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Topology optimization of microstructures of cellular materials and composites for macrostructures

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This paper introduces a topology optimization algorithm for the optimal design of cellular materials and composites with periodic microstructures so that the resulting macrostructure has the maximum stiffness (or minimum mean compliance). The effective properties of the heterogeneous material are obtained through the homogenization theory, and these properties are integrated into the analysis of the macrostructure. The sensitivity analysis for the material unit cell is established for such a two-scale optimization problem. Then, a bi-directional evolutionary structural optimization (BESO) approach is developed to achieve a clear and optimized topology for the material microstructure. Several numerical examples are presented to validate the proposed optimization algorithm and a variety of anisotropic microstructures of cellular materials and composites are obtained. The various effects on the topological design of the material microstructure are discussed.

1. Introduction

Many industrial and engineering materials are heterogeneous and consist of dissimilar constituents that are distinguishable at some length scales. The behavior of those heterogeneous materials is determined, on the one hand, by the relevant materials properties of the constituents and, on the other hand, by their topology at the micro-scale level. The theory of homogenization has been recognized as a rigorous modeling methodology for characterizing the mechanical behavior of cellular materials and composites with periodic microstructures [1–3]. The inverse problem is to design a new microstructure of the periodic representative unit cell (RUC) or representative volume element (RVE) so that the resulting material has desirable physical properties [4,5]. A systematic and scientific means of microstructural design is formulated as an optimization problem for the parameters that represent the material properties and topology of the material microstructure.

Over the last two decades, various topology optimization algorithms, e.g. homogenization method [6], solid isotropic material with penalization (SIMP) [7–9], evolutionary structural optimization (ESO) [10,11], and level set technique [12,13] have been developed. These topology optimization techniques have been used extensively to solve the design problems not only for macroscopic structures, but also for microstructures of materials/composites in recent years. For instance, tailoring microstructures of materials with prescribed constitutive properties has been investigated using the SIMP method [14,15], genetic algorithms [16], and the level set method [17]. Some attempts have also been made to design new materials with extraordinary physical properties, e.g., extremal thermal conductivity [18,19], maximum fluid permeability [20], maximum stiffness and fluid permeability [21], and maximum stiffness and thermal conductivity [22].

Unlike the continuous density-based topology methods, the ESO/BESO methods represent the structural topology and shape with discrete design variables (solid or void) with a clear structural boundary [11,23]. ESO was originally developed based upon a simple concept of gradually removing redundant or inefficient material from a structure so that the resulting topology evolves towards an optimum. A later version of the ESO method, namely the bi-directional evolutionary structural optimization (BESO) method, allows not only removing materials, but also adding materials to the design domain. It has been demonstrated that the current BESO method is capable of not only generating reliable and practical topologies for macrostructures [23,24], but also creating optimal microstructures for materials/composites with high computational efficiency [25,26].

Optimal microstructures obtained by the material design are only optimum in terms of desirable material properties. Hence, the structure constructed from the resulting materials may not be efficient or optimal since the service conditions such as applied loading, boundary condition are varied in practical use. Obviously,

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topology optimization of materials for structures is in essence a multi-scale problem which should consider the performance of macrostructures and topologies of material microstructures simultaneously. In this context, composite materials of two-dimensional structures are designed using the homogenization design method [27]. Rodrigues et al. [28] proposed a hierarchical computational procedure by integrating the macrostructures with local material microstructures using the SIMP method. Zhuang and Sun [29] studied the integrated optimization of cellular materials for sandwich panels. In order to design lightweight structures, Liu et al. [30] introduced a concurrent topology optimization of materials and structures using the porous anisotropic material with penalization (PAMP) model.

In this paper, a topology optimization approach based on BESO is proposed for designing microstructures of cellular materials and composites for structures. The optimization problem is formulated with the objective at the macro-scale level but with constraints at the micro-scale level. Two-scale finite element analyses of structures and materials are conducted and the sensitivity analysis is established for optimally designing microstructures of materials. Then, the BESO method is applied for iteratively updating the microstructures of the material. Finally, some examples are presented to illustrate the effectiveness of the proposed algorithm for designing cellular materials and composites for structures.

2. Optimization problem and material interpolation scheme

We consider a macrostructure with the known boundary conditions and external forces as illustrated by Fig. 1a. The macrostructure is composed of cellular material or two-phase composite with uniform microstructures (Fig. 1b) repeated periodically with the base cell (Fig. 1c). The optimization objective is to find the spatial distribution of each phase within the base cell so that the resulting macrostructure has the best load-carrying capability for the given weight or volume fraction of material(s). For such a two-scale optimization problem, there are two finite element models (namely macro model and micro model respectively), one for the structure and another for the material base cell. To seek optimal macrostructural performance (stiffness herein), the topology optimization is formulated in the macro finite element model with regard to the design variables in the micro level as

$$\begin{aligned}&\text{Minimize}: C=\frac{1}{2}\mathbf{F}^T\mathbf{U}\\&\text{Subject to}: V_1^*-\sum_{j=1}^{N}V_jx_j=0\\&x_j=0\quad\text{or}\quad 1\end{aligned}\tag{1}$$

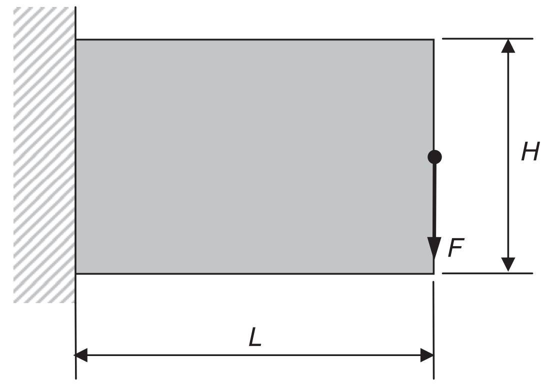


Fig. 2. A cantilever with length L and height H.

where C is the mean compliance of the macrostructure. $\mathbf{F}$ and $\mathbf{U}$ denote the external force vector and nodal displacement vector of the macrostructure respectively. $\mathbf{K}_i$ is the stiffness matrix of the ith element in the macrostructure model. M and N are the total number of elements in the macrostructure model and the micro base cell model respectively. V_j is the volume of element j in the micro base cell model and V_1^* is the prescribed volume for phase 1 in the material base cell. Note that different from the traditional topology optimization for periodic structures that is restricted in macro model to seek optimum using elemental design variables [31,32]; this study relates the macroscopic performance directly to the design variables, x_j in the micro model by using the homogenized properties of material base cell. The binary design variable x_j denotes the artificial relative density of element j in the micro FE model. If an element is made of phase 1, $x_j = 1$ and phase 2, $x_j = 0$.

