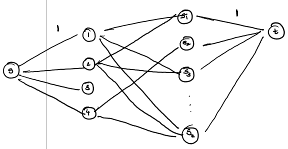
EXAM

27 October 2020 20:23

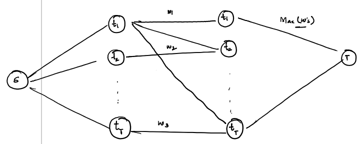
CS 6381 : Assignment #4k: Exam

Due Date:

1. Let Q be a finite set and let $(S_1, S_2, \ldots, S_k)$ be a family of subsets of Q. A **system of distinct representatives** (SDR) is a set $(q_1, q_2, \ldots, q_k)$ of distinct elements of Q, such that $q_i \in S_i; i = 1, 2, .., k$. Show how to test if an SDR exists and how to find one if it does using maximum flow.



2. Elimination of Sports Teams: Consider a scenario with a set T of teams in some league. At some point in the season, some games hav been palyed so far and their outcomes are known. More games are scheduled and we know the upcoming schedule. We want to know if some team is eliminated. A team is eliminated if regardless of the outcomes of the future games, this team has no chance of winning. Each game is won by one team and the other loses the game. [There are no ties]. Let w_i be the number of wins for team i so far [this is given]. Let $r_{i,j} = r_{j,i}$ be the number of games between teams i and j scheduled to be played in future. [this is also given]. We wish to determine whether or not a team s is eliminated. Show how to use max-flow to answer this question. [Hint: Check the book by Cunningham et al.]



3. In the following graph, we do not know the internal connections and capacities. However, we know the values of $F^*_{1,3}, F^*_{1,4}, F^*_{2,3}$, and $F^*_{2,4}$ where $F^*_{i,j}$ represents maximum flows with i as origin and j as destination.

We wish to get either exact values or tightest bounds on maximum flow $F^*_{0,5}$ for each of the following problems [you may use the exact solutions of the previous problems (call them F^*_a, F^*_b etc) in arriving at bounds for the later ones]:

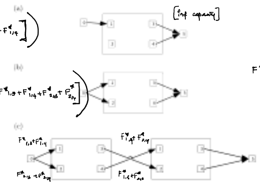
$\{ F^*_{1,3}, F^*_{2,3}, F^*_{1,4}, F^*_{2,4} \}$ $[F^*_{0,5}]$

(a) $\left(\max(F^*_{1,3}, F^*_{1,4}), [F^*_{1,3} + F^*_{1,4}] \right)$

(b) $\left(\max(F^*_{1,3}, F^*_{2,3}, F^*_{1,4}, F^*_{2,4}), [F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4}] \right)$

(?)

(c)



$F^*_{1,3} \to$ Min cut

i) 1) (2, 3, 4

ii) 1, 2) (3, 4

iii) 1, 2, 4) (3

Cases: $F^*_{1,3}$ min cut $\to$ 1) (3, 4 $\to F^*_{1,3}$

1, 4) (3

For (a) we can send a max of $F^*_{1,4}$

that gives a flow of $[F^*_{1,3} + F^*_{1,4}]$ 1) (3, 4

[MAX]

i) 1) (2, 3, 4 $F^*_{1,3}$

But 2 can always be reachable from $\boxed{0}$ Min cut:

$F^*_{2,3}$ 2) (1, 3, 4 $\longrightarrow$ 1, 2) (3, 4 $\leq F^*_{1,3} + F^*_{2,3}$

2, 1) (3, 4 $\longrightarrow$

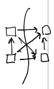
2, 1, 4) (3

4. Determine maximum weight perfect matching in the following graph: Start with the following dual solution:

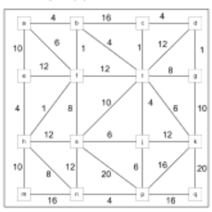
[a=4,b=0,c=16,d=0,e=12,f=2,t=12,g=1,h=0,s=12]

[j=0,k=12,m=16,n=8,p=16,q=8] for both parts.

(a) First remove all "diagonal edges" to get a bipartite graph and solve this problem.

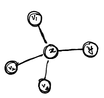


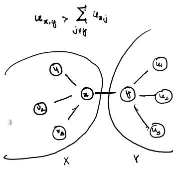
(b) Do this for the general graph.



