Martingales

18 February 2021 | Rohith Krishna

Lectures on Computational Finance by Rakesh Nigam

Discrete-time Filtrations

A stochastic process is an infinite collection of random variables $\{X_1, X_2, \ldots\}$. Martingales are a specific type of stochastic process that are used in finance. The basic idea is that a martingale is stochastic process where, under a certain type of probability measure, the expected value of the random variable in the future given all present and past information, is the value of the random variable in the present.

A stochastic process $X(t)$ is specified in $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is a field of non-empty subsets in Ω, also called σ-algebra. We use the idea of σ-algebra to describe the amount of information available at a given moment of time. In typical modelling of stochastic processes in finance, information about random events increases with time. In order to model this increasing information flow, we use the idea of **filtration.**

Definition. A family of fields $(\mathcal{F}_t)$ on probabilty space $(\Omega, \mathcal{F}, P)$ is called a filtration if $\mathcal{F}_s \subset \mathcal{F}_t$ for all $s \leq t$, where $0 \leq t \leq T$.

The filtration $(\mathcal{F}_t)$ are sub-σ-algebras of $\mathcal{F}$. It is interpreted as the information available to an agent at time t. Further, $\mathcal{F}_0$ is the initial information available at time $t = 0$ and is equivalent to the sample space, $\mathcal{F}_0 = \Omega$.

Suppose we have a stochastic process Y_n expressed as a deterministic function of stochastic processes $X_0, X_1, \ldots, X_n$, say $Y_n = f(X_0, \ldots, X_n)$. Here n indexes to discrete time and the process X_n evolves through discrete periods 0 to n.

1. Given information set $\mathcal{F}_n$, we know outcomes $X_0, \ldots, X_n$, using which Y_n can be computed deterministically.
2. Given initial information set $\mathcal{F}_0$ then only X_0, the initial state of the random process X_n is known. Y_n cannot be determined because the evolution of $\{X_n\}$ is yet to occur. However $E[Y_n|\mathcal{F}_0]$ can be calculated.

Example. Consider 2 tosses of a fair coin. The outcome strings are used to define the random variable S_t. The random process could be used to model the upward and downward movement of stocks in 2 discrete time periods. The sample space is $\Omega = \{w_1, w_2, w_3, w_4\}$ representing outcomes H_1H_2, H_1T_2, T_1H_2 and T_1T_2 respectively.

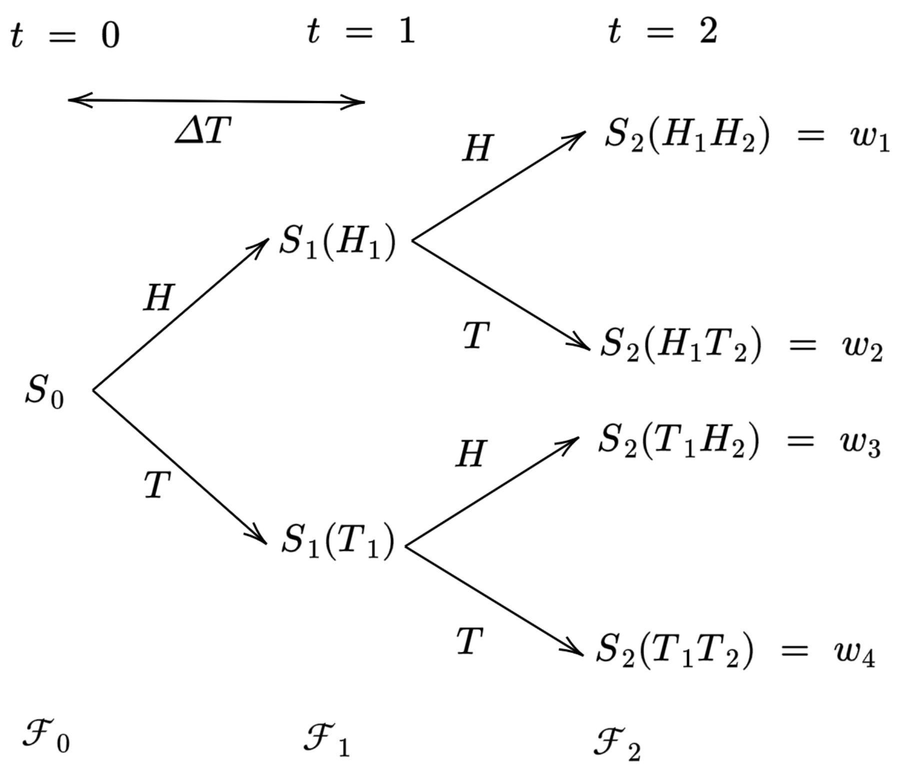

Figure. The tree depicts the evolution of the random process, and the information available at each time step.