The local material of an element in the material base cell can be treated to be isotropic and its physical properties are assumed to be a function of the elemental density, x_j. To ensure the existence of 0/1 solution, a SIMP (Solid Isotropic Material with Penalization) model is adopted here [9]. The elasticity matrix at a point with density value x_j can be expressed as

$$\mathbf{D}=x_j^p\mathbf{D}^1+(1-x_j^p)\mathbf{D}^2\tag{2}$$

where p is the exponent of penalization and $p = 3$ is used throughout this paper. $\mathbf{D}^1$ and $\mathbf{D}^2$ denote the elasticity matrices for phase 1 and phase 2 respectively. For the cellular material design, phase 2 is void with $\mathbf{D}^2 = \mathbf{0}$.

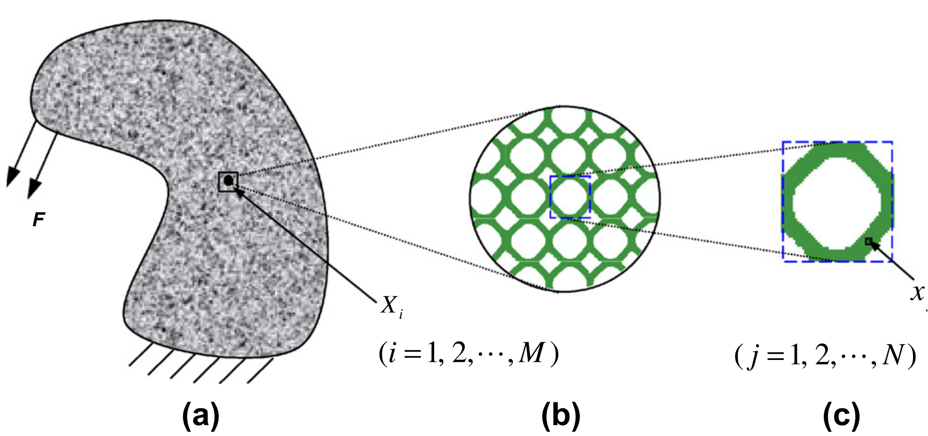


Fig. 1. A structure composed of cellular materials or composites (a) macrostructure; (b) microstructure; (c) a unit cell.

3. Finite element analysis and sensitivity analysis

The finite element analysis should be conducted both for the structure at the macro-scale and the material base cell at the micro-scale. Generally, the macro-scale analysis of the structure is to determine the structural deformation and evaluate the objective function. The static behavior of the structure is represented by the following macro FE equilibrium equation

$$\mathbf{KU} = \mathbf{F} \tag{3}$$

where $\mathbf{K}$ is the stiffness matrix of the macrostructure which is assembled by the elemental stiffness matrix, $\mathbf{K}_i$

$$\mathbf{K}_i = \int_{V_i} \mathbf{B}\mathbf{D}^H\mathbf{B}dV_i \tag{4}$$

where $\mathbf{B}$ is the strain/displacement matrix, $\mathbf{D}^H$ is the elasticity modulus matrix which should be computed from the micro-scale analysis of the material base cell according to the classical homogenization procedure [4,5]. This means that $\mathbf{D}^H$ relates the base cell microstructure of periodic material to the structural performance at the macro-scale level.

When the material base cell is very small compared with the size of the structural body, the homogenization theory can be applied for obtaining the elasticity matrix of the material. In the micro FE model, the material base cell is analyzed by imposing the periodic boundary conditions [33] as

$$\mathbf{ku} = \int_Y \mathbf{b}^T\mathbf{D}dY \tag{5}$$

where $\mathbf{k} = \int_Y \mathbf{b}^T\mathbf{D}\mathbf{b}dY$ is the stiffness matrix of the microstructure of the base cell, $\mathbf{b}$ is the strain/displacement matrix at the micro-scale level. The right hand side of Eq. (5) denotes the external forces caused by uniform strain fields e.g. $\{1,0,0\}^T$, $\{0,1,0\}^T$ and $\{0,0,1\}^T$ for 2D cases. $\mathbf{u}$ denotes the displacement fields of the unit cell caused by these uniform strain fields. As a result, the homogenized elasticity matrix can be expressed by

$$\mathbf{D}^H = \frac{1}{|Y|}\int_Y \mathbf{D}(\mathbf{I} - \mathbf{bu})dY \tag{6}$$

where $\mathbf{I}$ is an identity and $|Y|$ is the total area or volume of the material base cell.

To implement such topology optimization techniques as the BESO method, sensitivity analysis is necessary for guiding the search direction of the optimization algorithm. At the macro-scale level, the mean compliance depends on the material properties in each element. Thus, the derivation of the mean compliance with respect to the variation of material density in the ith element (X_i) can be expressed by

$$\frac{dC}{dX_i} = \frac{1}{2}\mathbf{F}^T\frac{\partial\mathbf{U}}{\partial X_i} = \frac{1}{2}\mathbf{F}^T\mathbf{K}^{-1}\left(\frac{\partial\mathbf{F}}{\partial X_i} - \frac{\partial\mathbf{K}}{\partial X_i}\mathbf{U}\right) = -\frac{1}{2}\mathbf{U}^T\frac{\partial\mathbf{K}}{\partial X_i}\mathbf{U}$$
$$= -\frac{1}{2}\mathbf{U}_i^T\frac{\partial\mathbf{K}_i}{\partial X_i}\mathbf{U}_i \tag{7}$$

