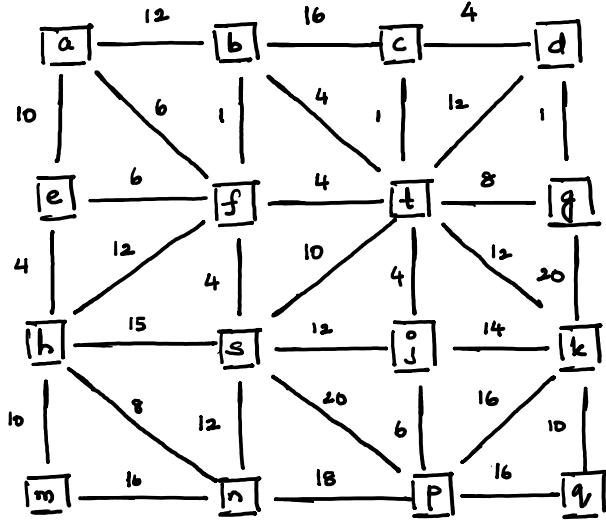


Maximum flows if edge capacities are given by  $\min [u_{ij}, \lambda]$

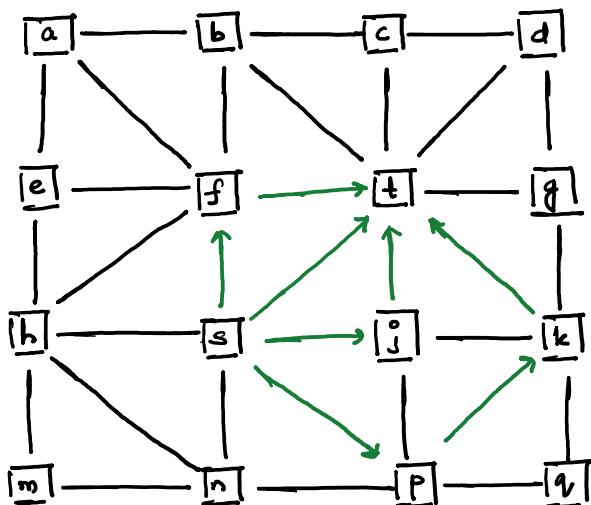
- a) Find maximum surviving flow
- b) Find maximum  $q$ -path flow where  $q=2$



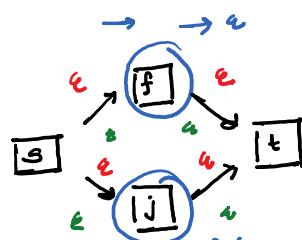
From HW1 when  $\lambda = \infty$

$F(\infty)$ : Max flow with current  $u_{ij} = \boxed{48}$

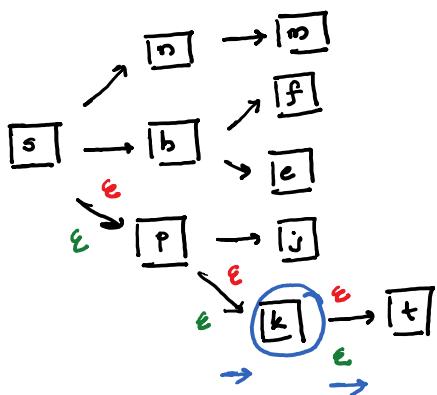
$F(\epsilon)$ : Max flow with  $u_{ij} = \epsilon + \underline{(i,j)} \epsilon$   
for small enough  $\epsilon$ .



Using MKM method for finding maxflow

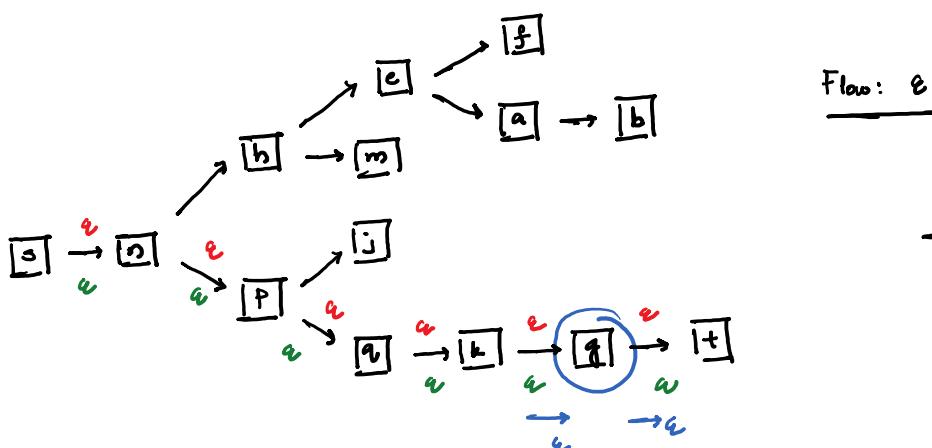
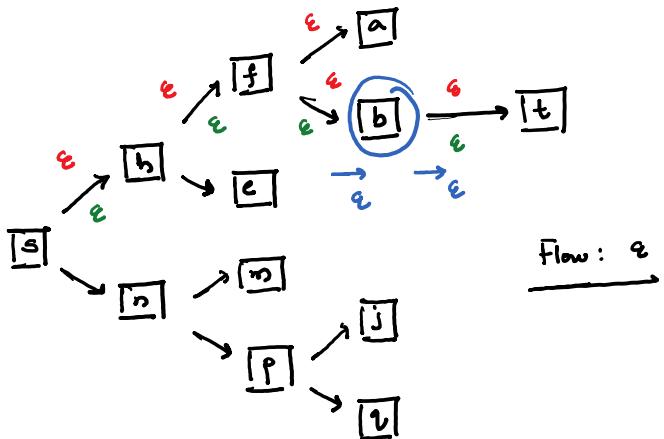
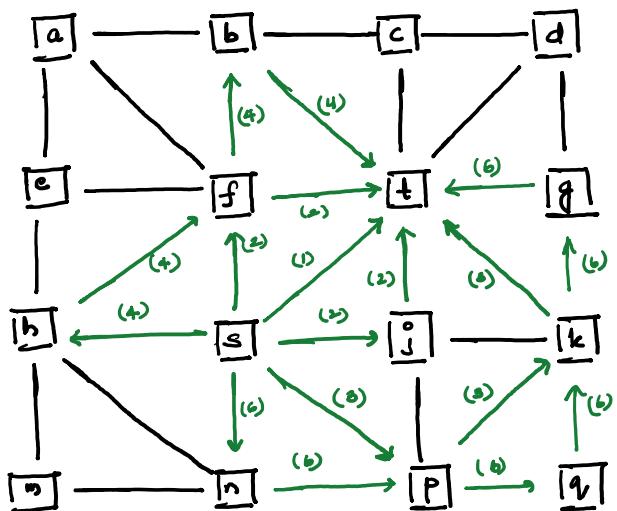


Flow:  $\epsilon + 2\epsilon$



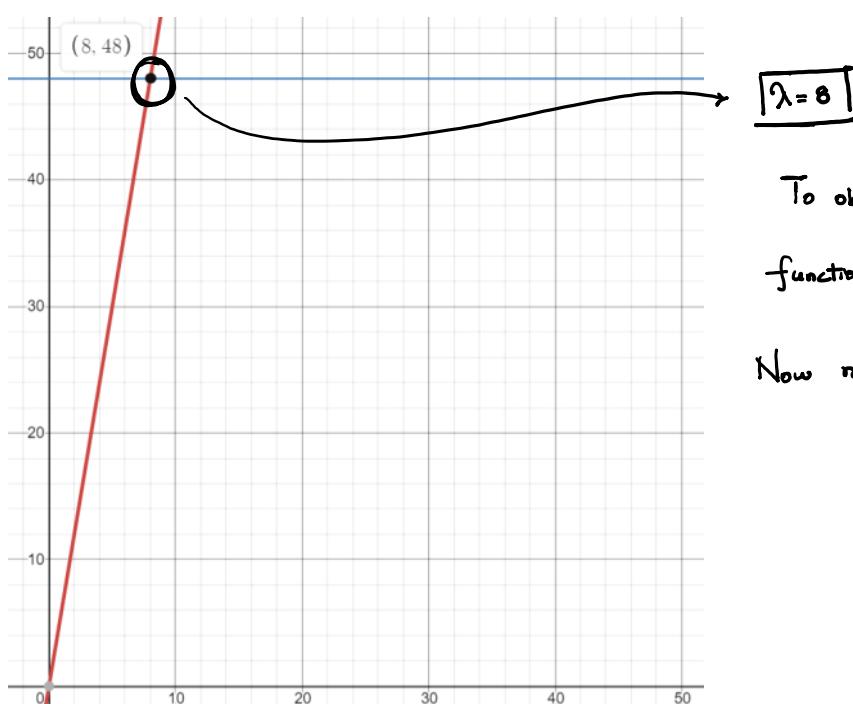
Flow:  $\epsilon$

4

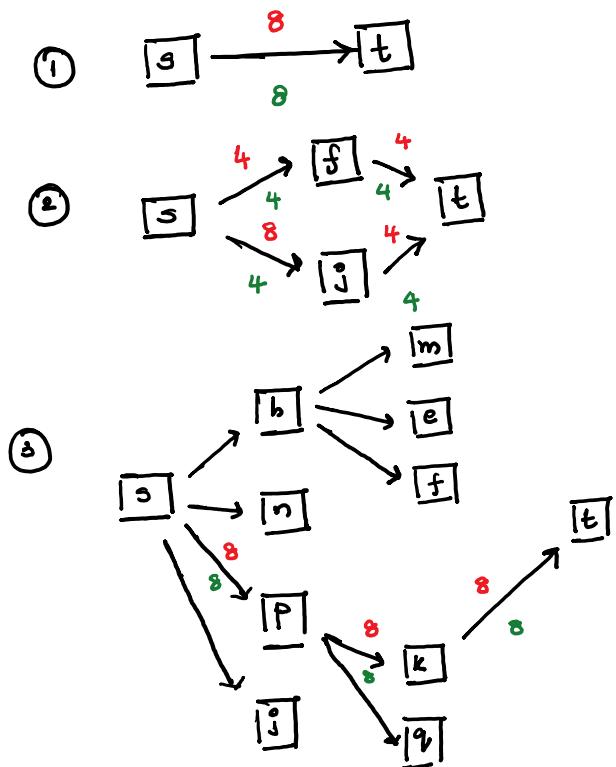
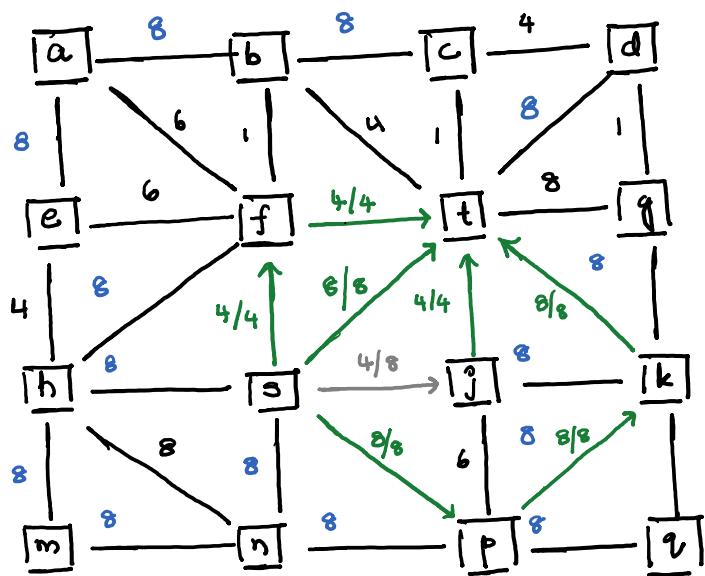
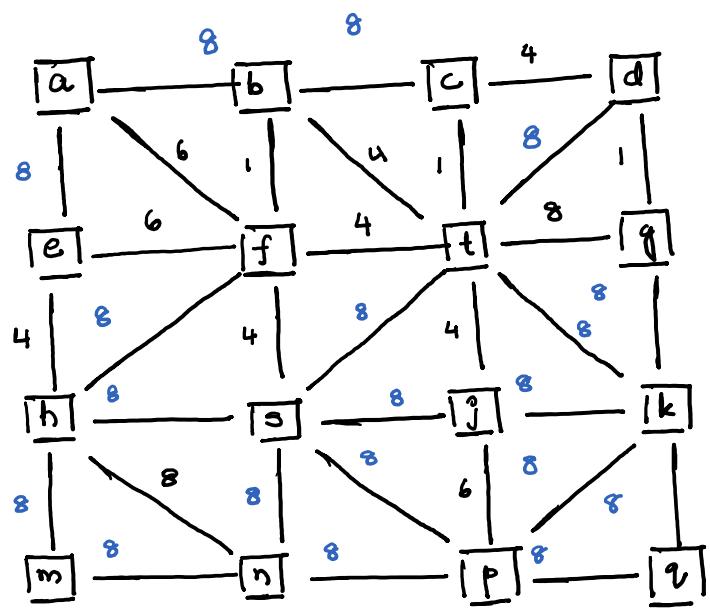
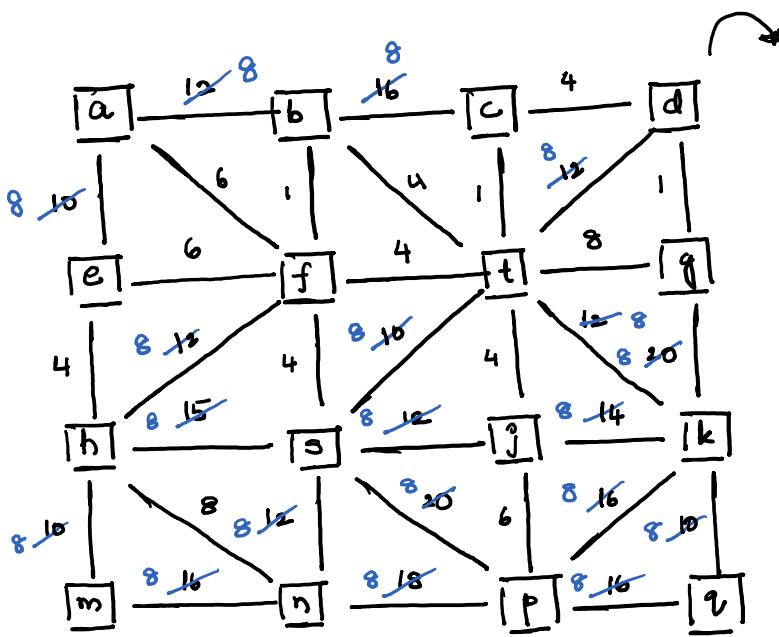


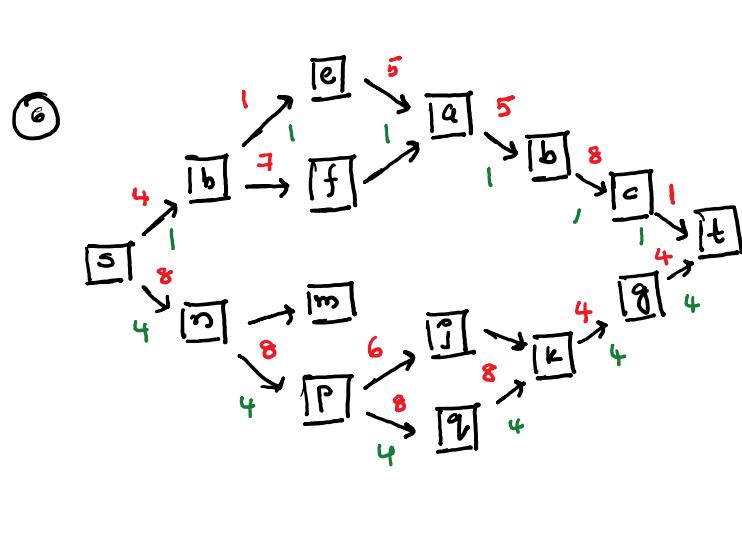
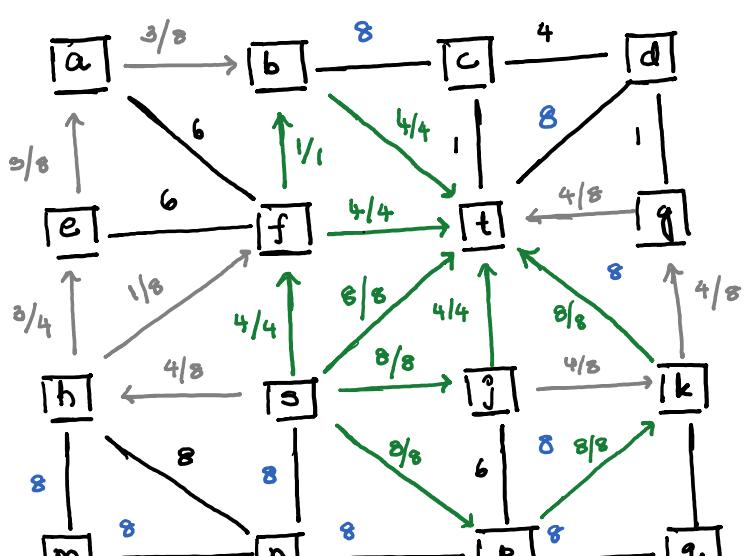
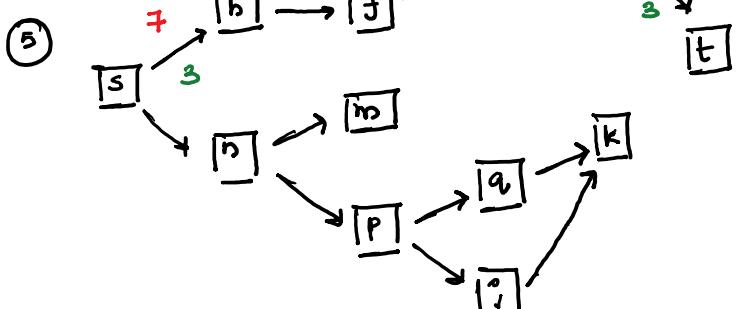
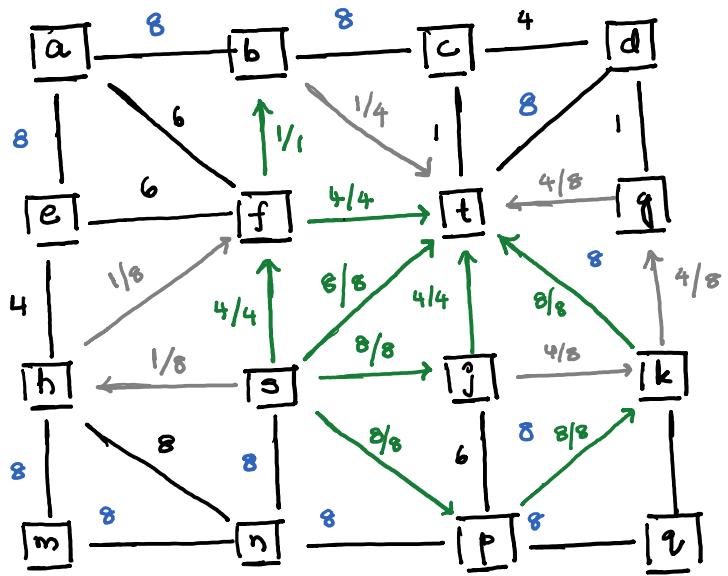
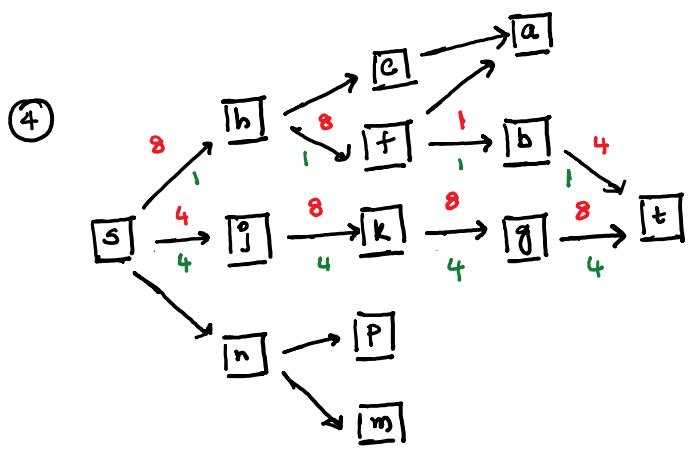
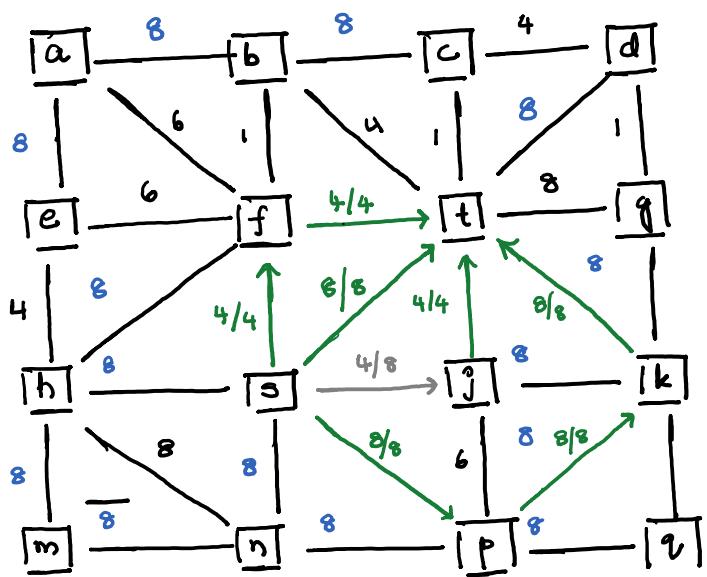
$$\text{Total flow} : 4\epsilon + 2\epsilon = 6\epsilon$$

