CS 7301 ADVANCED OPTIMIZATION IN ML - ASSIGNMENT 2

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(10 points total) In this assignment, you will prove the submodularity (or non-submodularity) of the following five functions. Denote $V = 1, , n$ as a ground set, $X \subseteq V$ as a subset and $f : 2^V \implies R$ as a set function.

PROPERTIES

A function $f : 2^V \implies R$ is submodular if for every $A \subseteq B \subseteq V$ and $e \epsilon V \setminus B$ it holds that

$$f(A \cup e) - f(A) \geq f(B \cup e) - f(B) \tag{1}$$

1.1

(2 Points) Modular Function $m(X) = \sum_{j \epsilon X} m(j)$

SOLUTION

From (1) for sets A, B such that $A \subseteq B \subseteq V$.

$$\begin{aligned} f(A \cup e) - f(A) &= \sum_{j \epsilon A \cup e} m(j) - \sum_{j \epsilon A} m(j) \\ &= \sum_{j \epsilon A} m(j) + m(e) - \sum_{j \epsilon A} m(j) \\ &= m(e) \end{aligned}$$

$$
\begin{aligned}
f(B \cup e) - f(B) &= \sum_{j \epsilon B \cup e} m(j) - \sum_{j \epsilon B} m(j) \\
&= \sum_{j \epsilon B} m(j) + m(e) - \sum_{j \epsilon B} m(j) \\
&= m(e)
\end{aligned}
$$

As $f(A \cup e) - f(A) \geq f(B \cup e) - f(B) \implies m(e) \geq m(e)$ (TRUE)

$f(S)$ satisfies the property of submodularity.

1.2

(2 Points) Set Cover Function $f(X) = w(\Gamma(X))$
where $\Gamma(i)$ is the set of items covered by item i, and $\Gamma(X) = sup_{i \epsilon X} \Gamma(i)$.

<u>SOLUTION</u>

From (1) for sets A, B such that $A \subseteq B \subseteq V$.

$$\begin{aligned}
f(A \cup e) - f(A) &= w(\Gamma(A \cup e)) - w(\Gamma(A)) \\
&= w(sup_{i \epsilon A \cup e} \Gamma(i)) - w(sup_{i \epsilon A} \Gamma(i)) \\
&= \sum_{i \epsilon \Gamma(A \cup e)} w(i) - \sum_{i \epsilon \Gamma(A)} w(i) \\
&= \sum_{i \epsilon \Gamma(e) \setminus \Gamma(A)} w(i) + \sum_{i \epsilon \Gamma(A)} w(i) - \sum_{i \epsilon \Gamma(A)} w(i) \\
&= \sum_{i \epsilon \Gamma(e) \setminus \Gamma(A)} w(i)
\end{aligned}$$

$$\begin{aligned}
f(B \cup e) - f(B) &= w(\Gamma(B \cup e)) - w(\Gamma(B)) \\
&= w(sup_{i \epsilon B \cup e} \Gamma(i)) - w(sup_{i \epsilon B} \Gamma(i)) \\
&= \sum_{i \epsilon \Gamma(B \cup e)} w(i) - \sum_{i \epsilon \Gamma(B)} w(i) \\
&= \sum_{i \epsilon \Gamma(e) \setminus \Gamma(B)} w(i) + \sum_{i \epsilon \Gamma(B)} w(i) - \sum_{i \epsilon \Gamma(B)} w(i) \\
&= \sum_{i \epsilon \Gamma(e) \setminus \Gamma(B)} w(i)
\end{aligned}$$

As $A \subseteq B \subseteq V$ and $|\Gamma(e) \setminus \Gamma(A)| \geq |\Gamma(e) \setminus \Gamma(b)|$

$$\sum_{i \epsilon \Gamma(e) \setminus \Gamma(A)} w(i) \geq \sum_{i \epsilon \Gamma(e) \setminus \Gamma(B)} w(i)$$

$f(A \cup e) - f(A) \geq f(B \cup e) - f(B)$ Assuming $\forall i, w(i) \geq 0$

Thus set cover function is submodular.

1.3

(2 Points) Facility Location $f(X) = \sum_{i \epsilon V} max_{j \epsilon X} s_{ij}$

SOLUTION

From (1) for sets A, B such that $A \subseteq B \subseteq V$.

