Continuous Time Markov Chains

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Lectures on Stochastic Calculus by Rakesh Nigam

Motivation for CTMC

Most dynamical systems are asynchronous in nature. That is events or measurements or both do not occur based on a global clock. An asynchronous system does not depend on strict arrival times. As a consequence, in such systems,

1. Events, measurements or durations are irregularly spaced.
2. Rates vary by several orders of magnitude.
3. Durations of continuous measurement need to be expressed explicitly.

In a Discrete Time Markov Chain (DTMC), computations proceed one time-step at a time. However for uneventful times, this is computationally expensive. In Continuous Time Markov Chains (CTMC), there is no natural time step. Hence these models jump over uneventful time periods.

Time Discreteness and Markov Property

In this section we show that sampling a DTMC at sub-intervals (or equivalently, sampling with a greater sampling rate) in order to extend them to CTMC does not work. We also determine those exact cases where this yields a meaningful solution and those cases where it does not.

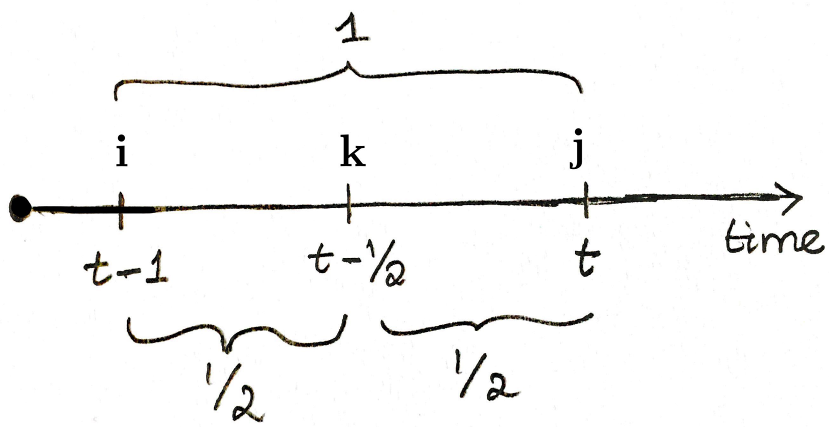
Proposition. A DTMC cannot be converted to a CTMC by sampling at sub-intervals.

Example. Consider the following 2-state DTMC with transition probability matrix T_1. Let us denote the states by $S = \{1, 2\}$.

$$T_1 = \begin{bmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{bmatrix} \tag{1}$$

Let the states be denoted by 1 and 2. Each entry in T_1 denotes the conditional probability that the system moved to state j in time t given that it is in state i in time $(t-1)$. That is $P\left[X_t = j | X_{t-1} = i\right]$.

If T_1 were to describe a continuous-time system sampled at period 1 unit, then $T_{1/2}$ describes the same system sampled at period of $1/2$ time unit (or equivalently twice the sampling rate). We can compute $T_{1/2}$ by matrix factorisation of T. This is because of the following reason:



At time t the system is at state j and at time $(t-1)$ is it as state i. Let k be the state of the system at time period $(t - \frac{1}{2})$. The state k can be either 1 or 2. Sampling at half time intervals would mean that the prior transition matrix T_1 would now represent 2-step transition probabilities, and retain the same values. The probability of two-step transition can now be decomposed in using the discrete Chapman-Kolmogorov equation.

$$P\big[(X_t = j|X_{t-1} = i)\big] = \sum_k P\big[(X_{t-\frac{1}{2}} = k|X_{t-1} = i)\big] P\big[(X_t = j|X_{t-\frac{1}{2}} = k)\big] \tag{2}$$

$$T_1(i,j) = \sum_k T_{\frac{1}{2}}(i,k).\,T_{\frac{1}{2}}(k,j) \tag{3}$$

$$T_1 = T_{\frac{1}{2}} T_{\frac{1}{2}} = \big[T_{\frac{1}{2}}\big]^2 \implies \boxed{T_{\frac{1}{2}} = \big[T_1\big]^{\frac{1}{2}}} \tag{4}$$

