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On the relation between agM, elliptic integrals and the time period of a simple pendulum

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Abstract

The algorithm for computing the arithmetic-geometric mean (agM) of two numbers is defined, and some properties are studied. A theorem of Gauss which establishes the relation between agM and elliptic integral is proved in three ways. Making use of this theorem, approximations to the time period of a simple pendulum are also derived, that fare much better than usual perturbation theories.

1 Introduction

The arithmetic-geometric mean of Gauss is a rapidly converging sequence. Given two numbers, a and b one could find their arithmetic mean and geometric mean. The latter two numbers lie in between the two given numbers on the number line. Now this process could be iterated, by finding the arithmetic mean and geometric mean of the newly obtained numbers. The resulting series converges rapidly into a number called the arithmetic-geometric mean, $M(a, b)$ and the iterative process is denoted by agM. The study of the arclength of ellipses and lemniscates resulted in elliptic integrals, which could not be analytically evaluated by standard functions. Gauss succeeded in proving the incredible relation between elliptic integrals and the arithmetic-geometric mean. This was used to find better approximations for π. Later, it was found that the period of the simple pendulum could be expressed like those non-elementary functions relating to lemniscates, and thus could be solved, exactly, using agM method. This is investigated here. The advantage this method has over power series expansion is also discussed.

2 The arithmetic geometric mean

2.1 Historical note

The algorithm for the agM was used by Lagrange in 1785, for calculating the approximations for elliptic integrals, although he did not discover its relation with elliptic integrals. It was in 1791, that Gauss discovered this peculiar relation. In 1818, his treatise *Determinatio attractionis* (On the attraction of the elliptic ring), contains a proof of his theorem relating agM and elliptic integrals, although he has recorded it in his diary in 1799, itself. It can also be seen from Gauss' diaries that he attached significance to the argument pair of $(\sqrt{2}, 1)$, in his work on agM. The book in [13] provides insight into the history of the arithmetic geometric mean.

2.2 Analysis

Definition 2.1. Let $a \geq 0$ and $b \geq 0$ be two numbers such that $a \geq b \geq 0$. Define the numbers a_0 and b_0 by

$$a_0 := a, \qquad b_0 := b; \tag{2.1}$$

then for $n = 0, 1, 2, ...$ define the sequences a_n and b_n by

$$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n} \tag{2.2}$$

It is to be noted that each a_{n+1} is the *arithmetic mean* of the previous a_n and b_n, while each b_{n+1} is the *geometric mean* of the same two numbers.

Definition 2.2. The sequences $\{a_n\}$ and $\{b_n\}$ define the **arithmetic-geometric mean algorithm**, which is abbreviated as **agM**.

Proposition 2.3. Now, the following properties of agM are valid:

1. $\{a_n\}$ is a decreasing sequence, while $\{b_n\}$ is a increasing sequence. More precisely,

$$a_0 \geq a_1 \geq ... \geq a_n \geq a_{n+1} \geq b_{n+1} \geq b_n \geq ... \geq b_1 \geq b_0 \tag{2.3}$$

2.

$$0 \leq a_n - b_n \leq \frac{a-b}{2^n} \tag{2.4}$$

3. The limits

$$A := \lim_{n\to\infty} a_n \quad \textit{and} \quad B := \lim_{n\to\infty} b_n \tag{2.5}$$

both exist and are equal,

$$A = B. \tag{2.6}$$

Proof. Of (2.3)
Since the square of a real number is always non-negative, it follows that, for $n = 0, 1, 2, ...$, $(a_n - b_n)^2 \geq 0$ and that there is a strict inequality unless $a_n = b_n$. Therefore,

$$\begin{gathered} a_n^2 + b_n^2 - 2a_n b_n \geq 0 \\ a_n^2 + b_n^2 - 2a_n b_n + 4a_n b_n \geq 4a_n b_n \\ a_n^2 + b_n^2 + 2a_n b_n \geq 4a_n b_n \\ (a_n + b_n)^2 \geq 4a_n b_n \\ \frac{(a_n + b_n)^2}{4} \geq 4a_n b_n \\ \frac{a_n + b_n}{2} \geq \sqrt{a_n b_n} \qquad (2.7) \end{gathered}$$