Let, $g(X) = \max\limits_{i \epsilon X} w_i$

If we can prove g(X) to be submodular, we can conclusively say Facility Location is submodular as it is the sum of submodular functions $g_j(X)$

Consider, $w_i \epsilon R^+$

$$
\begin{aligned}
g(A \cup e) - g(A) &= \max_{i \epsilon A \cup e} w_i - \max_{i \epsilon A} w_i \\
&= \max(\max_{i \epsilon A} w_i, w(e)) - \max_{i \epsilon A} w_i \\
&= \max(0, w(e) - \max_{i \epsilon A} w_i)
\end{aligned}
$$

$$
\begin{aligned}
g(B \cup e) - g(B) &= \max_{i \epsilon B \cup e} w_i - \max_{i \epsilon B} w_i \\
&= \max(\max_{i \epsilon B} w_i, w(e)) - \max_{i \epsilon B} w_i \\
&= \max(0, w(e) - \max_{i \epsilon B} w_i)
\end{aligned}
$$

It can be clearly seen that

$$\max(0, w(e) - \max_{i \epsilon A} w_i) \geq \max(0, w(e) - \max_{i \epsilon B} w_i) \text{ as } A \subseteq B \subseteq V$$

$$\implies g(A \cup e) - g(A) \geq g(B \cup e) - g(B)$$

1. So, $g(X)$ is submodular and hence $f(X)$ is submodular when $s_{ij} \geq 0$.
2. If $s_{ij} \epsilon R$ then submodularity is not guaranteed.

1.4

(2 Points) Dispersion sum $f(X) = \sum_{i,j \epsilon X} d_{ij}$

<u>SOLUTION</u>

From (1) for sets A, B such that $A \subseteq B \subseteq V$.

$$\begin{aligned} f(A \cup e) - f(A) &= \sum_{i,j \epsilon A \cup e} d_{ij} - \sum_{i,j \epsilon A} d_{ij} \\ &= 2 \sum_{i \epsilon A} d_{ie} \end{aligned}$$

$$\begin{aligned} f(B \cup e) - f(B) &= \sum_{i,j \epsilon B \cup e} d_{ij} - \sum_{i,j \epsilon B} d_{ij} \\ &= 2 \sum_{i \epsilon B} d_{ie} \end{aligned}$$

$d_{ij} \geq 0 \implies f(B \cup e) - f(B) \geq f(A \cup e) - f(A) \implies$ violates (1)
So, it is not submodular.

1.5

(2 Points) Feature based function $f(X) = \sum_{i\epsilon I} \psi(m_i(X))$ where ψ is a concave function

SOLUTION

If functions $m_i(X)$, considered in the composition are positive modular then the the function $f(X)$ is submodular.

From (1) for sets A, B such that $A \subseteq B \subseteq V$.

$$
\begin{aligned}
f(A \cup e) - f(A) &= \sum_{i\epsilon I} \psi(m_i(A \cup e)) - \sum_{i\epsilon I} \psi(m_i(A)) \\
&= \sum_{i\epsilon I} \psi(\sum_{j\epsilon A \cup e} mo_i(j)) - \sum_{i\epsilon I} \psi(\sum_{j\epsilon A} mo_i(j)) \\
&\text{Consider} \sum_{j\epsilon A} mo_i(j) \text{ as } X_i \\
&= \sum_{i\epsilon I} \psi(X_i + mo(e)) - \sum_{i\epsilon I} \psi(X_i) \\
&= \sum_{i\epsilon I} \psi(X_i + mo(e)) - \psi(X_i)
\end{aligned}
$$

$$
\begin{aligned}
f(B \cup e) - f(B) &= \sum_{i\epsilon I} \psi(m_i(B \cup e)) - \sum_{i\epsilon I} \psi(m_i(B)) \\
&= \sum_{i\epsilon I} \psi(\sum_{j\epsilon B \cup e} mo_i(j)) - \sum_{i\epsilon I} \psi(\sum_{j\epsilon B} mo_i(j)) \\
&\text{Consider} \sum_{j\epsilon B} mo_i(j) \text{ as } Y_i \\
&= \sum_{i\epsilon I} \psi(Y_i + mo(e)) - \sum_{i\epsilon I} \psi(Y_i) \\
&= \sum_{i\epsilon I} \psi(Y_i + mo(e)) - \psi(Y_i)
\end{aligned}
$$

If ψ is concave and mo_i is positive modular then

$Y_i \geq X_i$ as $A \subseteq B \subseteq V$

$\psi(X_i + mo(e)) - \psi(X_i) \geq \psi(Y_i + mo(e)) - \psi(Y_i)$

$$\sum_{i\epsilon I} \psi(X_i + mo(e)) - \psi(X_i) \geq \sum_{i\epsilon I} \psi(Y_i + mo(e)) - \psi(Y_i)$$

$f(A \cup e) - f(A) \geq f(B \cup e) - f(B)$

Thus feature based function is submodular.

2 CODE

Google colab link : Assignment 2