An Introduction to Chaotic Maps

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Abstract - Chaotic behavior can be observed in a wide range of systems around us. Being the most erratic, it turns out that chaotic behavior can arise even from very simple systems. These systems can be as simple as first-order difference equations which find applications in biology, physics and economics. In order to understand this random behavior, we analyze one-dimensional maps such as the logistic map and tent map. The trajectory produced in the cobweb plot by iterating the map, gives stable points, periodic oscillations and random behavior. The map gives a pitchfork bifurcation. We plotted the time series which provides a qualitative view of chaos. Further, the Lyapunov exponent is defined which provides the quantitative measure of chaos.

Keywords - Chaos, First-order difference equations, One-dimensional maps, Cobweb plot, Stable points, Iteration, Bifurcation, Time series, Fourier transformation, Lyapunov exponent

Introduction

Chaotic behavior can be observed in a variety of systems ranging from fluctuations in animal population to those in stock markets. In actual dynamical systems like these, there are a lot of variables which determine these fluctuations. Therefore, it may seem very obvious about utter disorderliness in these systems. However, it was found out that this complicated behavior could arise even from extremely simple mathematical models such as first-order difference equations. [1]

Since long, it has been one of the goals of science, to predict the state of a system as time evolves. [3] And time evolves continuously. However, for 'mathematical simplicity', we shall restrict ourselves and look the system at discrete time intervals.

The Logistic Map

Let us consider a simple ecological system given by the mathematical model, $f(x) = kx$, where the population of a species at a given instant depends on the already existing population.

$$x_{n+1} = f(x_n)$$

Here the variable n stands for time, and x_n designates the population at time n. The evolution of this dynamical process is reflected by composition of the function f. [3] Let us define $f^r(x)$, to be the result of applying the function f to the initial state r times. For this system, it is clear that if the initial value of x is greater than zero, the population will grow without bound. This type of expansion, in which the population is multiplied by a constant factor r per unit of time, is called **exponential growth.**

Nevertheless, real habitats have finite resources and the population cannot keep increasing exponentially. At some point of time the population will get saturated and the growth will slow to something less than exponential. [1, 3]

An improved model, to be used for such resource-limited population, might be given by

$$g(x_n) = x_{n+1} = kx_n(1 - x_n)$$

When the population x is small, the factor $(1 - x)$ is close to one, and $g(x)$ closely resembles the function $f(x)$. On the other hand, if the population x is far from zero, then $g(x)$ is no longer proportional to the population x but to the product of x and the "remaining space" $(1 - x)$. This is a nonlinear effect, and the model given by $g(x)$ is an example of a **logistic growth model**. [3]

Definition 1: A function a **map** when the system is viewed at discrete time intervals. Let x be a point and let f be a map. The **orbit** of x under f is the set of points $\{x, f(x), f^2(x), \dots\}$. The starting point x for the orbit is called the **initial value** of the orbit. A point p is a **fixed point** of the map f if, $f(p) = p$. [3]

Stability of fixed points

A stable fixed point has the property that points near it are moved even closer to the fixed point under the dynamical system. For an unstable fixed point, nearby points move away as time progresses. A good analogy is that a ball at the bottom of a valley is stable, while a ball balanced at the tip of a mountain is unstable. The question of stability is significant because a real-world system is constantly subject to small perturbations. Therefore, a steady state observed in a realistic system must correspond to a stable fixed point. If the fixed point is unstable, small errors or perturbations in the state would cause the orbit to move away from the fixed point, which would then not be observed. [3]

Cobweb Plot of the logistic map

The cobweb plot is a rough plot of an orbit. [3] It is plotted between the initial value and the consequent value of x. For the logistic map the cobweb plot shows a very interesting behavior. The logistic map (illustrated in Fig.1) has a practical disadvantage. It requires x to remain in the interval $0 < x < 1$. If x exceeds 1 then the iterations diverge towards $-\infty$ (which means that the population becomes extinct) (Fig.2 and Fig.3). Moreover $g(x)$ attains a maximum value $k/4$ (at $x = 1/2$). The equation possesses a non-trivial dynamical behavior for all, $0 < k < 4$. Otherwise the population becomes extinct.

