Project 2

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Contents

1 System Dynamics

1.1 The Coordinate Frame, Notations, and Conventions

In the given problem, we have a crane which behaves as a frictionless cart with mass M that can move along a one dimensional track. The crane is actuated by an external force F which is the input of the system. Two loads of masses m_1 and m_2 are connected to the crane by cables of length l_1 and l_2 respectively. The Figure 1 shown below describes the system and shows the coordinate convention used marked by x and y. The angles made by masses m_1 and m_2 are θ_1 and θ_2 respectively where F is a function of time. Displacement of the crane is given by x.

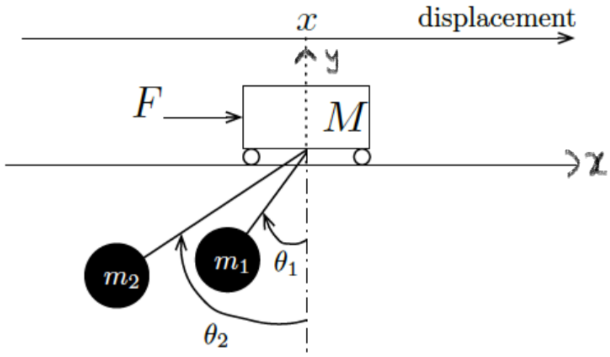

Fig. 1. The crane system.

For mass m_1; $x = -l_1 \sin\theta_1$, $y = -l_1 \cos\theta_1$, $\dot{x} = -l_1 \cos\dot{\theta}_1$, $\dot{y} = l_1 \sin\dot{\theta}_1$
For mass m_2; $x = -l_2 \sin\theta_2$, $y = -l_2 \cos\theta_2$, $\dot{x} = -l_2 \cos\dot{\theta}_2$, $\dot{y} = l_2 \sin\dot{\theta}_2$

1.2 Kinetic Energy

The kinetic energy of the system is given by the equation

$$K = \frac{1}{2}M\dot{x}^2(t) + \frac{1}{2}m_1[\dot{x}^2(t) + l_1^2\dot{\theta}_1^2(t) - 2l_1\dot{x}(t)\dot{\theta}_1(t)cos\theta_1(t)] + \frac{1}{2}m_2[\dot{x}^2(t) + l_2^2\dot{\theta}_2^2(t) - 2l_2\dot{x}(t)\dot{\theta}_2(t)cos\theta_2(t)] \quad (1)$$

1.3 Potential Energy

The potential energy of the system is given by the equation

$$P = -m_1 g l_1 cos\theta_1(t) - m_2 g l_2 cos\theta_2(t) \quad (2)$$

All calculations have been shown in Appendix.

1.4 Euler Lagrange Equation

The Lagrangian is given by the equation

$$L = K - P \tag{3}$$

Using the values of K and P from 1, 2 respectively and using it in 3, we get

$$L = \frac{1}{2}M\dot{x}^2(t) + \frac{1}{2}m_1[\dot{x}^2(t) + l1^2\ddot{\theta}_1(t)^2 - 2l_1\dot{x}(t)\dot{\theta}_1(t)cos\theta_1(t)] \\ + \frac{1}{2}m_2[\dot{x}^2(t) + l2^2\ddot{\theta}_2(t)^2 - 2l_2\dot{x}(t)\dot{\theta}_2(t)cos\theta_2(t) + m_1gl_1cos\theta_1(t) \\ + m_2gl_2cos\theta_2(t) \tag{4}$$

The equations of motion given by the Euler Lagrange equation are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = F \tag{5}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} - \frac{\partial L}{\partial \theta_1} = 0 \tag{6}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} - \frac{\partial L}{\partial \theta_1} = 0 \tag{7}$$

From Equation 4, 5, 6, and 7 we get the three equations of motion as

$$\ddot{x} = \frac{F - m_1gsin\theta_1(t)cos\theta_1(t) - m_1l_1\dot{\theta}_1^2(t)sin\theta_1(t) - m_2gsin\theta_2(t)cos\theta_2(t) - m_2l_2\dot{\theta}_2^2(t)sin\theta_2(t)}{M + m_1sin^2\theta_1(t) + m_2sin^2\theta_2(t)} \tag{8}$$

$$\ddot{\theta}_1 = [\frac{F - m_1gsin\theta_1(t)cos\theta_1(t) - m_1l_1\dot{\theta}_1^2(t)sin\theta_1(t) - m_2gsin\theta_2(t)cos\theta_2(t) - m_2l_2\dot{\theta}_2^2(t)sin\theta_2(t)}{M + m_1sin^2\theta_1(t) + m_2sin^2\theta_2(t)}]\frac{cos\theta_1(t)}{l_1} \\ - \frac{gsin\theta_1(t)}{l_1} \tag{9}$$

$$\ddot{\theta}_2 = [\frac{F - m_1gsin\theta_1(t)cos\theta_1(t) - m_1l_1\dot{\theta}_1^2(t)sin\theta_1(t) - m_2gsin\theta_2(t)cos\theta_2(t) - m_2l_2\dot{\theta}_2^2(t)sin\theta_2(t)}{M + m_1sin^2\theta_1(t) + m_2sin^2\theta_2(t)}]\frac{cos\theta_2(t)}{l_2} \\ - \frac{gsin\theta_2(t)}{l_2} \tag{10}$$

All calculations have been shown in Appendix.

1.5 Non Linear Space Representation

The state choice is given as:

$$\vec{X(t)} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{pmatrix} \tag{11}$$

The non linear space representation is given by

$$\vec{\dot{X}(t)} = f(\vec{X(t)}, \vec{U(t)}) = \begin{pmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{pmatrix} \tag{12}$$

where the values of $\ddot{x}, \ddot{\theta}_1$, and $\ddot{\theta}_2$ are the values from equations 8, 9 and 10 respectively and $\vec{U(t)} = F$

2 Linearization of the System Around an Equilibrium Point

The linearization around an equilibrium point can be obtained as follows:

$$\begin{aligned} A_F &= \nabla_{\vec{X(t)}} | f(\vec{X(t)}, \vec{U(t)}) \\ B_F &= \nabla_{\vec{U(t)}} | f(\vec{X(t)}, \vec{U(t)}) \end{aligned} \tag{13}$$

where $f(\vec{X(t)}, \vec{U(t)})$ is given in equation 12.

The equilibrium condition around which the system is linearized is given by $x = 0, \theta_1 = 0, \theta_2 = 0$. Solving the partial differentials and using the equilibrium condition, we get

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{g m_1}{M} & 0 & -\frac{g m_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g\left(\frac{m_1}{M}+1\right)}{l_1} & 0 & -\frac{g m_2}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g m_1}{M l_2} & 0 & -\frac{g\left(\frac{m_2}{M}+1\right)}{l_2} & 0 \end{pmatrix} \tag{14}$$

$$B = \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{pmatrix} \tag{15}$$

If we take the outputs as x, θ_1, θ_2;

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{16}$$

$$D = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{17}$$

Therefore, the state space representation for the linearized system is given by:

$$\vec{X(t)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{gm_1}{M} & 0 & -\frac{gm_2}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g\left(\frac{m_1}{M}+1\right)}{l_1} & 0 & -\frac{gm_2}{Ml_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{gm_1}{Ml_2} & 0 & -\frac{g\left(\frac{m_2}{M}+1\right)}{l_2} & 0 \end{pmatrix} \vec{X(t)} + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{Ml_1} \\ 0 \\ \frac{1}{Ml_2} \end{pmatrix} U(t)$$

$$\vec{Y(t)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \vec{X(t)} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} U(t) \tag{18}$$

where;
$\vec{X(t)}$ is a 6x1 (nxl) state vector,
A is a 6x6 matrix (nxn),
B is a 6x1 matrix (nxm),
$\vec{U(t)}$ is a 1x1 (mxl) input vector,
$\vec{Y(t)}$ is a 6x1 (pxl) output vector,
C is a 3x6 (pxn) matrix,
D is a 3x1 (pxm) matrix.

3 Controllability

For a LTI system the Gramian of controllability $W_c(0,t^*)$ is controllable if and only if the *nxnam* controllability matrix satisfies

$$rank([B_K \quad AB_K \quad A^2B_K \quad A^3B_K \quad .... \quad A^{n-1}B_K]) = n \tag{19}$$

Substituting the values of A and B from Equation 14 and 15 respectively, we get the result that

$$rank([B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B \quad A^6B \quad]) = 6 \tag{20}$$

For our system, n = 6, hence the system is controllable.
But along with the condition that the controllability matrix must be full rank, it must also be invertible i.e. the determinant value of the controllability matrix must not be zero. Applying

this condition, we get

$$determinant([B \quad AB \quad A^2B \quad A^3B \quad A^4B \quad A^5B \quad A^6B \quad]) \neq 0$$
$$-\frac{g^6\,(l_1-l_2)^2}{M^6\,{l_1}^6\,{l_2}^6} \tag{21}$$

From Equation 21 we get a specific condition which must hold for the linearized system to be controllable

$$l_1 \neq l_2 \tag{22}$$

Hence, the linearized system is controllable for all conditions except $l_1 = l_2$ i.e $l_1 \neq l_2$ is the only condition that should be satisfied for the system to be controllable.

4 Linear Quadratic Regulator (LQR) Controller Design

Choosing the following values;
$M = 1000Kg$,
$m_1 = 100Kg$,
$m_2 = 100Kg$,
$l_1 = 20m$,
$l_2 = 10m$,
$g = 9.8ms^-2)$
our A, B, C and D matrices are as follows

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{49}{50} & 0 & -\frac{49}{50} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{539}{1000} & 0 & -\frac{49}{1000} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{49}{500} & 0 & -\frac{539}{500} & 0 \end{pmatrix}$$
$$B = \begin{pmatrix} 0 \\ \frac{1}{1000} \\ 0 \\ \frac{1}{20000} \\ 0 \\ \frac{1}{10000} \end{pmatrix}$$
$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$
$$D = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{23}$$

Also, as rank of the controllability matrix is 6 and $l_1(=20m) \neq l_2(=10m)$, the system is controllable.

The aim of designing a Linear Quadratic Regualtor Controller is to look for **K** that minimizes the following cost function if the pair *(A,B_K)* is stabilizable

$$J(\mathbf{K},\vec{X(0)}) = \int_0^{\infty} \vec{X^T}(t)Q\vec{X(t)} + \vec{U^T}(t)R\vec{U_K}(t)dt \tag{24}$$

such that

1. The closed loop system is guaranteed to be stabilized.
2. A good way to tradeoff speed of response with control effort is provided.

The MATLAB function of LQR is used to find the value of gain **K**. The arguments to the lqr() function in MATLAB are the matrices A,B,Q,R where A,B are the values given in Equation 23. Q is obtained from perfoming the operation $C^T * C$ where C is the value obtained in Equation 23. But, the values in elements $Q(1,1),Q(2,2),Q(3,3),Q(4,4),Q(5,5),Q(6,6)$ are changed to improve the system such that

$$Q = \begin{pmatrix} 100000000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2072430287367949}{65536} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{2072430287367949}{65536} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{25}$$

and the value of R = 1. Only the value of Q is changed keeping the value of R fixed as only their relative values are important. From these values of arguments, the gains obtained is **K** = 1.0e+05 * $\begin{pmatrix} 0.1000 & 0.2360 & 1.7674 & -0.0907 & 0.8437 & -1.5759 \end{pmatrix}$

The eigen values of **A-BK** is $\begin{pmatrix} -3.2718+3.3994\mathrm{i} \\ -3.2718-3.3994\mathrm{i} \\ -0.2852+0.5197i\mathrm{i} \\ -0.2852-0.5197\mathrm{i} \\ -0.1359+0.7716\mathrm{i} \\ -0.1359-0.7716\mathrm{i} \end{pmatrix}$

Using Lyapunov's Indirect Method, we can guarantee that the closed loop system is **locally stable** as the real parts of the all the eigen values of **A-BK** are negative.

