Mixed Random Variables

Stochastic Processes. Lecture 16.

Mixed Random Variables

- Defining the **CDF of a mixed random variable** as a sum of a continuous and a discrete function, $F_Y(y) = C(y) + D(y)$.
- **Expected value of a mixed random variable** is the sum of expected values of its continuous and discrete components.
- **Example.** Let X be a continuous random variable with PDF $f_X(x)$ and let Y be a discrete function of this the RV X. Find the CDF of $Y \implies F_Y(y)$ if,

$$f_X(x) = \begin{cases} 2x, & 0 \leq X \leq 1 \\ 0, & \text{otherwise} \end{cases} \qquad Y = g(X) = \begin{cases} X, & 0 \leq X \leq 1/2 \\ 1/2, & X > 1/2 \end{cases}$$

  Also find $P(1/4 \leq Y \leq 3/8)$ and $E(Y)$.

Dirac Delta Function

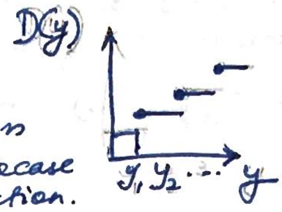
- Defining the **Dirac Delta Function** as a limiting case of the derivative of the parameterized **unit step** function.
- **Shifting property** of the Dirac delta and its proof.
- Properties of the Dirac Delta function.
- Definition of the **generalized PDF** for a discrete random variable X.
- **Example.** Obtain mean and variance of the distribution of a mixed random variable X, whose CDF is given by:

$$F_X(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}(1 - e^{-x}), & x \geq 1 \\ \frac{1}{4} + \frac{1}{2}(1 - e^{-x}), & 0 \leq x < 1 \\ 0, & x < 0 \end{cases}$$

Mixed Random Variables

The function Y is a mixed random variable.

$$\underbrace{F_Y(y)}_{\text{CDF of mixed RV } Y} = \underbrace{C(y)}_{\text{continuous function of } y} + \underbrace{D(y)}_{\text{discrete function of } y \Rightarrow \text{ staircase function.}}$$



Recall: if $F_X(x)$ is ~~the~~ CDF of a continuous RV X, then to obtain the PDF, we differentiate the CDF.

$$f_X(x) = \frac{d\,F_X(x)}{dx}$$

Here $f_X(x)$ is the pdf.
The pdf sums to 1.

However, here, we find that:

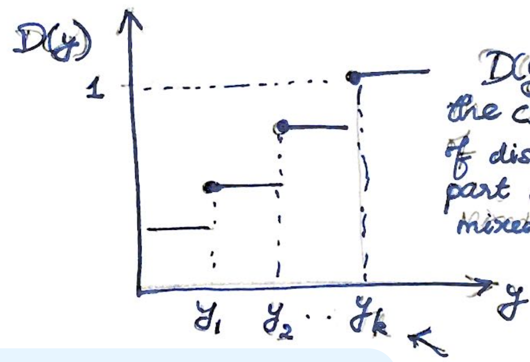
$$\underbrace{c(y)}_{} = \frac{dC(y)}{dy} \quad \text{where } C(y) \text{ is differentiable.}$$

$c(y)$ is not a valid PDF as it does not sum to 1.
$c(y)$ is a kind of pseudo PDF as $C(y)$ forms a part of the mixed Random variable.

Let $\{y_1, y_2, \dots\}$ be the set of jump points of $D(y)$
that is, at these points, $P(Y = y_k) > 0$, $k = 1, 2, 3, \dots$

The probability masses at non-jump points is zero. (CDF remains constant here).

Probability masses at jump points is greater than '0'.



D(y) is the CDF of discrete part of mixed RV

Jump points.

∴ Expected value of mixed RV: $k = 1, 2, 3, \dots$

$$E[Y] = \int_{-\infty}^{\infty} y\, c(y)\, dy + \sum_{y_k} y_k\, P[Y = y_k]$$

Example. Let X be a continuous RV with PDF $f_X(x)$

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = g(X) = \begin{cases} X & 0 \le X \le 1/2 \\ 1/2 & X > 1/2 \end{cases}$ ← This function Y is a fn of RV X. It models different regimes.