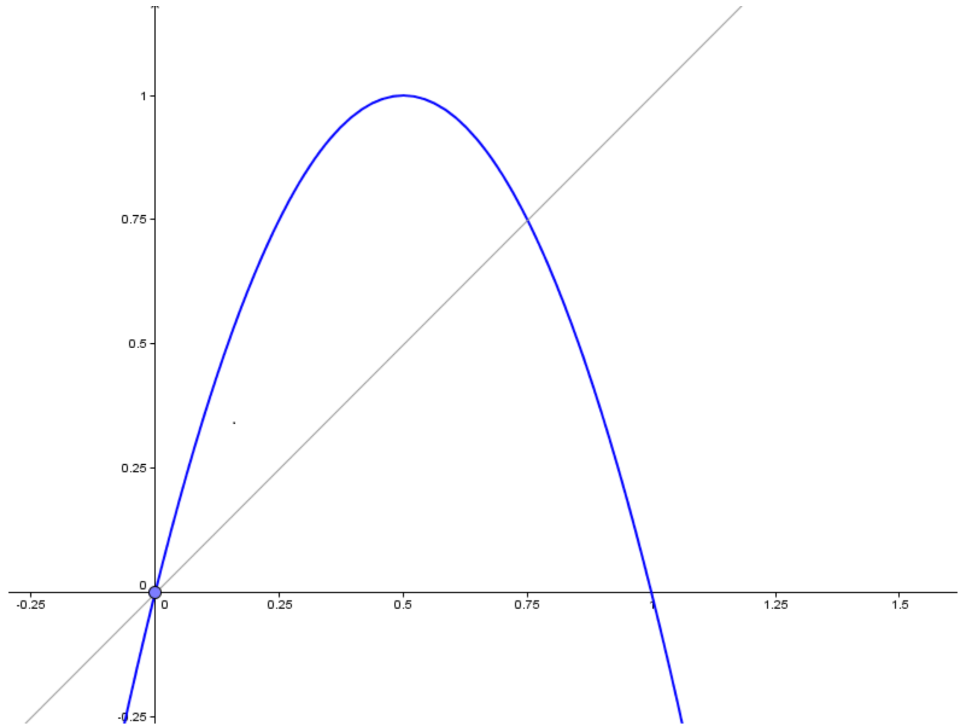

Figure 1

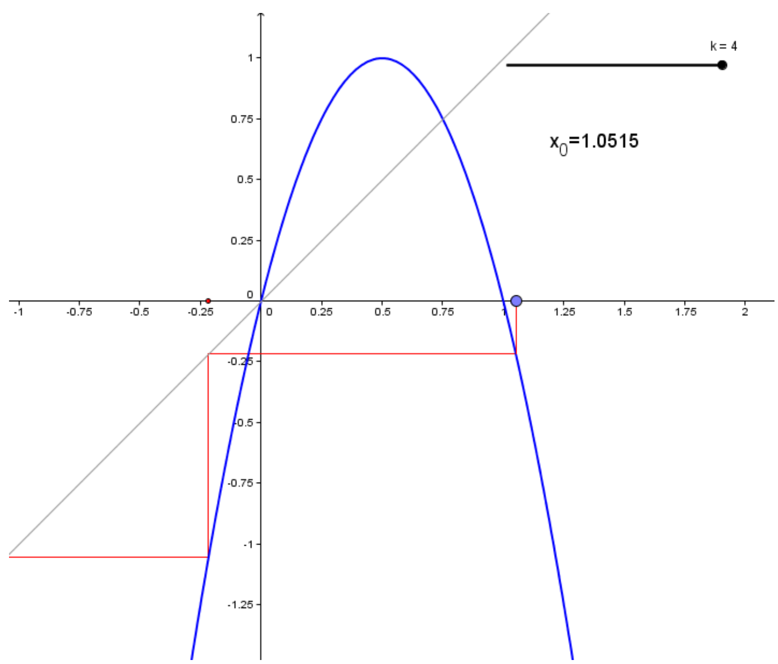

Figure 2

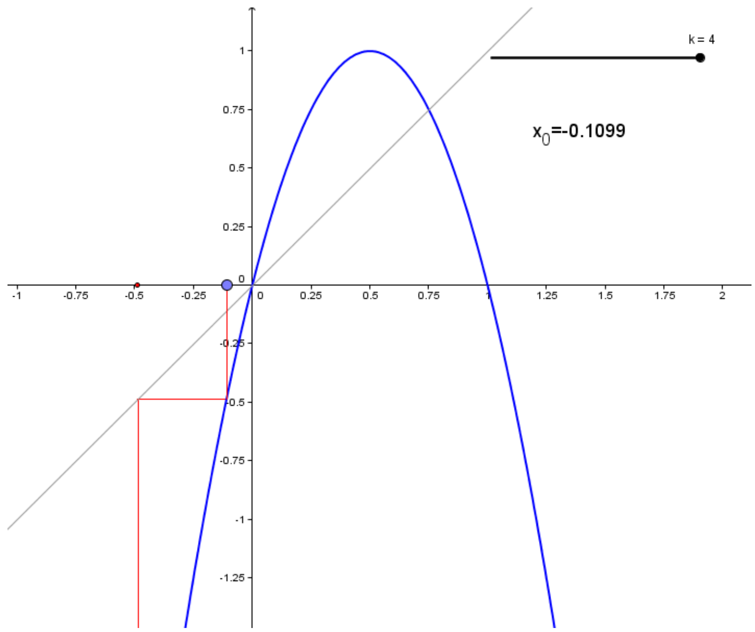

Figure 3

For $k < 1$, the origin, acts as a stable and attracting fixed point (Fig.4). However, for $1 < g(x) < 3$, the origin loses its stability and $(k-1)/k$ becomes the fixed point (Fig.5). Then, for $3 < k < 3.45$, we observe that it begins to oscillate between two points (Fig.6). For $3.45 < k < 4$, strange and complicated behavior occurs (Fig.7 and Fig.8). It shows the birth of chaos in the system.