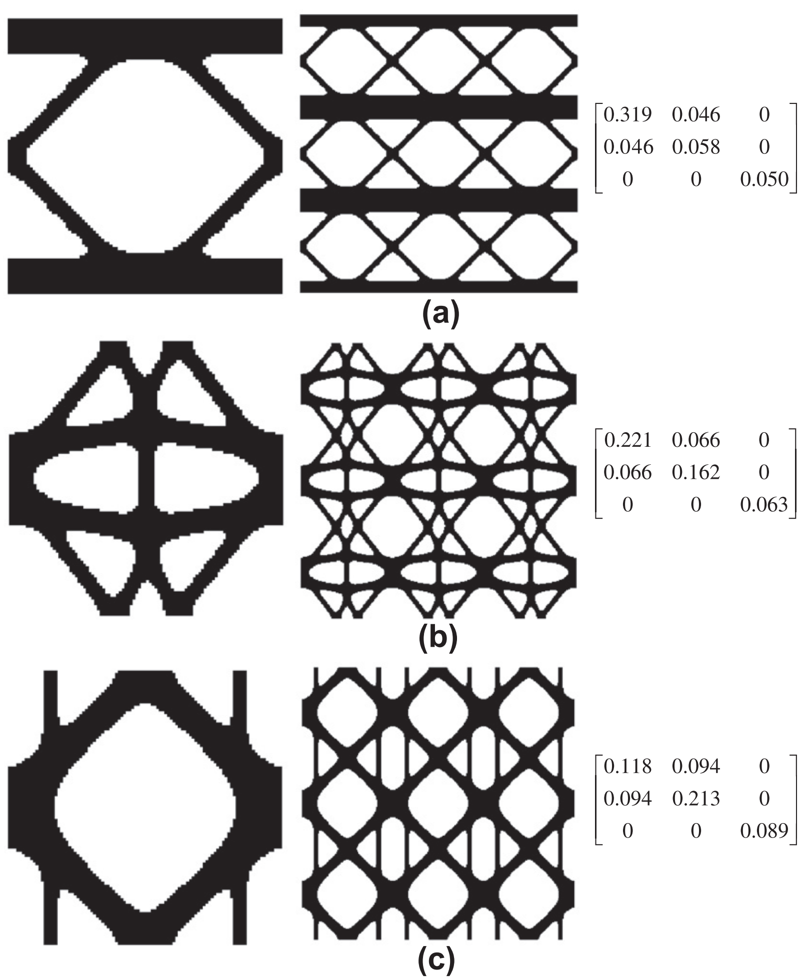


Fig. 3. Optimized microstructures of cellular materials starting from initial design 1: base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) for cantilevers with L and H (a) L = 40 and H = 20; (b) L = 40 and H = 40, and (c) L = 20 and H = 40 .

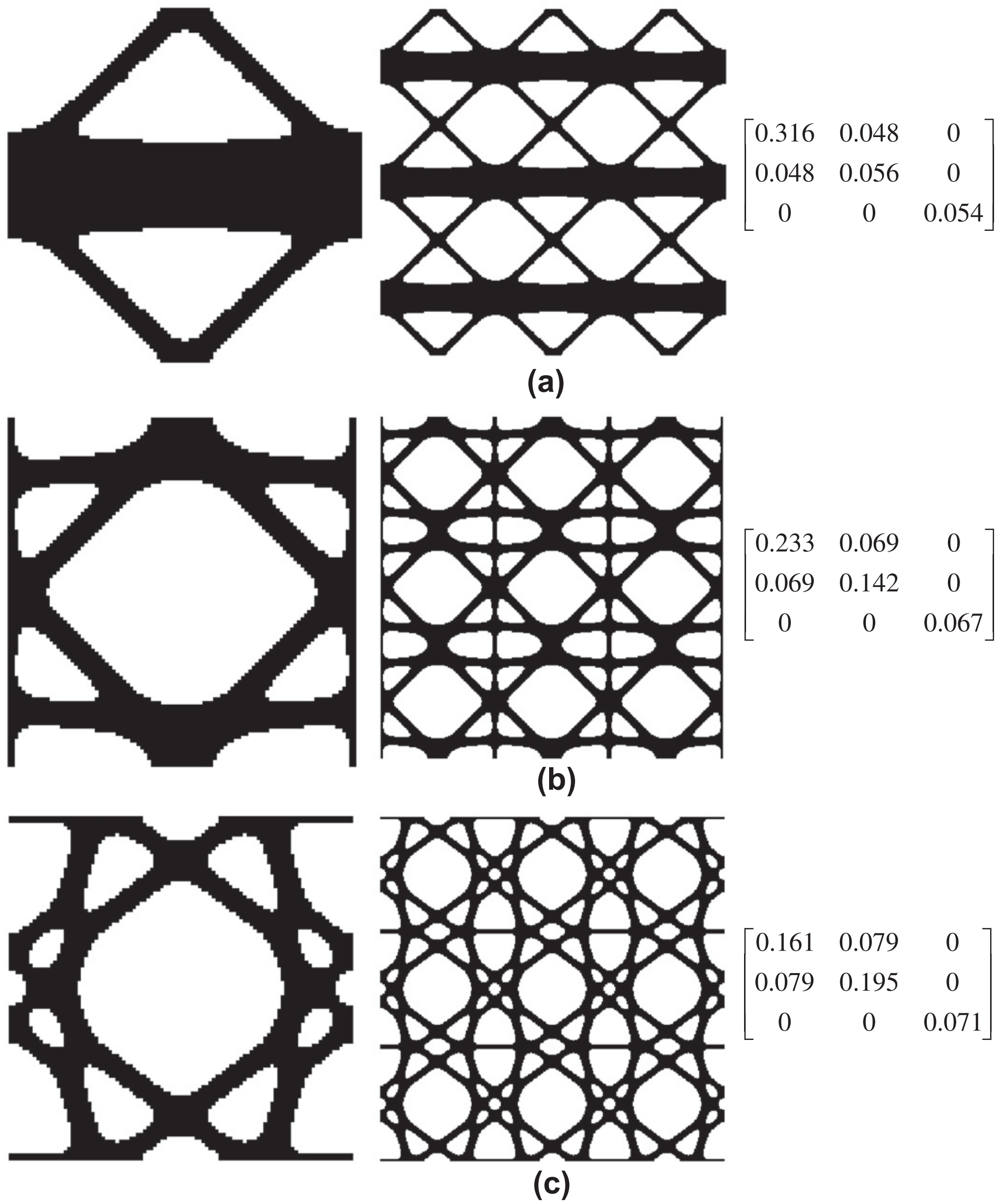


Fig. 4. Optimized microstructures of cellular materials starting from initial design 2: base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) for cantilevers with L and H (a) L = 40 and H = 20; (b) L = 40 and H = 40, and (c) L = 20 and H = 40 .