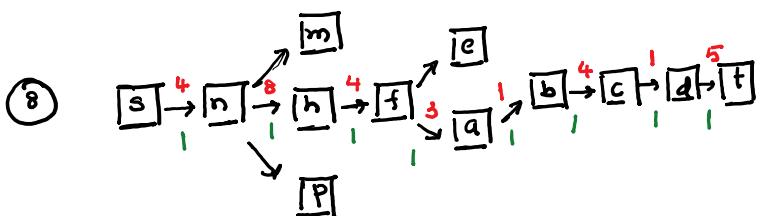
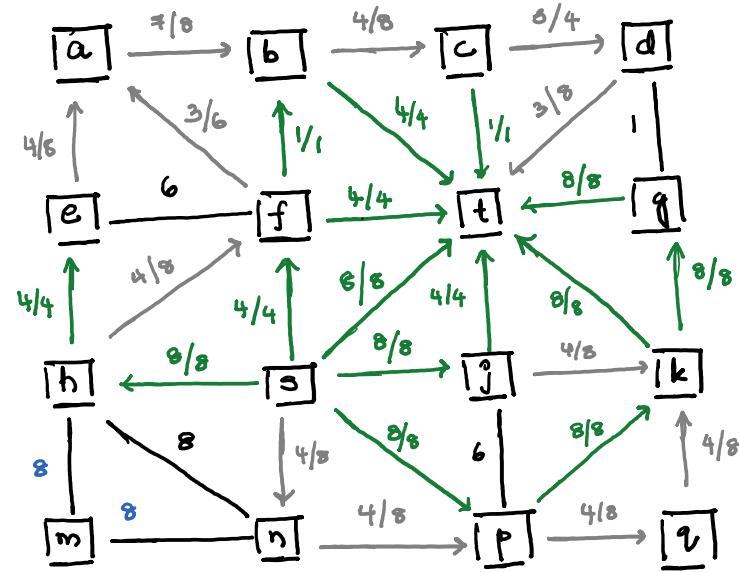
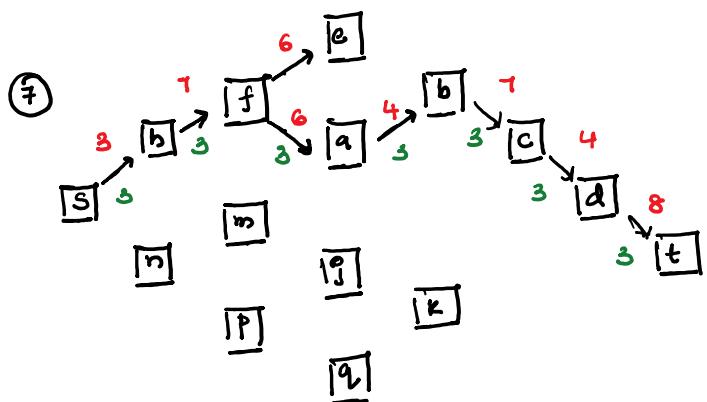
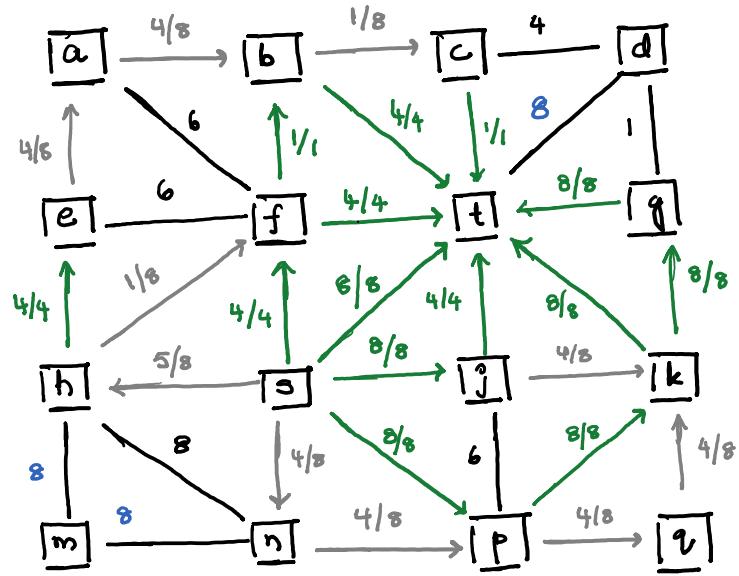
Plotting  $F(\lambda)$  curve with values of  $\lambda = \infty, \lambda = 0$  gives.



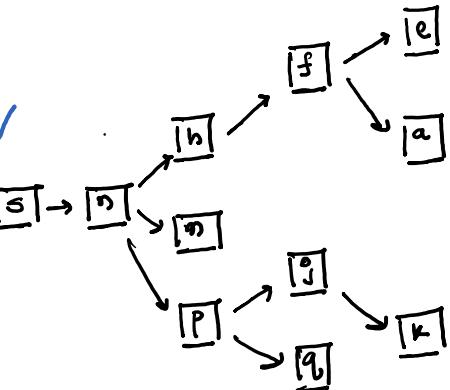
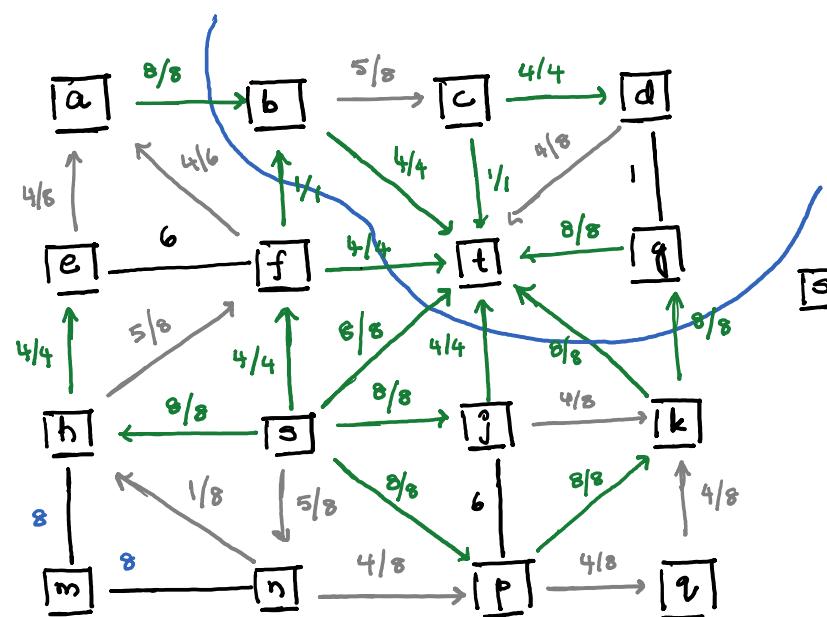
MAX FLOW for min[u, 8]







MIN CUT :



$(b, c, d, t, q)$

$$\text{Flow obtained : } \underline{1+3} + \underline{1+4+3} + \underline{1+4} + \underline{8+8+8} = 24 + 8 + 9 = \boxed{41}$$

No. of edges in min cut with  $\lambda = 8 \Rightarrow \underbrace{(a,b), (s,t), (r,t), (k,q)}_6 \rightarrow \boxed{41} \leftarrow \boxed{\text{Slope}}$

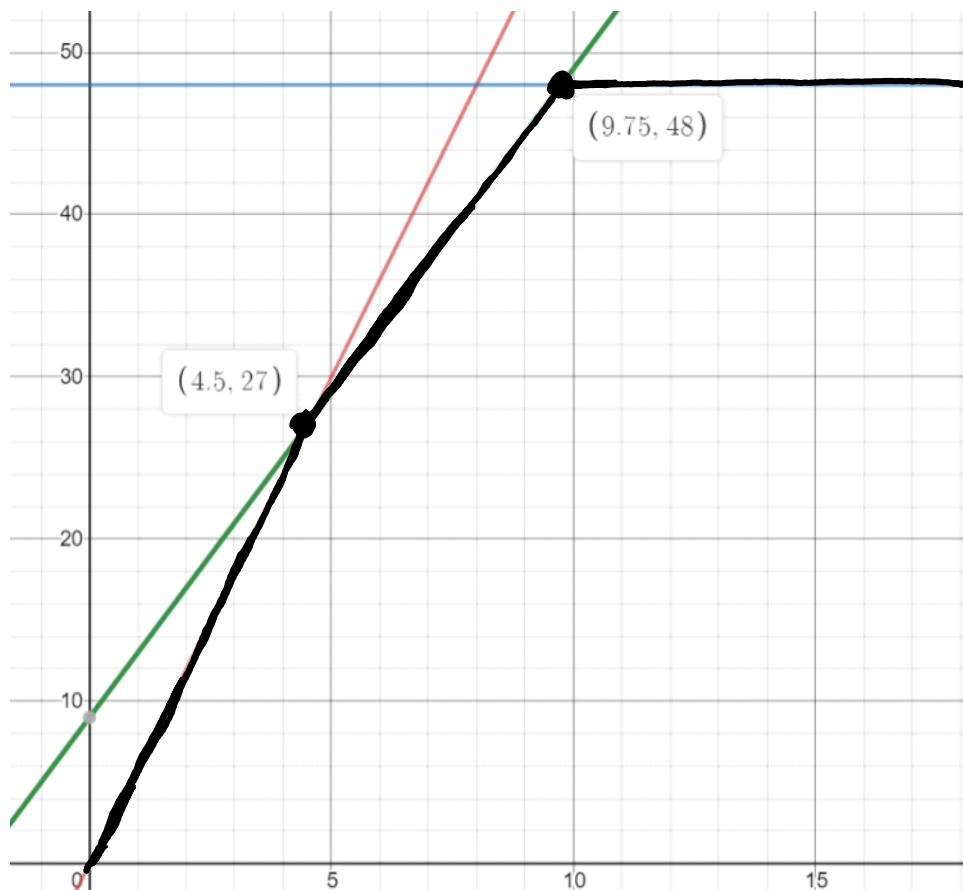
Equations of line :  $y = 4x + b$

$$\left. \begin{array}{l} \\ \end{array} \right\} \quad \begin{array}{l} \\ y = 4x + 9 \end{array}$$

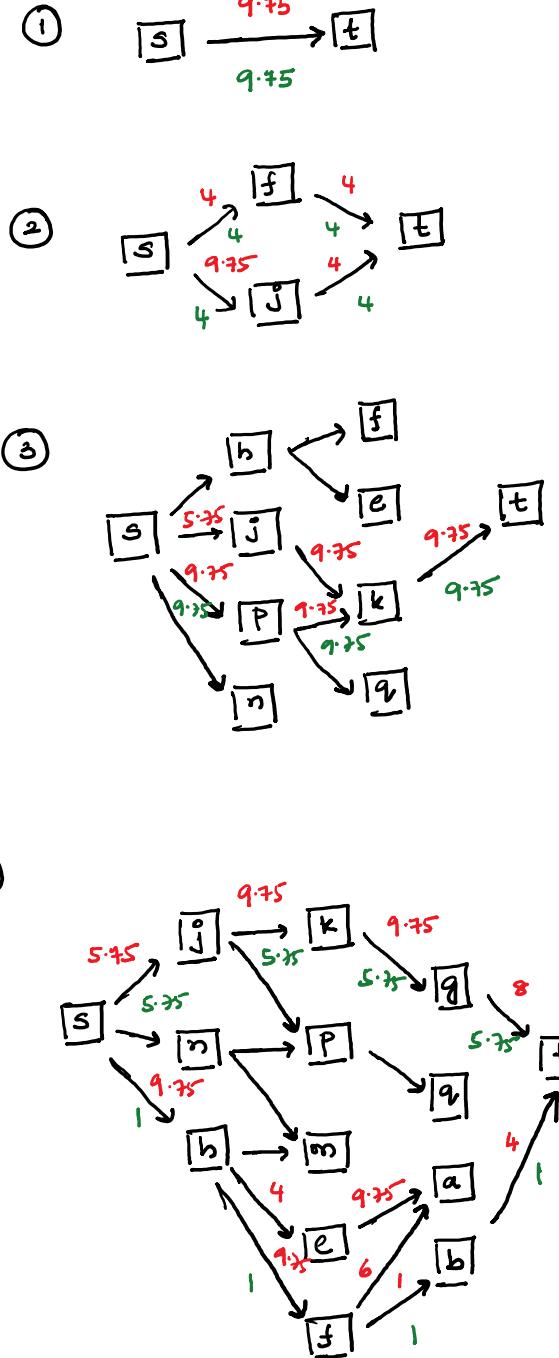
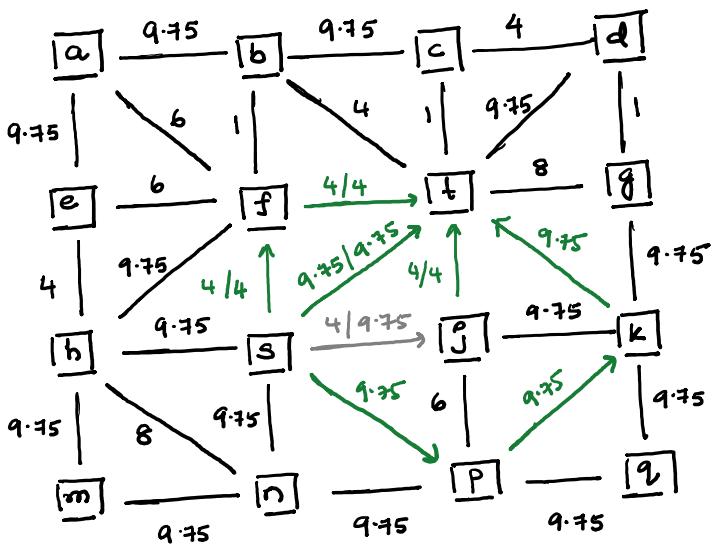
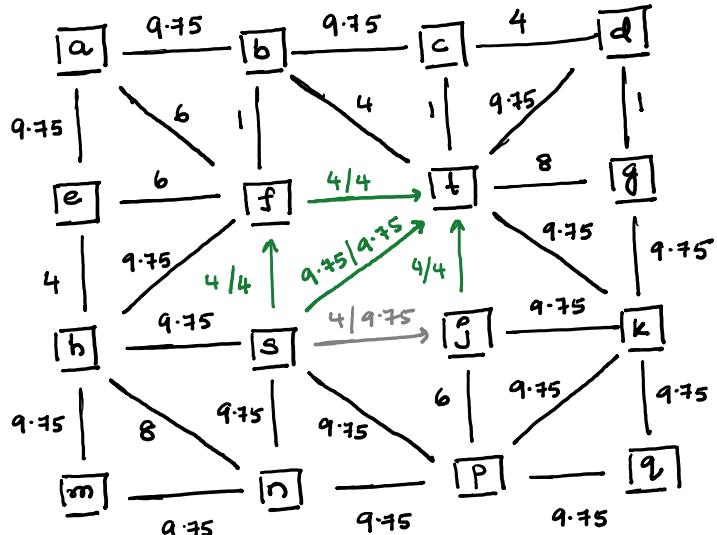
$$\textcircled{C} \quad x = 8, \quad y = 41 \Rightarrow b = 9$$

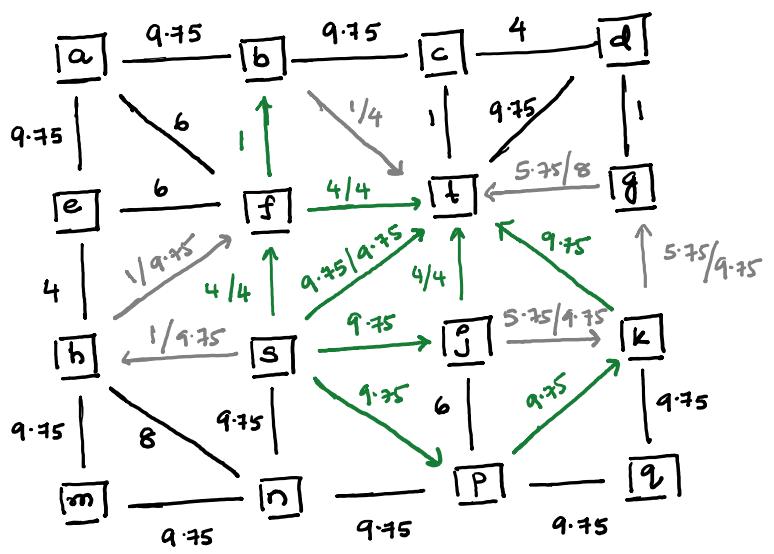
Point of intersection @  $y = 6x \rightarrow 6x = 4x + 9 \Rightarrow x = \frac{9}{2} = \underline{4.5}$

Point of intersection @  $y = 4x \Rightarrow 4x + 9 = 48 \rightarrow x = \frac{39}{4} = \underline{9.75}$

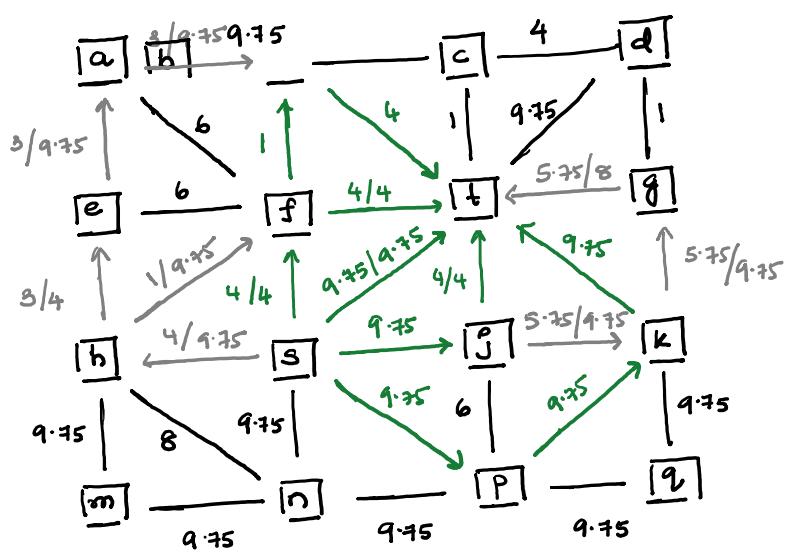
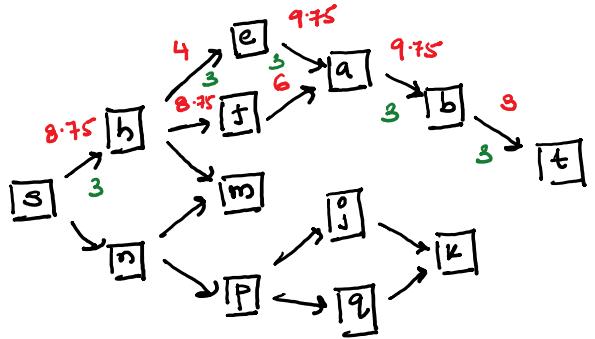