The equation (2.7), of course, is the arithmetic-mean geometric mean inequality. On applying this to a_{n+1} and b_{n+1}, one obtains $a_{n+1} \geq b_{n+1}$. Thus, from $a_n \geq b_n$ and $a_{n+1} \geq b_{n+1}$,

$$a_n \geq \frac{a_n + b_n}{2} =: a_{n+1} \geq b_{n+1} := \frac{a_n - b_n}{2} \tag{2.8}$$

which is (2.3). ■

Proof. Of (2.4)
From $b_{n+1} \geq b_n$, one could conclude that

$$a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = \frac{a_n + b_n}{2} - b_n = \frac{a_n - b_n}{2}$$

and (2.4) follows by induction. ■

Proof. Of (2.5)
By (2.3), the sequence $\{a_n\}$ decreases monotonically and is bounded from below by b_0; therefore, A exists. Similarly, by (2.3), the sequence $\{b_n\}$ increases monotonically and is bounded from above by a_0; therefore, B exists. Now, letting n to infinity in (2.4) and using the limits in (2.5), one obtains, $0 \leq A - B \leq 0$; therefore,by the 'sandwich theorem', $A = B$. ■

Now, based on the aforementioned properties, the definition for agM can be established as follows.

Definition 2.4. The **arithmetic-geometric mean**, $M(a, b) \equiv \mu$ of the numbers a and b is defined to be the common limit

$$M(a, b) \equiv \mu := A := \lim_{n\to\infty} a_n \equiv B := \lim_{n\to\infty} b_n \tag{2.9}$$

of the agM as applied to the numbers a and b.

From this definition, it is obvious that the following properties hold true:

$$M(a, a) = a \tag{2.10}$$

$$M(a, b) = M(b, a) \tag{2.11}$$

$$M(a, b) = M(a_0, b_0) = M(a_1, b_1) = ... = M(a_n, b_n) \tag{2.12}$$

$$M(a, b) = M\Big(\frac{a + b}{2}, \sqrt{ab}\Big) \tag{2.13}$$

$$M(a, b) = M(a + \sqrt{a^2 - b^2}, a - \sqrt{a^2 - b^2}) \tag{2.14}$$

$$M(ka, kb) = kM(a.b) \tag{2.15}$$

Proposition 2.5. The geometric mean b_n is a closer approximation to μ than a_n, that is,

$$0 < \frac{\mu - b_n}{a_n - \mu} < 1 \tag{2.16}$$

Proof. It can be observed that,

$$\mu < a_{n+1} = \frac{a_n + b_n}{2} \Rightarrow 2\mu < a_n + b_n \Rightarrow \mu - b_n < a_n - \mu$$

Since $0 < \mu - b_n < a_n - \mu$, one could divide by $a_n - \mu$ in order to complete the proof. ■

Table 1: The first four iterations of the arithmetic-geometric mean for $a = \sqrt{2}$ and $b = 1$ are shown. The last column shows the number of digits to the right of the decimal point for which the means match each other.

n	a_n	b_n	Digits of Accuracy
0	1.414213562373905048802	1.000000000000000000000	0
1	1.207106781186547524401	1.189207115002721066717	0
2	1.198156948094634295559	1.198123521493120122607	4
3	1.198140234793877209083	1.198140234677307205798	9
4	1.198140234735592207441	1.198140234735592207439	19

$$M(\sqrt{2}, 1) = 1.1981402347355922074 \tag{2.17}$$

Table 1 illustrates the solution of the recursion relations in 2.2 for $(a, b) = (\sqrt{2}, 1)$. Just after four iterations an agreement of 19 decimal places between the two means is obtained. This suggests that the agM rapidly converges. The work by Carvalhaes and Suppes shows the convergence rate. If a measure of of error c_n is introduced in the nth iteration as

$$c_n = \sqrt{a_n^2 - b_n^2} \tag{2.18}$$

it is observed that

$$c_{n+1} = \sqrt{a_{n+1}^2 - b_{n+1}^2}$$
$$c_{n+1} = \sqrt{\frac{(a_n + b_n)^2}{4} - a_n b_n}$$
$$c_{n+1} = \sqrt{\frac{a_n^2 + b_n^2 + 2a_n b_n - 4a_n b_n}{4}}$$
$$c_{n+1} = \frac{a_n - b_n}{2} \cdot \frac{a_{n+1}}{a_{n+1}}$$
$$c_{n+1} = \frac{c_n^2}{4a_{n+1}}. \tag{2.19}$$

Since the sequence $\{c_n\}$ goes to 0 quadratic manner, the convergence of the arithmetic-geometric means is quadratic. Therefore, an agreement of about 2^n digits between the means is observed after n iterations.

