial x_j} \\ &= \frac{\partial v_{0i}}{\partial x_j} + \varepsilon\left\{\frac{\partial v_{1i}}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial v_{1i}}{\partial y_j}\right\} \\ &\approx \frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j} \qquad \leftarrow \{\varepsilon << 1\} \end{aligned} \tag{44}$$

Therefore, equation (40) can be written as

$$F^{\varepsilon}(\mathbf{v}^{\varepsilon}) = \frac{1}{2}\int_{\Omega} E^{\varepsilon}_{ijkl}\left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j}\right)dx - \int_{\Omega} \mathbf{f}\cdot\mathbf{v}_0 dx - \int_{\Gamma_t} \mathbf{t}\cdot\mathbf{v}_0 ds + \varepsilon R^{\varepsilon}(\mathbf{v}_0, \mathbf{v}_1) \quad (45)$$

Here, R^{ε} is the contribution of $\varepsilon\mathbf{v}_1$ in the calculation of energy from body force and traction. Using

$$\lim_{\varepsilon\to 0}\int_{\Omega}\Phi(x, x/\varepsilon)dx = \frac{1}{|Y|}\int_{\Omega}\int_{Y}\Phi(x,y)dydx, \quad (46)$$

we get,

$$\lim_{\varepsilon\to 0} F^{\varepsilon}(\mathbf{v}^{\varepsilon}) = F(\mathbf{v}_0, \mathbf{v}_1) = \frac{1}{2|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial v_{0k}}{\partial x_l} + \frac{\partial v_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j} + \frac{\partial v_{1i}}{\partial y_j}\right)dy\,dx - \int_{\Omega}\mathbf{f}\cdot\mathbf{v}_0 dx - \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds \quad (47)$$

A minimizer $\{\mathbf{u}_0, \mathbf{u}_1\}$ of the functional F, follow the following equations:

$$\frac{1}{|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}\right)dy\,dx = \int_{\Omega}\mathbf{f}\cdot\mathbf{v}_0 dx + \int_{\Gamma_t}\mathbf{t}\cdot\mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0 \quad (48)$$

$$\frac{1}{|Y|}\int_{\Omega}\int_{Y} E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_i}{\partial x_j}\right)dy\,dx = 0, \qquad \text{for every } \mathbf{v}_1 \quad (49)$$

Now, from localizing u_{1k}

$$u_{1k}(x,y) = -\chi_k^{pq}(y)\frac{\partial u_{0p}}{\partial x_q}(x), \quad (50)$$

$$
\begin{aligned}
\Rightarrow \int_\Omega \int_Y E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} - \frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijkl}\frac{\partial \chi_k^{pq}}{\partial y_l}\frac{\partial u_{0p}}{\partial x_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
\int_\Omega \int_Y \frac{\partial u_{0k}}{\partial x_l}\left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy\, dx = 0 \\
\int_\Omega \frac{\partial u_{0k}}{\partial x_l} dx \cdot \int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy\ = 0
\end{aligned}
$$

$$
\Rightarrow \int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right)\frac{\partial v_i}{\partial x_j} dy = 0 \qquad \text{for k, l = 1 and 2,} \tag{51}
$$

Similarly, substituting equation (50) in (48) gives the homogenized equation.

$$
\begin{aligned}
\text{LHS} &= \frac{1}{|Y|}\int_\Omega \int_Y E_{ijkl}(x,y)\left(\frac{\partial u_{0k}}{\partial x_l} + \frac{\partial u_{1k}}{\partial y_l}\right)\left(\frac{\partial v_{0i}}{\partial x_j}\right) dy\, dx \\
&= \frac{1}{|Y|}\int_\Omega \int_Y \left(E_{ijkl}\frac{\partial u_{0k}}{\partial x_l} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial u_{0k}}{\partial x_l}\right)\frac{\partial v_{0i}}{\partial x_j} dy\, dx \\
&= \frac{1}{|Y|}\int_\Omega \left\{\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy\right\}\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j} dx \\
&= \int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx
\end{aligned}
$$

Homogenized equation

$$
\int_\Omega E_{ijkl}^H(x)\frac{\partial u_{0k}}{\partial x_l}\frac{\partial v_{0i}}{\partial x_j}\, dx = \int_\Omega \mathbf{f}\cdot \mathbf{v}_0 dx + \int_{\Gamma_t} \mathbf{t}\cdot \mathbf{v}_0 ds \qquad \text{for every } \mathbf{v}_0 \tag{52}
$$

where $E_{ijkl}^H(x)$ is

$$
\boxed{E_{ijkl}^H = \frac{1}{|Y|}\int_Y \left(E_{ijkl} - E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\right) dy} \tag{53}
$$