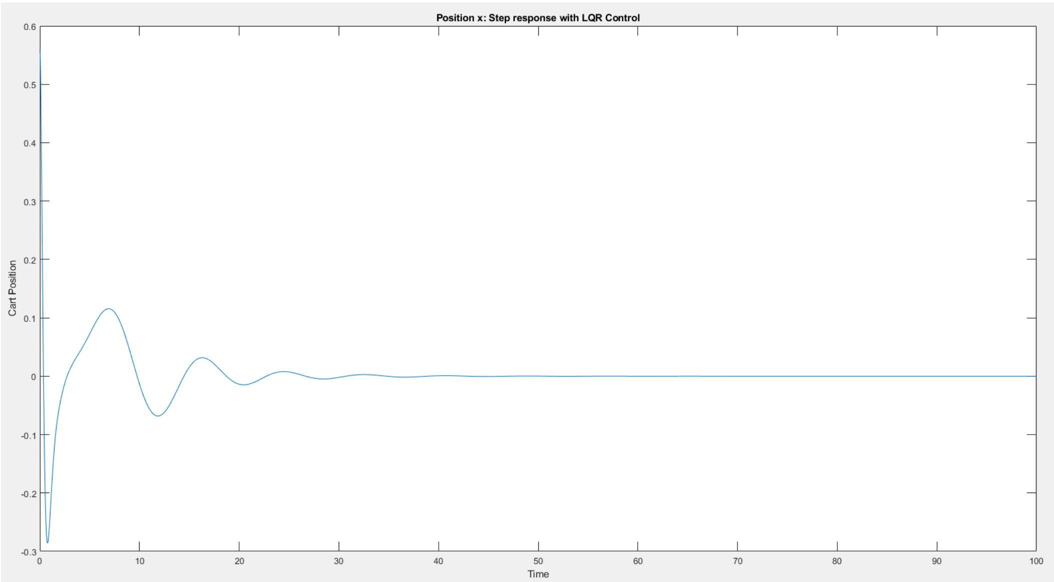

Fig. 2. Position x: Response of LQR controller to initial conditions for linearized system.

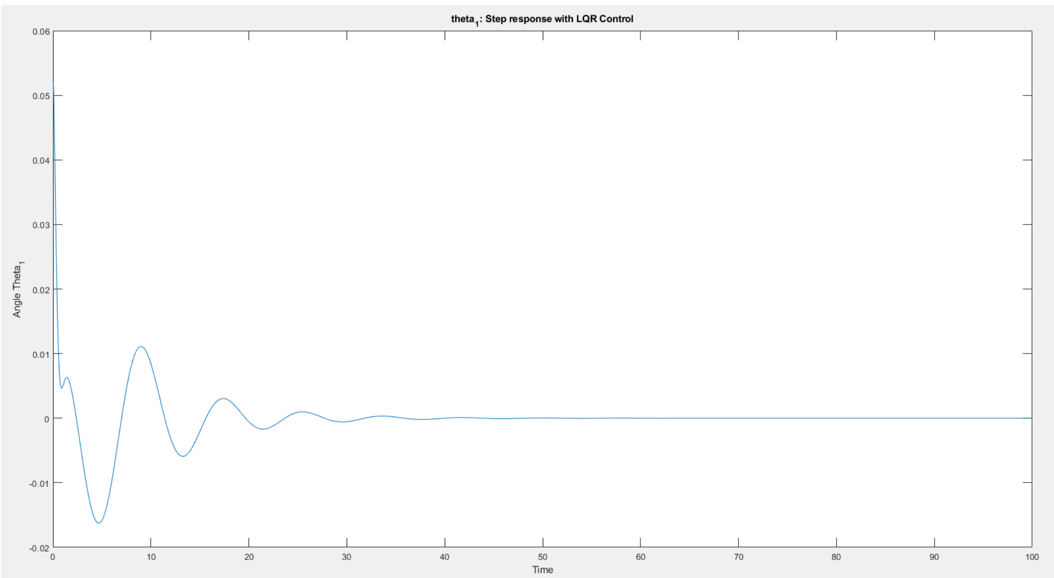

Fig. 3. Angle1 Response of LQR controller to initial conditions for linearized system.

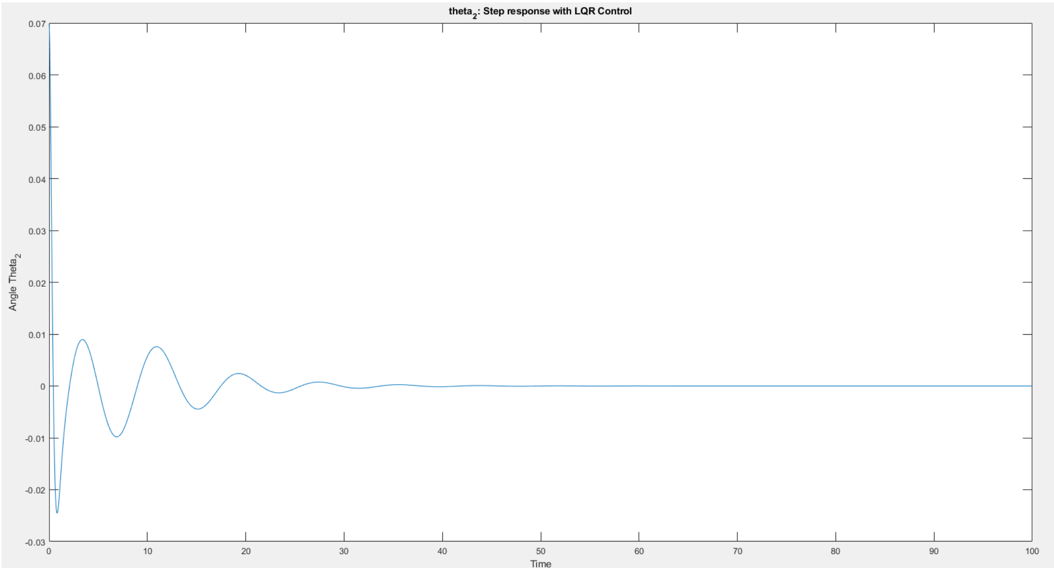

Fig. 4. Angle2 Response of LQR controller to initial conditions for linearized system.

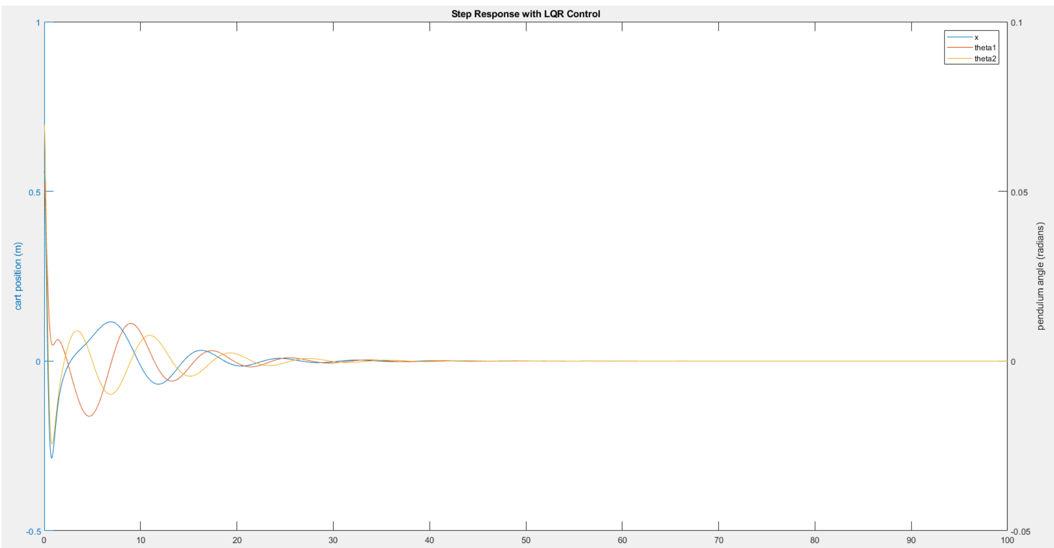

Fig. 5. Response of LQR controller to initial conditions for linearized system.

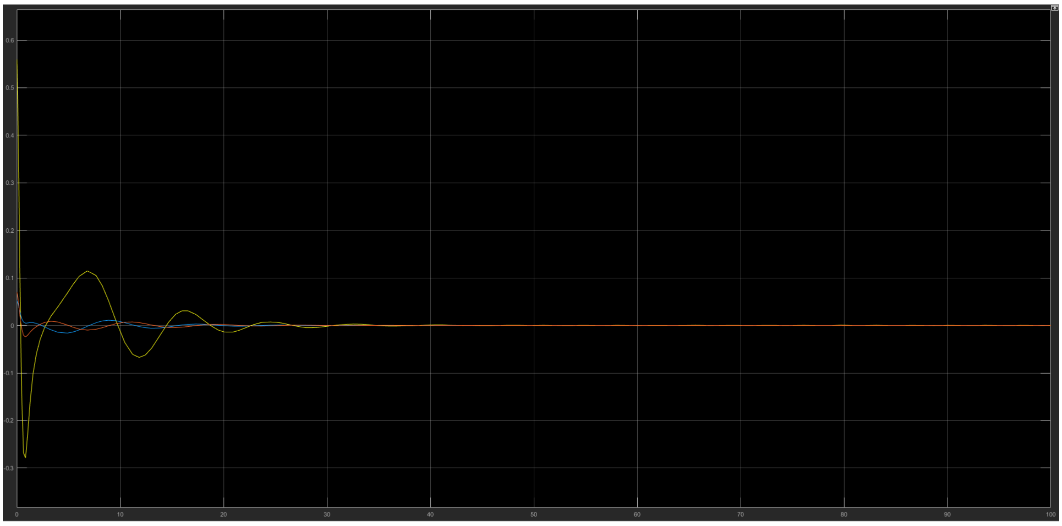

Fig. 6. Response of LQR controller to initial conditions for non-linear system.

The Simulink block diagram for LQR control of the non linear system is shown in Figure 7

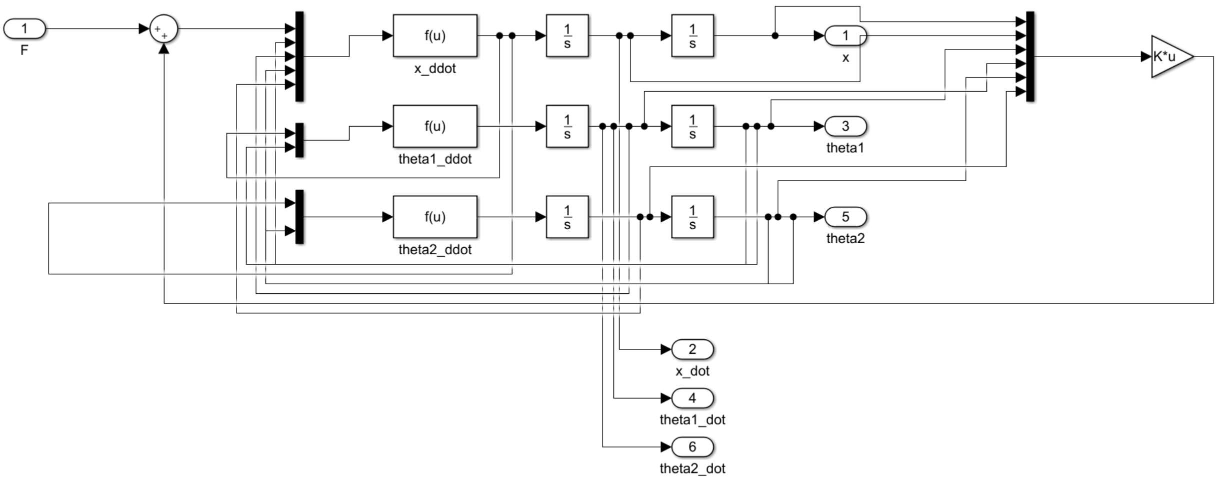

Fig. 7. Simulink block diagram to implement LQR on non linear system.