3 The lemniscate

Definition 3.1. A lemniscate is the locus of points the product of whose distances from two fixed points (called the foci) a distance $2m$ away is the constant m^2. Its equation in the Cartesian form is

$$x^2 + y^2 = \sqrt{2}m\sqrt{(x^2 - y^2)}, \qquad m > 0 \tag{3.1}$$

Note: In this discussion, a unit lemniscate is considered, that is, $m = 1/\sqrt{2}$.

3.1 Arclength of a lemniscate

Expressing (3.1) in polar coordinates requires the substitution $x = z\cos x$ and $y = z\sin x$. Setting m to $1/\sqrt{2}$, the lemniscate equation is

$$z^2 = \cos 2\theta \tag{3.2}$$

The 1/4th arclength of the lemniscate, q can be determined as

$$\begin{aligned} q &= \int_0^z \sqrt{dz^2 + z^2\,d\theta^2} = \int_0^z \sqrt{1 + z^2\Big(\frac{d\theta}{dz}\Big)^2}\,dz = \int_0^z \sqrt{1 + z^2\Big(\frac{-z}{\sin 2\theta}\Big)^2}\,dz \\ &= \int_0^z \sqrt{1 + \frac{z^4}{(1-\cos^2 2\theta)}}\,dz = \int_0^z \sqrt{1 + \frac{z^4}{1-z^4}}\,dz = \int_0^z \sqrt{\frac{1}{1-z^4}}\,dz \end{aligned}$$

$$q = \int_0^z \frac{dz}{\sqrt{1-z^4}} \tag{3.3}$$

Note that this is an **elliptical integral** of the first kind. This and various integrals in the subsequent sections cannot be evaluated in finite terms with elementary functions.

3.2 Historical note

Historically, the arclength of a lemniscate was of interest to many *natural philosophers* of the 17th and 18th century because of its relation to the circumference of an ellipse, which of course was the path of motion of planets.[13]

In 1691, Jakob Bernoulli was working on the equation for the elastic curve, which is formed if a rod is forced to bend so far that its ends are perpendicular to an imaginary line connecting their endpoints. He found the equation of the elastic curve to be

$$p = \int_0^1 \frac{z^2}{\sqrt{1-z^4}}\,dz \tag{3.4}$$

Later in 1694, Jakob and his brother Johann Bernoulli independently discovered the lemniscate equation in (3.1) and published their findings in *Acta Eruditorium*.

In 1730, James Stirling [18] specified approximate values for the integrals in (3.1) and(3.4) on the interval [0,1] which were accurate to 17 places.

$$p = \int_0^1 \frac{z^2}{\sqrt{1-z^4}}\,dz = 0.59907011736779611 \tag{3.5}$$

$$q = \int_0^1 \frac{1}{\sqrt{1-z^4}}\,dz = 1.21102877714605987 \tag{3.6}$$

It is to be noted now, that

$$2 \cdot p = 1.19814023473559222 \tag{3.7}$$

which agrees with $M(\sqrt{2},1)$ in (2.17) to 16 decimal places. This was first observed by Euler who also worked on the elastic curve. In a remarkable paper in 1786, published posthumously, he proved the following result:

$$\int_0^1 \frac{1}{\sqrt{1-z^4}}\,dz \cdot \int_0^1 \frac{z^2}{\sqrt{1-z^4}}\,dz = \frac{\pi}{4}, \qquad p\cdot q = \frac{\pi}{4}; \tag{3.8}$$

On combining (3.7),(2.17),(3.8), it is found that

$$M(\sqrt{2},1) = 2\int_0^1 \frac{z^2}{\sqrt{1-z^4}}\,dz \tag{3.9}$$

The work on the elastic curve is quoted, because Gauss is known to have read them [19]. Between 1797 and 1798, Gauss had started his study of the lemniscate. He was particularly encouraged by the striking similarity between $\bar{\omega}$ the half-circumference of the unit lemniscate and π, the half-circumference of a unit circle.

$$\bar{\omega} = 2\int_0^1 \frac{1}{\sqrt{1-z^4}}\,dz \qquad\qquad \pi = 2\int_0^1 \frac{1}{\sqrt{1-z^2}}\,dz \tag{3.10}$$

Gauss studied the expression $\pi/\bar{\omega}$, which turned out to be the key to further discoveries. Among other things, he found a series for the inverse of this expression, and with it he calculated $\pi/\bar{\omega}$ to 15 decimal places.
Then in 1799, came his most ingenious discovery - a merger of two fields. He noted that:

> *The agM is equal to $\pi/\bar{\omega}$ between 1 and $\sqrt{2}$; we have confirmed up to the 11th decimal digit; if this is proven, then a new field of analysis stands before us, namely the investigation of functions etc.*

Note that the work by the previous two authors has something to contribute to his theorem.