5. **Let $G = (V, E)$ be an undirected graph. Let $\{u_{i,j}\}; (i,j) \in E$ be non-negative capacities. Suppose $u_{x,y} > \sum_{j \neq y} u_{x,j}$. Show that (x, y) is an edge in Gomory Hu cut-tree.

$$u_{x,y} > \sum_{j \neq y} u_{x,j}$$



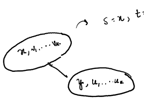


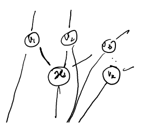
In find cut-tree

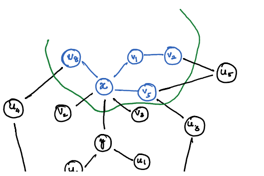
$$\lambda_{x,y} = \min_{e \in P_{x,y}}$$

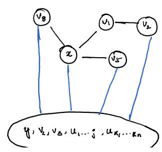
$$u_{x,y} \leq C(X,Y) \leq u_{x,y} + \sum_{j \neq y} u_{x,j} \quad - ①$$

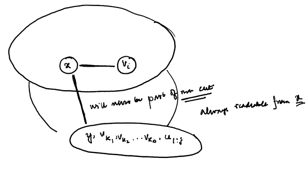
$s = x, \ t = v_i$

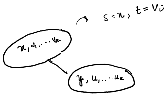


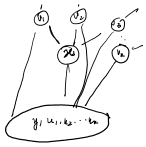


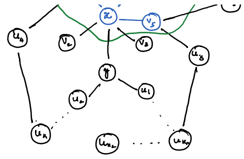


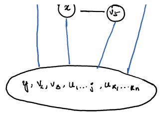


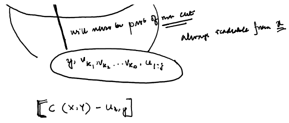


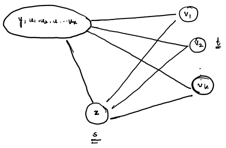

s = x_i , t = v_i








will never be part of min cut
always reachable from



1) Given Q be a finite set G $(S_1, S_2, S_3 \cdots S_k)$ family of subsets of Q

SDR system of distinct representatives $(q_1, q_2, \cdots q_k)$ of distinct elements of Q

$$q_i \in S_i$$

TRIVIAL CASE :

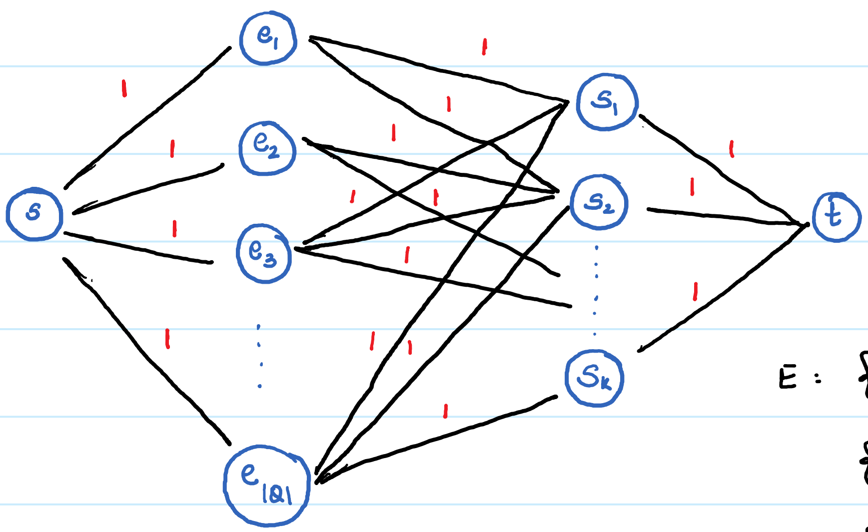
if $k > |Q|$

SDR doesnot exist.

Using MAX FLOW to find one :

We build the following NETWORK. $G(V, E)$, $V = \{s, t\} \cup \{S_i\}$.

$\cup \{ e_i : e_i \in Q \}$



$E : \{ (s, e_i) : e_i \in Q \} \cup$

$\{ (S_j, t) : \forall j < k \} \cup$

$\{ (e_i, S_j) : \forall e_i \in S_j , j < k \}$

Here e_i corresponds to an element in finite set Q.