Find CDF of $Y \Rightarrow F_Y(y)$

Range of $X = R_X = [0, 1]$ $x \in [0, 1]$. y

Range of $Y = R_Y = [0, 1/2$ since $0 \le g(x) \le 1/2$

Between 0 to $1/2$ $g(x)$ takes value of x.
$> 1/2$, $g(x)$ takes value of $1/2$.

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ 1 & \text{for } y \ge 1/2 \end{cases}$$

$$P[Y = 1/2] = P[X > 1/2] = \int_{x=1/2}^{x=1} 2x \, dx = 2\left(\frac{x^2}{2}\right)\Bigg|_{1/2}^{1}$$

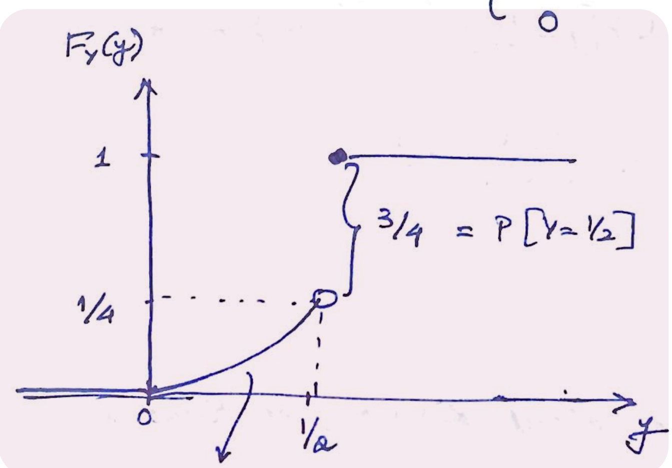
$$= x^2\Big|_{1/2}^{1} = 1^2 - 1/2^2 = 3/4 \quad \Rightarrow \quad P[Y = 1/2] = 3/4$$

For $0 \le y \le 1/2 \Rightarrow F_Y(y) = P[Y \le y] = P[X \le y]$

Because in $[0, 1/2]$ $Y = g(X) = X$

$$\Rightarrow F_Y(y) = P[X \le y] = \int_{x=0}^{x=y} 2x \, dx = y^2$$

CDF of Y is $F_Y(y) = \begin{cases} 1 & y \ge 1/2 \\ y^2 & 0 \le y < 1/2 \\ 0 & y < 0 \end{cases}$



→ Discrete part of Y.
Jump at $Y = 1/2$
$= P[Y = 1/2] = 1 - 1/4 = 3/4$.

CDF here is the continuous part of Y

Note. CDF of Y is not continuous. Thus Y cannot be a continuous R.V. Also, CDF is not in a stair case form. It cannot be a discrete RV either.

$$F_Y(y) = C(y) + D(y)$$

$$C(y) = \begin{cases} 1/4 & y \ge 1/2 \\ y^2 & 0 \le y < 1/2 \\ 0 & y < 0 \end{cases} \qquad D(y) = \begin{cases} 3/4, & y \ge 1/2 \\ 0, & y < 1/2 \end{cases}$$

Find $P[1/4 \le Y \le 3/8]$

$Y \in [1/4, 3/8]$ is the interval event whose probability is to be found.

CDF of Y

$$F_Y(y) = \begin{cases} 1 & y \ge 1/2 \\ y^2 & 0 \le y < 1/2 \\ 0 & y < 0 \end{cases}$$

$$C(y) = \begin{cases} 1/4, & y \ge 1/2 \\ y^2, & 0 \le y < 1/2 \\ 0, & y < 0 \end{cases}$$

$$F_Y(y) = \underbrace{C(y)}_{\text{continuous}} + \underbrace{D(y)}_{\text{discrete}}$$

$$D(y) = \begin{cases} 3/4, & y \ge 1/2 \\ 0, & y < 1/2 \end{cases}$$

$$P[1/4 \le Y \le 3/8] = \underbrace{F_Y(3/8) - F_Y(1/4)}_{0 \le y \le 1/2 \text{ case}} + \underbrace{P(Y = 1/4)}$$

for $0 \le y \le 1/2$ y is continuous. point probability is '0'.