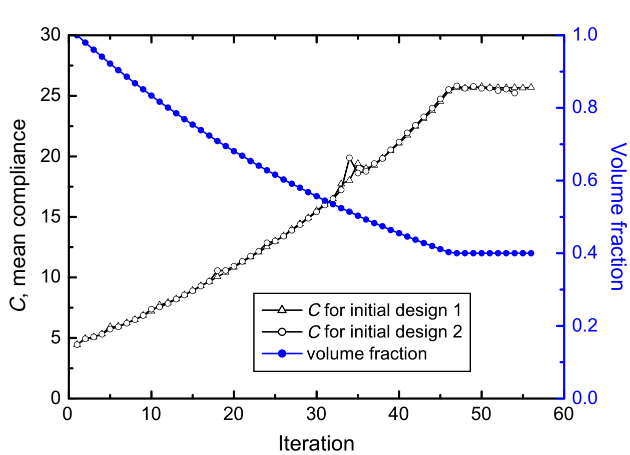


Fig. 5. Evolution histories of mean compliance and volume fraction of solid phase.

where $\mathbf{U}_i$ is the displacement vector of the ith element in the macrostructure. In the proposed optimization framework, it is assumed that the macrostructure is composed of a uniform cellular material or composite. Therefore, the design variable, x_j in the material base cell is relevant to all elements in the macrostructure. The sensitivity of the mean compliance against x_j is equal to the derivative summation of the mean compliance over all elements in the macrostructure as

$$\frac{dC}{dx_j} = -\frac{1}{2}\sum_{i=1}^{M}\mathbf{U}_i^T\frac{\partial \mathbf{K}_i}{\partial x_j}\mathbf{U}_i = -\frac{1}{2}\sum_{i=1}^{M}\mathbf{U}_i^T\int_{V_i}\mathbf{B}^T\frac{\partial \mathbf{D}^H}{\partial x_j}\mathbf{B}dV_i\mathbf{U}_i \tag{8}$$

where M is the total number of elements in the macrostructural model.

At the micro-scale level, the derivative of $\mathbf{D}^H$ with respect to x_j can be obtained according to the adjoint variable method [34], as

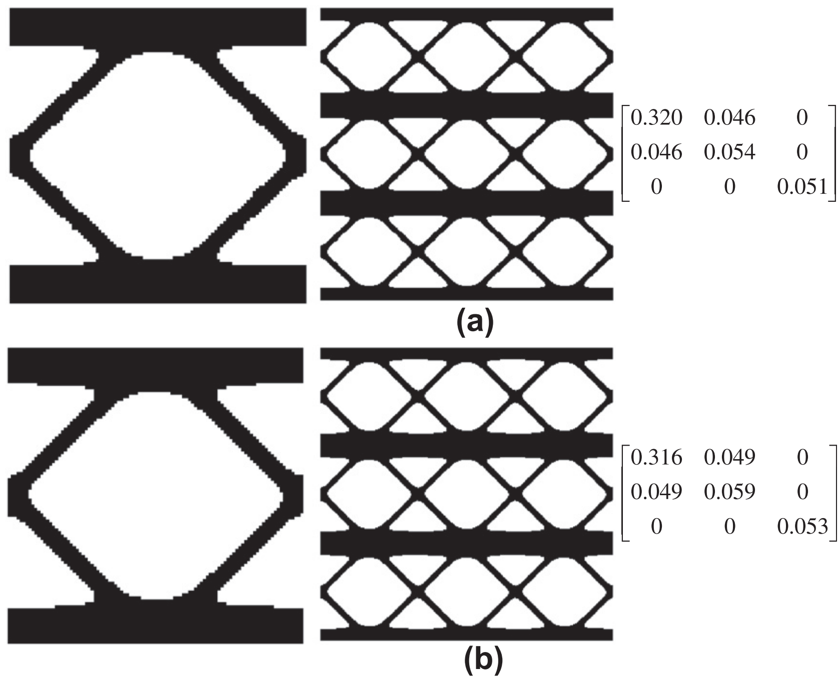


Fig. 6. Optimized microstructures of cellular materials when the macrostructure uses a different mesh: base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) (a) 20 × 10, and (b) 60 × 30.

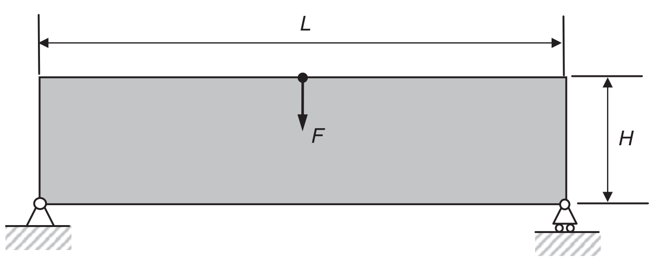


Fig. 7. A MBB beam with length L and height H.

$$\frac{\partial \mathbf{D}^H}{\partial x_j} = \frac{1}{|Y|}\int_Y (\mathbf{I} - \mathbf{bu})^T \frac{\partial \mathbf{D}}{\partial x_j}(\mathbf{I} - \mathbf{bu})dY \tag{9}$$

With the help of the material interpolation scheme in Eq. (2), the above equation can be rewritten as,

$$\frac{\partial \mathbf{D}^H}{\partial x_j} = \frac{p x_j^{p-1}}{|Y|}\int_{Y_j} (\mathbf{I} - \mathbf{bu}_j)^T (\mathbf{D}_1 - \mathbf{D}_2)(\mathbf{I} - \mathbf{bu}_j)dY_j \tag{10}$$

where $\mathbf{u}_j$ denotes the displacement vectors of the jth element in the material base cell. It can be seen that Eq. (10) can be easily calculated at the element level. Combining Eq. (10) with Eq. (8), the sensitivity of the mean compliance is

$$\frac{dC}{dx_j} = -\frac{p x_j^{p-1}}{2|Y|}\sum_{i=1}^{M}\mathbf{U}_i^T\left\{\int_{V_i}\mathbf{B}^T\left[\int_{Y_j}(\mathbf{I} - \mathbf{bu})^T(\mathbf{D}_1 - \mathbf{D}_2)(\mathbf{I} - \mathbf{bu})dY_j\right]\mathbf{B}dV_i\right\}\mathbf{U}_i \tag{11}$$

It can be seen that the sensitivities of elements in the material base cell is also related to the displacement field of the structure. In other word, the structure in turn affects the design of materials.

4. Numerical implementation and BESO procedure

In the BESO method, the sensitivity numbers which denote the relative ranking of the elemental sensitivities will be used to update the design variables of the material unit cell. Therefore, the sensitivity number of the jth element in the material base cell for minimizing the mean compliance of the macrostructure can be defined with the elemental sensitivity multiplying a constant, $-1/p$, as

$$\begin{aligned}\alpha_j &= -\frac{1}{p}\frac{dC}{dx_j} \\ &= \frac{x_j^{p-1}}{2|Y|}\sum_{i=1}^{M}\mathbf{U}_i^T\left\{\int_{V_i}\mathbf{B}^T\left[\int_{Y_j}(\mathbf{I} - \mathbf{bu})^T(\mathbf{D}_1 - \mathbf{D}_2)(\mathbf{I} - \mathbf{bu})dY_j\right]\mathbf{B}dV_i\right\}\mathbf{U}_i\end{aligned} \tag{12}$$