S_j Corresponds to a subset of Q

Add edges of [CAPACITY = 1] from s to all the vertices e_i

Add edges (e_i, S_j) such that $e_i \in S_j$ with [CAPACITY = 1]

Add edges (S_j, t) with [CAPACITY = 1]

For given collection of subsets $(S_1, S_2, \dots S_k)$ of set Q.

SDR exists only when a network constructed as above for the collection admits a maximum flow of $\boxed{k}$

Any max flow gives us a SDR $(q_1, q_2 \cdots q_k)$.

Now, in order to find $(q_1, q_2 \dots q_k)$

PROCEDURE :

1) Find max flow, if max-flow $< k$ SDR does not exist

2) Apply PATH DECOMPOSITION to find $\underline{k}$ paths.

3) In each path, consider the arc (e_i, s_j)

4) Add. $[q_j = e_i]$

[By the nature of graph considered. As the capacity of each arc considered is $\boxed{1}$ There shall be no overlap of arcs in all the k paths]

This procedure gives a REPRESENTATIVE !!

2) Given set of Teams $\mathcal{T}$,

No. of wins so far for team i - w_i,

No. of games scheduled to be played in future : $[r_{ij} = r_{ji}]$

Determine team s is eliminated (or) not.

Following the notation from book Cunningham et al. :

Let, T be $\mathcal{T} \setminus \{s\}$

w_i (Wins for team i) , r_{ij} (Remaining games to be played)

$P = \{ \{i,j\} : \{i,j\} \subseteq T , i \neq j , r_{ij} > 0 \}$.

$M \rightarrow$ No. of wins for s if they win all their remaining games

$A \subseteq T$

Total no. of wins for teams in A at the end of the season is atleast

$w(A) + \sum (r_{ij} : \{i,j\} \subseteq A , \{i,j\} \in P)$

if $> M|A|$

$\Rightarrow$ Avg. no. of wins of teams in A at the end of season

is $> M$

As M is the most no. of wins for team $\boxed{s}$

So, atleast one team in A shall finish with more wins than $\boxed{s}$

$\Rightarrow$ s is eliminated if $\exists\ A \subseteq T$ s.t

$$w(A) + \sum (r_{ij} : \{i,j\} \subseteq A , \{i,j\} \in P) > M|A|$$

Considering $\boxed{S}$ is not eliminated

For all the other games $\exists\ y_{ij}, y_{ji}$ s.t $y_{ij} + y_{ji} = r_{ij}$

where $y_{ij} \rightarrow$ No. of wins for <u>team i</u> over <u>team j</u>

So, following should be satisfied.

$$y_{ij} + y_{ji} = r_{ij} \quad , \forall \{i,j\} \in P$$

$$w_i + \sum' (y_{ij} : j \in T,\ j \neq i) \leq M \quad \forall i \in T$$

$$y_{ij} \geq 0 \quad \forall \{i,j\} \in P$$

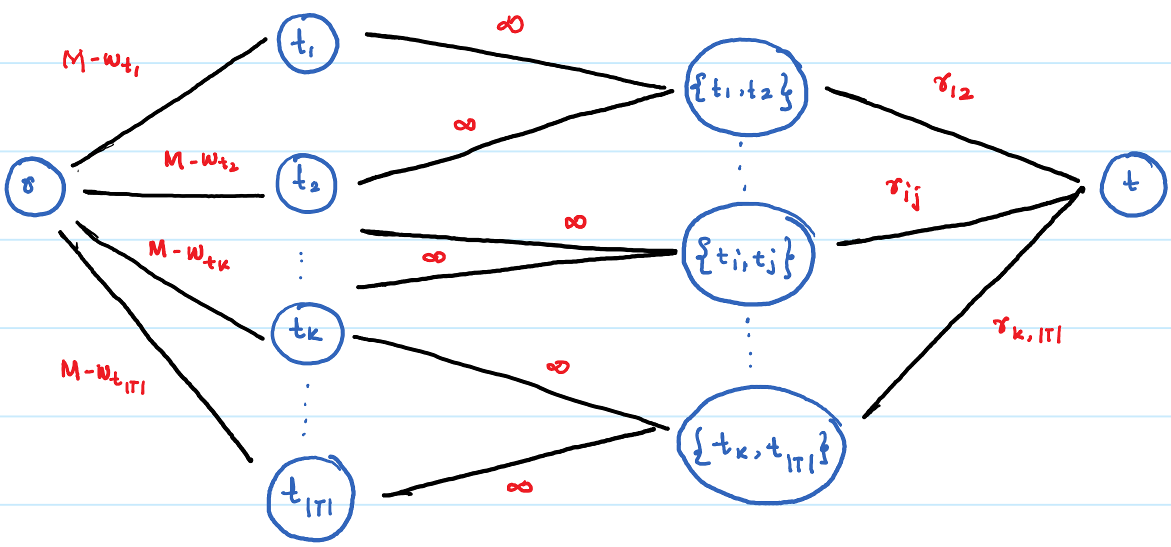
$$y_{ij} \text{ integral } \forall \{i,j\} \in P.$$