5 Observability

Given four output vectors the output vectors for which the linearized system is observable have to be found out. The condition for observability is that the observability matrix is full ranked i.e.
rank([$C^T \quad A^TC^T \quad (A^T)^2C^T \quad (A^T)^3C^T \quad .... \quad (A^T)^{n-1}C^T]) = n$
where n=6 for our system

1. Output $x(t)$ Hence the value of C for this output vector is $C1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Calculation from MATLAB yields the result that

$rank([C_1^T \quad A^TC_1^T \quad (A^T)^2C_1^T \quad (A^T)^3C_1^T \quad (A^T)^4C_1^T \quad (A^T)^5C_1^T \quad]) = 6$

Hence, the linearized system is **observable** for the output $x(t)$

2. Output $\theta_1(t), \theta_2(t)$ Hence the value of C for this output vector is $C2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Calculation from MATLAB yields the result that

rank([$C_2^T \quad A^TC_2^T \quad (A^T)^2C_2^T \quad (A^T)^3C_2^T \quad (A^T)^4C_2^T \quad (A^T)^5C_2^T \quad]) = 4$

Hence, the linearized system is **not observable** for the output $\theta_1(t), \theta_2(t)$

3. Output $x(t), \theta_2(t)$ Hence the value of C for this output vector is $C3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Calculation from MATLAB yields the result that

rank([$C_3^T \quad A^TC_3^T \quad (A^T)^2C_3^T \quad (A^T)^3C_3^T \quad (A^T)^4C_3^T \quad (A^T)^5C_3^T \quad]) = 6$

Hence, the linearized system is **observable** for the output $x(t), \theta_2(t)$

4. Output $x(t), \theta_1(t), \theta_2(t)$ Hence the value of C for this output vector is $C4 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Calculation from MATLAB yields the result that

rank([$C_4^T \quad A^TC_4^T \quad (A^T)^2C_4^T \quad (A^T)^3C_4^T \quad (A^T)^4C_4^T \quad (A^T)^5C_4^T \quad]) = 6$

Hence, the linearized system is **observable** for the output $x(t), \theta_1(t), \theta_2(t)$

6 Luenberger Observer

Using the theory from Luenerger Observer and the equations 26, 27

$$\dot{X}\hat{(t)} = A\hat{X}(t) + B_K U_K(t) + L(Y(t) - C\hat{X}(t)(26)$$

$$\dot{X}_e(t) = (A - LC)X_e(t) + B_D U_D(t) \tag{27}$$

We get the value of L by using the pole placement technique which MATLAB provides in the function $place(A^T, C^T, P)$ where P is the poles that we have selected. Using these values and the equations we have we get the observer running.

Taking an initial state conditions as $\begin{pmatrix} \frac{14}{25} \\ 0 \\ \frac{\pi}{45} \\ 0 \\ \frac{\pi}{45} \\ 0 \end{pmatrix}$ with poles at $\begin{pmatrix} -\frac{103}{250} & -\frac{41}{100} & -\frac{21}{50} & -\frac{43}{100} & -\frac{11}{25} & -\frac{9}{20} \end{pmatrix}$, $\begin{pmatrix} -\frac{13}{10} & -\frac{7}{5} & -\frac{3}{2} & -\frac{8}{5} & -\frac{17}{10} & -\frac{9}{5} \end{pmatrix}$, $\begin{pmatrix} -\frac{13}{10} & -\frac{7}{5} & -\frac{3}{2} & -\frac{8}{5} & -\frac{17}{10} & -\frac{9}{5} \end{pmatrix}$ for outputs with C as C_1, C_2, C_3 respectively and a force input of 50*unitstep i.e. 50 units of force we get the following results for the values of C as C_1, C_3, C_4 we found out in the observabilty checks for previous section.

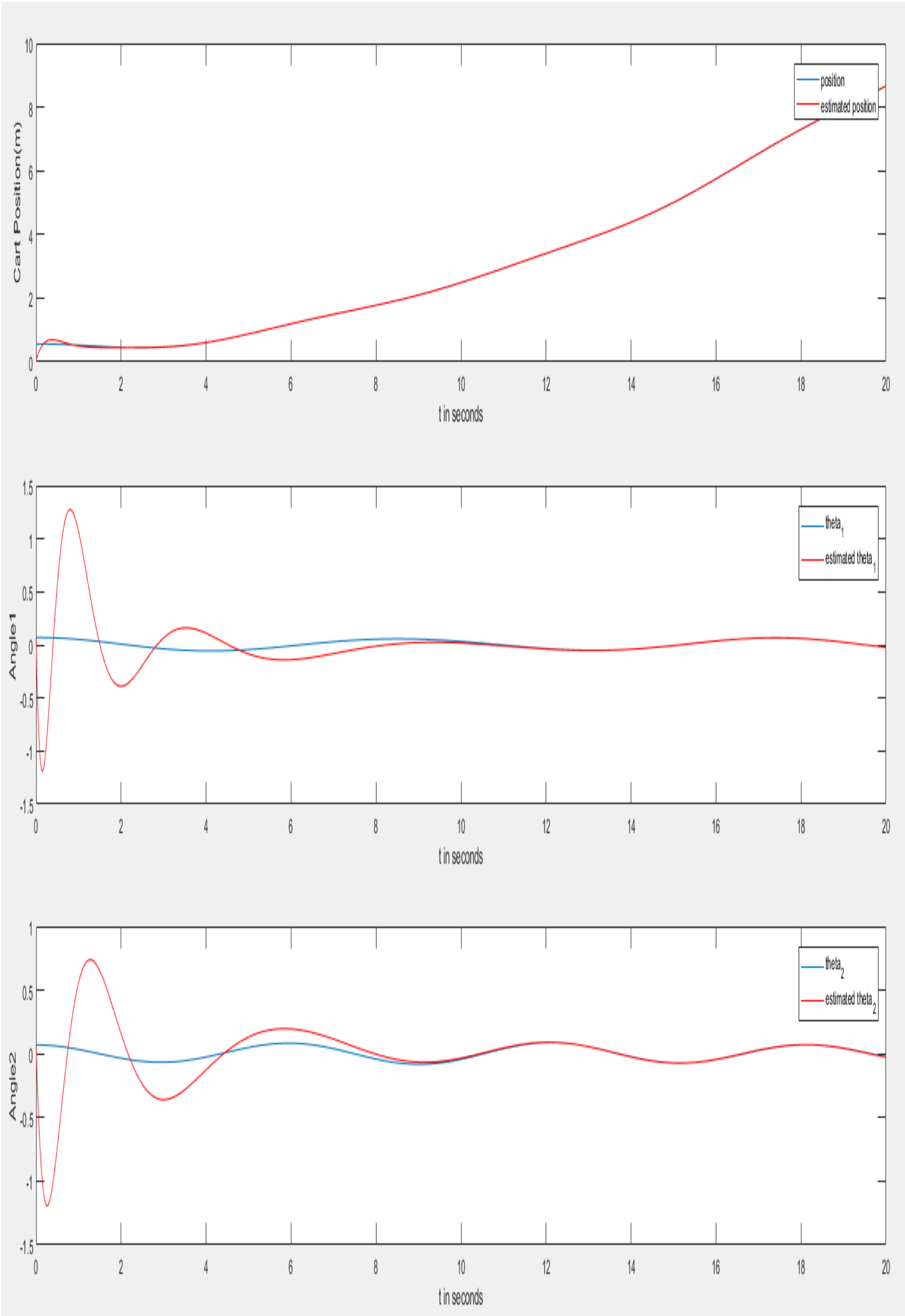

Fig. 8. Actual and estimated state with initial conditions for linearized system where C is C_1.

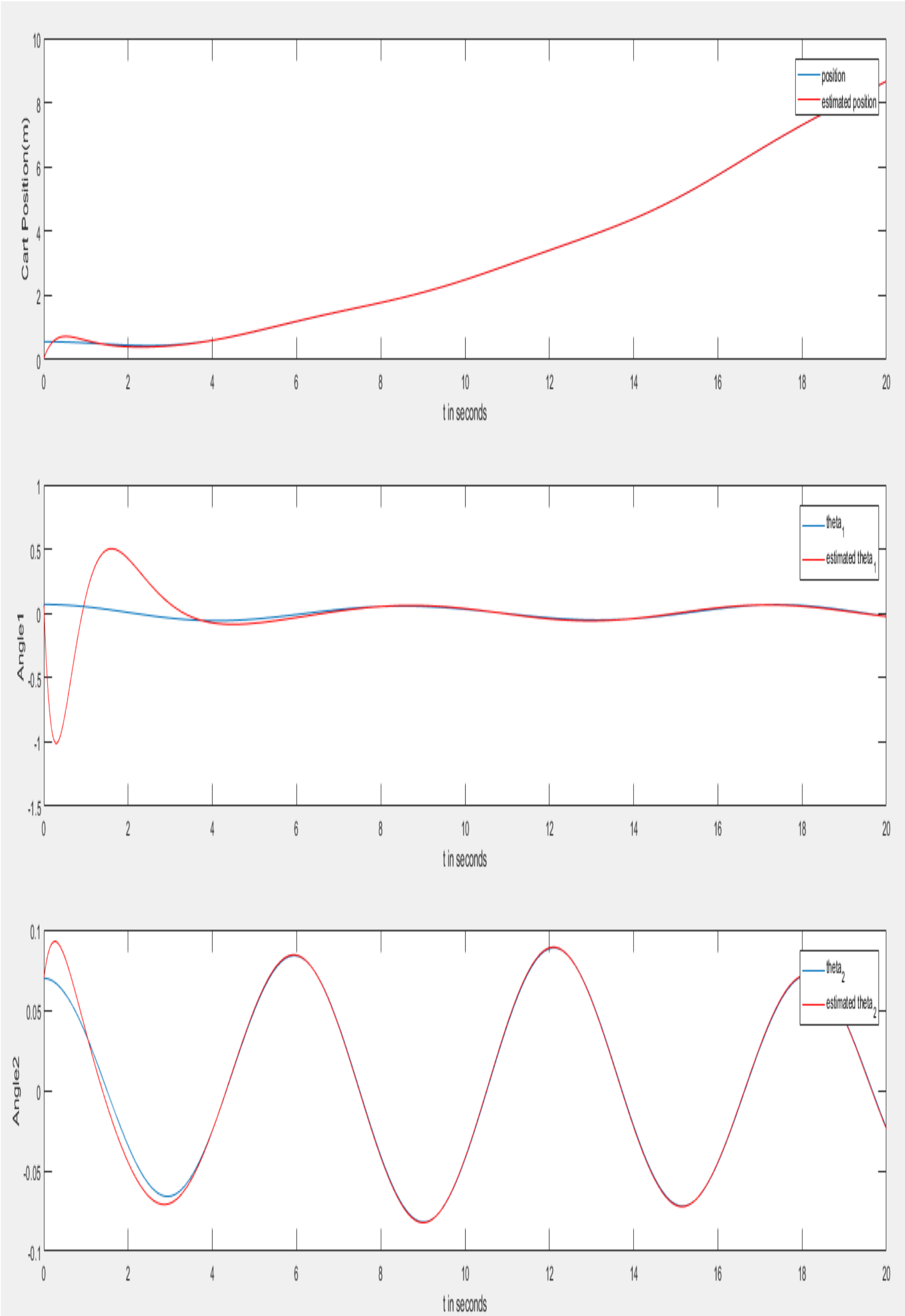

Fig. 9. Actual and estimated state with initial conditions for linearized system where C is C_3.