$$P[1/4 \le Y \le 3/8] = (3/8)^2 - (1/4)^2 + 0 = 9/64 - 1/16 = 5/64$$

$$P[Y \ge 1/4] = 1 - F_Y(1/4) + P[Y = 1/4]$$

$$= 1 - (1/4)^2 + 0 = 1 - 1/16 = \frac{15}{16} = 60/64$$

$$c(y) = \frac{dC(y)}{dy} = \begin{cases} 2y & 0 \le y < 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[Y] = \int_{-\infty}^{\infty} y\, c(y)\, dy + \sum_{y_k} y_k P[y = y_k]$$

$$E[Y] = \int_0^{1/2} y(2y)\, dy + \frac{1}{2} P[Y = 1/2]$$

corresponds to the jump at $y = 1/2$ by $3/4$.

$$E[Y] = \frac{2y^3}{3}\Big|_0^{1/2} + \frac{1}{2}\left(\frac{3}{4}\right) = \frac{2}{3}\left(\frac{1}{8}\right) + \frac{3}{8}$$

$$E[Y] = 11/24$$

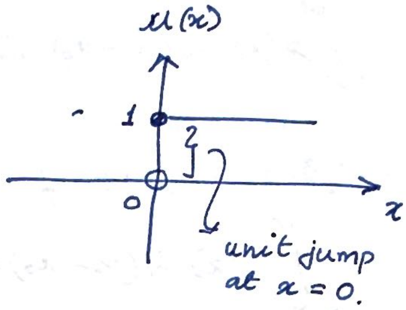
Note.

1. CDF can answer questions about discrete, continuous and mixed RVs.
2. PDF is defined only for continuous RV.
3. PMF is defined only for discrete RV.
4. We cannot define PDF for discrete RVs since its CDF has jumps and we cannot differentiate CDF at jump points.

Dirac Delta Function

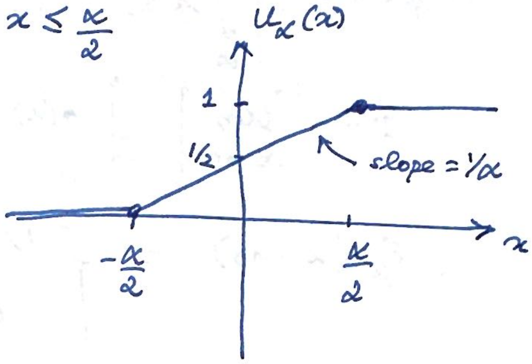
We define the unit step function as:

$$u(x) = \begin{cases} 1 & x \geqslant 0 \\ 0 & \text{otherwise} \end{cases}$$



How does one remove this jump?
Define.

$$u_\alpha(x) = \begin{cases} 1, & x > \alpha/2 \\ \frac{1}{\alpha}(x + \alpha/2), & -\frac{\alpha}{2} \leq x \leq \frac{\alpha}{2} \\ 0, & x < -\alpha/2 \end{cases}$$



$$u = \frac{x}{\alpha} + \frac{1}{2}$$

slope $= 1/\alpha$
$u \to$ intercept $= 1/2$

$$u_\alpha(x) = \frac{x}{\alpha} + 1/2 \quad , \quad -\alpha/2 \leq x \leq \alpha/2$$

$$u = x/\alpha + 1/2$$

$$u = \underbrace{(1/\alpha)}_{\text{slope} = \frac{du_\alpha}{dx} = \frac{1}{\alpha}} x + \underbrace{\frac{1}{2}}_{\text{intercept.}}$$