$\mathcal{F}_0$ is the initial information and $\mathcal{F}_2$ is that in the last time step. At $t=0$, the process hasn't taken place yet and given information set $\mathcal{F}_0$, it is impossible to distinguish any of the 4 paths - each of them are equally likely outcomes. When we are at $t=1$, the given information set can be $\mathcal{F}_1 = \{w_1, w_2\}$ or $\mathcal{F}_1 = \{w_3, w_4\}$, and upper set of outcomes $\{w_1, w_2\}$ can be distinguished from the lower set $\{w_1, w_2\}$. The paths themselves cannot be distinguished at this point in time. However, when we are at $t=2$, with the given information set $\mathcal{F}_2 = \{\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}\}$ we can distinguish all the 4 paths. To summarize,

$$\begin{aligned}
&\text{At } \ t=0, \ \ E[S_n|\mathcal{F}_o] = E[S_n|\Omega] = E[S_n] = (\text{constant}) \\
&\text{At } \ t=n, \ \ E[S_n|\mathcal{F}_n] = S_n \ (\text{deterministic})
\end{aligned}$$

Constructing Martingales

Obtaining the martingale condition

In this section we define martingales as stochastic processes which obey a certain property. We will also observe how this property facilitates their use in finance, specifically in the context of risk neutral measures.

Let the stochastic process $X_1, X_2, \ldots, X_n$ be iid Bernoulli random variables evolving in discrete time n, with probability of success p. This could refer to the upward movement of a stock price by an amount 1. Let S_n be the sum of all $X_i, i \in \{1, 2, \ldots, n\}$, and could refer to the value of the stock at time n. Let $X_0 = 0$.

$$\begin{aligned}
X_n &= \begin{cases} +1 & wp \ \ p \\ -1 & wp \ \ 1-p \end{cases} && (1) \\
E[X_n] &= p(+1) + (1-p)(-1) = 2p - 1 && (2) \\
S_n &= X_1 + \cdots + X_n = \Sigma_n X_n && (3) \\
S_n &= S_{n-1} + X_n && (4)
\end{aligned}$$

Objective. To find the expected value of the stock at time n, given the information set available till period m.

When $n \leq m$, $E[S_n|\mathcal{F}_m] = S_n$ obtained deterministically. We are specifically interested in the case when $n > m$, that is expected value at a future period given information till present. The Information set available at m and n for non-deterministic case $n > m$:

$$S_n = \underbrace{X_1 + X2 + \cdots + X_m}_{S_m\text{ : m terms} \qquad\qquad \mathcal{F}_m} + \underbrace{X_{m+1} + X_{m+2} + \cdots + X_n}_{\text{(n-m) terms, } n>m \qquad\qquad \mathcal{F}_n} \tag{5}$$

When $n > m$, given $\mathcal{F}_m$ have lesser information about the future outcome than $\mathcal{F}_n$. There are two components to $E[S_n|\mathcal{F}_m]$, a deterministic component because of the past information, and a non-deterministic component about the future.

$$\begin{aligned}
E[S_n|\mathcal{F}_m] &= E[S_m + \underbrace{(X_{m+1} + \cdots + X_n)}_{(n-m)\text{ terms}}|\mathcal{F}_m] \\
&= \underbrace{E[S_m|\mathcal{F}_m]}_{\text{deterministic}} + \underbrace{E[(X_{m+1} + \cdots + X_n)|\mathcal{F}_m]}_{\text{stochastic component}} \quad \begin{bmatrix}\text{linearity of} \\ \text{expectations}\end{bmatrix} \\
&= S_m + (n-m)E[X_i] \\
&= S_m + (n-m)(2p-1)
\end{aligned} \tag{6}$$

