1) Given Q be a finite set of $(S_1, S_2, S_3 \cdots S_k)$ family of subsets of Q

SDR system of distinct representatives $(q_1, q_2, \cdots q_k)$ of distinct elements of Q

$$q_i \in S_i$$

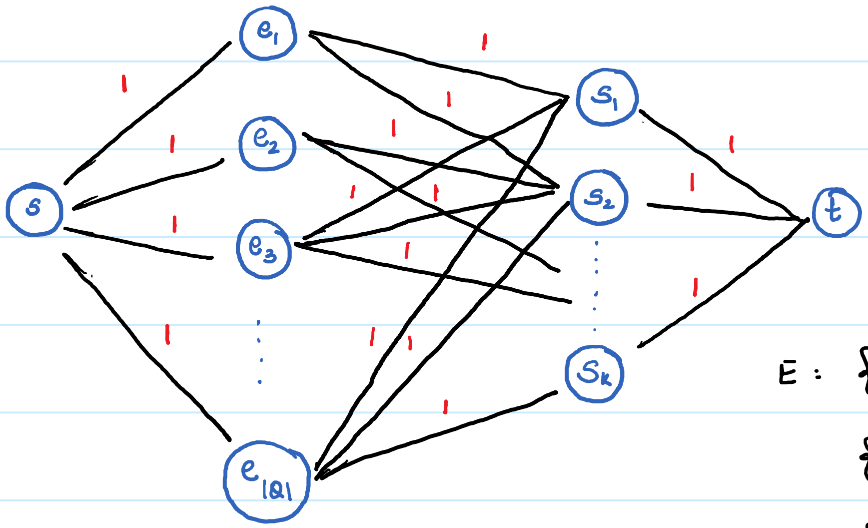
TRIVIAL CASE :

if $k > |Q|$

SDR doesnot exist.

Using MAX FLOW to find one :

We build the following NETWORK. $G(V,E)$, $V = \{s,t\} \cup \{S_i\}$.
$\cup \{ e_i : e_i \in Q \}$



$E : \{ (s, e_i) : e_i \in Q \} \cup$
$\{ (S_j, t) : \forall j < k \} \cup$
$\{ (e_i, S_j) : \forall e_i \in S_j , j < k \}$

Here e_i corresponds to an element in finite set Q.

S_j Corresponds to a subset of Q

Add edges of [CAPACITY = 1] from s to all the vertices e_i

Add edges (e_i, S_j) such that $e_i \in S_j$ with [CAPACITY = 1]

Add edges (S_j, t) with [CAPACITY = 1]

For given collection of subsets $(S_1, S_2, \dots S_k)$ of set Q.

SDR exists only when a network constructed as above for the collection admits a maximum flow of $\boxed{k}$

Any max flow gives us a SDR $(q_1, q_2 \cdots q_k)$.

Now, in order to find $(q_1, q_2 \dots q_k)$

PROCEDURE:

1) Find max flow, if max-flow < k SDR does not exist

2) Apply PATH DECOMPOSITION to find $\underline{\underline{k}}$ paths.

3) In each path, consider the arc (e_i, S_j)

4) Add. $[q_j = e_i]$

[By the nature of graph considered. As the capacity of each arc considered is $\boxed{1}$ There shall be no overlap of arcs in all the k paths]

This procedure gives a REPRESENTATIVE !!

2) Given set of Teams $\mathcal{T}$,

No. of wins so far for team i - w_i,

No. of games scheduled to be played in future : $[r_{ij} = r_{ji}]$

Determine team s is eliminated (or) not.

Following the notation from book Cunningham et al. :

Let, T be $\mathcal{T} \setminus \{s\}$

w_i (Wins for team i) , r_{ij} (Remaining games to be played)

$P = \{ \{i,j\} : \{i,j\} \subseteq T , i \neq j , r_{ij} > 0 \}$.