$$u_\alpha(-\alpha/2) = \frac{-\alpha/2}{\alpha} + 1/2 = 0$$

$$u_\alpha(\alpha/2) = \frac{\alpha/2}{\alpha} + 1/2 = 1$$

$$u_\alpha(0) = 1/2$$

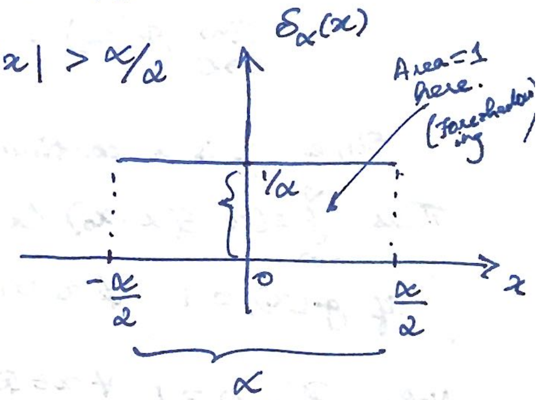
Define $\delta_\alpha(x)$ as the derivative of $u_\alpha(x)$ wrt x.

$$\delta_\alpha(x) = \frac{d u_\alpha(x)}{dx} = \begin{cases} 1/\alpha, & |x| < \alpha/2 \\ 0, & |x| > \alpha/2 \end{cases}$$

Now consider the limit of $u_\alpha(x)$ as $\alpha \to 0$

$$u(x) = \lim_{\alpha \to 0} u_\alpha(x)$$

Except now the height $\to \infty$.



Dirac Delta function:

$$\delta(x) = \lim_{\alpha \to 0} \delta_\alpha(x) = \begin{cases} \infty, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

As $\alpha \to 0$ $\delta_\alpha(x) \to \infty$
width $\to 0$ height $\to \infty$

$$\delta(x) = \frac{d}{dx} u(x)$$

Note: When using $\delta(x)$ have $\delta_\alpha(x)$ in mind with small α.

Let the continuous function $g:\mathbb{R}\to\mathbb{R}$.

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = \lim_{\alpha\to 0}\left[\int_{-\infty}^{\infty} g(x)\,\delta_\alpha(x-x_0)\,dx\right]$$

Shifting property of δ function.

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0)$$

Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function

Proof: $$I = \int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx$$

$$I = \lim_{\alpha\to 0}\left[\int_{-\infty}^{\infty} g(x)\,\delta_\alpha(x-x_0)\,dx\right]$$

$$\delta_\alpha(x-x_0) = \begin{cases} 1/\alpha, & |x-x_0| < \alpha/2 \\ 0, & |x-x_0| > \alpha/2 \end{cases}$$

$$I = \lim_{\alpha\to 0}\int_{x_0-\alpha/2}^{x_0+\alpha/2} \frac{g(x)}{\alpha}\,dx = \frac{g(x_\alpha)}{\alpha}\left[\left(x_0+\frac{\alpha}{2}\right)-\left(x_0-\frac{\alpha}{2}\right)\right]$$

mean value theorem of calculus.

$$I = \lim_{\alpha\to 0}\frac{g(x_\alpha)}{\alpha}\left[\left(x_0+\frac{\alpha}{2}\right)-\left(x_0-\frac{\alpha}{2}\right)\right] \Rightarrow \frac{g(x_\alpha)}{\alpha}\cdot\alpha \Rightarrow g(x_\alpha)$$

$$I = \lim_{\alpha\to 0} g(x_\alpha) = g(x_0)$$

$$\lim_{\alpha\to 0} g(x_\alpha) = g\left(\lim_{\alpha\to 0} x_\alpha\right) = g(x_0)$$

Since g is a continuous function and $\lim_{\alpha\to 0} x_\alpha = x_0$.

Thus $\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = g(x_0)$

If $g(x)=1 \;\forall x\in\mathbb{R} \Rightarrow \int_{-\infty}^{\infty}\delta(x)\,dx = 1$

Note: If $g(x)=1 \;\forall x\in\mathbb{R} \Rightarrow \int_{-\infty}^{\infty}\delta(x)\,dx = 1 \Rightarrow$ Area under the Dirac Delta distribution is 1.