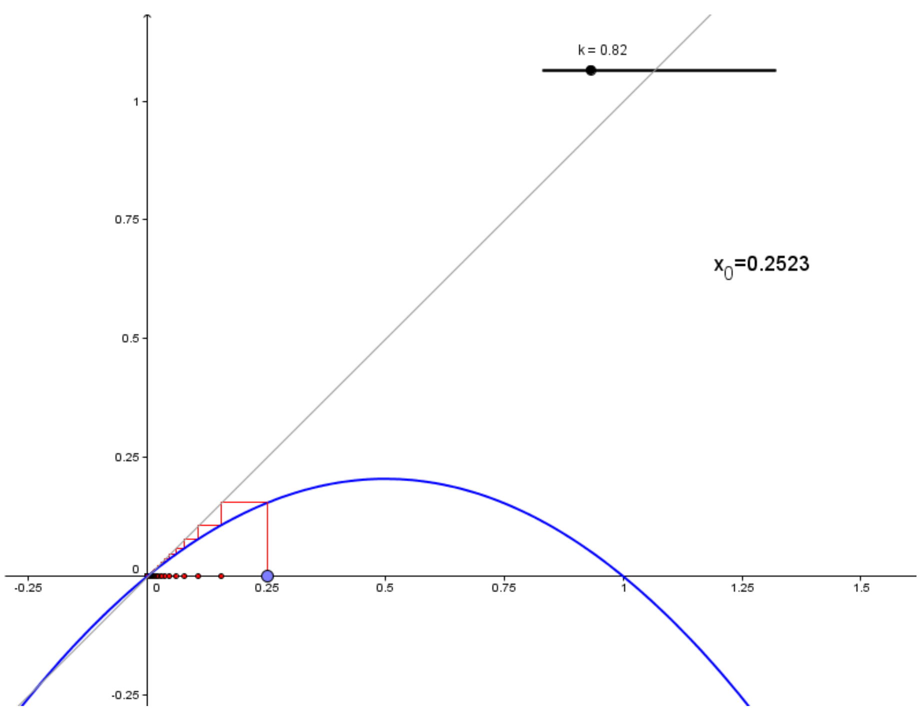

Figure 4

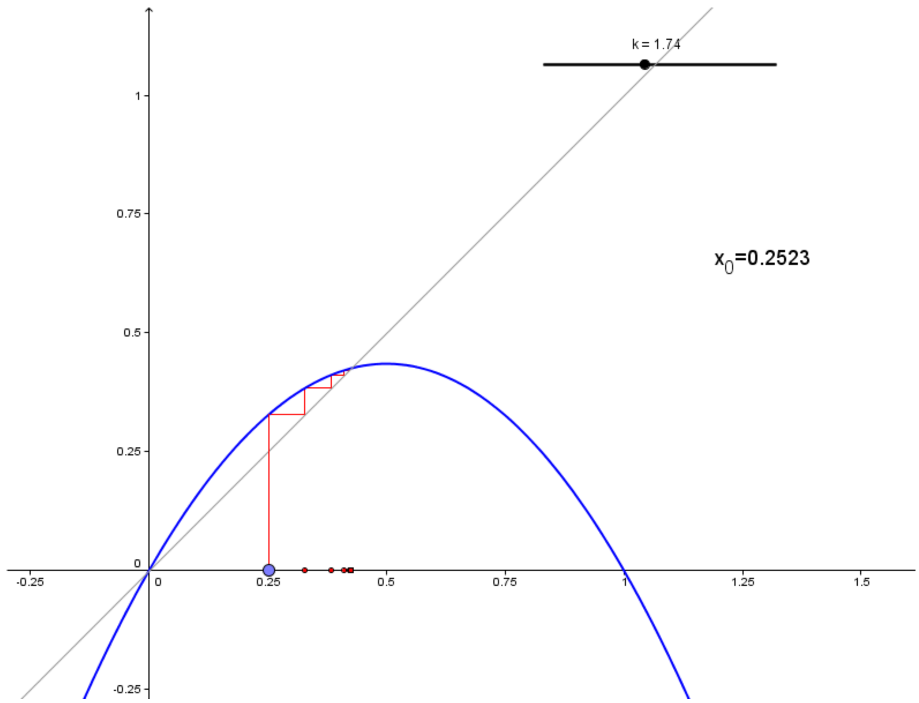

Figure 5

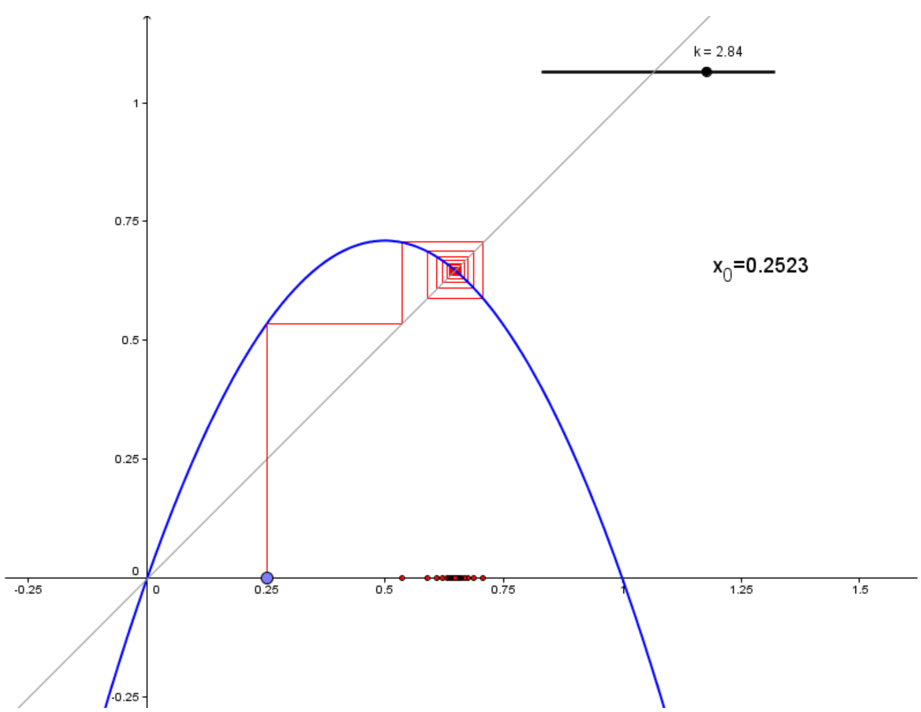

Figure 6

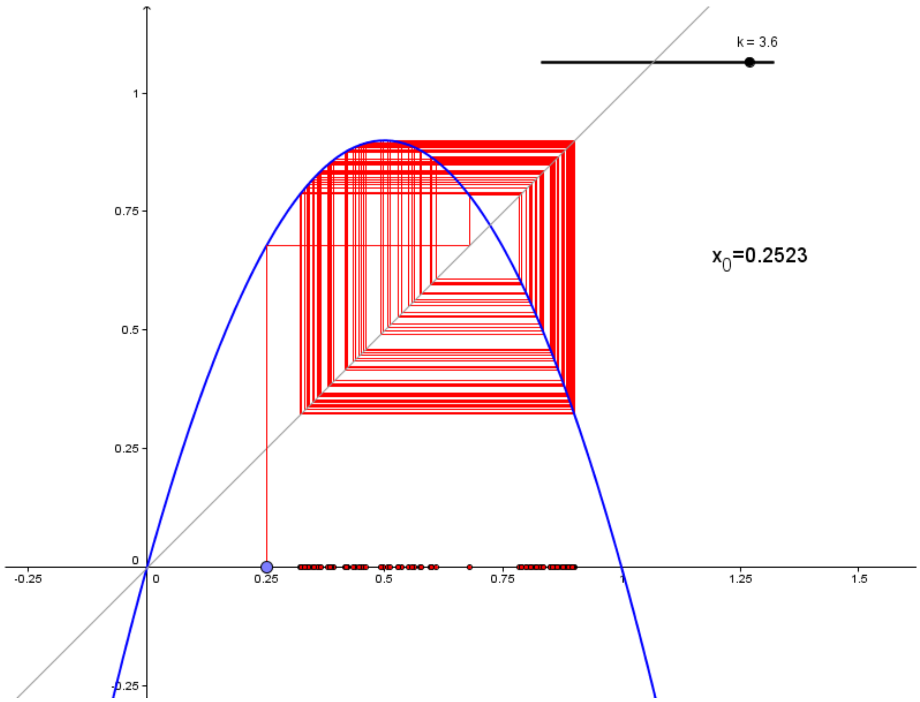

Figure 7

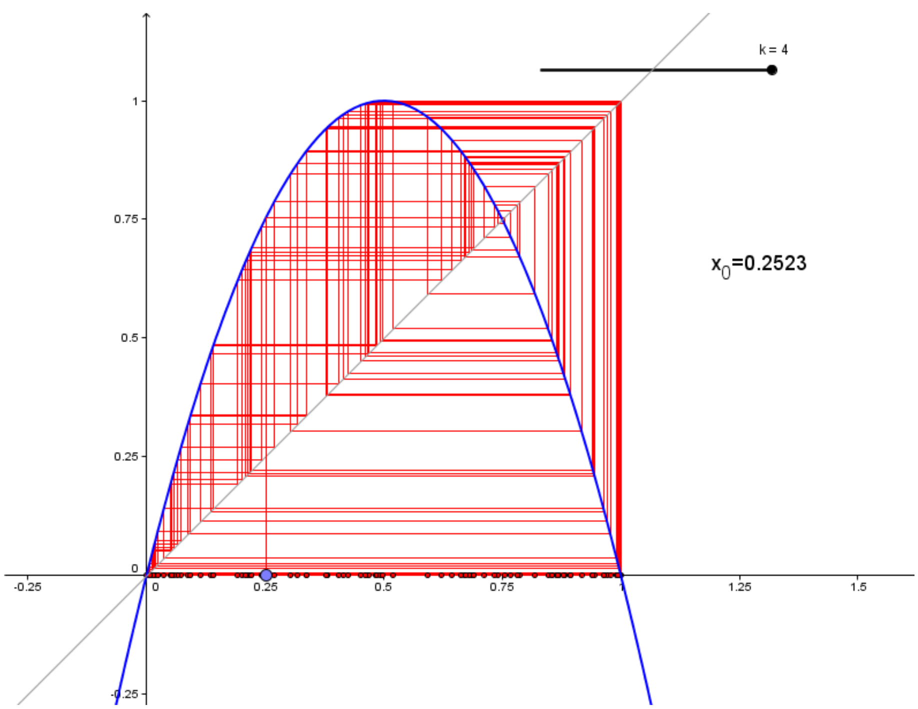

Figure 8

Bifurcation diagram of the logistic map

Every map has some relevant parameter. A bifurcation is said to occur, when at some specific value of this relevant parameter, there is a qualitative change in the behavior of the map. For the logistic map, there is one relevant parameter k.

On changing k from 0 to 4, there is a qualitative change in the behavior of $f(x)$.This bifurcation in the logistic map is classified on the basis of the qualitative change.