Thus we have,

$$E[S_n|\mathcal{F}_m] = \begin{cases} S_n & \text{if } n \le m \\ S_m + (n-m)(2p-1) & \text{if } n > m \end{cases} \tag{7}$$

In Finance we are often interested in the risk neutral measure, where the probability of the stock value going upward is the same as that going downward. Under a risk neutral measure P with $p = 1/2$. The above equation reduces to,

$$E^P[S_n|\mathcal{F}_m] = \begin{cases} S_n & \text{if } n \le m \\ S_m & \text{if } n > m \end{cases} \tag{8}$$

In the second case where $n > m$, we have information till present $\mathcal{F}_m$, and we expect future outcome $E[S_n]$ conditioned on the present information. This equation tells us that under the risk neutral measure, the expected value of the stock in future given present information, is the present value itself. For notational convenience we have dropped the measure P.

This condition, $E[S_n|\mathcal{F}_m] = S_m$ for $n > m$ is called the **martingale condition.** And any stochastic process that obeys this property is called a **martingale**.

$$\begin{aligned}
E[S_n|\mathcal{F}_m] &= S_m \qquad \text{for } n > m \\
E[S_n|\mathcal{F}_m] - S_m &= 0 \\
E[S_n|\mathcal{F}_m] - E[S_m|\mathcal{F}_m] &= 0 \\
E[(S_n - S_m)|\mathcal{F}_m] &= 0
\end{aligned} \tag{9}$$

Or in other words in a risk neutral world, given all the information till present, the expected stock price for the future is the same as the current stock price. In the random walk analogy, this can be attributed to the fact that there exists no drift because the stock price is as likely to move upward as it can downward.

Converting any stochastic process to a martingale

Consider a stochastic process X_n evolving in discrete time. X_i takes values +1 or -1, with probability p and $1-p$ respectively. That is, under this measure P, we have $X_i \sim Bern(p)$. If $S_n = \Sigma_n X_n$, then we have derived the expected value of S_n given information set $\mathcal{F}_m$.

$$E^P[S_n|\mathcal{F}_m] = \begin{cases} S_n & \text{if } n \leq m \\ S_m + (n-m)(2p-1) & \text{if } n > m \end{cases} \tag{10}$$

Under the risk neutral measure RN, $p = 1/2$, we realized that S_n to be a martingale, since $E^{RN}[S_n|\mathcal{F}_m] = S_m$. We have constructed a martingale with respect to the risk neutral measure.

Objective. To convert a stochastic process to a martingale under the specified measure. Specifically, given the Bernoulli distribution of the process in measure P, we wish to construct a new stochastic process M_n such that it satisfies the martingale condition.

Let such a stochastic process M_n be defined by:

$$M_n = S_n - E[S_n] \tag{11}$$
$$M_n = S_n - n(2p-1) \tag{12}$$

Now for the case $n > m$, we require the expected value of M_n (future) conditioned on the present information set $\mathcal{F}_m$, under the same measure P.

$$\begin{aligned} E^P[M_n|\mathcal{F}_m] &= E^P[S_n - n(2p-1) \mid \mathcal{F}_m] \\ &= E^P[S_n \mid \mathcal{F}_m] - E^P[n(2p-1) \mid \mathcal{F}_m] \\ &= S_m + (n-m)(2p-1) - n(2p-1) \\ &= S_m - m(2p-1) = M_m \end{aligned} \tag{13}$$

Since under measure P, the condition $E^P[M_n|\mathcal{F}_m] = M_m$ is satisfied, the stochastic process M_m is a martingale.