$\delta(x)$ is NOT a valid function since there is no condition function that can satisfy the conditions below:

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0$$

$$\int_{-\infty}^{\infty}\delta(x)\,dx = 1$$

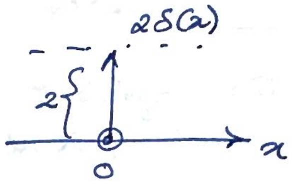
Properties of the Dirac delta function

① $\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$

② $\delta(x) = \dfrac{d}{dx}(u(x))$ where $u(x)$ is the unit step function

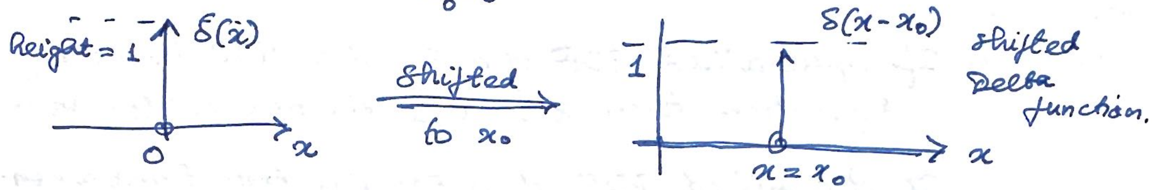
$u(x) = \begin{cases} 1, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$

③ $\int_{-\epsilon}^{+\epsilon} \delta(x)\, dx = 1$ for any $\epsilon > 0$



④ For any $\epsilon > 0$ and $g(x)$ that is continuous over $(x_0 - \epsilon, x_0 + \epsilon)$ then, SHIFTING PROPERTY

$$\int_{-\infty}^{\infty} g(x)\, \delta(x - x_0)\, dx = \int_{x_0-\epsilon}^{x_0+\epsilon} g(x)\, \delta(x - x_0)\, dx = g(x_0)$$