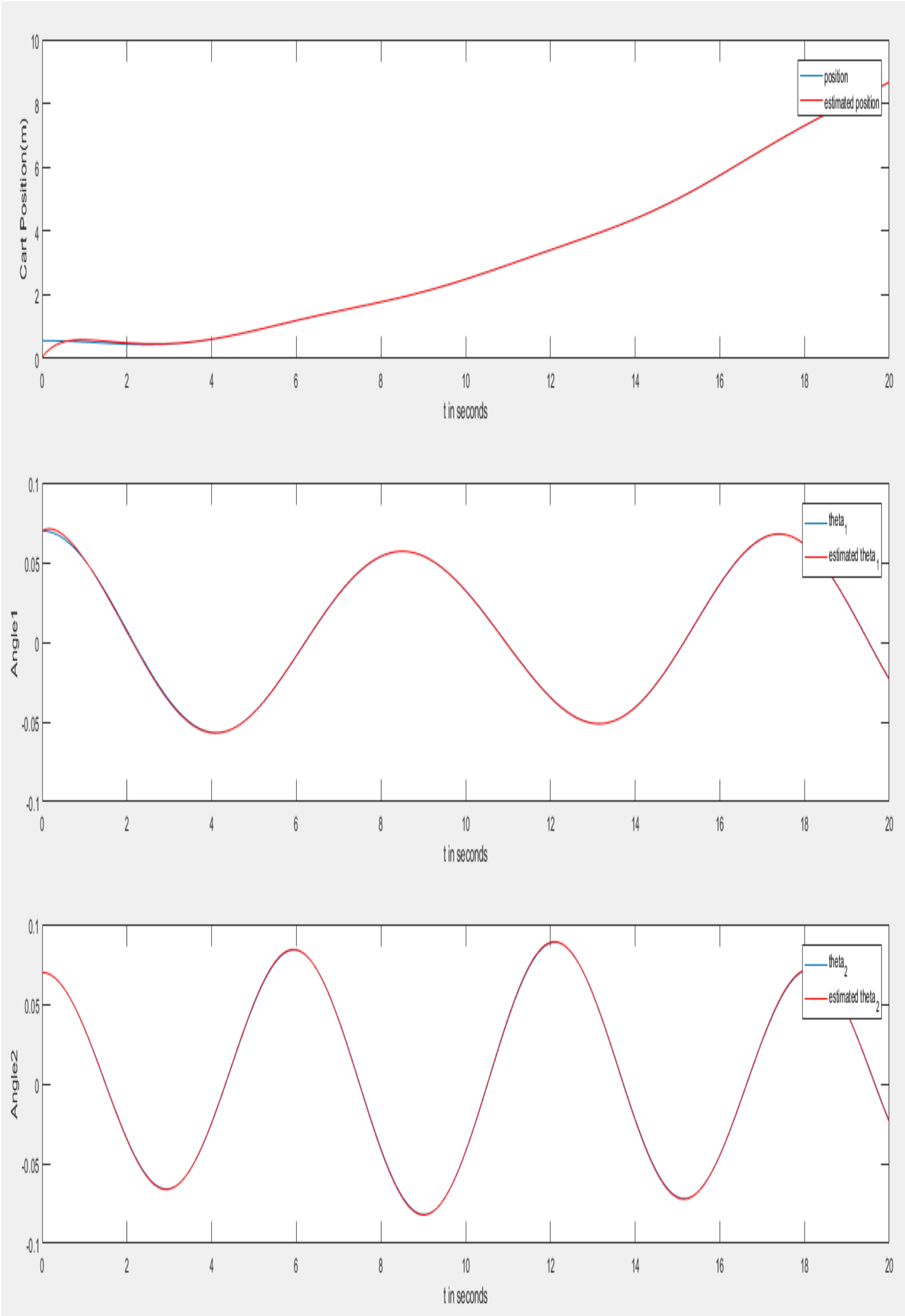

Fig. 10. Actual and estimated state with initial conditions for linearized system where C is C_4.

For simulating the results for the non linear system, the following simulink block was used

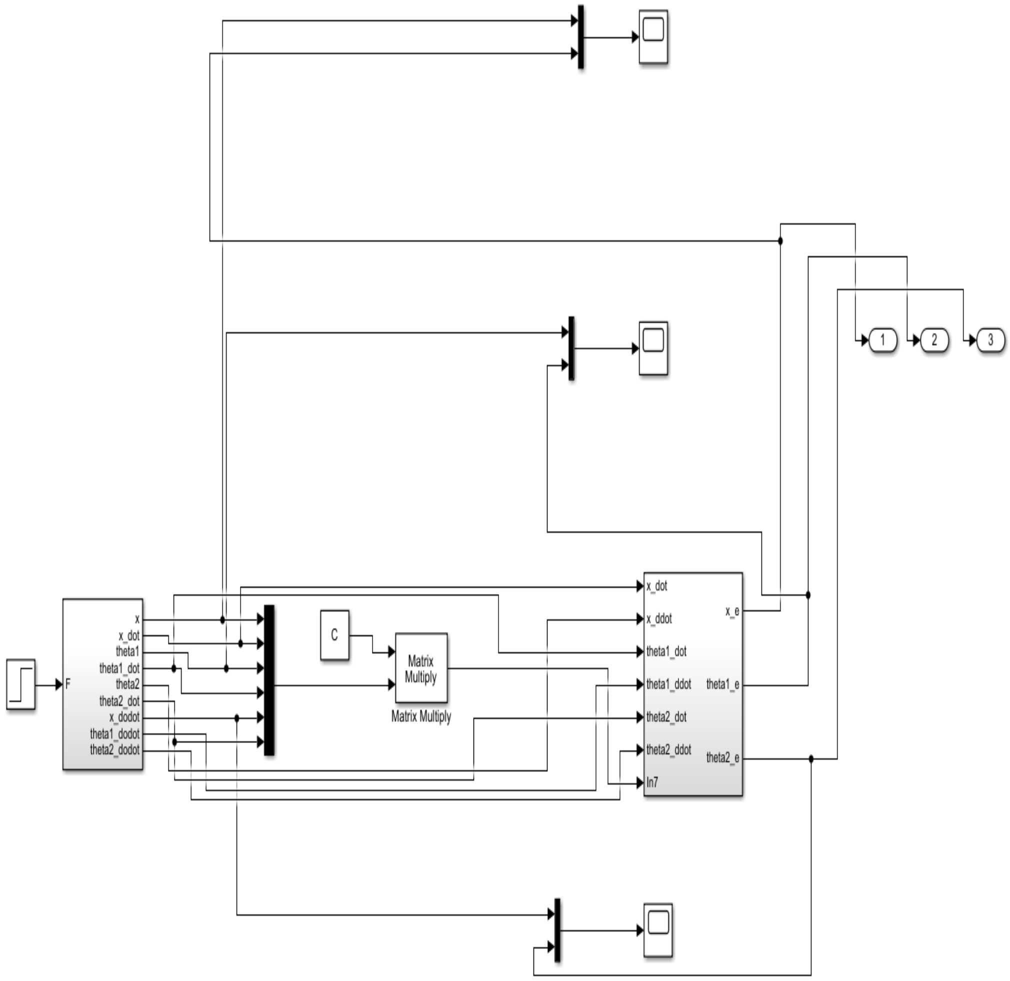

Fig. 11. Simulink model for non linear system with luenberger observer.

The estimator subsystem is shown in the below given figure

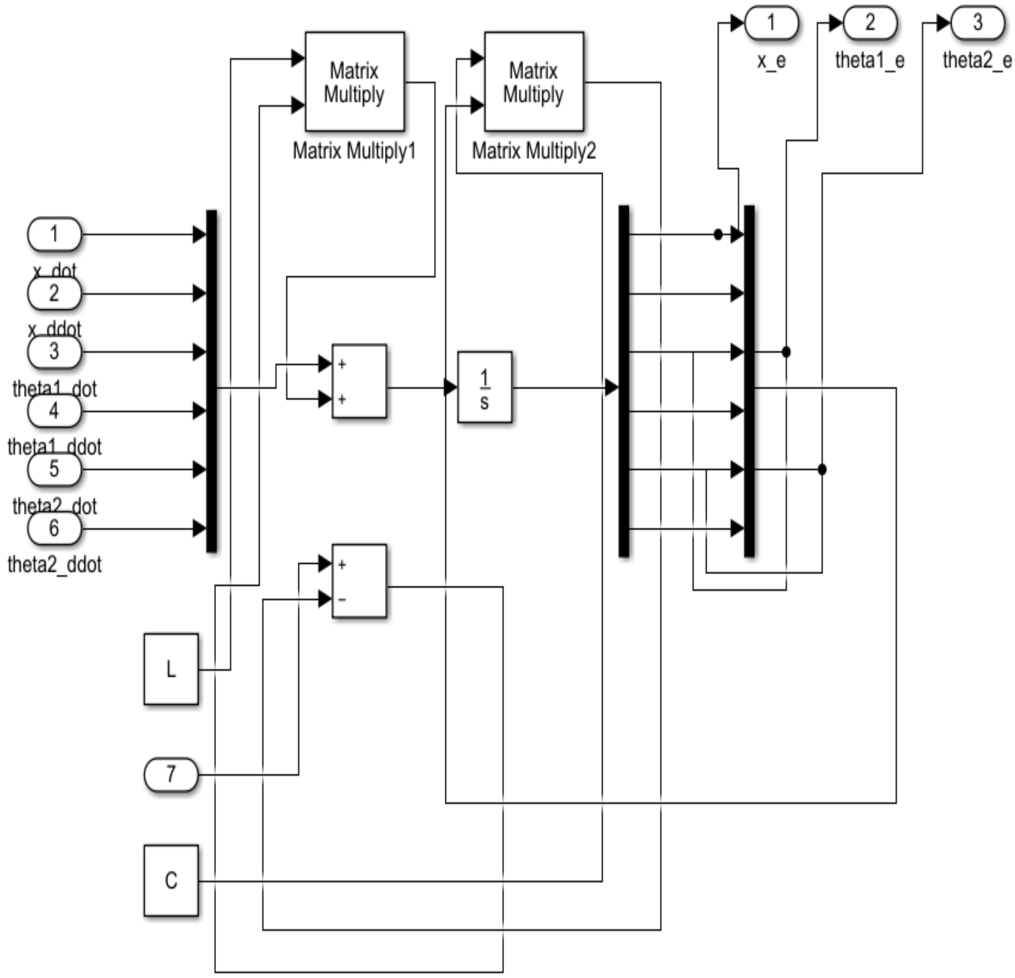

Fig. 12. Simulink model for estimator subsystem.