Conservation of expectations

For a random variable X and a partition set Y on sample space Ω, the **law of iterated expectations** states that

$$E\big[X \mid \Omega\big] = E\Big[E\big[X \mid Y\big] \mid \Omega\Big] \tag{14}$$

Likewise, we can extend this to any random variable Z, conditioned on different information sets $\mathcal{F}_n$, iteratively with ($n > m$),

$$E\big[Z \underbrace{\mid \ \mathcal{F}_m \ }_{\text{less info.}}\big] = E\Big[\ E\big[\ Z \underbrace{\mid \ \mathcal{F}_n \ }_{\text{more info.}}\big] \underbrace{\mid \ \mathcal{F}_m \ }_{\text{less info.}}\Big] \tag{15}$$

For the martingale defined as $M_n = S_n - E\big[S_n\big]$, we had seen that $E^P\left[M_n \middle| \mathcal{F}_m\right] = M_m$. Take expectations on both sides of this equation,

$$\begin{aligned} E^P\left[M_n \middle| \mathcal{F}_m\right] &= M_m \\ E\Big[E\big[M_n \big| \mathcal{F}_m\big] \mid \mathcal{F}_0\Big] &= E\big[M_m \mid \mathcal{F}_0\big] \quad \left\{\begin{matrix}\text{taking expectations} \\ \text{on both sides.}\end{matrix}\right\} \\ E\big[M_n \mid \mathcal{F}_0\big] &= E\big[M_m \mid \mathcal{F}_0\big] \quad \left\{\begin{matrix}\text{using iterated} \\ \text{expectations.}\end{matrix}\right\} \\ E\big[M_n\big] &= E\big[M_m\big] \quad \left\{\begin{matrix}\text{conservation of} \\ \text{expectations.}\end{matrix}\right\} \end{aligned} \tag{16}$$

Thus the expectation of martingale M_n is a conserved quantity for it does not change with repect to the period of interest n (future) or m (present).

Example. The flow of information and its effect on conditional probability is studied in this example. Consider the movement of asset price movements in a two-step set trajectory that results in payoff Y at $t = 2$. The paths taken to achieve the payoff is the sample space, or the initial information set, $\mathcal{F}_0 = \{w_1, w_2, w_3, w_4, w_5\}$. Assume a uniform probability measure P, that is each path is equally likely.

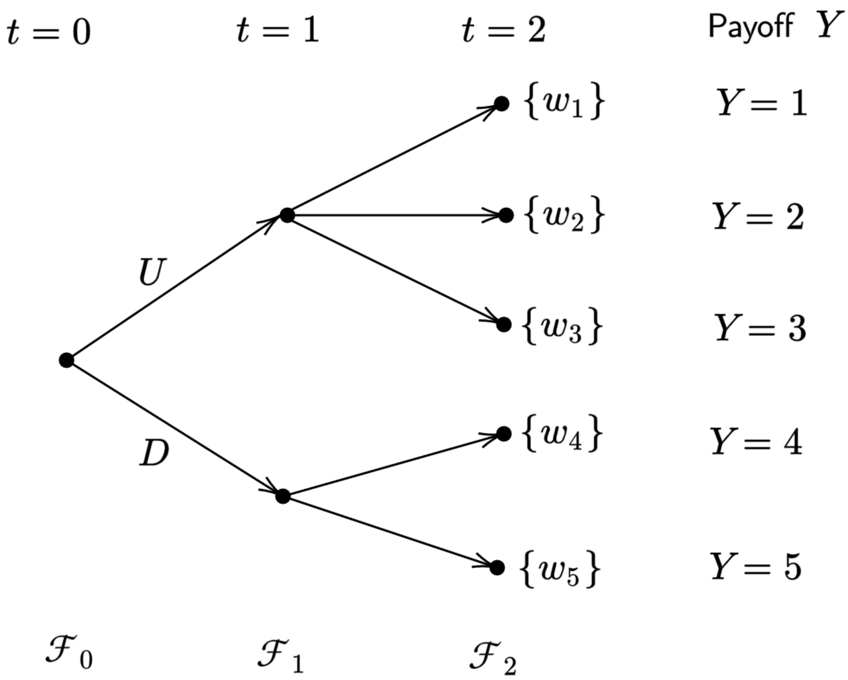

Figure. Paths taken and terminal payoffs received.