MAX FLOW for min[u, 9.75]

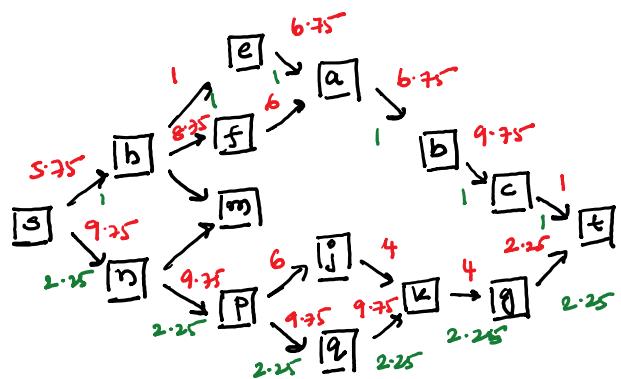


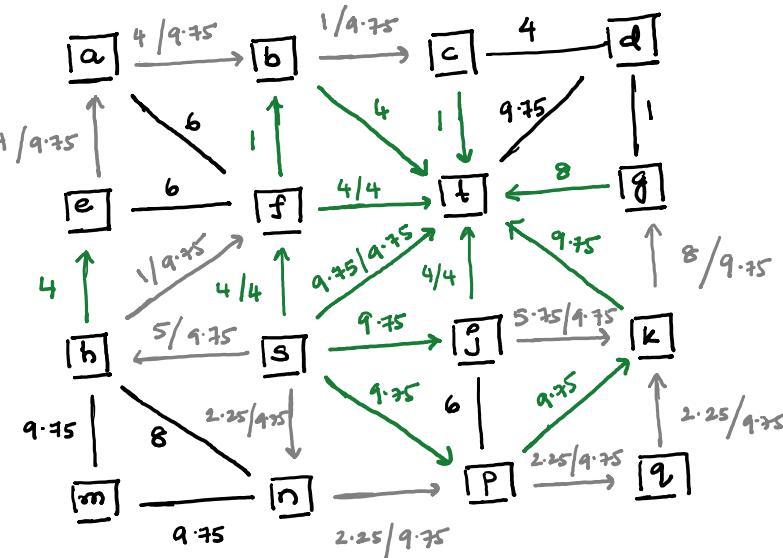


⑤

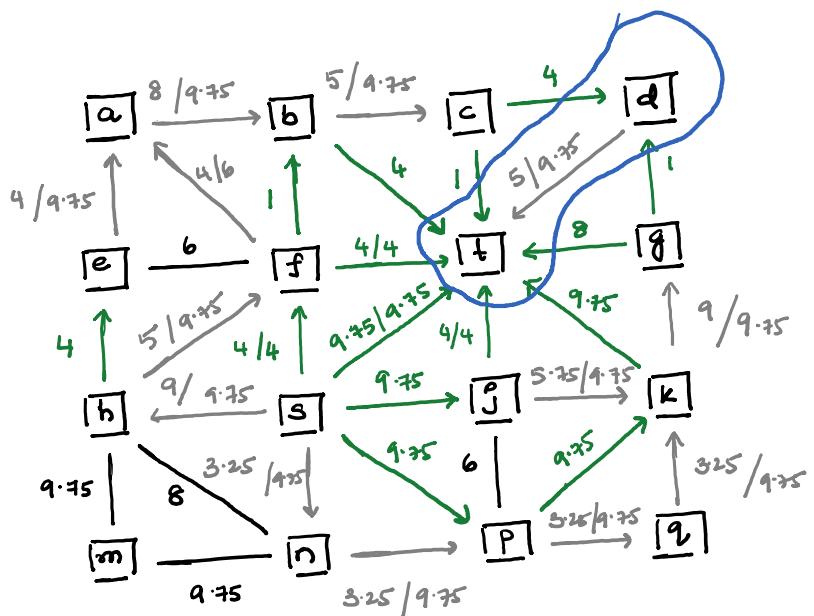
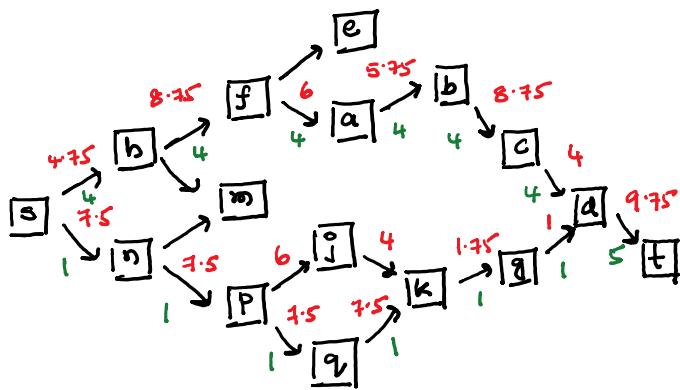


⑥



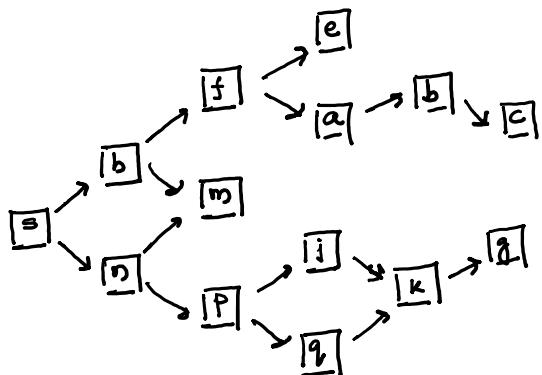


⑦



⑧

Min-cut :



Flow : 45.5

No. of edges with  $\lambda = 9.75 \Rightarrow (s, t), (k, t)$

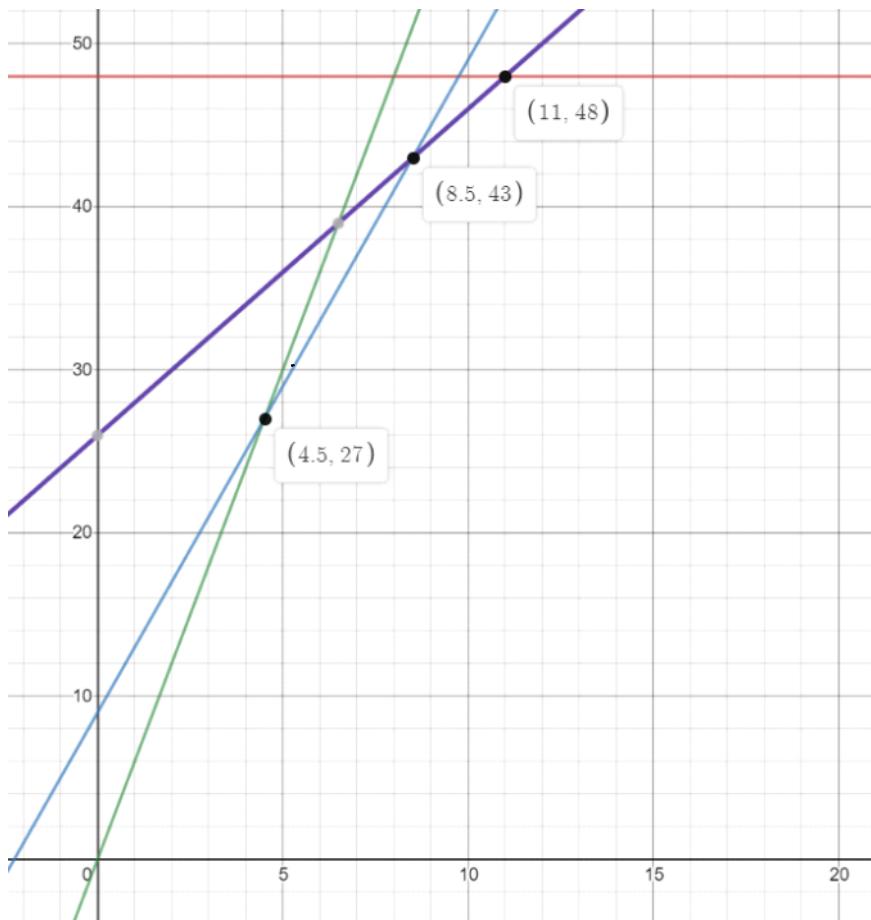
$$\Rightarrow y = 2x + b \quad @ \quad x = 9.75 \Rightarrow \underline{\text{Flow} = 45.5}$$

$$45.5 = 2(9.75) + b \Rightarrow \boxed{b = 26}$$

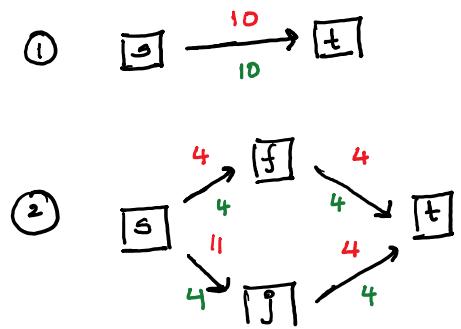
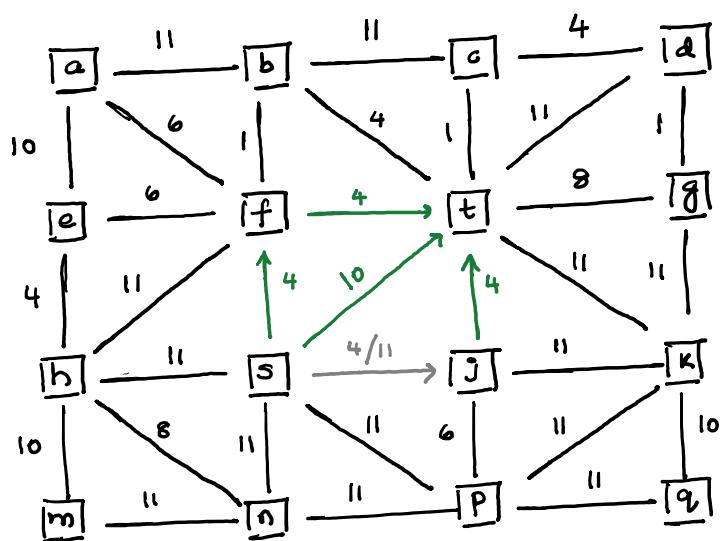
Points of intersection:  $y = 48 \Rightarrow 2x + 26 = 48 \Rightarrow \boxed{x = 11}$

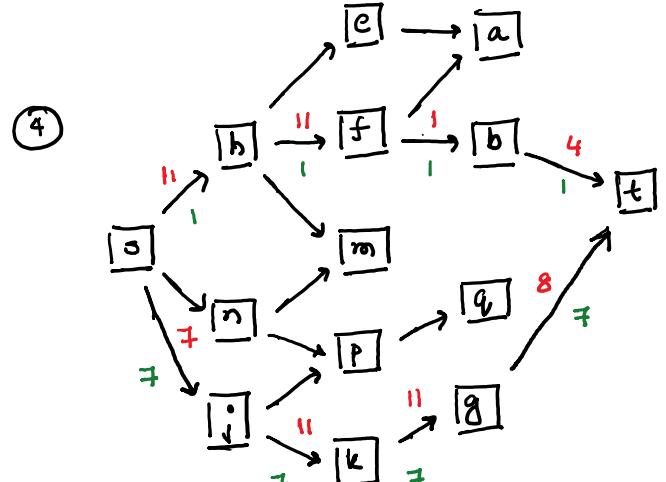
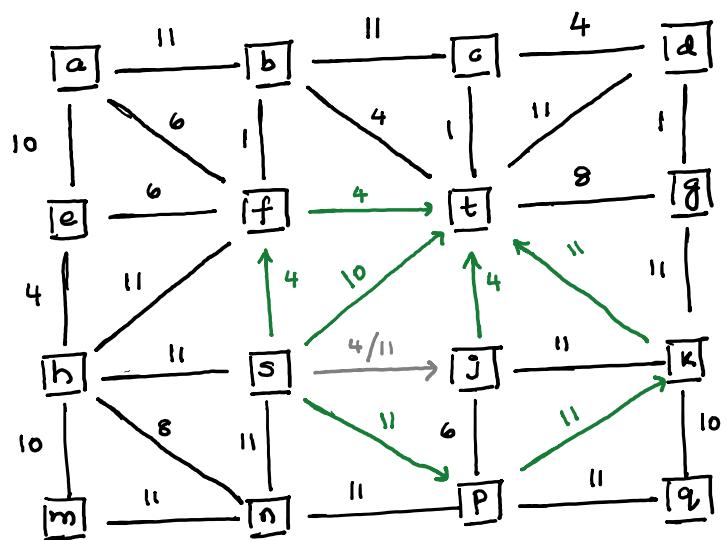
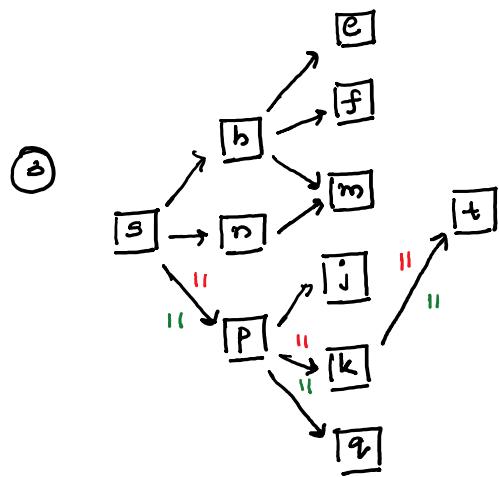
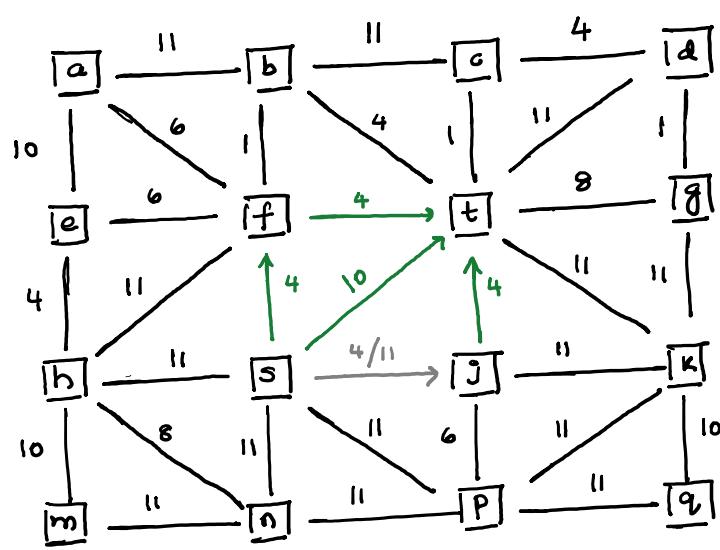
$$4x + 9 = 2x + 26 \Rightarrow 2x = 17 \Rightarrow \boxed{x = 8.5}$$

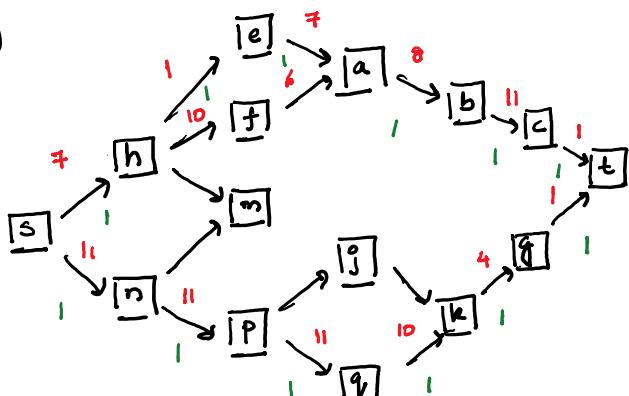
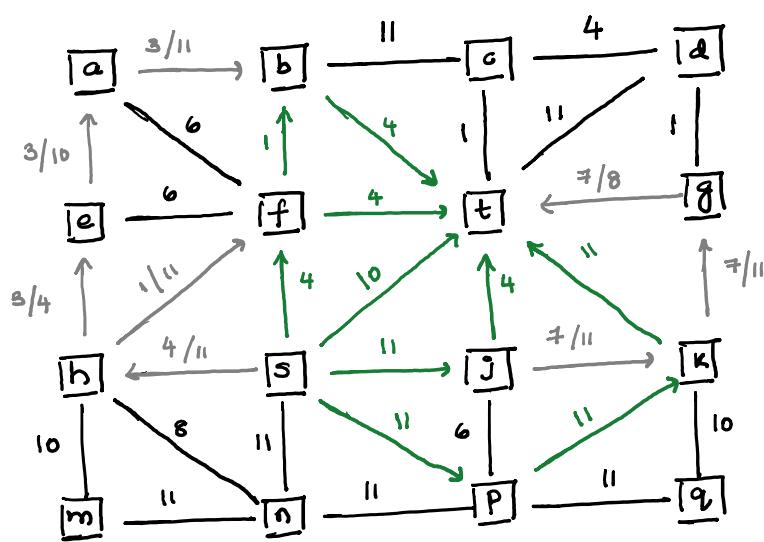
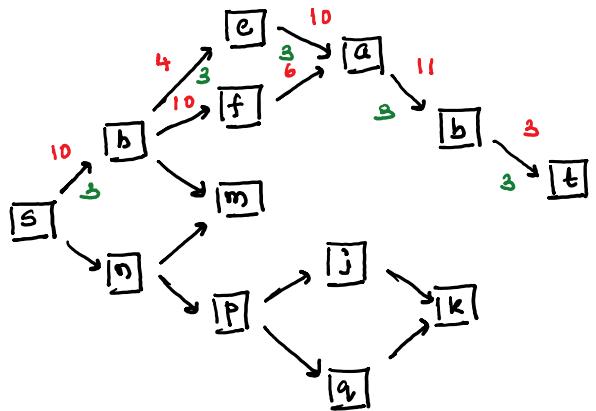
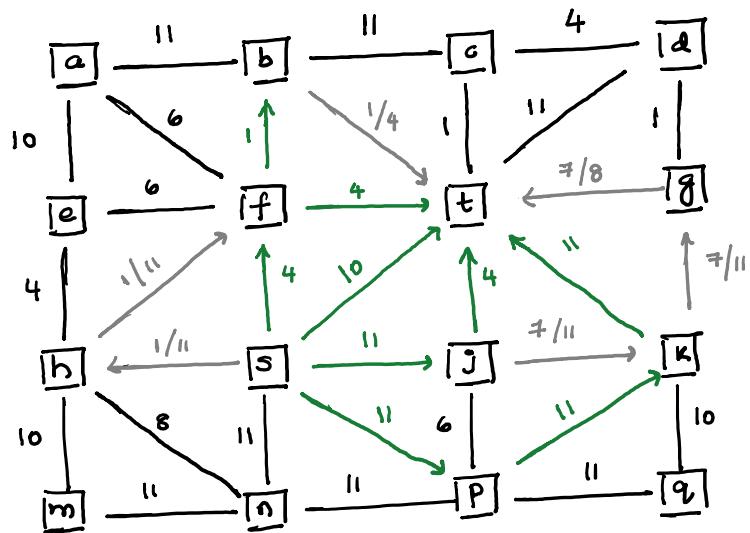
### MAX FLOW for $\min[u, 11]$