4 Theorem of Gauss

Theorem 4.1. Let $\mid x \mid < 1$, and define

$$K(x) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-x^2\sin^2\phi}}\,d\phi \tag{4.1}$$

Then

$$M(1+x,1-x) = \frac{\pi}{2K(x)} \tag{4.2}$$

The integral $K(x)$ is an elliptic integral of the first kind and were already studied by Legendre before Gauss. In order to prove the above theorem, a reformulation of it is done as follows. Define

$$I(a,b) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} \tag{4.3}$$

Proposition 4.2.

$$I(a,b) = \frac{1}{a}K(x) \tag{4.4}$$

Proof.

$$a \cdot I(a,b) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\frac{1}{a}\sqrt{a^2\cos^2\phi + b^2\sin^2\phi}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\frac{a^2\cos^2\phi + b^2\sin^2\phi}{a^2}}} \quad \Rightarrow$$

$$a \cdot I(a,b) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\cos^2\phi + \frac{b^2}{a^2}\sin^2\phi}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{\cos^2\phi + \sin^2\phi + \left(\frac{b^2}{a^2} - 1\right)\sin^2\phi}} \Rightarrow$$

$$a \cdot I(a,b) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 + \left(\frac{b^2}{a^2} - 1\right)\sin^2\phi}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2\phi}} = K(x)$$

where

$$x = \frac{1}{a}\sqrt{a^2 - b^2} \tag{4.5}$$

■

Proposition 4.3.

$$M(1+x, 1-x) = \frac{1}{a}M(a,b) \tag{4.6}$$

Proof. Using properties (2.10) through (2.15), following can be asserted.

$$M(1+x, 1-x) = M(1, \sqrt{1-x^2}) = M\left(1, \sqrt{1 - \frac{a^2 - b^2}{a^2}}\right) = M\left(1, \frac{b}{a}\right) = \frac{1}{a}M(a,b)$$

■

The following reformulation of equation (4.2) is now immediate

Theorem 4.4 (Theorem of Gauss)**.** Let $a > b > 0$. Then

$$I(a,b) = \frac{\pi}{2M(a,b)} \tag{4.7}$$

4.1 The first proof

Proof. (By Gauss) Since $M(1+x, 1-x)$ is an even function of x, Gauss assumes that it can be represented in the form of a power series in even powers of x.

$$\frac{1}{M(1+x, 1-x)} = \sum_{k=0}^{\infty} A_k x^{2k} \tag{4.8}$$

Now substituting for x as,

$$x = \frac{2t}{1+t^2}$$

,

$$M(1+x,1-x) = M\Big(1+\frac{2t}{1+t^2}, 1-\frac{2t}{1+t^2}\Big) = \frac{1}{1+t^2} M\Big((1+t)^2,(1-t)^2\Big)$$

$$M(1+x,1-x) = \frac{1}{1+t^2} M\Big(1+t^2, 1-t^2\Big) \tag{4.9}$$

Substituting (4.9) in (4.8),

$$\sum_{k=0}^{\infty} A_k \left(\frac{2t}{1+t^2}\right)^{2k} = (1+t^2)\sum_{k=0}^{\infty} A_k t^{4k}$$

Multiplying on both sides by $2t/(1+t^2)$,

$$\sum_{k=0}^{\infty} A_k \left(\frac{2t}{1+t^2}\right)^{2k+1} = \sum_{k=0}^{\infty} A_k (2t) t^{4k}$$

A comparison of the coefficients of powers of t gives an infinite system of equations in $\{A_k\}$. Gauss finds the relation between A_k and A_{k-1} and thus finds the value for A_k.

$$\frac{A_k}{A_{k-1}} = \left(\frac{2k-1}{2k}\right)^2, \qquad A_k = \left(\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\right)^2$$

Gauss does not stop here and justifies this pattern by algebraic manipulations while equating coefficients of the two power series, and finally arrives at the following result.

$$\frac{1}{M(1+x,1-x)} = \sum_{k=0}^{\infty} \left(\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\right)^2 x^{2k} \tag{4.10}$$

The integrand in (4.1) expands in a power series using the binomial theorem as follows.

$$K(x) = \sum_{k=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k k!} x^{2k} \int_0^{\frac{\pi}{2}} \sin^{2k}\theta \, d\theta$$

The integral for the powers of sines is evaluated using the formula

$$\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\, d\theta = \frac{2k-1}{2k}\int_0^{\frac{\pi}{2}} \sin^{2k-2}\theta\, d\theta$$