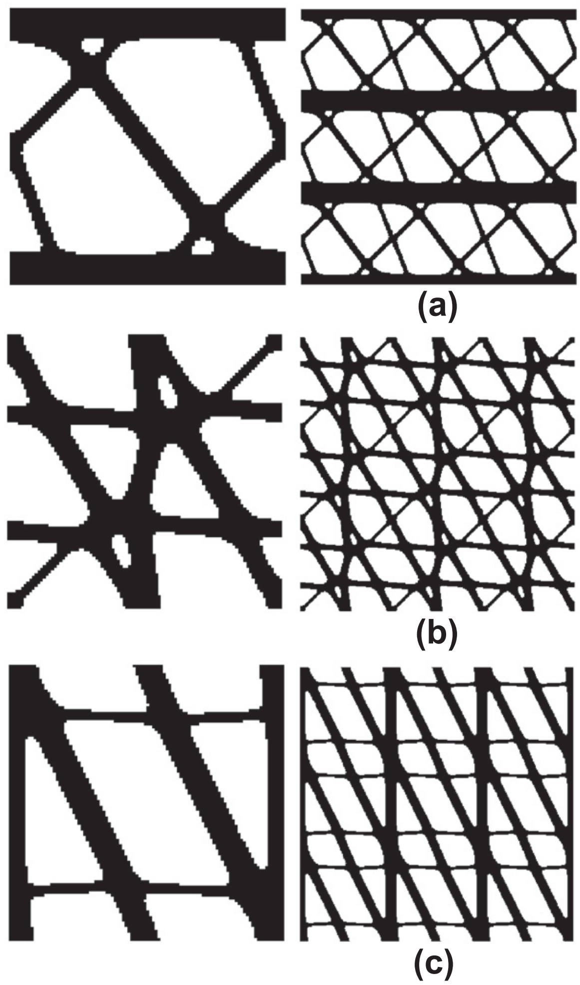
Because the design variable x_j is restricted to be either 0 or 1, the optimality criterion can be stated as that the sensitivity numbers of elements with $x_j = 1$ should be higher than those of elements with $x_j = 0$ [23]. Accordingly, we devise a simple scheme for updating the design variable $x_j = 0$ for elements with the lowest sensitivity numbers and $x_j = 1$ for elements with the highest sensitivity numbers.

Numerical instabilities such as checkerboard pattern and mesh-dependency problem are common phenomenon in the topology optimization techniques based on the finite element analysis. Here, a mesh-independent filter for discrete design variables [35] is applied for the elemental sensitivity numbers as

$$\hat{\alpha}_j = \frac{\sum_{r=1}^{N} w(r_{jr})\alpha_r}{\sum_{r=1}^{N} w(r_{jr})} \tag{13}$$

where r_{jr} denotes the distance between the center of elements j and r. $w(r_{jr})$ is the weight factor given as

$$w(r_{jr}) = \begin{cases} r_{\min} - r_{jr} & \text{for } r_{jr} < r_{\min} \\ 0 & \text{for } r_{jr} \geqslant r_{\min} \end{cases} \tag{14}$$



$$\begin{bmatrix} 0.272 & 0.054 & 0.007 \\ 0.054 & 0.123 & 0.029 \\ 0.007 & 0.029 & 0.054 \end{bmatrix}$$

$$\begin{bmatrix} 0.174 & 0.059 & 0.017 \\ 0.059 & 0.210 & 0.045 \\ 0.017 & 0.045 & 0.054 \end{bmatrix}$$

$$\begin{bmatrix} 0.107 & 0.043 & 0.021 \\ 0.046 & 0.275 & 0.073 \\ 0.021 & 0.073 & 0.041 \end{bmatrix}$$

Fig. 8. Optimized microstructures of cellular materials starting from initial design 1: base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) for MBB beams with L and H (a) L = 80 and H = 20; (b) L = 80 and H = 40, and (c) L = 40 and H = 40 .

where $r_{\min}$ is the filter radius which can be specified by the user.

Due to the discrete design variables used in the BESO algorithm, Huang and Xie [35] proposed that the elemental sensitivity number should be further modified by averaging with its historical information to improve the convergence of the solution. Thus, the sensitivity number after the first iteration is calculated by

$$\tilde{\alpha}_j = \frac{1}{2}(\hat{\alpha}_{j,k} + \hat{\alpha}_{j,k-1}) \tag{15}$$

where k is the current iteration number. Then let $\hat{\alpha}_{j,k} \leftarrow \tilde{\alpha}_j$ for the next iteration. Therefore the modified sensitivity number takes into consideration of the sensitivity information from the previous iterations.

The whole BESO procedure for designing microstructures of cellular material or composite for a structure is outlined as follow.

Step 1: Define BESO parameters: the target volume of phase 1, V_1^*, the evolutionary ratio ER (normally ER = 2%) and the filter radius $r_{\min}$. Construct an initial design of the material base cell.
Step 2: Carry out finite element analysis for the material base cell.
Step 3: Calculate the homogenized material properties $\mathbf{D}^H$ according to Eq. (6).
Step 4: Construct the finite element model for the macrostructure and substitute the resulting material properties $\mathbf{D}^H$ into the model. Then carry out finite element analysis for the structure at the macro-scale level.
Step 5: Calculate the elemental sensitivity numbers α_j according to Eq. (12).
Step 6: Filter elemental sensitivity number using Eq. (13) and then average with its historical information using Eq. (15).
Step 7: Determine the target volume of phase 1 for the next design. When the current volume of phase 1, V_k^1, is larger than the target volume V_1^*, reduce the volume of phase 1 as

$$V_{k+1}^1 = V_k^1(1 - ER) \tag{16a}$$

If the resulting V_{k+1}^1 is less than V_1^*, then V_{k+1}^1 is set to V_1^*. Similarly, the volume of phase 1 should be increased when V_k^1 is less than the target volume V_1^*

$$V_{k+1}^1 = V_k^1(1 + ER) \tag{16b}$$

If the resulting V_{k+1}^1 is larger than V_1^*, then V_{k+1}^1 is set to V_1^*.

Step 8: According to the relative ranking of the elemental sensitivity numbers, reset the design variable x_j to 1 (phase 1) for elements with highest sensitivity numbers and to 0 (phase 2) for elements with lowest sensitivity numbers so that the resulting volume of phase 1 is equal to V_{k+1}^1. As a result, the topology of the material base cell is updated.