The information sets available at different periods are:

$$
\begin{aligned}
\mathcal{F}_0 &= \{w_1, w_2, w_3, w_4, w_5\} \\
\mathcal{F}_2 &= \{\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}, \{w_5\}\} \\
\mathcal{F}_1 &= \{\underbrace{\{w_1, w_2, w_3\}}_{A_U}, \underbrace{\{w_4, w_5\}}_{A_D}\}
\end{aligned}
\tag{17}
$$

The conditional probabilities for each path given different information sets are:

$$
\begin{aligned}
P(w_i | \mathcal{F}_0) &= P(w_i) = 1/5 \;\; \forall i \in \{1,2,3,4,5\} \\
P(A_U | \mathcal{F}_0) &= P(A_U) = 3/5 \\
P(A_D | \mathcal{F}_0) &= P(A_D) = 2/5 \\
P(w_i | \mathcal{F}_1) &= \begin{cases} \begin{cases} 1/3 & \text{if } i \in \{1,2,3\} \\ 0 & \text{if } i \in \{4,5\} \end{cases} & \text{if } \mathcal{F}_1 = A_U \\ \begin{cases} 0 & \text{if } i \in \{1,2,3\} \\ 1/2 & \text{if } i \in \{4,5\} \end{cases} & \text{if } \mathcal{F}_1 = A_D \end{cases} \\
P(w_i | \mathcal{F}_2) &= \begin{cases} 1 & \text{if } i \text{ is realized} \\ 0 & \text{if } i \text{ is not realized} \end{cases}
\end{aligned}
\tag{18}
$$

Using these, the conditional expectations are also now computed,

$$
\begin{aligned}
&E\big[Y\big|\mathcal{F}_0\big] = E[Y] = \frac{1}{5}(1+2+3+4+5) = 3 \\
&E\big[Y\big|\mathcal{F}_1\big] = \begin{cases} E\big[Y\big|A_U\big] = \frac{1}{3}(1+2+3) = 2 & \text{wp } \; P(A_U\big|\mathcal{F}_0) = 3/5 \\ E\big[Y\big|A_D\big] = \frac{1}{2}(4+5) = 4.5 & \text{wp } \; P(A_D\big|\mathcal{F}_0) = 2/5 \end{cases}
\end{aligned} \tag{19}
$$

We notice that conditional expectation in the intermediate time-step, $E[Y\big|\mathcal{F}_1]$ behaves like a random variable, say Z,

$$
Z = E\big[Y\big|\mathcal{F}_1\big] = \begin{cases} 2 & \text{wp } 3/5 \\ 4.5 & \text{wp } 2/5 \end{cases} \tag{20}
$$

Depending on the information set we have in intermediate time periods, we encounter these random variables. In the context of pricing an asset based on its terminal (or maturity) payoffs, we encounter stochastic processes over intermediate time periods.

Using the law of iterated expectation,

$$
\begin{aligned}
\text{Law of Iterated Expectations:} \quad & E\Big[E\big[Y\big|\mathcal{F}_1\big] \; \big|\mathcal{F}_0\Big] = E\big[Y\big|\mathcal{F}_0\big] \\
\text{LHS:} \quad & E\Big[\underbrace{E\big[Y\big|\mathcal{F}_1\big]}_{Z} \; \big|\mathcal{F}_0\Big] = E\big[Z|\mathcal{F}_0\big] = E[Z] = 2\left(\frac{3}{5}\right) + 4.5\left(\frac{2}{5}\right) = 3 \\
\text{RHS:} \quad & E\big[Y\big|\mathcal{F}_0\big] = E[Y] = \frac{1}{5}(1+2+3+4+5) = 3
\end{aligned} \tag{21}
$$

Martingales in Finance

In the context of Finance, we observe that the discounted stock and value processes are martingales under the risk neutral measure. If S_n denotes the stock process, then we define the discounted stock process as $\widetilde{S}_n = S_n e^{-rn\Delta T}$, where r is the risk-free interest rate, n is the number periods in the discrete-time model and ΔT is the intervale between two successive periods. The nature of the original stock process is not a martingale. Under risk-neutral measure $\mathcal{Q}$ we see that,

$$
\mathbb{E}^{\mathcal{Q}}\left[S_{n+1}|\mathcal{F}_n\right] = \mathbb{E}^{\mathcal{Q}}\left[S_{n+1}|S_n\right] = S_n e^{r\Delta T} \tag{22}
$$

However on multiplying the above equation throughout by $e^{-r(n+1)\Delta T}$, we observe that the discounted stock process satisfies the martingale condition.