The matrix $T_{\frac{1}{2}}$ is the *matrix square root* of T_1. By decomposing the matrix, T, we get the value of $T_{\frac{1}{2}}$.

$$T_{\frac{1}{2}} = \begin{bmatrix} 0.8334 & 0.1667 \\ 0.3334 & 0.6667 \end{bmatrix} \tag{5}$$

We observe that $T_{\frac{1}{2}}$ is a stochastic matrix, with rows summing to 1. Although it appears to be a reasonable method to convert DTMC to CTMC by sub-interval sampling, it is actually no so because this holds only in the case when T_1 is positive definite (has positive eigenvalues). In the following example, we consider a transition matrix with at least one negative eigenvalue and show that it does not yield a real-valued stochastic matrix.

Example. Consider the DTMC with the following transition matrix T_1.

$$T_1 = \begin{bmatrix} 0.1 & 0.9 \\ 0.9 & 0.1 \end{bmatrix} \tag{6}$$

The eigenvalues of T_1 are 1 and -0.8, and is thus not positive definite. Like the previous example, decomposing T_1 using matrix square root gives the transition matrix $T_{\frac{1}{2}}$ for half sub-interval sampling of the Markov Chain.

$$T_{\frac{1}{2}} = \begin{bmatrix} 0.5 + 0.447i & 0.5 - 0.447i \\ 0.5 - 0.447i & 0.5 + 0.447i \end{bmatrix} \tag{7}$$

Thus we see that there is no real valued stochastic matrix describing the same procss as T_1 but at half the sampling periodicity. Put differently, there is no 2-state CTMC, which when sampled at rate of 1 time unit produces a Markov Chain with matrix T_1. The problem in generating $T_{\frac{1}{2}}$ arises because T_1 has negative eigenvalues.

Proposition. Only stochastic transition matrices with all positive eigenvalues correspond to a CTMC process sampled at a given periodicity. This means that:

1. The set of CTMC is smaller than the set of DTMC.

2. These processes are Markovian only when sampled at a particular periodicity and the only method of extension to points of time outside the periodicity would be to construct non-Markovian and non-stationary processes.
3. Many systems do not have a natural sampling rate. The rate is chosen for computational or measurement convenience.

Properties of CTMC

Definition. A Continuous Time Markov Chain is a stochastic process $X(t)$ that evolves in continuous time $(t \geq 0)$ on discrete state space. It obeys the Markov property and time homogeneity.

Markov Property

The Markov property states that the conditional probability of the process to be in a future state j depends only on the current state and is independent of the past path taken by the process.

$$P\big[X(t) = j | X(t_1) = i_1, \ldots, X(t_n) = i_n\big] = P\big[X(t) = j | X(t_n) = i_n\big] \tag{8}$$

Time Homogeneity

This refers to the property that the conditional probability of the process being in a future state j given a current state i remains the same so long as the time interval between transition is the same. That is, for example 2-period transitional probability remains the same no matter what the time point of the initial state - transition from i to j in interval $t = 1$ to $t = 3$ is same as that in the interval $t = 5$ to $t = 7$.

$$P\big[X(t) = j | X(s) = i\big] = P\big[X(t-s) = j | X(0) = i\big] \tag{9}$$

Notation

Define the notation $p_{ij}(s, t+s)$ as the following.

$$p_{ij}(s, t+s) = P\big[X(t+s) = j | X(s) = i\big] \tag{10}$$
$$p_{ij}(0, t) = p_{ij}(t) = P\big[X(t) = j | X(0) = i\big] \tag{11}$$

Thus we have the transition probability matrix given by,

$$P[t] = \begin{bmatrix} & & \\ & p_{ij}(t) & \\ & & \end{bmatrix}, \qquad t \geq 0 \tag{12}$$

For a transition at a single instant $t = 0$, note that,

$$p_{ij}(0) = P\big[X(0) = j | X(0) = i\big] = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{13}$$

$$P(t) = \begin{bmatrix} & & \\ & p_{ij}(t) & \\ & & \end{bmatrix}, t \geq 0 \implies \boxed{P(0) = I} \tag{14}$$

This implies that the probability for same instant transitions 1 if it remains in the same state and 0 if it moves to another state. Thus single-instant transitions from one state to a different state are not allowed. Consequently, the transition matrix $P(0)$ is an Identity matrix.