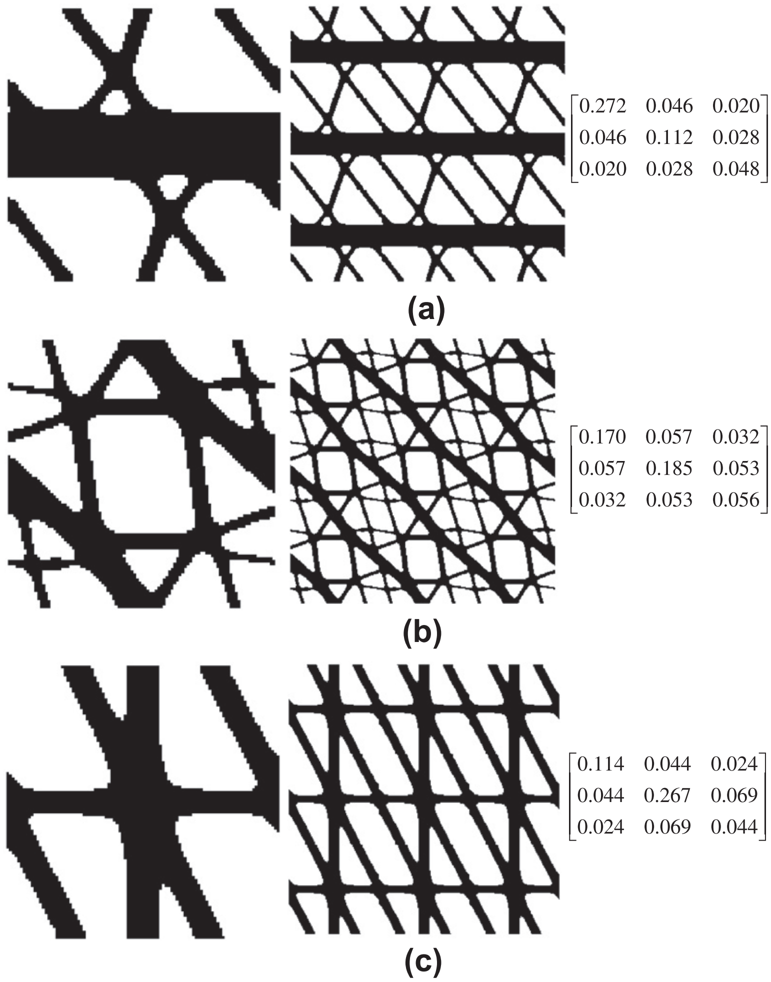


Fig. 9. Optimized microstructures of cellular materials starting from initial design 2: base cell (left); 3×3 unit cells (middle); elasticity matrix (right) for MBB beams with L and H (a) $L = 80$ and $H = 20$; (b) $L = 80$ and $H = 40$, and (c) $L = 40$ and $H = 40$.

Step 9: Repeat Steps 2–8 until both the volume constraint in the material base cell is satisfied and the mean compliance of the macrostructure is convergent.

5. Results and discussion

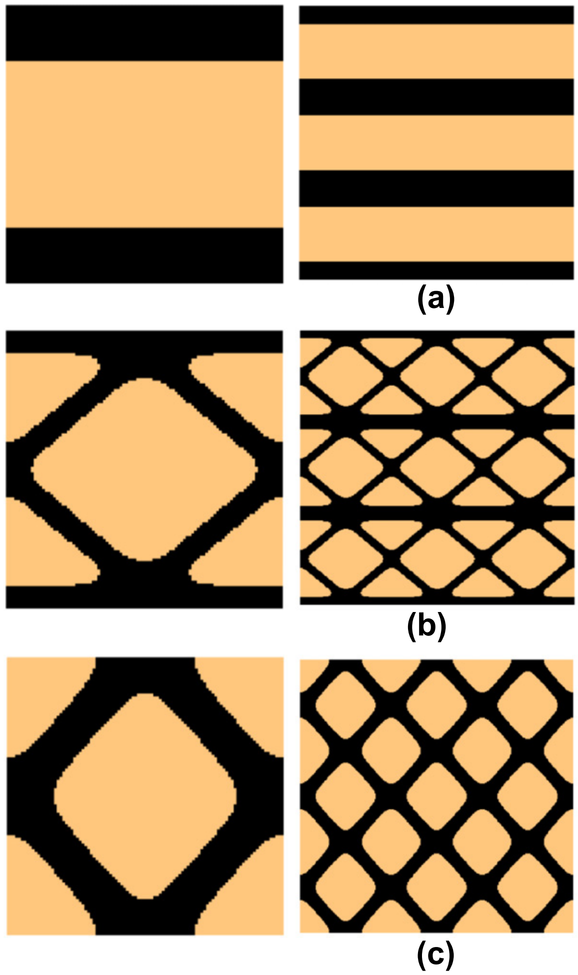
In this section, we will present some examples for designing microstructures of cellular materials or two-phase composites for structures. The square base cell is discretized into 100×100 4-node quadrilateral elements which represents the microstructure of the materials. To start the optimization procedure, two different initial designs are considered: initial design 1 is full with phase 1 material except for four elements at the center of the base cell with phase 2 material; initial design 2 is full with phase 1 material except for four corner elements for phase 2 material. The phase 2 material is void for cellular materials. Simultaneously, finite element analysis will be conduct for the given structure. For simplicity, a uniform square mesh with size 1×1 is assigned for all macrostructures.

5.1. Optimized microstructures of cellular materials for cantilevers

It is assumed that the cantilever as shown in Fig. 2 is composed of cellular materials. The volume fraction of solid phase for the cellular material is 40% of the total volume and solid material is assumed to be isotropic with Young's modulus $E_1 = 1$ and Poisson's ratio $\nu = 0.3$. The problem is to find the best topology of the solid phase within the material base cell so that the resulting cantilever has the maximum stiffness (or minimum mean compliance). The proposed BESO technique will be adopted by using parameters: evolution rate $ER = 2\%$ and filter radius $r_{\min} = 7$.

BESO starts from initial design 1, the resulting microstructures of cellular materials and their elasticity matrices for cantilevers with various L and H are shown in Fig. 3. As expected, the dimensions of the macrostructure affect the optimized microstructure of the material. The results also identify that the structural stiffness of x-direction is higher than that of y-direction for a long cantilever ($L = 40$, $H = 20$) and is lower than that of y-direction for a short cantilever ($L = 20$, $H = 40$). With the increment of the length of a cantilever, the structural stiffness in x-direction increases to improve the structural capacity to resist bending. With the decrement of the length of a cantilever, the shear modulus increases so as to improve the structural capacity to resist shearing. Those conclusions are consistent with our common knowledge in structural analysis. As for these microstructures, it is interestingly noted that the resulting materials are orthotropic and their topologies are symmetric about x and y axes. although there is no such symmetric constraints imposed by the optimization algorithm.