M → No. of wins for s if they win all their remaining games

$A \subseteq T$

Total no. of wins for teams in A at the end of the season is atleast

$$w(A) + \sum (r_{ij} : \{i,j\} \subseteq A , \{i,j\} \in P)$$

if $> M|A|$

⇒ Avg. no. of wins of teams in A at the end of season is $> M$

As M is the most no. of wins for team $\boxed{s}$

So, atleast one team in A shall finish with more wins than $\boxed{s}$

⇒ s is eliminated if $\exists\, A \subseteq T$ s.t

$$w(A) + \sum (r_{ij} : \{i,j\} \subseteq A , \{i,j\} \in P) > M|A|$$

Considering $\boxed{S}$ is not eliminated

For all the other games $\exists\ y_{ij}, y_{ji}$ s.t $y_{ij} + y_{ji} = r_{ij}$

where $y_{ij} \rightarrow$ No. of wins for <u>team i</u> over <u>team j</u>

So, following should be satisfied.

$$y_{ij} + y_{ji} = r_{ij} \quad , \forall \{i,j\} \in P$$

$$w_i + \sum' (y_{ij} : j \in T,\ j \neq i) \leq M \quad \forall i \in T$$

$$y_{ij} \geq 0 \quad \forall \{i,j\} \in P$$

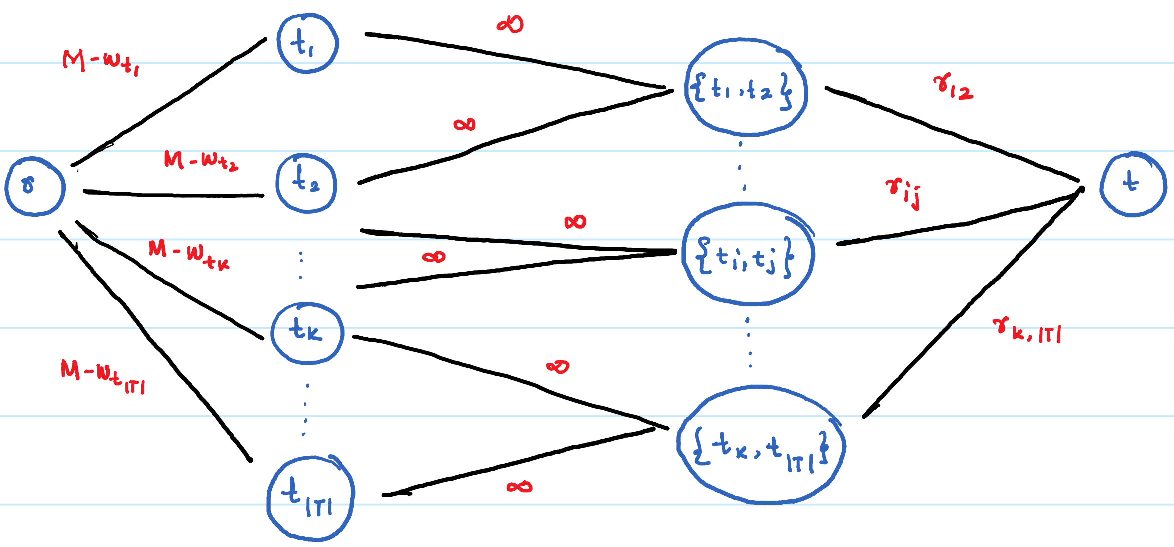
$$y_{ij} \text{ integral } \forall \{i,j\} \in P.$$

Create a flow network $G = (V,E)$, $V = T \cup P \cup \{r,t\}$

$\forall i \in T$ $\exists$ arc (r,i) with capacity $M - w_i$

$\forall i \in T, j \in T$ there are arcs $\{i, \{i,j\}\}$, $\{j, \{j,i\}\}$ with capacity ∞

and there is an arc $(\{i,j\}, t)$ with capacity r_{ij}



We can determine if s is eliminated (or) not solving a MAXIMUM FLOW problem.

1) Not eliminated : Maximum flow will determine set of outcomes for games in which $\boxed{s}$ finish first.

2) Eliminated : Min cut determines a set A satisfying

$$w(A) + \sum (r_{ij} : \{i,j\} \subseteq A,\ \{i,j\} \in P) > M|A|$$

Consider $\delta(C)$ be min (r,t)-cut

By Max-flow min cut : $\delta(C) \leq \sum (r_{ij} : \{i,j\} \in P)$

Let $A = T \setminus C$

Claim: $C = \{r\} \cup (T \setminus A) \cup \{\{i,j\} \in P : i \text{ (or) } j \notin A\}$.