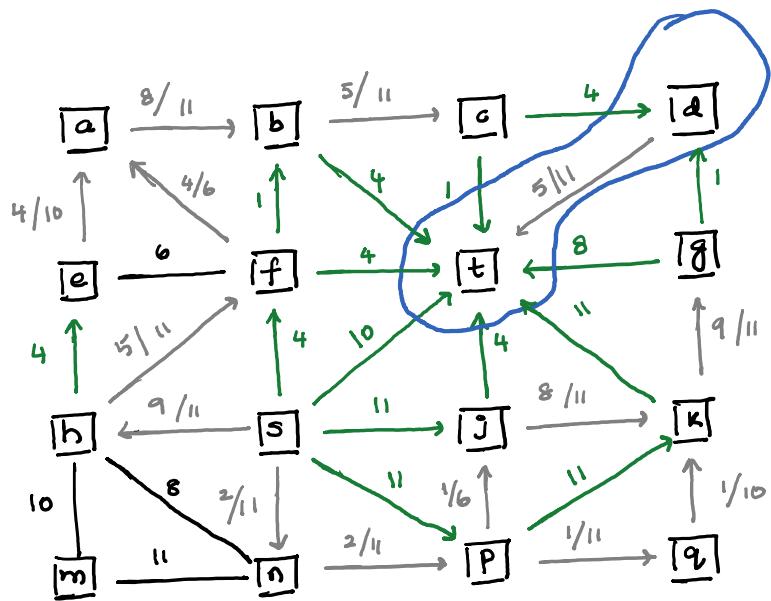
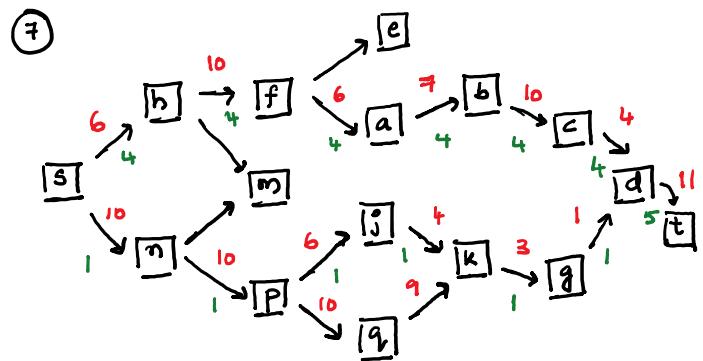
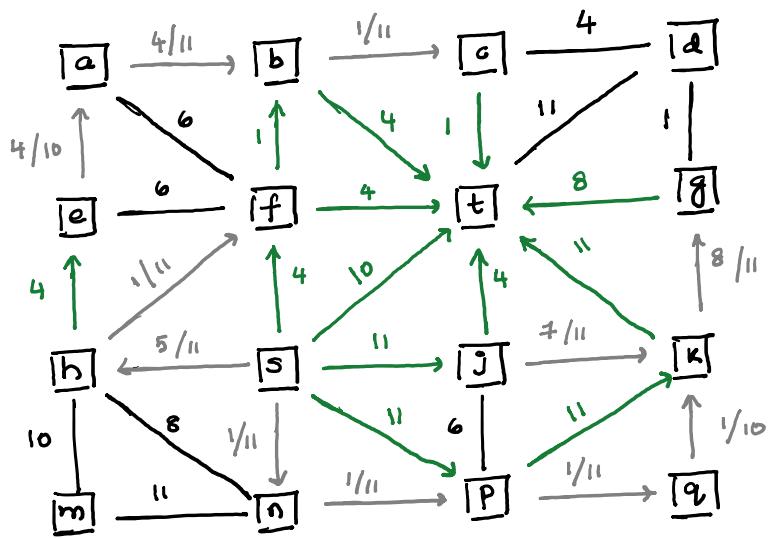


Consider  $\min [u_{ij}, 11]$









$$y = x + b \Rightarrow @ z=11, \text{Flow: } 47$$

$$\Rightarrow \boxed{b = 36}$$

Points of intersection :

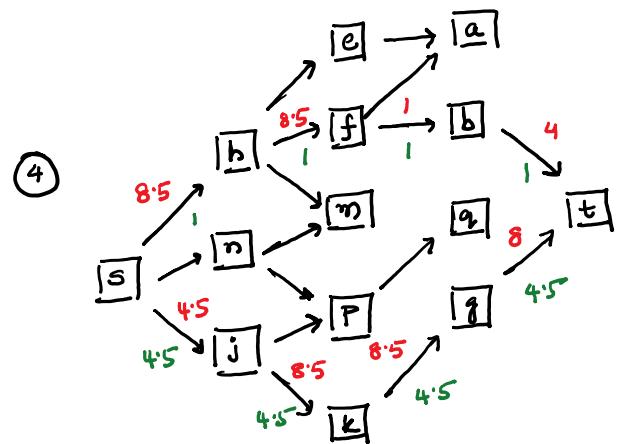
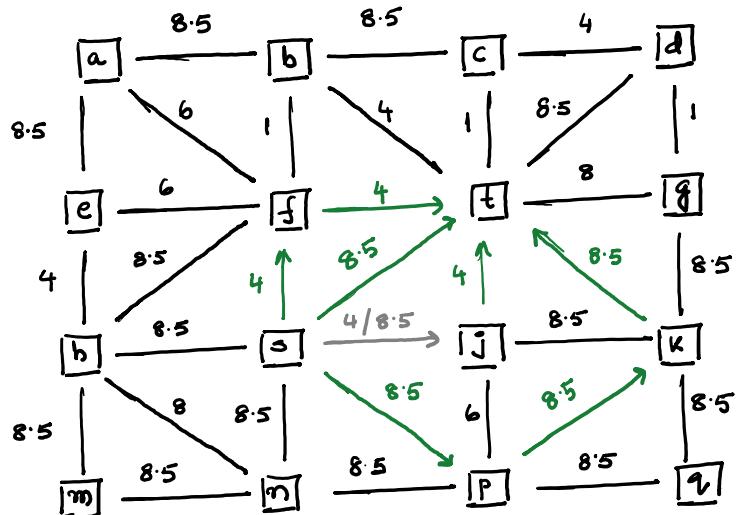
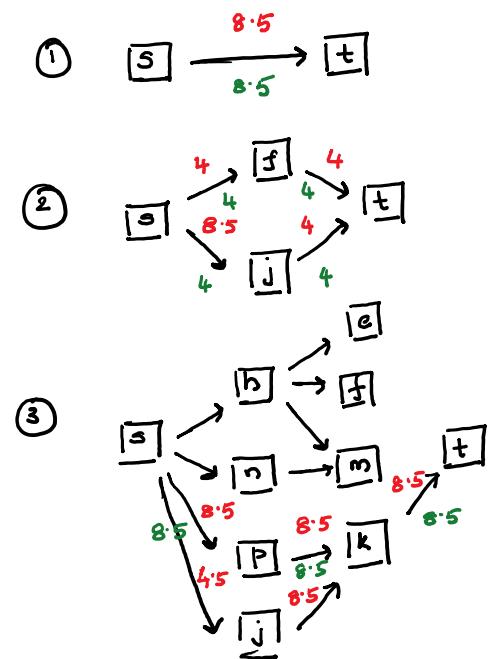
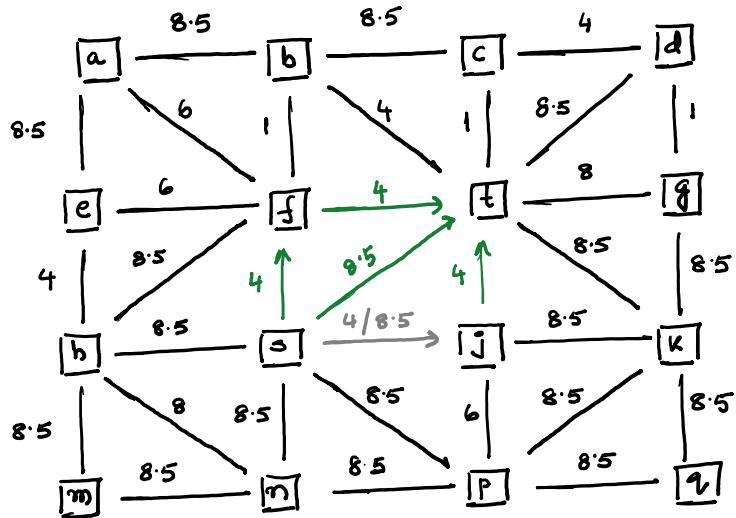
No. of edges with  $\lambda = 11$  :  $(k, t)$

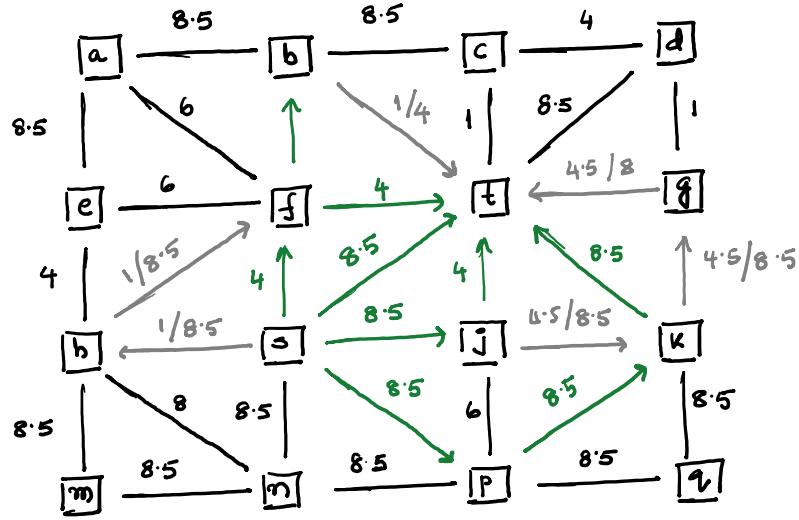
Flow : 47

$$\textcircled{1} \underset{\lambda=12}{\Leftrightarrow} F(\infty)$$

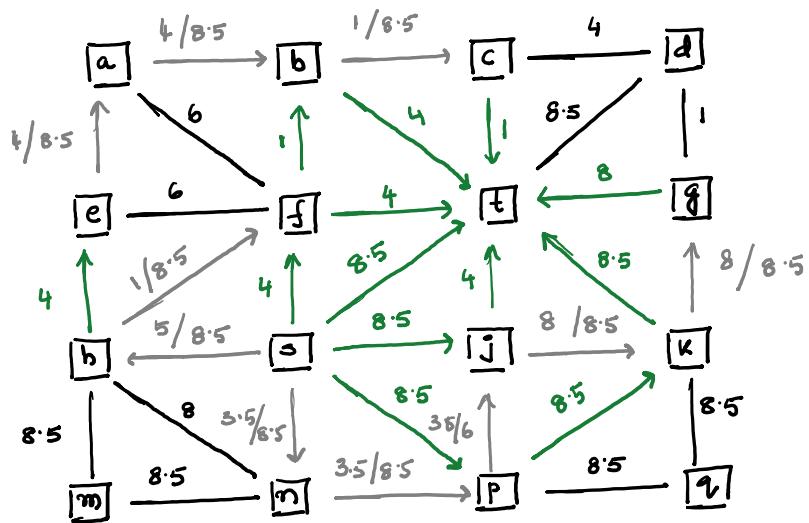
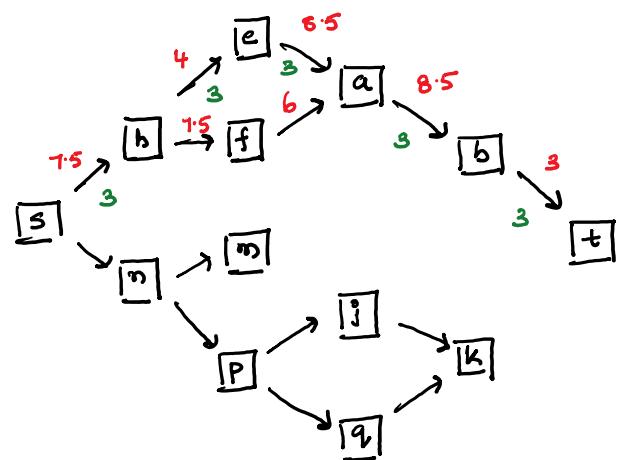
$$2x+26 = x+36 \Rightarrow \boxed{x=10}$$

MAX FLOW for min[u, 8.5]

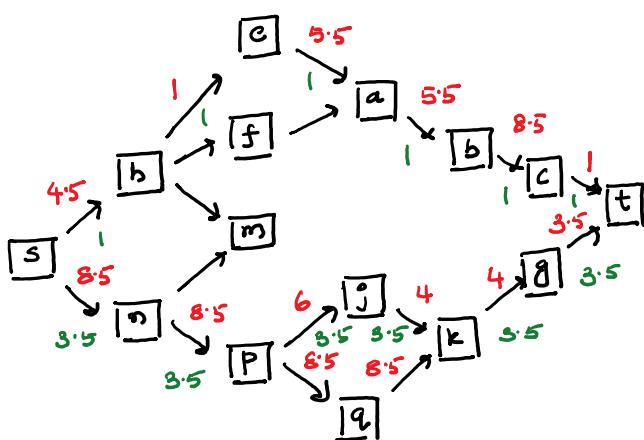


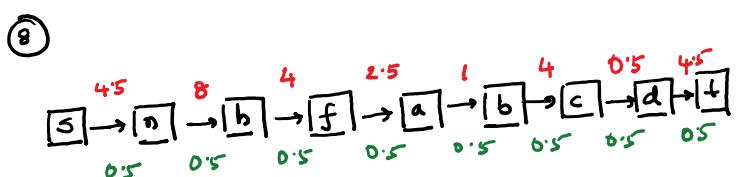
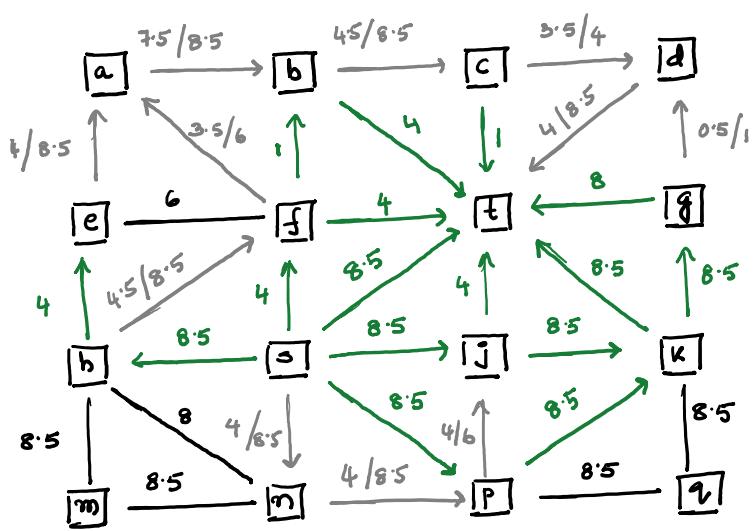
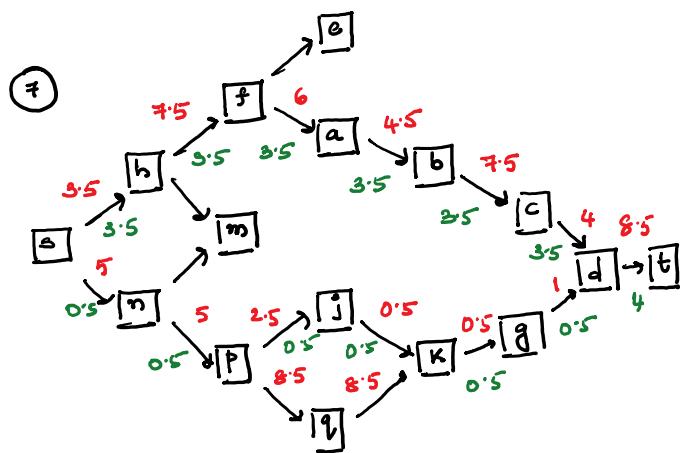
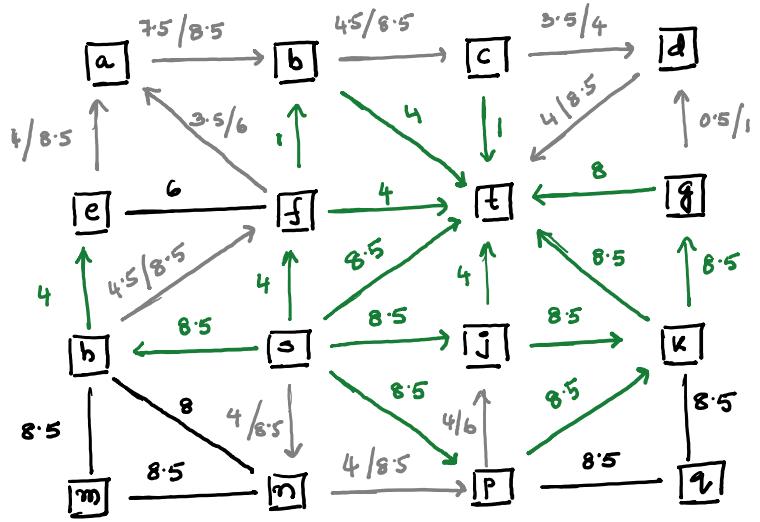


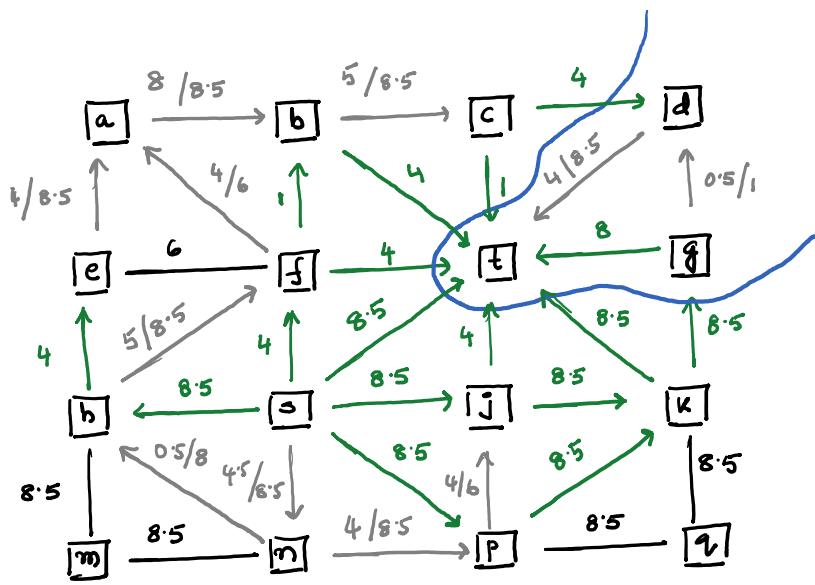
⑤



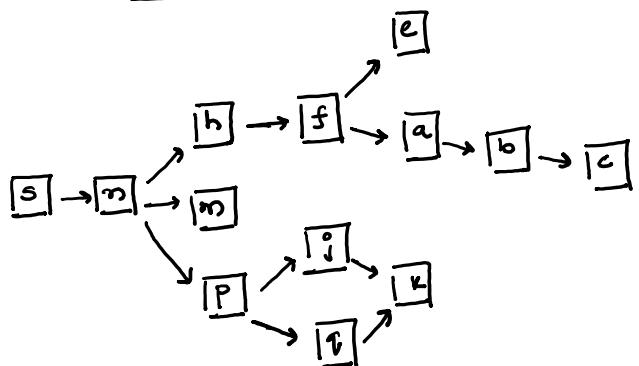
⑥







Min-cut :



No. of edges with  $\lambda = 8.5$

$$(s,t), (k,t), (k,g) \Rightarrow 3$$

$$\text{Flow: } 42.5 \quad | \quad 0.5 + 4 + 3.5 + 1 + 3 + 4.5 + 1 + 8.5 \\ + 4 + 4 + 8.5 = 42.5$$