Chapman Kolmogorov Equation

Recall that the discrete time Chapman-Kolmogorov equation establishes the relationship between multi-step transitions as a product of sub-step transition matrices. That is,

$$P^{(n+m)} = P^{(n)}.\, P^{(m)} \tag{15}$$

$$P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)}.\, P_{kj}^{(m)}, \qquad \forall i, j \in S \text{ and } n, m \geq 0, \tag{16}$$

The continuous time analogue of the Chapman-Kolmogorov equation is given by

$$P(0, s+t) = \sum_{k} P_{ik}(0, s).\, P_{kj}(s, s+t), \quad 0 < s < t \tag{17}$$

$$P(s+t) = P(s).\, P(t) \tag{18}$$

Note that $P_{ij}(t)$ is a continous and differentiable function of t.

Rate Matrix

The CTMC does not have a natural sampling period. Arrivals in a CTMC process may occur at irregularly spaced intervals. There is thus a difficulty in specifying the CTMC with transition probability matrix P. The time-step of P could potentially be a real value, given no periodicity in sampling. Therefore we resort to the use of a *rate matrix* Q to specify the CTMC. We had noted that the $P_{ij}(0,t)$ matrix is continuous and differentiable with respect to t. Obtaining the right derivative of $p_{ij}(t)$ we have,

$$q_{ij} = \left.\frac{dp_{ij}(t)}{dt}\right|_{t=0} \tag{19}$$

$$q_{ij} = \lim_{h\to 0}\left.\left[\frac{p_{ij}(t+h) - p_{ij}(t)}{h}\right]\right|_{t=0} \tag{20}$$

$$q_{ij} = \lim_{h\to 0}\left[\frac{p_{ij}(h) - p_{ij}(0)}{h}\right] \tag{21}$$

Proposition. The elements of the transition probability matrix p_{ij} are related to those of the rate matrix by the following equation, where $q_{ii} \leq 0$, $q_{ij} \geq 0$ and h takes small non-negative values.

$$p_{ij}(h) = \begin{cases} 1 + hq_{ii} + o(h), & i = j, \ \text{for small } h \\ 0 + hq_{ij} + o(h), & i \neq j, \quad h \to 0 \end{cases} \tag{22}$$

Proof. We first note that the transition matrix for $p_{ij}(0) = \delta_{ij}$, which is the Kronecker Delta function taking the value of 1 when $i = j$ and 0 otherwise. Differentiating $p_{ii}(t)$ with respect to t,

$$\frac{dp_{ii}(t)}{dt}_{t=0} = \lim_{h\to 0}\left[\frac{p_{ii}(h) - p_{ii}(0)}{h}\right] = \lim_{h\to 0}\left[\frac{1 + hq_{ii} + o(h) - 1}{h}\right] \tag{23}$$

$$= \lim_{h\to 0}\left[q_{ii} + \frac{o(h)}{h}\right] = q_{ii} + 0 = q_{ii} \tag{24}$$

Likewise for the case $i \neq j$, differentiating wrt. t,

$$\frac{dp_{ij}(t)}{dt}_{t=0} = \lim_{h\to 0}\left[\frac{p_{ij}(h) - p_{ij}(0)}{h}\right] = \lim_{h\to 0}\left[\frac{hq_{ij} + o(h) - 0}{h}\right] \tag{25}$$

$$= \lim_{h\to 0}\left[q_{ij} + \frac{o(h)}{h}\right] = q_{ij} + 0 = q_{ij} \tag{26}$$

Proposition. The row sum of the rate matrix is zero.