The plots of the state and estimated state with initial conditions for the non linear system are:

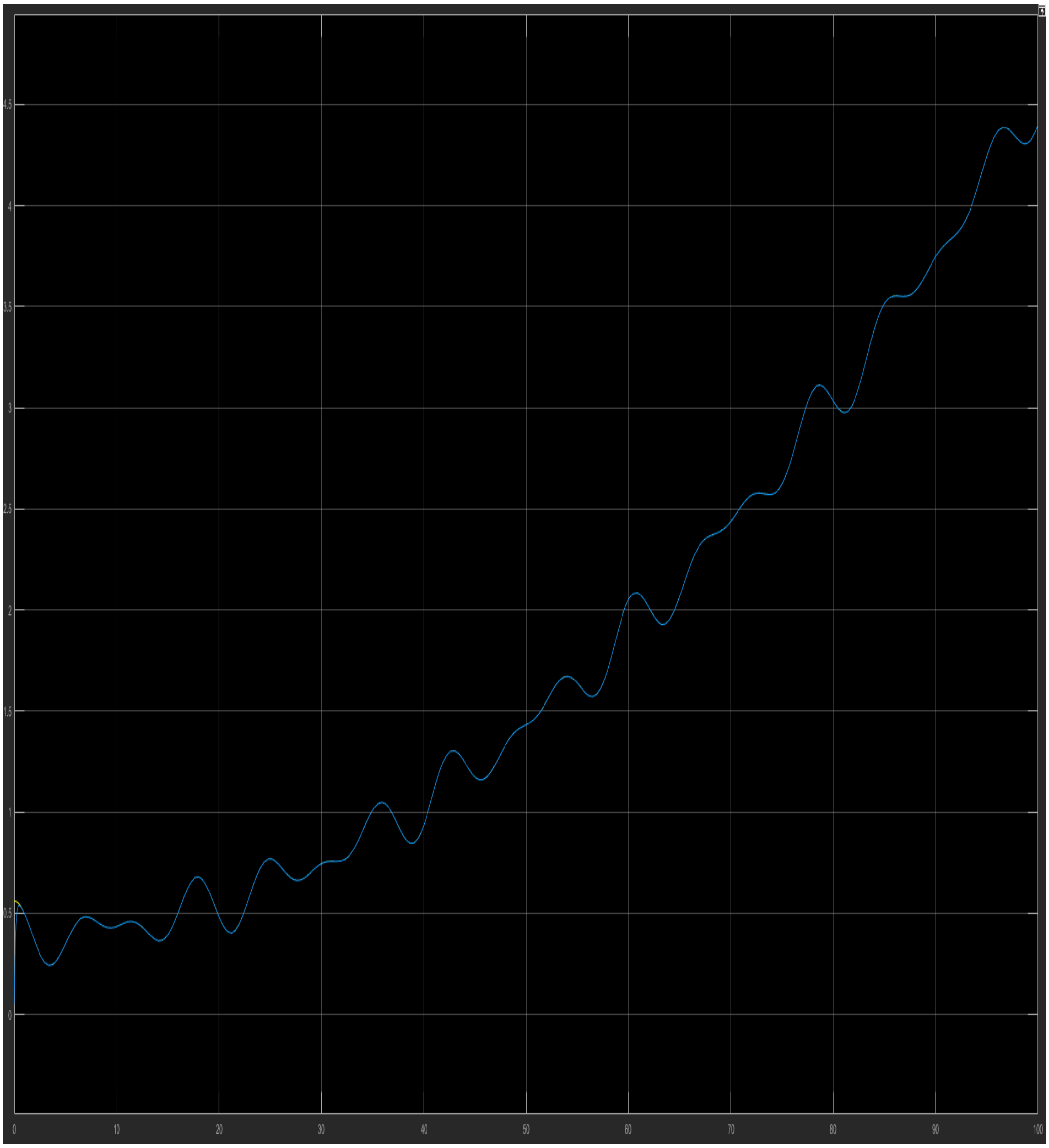

Fig. 13. Actual and estimated position for linearized system where C is C_1.

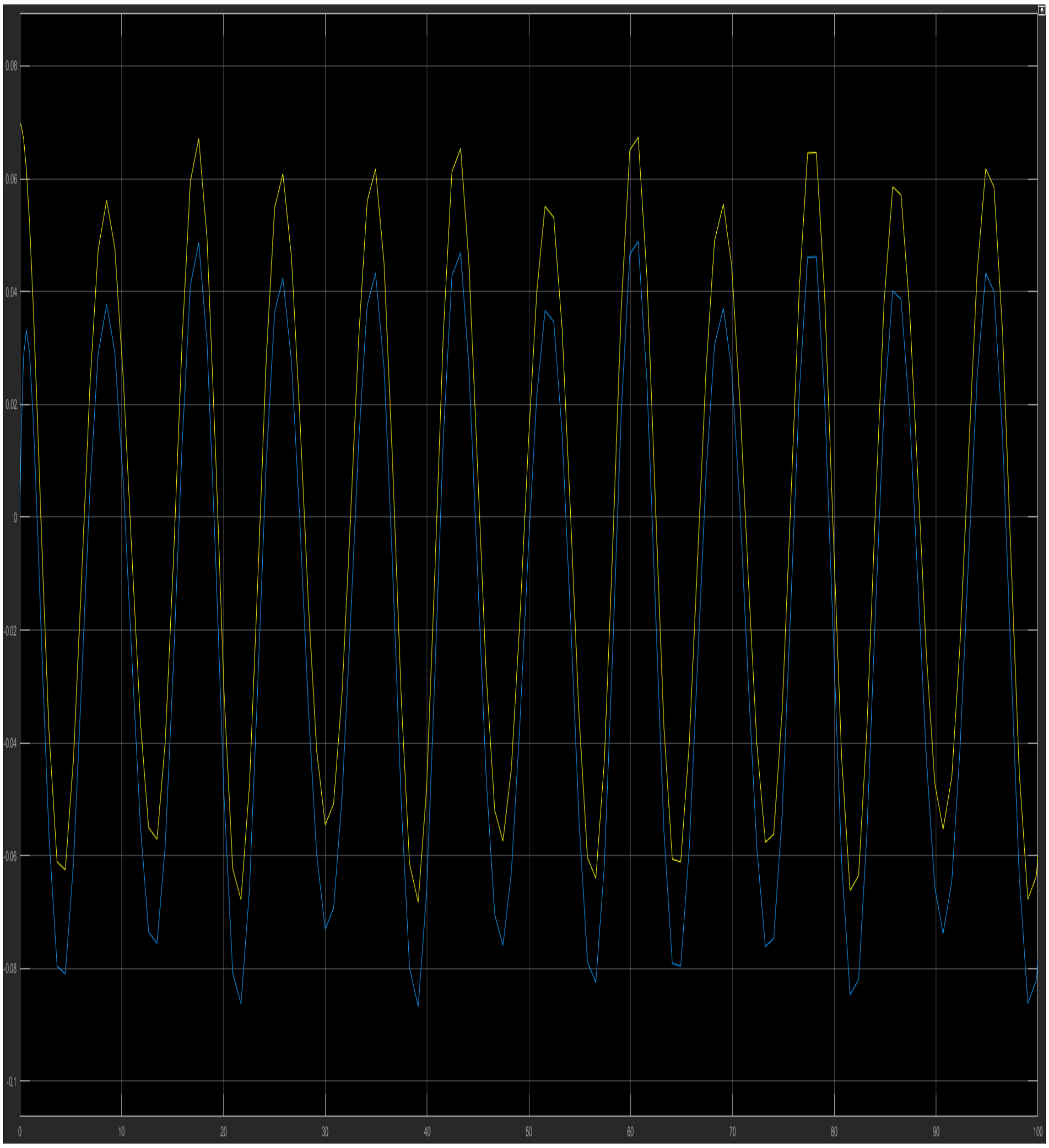

Fig. 14. Actual and estimated angle1 for linearized system where C is C_1.

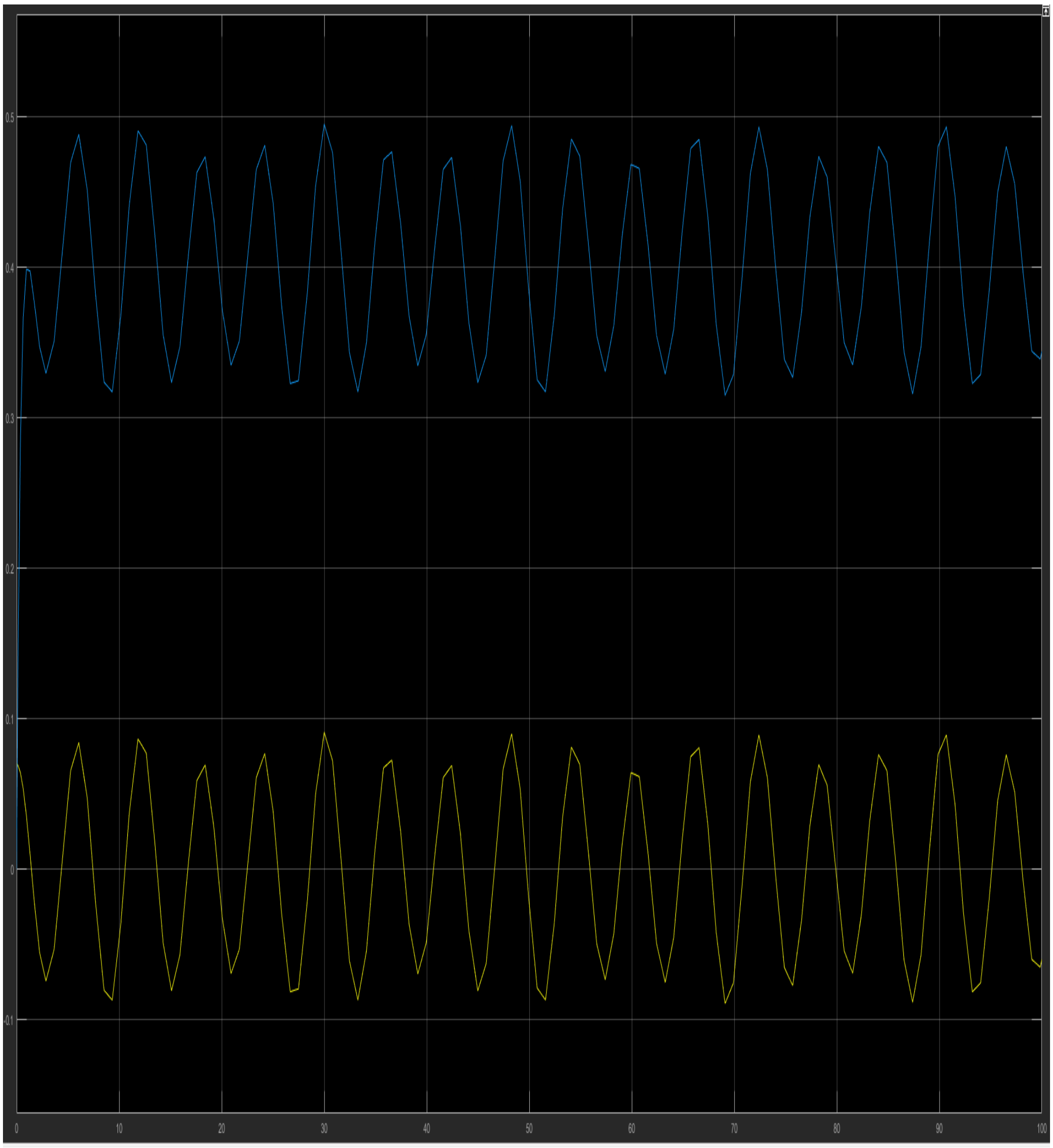

Fig. 15. Actual and estimated angle2 for linearized system where where C is C_1.