For $k < 1$, the parabola lies below the diagonal, and the origin is the only fixed point. As $\boldsymbol{k}$ increases, the parabola gets taller, becoming tangent to the diagonal at $k = 1$. For $k > 1$ the parabola intersects the diagonal, in a second fixed point $(k-1)/k$, while the origin loses its stability. Thus, we see that x bifurcates from the origin to $(k-1)/k$ in a ***transcritical bifurcation*** at $k = 1$.

Whereas at all other points, ***flip bifurcation*** occurs. This means that at all these points period doubling occurs (Fig 9). A closer observation of the bifurcation diagram would show period-3 windows.

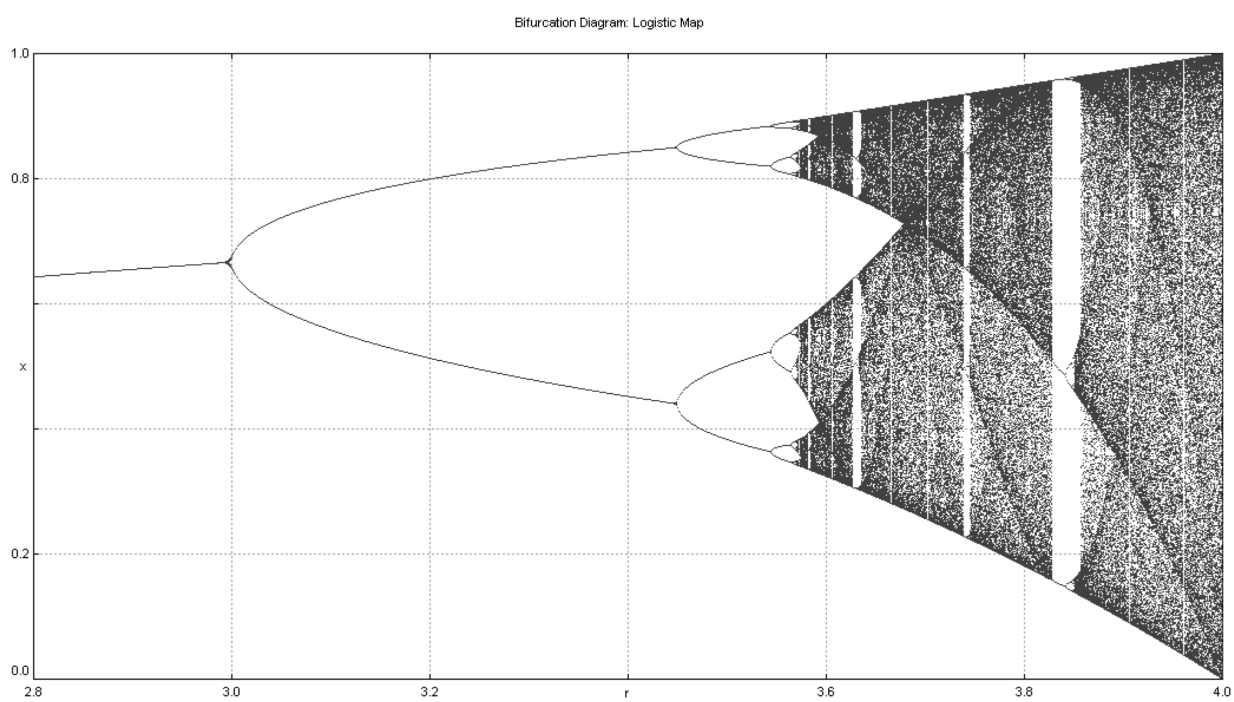

Figure 9

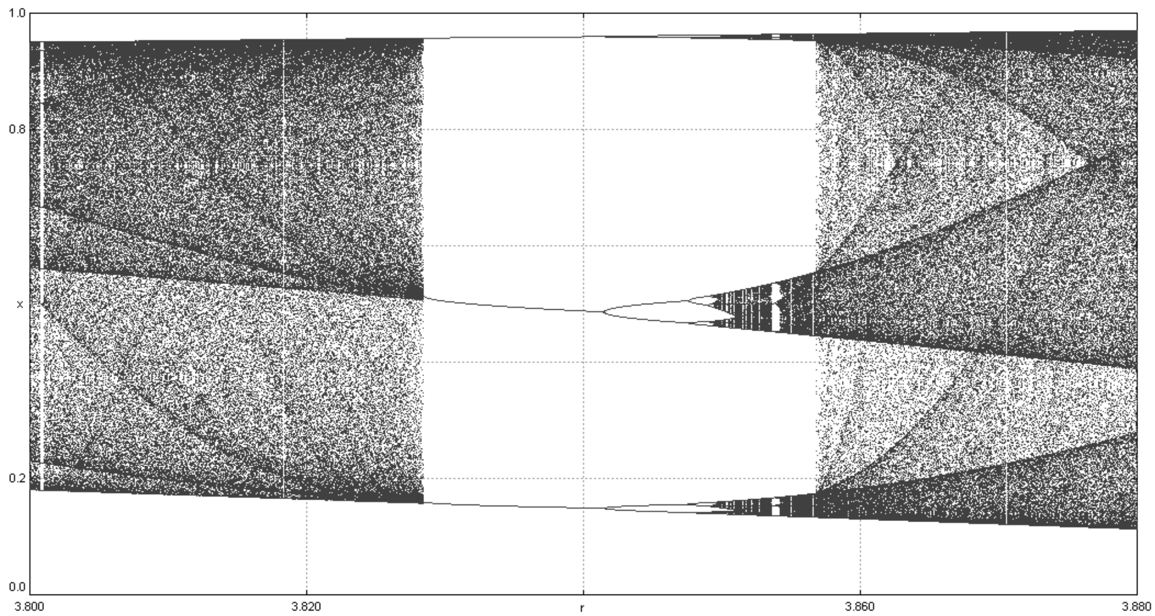

Figure 10

Intermittency

The bifurcation diagram of the logistic map shows the occurrence of period-3 windows. This leads to the phenomenon called intermittency. It can be defined as the sudden burst of chaos after long patches of roughly periodic motion. In the logistic map intermittency was found to be occurring at 3.8282 by using the time series.

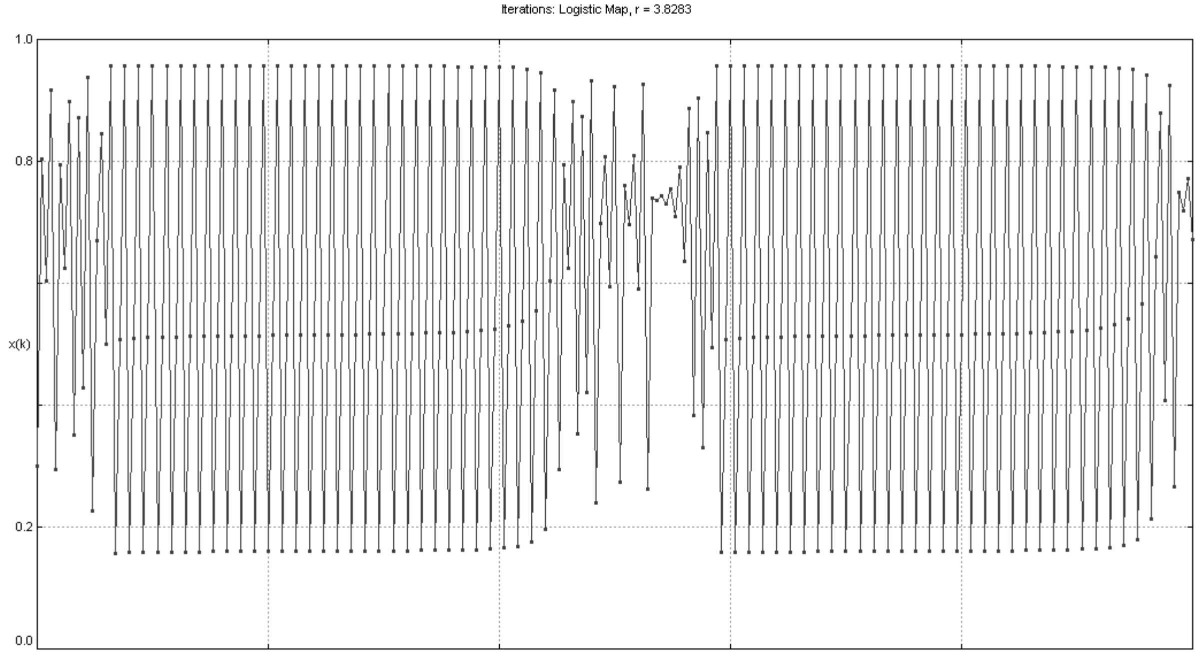

Figure 11

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