Proof. Let Q denote the rate matrix such that $Q = \{q_{ij}\}$. Also, $\Sigma_j p_{ij}(t) = 1, \;\; \forall i$, since the row sum of stochastic matrix $P(t)$ is 1. Differentiating this with respect to t, the RHS turns out to be 0.

$$\frac{d}{dt}(\Sigma_j p_{ij}(t))\Big|_{t=0} = \frac{d(1)}{dt}\Big|_{t=0} \tag{27}$$

$$\sum_j \frac{dp_{ij}(t)}{dt} = 0 \implies \boxed{\Sigma_j q_{ij} = 0} \tag{28}$$

Description of the CTMC

In the case of a DTMC, the transition probability matrix answers all questions about the description of the process. Given the single time-step transition matrix, we know the probability that it jumps from one state to another state. Further, using the Chapman-Kolmogorov equation, multiple time-step transitions can be computed by simply applying the matrix products. However, the same cannot be said for the CTMC. In this section, we attempt to answer two main questions:

1. How long does the CTMC stay at a particular state before jumping to the next state?
2. With what probability does the CTMC jump to the given next state?

Let T_i denote the time spent by the CTMC in state i before moving to another state. For $i \geq 1$, the probability that time spent in state i greater than an arbitrary value t, is the probability of intersection between events: that it stays in i for period s for all values of s betwee 0 and t, conditioned on initial state i.

$$P\big[T_i > t\big] = P\big[X(s) = i,\; 0 \leq s \leq t \mid X(0) = i\big] \tag{29}$$

Applying the Markov chain rule, the probability of being in state i for sub-intervals (fractions of total time) depends only on the current state information and not the past.

$$\begin{aligned} P\big[T_i > t\big] = P&\Big[X(s) = i,\; 0 \leq s \leq \tfrac{t}{n} \mid X(0) = i\Big]. \\ \times P&\Big[X(s) = i,\; \tfrac{t}{n} \leq s \leq \tfrac{2t}{n} \mid X\Big(\tfrac{t}{n}\Big) = i\Big]. \\ \times \cdots \times P&\Big[X(s) = i,\; \tfrac{(n-1)t}{n} \leq s \leq t \mid X\Big(\tfrac{(n-1)t}{n}\Big) = i\Big] \end{aligned} \tag{30}$$

Using time homogeneity property, the conditional probability in each of these n intervals is the same because their same interval size $\frac{t}{n}$.

$$P\Big[T_i > t\Big] = \Big(P\Big[X(s) = i,\ 0 \le s \le \tfrac{t}{n} \mid X(0) = i\Big]\Big)^n \quad \forall\, n$$
$$= \lim_{n\to\infty} \Big(\Big[X(s) = i,\ 0 \le s \le \tfrac{t}{n} \mid X(0) = i\Big]\Big)^n$$
$$= \lim_{n\to\infty} \Big(p_{ii}(\tfrac{t}{n})\Big)^n = \underbrace{\lim_{n\to\infty} \Big(1 + q_{ii}.\, \tfrac{t}{n} + o(\tfrac{t}{n})\Big)^n}_{\text{exponential function}} = e^{tq_{ii}} \tag{31}$$

The probability that the time spent by the CTMC is state i greater than t follows an exponential distribution, with the rate matrix diagonal element, $-q_{ii}$ as the parameter. Also note that $q_{ii} \le 0$. Thus,

$$P\Big[T_i > t\Big] = e^{tq_{ii}}\ , \qquad P\Big[T_i \le t\Big] = 1 - e^{tq_{ii}}$$
$$\implies T_i \sim Exp(-q_{ii})\ , \qquad E\Big[T_i\Big] = \frac{-1}{q_{ii}} \tag{32}$$