Create a flow network $G = (V,E)$, $V = T \cup P \cup \{r,t\}$

$\forall i \in T \ \exists$ arc (r,i) with capacity $M - w_i$

$\forall i \in T, j \in T$ there are arcs $\{i, \{i,j\}\}$, $\{j, \{j,i\}\}$ with capacity ∞

and there is an arc $(\{i,j\}, t)$ with capacity r_{ij}



We can determine if s is eliminated (or) not solving a MAXIMUM FLOW problem.

1) Not eliminated : Maximum flow will determine set of outcomes for games in which $\boxed{s}$ finish first.

2) Eliminated : Min cut determines a set A satisfying

$$w(A) + \sum (r_{ij} : \{i,j\} \subseteq A,\ \{i,j\} \in P) > M|A|$$

Consider $\delta(C)$ be min (r,t)-cut

By Max-flow min cut : $\delta(C) \leq \sum (r_{ij} : \{i,j\} \in P)$

Let $A = T \setminus C$

Claim: $C = \{r\} \cup (T \setminus A) \cup \{ \{i,j\} \in P : i \text{ (or) } j \notin A \}$.

1) if i (or) j is not in A but $\{i,j\} \notin C$ then $\delta(C) = \infty$

2) if $\{i,j\} \in C$ and $i, j \in A$ then deleting $\{i,j\}$ from C decreases capacity of $\delta(C)$ by r_{ij}.

$\Rightarrow$ In either case $\delta(C)$ is not a min-cut [CONTRADICTION !!]

$$\therefore \delta(C) = M|A| - w(A) + \sum (r_{ij} : \{i,j\} \in P,\ \{i,j\} \notin A)$$

5) Let $G = (V, E)$ be an undirected graph, $\{u_{i,j}\}$; $(i,j) \in E$ Non negative capacities

Suppose $u_{x,y} > \sum_{j \neq y} u_{x,j}$ then (x,y) is an edge in Gomory-Hu CUT TREE

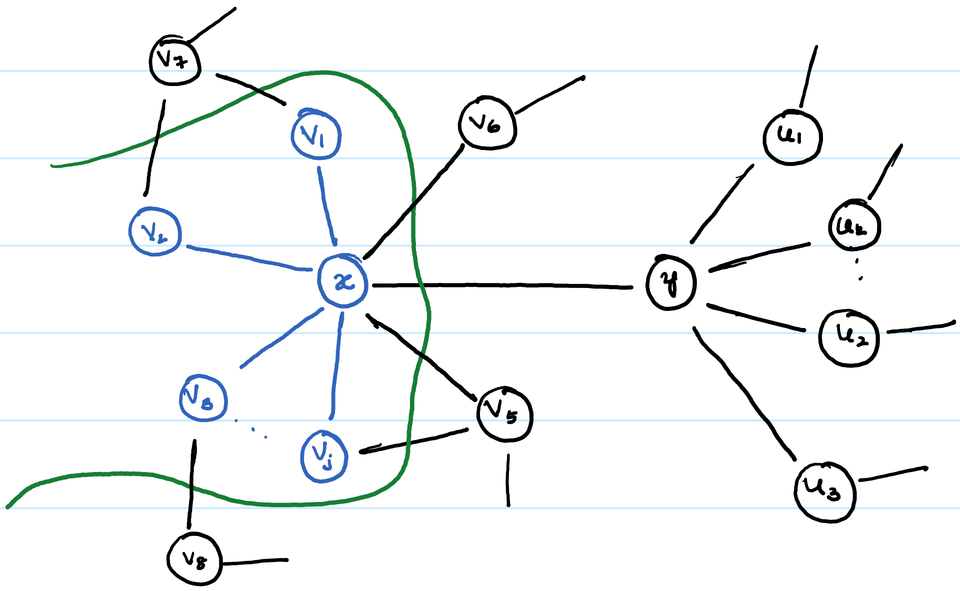
(I)