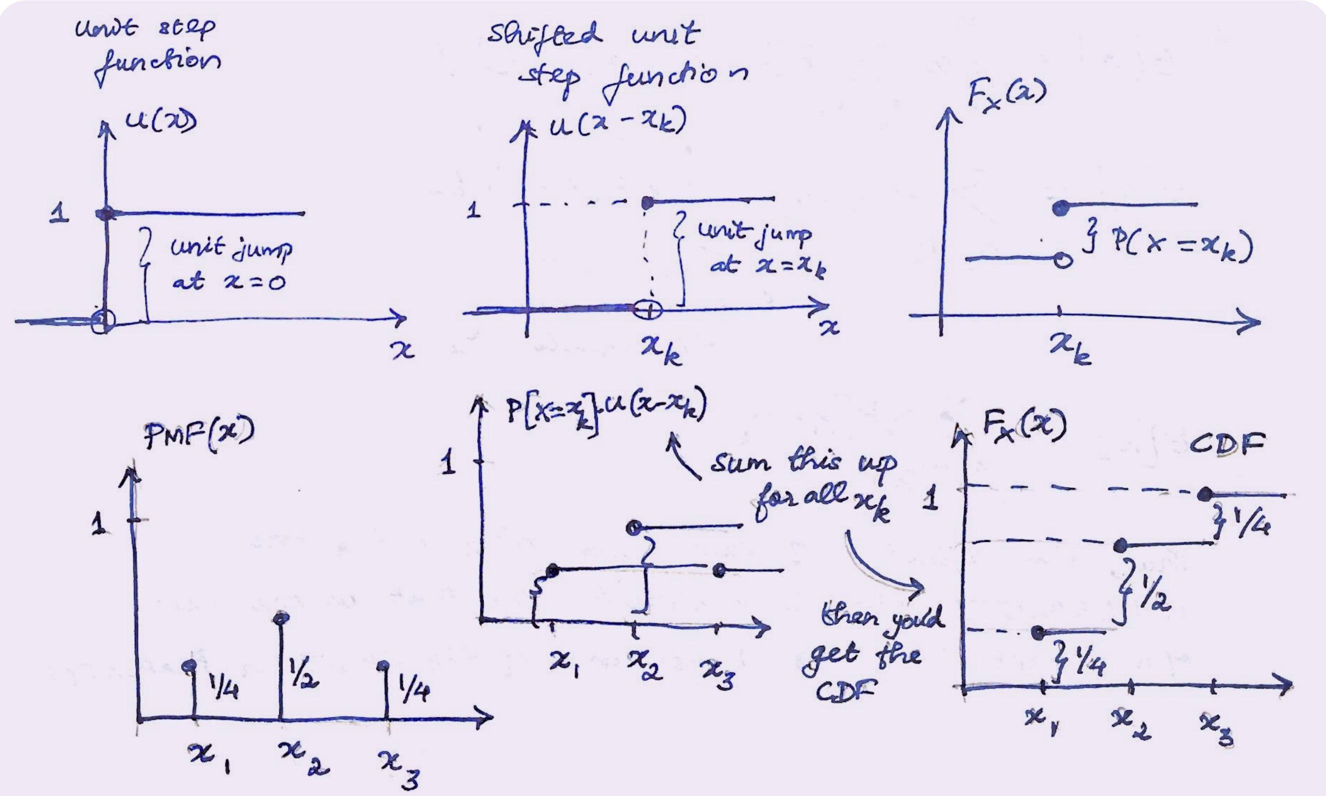


Idea: Use Dirac delta function to extend the definition of PDF to mixed and discrete RVs.

Say, X is a discrete RV with range $= R_X = \{x_1, x_2, \ldots\}$

and PMF : $P_X(x_k) = P[X = x_k]$

CDF : $F_X(x) = P[X \leq x] = \sum_{x_k \in R_X} P[X = x_k]\, u(x - x_k)$



Now, in order to obtain the PDF of X,

$$f_X(x) = \frac{dF_X(x)}{dx} = \sum_{x_k \in R_X} \underbrace{P_X(x_k)}_{\text{PMF of } X} \underbrace{\frac{d}{dx}\left[u(x-x_k)\right]}_{\delta(x-x_k)}$$

Recall $\delta(x-x_0) = \frac{d}{dx}\left(u(x-x_0)\right)$

$$\boxed{f_X(x) = \sum_{x_k \in R_X} P_X(x_k)\, \delta(x-x_k)}$$

GENERALIZED PDF for the discrete RV 'X'

The coefficient of $\delta(x-x_k)$ is the PMF of X at x_k $P[X=x_k]$.

Note: All RVs have a generalized PDF.

If generalized PDF of a RV is a sum of $\delta(x)$ functions then X is a discrete Random variable

If generalized PDF of a RV does NOT include any δ function then X is a continuous random variable.

If Generalized PDF of a RV has both δ and a non-δ func. then X is a mixed random variable.

Expectation Value

The definition of expected value & variance are the same for all kinds of RV X

$$E[X] = \int_{-\infty}^{\infty} x \underbrace{f_X(x)}_{\text{PDF}}\, dx$$

Suppose we have a discrete RV X, we can write

$$f_X(x) = \sum_{x_k \in R_X} P[X=x_k]\, \delta(x-x_k).$$

$$\therefore \quad E[X] = \int_{-\infty}^{\infty} x \sum_{x_k \in R_X} P[X=x_k]\, \delta(x-x_k)\, dx$$

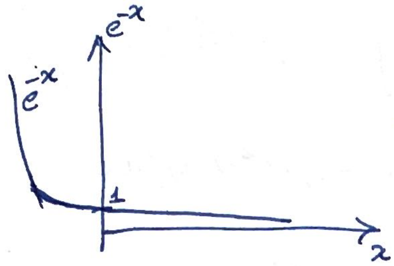
$$E[X] = \sum_{x_k \in R_X} P[X=x_k] \underbrace{\int_{-\infty}^{\infty} \overbrace{x}^{g(x)=x}\, \delta(x-x_k)\, dx}_{\text{By shifting property this equals } x_k}$$