Now, Define

$$
a_H(\mathbf{u},\mathbf{v}) = \int_\Omega E_{ijkl}^H(\mathbf{x})\frac{\partial u_k}{\partial x_l}\frac{\partial v_i}{\partial x_j}\, dx, \tag{54}
$$

$$
a_Y(\chi^{kl},\mathbf{v}) = \int_Y E_{ijpq}(\mathbf{x},\mathbf{y})\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial v_i}{\partial y_j}\, dy, \tag{55}
$$

$$
L_Y^{kl}(\mathbf{v}) = \int_Y E_{ijkl}\frac{\partial v_i}{\partial y_j}\, dy \tag{56}
$$

At microscopic level, we have

$$a_Y(\chi^{kl}, \mathbf{v}) = L_Y^{kl}(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{U}_Y, \tag{57}$$

At macroscopic level, we have

$$a_H(\mathbf{u}, \mathbf{v}) = L(\mathbf{v}) \qquad \forall \mathbf{v} \in \mathcal{U}_0 \tag{58}$$

where $\mathcal{U}_0$ is homogeneous case of $\mathcal{U}$, i.e., $\mathbf{g} = 0$.

4.4 Implementation 2D Homogenization

Basic homogenization equation,

$$u_{1i}(\mathbf{x}, \mathbf{y}) = -\chi_i^{pq} \frac{\partial u_{0p}(\mathbf{x})}{\partial x_q} \tag{59}$$

Solve χ_p^{kl} from:

$$\int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) \frac{\partial v_{1i}}{\partial y_j} \, dy = 0 \tag{60}$$

Compute:

$$E_{ijkl}^H = \frac{1}{|Y|} \int_Y \left(E_{ijkl} - E_{ijpq} \frac{\partial \chi_p^{kl}}{\partial y_q} \right) dy \tag{61}$$

4.5 Examples

Consider: k=1, l=1

$$\begin{aligned} \int_Y E_{ijkl} \frac{\partial v_i}{\partial y_j} \, dy &= \int_Y E_{ij11} \frac{\partial v_i}{\partial y_j} \, dy \\ &= \int_Y \left(E_{1111} \frac{\partial v_1}{\partial y_1} + E_{2211} \frac{\partial v_2}{\partial y_2} \right) dy \end{aligned} \tag{62}$$

$$
\begin{aligned}
\int_Y E_{ijpq}\frac{\partial \chi_p^{kl}}{\partial y_q}\frac{\partial v_i}{\partial y_j}\,dy &= \int_Y E_{ijpq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_i}{\partial y_j}\,dy \\
&= \int_Y \left\{ E_{11pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_1}{\partial y_1} + E_{12pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_1}{\partial y_2} \right. \\
&\qquad\quad \left. + E_{21pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_2}{\partial y_1} + E_{22pq}\frac{\partial \chi_p^{11}}{\partial y_q}\frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{1112}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{1121}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{1122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_1} \right. \\
&\quad + \left(E_{1211}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{1212}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{1222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_2} \\
&\quad + \left(E_{2111}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{2112}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{2121}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{2122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_1} \\
&\quad \left. + \left(E_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{2212}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{2221}\frac{\partial \chi_2^{11}}{\partial y_1} + E_{2222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} + \cancelto{0}{E_{1112}\frac{\partial \chi_1^{11}}{\partial y_2}} + \cancelto{0}{E_{1121}\frac{\partial \chi_2^{11}}{\partial y_1}} + E_{1122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_1} \right. \\
&\quad + \left(\cancelto{0}{E_{1211}\frac{\partial \chi_1^{11}}{\partial y_1}} + E_{1212}\frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221}\frac{\partial \chi_2^{11}}{\partial y_1} + \cancelto{0}{E_{1222}\frac{\partial \chi_2^{11}}{\partial y_2}} \right)\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(E_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} + \cancelto{0}{E_{2212}\frac{\partial \chi_1^{11}}{\partial y_2}} + \cancelto{0}{E_{2221}\frac{\partial \chi_2^{11}}{\partial y_1}} + E_{2222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_2} \right\} dy \\
&= \int_Y \left\{ \left(E_{1111}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_1}{\partial y_1} \right. \\
&\quad + E_{1212}\left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1} \right)\left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\quad \left. + \left(E_{2211}\frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222}\frac{\partial \chi_2^{11}}{\partial y_2} \right)\frac{\partial v_2}{\partial y_2} \right\} dy
\end{aligned}
\tag{63}
$$