Repeated application of this formula give

$$\int_0^{\frac{\pi}{2}} \sin^{2k}\theta\, d\theta \;=\; \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\int_0^{\frac{\pi}{2}} d\theta \;=\; \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\frac{\pi}{2}$$

Thus,

$$K(x) = \frac{\pi}{2}\sum_{k=0}^{\infty}\left(\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\right)^2 x^{2k} \tag{4.11}$$

From (4.4), (4.6), (4.10), (4.11) it is evident that

$$I(a,b) = \frac{\pi}{2M(a,b)}$$

■

4.2 The second proof

In order to prove the theorem in (4.7) one has to prove the remarkable property that

$$I(a,b) = I(a_1,b_1) = I(a_2,b_2) = I(a_3,b_3) = \cdots$$

Then, it can be concluded that

$$I(a,b) = \lim_{n\to\infty} I(a_n,b_n) = I(M(a,b),M(a,b)) = \frac{\pi}{2M(a,b)} \tag{4.12}$$

Each of the equality in (4.12) is proved separately. For proving the second equality, the interchangeability of the limit and the integral must be proved. That is

$$\lim_{n\to\infty} I(a_n,b_n) = I(\lim_{n\to\infty} a_n, \lim_{n\to\infty} b_n) = I(M(a,b),M(a,b)) \tag{4.13}$$

For, it is required to prove the following.

Proposition 4.5. The sequence $\left\{\frac{1}{\sqrt{a_n^2\cos^2\phi+b_n^2\sin^2\phi}}\right\}$, $n=0,1,2,\cdots$, converges uniformly to $\frac{1}{M(a,b)}$. This means that the convergence rate of the sequence in independent of ϕ.

Proof. That means given any $\epsilon > 0$ one must prove there exists a positive number $N(\epsilon)$, which is independent of the variable ϕ such that the following implication is true.

$$n > N(\epsilon) \Longrightarrow \left|\frac{1}{\sqrt{a_n^2\cos^2\phi+b_n^2\sin^2\phi}} - \frac{1}{M(a,b)}\right| < \epsilon \tag{4.14}$$

The identity $\sin^2\phi + \cos^2\phi = 1$ as well as the inequalities (2.3) and

$$b_n \le \sqrt{a_n^2\cos^2\phi + b_n^2 sin^2\phi} \le a_n$$

show that

$$-(a_n-b_n) = b_n-a_n < b_n-M(a,b) < \sqrt{a_n^2\cos^2\phi+b_n^2\sin^2\phi}-M(a,b) < a_n-M(a,b) < a_n-b_n$$

Applying (2.4)

$$\Longrightarrow \left|\sqrt{a_n^2\cos^2\phi + b_n^2\sin^2\phi} - M(a,b)\right| < a_n - b_n < \frac{a-b}{2^n}$$

$$\Longrightarrow \left|\frac{1}{\sqrt{a_n^2\cos^2\phi + b_n^2\sin^2\phi}} - \frac{1}{M(a,b)}\right| = \left|\frac{\sqrt{a_n^2\cos^2\phi + b_n^2\sin^2\phi} - M(a,b)}{M(a,b)\cdot\sqrt{a_n^2\cos^2\phi + b_n^2\sin^2\phi}}\right| < \frac{a-b}{2^n}\frac{1}{b^2}$$

For the implication in (4.14) to be true it is sufficient that the following inequality be true.

$$\frac{a-b}{2^n}\frac{1}{b^2} < \epsilon \Leftrightarrow 2^n > \frac{a-b}{b^2\epsilon} \Leftrightarrow n > \frac{\ln\frac{a-b}{b^2\epsilon}}{\ln 2},$$

That is, by choice one could define $N(\epsilon)$ to

$$N(\epsilon) := \frac{\ln\frac{a-b}{b^2\epsilon}}{\ln 2}$$

and thus prove the implication in (4.14). This means that the limit and integral signs in (4.13) are interchangeable. This proves the second equality in (4.12). For proving the first equality in (4.12), $I(a,b) = I(a_1,b_1)$ has to be proved. Gauss proves this based on the following change of variable in the integral $I(a,b)$. A new variable $\phi\prime$ is used.

$$\sin\phi := \frac{2a\sin\phi\prime}{a+b+(a-b)\sin^2\phi\prime} \tag{4.15}$$

■

Proposition 4.6. The function in (4.15) is bijective from the interval $0 \le \phi \le \frac{\pi}{2}$ to the interval $0 \le \phi\prime \le \frac{\pi}{2}$.