$$\mathbb{E}^{\mathcal{Q}}\left[S_{n+1}|\mathcal{F}_n\right] = S_n e^{r\Delta T} \tag{23}$$

$$\mathbb{E}^{\mathcal{Q}}\left[e^{-r(n+1)\Delta T}S_{n+1}|\mathcal{F}_n\right] = S_n e^{r\Delta T}e^{-r(n+1)\Delta T} \tag{24}$$

$$\mathbb{E}^{\mathcal{Q}}\left[\widetilde{S}_{n+1}|\mathcal{F}_n\right] = S_n e^{-rn\Delta T} \tag{25}$$

$$\mathbb{E}^{\mathcal{Q}}\left[\widetilde{S}_{n+1}|\mathcal{F}_n\right] = \widetilde{S}_n \tag{26}$$

Therefore, we say that the discounted stock-process is a $\mathcal{Q}$-martingale. Likewise, we also observe that the discounted value process $\widetilde{V}_n$ is also a $\mathcal{Q}$-martingale, where $\widetilde{V}_n$ is defined much in the same way as $\widetilde{S}_n$. This property of such martingales in finance becomes very helpful in pricing options using the method of replication and rebalancing of a self-financing strategy.

Martingale Representation Theorem

Since Finance deals with a host of different martingale processes, we find it pertinent to characterize the set of all martingales. In this context the Martingale Representation Theorem (MRT) in a discrete-time, binary-outcome framework turns out to be a useful result. It is essentially a martingale transform of the discounted stock price process. We shall state and prove this theorem here.

Theorem. (Martingale Representation) Let $(M_n)_{n=0,1,...N}$ be a discrete-time martingale and let $\mathcal{F}_n$ denote the filtration generated by M_n. If $(X_n)_{n=1,...N}$ is another discrete-time martingale with respect to the same filtration $\mathcal{F}_n$, then there exists a predictable process $(\phi_n)_{n=1,...N}$ such that,

$$X_n = X_0 + \sum_{k=1}^{n} \phi_k \left[M_k - M_{k-1}\right] \tag{27}$$

Here, ϕ_k is a predictable stochastic process. It means that when $\mathcal{F}_{n-1}$ is specified, we can predict the value for ϕ_n under the specified measure $\mathcal{Q}$. The martingales X_n and M_n and their evolution through time are specified by the following stencil.

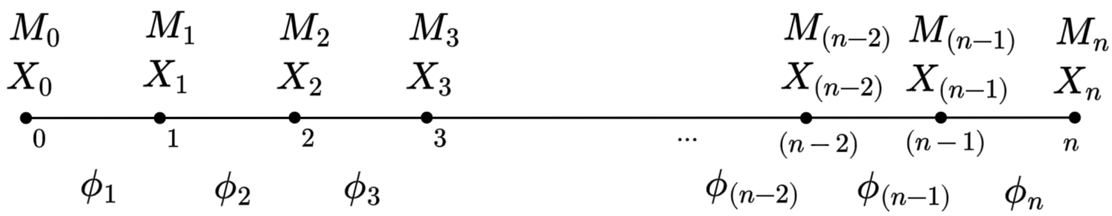

Figure. Stencil for martingale evolution.

Proof. Consider a given market and an n-period discrete-time process where at each step there exists a binary outcome $\Omega = \{H, T\}$. The martingale process comprises of a set of sequences of H and T of length n. We characterize this process by:

$$M_{n+1}(w_1 w_2 \dots w_n H) = M_n(w_1 w_2 \dots w_n).\, u \tag{28}$$
$$M_{n+1}(w_1 w_2 \dots w_n T) = M_n(w_1 w_2 \dots w_n).\, d \tag{29}$$

where, u and d denote the ratio by which the value goes up or down respectively, on the occurrence of H or T. This also referred to as the price evolution rule, and it is linked to the underlying stock process that generates the filtration $\mathcal{F}_n$. We set the risk-free rate $r = 0$. Under the risk-neutral measure $\mathcal{Q}$, we have $P[w = H] = p$ and $P[w = T] = 1 - p = q$.