First consider the arc (x,y) in $G(V,E)$

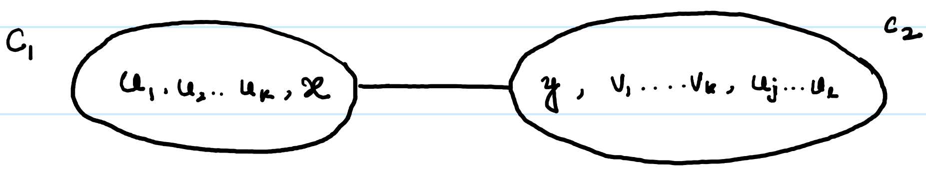
Now, let us begin the development of CUT TREE from $s = x$, $t = y$.

Finding max flow between terminals x and y gives us a MIN-CUT.

that helps in forming first set of nodes/condensed nodes in graph.



Let this min cut separate x, y nodes as follows.



[All u_i are reachable from x in G]

i,e from (I), All the nodes reachable from x in G are part of condensed nodes C_1 & C_2

Claim : As we proceed further in development of cut tree

The arc between $[C_1, C_2]$ develops into arc between $[x, y]$

i.e $\nexists$ node A that shall be between $[x, y]$ in cut tree

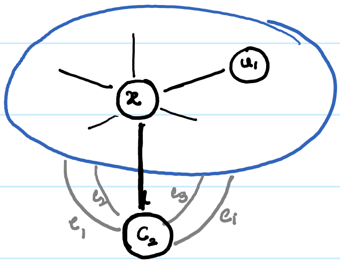
Proof : **Part-1** : Considering condensed node C_1

let us consider $s = x$, $t = u_i$ for some i s.t $u_i \in C_1$

if the condensed node C_2 lies to the side of x in the (x, u_i) min cut

we can say that in this step

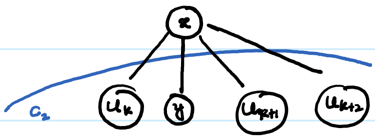
$\nexists$ node A that lies between x & C_2 in cut tree



Consider the graph

Let e_i be arcs between C_2 and nodes in C_1

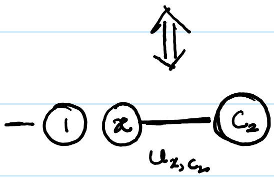
We know $u_{x,y} > \sum_{j \neq y} u_{x,j}$



Now, consider arc $[x, C_2]$

Capacity of this arc in condensed graph

$$u_{x,C_2} = u_{x,y} + \sum_{j} (u_{x,j} : j \in C_2) \geq u_{x,y} \quad -①$$



In development of cut tree for C_1

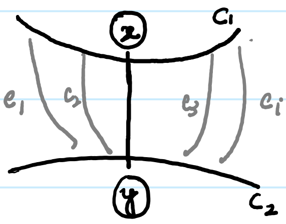
for arc $[x, C_2]$ to be saturated , $f_{x,C_2} = u_{x,C_2}$

In any step of

At condensed node c_2, by flow conservation

$$u_{x,y} \leqslant u_{x,c_2} = f_{x,c_2} \leq \sum_1^l f_{e_i} \leqslant \sum_1^l u_{e_i} \longrightarrow (2)$$

$$\sum_1^l u_{e_i} = c\,[c_1, c_2] - u_{x,c_2} \longrightarrow (3)$$



[Cut capacity of nodes in c_1, c_2]