Proof. [Also by Gauss] Define the function

$$f(t) := \frac{2at}{a+b-(a-b)t^2}$$

Then

$$f\prime(t) = \left(2a\frac{a+b-(a-b)t^2}{(a+b-(a-b)t^2)^2}\right) \ge \left(\frac{2ab}{(a+b-(a-b)t^2)^2}\right) > 0$$

Therefore $f(t)$ is increasing on $[0,1]$. Note $f(0) = 0$ and $f(1) = 1$. This proves that $f(t)$ maps $[0,1]$ bijectively onto itself.
Now, based on the Gauss' substitution the proof for $I(a,b) = I(a_1,b_1)$ is immediate.

Proof. The value for $\cos\phi$ is found as follows.

$$\begin{aligned}
\cos^2\phi &= 1-\sin^2\phi \\
&= 1-\frac{4a^2\sin^2\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= \frac{(a+b)^2+2(a^2-b^2)\sin^2\phi\prime+(a-b)^2\sin^4\phi\prime-4a^2\sin^2\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= \frac{4a_1^2-4(2a_1^2-b_1^2)\sin^2\phi\prime+4(a_1^2-b_1^2)\sin^4\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= \frac{4a_1^2\cos^4\phi\prime+4b_1^2\sin^2\phi\prime\cos^2\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2}
\end{aligned}$$

Factoring out $4\cos^2\phi\prime$ and taking square root on both sides gives,

$$\cos\phi=\frac{2\cos\phi\prime\sqrt{a_1^2\cos^2\phi\prime+b_1^2\sin^2\phi\prime}}{(a+b)+(a-b)\sin^2\phi\prime} \tag{4.16}$$

By substituting the values for $\sin\phi$ and $\cos\phi$, in the following expression,

$$\begin{aligned}
a^2\cos^2\phi+b^2\sin^2\phi &= a^2\left\{\frac{2\cos\phi\prime\sqrt{a_1^2\cos^2\phi\prime+b_1^2\sin^2\phi\prime}}{(a+b)+(a-b)\sin^2\phi\prime}\right\}^2+\frac{4a^2b^2\sin^2\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= \frac{4a^2\cos^2\phi\prime(a_1^2\cos^2\phi\prime+b_1^2\sin^2\phi\prime)+4a^2b^2\sin^2\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= 4a^2\frac{a_1^2(1-\sin^2\phi\prime)^2+b_1^2\sin^2\phi\prime(1-\sin^2\phi\prime)+b^2\sin^2\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= a^2\frac{(a+b)^2(1-\sin^2\phi\prime)^2+4ab\sin^2\phi\prime(1-\sin^2\phi\prime)+(a-b)^2\sin^4\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= a^2\frac{(a+b)^2-2(a-b)(a+b)\sin^2\phi\prime+4b^2\sin^2\phi\prime}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2} \\
&= \left\{a\frac{(a+b)+(a-b)\sin^2\phi\prime}{(a+b)-(a-b)\sin^2\phi\prime}\right\}^2
\end{aligned}$$

Taking square root on both sides gives

$$\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}=a\frac{(a+b)+(a-b)\sin^2\phi\prime}{(a+b)-(a-b)\sin^2\phi\prime} \tag{4.17}$$

The differential of the left hand side of (4.15) after using (4.16) gives

$$\cos\phi\,d\phi=\frac{2\cos\phi\prime\sqrt{a_1^2\cos^2\phi+b_1^2\sin^2\phi}}{(a+b)+(a-b)\sin^2\phi\prime}\,d\phi$$

Now the differential of the right hand side of (4.15) gives

$$d\left\{\frac{2a\sin\phi\prime}{a+b+(a-b)\sin^2\phi\prime}\right\}=\frac{2a\cos\phi\prime\left\{(a+b)-(a-b)\sin^2\phi\prime\right\}}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2}\,d\phi\prime$$

Equating the two differentials according to (4.15) the following is obtained.

$$\frac{2\cos\phi\prime\sqrt{a_1^2\cos^2\phi+b_1^2\sin^2\phi}}{(a+b)+(a-b)\sin^2\phi\prime}\,d\phi=\frac{2a\cos\phi\prime\left\{(a+b)-(a-b)\sin^2\phi\prime\right\}}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2}\,d\phi\prime$$

$$\Longrightarrow\frac{d\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\frac{a\left\{(a+b)-(a-b)\sin^2\phi\prime\right\}}{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2}\,d\phi\prime\cdot\frac{\frac{\left\{(a+b)+(a-b)\sin^2\phi\prime\right\}^2}{a\left\{(a+b)-(a-b)\sin^2\phi\prime\right\}}}{\sqrt{a_1^2\cos^2\phi+b_1^2\sin^2\phi}}$$