Therefore, using equations (57) , (63) and (62) for k=1, l=1 we have:

$$\begin{aligned}
&\int_Y \left\{ \left(E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\qquad + E_{1212} \left(\frac{\partial \chi_1^{11}}{\partial y_2} + \frac{\partial \chi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\qquad \left. + \left(E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} + E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
&\int_Y \left(E_{1111} \frac{\partial v_1}{\partial y_1} + E_{2211} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{64}$$

From equation (61), we can write

$$E_{1111}^H = \frac{1}{|Y|} \int_Y \left(E_{1111} - E_{1111} \frac{\partial \chi_1^{11}}{\partial y_1} - E_{1122} \frac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{65}$$

$$E_{2211}^H = \frac{1}{|Y|} \int_Y \left(E_{2211} - E_{2211} \frac{\partial \chi_1^{11}}{\partial y_1} - E_{2222} \frac{\partial \chi_2^{11}}{\partial y_2} \right) dy \tag{66}$$

$$E_{1211}^H = -\frac{1}{|Y|} \int_Y \left(E_{1212} \frac{\partial \chi_1^{11}}{\partial y_2} + E_{1221} \frac{\partial \chi_2^{11}}{\partial y_1} \right) dy \tag{67}$$

Let $\chi_1^{11} = \Phi_1, \chi_2^{11} = \Phi_2$ and $E_{1111} = D_{11}, E_{2222} = D_{22}, E_{1212} = D_{66}, E_{1122} = E_{2211} = D_{12}$

$$\begin{aligned}
&\int_Y \left\{ \left(D_{11} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{12} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_1}{\partial y_1} \right. \\
&\qquad + D_{66} \left(\frac{\partial \Phi_1^{11}}{\partial y_2} + \frac{\partial \Phi_2^{11}}{\partial y_1} \right) \left(\frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right) \\
&\qquad \left. + \left(D_{12} \frac{\partial \Phi_1^{11}}{\partial y_1} + D_{22} \frac{\partial \Phi_2^{11}}{\partial y_2} \right) \frac{\partial v_2}{\partial y_2} \right\} dy = \\
&\int_Y \left(D_{11} \frac{\partial v_1}{\partial y_1} + D_{12} \frac{\partial v_2}{\partial y_2} \right) dy
\end{aligned} \tag{68}$$

Also,

$$D_{11}^H = \frac{1}{|Y|} \int_Y \left(D_{11} - D11 \frac{\partial \Phi_1}{\partial y_1} - D_{12} \frac{\partial \Phi_2}{\partial y_2} \right) dy \tag{69}$$

Rearranging Eq. (68)

$$
\begin{aligned}
&\int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D66 \end{bmatrix} \\
&\qquad \times \begin{bmatrix} \frac{\partial \Phi_1}{\partial y_1} \\ \frac{\partial \Phi_2}{\partial y_2} \\ \frac{\partial \Phi_1}{\partial y_2} + \frac{\partial \Phi_2}{\partial y_1} \end{bmatrix} dY \qquad (70) \\
&= \int_Y \left\{ \frac{\partial v_1}{\partial y_1} \quad \frac{\partial v_2}{\partial y_2} \quad \frac{\partial v_1}{\partial y_2} + \frac{\partial v_2}{\partial y_1} \right\} \begin{bmatrix} D_{11} \\ D_{12} \\ 0 \end{bmatrix} dY
\end{aligned}
$$

Let us define

$$
\mathbf{b} = \begin{bmatrix} \frac{\partial}{\partial y_1} & 0 \\ 0 & \frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} \end{bmatrix} \tag{71}
$$

and

$$
\mathbf{D} = \begin{bmatrix} \mathbf{d}_1 & \mathbf{d}_2 & \mathbf{d}_3 \end{bmatrix} \tag{72}
$$

Then Eq (68), can be written as

$$
\int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{D} \mathbf{b} \Phi \, dY = \int_Y \mathbf{v}^T \mathbf{b}^T \mathbf{d}_1 \qquad \forall \mathbf{v} \in \mathbf{V}_Y \tag{73}
$$

and eq. (69) becomes:

$$
\boxed{D_{11}^H = \frac{1}{|Y|} \int_Y \left(D_{11} - \mathbf{d}_1^T \mathbf{b} \Phi \right) dy} \tag{74}
$$

5 Results

[1] [2]

6 References

References

[1] John Doe. Title. *Journal*, 2017.

[2] Intel. Example website. `http://example.com`, Dec 1988. Accessed on 2012-11-11.