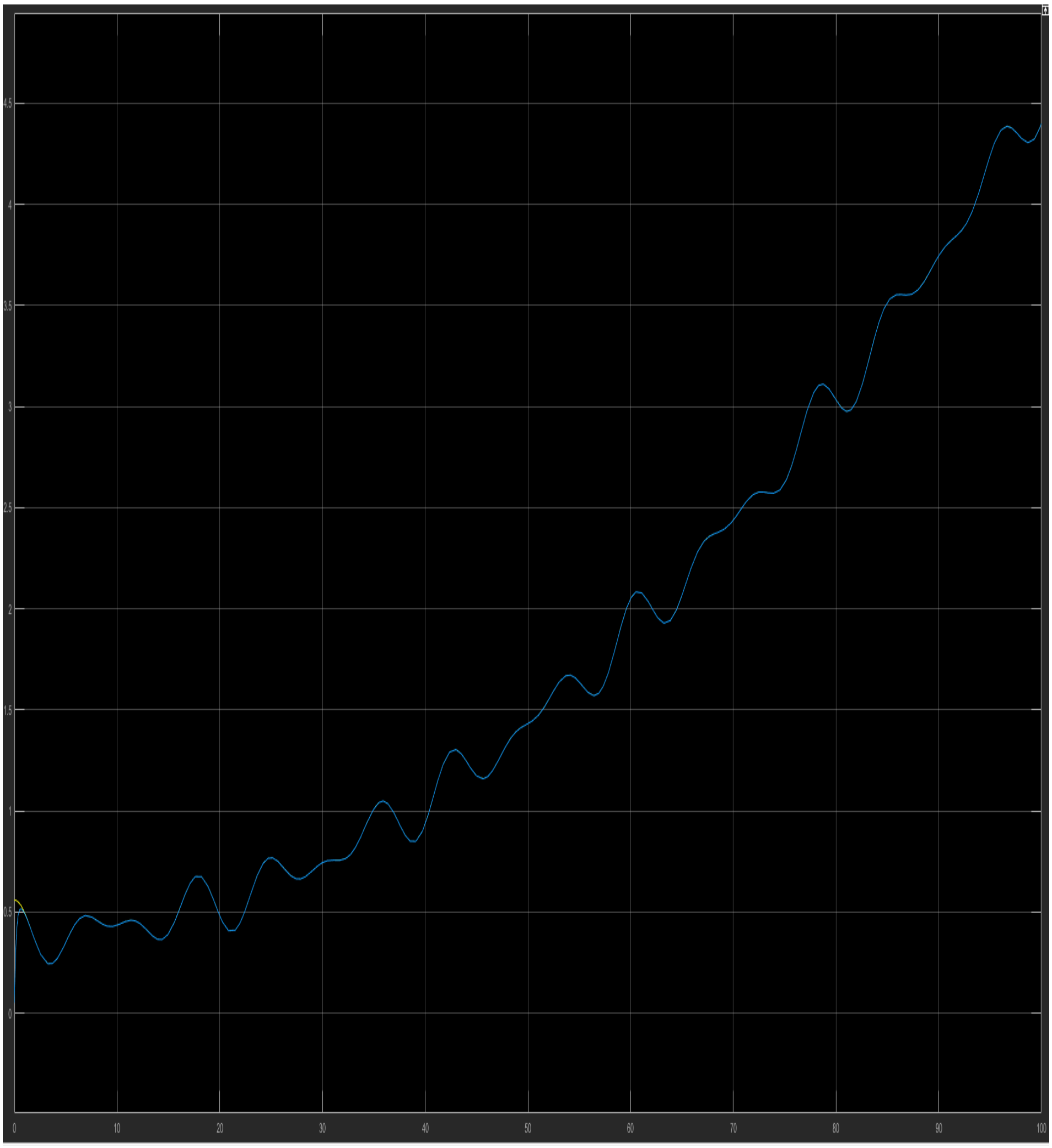

Fig. 16. Actual and estimated position for linearized system where C is C_3.

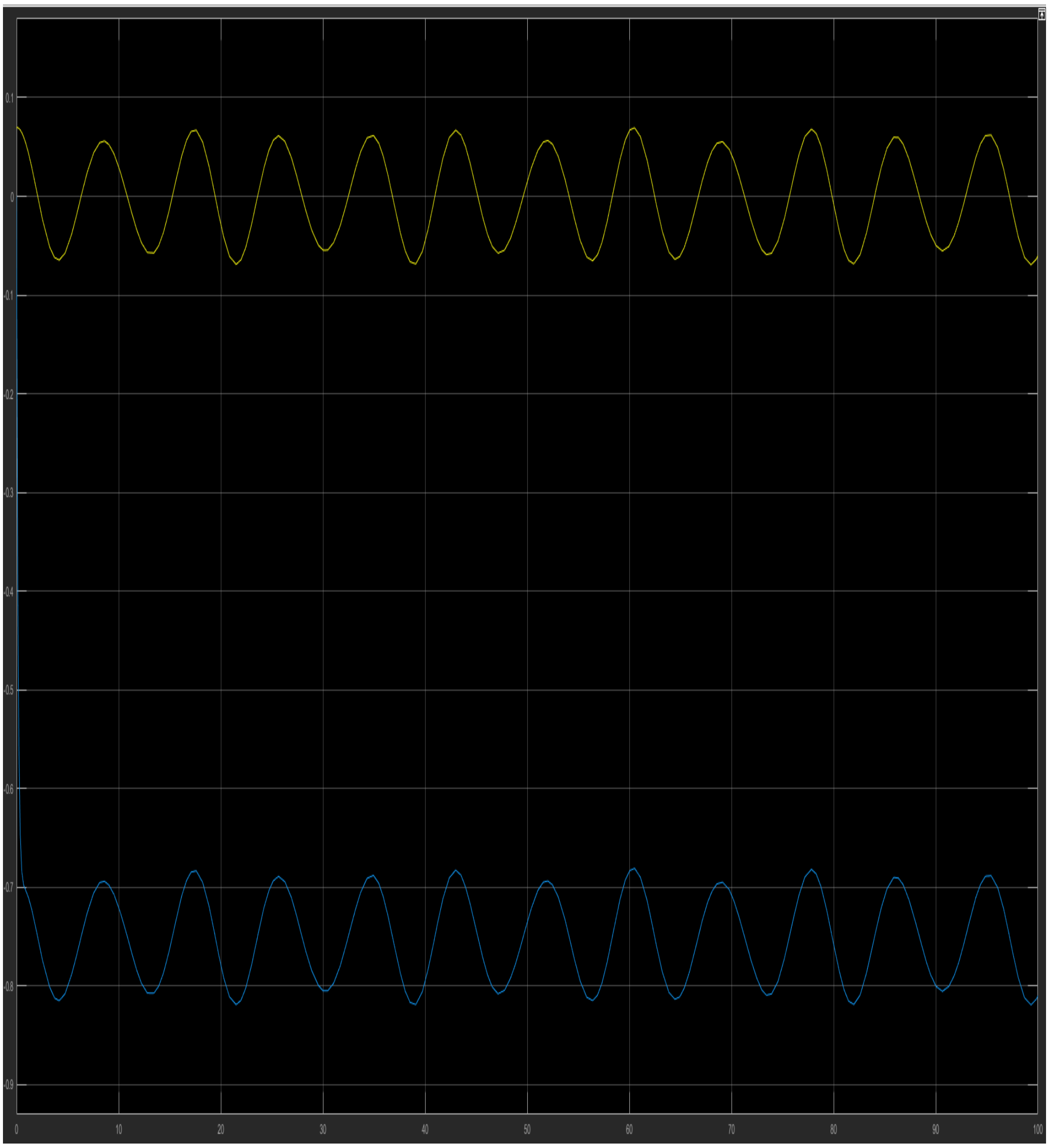

Fig. 17. Actual and estimated angle1 for linearized system where C is C_3.

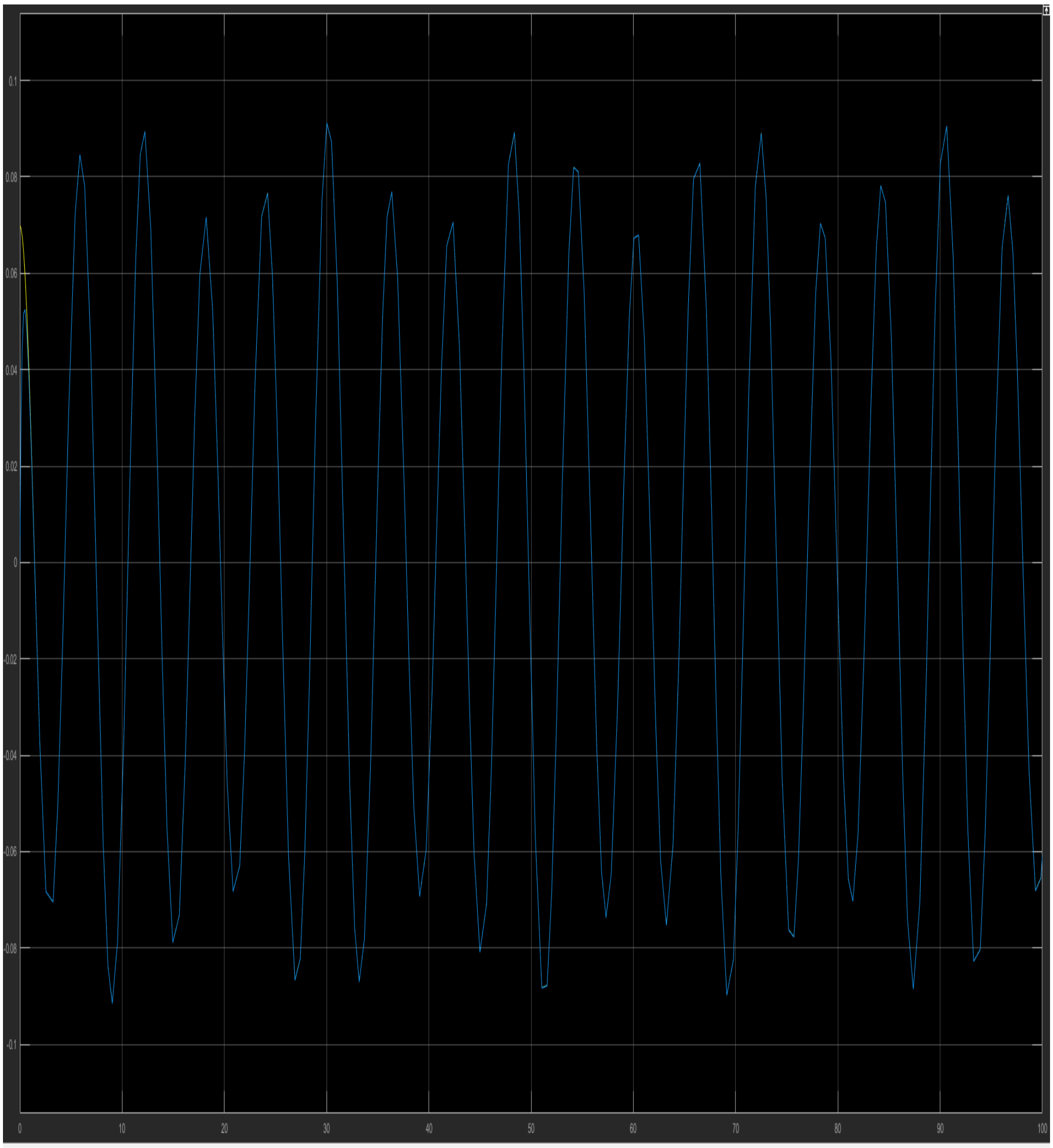

Fig. 18. Actual and estimated angle2 for linearized system where C is C_3.