1) if i (or) j is not in A but $\{i,j\} \notin C$ then $\delta(C) = \infty$

2) if $\{i,j\} \in C$ and $i,j \in A$ then deleting $\{i,j\}$ from C decreases capacity of $\delta(C)$ by r_{ij}.

$\Rightarrow$ In either case $\delta(C)$ is not a min-cut [CONTRADICTION !!]

$$\therefore \delta(C) = M|A| - w(A) + \sum (r_{ij} : \{i,j\} \in P,\ \{i,j\} \nsubseteq A)$$

5) Let $G = (V, E)$ be an undirected graph, $\{u_{i,j}\}$; $(i,j) \in E$ Non negative capacities

Suppose $u_{x,y} > \sum_{j \neq y} u_{x,j}$ then (x,y) is an edge in Gomory-Hu cut tree

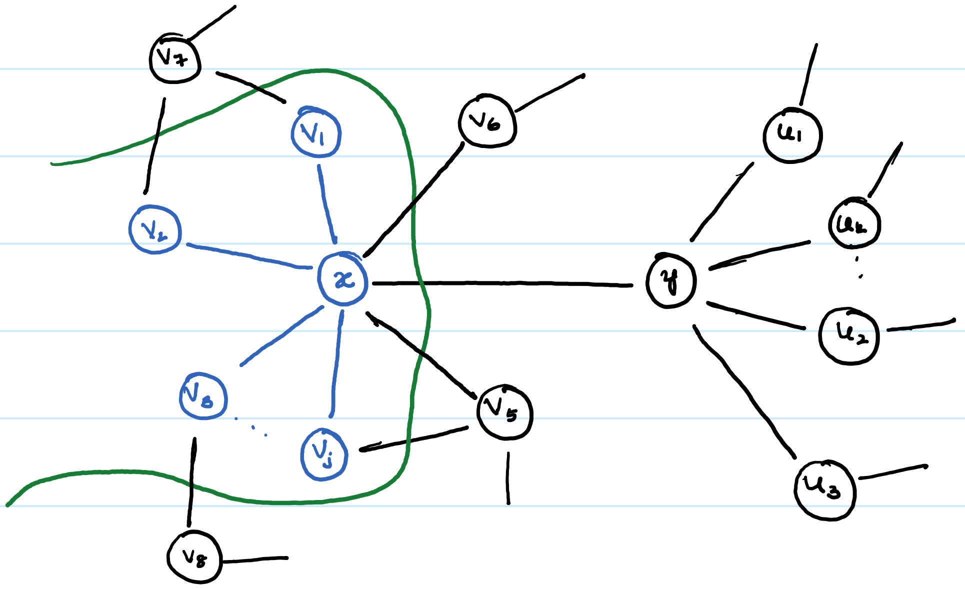
(I)

First consider the arc (x,y) in $G(V,E)$

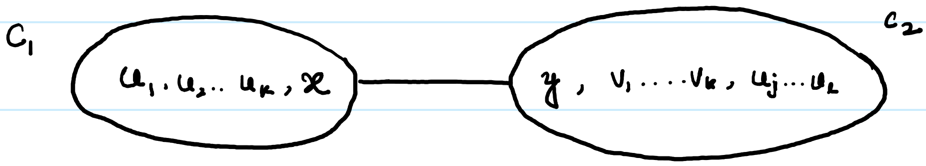
Now, let us begin the development of CUT TREE from $s = x$, $t = y$.

Finding max flow between terminals x and y gives us a MIN-CUT.

that helps in forming first set of nodes/condensed nodes in graph.



Let this min cut separate x, y nodes as follows.