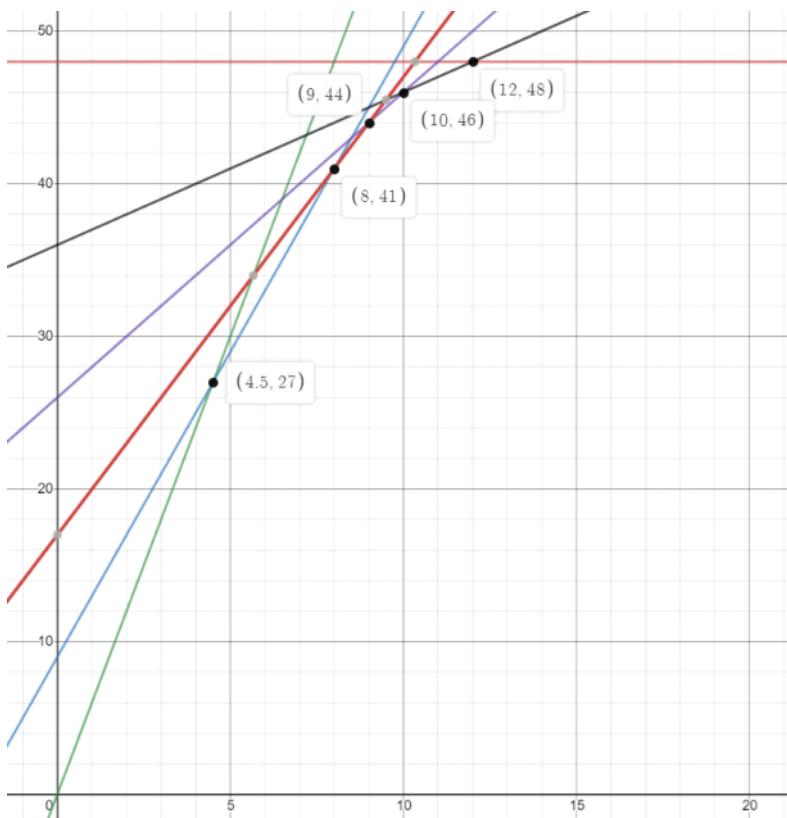
$$y = 3x + b \quad @ \quad x = 8.5 \quad \text{Flow} = 42.5$$

$$\Rightarrow \boxed{b = 17}$$

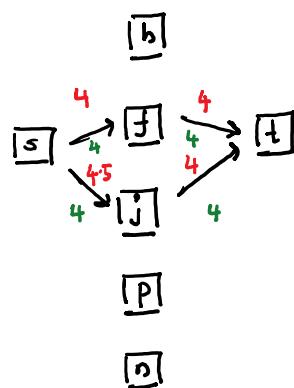
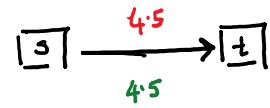
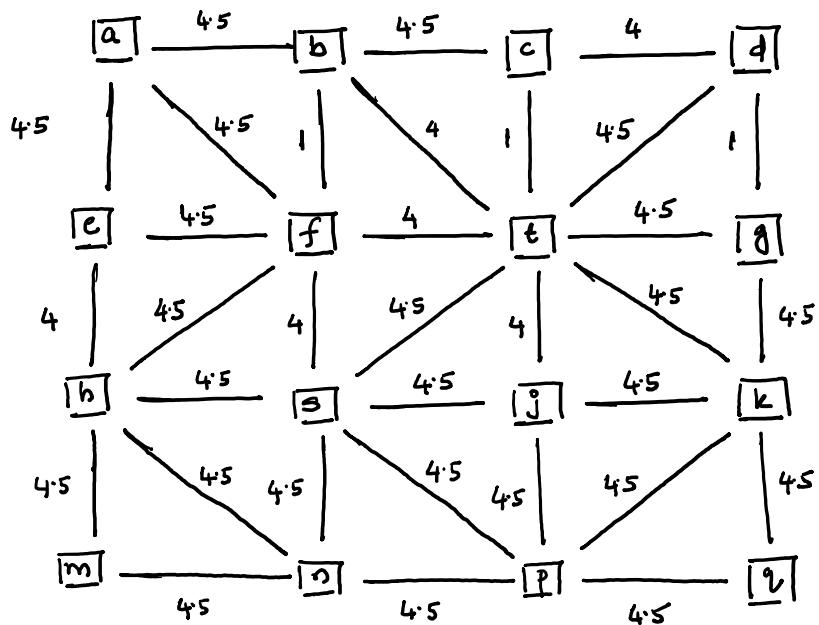
Points of intersection :

$$2x + 2b = 3x + 17 \Rightarrow \boxed{x = 9}$$

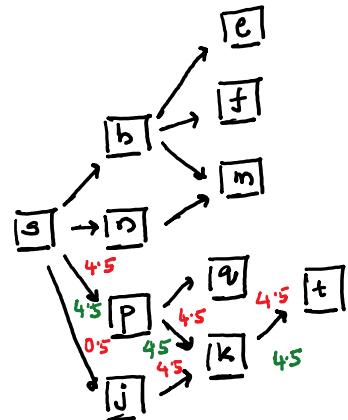
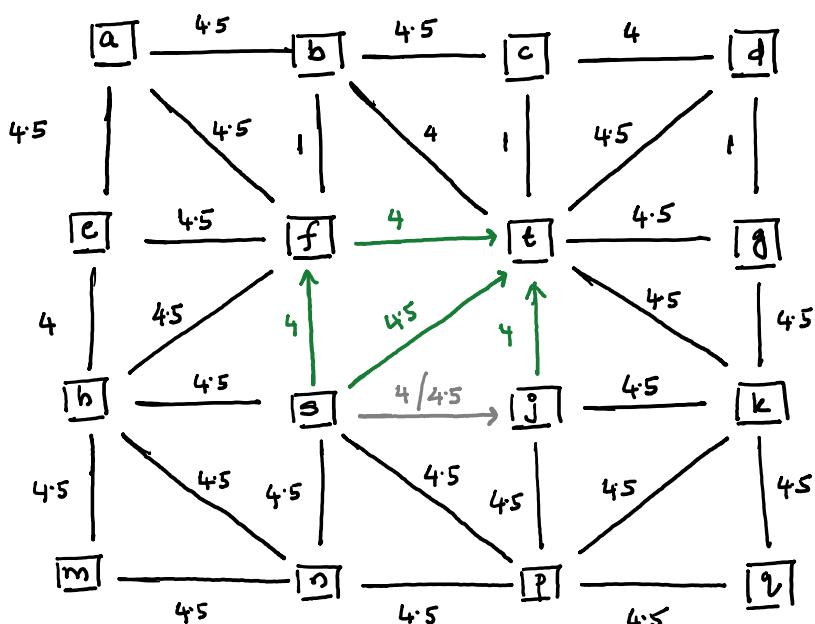
$$4x + 9 = 3x + 17 \Rightarrow \boxed{x = 8}$$



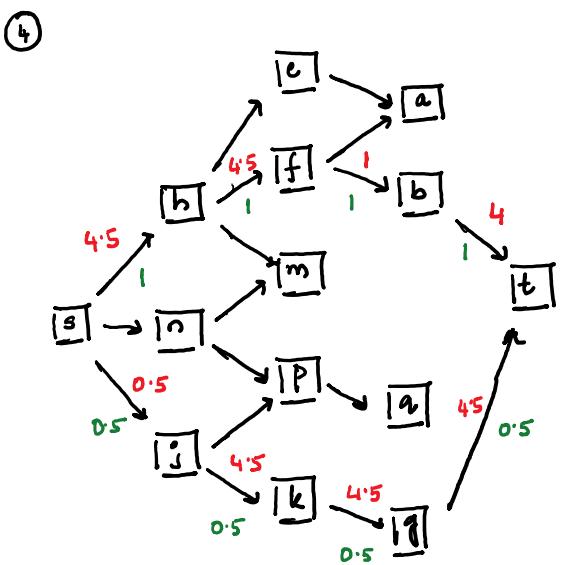
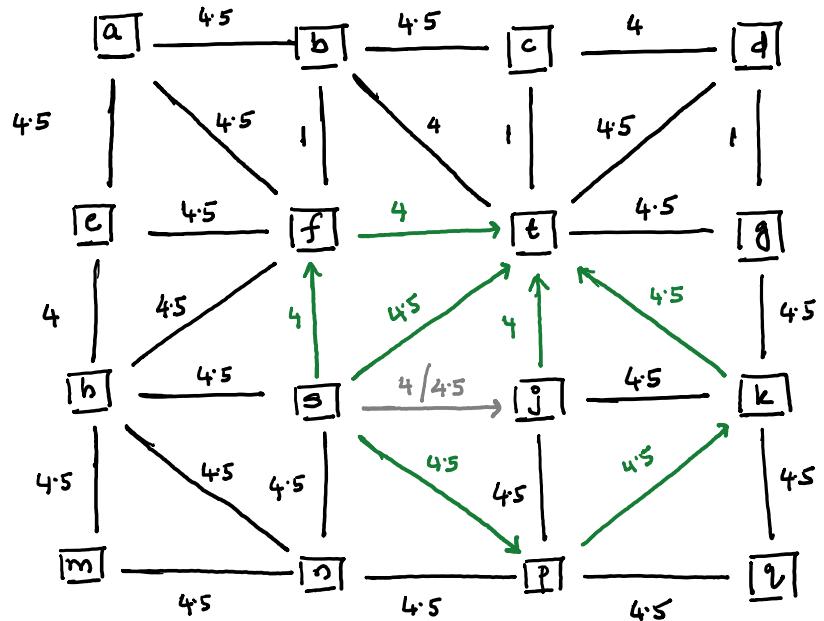
MAX FLOW for min[u, 4.5]



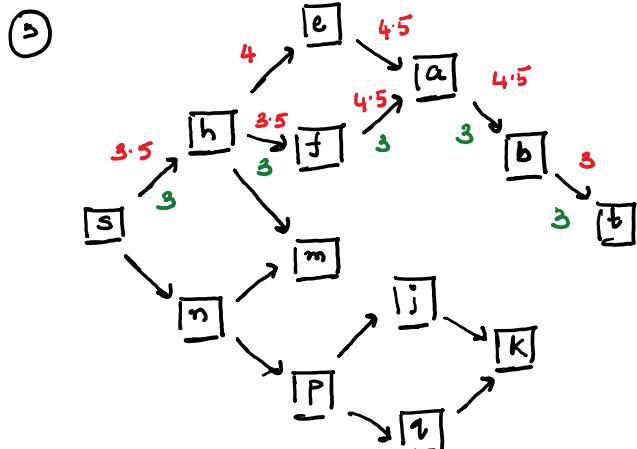
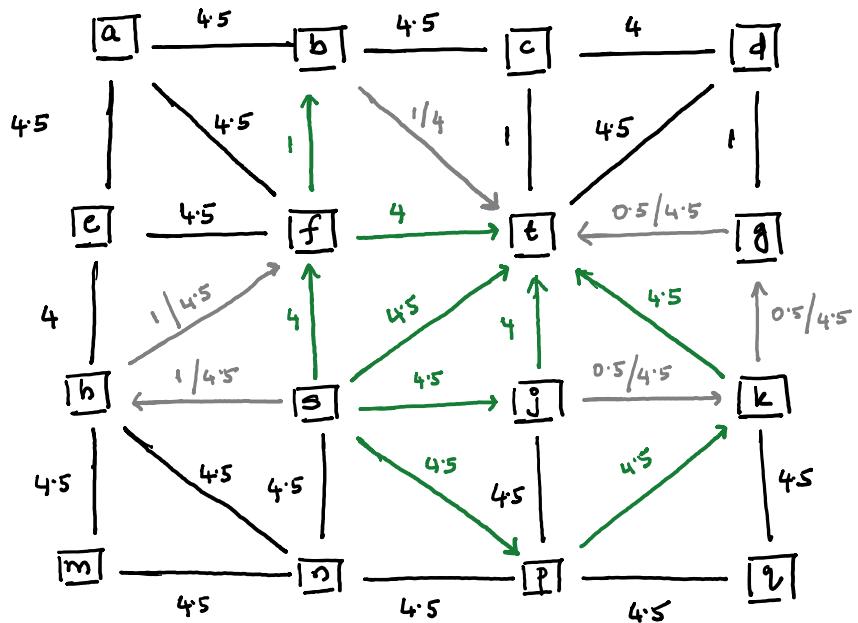
$$\text{Flow: } \underline{4.5 + 8 = 12.5}$$

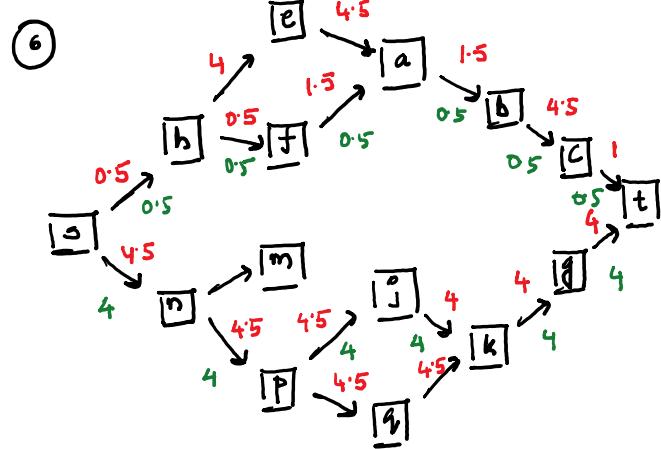
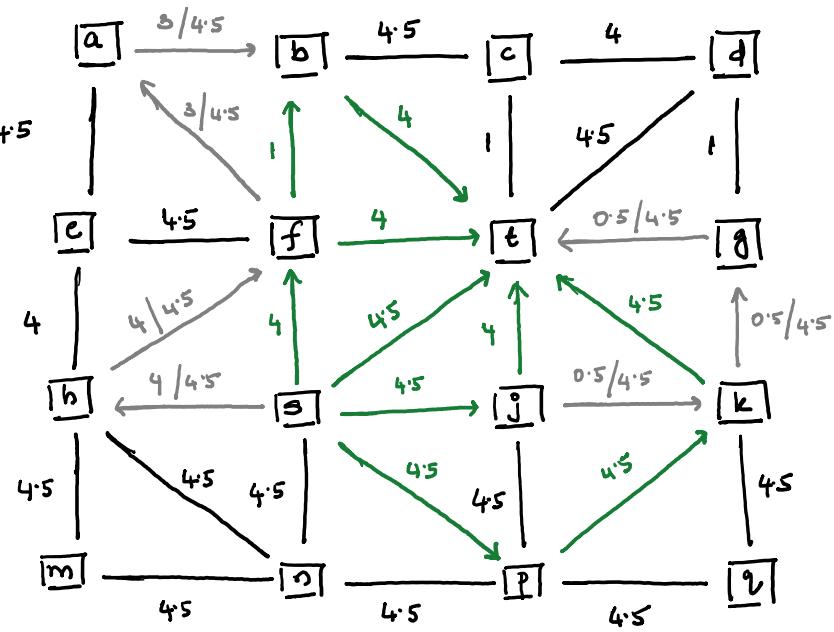


$$\text{Flow: } \underline{12.5 + 4.5 = 17}$$

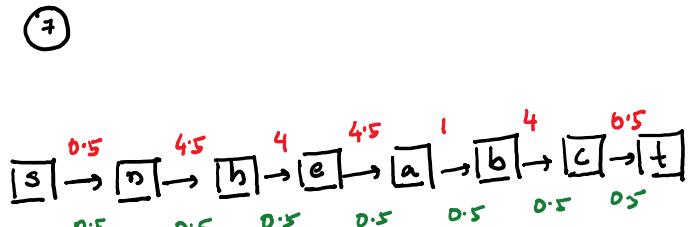
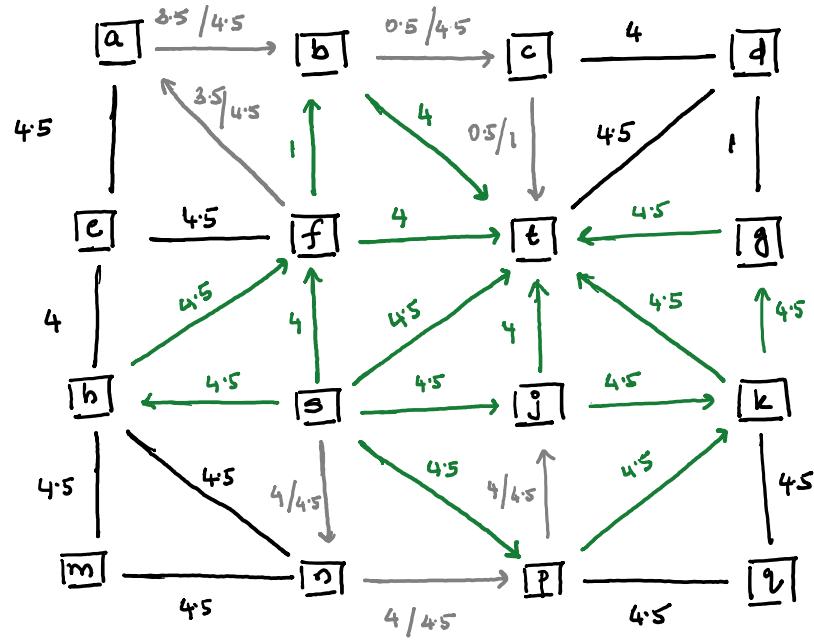


$$\text{Flow: } 17 + 1.5 = \underline{18.5}$$

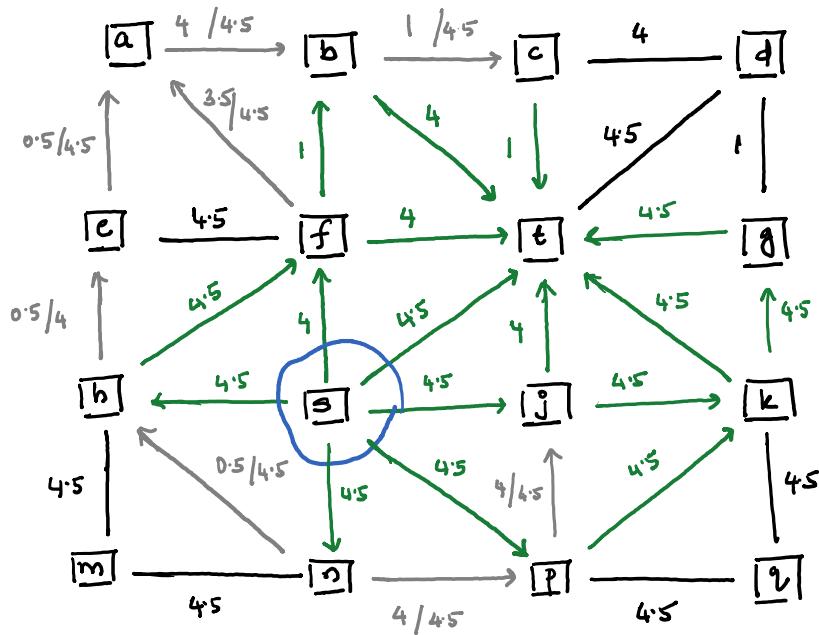




$$\underline{\text{Floor:}} \quad 21.5 + 4 + 0.5 = \underline{26}$$



Flow : 26.5



Total flow:  $26.5$

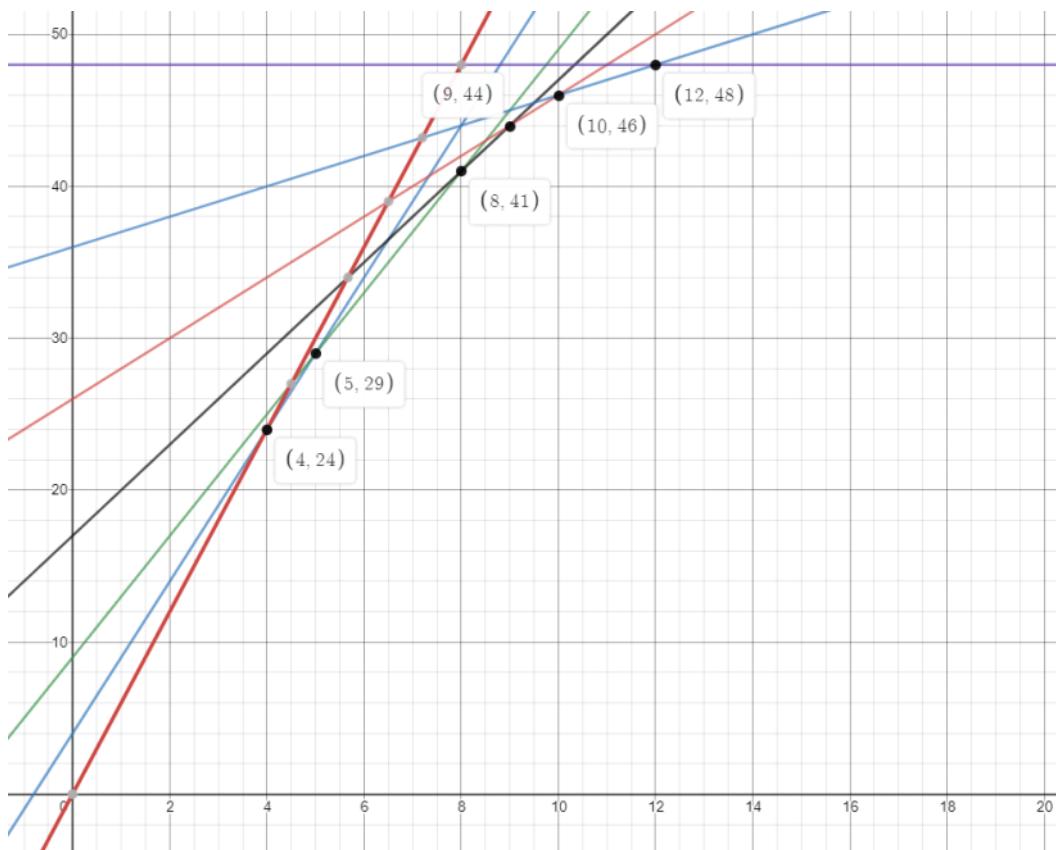
# edges with  $\lambda = 4.5$  (5)