As $c\,[c_1, c_2]$ is the min cut for (x,y)

$$c\,[c_1, c_2] \leq \text{Any cut } (x,y)$$

Consider cut $(x, V \setminus \{x\})$

$$c\,[c_1, c_2] \leq u_{x,y} + \sum_{j \neq y} u_{x,j} \longrightarrow (4)$$

Using (4) in (3) we get:

$$\sum^l u_{e_i} = \left(u_{xy} + \sum_{j \neq y} u_{x,j}\right) - u_{x,c_2} \longrightarrow (5)$$

From (1) we know $u_{x,c_2} \geqslant u_{x,y}$

(5) can be written as

$$\sum_1^l u_{e_i} \leqslant \sum_{j \neq y} u_{x,j} \longrightarrow (6)$$

Using (6) in (2) gives

[CONTRADICTION by (I) !!]

$$u_{x,y} \leqslant u_{x,c_2} = f_{x,c_2} \leqslant \sum_1^l f_{e_i} \leqslant \sum_1^l u_{e_i} \leqslant \sum_{j \neq y} u_{x,j}$$

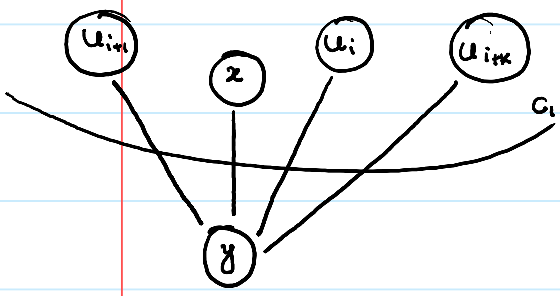
$f_{x,c_2} \neq u_{x,c_2} \Rightarrow f_{x,c_2} < u_{x,c_2}$

$\Rightarrow$ C_2 is always reachable from x in any min cut of (x, u_i) s.t $\underline{u_i \in C_1}$

PART-2 : Now consider the development of condensed node C_2 by keeping C_1

If we can say that C_1 is always reachable from y in any min cut (y, v_i) s.t $v_i \in C_2$ then $\nexists$ node A which shall be between $[y, C_1]$ in cut-tree..

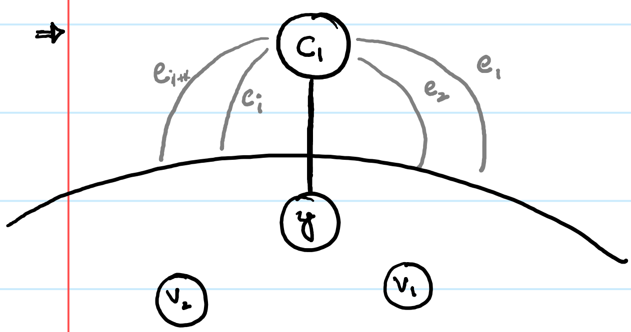
In the condensed graph with nodes $\longrightarrow$ $C_1 \cup \{u_i : u_i \in C_2\}$



$$\Rightarrow u_{C_1,y} = u_{x,y} + \sum{}' (u_{j,y} : j \in C_1)$$

$$\Rightarrow u_{C_1,y} \geq u_{x,y} \longrightarrow 1$$

$\Rightarrow$



Constructing similar argument as PART-1

For arc (y, C_1) to be saturated.

$$u_{y,C_1} = f_{y,C_1} \longrightarrow 2$$

By flow conservation, @ C_1 $\quad f_{y,C_1} \leq \sum{}' f_{e_i}$

$$u_{x,y} \leq u_{y,C_1} = f_{y,C_1} \leq \sum{}' f_{e_i} \leq \sum{}' u_{e_i} \longrightarrow 3$$

$$\sum u_{e_i} = c[C_1, C_2] - u_{y,C_1} \longrightarrow 4$$

$$c[C_1, C_2] \leq u_{x,y} + \sum_{j \neq y} u_{x,j} \longrightarrow 5$$

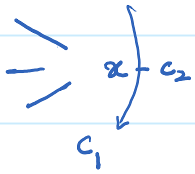
$$\sum' u_{e_i} = \left(u_{x,y} + \sum_{j \neq y}' u_{x,j}\right) - u_{y,c_1}$$

$$\leq \sum_{j \neq y}' u_{x,j} \longrightarrow (5)$$

From (5), (3) we have $u_{x,y} \leq \sum_{j \neq y}' u_{x,j}$ [CONTRADICTION!!]