[All u_i are reachable from x in G]

i.e from (I), All the nodes reachable from x in G are part of condensed nodes C_1 & C_2

Claim : As we proceed further in development of cut tree

The arc between $[C_1, C_2]$ develops into arc between $[x, y]$

i.e $\nexists$ node A that shall be between $[x, y]$ in cut tree

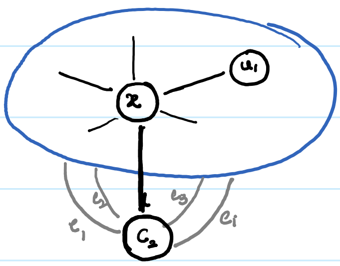
PROOF : PART-1 : Considering condensed node C_1

let us consider $s = x$, $t = u_i$ for some i s.t $u_i \in C_1$

if the condensed node C_2 lies to the side of x in the (x, u_i) min cut

we can say that in this step

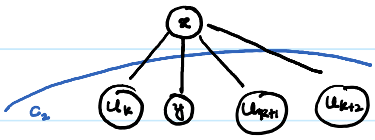
$\nexists$ node A that lies between x & C_2 in cut tree



Consider the graph

Let e_i be arcs between C_2 and nodes in C_1

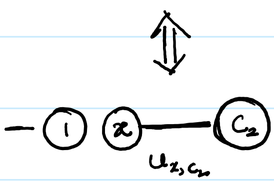
We know $u_{x,y} > \sum_{j \neq y} u_{x,j}$



Now, consider arc $[x, C_2]$

Capacity of this arc in condensed graph

$$u_{x,C_2} = u_{x,y} + \sum_j (u_{x,j} : j \in C_2) \geq u_{x,y} \quad -(1)$$



In development of cut tree for C_1

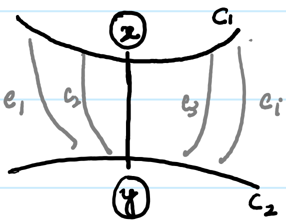
for arc $[x, C_2]$ to be saturated , $f_{x,C_2} = u_{x,C_2}$

At condensed node c_2, by flow conservation

$$u_{x,y} \leq u_{x,c_2} = f_{x,c_2} \leq \sum_1^{l} f_{e_i} \leq \sum_1^{l} u_{e_i} \longrightarrow (2)$$

$$\sum_1^{l} u_{e_i} = c\,[c_1, c_2] - u_{x,c_2} \longrightarrow (3)$$

[Cut capacity of nodes in c_1, c_2]



As $c\,[c_1, c_2]$ is the min cut for (x,y)

$c\,[c_1, c_2] \leq$ Any cut (x,y)

Consider cut $(x, V \setminus \{x\})$

$$c\,[c_1, c_2] \leq u_{x,y} + \sum_{j \neq y} u_{x,j} \longrightarrow (4)$$

Using (4) in (3) we get:
$$\sum_1^{l} u_{e_i} = \left(u_{xy} + \sum_{j \neq y} u_{x,j}\right) - u_{x,c_2} \longrightarrow (5)$$

From (1) we know $u_{x,c_2} \geq u_{x,y}$

(5) can be written as $\sum_1^{l} u_{e_i} \leq \sum_{j \neq y} u_{x,j} \longrightarrow (6)$

Using (6) in (2) gives

[CONTRADICTION by (I)!!]

$$u_{x,y} \leq u_{x,c_2} = f_{x,c_2} \leq \sum_1^{l} f_{e_i} \leq \sum_1^{l} u_{e_i} \leq \sum_{j \neq y} u_{x,j}$$

$f_{x,c_2} \neq u_{x,c_2} \Rightarrow f_{x,c_2} < u_{x,c_2}$

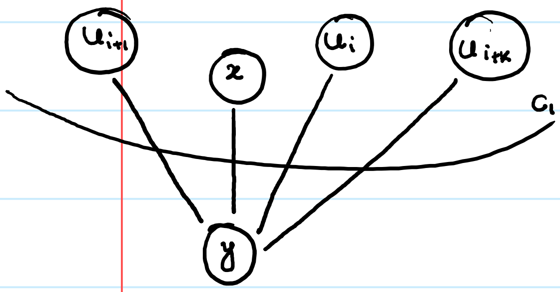
$\Rightarrow$ C_2 is always reachable from x in any min cut of (x, u_i) s.t. $u_i \in C_1$

PART-2 : Now consider the development of condensed node C_2 by keeping C_1

If we can say that C_1 is always reachable from y in any min cut (y, v_i) s.t $v_i \in C_2$ then

$\nexists$ node A which shall be between $[y, c_1]$ in cut-tree..