Finding points of intersection

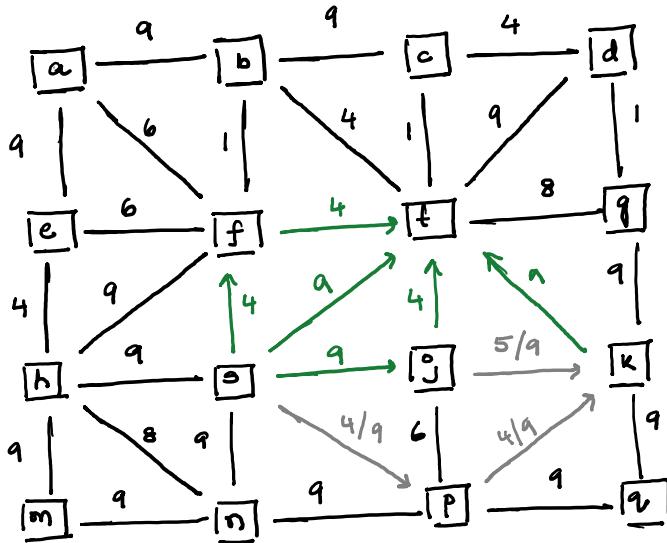
$$\text{at } x=4.5, \quad y = 5x + b \Rightarrow y = 26.5$$

$$\Rightarrow 6x = 5x + 4 \Rightarrow \boxed{x=4}$$

$$\Rightarrow b = 26.5 - 5(4.5) = \boxed{4} \Rightarrow 5x + 4 = 4x + 9 \Rightarrow \boxed{x=5}$$



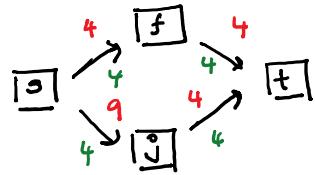
MAX FLOW for min[u, 9]



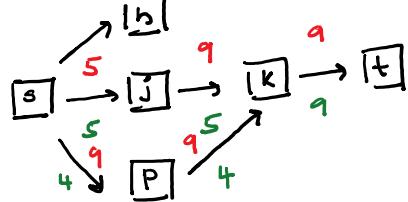
①



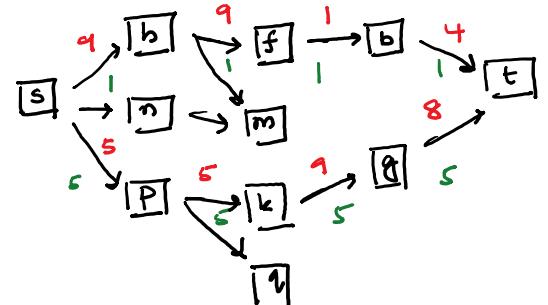
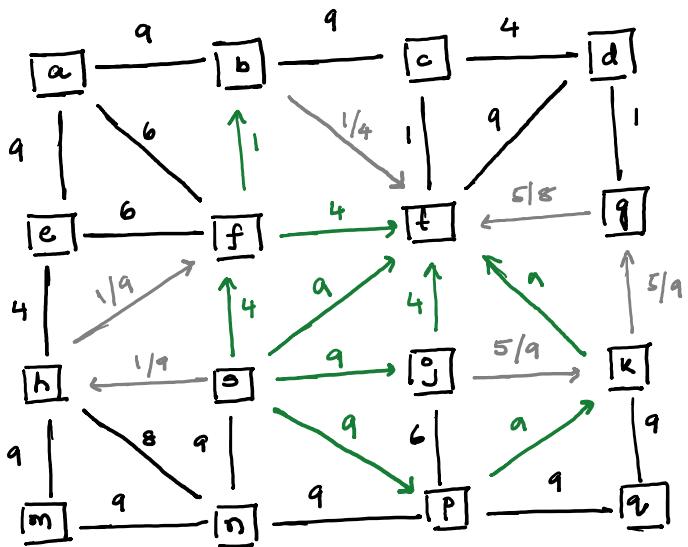
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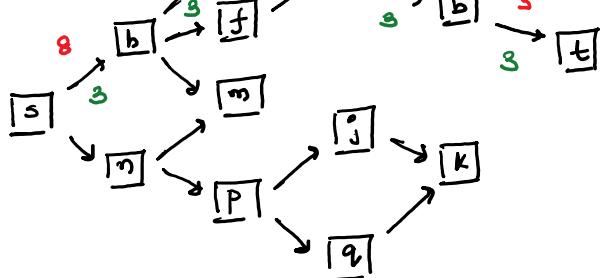
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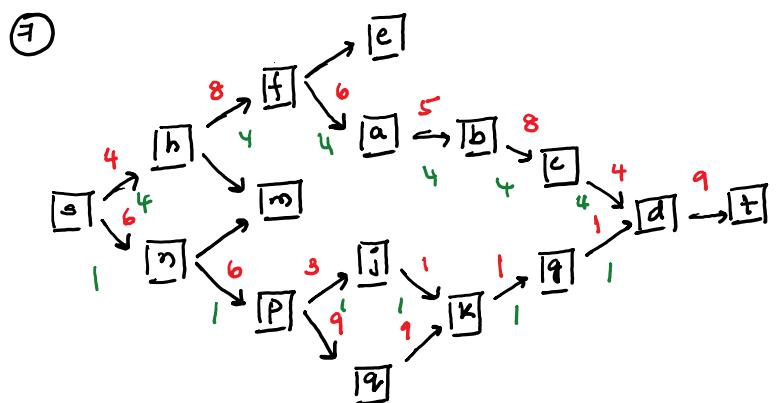
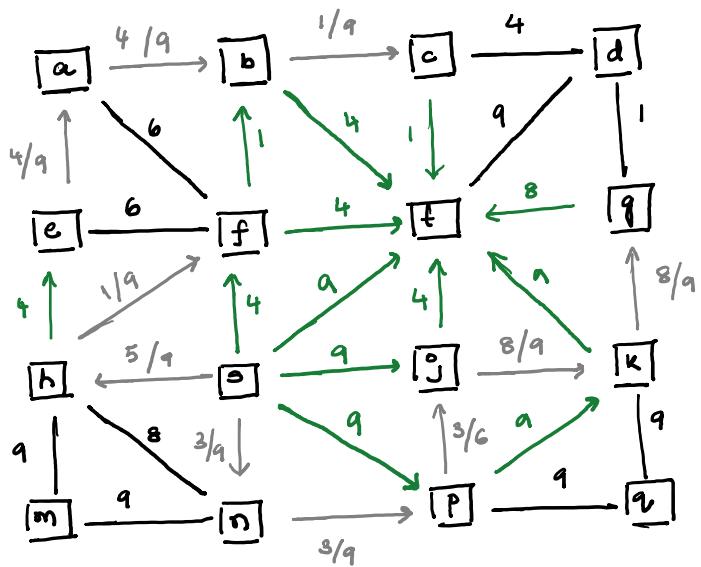
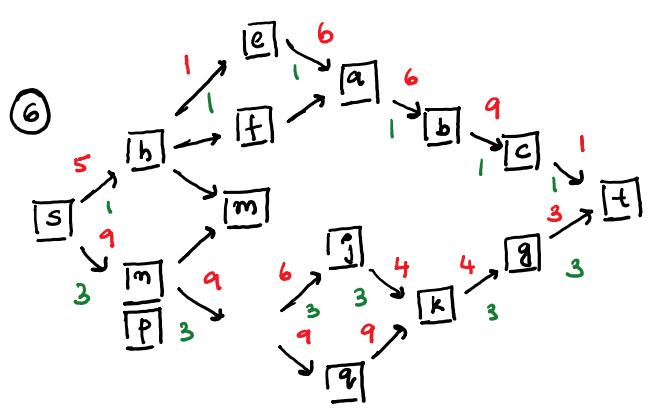
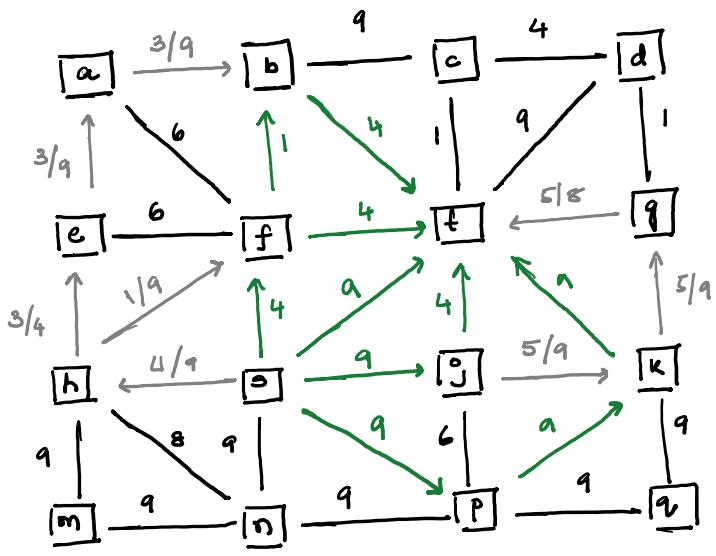


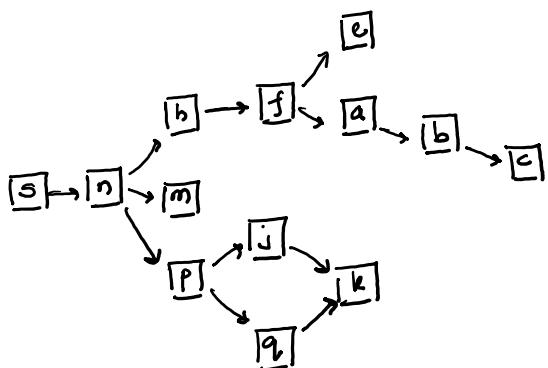
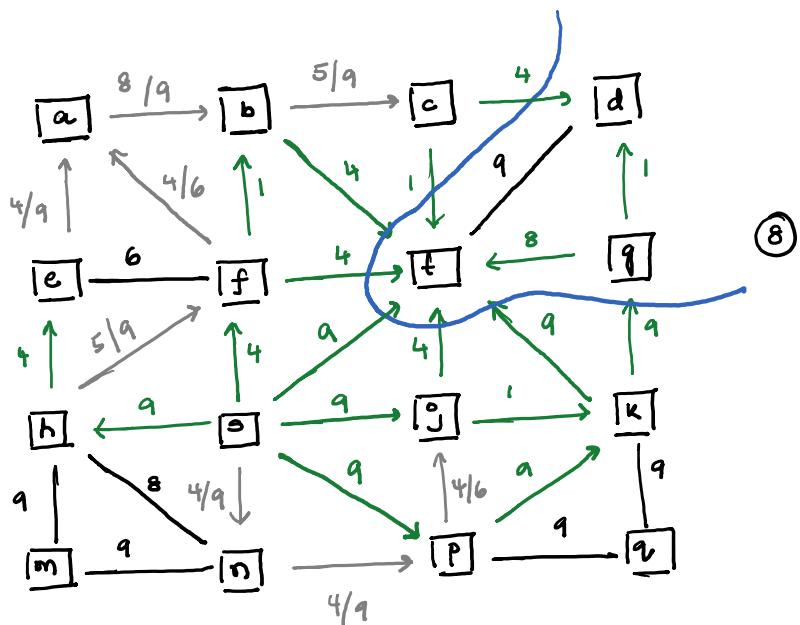
④



⑤





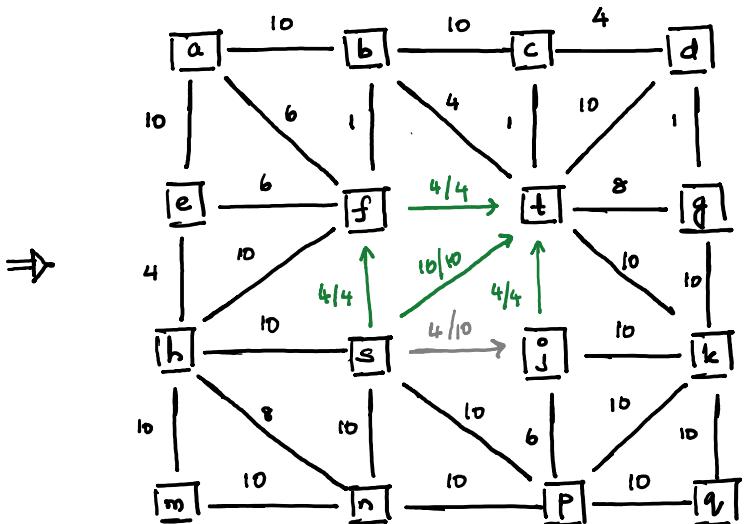
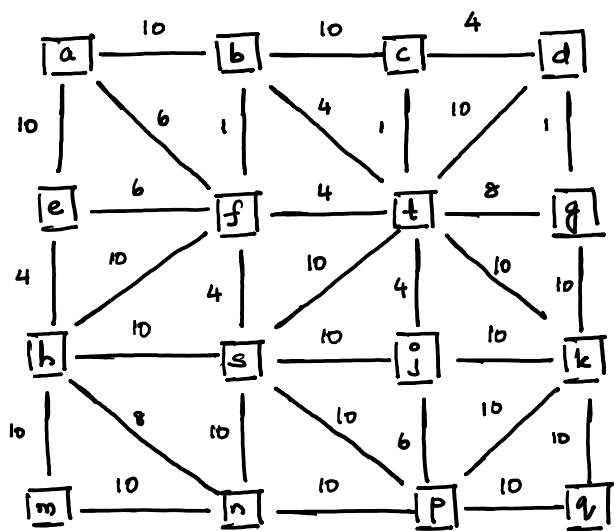


Current flow @  $\lambda=9$

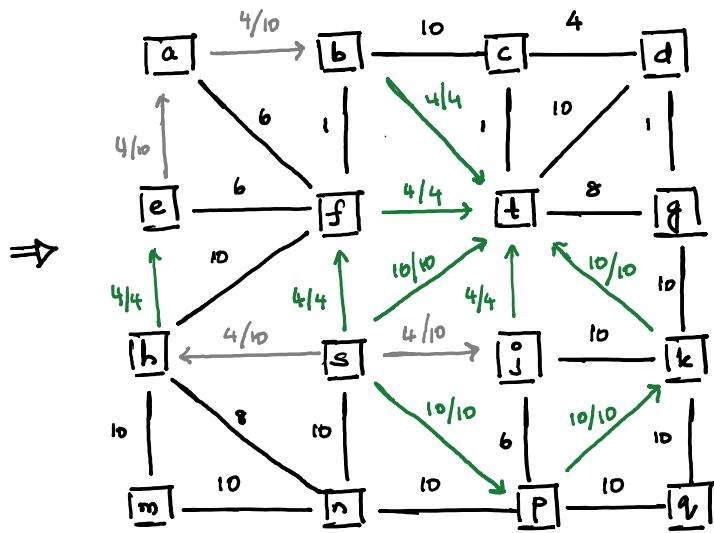
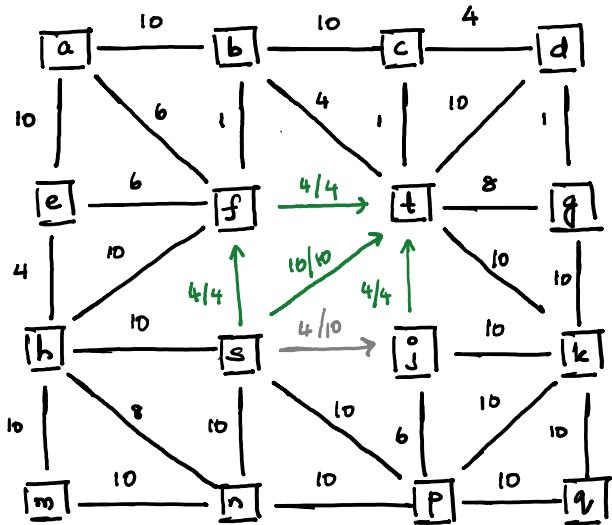
in graph  $\Rightarrow 44$

## MAX FLOW for min[u, 10]

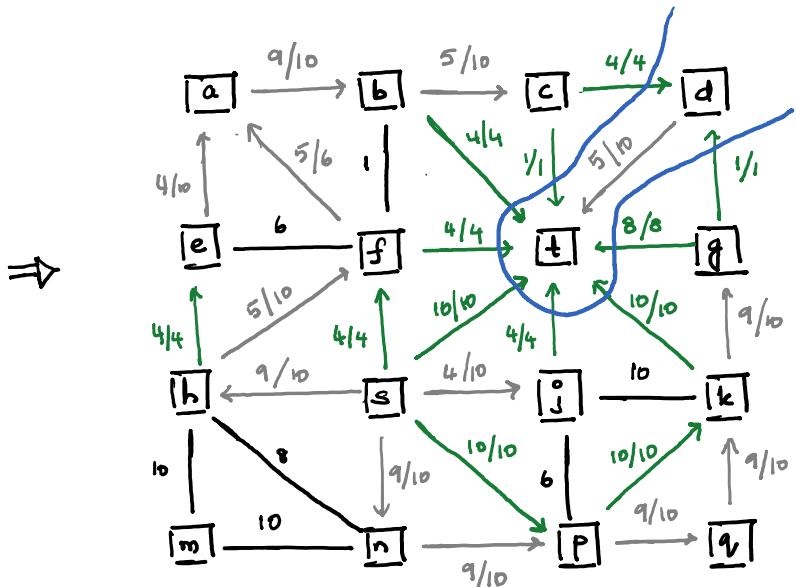
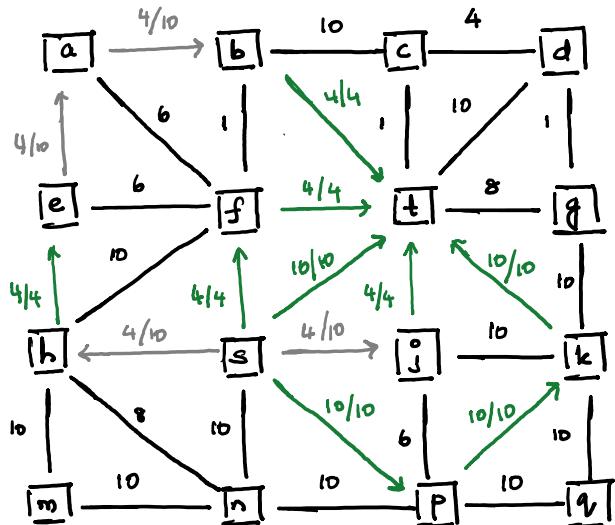
Consider graph with  $\min[u_{ij}, 10]$



$$[10 + 4 + 4]$$



$[10 + 4]$

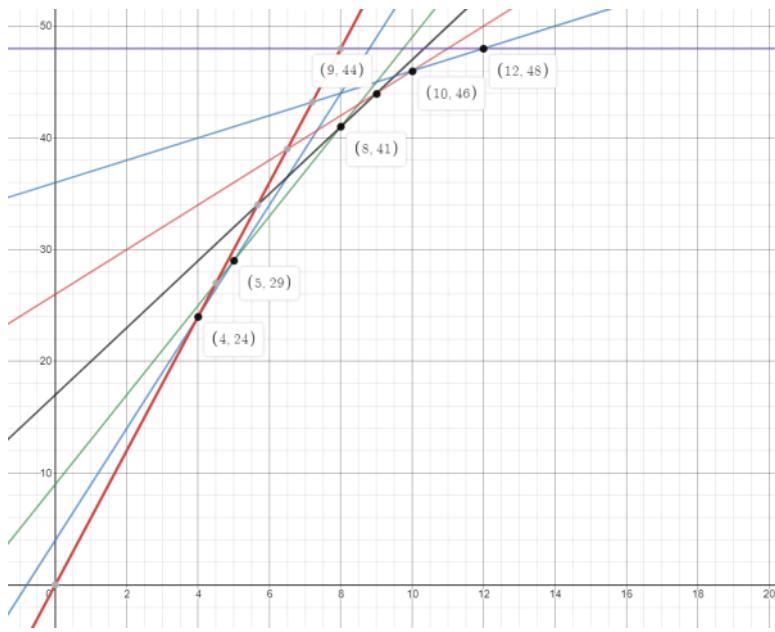


$[9 + 5]$

Max flow:  $18 + 14 + 14 = 46$ .