C_1 shall always be reachable from y (For any min cut (y, v_i) s.t. $v_i \in C_2$)

From PART-1:



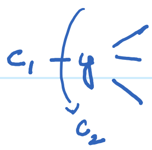
Tree for $C_1 \rightarrow$ has arc $[x - C_2]$

$c_1 - c_2$

Original cut tree has $[C_1 - C_2]$

From PART-2:

Tree for $C_2 \rightarrow$ has arc $[y - C_1]$

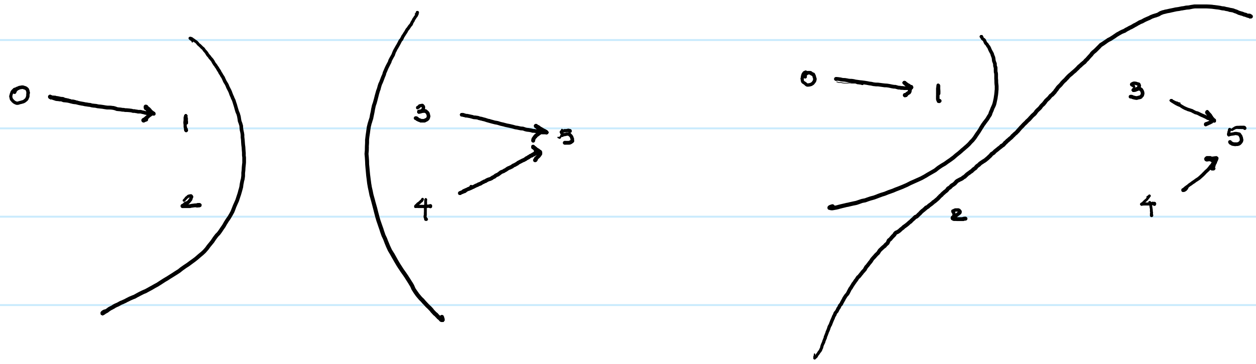


So, we can conclude arc $[x-y]$ is an edge in Gomory Hu Cut tree

(a) Arcs (0,1), (3,5), (4,5) are not part of any (0,5) min-cut.

So, for $F^*_{0,5}$ possible choices for min-cut

1) $(\{0,1,2\}, \{3,4,5\})$ 2) $(\{0,1\}, \{2,3,4,5\})$



All paths through $\boxed{2}$ to 3 (or) 4 originate from $\boxed{1}$ $\Rightarrow$ They shall be part of $F^*_{1,3}$ (or) $F^*_{1,4}$ (or) both

$$\Rightarrow \quad F^*_{0,5} \leq F^*_{1,3} + F^*_{1,4}$$

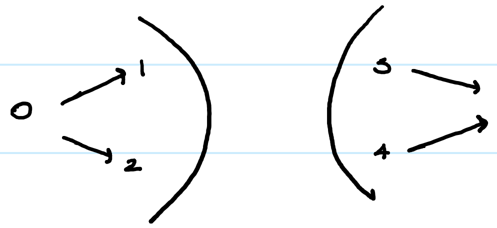
We know $F^*_{1,3}$, $F^*_{1,4}$ both are feasible flows for $F^*_{0,5}$

$$\Rightarrow \quad F^*_{0,5} \geq F^*_{1,3}, \quad F^*_{0,5} \geq F^*_{1,4}$$

$$\Rightarrow \quad F^*_{0,5} \in \left[\max\left(F^*_{1,3}, F^*_{1,4}\right), F^*_{1,3} + F^*_{1,4}\right]$$

(b) For this graph, we know arcs $(0,1)$, $(0,2)$, $(3,5)$, $(4,5)$ cannot be part of any $(0,5)$ min cut

So, nodes $\{0,1,2\}$ shall always be on one side & $\{3,4,5\}$ on the other.