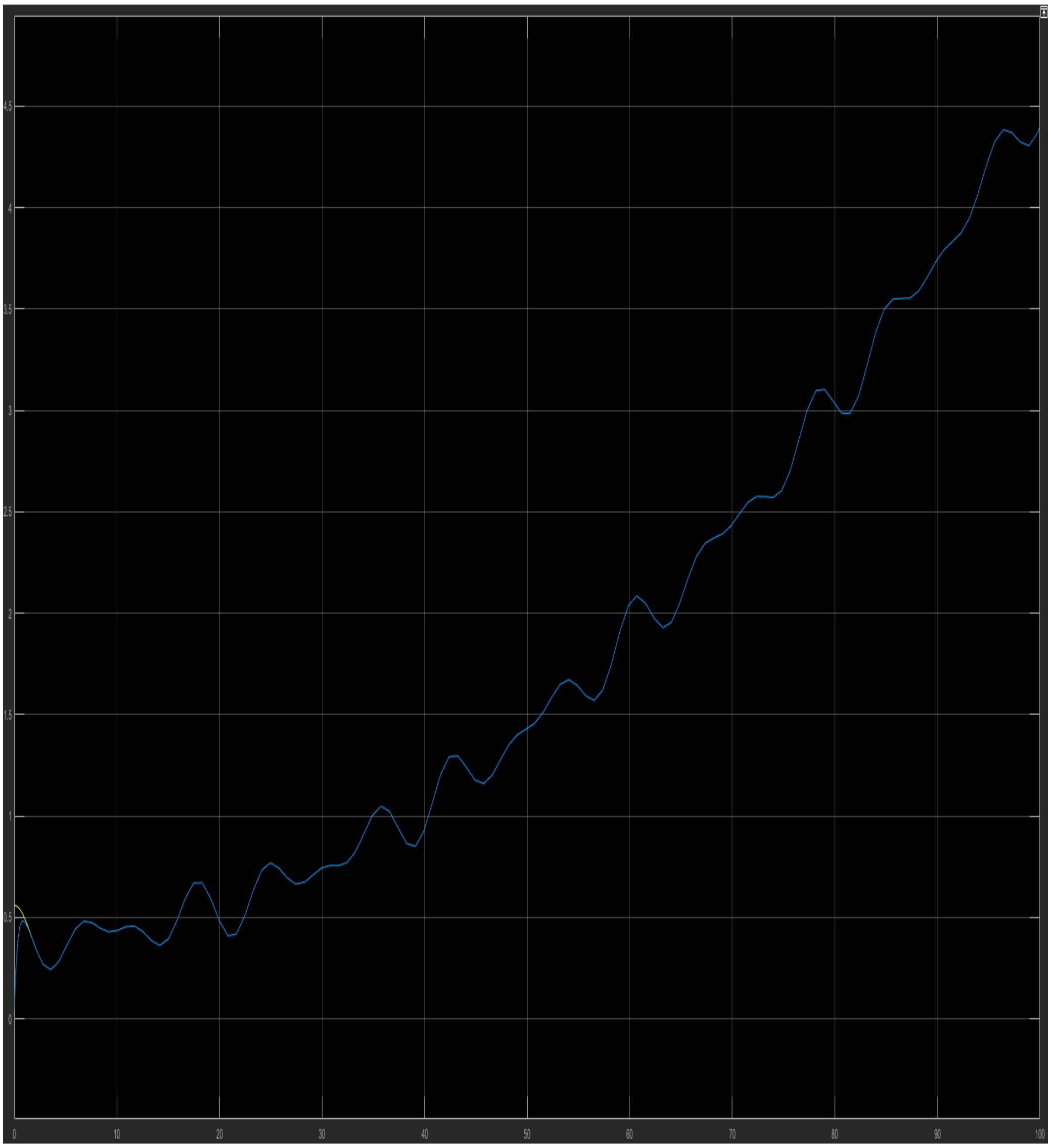

Fig. 19. Actual and estimated position for linearized system where C is C_4.

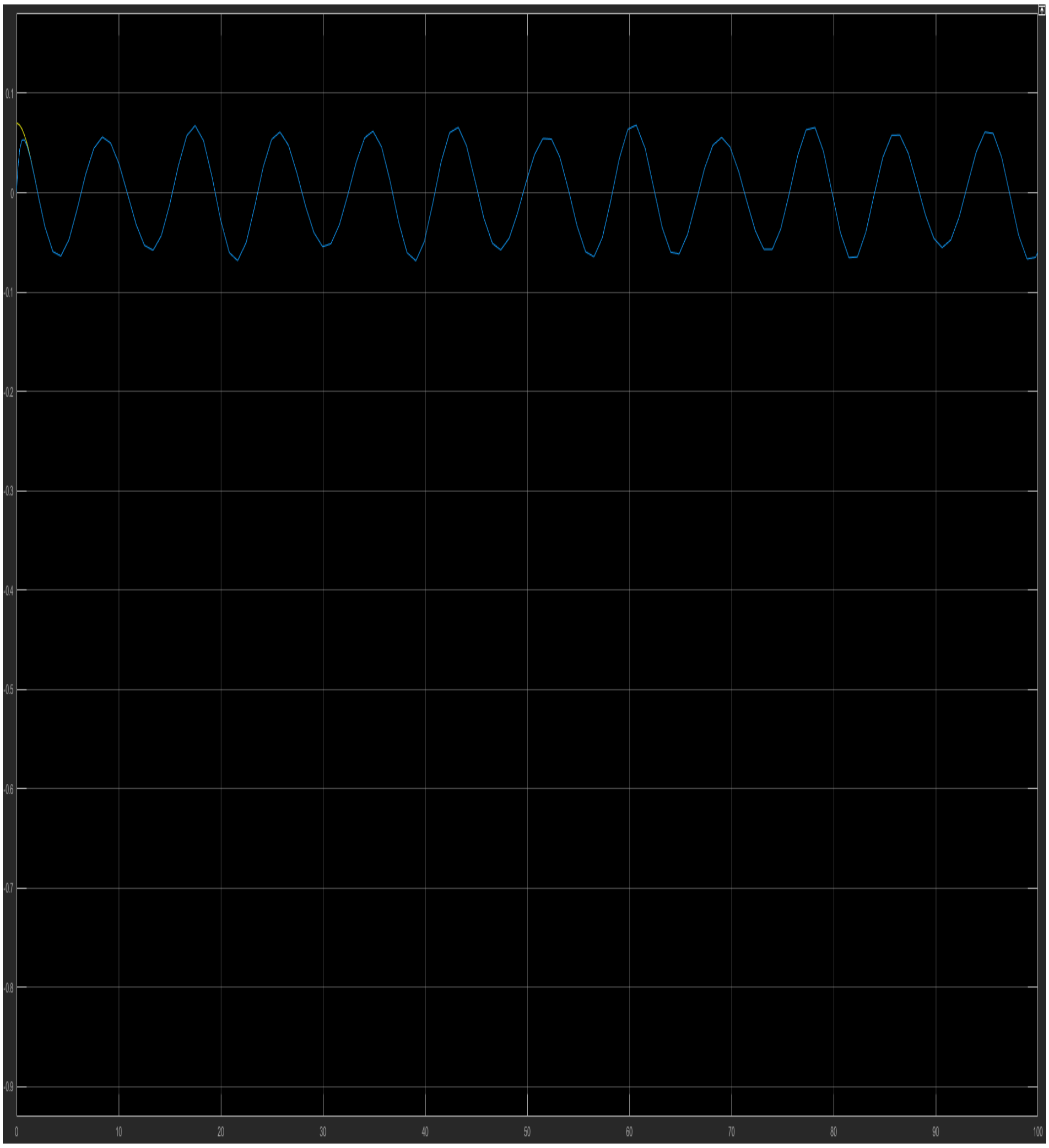

Fig. 20. Actual and estimated angle1 for linearized system where C is C_4.

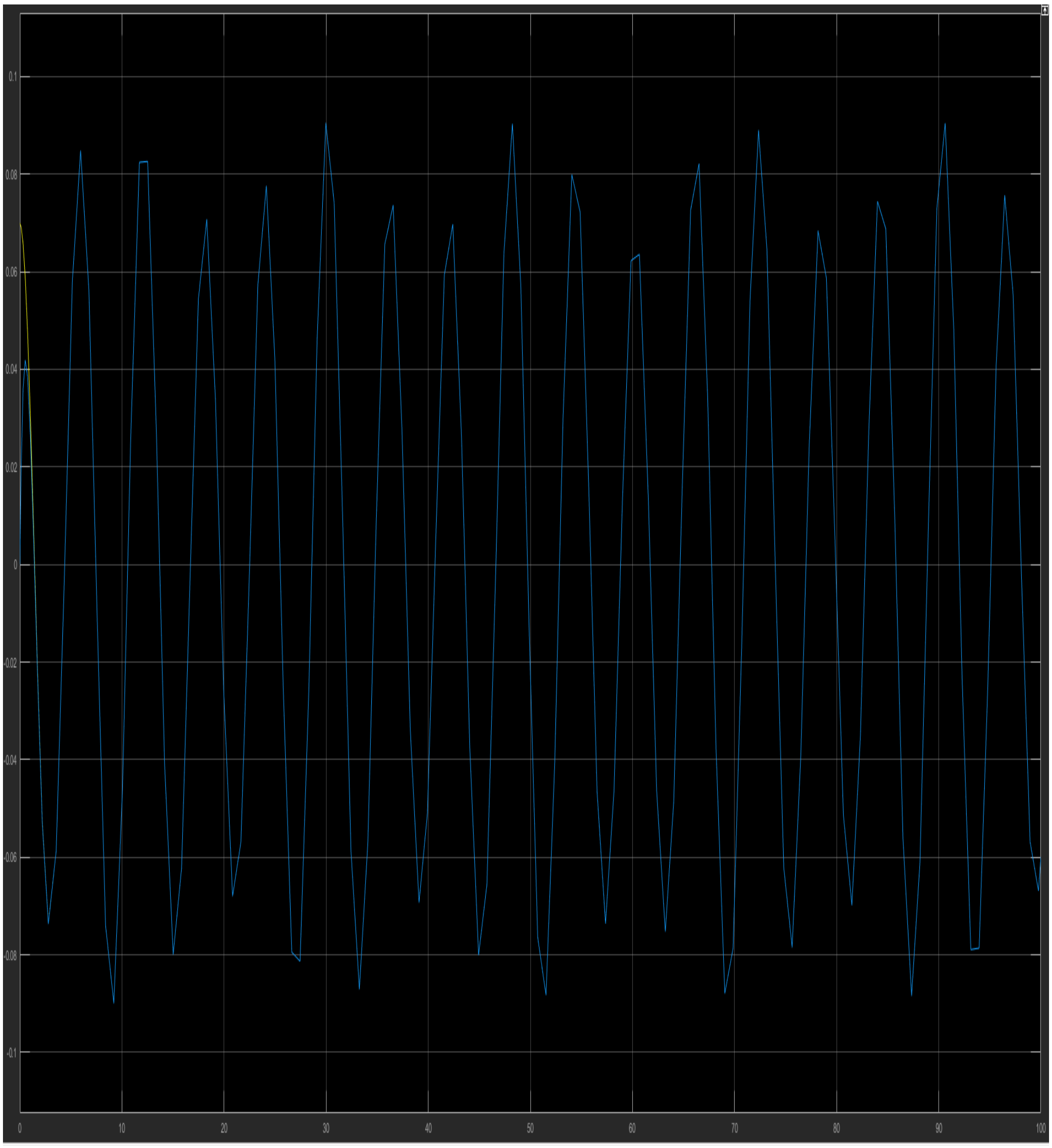

Fig. 21. Actual and estimated angle2 for linearized system where C is C_4.

For Figures 15, 17 we observe that the state estimation differs by a constant offset. This is because we use the technique derived for a linearized system directly on the non linearized version. To get better estimates of the states for a non linear system, other advanced techniques specific to non linear systems must be used.