This proves $I(a,b)=I(a_1,b_1)$ which is the first equality in (4.12). Now coming to the last equality in the same equation,

$$I(M(a,b),M(a,b))=\int_0^{\frac{\pi}{2}}\frac{d\phi}{M(a,b)}=\frac{\pi}{2M(a,b)}\tag{4.18}$$

■

4.3 A third proof

This third proof is due to Newman, who suggested another method (without as much algebraic manipulation as the previous one), a change of variables (Landen Transformation) that proves that the integral $I(a,b)$ is invariant under the operation, $a\to\frac{a+b}{2}$ and $b\to\sqrt{ab}$. This simplified occurs in two parts and $I(a,b)=I(a_1,b_1)$ is proved.

Proof. A substitution of $t=b\tan\phi$ is made. Hence $dt=b\sec^2\phi\,d\phi$. The limits therefore, change to $-\infty$ to ∞. This reduces the integral to

$$I(a,b)=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{a^2\cos^2\phi+b^2\sin^2\phi}}=\int_{-\infty}^{\infty}\frac{\cos\phi\,dt}{b\sec^2\phi\sqrt{a^2+b^2\tan^2\phi}}$$

$$I(a,b)=\int_{-\infty}^{\infty}\frac{dt}{\sqrt{(b^2\sec^2\phi)(a^2+t^2)}}=\int_{-\infty}^{\infty}\frac{dt}{\sqrt{(b^2+b^2\tan^2\phi)(a^2+t^2)}}$$

$$I(a,b)=\int_{-\infty}^{\infty}\frac{dt}{\sqrt{(t^2+a^2)(t^2+b^2)}}\tag{4.19}$$

Now using the values for a_1 and b_1 and substituting

$$t=\frac{x-ab/x}{2}$$

one gets,

$$\begin{aligned}
t^2 + a_1^2 &= \frac{(x^2 - ab)^2}{4x^2} + \frac{(a+b)^2}{4} \\
&= \frac{x^4 - 2abx^2 + a^2b^2 + (a^2+b^2)x^2 + 2abx^2}{4x^2} \\
&= \frac{x^4 + (a^2+b^2)x^2 + a^2b^2}{4x^2} \\
&= \frac{(x^2+a^2)(x^2+b^2)}{4x^2}
\end{aligned}$$

and

$$\begin{aligned}
t^2 + b_1^2 &= \frac{(x^2 - ab)^2}{4x^2} + \left(\sqrt{ab}\right)^2 \\
&= \frac{x^4 - 2abx^2 + a^2b^2 + 4abx^2}{4x^2} \\
&= \frac{x^4 + 2abx^2 + a^2b^2}{4x^2} \\
&= \frac{(x^2+ab)^2)}{4x^2}
\end{aligned}$$

Now,

$$dt = \frac{x^2 + ab}{2x^2} dx$$

Finally, the above can be substituted in the integral. Also note that the transformation from t to x maps the interval $(-\infty, \infty)$ to $(0, \infty)$

$$\begin{aligned}
I(a_1, b_1) &= \int_{-\infty}^{\infty} \frac{dt}{\sqrt{(t^2 + a_1^2)(t^2 + b_1^2)}} \\
&= \int_0^{\infty} \frac{2x}{\sqrt{(x^2+a^2)(x^2+b^2)}} \cdot \frac{1}{x^2+ab} \cdot \frac{x^2+ab}{2x} dx \\
&= \int_0^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = I(a,b)
\end{aligned}$$

Now that the invariance of the integral under the arithmetic-geometric transformation is established, the theorem of Gauss can be proved as in (4.18). ■

5 The time period of a simple pendulum

A simple pendulum is a particle of mass m, suspended from a fixed end by a massless rigid rod of length l. There are no dissipative forces acting in the system. The time period (T) of a simple pendulum can be evaluated as an elliptic integral of the first kind. The linearization of the equation of motion gives an approximate solution (referred to as the Huygen's small-angle approximation). [1, 3] The pendulum starts from an initial amplitude of θ_0, while θ is its amplitude at any given time t. The potential zero is set a distance l from the fixed end. The equation of motion is