Min cut:  $\{(s, j, n, b, p, q, k, g, f, c, a, b, c), (d, t)\}$

No. of edges in min-cut with flow = 10: 2

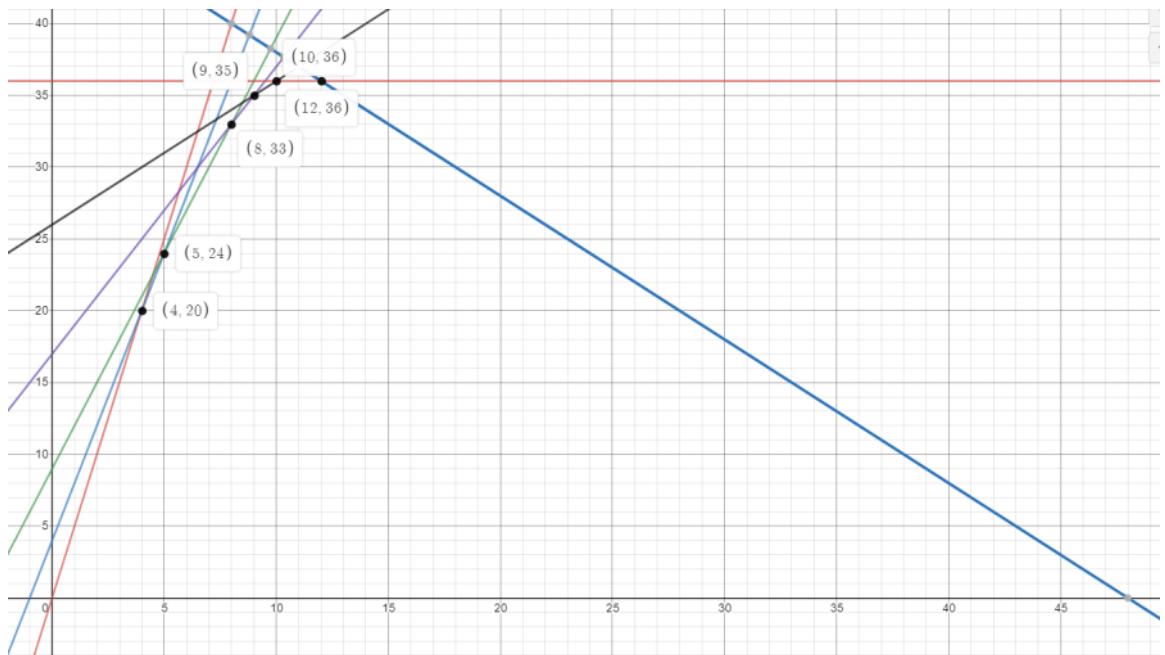


Piece wise linear function ( $F(\lambda)$ ):

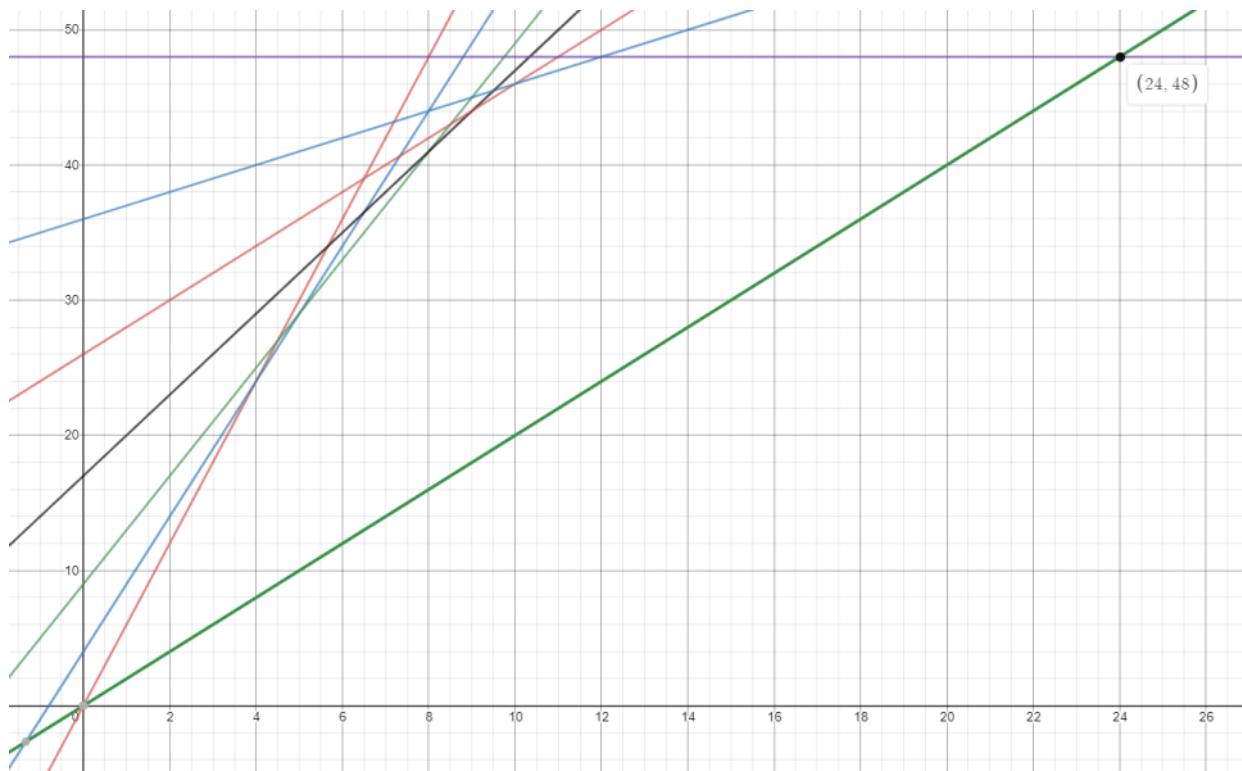
- ①  $6\lambda$
- ②  $5\lambda + 4$
- ③  $4\lambda + 9$
- ④  $3\lambda + 17$
- ⑤  $2\lambda + 26$
- ⑥  $\lambda + 36$
- ⑦  $48$

To find maximum surviving flow:  $F(\lambda) - \lambda$  [curve]  $\rightarrow$  Equations shall be

- ①  $5\lambda$
- ②  $4\lambda + 4$
- ③  $3\lambda + 9$
- ④  $2\lambda + 17$
- ⑤  $\lambda + 26$
- ⑥  $36$
- ⑦  $48 - \lambda$

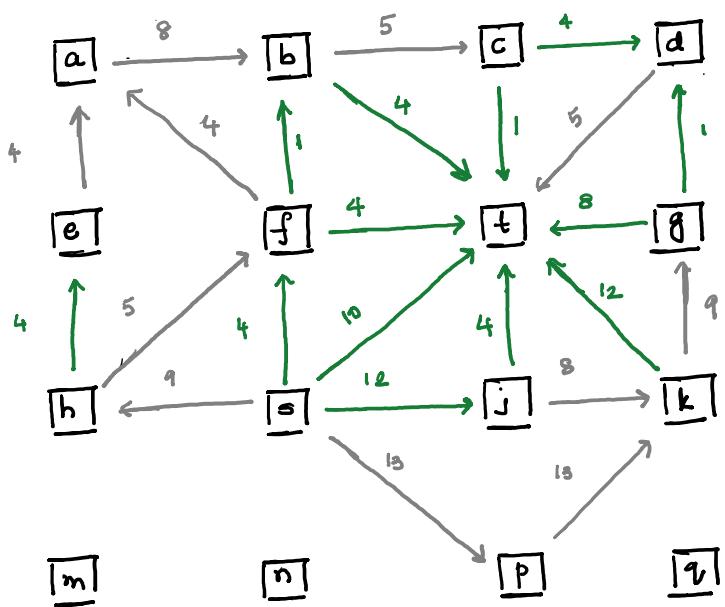


For maximum  $q$ -path flows  $\underline{q=2}$



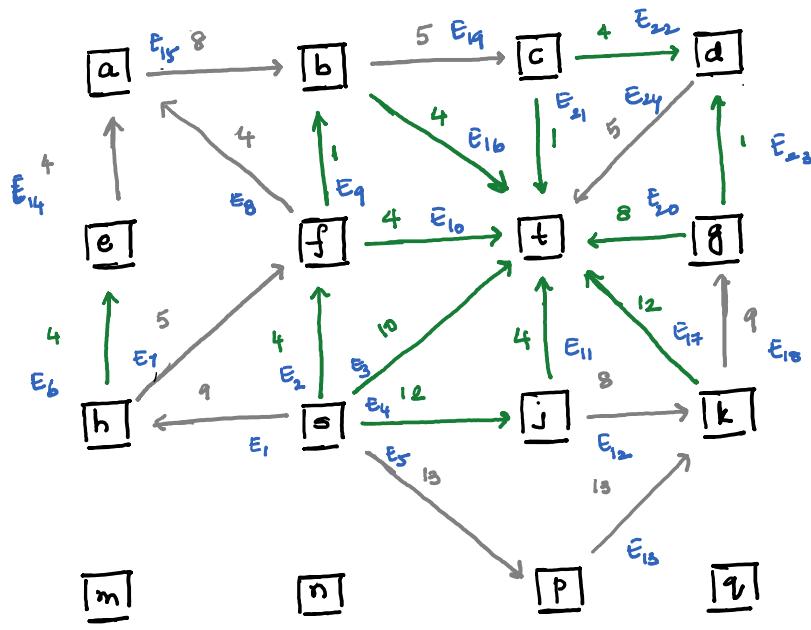
$$\lambda^* = 24 \rightarrow \text{Consider } \min [w_{ij}, 24]$$

Flow with this  $\leftarrow$  Equivalent to flow with  $\underline{\lambda=\infty}$  (As all edge capacities  $\leq 24$ )



Considering only  
positive flows

To find the  $q$ -path decomposition. We use Kishimoto's work



Satisfies

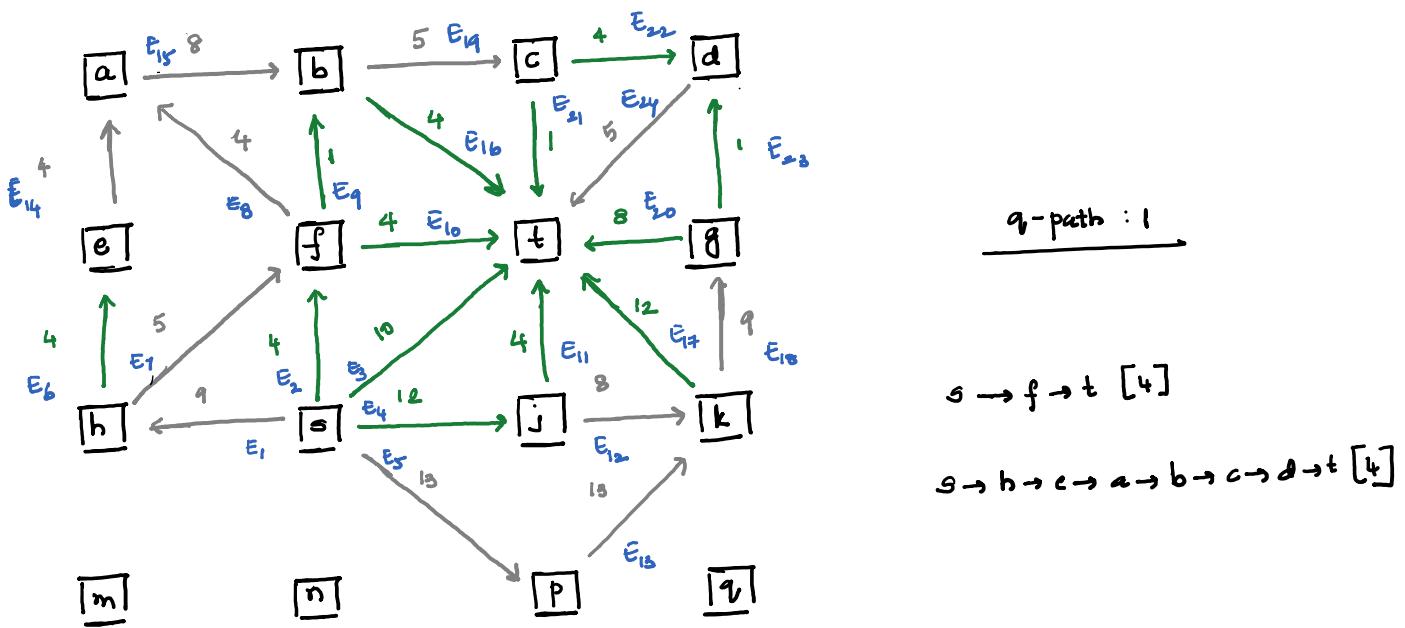
$$\forall (i, j) \in E \quad 0 \leq f_{i,j} \leq u_{i,j}$$

$$f_{i,j} \leq F/2 \leq 24$$

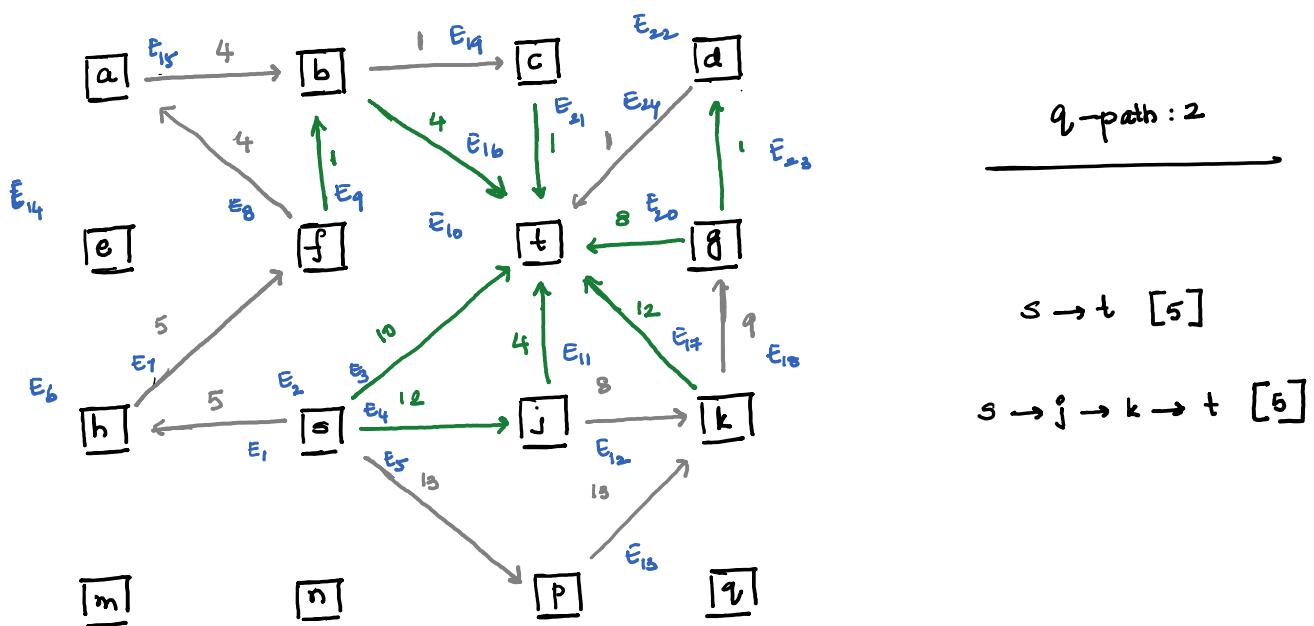
It has 24 edges that carry positive flows and  $\underline{q}=2$

So, we create  $26 \times 26$  table with 24 rows corresponding to edges

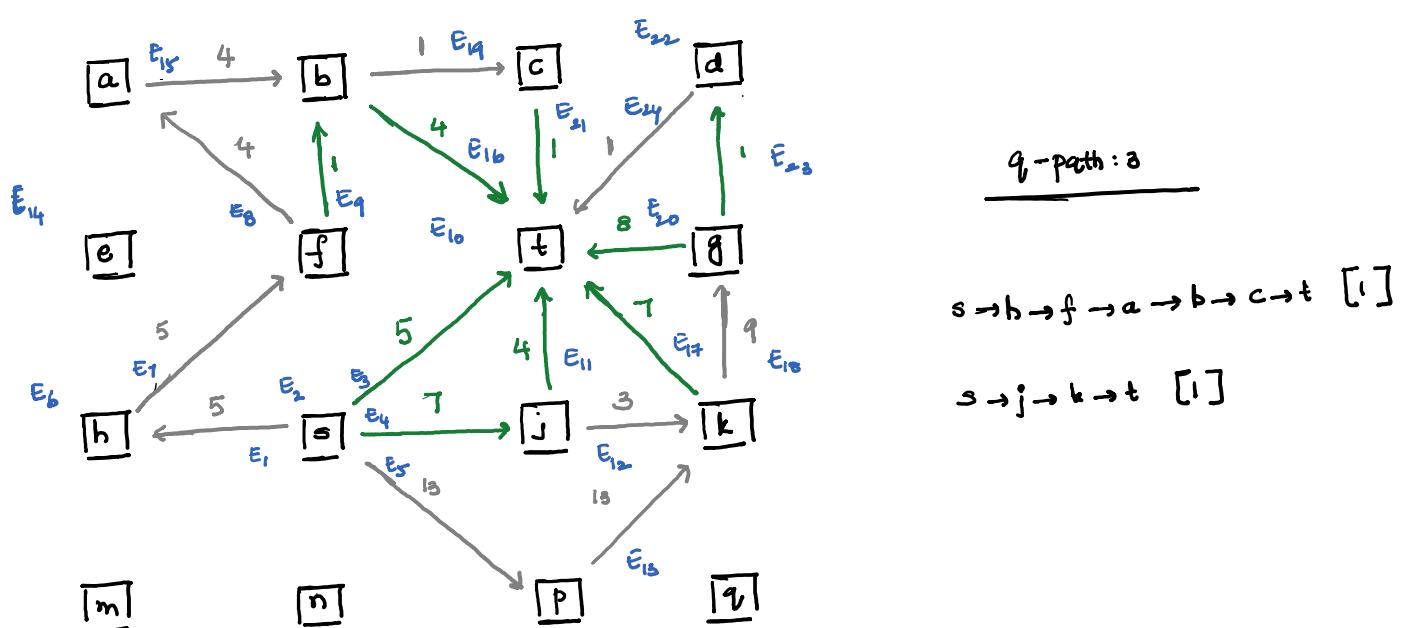
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2	a1	9		5	6	4																						24	
3	a2		4	5	6	9																						24	
4	U1	15					4	5																				24	
5	U2	20									4																	24	
6	U3	14			12							4	8													10		24	
7	U4			11										13														24	
8	U5				20									4														24	
9	U6					20									4													24	
10	U7					19	4	1																				24	
11	U8					20										4												24	
12	U9					23											1											24	
13	U10						20																				4	24	
14	U11							20																			4	24	
15	U12								16																			24	
16	U13								11																			24	
17	U14									20	4																	24	
18	U15										16	3																24	
19	U16											20															4	24	
20	U17												12														12	24	
21	U18													15													1	24	
22	U19														19		1	4										24	
23	U20															16											8	24	
24	U21																23										1	24	
25	U22																	20									4	24	
26	U23																	23	1									24	
27	U24																		19	1	4							24	
28		24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24	24		
29																													