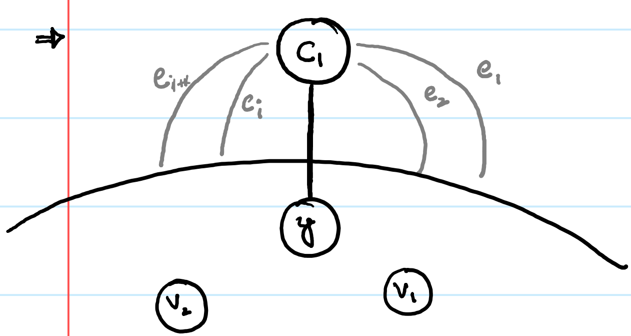
In the condensed graph with nodes $\longrightarrow$ $C_1 \cup \{u_i : u_i \in C_2\}$



$$\Rightarrow u_{c_1, y} = u_{x,y} + \sum (u_{j,y} : j \in C_1)$$

$$\Rightarrow u_{c_1, y} \geq u_{x,y} \longrightarrow (1)$$

$\Rightarrow$



Constructing similar argument as PART-1

For arc (y, c_1) to be saturated.

$$u_{y, c_1} = f_{y, c_1} \rightarrow (2)$$

By flow conservation, @ c_1 $\quad f_{y,c_1} \leq \sum f_{e_i}$

$$u_{x,y} \leq u_{y,c_1} = f_{y,c_1} \leq \sum f_{e_i} \leq \sum u_{e_i} \rightarrow (3)$$

$$\sum u_{e_i} = c[c_1, c_2] - u_{y,c_1} \longrightarrow (4)$$

$$c[c_1, c_2] \leq u_{x,y} + \sum_{j \neq y} u_{x,j} \longrightarrow (5)$$

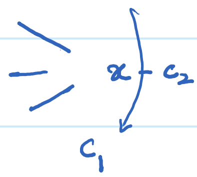
$$\sum_{i=1} u_{e_i} = \left(u_{x,y} + \sum_{j \neq y} u_{x,j} \right) - u_{y,c_1}$$

$$\leq \sum_{j \neq y} u_{x,j} \longrightarrow (5)$$

From (5), (3) we have $u_{x,y} \leq \sum_{j \neq y} u_{x,j}$ [CONTRADICTION!!]

C_1 shall always be reachable from y (For any min cut (y, v_i) s.t. $v_i \in C_2$)

From PART-1 :



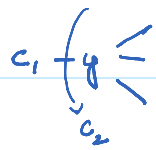
Tree for $C_1 \rightarrow$ has arc $[x - c_2]$

$c_1 - c_2$

Original cut tree has $[c_1 - c_2]$

From PART-2 :

Tree for $C_2 \rightarrow$ has arc $[y - c_1]$

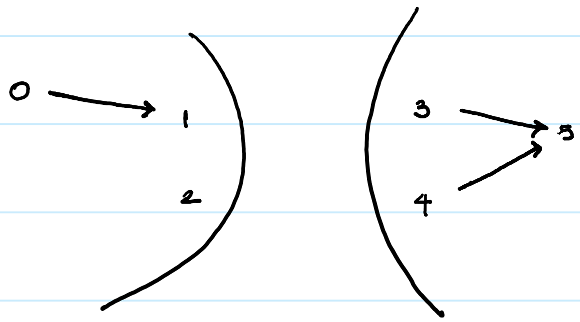


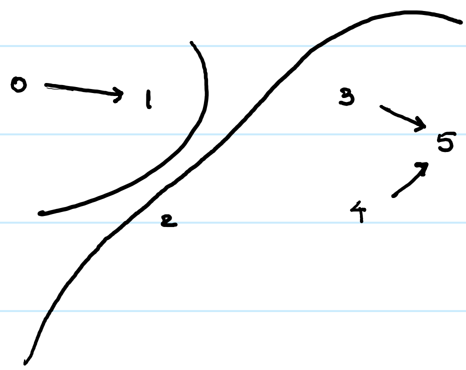
So, we can conclude arc $[x-y]$ is an edge in Gomory Hu Cut tree

(a) Arcs $(0,1)$, $(3,5)$, $(4,5)$ are not part of any $(0,5)$ min-cut.