Fig. 4 shows the resulting microstructures of cellular materials and their elasticity matrices starting from initial design 2. The



(a) $\begin{bmatrix} 0.548 & 0.097 & 0 \\ 0.097 & 0.323 & 0 \\ 0 & 0 & 0.113 \end{bmatrix}$

(b) $\begin{bmatrix} 0.476 & 0.136 & 0 \\ 0.136 & 0.364 & 0 \\ 0 & 0 & 0.153 \end{bmatrix}$

(c) $\begin{bmatrix} 0.392 & 0.160 & 0 \\ 0.160 & 0.412 & 0 \\ 0 & 0 & 0.170 \end{bmatrix}$

Fig. 10. Optimized microstructures of composites (dark for phase 1 and grey for phase 2): base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) for cantilevers with L and H (a) $L = 40$ and $H = 20$; (b) $L = 40$ and $H = 40$, and (c) $L = 20$ and $H = 40$.

results give the similar elasticity matrices but different microstructures comparing with those starting from initial design 1. Even so, the resulting mean compliances of the macrostructures are very close, e.g. the mean compliances for the cantilever with $L = 40$, $H = 20$ are 25.69 from initial design 1 and 25.55 from initial design 2. Therefore, similar to the pure material design, there are multiple solutions of microstructures that produce the structure with a minimum compliance.

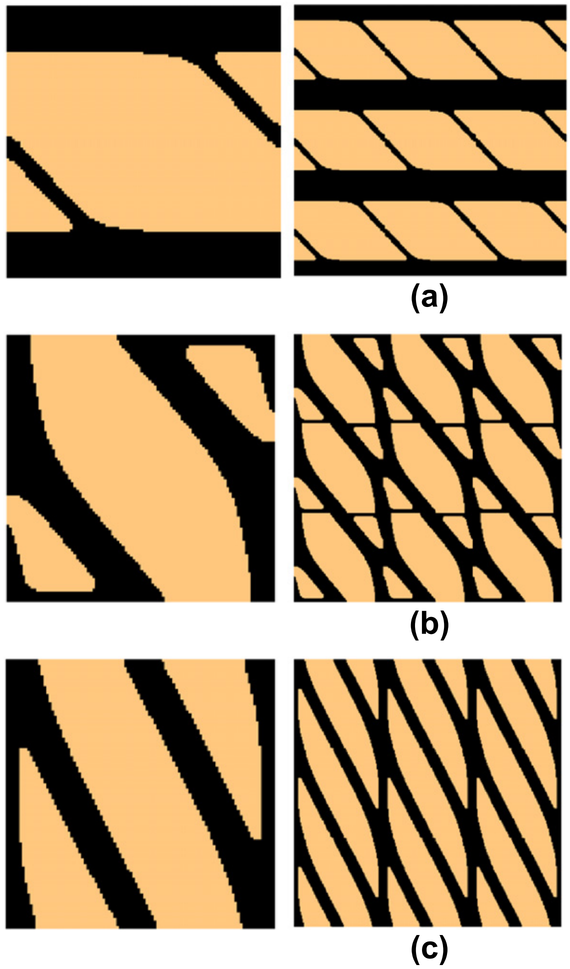
Fig. 5 plots the evolution histories of the mean compliance of the cantilever with $L = 40$, $H = 40$ and volume fraction of solid phase within the material base cell. Obviously, the histories of the mean compliance from initial design 1 and initial design 2 are very similar. The mean compliance converges after 56 and 54 iterations respectively, which indicates that the proposed topology optimization algorithm is of rather high computational efficiency.

As the asymptotic homogenization theory implies that the size of microstructure is infinitesimal and microstructural size effects are not taken into account [27], hence the optimized microstructures should be the same for a given macrostructure regardless of its mesh size. To verify this, it is necessary to numerically investigate the effect of the mesh refinement of the macrostructure. Here, different meshes 20 × 10 and 60 × 30 are assigned for the cantilever with $L = 40$ and $H = 20$. The resulting microstructures and elasticity matrices are displayed in Fig. 6, they have no significant difference from the ones shown in Fig. 3a. As expected, the numerical tests indicate that the mesh refinement of the macrostructure has very little effect on designing microstructures of materials.

5.2. Optimized microstructures of cellular materials for MBB beams

In this example, we will design microstructures of cellular materials for a MBB beam as shown in Fig. 7. The volume fraction of solid phase is supposed to be 40% of the total volume. The solid is assumed to be isotropic with Young's modulus $E_1 = 1$ and Poisson's ratio $\nu = 0.3$. The BESO parameters are evolution rate $ER = 2\%$ and filter radius $r_{\min} = 5$.

When BESO starts from initial design 1 for MBB beams with various L and H, the resulting microstructures of cellular materials and their elasticity matrices are given in Fig. 8. Similar to the microstructures for cantilevers, the structural stiffness of x-direction increases to improve the structural capacity to resist bending for a long-span beam. Differing from the microstructures for cantilevers, the microstructures of MBB beams are totally anisotropic as shown with their elasticity matrices. The anisotropy or orthotropy of the resulting materials depends on the macrostructure and the optimization objective. The numerical results indicate that constructing a MBB beam with an anisotropic material would result in a minimum mean compliance. When BESO starts from initial design 2,



(a) $$\begin{bmatrix} 0.526 & 0.108 & 0.011 \\ 0.108 & 0.340 & 0.014 \\ 0.011 & 0.014 & 0.127 \end{bmatrix}$$

(b) $$\begin{bmatrix} 0.378 & 0.131 & 0.029 \\ 0.131 & 0.452 & 0.045 \\ 0.029 & 0.045 & 0.149 \end{bmatrix}$$

(c) $$\begin{bmatrix} 0.333 & 0.123 & 0.014 \\ 0.123 & 0.494 & 0.052 \\ 0.014 & 0.052 & 0.142 \end{bmatrix}$$

Fig. 11. Optimized microstructures of composites (dark for phase 1 and grey for phase 2): base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) for MBB beams with L and H (a) $L = 80$ and $H = 20$; (b) $L = 80$ and $H = 40$, and (c) $L = 40$ and $H = 40$.

the different anisotropic microstructures of cellular materials but with similar elasticity matrices are obtained as shown in Fig. 9.

5.3. Optimized microstructures of composites for structures

When structures are composed of two-phase composites with Young's modulus $E_1 = 1$, and Poisson's ratio $\upsilon_1 = 0.3$ for phase 1 and $E_2 = 0.2$ and $\upsilon_2 = 0.3$ for phase 2, the goal of the problem is to find optimal distribution of two phase materials so that the resulting macrostructure has the maximum stiffness. The target volume fraction of phase 1 within the base cell is 40% of the total volume. The BESO parameters in the following examples are the same as those used for designing cellular materials but the optimization procedure only starts from initial design 1.