7 Linear Quadratic Gaussian (LQG) Regulator

Linear Quadratic Gaussian problems in control systems is one of the core optimal control concepts. It consists of linear systems having some state information (all the state variables are not visible for feedback) and the system is disturbed by additive white Gaussian noise. To solve this problem, the LQG controller uses a combination of the estimator and the linear quadratic regulator. The separation principal allows us to combine these two independently and then design the LQG regulator to solve the problem.

For a continuous system given by;

$$\begin{aligned} \dot{X} &= Ax + Bu + w \\ y &= Cx + Du + v \end{aligned} \tag{28}$$

where w is the process noise and v is the Gaussian white noise, the LQG regulator minimizes the below given cost function:

$$J = E \lim_{\tau \to \infty} \int_0^{\tau} [x^T, u^T] Q_{xu} [x, u]^T \, dt \tag{29}$$

and v and w have co-variance

$$E([w, v]^T . [w^T, v^T]) = QWV \tag{30}$$

To track a constant reference on x, the we use the Servo Controller Design with the Linear Quadratic Gaussian (LQG) Regulator we already have. This is done by basically adding an integral component.

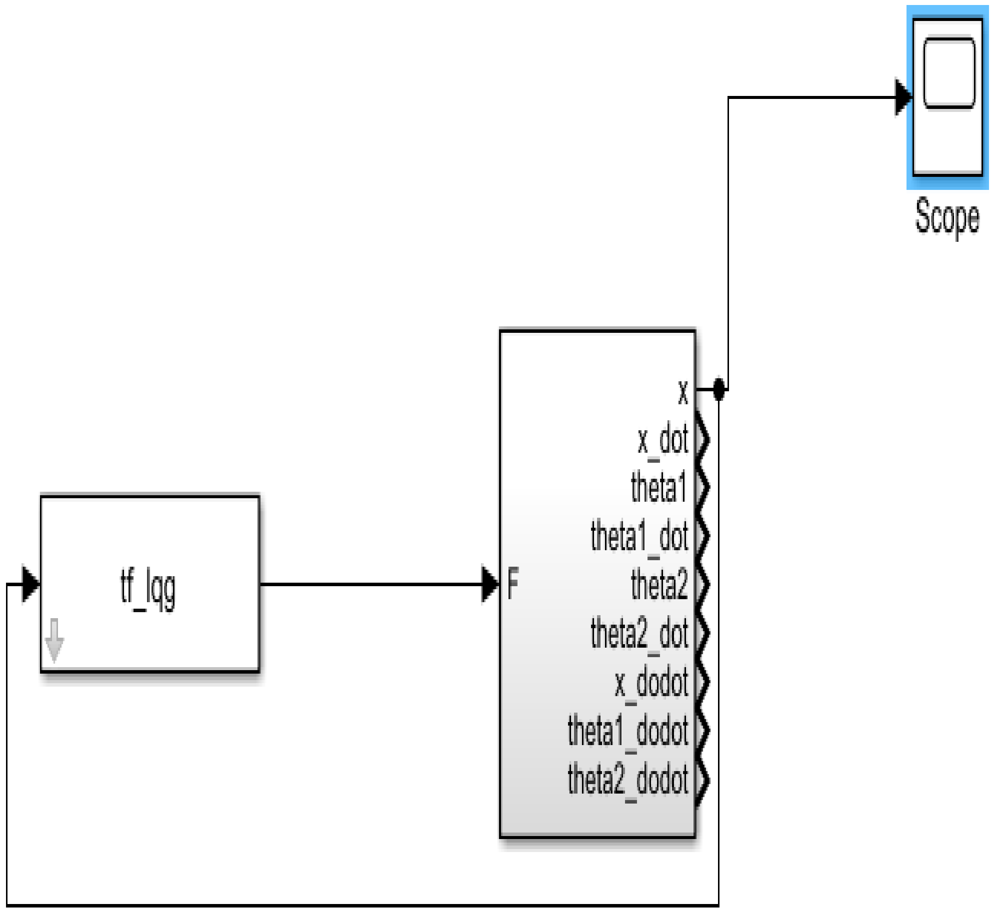

Fig. 22. LQG simulink diagram.

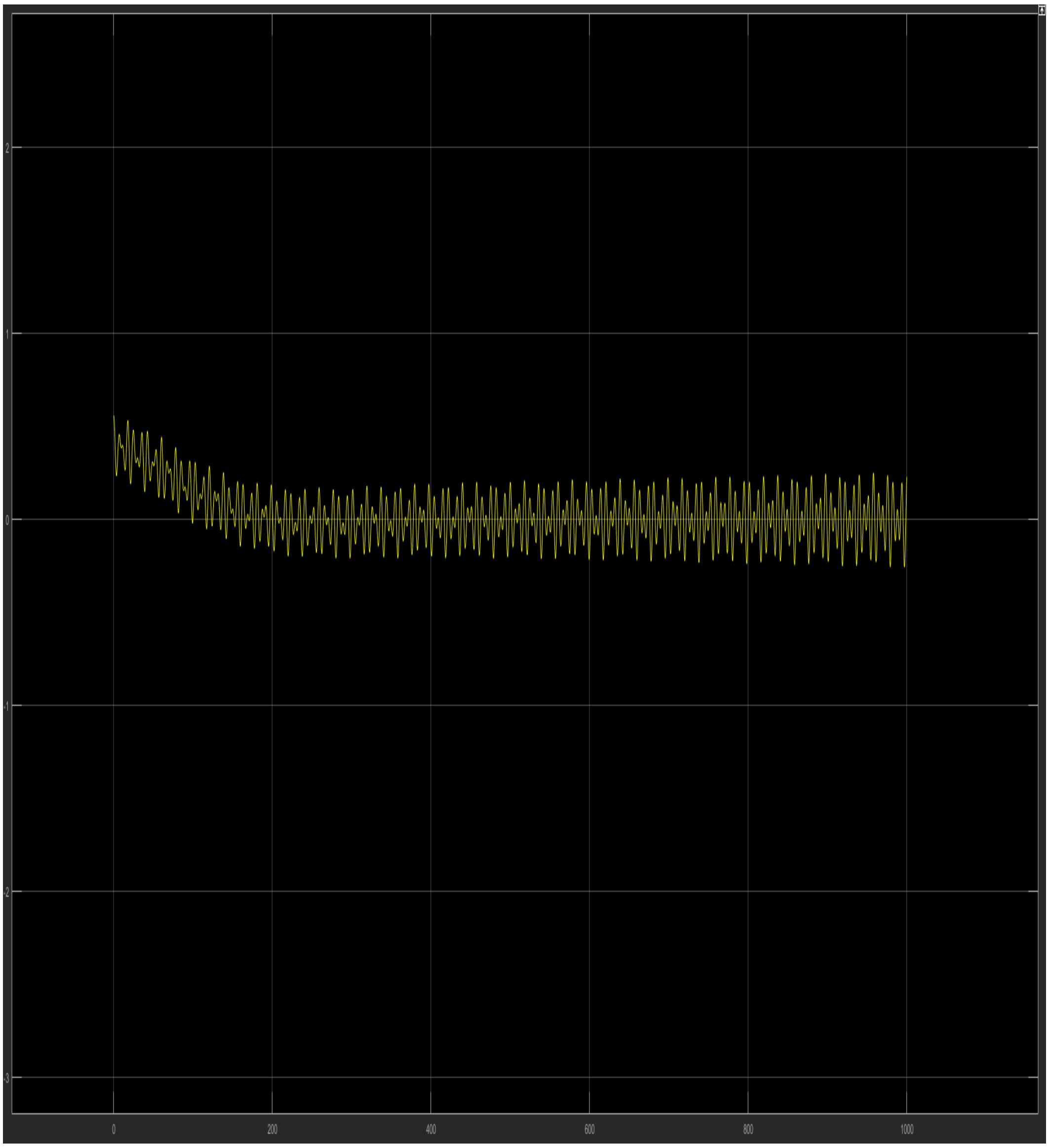

Fig. 23. LQG output.

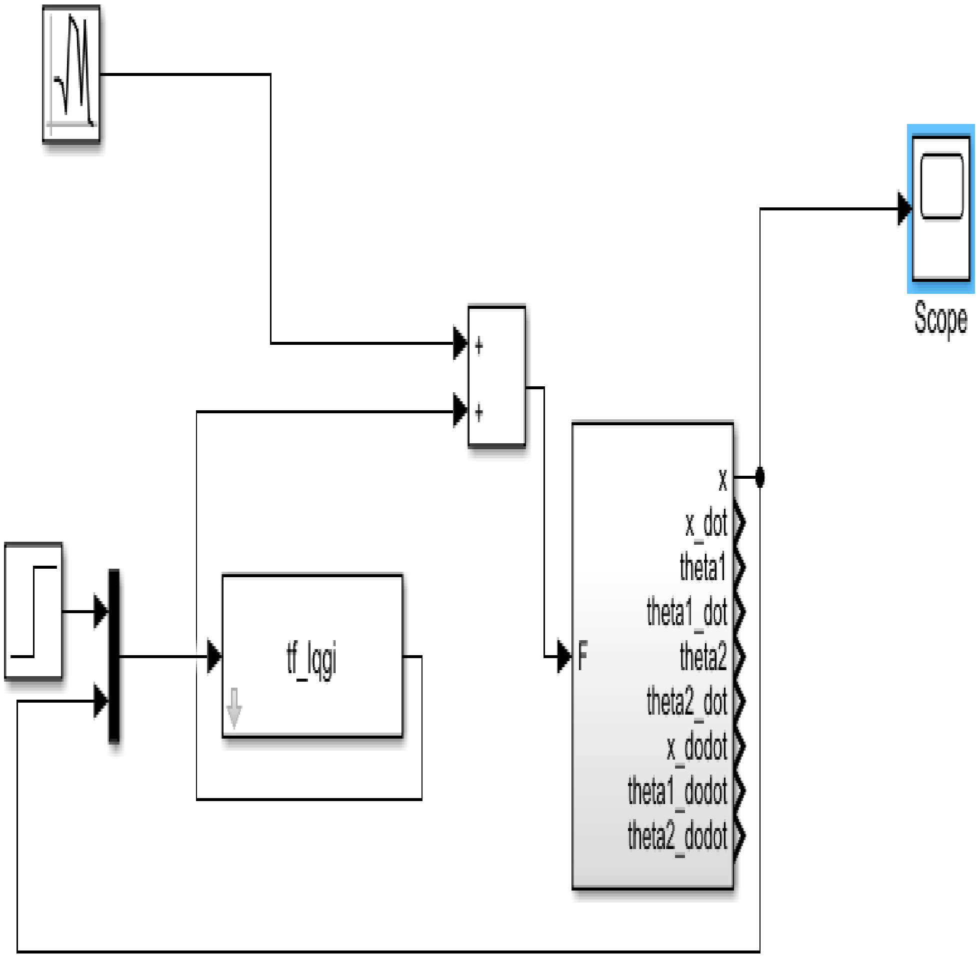

Fig. 24. LQG with servo control simulink diagram.

Fig. 25. LQG with servo control output.

What we can understand from these plots is that the reference **is being tracked** but the design needs a finer tuning (the Qn needs finer tuning as currently it is a random 6x6 matrix) so that the oscillations cease to exist. In Figure 25 we see that the reference on x is still being tracked and on an Gaussian noise fed to the input Force, the system is still being able to track a constant reference for x. Also, using the techniques we implemented on linearized systems for non linear systems is not a good practice for best results.

References

[1] http://ctms.engin.umich.edu/CTMS/

[2] ENPM 667 Lecture Slides and Notes.

[2] MATLAB Control System Toolbox Documentation.