So, for $F^*_{0,5}$ possible choices for min-cut

1) $(\{0,1,2\}, \{3,4,5\})$ 2) $(\{0,1\}, \{2,3,4,5\})$





All paths through $\boxed{2}$ to 3 (or) 4 originate from $\boxed{1}$ $\Rightarrow$ They shall be part of $F^*_{1,3}$ (or) $F^*_{1,4}$ (or) both

$$\Rightarrow \quad F^*_{0,5} \leq F^*_{1,3} + F^*_{1,4}$$

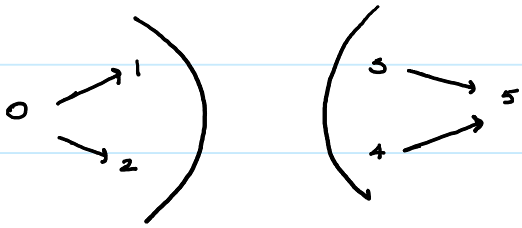
We know $F^*_{1,3}$, $F^*_{1,4}$ both are feasible flows for $F^*_{0,5}$

$$\Rightarrow \quad F^*_{0,5} \geq F^*_{1,3}, \quad F^*_{0,5} \geq F^*_{1,4}$$

$$\Rightarrow \quad F^*_{0,5} \in \left[\max\left(F^*_{1,3}, F^*_{1,4}\right), F^*_{1,3} + F^*_{1,4}\right]$$

(b) For this graph, we know arcs $(0,1)$, $(0,2)$, $(3,5)$, $(4,5)$ cannot be part of any $(0,5)$ min cut

So, nodes $\{0,1,2\}$ shall always be on one side & $\{3,4,5\}$ on the other.



$F^*_{0,5} \geq F^*_{1,3}$, $F^*_{0,5} \geq F^*_{1,4}$

$F^*_{0,5} \geq F^*_{2,3}$, $F^*_{0,5} \geq F^*_{2,4}$

We know $F^*_{1,3} \geq u_{1,3}$, $F^*_{1,4} \geq u_{1,4}$, $F^*_{2,3} \geq u_{2,3}$, $F^*_{2,4} \geq u_{2,4}$

Capacity of this cut $(\{0,1,2\}, \{3,4,5\})$

$$\leq u_{1,3} + u_{1,4} + u_{2,3} + u_{2,4}$$

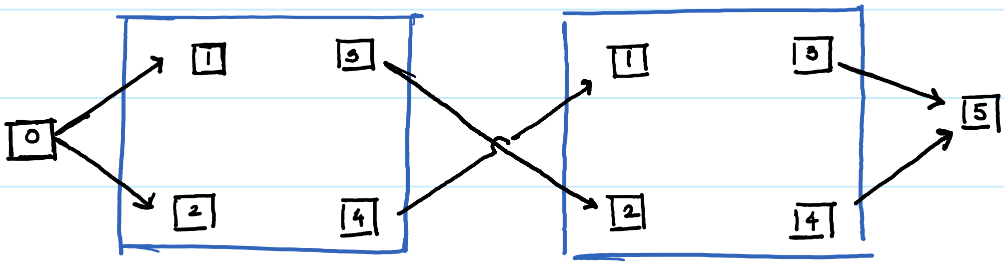
$$\leq F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4}$$

Since $F^*_{1,3}$, $F^*_{1,4}$, $F^*_{2,3}$, $F^*_{2,4}$ all form feasible flows for $F_{0,5}$

$$F^*_{0,5} \geq \max(F^*_{1,3}, F^*_{1,4}, F^*_{2,3}, F^*_{2,4})$$

$$F^*_{0,5} \in \left[\max(F^*_{1,3}, F^*_{1,4}, F^*_{2,3}, F^*_{2,4}) , F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4}\right]$$

(c)



Previous bounds are still valid for this problem i.e

$$\max\left(F^*_{1,3},\ F^*_{1,4},\ F^*_{2,3},\ F^*_{2,4}\right) \le F^*_{0,5} \le \underbrace{F^*_{1,3} + F^*_{1,4} + F^*_{2,3} + F^*_{2,4}}$$

Checking for more tighter bounds :