Fig. 10 shows the resulting microstructures and their elasticity matrices of composites for the cantilevers with various L and H (dark for phase 1 and grey for phase 2). The resulting composites are orthotropic with clear reinforcement orientation, 0° for the long cantilever $L = 40$ and $H = 20$, 0° and ±45° for the cantilever $L = 40$ and $H = 40$ and ± 45° for the short cantilever $L = 20$ and $H = 40$. For the MBB beams, the microstructures and their elasticity matrices of composites are shown in Fig. 11. In general, all reinforcements are clearly oriented to efficiently carry the applied load in the macrostructure.

An important parameter for designing composites is the level of inhomogeneity of the homogenous constituents' behavior, which is often described by the phase contrast. The elastic contrast of a two-phase composite takes the form $c_{el} = E_1/E_2$. Therefore, a cellular material can be looked as an extreme case when c_{el} tends to infinity. Fig. 12 shows the optimized microstructures and their elasticity matrices of composites for the cantilever with $L = 40$ and $H = 40$ when $c_{el} = 20$, 10 and 2.5, respectively. The examples well demonstrate that the optimized microstructures of composites highly depend on the elastic contrast of the constituents, c_{el}. The optimized microstructures of composites with various c_{el} are shown in Fig. 13 for the MBB beam with $L = 80$ and $H = 40$. It is reasonable that the constituents are distributed in a different manner within the base cell when their load-carrying capacities are changed.

To transfer the load effectively, all resulting composites are continuously reinforced. It might be however untrue when multiple optimization objectives are considered [22]. It is also worth mentioning that the optimized microstructure of the composite is not a sole solution because it may depend on the selection of the initial design as designing cellular materials. This work is to provide one of optimized solutions which can be used for the microstructural design of composites.

6. Conclusions

This paper has developed an optimization approach for a two-scale optimization problem which designs a microstructure of a cellular material or composite for a given volume fraction so that

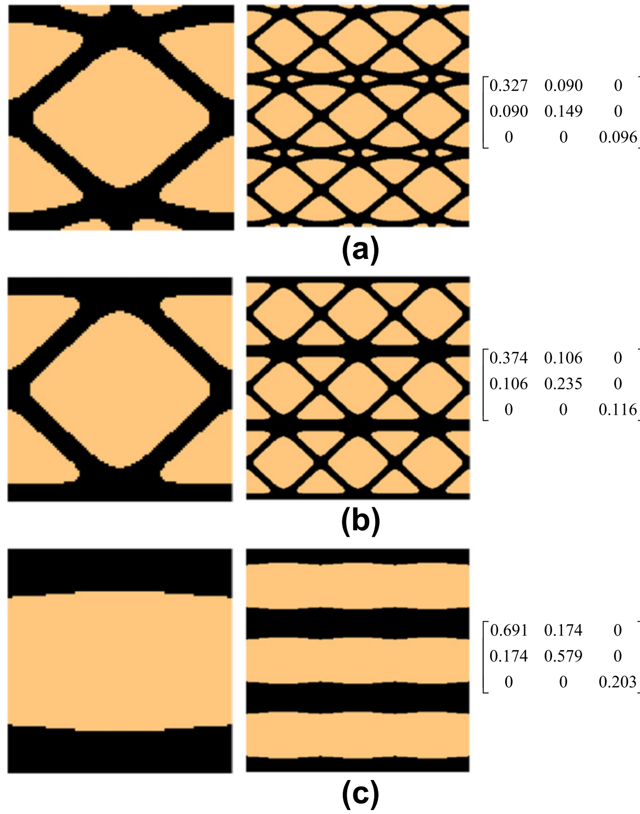


Fig. 12. Optimized microstructures of composites (dark for phase 1 and grey for phase 2): base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) for cantilevers (a) $c_{el} = 20$; (b) $c_{el} = 10$, and (c) $c_{el} = 2.5$.

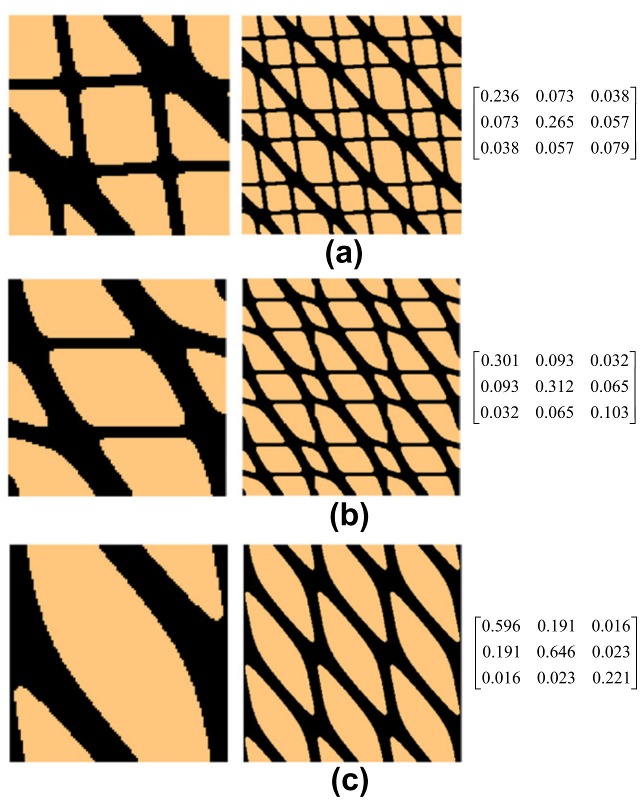


Fig. 13. Optimized microstructures of composites (dark for phase 1 and grey for phase 2): base cell (left); 3 × 3 unit cells (middle); elasticity matrix (right) for MBB beams (a) $c_{el} = 20$; (b) $c_{el} = 10$, and (c) $c_{el} = 2.5$.

the resulting macrostructure has the maximum stiffness. The effective elastic properties of the material with periodic microstructure are integrated into the finite element analysis of the macrostructure. Then, the resulting displacement field of the structure is integrated into the microstructural design of the material. This two-scale interaction is realized through the homogenization theory and the sensitivity analysis. Finally, a BESO method is developed by gradually re-distributing the constituents within the material base cell until a convergent solution is achieved. The given numerical examples clearly demonstrate the effectiveness of the developed approach and provide various optimized microstructures of cellular materials and composites. It reveals that the optimized microstructure of the material for a structure depends on the loading and boundary conditions of the macrostructure itself, the initial design of the unit cell and the difference in stiffness of the constituents.

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