$F^*_{0,5} \geq F^*_{1,3}$, $F^*_{0,5} \geq F^*_{1,4}$

$F^*_{0,5} \geq F^*_{2,3}$, $F^*_{0,5} \geq F^*_{2,4}$

We know $F^*_{1,3} \geq u_{1,3}$, $F^*_{1,4} \geq u_{1,4}$, $F^*_{2,3} \geq u_{2,3}$, $F^*_{2,4} \geq u_{2,4}$

Capacity of this cut $(\{0,1,2\}, \{3,4,5\})$

$\leq u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4}$

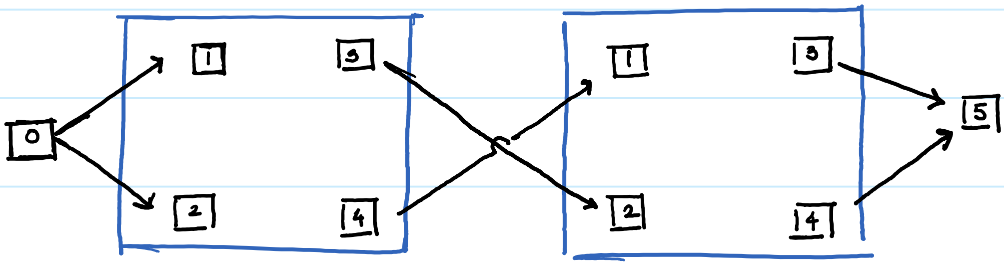
$\leq F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4}$

Since $F^*_{1,3}$, $F^*_{1,4}$, $F^*_{2,3}$, $F^*_{2,4}$ all form feasible flows for $F_{0,5}$

$F^*_{0,5} \geq \max(F^*_{1,3}, F^*_{1,4}, F^*_{2,3}, F^*_{2,4})$

$$F^*_{0,5} \in \left[\max(F^*_{1,3}, F^*_{1,4}, F^*_{2,3}, F^*_{2,4}) \,,\, F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4}\right]$$

(c)



Previous bounds are still valid for this problem i.e

$$\max\left(F^*_{1,3},\ F^*_{1,4},\ F^*_{2,3},\ F^*_{2,4}\right) \le F^*_{0,5} \le \underbrace{F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4}}$$

Checking for more tighter bounds :

Considering notation of $u_{i,j}$ as capacity of edge i,j. [$u_{i,j} = 0$ if arc (i,j) does not exist]

So, for each $F^*_{i,j}$ for known values we know that

$$F^*_{i,j} \geqslant u_{i,j}$$

So, we have

$$F^*_{1,3} \geqslant u_{1,3} \quad , \quad F^*_{1,4} \geqslant u_{1,4} \quad , \quad F^*_{2,3} \geqslant u_{2,3} \quad , \quad F^*_{2,4} \geqslant u_{2,4}$$

Let us start by considering min cuts for $F^*_{1,3}$

1) $(\{1\}, \{2,3,4\})$

2) $(\{1,2\}, \{3,4\})$

3) $(\{1,2,4\}, \{3\})$

We know $F^*_{0,5} \geqslant F^*_{1,3}$ and $F^*_{0,5} \geqslant F^*_{1,4}$.

Here we know arcs $(0,1)$, $(3,5)$, $(4,5)$ cannot be part of Minimum cut for $(0,5)$ as they are of '∞' capacity

For $F^*_{0,5}$ let us start by considering $F^*_{1,3}$ as feasible flow.

Based on the minimum cuts for $F^*_{1,3}$ we can decide on augmenting paths.

CASE-1 : $(\{1\}, \{2,3,4\})$

Flow $F^*_{1,3}$ has saturated all paths from $\boxed{0}$ to $\boxed{5}$

$$F^*_{0,5} = F^*_{1,3}$$

Case-2 : $(\{1,2\},\{3,4\})$

Even in this case $F^*_{1,3}$ has saturated all paths from $\boxed{0}$ to $\boxed{5}$

$F^*_{0,5} = F^*_{1,3}$

Case-3 : $(\{1,2,4\},\{3\})$

Now for feasible flow $F^*_{1,3}$

We have augmenting paths from 0 to 5 through $(1,2,4)$

All the paths through $\boxed{2}$ reach $\boxed{5}$ either from $\boxed{3}$ (or) $\boxed{4}$

So, they shall be part of combination of feasible flows $F^*_{1,3}$, $F^*_{1,4}$

$$\Rightarrow \quad F^*_{0,5} \leq F^*_{1,3} + F^*_{1,4} \quad — (1)$$