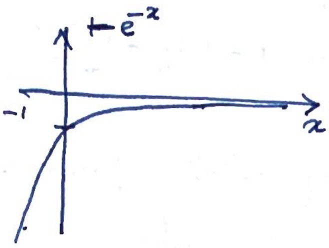
$$E[X] = \sum_{x_k \in R_X} x_k P[X=x_k]$$

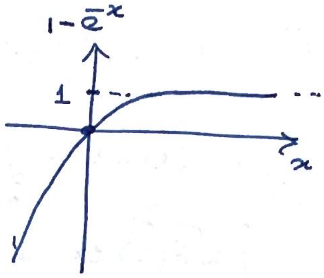
Thus, the definition of expectation value using the GENERALIZED PDF is consistent with that in the case of a discrete RV. $\Rightarrow$ Importance of the SHIFTING PROPERTY!

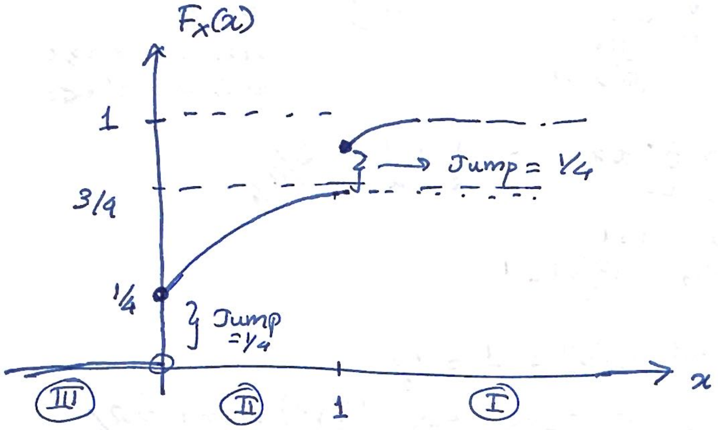
Example. $F_X(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}(1-e^{-x}), & x \geq 1 & \text{I} \\ \frac{1}{4} + \frac{1}{2}(1-e^{-x}), & 0 \leq x < 1 & \text{II} \\ 0, & x < 0 & \text{III} \end{cases}$

Obtain mean & variances.









CDF has two jumps at $x=0$ and $x=1$ of magnitude $1/4$.

Thus, it has two unit step functions of magnitude $1/4$ at $x=0$ and $x=1$

$\therefore \frac{1}{4}u(x) + \frac{1}{4}u(x-1)$

The continuous part of the CDF is $\frac{1}{2}(1-e^{-x})$ for $x>0$

$$\frac{d}{dx}\left[\frac{1}{2}(1-e^{-x})\right] = \frac{1}{2}e^{-x}, \quad x>0$$

Note: $\delta(x) = \frac{d\,u(x)}{dx}$

$f_X(x) = \frac{dF_X(x)}{dx}$

$$F_X(x) = \frac{1}{4}u(x) + \frac{1}{4}u(x-1) + \frac{1}{2}(1-e^{-x}).$$

$$f_X(x) = \frac{1}{4}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{2}e^{-x}u(x)$$ for $x>0$

$$F_X(0.5) = \underbrace{\frac{1}{4}u(0.5)}_{=1} + \underbrace{\frac{1}{4}u(0.5-1)}_{=0} + \underbrace{\frac{1}{2}(1-e^{-0.5})}_{=-\frac{1}{2}e^{-0.5}+\frac{1}{2}}$$

$$F_X(0.5) = \frac{1}{4} + \frac{1}{2} - \frac{1}{2}e^{-0.5}$$

$$F_X(0.5) = 3/4 - \frac{1}{2}e^{-0.5}$$

$$P[X>0.5] = 1 - P[X \leq 0.5] = [1 - F_X(0.5)]$$

$$P[X>0.5] = \int_{0.5}^{\infty} f_X(x)\,dx = \int_{0.5}^{\infty}\left[\frac{1}{4}\delta(x) + \frac{1}{4}\delta(x-1) + \frac{1}{2}e^{-x}u(x)\right]dx$$