Suppose the CTMC changes state at time t, from state i to state j. The probability of this jump is given by $\lim_{h\to 0} P\Big[X(t+h) = j \mid X(t) = i, X(t+h) \neq i\Big]$. Expanding this term within the limit, we obtain,

$$P\Big[X(t+h) = j \mid X(t) = i, X(t+h) \neq i\Big] \tag{33}$$
$$= \frac{P\Big[X(t+h) = j\ ,\ X(t) = i, X(t+h) \neq i\Big]}{P\Big[X(t) = i, X(t+h) \neq i\Big]}$$
$$= \frac{P\Big[X(t+h) = j\ ,\ X(t) = i\Big]}{P\Big[X(t+h) \neq i\ ,\ X(t) = i\Big]},\ \ j \neq i \tag{34}$$

Also note that,

$$P\Big[X(t+h) = j \mid X(t) = i\Big] = \frac{P\Big[X(t+h) = j\ ,\ X(t) = i\Big]}{P\Big[X(t) = i\Big]}$$
$$P\Big[X(t+h) \neq i \mid X(t) = i\Big] = \frac{P\Big[X(t+h) \neq i\ ,\ X(t) = i\Big]}{P\Big[X(t) = i\Big]}$$
$$\implies \frac{P\Big[X(t+h) = j \mid X(t) = i\Big]}{P\Big[X(t+h) \neq i \mid X(t) = i\Big]} = \frac{P\Big[X(t+h) = j\ ,\ X(t) = i\Big]}{P\Big[X(t+h) \neq i\ ,\ X(t) = i\Big]},\ \ j \neq i \tag{35}$$

Substituting the above equations in that for probability of jump in states,

$$\begin{aligned}\lim_{h\to 0} P\Big[X(t+\ h) = j \mid X(t) = i, X(t+h) \neq i\Big] \\ &= \lim_{h\to 0} \frac{P\Big[X(t+h) = j\ ,\ X(t) = i\Big]}{P\Big[X(t+h) \neq i\ ,\ X(t) = i\Big]},\ \ j \neq i \\ &= \lim_{h\to 0} \frac{p_{ij}(h)}{\sum_{k\neq i} p_{ik}(h)} = \lim_{h\to 0} \frac{hq_{ij}}{\sum_{k\neq i} hq_{ik}} \\ &= \frac{q_{ij}}{\sum_{k\neq i} q_{ik}} = \frac{q_{ij}}{-q_{ii}}\end{aligned} \tag{36}$$

Since we know that the row sum of the Q matrix is zero, $\Sigma_k q_{ik} = 0$, and that $q_{ii} \leq 0$, it implies that for all $k \neq i$ (off diagonal entries), $q_{ii} = -\sum_{k\neq i} q_{ik}$.

We have now set up the premise to answer the two questions intially raised. The CTMC remains in state i for a period T_i, such that $T_i \sim Exp(q_{ii})$ with mean $E\big[T_i\big] = \frac{-1}{q_{ii}}$. Then it jumps to another state $j \neq i$, with probability $\frac{q_{ij}}{-q_{ii}}$. Thus we have proved that the CTMC process depends only on the rate matrix Q rather than on the transition probability matrix P. If the CTMC process is only observed at jumps then a Markov Chain is obtained with Transition Matrix P. This MC with P as the transition matrix is called the *Embedded Markov Chain*.

$$P = \begin{bmatrix} & & \\ & p_{ij} & \\ & & \end{bmatrix}, = \begin{bmatrix} & & \\ & \frac{q_{ij}}{-q_{ii}} & \\ & & \end{bmatrix} \tag{37}$$

The states of the CTMC are defined to be recurrent or transient in accordance with their properties in the Embedded Markov Chain. The exception to this is periodicity, which is not applicable to a continuous process.