Consider a particular fixed path taken by M_n can take, $w = w_1 w_2 \dots w_N \in \Omega$. Let $G_n(w)$ refer to the set of all complete paths where the first n outcomes are given by $w_1 w_2 \dots w_n$. We know that X_n is a martingale. Hence for every $n < N$ and path $w \in \Omega$,

$$\mathbb{E}\left[X_{n+1}\mathcal{I}_{G_n(w)}\right] = \mathbb{E}\left[X_n \mathcal{I}_{G_n(w)}\right] \tag{30}$$

where $\mathcal{I}_{G_n(w)}$ denotes the indicator random variable for $G_n(w)$. The above result follows directly from the definition of martingale process. Since we also know that $(X_n)_{0 \le n \le N}$ is adapted to the filtration F_n, the value of $X_{n+1}(w) - X_n(w)$ depends on the path $w = w_1 w_2 \dots w_N$ only for the first $n + 1$ terms. On expanding this from the martingale defintion under risk-neutral measure $\mathcal{Q}$,

$$\mathbb{E}^{\mathcal{Q}}\left[X_{n+1} | \mathcal{F}_n\right] = X_n \tag{31}$$
$$\underbrace{pX_{n+1}(w_1 w_2 \dots w_n H)}_{\text{when } H \text{ in } (n+1)^{th} \text{ period}} + \underbrace{qX_{n+1}(w_1 w_2 \dots w_n T)}_{\text{when } T \text{ in } (n+1)^{th} \text{ period}} = X_n(w_1 w_2 \dots w_n) \tag{32}$$

Since the filtration is generated by martingale $(M_n)_{0 \le n \le N}$, the above equations hold when X_{n+1} and X_n is replace with S_{n+1} and S_n respectively.

$$\mathbb{E}^{\mathcal{Q}}\left[M_{n+1} | \mathcal{F}_n\right] = M_n \tag{33}$$
$$\underbrace{pM_{n+1}(w_1 w_2 \dots w_n H)}_{\text{when } H \text{ in } (n+1)^{th} \text{ period}} + \underbrace{qM_{n+1}(w_1 w_2 \dots w_n T)}_{\text{when } T \text{ in } (n+1)^{th} \text{ period}} = M_n(w_1 w_2 \dots w_n) \tag{34}$$

Rearranging the terms and solving for $-\frac{q}{p}$ we get,

$$\frac{X_{n+1}(w_1 w_2 \dots w_n H) - X_n(w_1 w_2 \dots w_n)}{X_{n+1}(w_1 w_2 \dots w_n T) - X_n(w_1 w_2 \dots w_n)} = -\frac{q}{p} \tag{35}$$

$$\frac{M_{n+1}(w_1 w_2 \dots w_n H) - M_n(w_1 w_2 \dots w_n)}{M_{n+1}(w_1 w_2 \dots w_n T) - M_n(w_1 w_2 \dots w_n)} = -\frac{q}{p} \tag{36}$$

This implies,

$$\frac{X_{n+1}(w_1 w_2 \dots w_n H) - X_n(w_1 w_2 \dots w_n)}{M_{n+1}(w_1 w_2 \dots w_n H) - M_n(w_1 w_2 \dots w_n)} \tag{37}$$

$$= \frac{X_{n+1}(w_1 w_2 \dots w_n T) - X_n(w_1 w_2 \dots w_n)}{M_{n+1}(w_1 w_2 \dots w_n T) - M_n(w_1 w_2 \dots w_n)} \tag{38}$$

$$:= \phi_n(w_1 w_2 \dots w_n) \tag{39}$$

Since the fractions above depend only on the value of $w = w_1 w_2 \dots w_n$, the definition of ϕ_n as a predictable process is valid. We reindex ϕ_n and M_n on the RHS such that k runs from $n = 1$ to N. On summing up individual equations for all $0 \leq n \leq N$, we get the martingale representation relation,

$$X_n - X_0 = \sum_{k=1}^{n} \phi_k \left[M_k - M_{k-1} \right] \tag{40}$$

The intuition in finance is that M_k is akin to the discounted stock process $\widetilde{S}_n$ which generates the filtration $\mathcal{F}_n$. The predictable sequance ϕ_k turns out to be a_k which is the number of units of stocks invested in the replicating portfolio. The martingale representation theorem therefore provides a self-financing portfolio with value process X_n for any option. This self-financing portfolio is defined using the predictable sequence ϕ_n in the martingale transform.