$$P[X>0.5] = \underbrace{\frac{1}{4}\int_{0.5}^{\infty}\delta(x)\,dx}_{\substack{=0 \\ \text{peak at } x=0}} + \underbrace{\frac{1}{4}\int_{0.5}^{\infty}\delta(x-1)\,dx}_{\substack{=1 \\ \text{peak at } x=1}} + \frac{1}{2}\int_{0.5}^{\infty}e^{-x}\underbrace{u(x)}_{=1\ \forall x \in [0.5,\infty)}\,dx.$$

$$P[X>0.5] = 0 + \frac{1}{4} + \frac{1}{2}\int_{0.5}^{\infty} e^{-x}\,dx = \frac{1}{4} + \frac{1}{2}\left[\frac{e^{-x}}{-1}\right]_{0.5}^{\infty}$$

$$P[X>0.5] = \frac{1}{4} - \frac{1}{2}\left[\underbrace{e^{-\infty}}_{=0} - e^{-0.5}\right] = 1/4 + 1/2\, e^{-0.5} \simeq \underline{\underline{0.55}}$$ Equal

$$P[X>0.5] = 1 - F_X(0.5) = 1 - \left[3/4 - 1/2\, e^{-0.5}\right] = 1/4 + 1/2\, e^{-0.5} \simeq 0.55 /\!/$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x\left[\frac{1}{4}\delta(x) + \frac{1}{4}\delta(x-1) + 1/2\, e^{-x}u(x)\right]dx$$

$$E[X] = \frac{1}{4}\int_{-\infty}^{\infty} \overbrace{x\,\delta(x)}^{0}\,dx + \frac{1}{4}\int_{-\infty}^{\infty} \overbrace{x\,\delta(x-1)}^{1}\,dx + \frac{1}{2}\int_{-\infty}^{\infty} x e^{-x} u(x)\,dx$$

$g(x) = x$, $g(0) = 0$ $\qquad$ $g(x) = x$, $g(1) = 1$ $\qquad$ $u(x) = 1 \;\forall x \geq 0$, $u(x) = 0 \;\forall x \leq 0$

$$E[X] = \frac{1}{4}(0) + \frac{1}{4}(1) + \frac{1}{2}\int_{0}^{\infty} x e^{-x}\,dx = \frac{1}{4} + \frac{1}{2} = 3/4 /\!/$$

$Y \sim Exp(\lambda)$ $\Leftarrow$ Exponential distribution $\quad \lambda = 1 \;\therefore E[Y] = 1$

$E[Y] = 1/\lambda$

why? If $\lambda = 1$, $f_Y(y) = e^{-y}$, $E[Y] = \frac{1}{1} = 1 /\!/$.

If $\lambda = 1$, $\underbrace{Var(Y)}_{=1} = E[Y^2] - (E(Y))^2 \Rightarrow E[Y^2] = 1 + 1 = 2 /\!/$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \frac{1}{4}\underbrace{\int_{-\infty}^{\infty} x^2\delta(x)\,dx}_{=0} + \frac{1}{4}\underbrace{\int_{-\infty}^{\infty} x^2\delta(x-1)\,dx}_{=1} + \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-x} u(x)\,dx$$

$g(x) = x^2$, $g(0) = 0$ $\qquad$ $g(x) = x^2$, $g(1) = 1^2 = 1$

$$E[X^2] = \frac{1}{4} + \frac{1}{2}\underbrace{\int_{0}^{\infty} x^2 e^{-x}\,dx}_{E[Y^2] = 2} = \frac{1}{4} + \frac{1}{2}(2) = 5/4$$

$$Var[X] = E[X^2] - (E[X])^2 = 5/4 - (3/4)^2 = \frac{5}{4} - 9/16 = 11/16$$

Note: Generalized PDF of mixed RV X is

$$f_X(x) = \sum_k a_k\,\delta(x - x_k) + g(x)$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Rightarrow \underbrace{\int_{-\infty}^{\infty}\sum_k a_k\delta(x - x_k)\,dx}_{\sum_k a_k} + \int_{-\infty}^{\infty} g(x)\,dx = 1$$