Forward Kolmogorov Equation

The Forward Kolmogorov equation is a first order differential equation that describes the dynamics of the CTMC. In order to determine its dynamics, we ask the question, given Q, how do we get $P(t)$ for any $t \geq 0$? From the continuous time analogue of the Chapman-Kolmogorov equation $P(t+h) = P(t)P(h)$ for $t \geq 0, h \geq 0$, we have,

$$
\begin{aligned}
\frac{P(t+h)-P(t)}{h} &= \frac{P(t)P(h)-P(t)}{h} \\
&= \frac{P(t)\big(P(h)-I\big)}{h} \\
&= \frac{P(t)\big(P(h)-P(0)\big)}{h}
\end{aligned} \tag{38}
$$

Taking the derivative of $P(t)$ with respect to time t,

$$
\begin{aligned}
\frac{dP(t)}{dt} &= \lim_{h\to 0}\left[\frac{P(t+h)-P(t)}{h}\right] \\
\frac{dP(t)}{dt} &= \lim_{h\to 0}\left[\frac{P(t)\big(P(h)-P(0)\big)}{h}\right] \\
\frac{dP(t)}{dt} &= P(t)\lim_{h\to 0}\left[\frac{\big(P(h)-P(0)\big)}{h}\right] \\
\frac{dP(t)}{dt} &= P(t)\left[\frac{dP(t)}{dt}\right]_{t=0} = P(t)Q
\end{aligned} \tag{39}
$$

This results in the famous *Forward Kolmogorov Equation* for determining the transition probabilities of a CTMC given the rate matrix. In the matrix form, we have,

$$
\frac{dP(t)}{dt} = P(t)Q \tag{40}
$$

$$
\frac{dp_{ij}(t)}{dt} = \sum_k p_{ik}(t)q_{kj} \quad \forall\, i,j \tag{41}
$$

The FKE can also be expressed in its element-wise form. Express the Chapman-Kolmogorov equation in element-wise form:

$$
p_{ij}(s,t+h) = \sum_{k\in S} p_{ik}(s,t).\, p_{kj}(t,t+h) \tag{42}
$$

Taking the derivative with respect to t gives: (for $s < t$)

$$
\begin{aligned}
\frac{\partial p_{ij}(s,t)}{\partial t} &= \lim_{h\to 0}\frac{p_{ij}(s,t+h)-p_{ij}(s,t)}{j} \\
\frac{\partial p_{ij}(s,t)}{\partial t} &= \sum_{k\in S} p_{ik}(s,t) \\
\frac{\partial p_{ij}(s,t)}{\partial t} &= \frac{\partial p_{kj}}{\partial t} = \sum_{k\in S} p_{ik}(s,t)q+kj(t) \qquad (43)
\end{aligned}
$$

Backward Kolmogorov Equation

The Backward Kolmogorov Equation (BKE) can be derived much in the same manner as the forward. From the continuous time analogue of the Chapman-Kolmogorov equation $P(t+h)=P(t)P(h)$ for $t\geq 0, h\geq 0$, we have,

$$
\frac{P(t+h)-P(t)}{h} = \frac{P(t)P(h)-P(t)}{h} = \frac{P(t)\big(P(h)-P(0)\big)}{h} \qquad (44)
$$

Taking the limit as $h\to 0$,

$$
\begin{aligned}
\lim_{h\to 0}\frac{P(t+h)-P(t)}{h} &= \lim_{h\to 0}\frac{P(h)-P(0)}{h}P(t) \\
\implies \frac{dP(t)}{dt} &= \left[\frac{dP(t)}{dt}\right]_{t=0} P(t) = QP(t) \qquad (45)
\end{aligned}
$$

Thus, we have the BKE given by:

$$
\begin{aligned}
\frac{dP(t)}{dt} &= QP(t) \qquad (46) \\
\frac{dp_{ij}(t)}{dt} &= \sum_k q_{ik}p_{kj}(t) \quad \forall\, i,j \qquad (47)
\end{aligned}
$$

In Finance, we use the BKE in pricing financial products such as options, futures and other derivatives. Suppose we have a payoff from a derivative at maturity period $t=T$. We wish to compute the initial price of the derivative at $t=0$, which can be done using the model the dynamics of its underlying asset given by the Backward Kolmogorov equation.