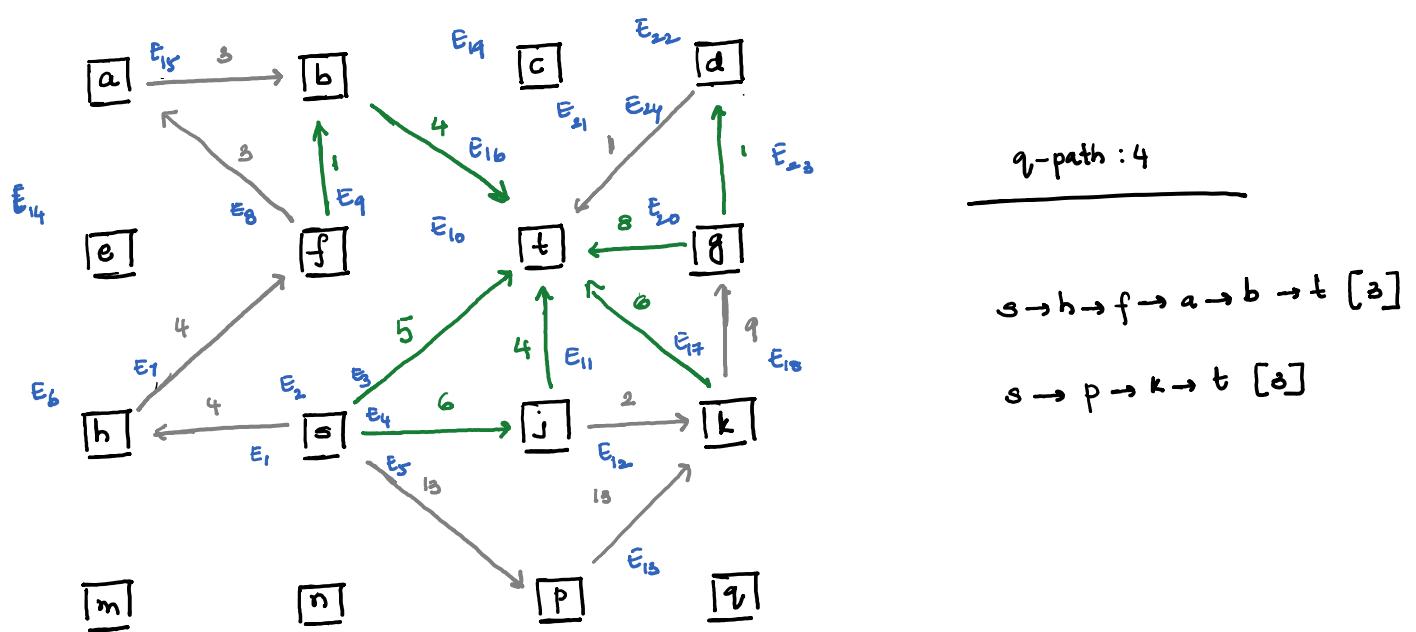
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2	a1	5		5	6	4																					20	20					
3	a2			5	6	9																					20	20					
4	U1	15						5																			20	20					
5	U2		20																								20	20					
6	U3			10																							10	20					
7	U4				8														4	8							20	20					
8	U5					7														13							20	20					
9	U6						20																				20	20					
10	U7							15	4	1																	20	20					
11	U8							16													4						20	20					
12	U9								19													1					20	20					
13	U10									20																	20	20					
14	U11										16																4	20					
15	U12											12															20	20					
16	U13												7									4	9				20	20					
17	U14													20													20	20					
18	U15														16	3											20	20					
19	U16															16											4	20					
20	U17																8											12	20				
21	U18																	11		8									20				
22	U19																	19		1									20				
23	U20																		12										8	20			
24	U21																		19										1	20			
25	U22																			20										20			
26	U23																				19	1									20		
27	U24																				19	1									20		
28		20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20	20					
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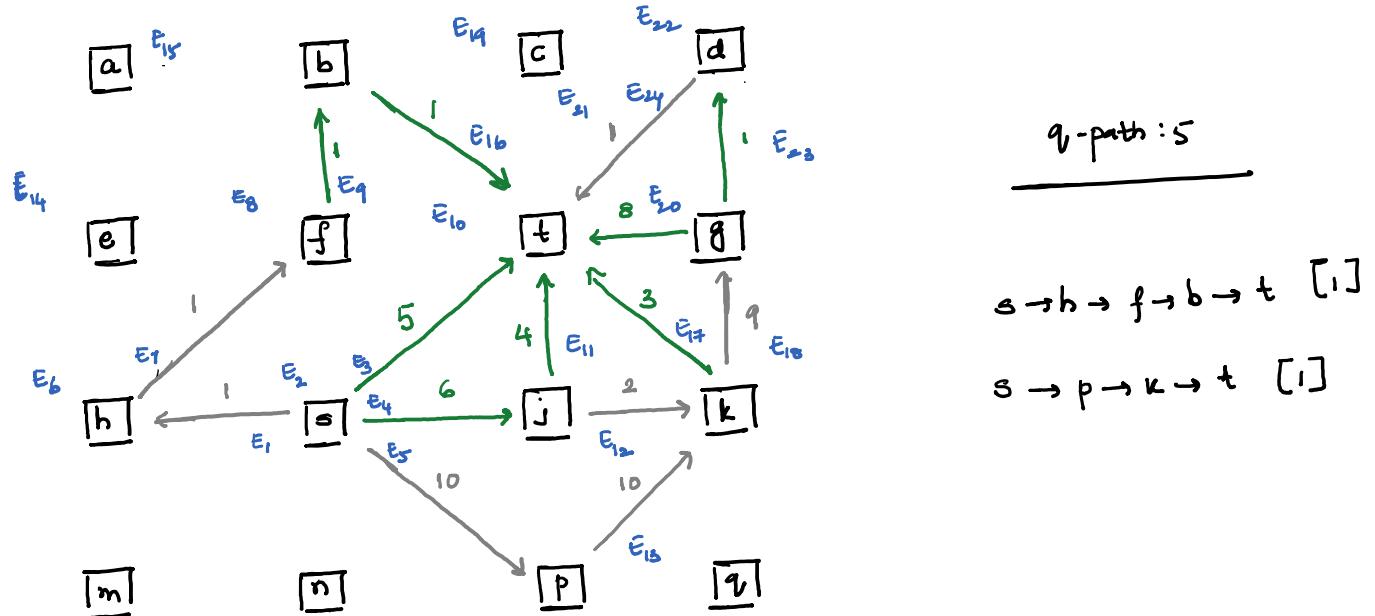
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2	a1	5			6	4																						15	
3	a2		5	1	9																							15	
4	U1	10					5																					15	
5	U2	15																										15	
6	U3		10																								5	15	
7	U4		8										4	3														15	
8	U5			2										13														15	
9	U6			15																								15	
10	U7				10	4	1																					15	
11	U8					11														4								15	
12	U9					14														1								15	
13	U10						15																					15	
14	U11							11																				15	
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18	U15											11	3							1								15	
19	U16												11														4	15	
20	U17													8														7	15
21	U18																		6	8									15
22	U19																		14	1									15
23	U20																		7									8	15
24	U21																		14									1	15
25	U22																		15										15
26	U23																		14	1									15
27	U24																		14	1									15
28		15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15	15		
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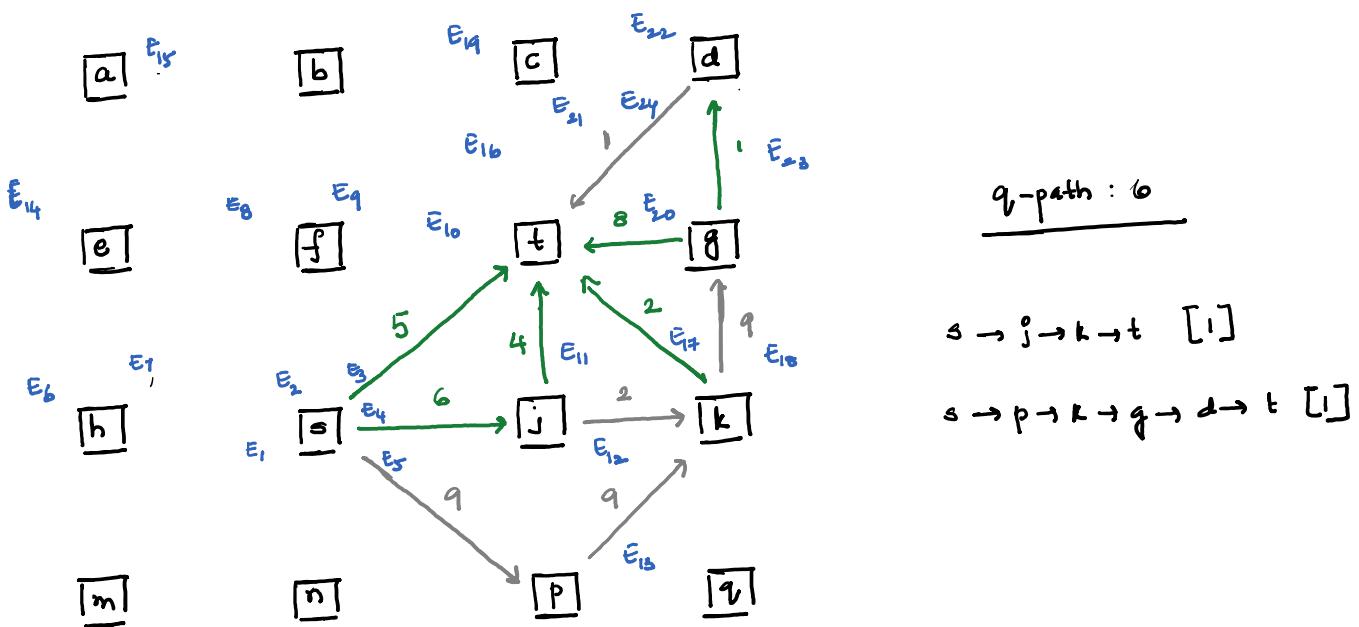
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2	a1	4			6	4																					14		
3	a2		5		9																						14		
4	U1	10						4																			14		
5	U2		14																								14		
6	U3		9																								5		
7	U4		8								4	2															14		
8	U5		1											13													14		
9	U6		14																								14		
10	U7				10	3	1																			14			
11	U8				11												3										14		
12	U9				13												1										14		
13	U10					14																					14		
14	U11						10																				4	14	
15	U12							12									2										14		
16	U13								1							4	9										14		
17	U14									14																	14		
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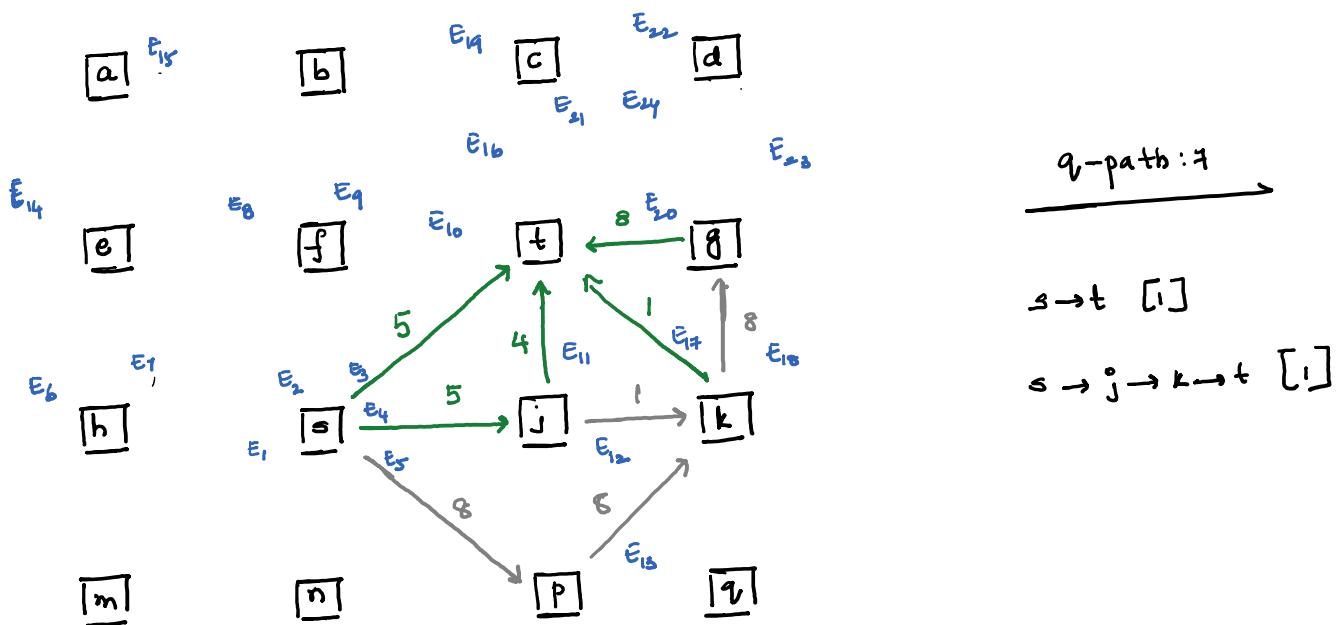
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4	U1	10							1																			11	
5	U2	11																										11	
6	U3	6																										5	
7	U4		5							4	2																11		
8	U5		1									10															11		
9	U6		11																									11	
10	U7			10		1																					11		
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12	U9				10														1									11	
13	U10					11																						11	
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15	U12							9												2								11	
16	U13								1										1	9								11	
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18	U15										11																	11	
19	U16											10																11	
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21	U18													2														11	
22	U19														11													11	
23	U20															3												8	
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28		11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11		
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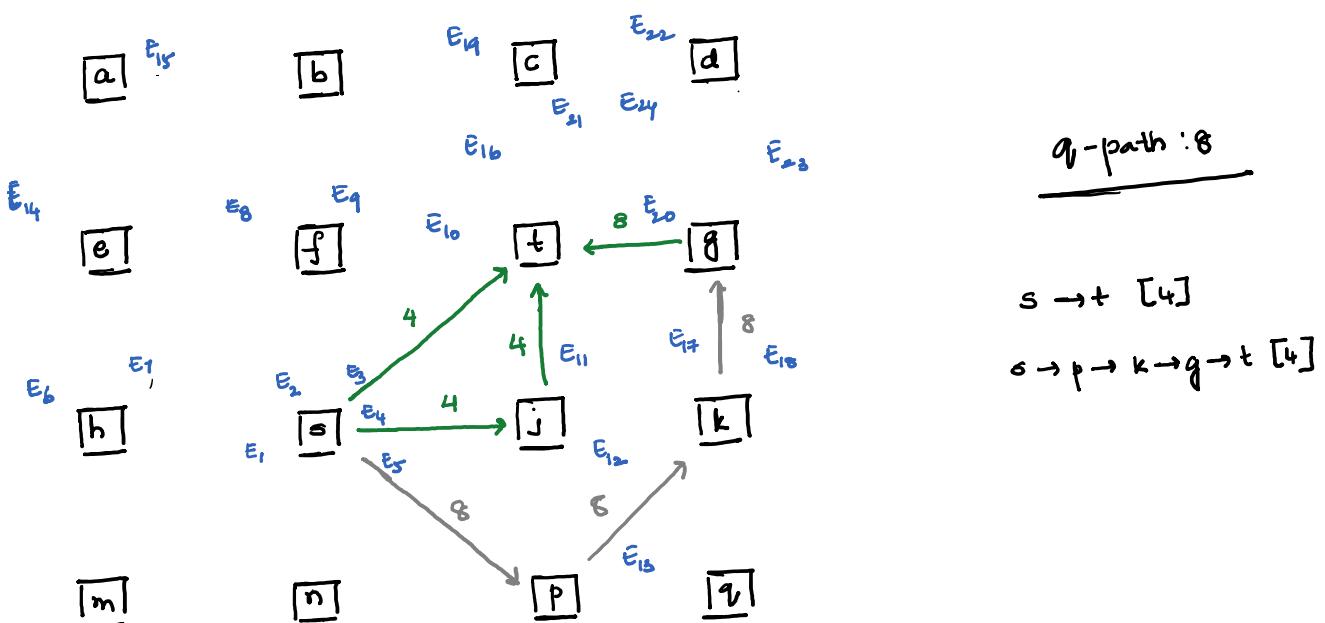


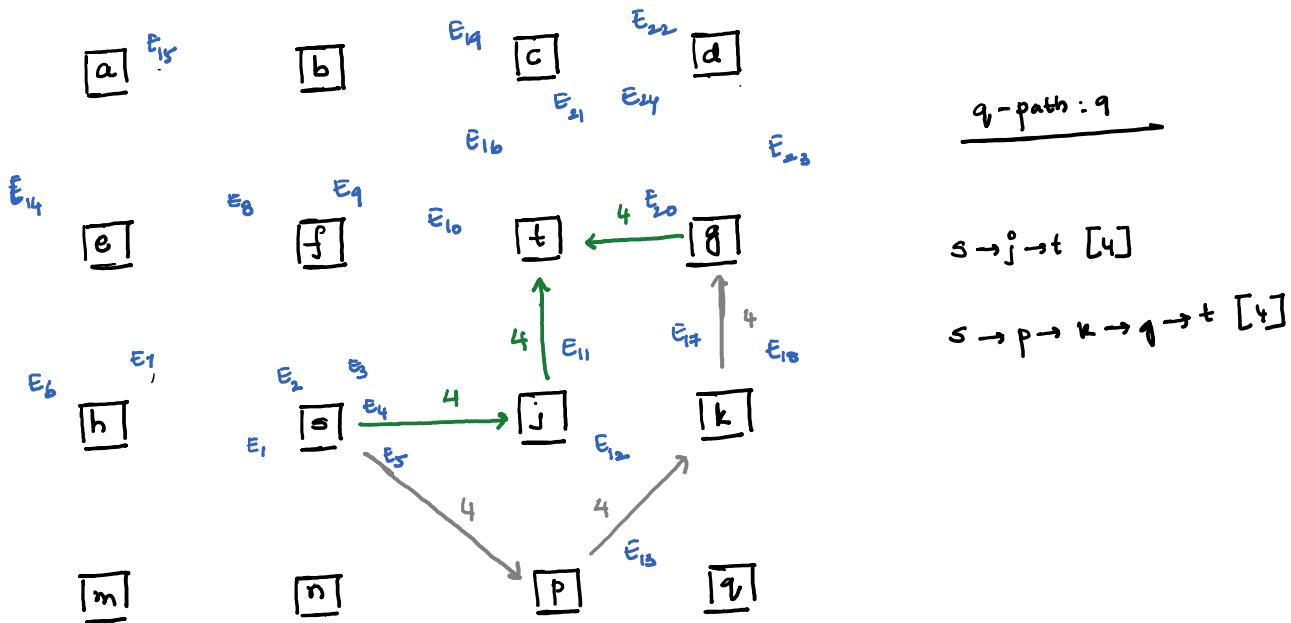
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3	a2			5		5																						10	
4	U1	10																										10	
5	U2		10																									10	
6	U3			5																								5	
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25	U22																						10					10	
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27	U24																								9	1		10	
28		10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10		
29																													



	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	
1		W1	W2	W3	W4	W5	W6	W7	W8	W9	W10	W11	W12	W13	W14	W15	W16	W17	W18	W19	W20	W21	W22	W23	W24	b1	b2			
2	a1				5	4																					9			
3	a2			5		4																					9			
4	U1	9																										9		
5	U2		9																									9		
6	U3		4																									5		
7	U4		4														4	1										9		
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28		9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	9	
29																														







As the slope jump for piece wise linear functions for lambda values of 4, 5, 12, 10 is 1.  
It is implicit that there shall be no change in curve.

But we still verify the max flow associated with the following lambda values :

Following are the flow values obtained in each individual case:

1. Lambda = 4 --> Max flow : 24 ---> Min cut :  $(s, V \setminus \{s\})$  --> Edges {6}
2. Lambda = 5 --> Max flow : 29 ---> Min-cut :  $(s, V \setminus \{s\})$  --> Edges : {5}
3. Lambda = 12 --> Max flow : 48 --> Min cut :  $(V \setminus \{d,t\}, \{d,t\})$

It can be observed that there is no change in the values of the lambda curve.