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \tag{5.1}$$

The energy equation is

$$\begin{aligned}
\frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta) &= mgl(1-\cos\theta_0) \\
\implies \frac{1}{2}ml^2\dot{\theta}^2 &= mgl(\cos\theta - \cos\theta_0) \\
\dot{\theta}^2 &= \frac{2g}{l}(\cos\theta - \cos\theta_0) \\
\frac{d\theta}{dt} &= \pm\sqrt{\frac{2g}{l}(\cos\theta - \cos\theta_0)} \\
dt &= \sqrt{\frac{l}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \\
T &= 4\int_0^{\theta_0}\sqrt{\frac{l}{2g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \\
T &= 2\int_0^{\theta_0}\sqrt{\frac{2l}{g}}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}
\end{aligned}$$

Using $\cos\theta = 1 - 2\sin^2(\theta/2)$ and changing the variable based on the implicit function $\sin x = \left(\frac{\sin\theta/2}{\sin\theta_0/2}\right)$ and assuming $k = \sin\theta_0/2$,

$$T = 4\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-k^2\sin^2 x}} \tag{5.2}$$

Using the definition for $K(k)$ in (4.1) and consequently using its expansion in (4.11), the time period of the pendulum is given exactly by,

$$\begin{aligned}
T &= 4\sqrt{\frac{l}{g}}K\left(\sin\frac{\theta_0}{2}\right) \\
&= 4\sqrt{\frac{l}{g}}\cdot\frac{\pi}{2}\left\{1 + \sum_{n=1}^{\infty}\left[\frac{1\cdot3\cdot5\cdots(2n-1)}{2\cdot4\cdot6\ldots2n}\right]^2 k^{2n}\right\} \\
&= 2\pi\sqrt{\frac{l}{g}}\left\{1 + \left(\frac{1}{2}\right)^2\left(\sin\frac{\theta_0}{2}\right)^2 + \left(\frac{1\cdot3}{2\cdot4}\right)^2\left(\sin\frac{\theta_0}{2}\right)^4 + \left(\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\right)^2\left(\sin\frac{\theta_0}{2}\right)^6 + \cdots\right\} \\
&= 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{1}{16}\theta_0^2 + \frac{11}{3072}\theta_0^4 + \cdots\right\}
\end{aligned}$$

The last equation comes from substituting the Maclaurin expansion of $\sin\theta_0/2$ into the previous series and rearranging in increasing powers of θ_0. If the series is truncated after the first itself, it gives the small angle approximation. Let T_0 be defined to be this approximate time period. Then,

$$\frac{T}{T_0} = \frac{T}{2\pi}\sqrt{\frac{g}{l}} = \frac{2}{\pi}K\left((\sin\frac{\theta_0}{2}\right) \tag{5.3}$$

It is now possible to appreciate Gauss' powerful agM method. The theorem (4.7) allows on to exactly calculate the period of the pendulum. Its quadratic convergence rate [as was proved in (2.19)] allows one to get 'good' approximations after only 4 iterations. The time period of the pendulum turns out to be,

$$T = T_0\frac{1}{M(1,\sqrt{1-k^2})} = T_0\frac{1}{M(1,\cos\frac{\theta_0}{2})} \tag{5.4}$$

■

If $q = \cos\frac{\theta_0}{2}$, then the first three iterations give:

$$\begin{aligned} T_1 &= \frac{2T_0}{1+q} \\ T_2 &= \frac{4T_0}{1+q+2q^{1/2}} \\ T_3 &= \frac{8T_0}{1+q+2q^{1/2}+2^{3/2}q^{1/4}(1+q)^{1/2}} \end{aligned}$$

Now, for example, if we consider the case when the pendulum starts horizontally,

$$\theta_0 = \frac{\pi}{2}, \frac{T_1}{T_0} = \frac{2\sqrt{2}}{1+\sqrt{2}} \approx 1.17157. \quad \frac{T_2}{T_1} = \frac{4\sqrt{2}}{(1+2^{1/4})^2} \approx 1.18032.$$

The exact value of T/T_0 at this amplitude is $2K(\sin(\pi/4))/\pi = 1.18034$. The value of T/T_0 using the perturbation method, truncated after three terms is 1.17601. Therefore even at such large amplitudes, the simple expression for T_2 provides an excellent approximation.

6 Conclusion

The arithmetic-geometric mean and its properties were discussed. The advent of elliptical integrals and their relation with the arithmetic-geometric mean was established. Gauss' phenomenal theorem was proved extensively in three different ways. Then the agM algorithm was used to calculate a sequence of approximate solutions for the period of the simple pendulum. This sequence has a second-order convergence rate. It converges rapidly to the exact solution of the time period. Power series expansions only provide approximate solutions after a huge number of terms are considered. Therefore this method has a dual advantage over other methods.

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