The forward and backward Kolmogorov equations give the dynamics of the system $P(t)$. We know that $P(0)=I$ and $P(t)=e^{Qt}$. For a finite state CTMC the stationary solution of both equations are the same.

$$\text{FKE:} \quad \frac{dP(t)}{dt} = P(t)Q \implies PQ = 0 \text{ gives stationary solution.} \tag{48}$$

$$\text{BKE:} \quad \frac{dP(t)}{dt} = QP(t) \implies QP = 0 \text{ gives stationary solution.} \tag{49}$$

Both the equations result in the same stationary solution for $P(t)$.

Stationary Distribution

The stationary distribution of the CTMC process is given by the vector $\boldsymbol{\pi}$, such that $\boldsymbol{\pi} = \boldsymbol{\pi} P(t)$, with $\Sigma_j \pi_j = 1, \pi_j \geq 0, \forall t$. Since $\boldsymbol{\pi}$ denotes the stationary or equilibrium distribution, we have,

$$P\Big[X(0) = i\Big] \quad = \quad \pi_i \ \text{ then } \ P\Big[X(t) = i\Big] = \pi_i \ \ \forall i, \forall t \tag{50}$$

$$\implies \boxed{\boldsymbol{\pi} P(t) = \boldsymbol{\pi}} \quad \forall\, t \tag{51}$$

Taking the derivative on both sides of the equation,

$$\frac{d\big(\boldsymbol{\pi} P(t)\big)}{dt} = \frac{d\big(\boldsymbol{\pi}\big)}{dt} \implies \boldsymbol{\pi} \frac{dP(t)}{dt} = \mathbf{0} \quad \forall\, t \tag{52}$$

$$\boldsymbol{\pi} \frac{dP(t)}{dt}\bigg|_{t-0} = \mathbf{0} \implies \boxed{\boldsymbol{\pi} Q = \mathbf{0}} \tag{53}$$

Thus to obtain $\boldsymbol{\pi}$, solve equations $\boldsymbol{\pi} Q = \mathbf{0}$ and $\Sigma_j \pi_j = 1$.

Theorem. For an irreducible process the stationary distribution $\boldsymbol{\pi}$ is unique if it exists. And if $\boldsymbol{\pi}$ exists the process is positive recurrent and all rows of $P(t)$ converge to $\boldsymbol{\pi}$.

Mental image of CTMC

The mental image of CTMC is a DTMC in which transitions can happen at any time (because time is continuous). Let S denote the discrete state space. It can be finite or countably infinite. The Markov property means that the jump times are exponentially distributred. We denote all information pertaining to the history of the process X upto time s by the filtration $\mathcal{F}_{X(s)}$. A family of σ-fields $\mathcal{F}_t$ is defined to be a filtration if $\mathcal{F}_{t_1} \subset \mathcal{F}_{t_2}$ whenever $t_1 \leq t_2$.

$$\begin{aligned}\mathcal{F}_{X(s)} :\ & \text{all information pertaining to the} \\ & \text{history of } X \text{ upto time } s.\end{aligned} \tag{54}$$

Let state $j \in S$ and let $s \leq t$, then Markov Property can be stated as

$$P\Big[X(t) = j \mid \mathcal{F}_{X(s)}\Big] = P\Big[X(t) = j \mid X(s)\Big] \tag{55}$$

Since we also want the process to be time homogeneous,

$$P\Big[X(t) = j \mid X(s)\Big] = P\Big[X(t-s) = j \mid X(0)\Big] \tag{56}$$

Any process satifying the above two equations is said to be a Time Homogoenous, Continuous Time Markov Chain. Equivalently, the CTMC can also be defined in terms of the transition rate matrix, as we have seen previously.